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I. MATHEMATICA

DISSERTATIONES

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HOMOGENEOUS AND CONFORMALLY INVARIANT VARIATIONAL INTEGRALS

TERO KILPELÄINEN

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TERO KILPELÄINEN

Contents

1. Introduction

We consider variational integrals of the form

$$I(u) = \int_G F(x,\nabla u(x))\, dx ,$$

where G is an open set in $\mathbb{R}^n$, u is in Sobolev space $W^1_p(G)$, $1 \leq p < \infty$ and $F(x,h) \approx |h|^p$. Usually the kernels F are assumed to satisfy certain natural conditions of measurability, convexity and growth, see e.g. [GLM, M, R3]. In Section 3 we shall determine the structure of the kernels which are assumed to be homogeneous in addition. The fundamental example $|h|^p$ of a kernel of this type turns out to be typical, i.e., $h \mapsto F(x,h)^{1/p}$ is always a norm. This fact will be used in Section 4, where we introduce a new norm in $L^p(G)^n$,

$$\|u\|_F = \left(\int_G F(x,u(x))\, dx \right)^{1/p} ,$$

called an F-norm. The concept of an F-norm leads to a simple proof for the lower semicontinuity of the variational integral I with respect to weak convergence, a fundamental tool in Calculus of Variations introduced already by L. Tonelli [T], see also [L, M, S, R1, R3]. In weak topology the lower semicontinuity of a norm is an elementary fact and it gives a proof for the lower semicontinuity of a variational integral in our case. At the end of Section 4 we shall characterize the kernels F whose induced norm makes $L^p(G)^n$ uniformly convex.

The rest of this paper deals with the conformal invariance of variational integrals. Section 5 paves the way for the general case in Section 6. It is well-known that the conformal capacity is conformally invariant. We show in Section 5 that its natural generalization, the F-capacity, is not necessarily conformally invariant.

In the last section we study the conformal invariance of the variational integral

$$I(u) = \int_G F(x,u(x),\nabla u(x))\, dx ,$$

where the kernel F satisfies only the familiar measurability conditions of Carathéodory and the natural growth restrictions. The well-known example is the n-Dirichlet integral

$$\int_G |\nabla u|^n\, dm .$$

We will show that it is a good prototype for all conformally invariant variational integrals, since every such integral in $\mathbb{R}^n$ is of the form

$$I(u) = \int_G k(x,u(x))\, |\nabla u(x)|^n\, dx .$$

M. Grüter [Gr] has obtained the same result in the plane case under additional assumptions on the regularity of F. He also makes use of the fact that F does not depend on x. Our proof uses the norm characterization of the kernels (Section 3) as a model and is based on the fact that only the euclidean balls remain invariant under all orthogonal maps.

2. Preliminaries

2.1. *Notation.* For each set A in the euclidean space $\mathbb{R}^n$, $n \geq 2$, we let $\bar{A}$ and ∂A denote the closure and boundary of A, both taken with respect to $\mathbb{R}^n$. Given two sets $A \subset B$ in $\mathbb{R}^n$, $A \subset\subset B$ means that $\bar{A}$ is compact in B. By $x \cdot y$ we denote the usual inner product of two vectors x and y in $\mathbb{R}^n$, the euclidean norm of x is $|x| = (x \cdot x)^{1/2}$. For $x \in \mathbb{R}^n$ we use representations $x = (x_1,x_2,\ldots,x_n) = \Sigma_{i=1}^n x_i e_i$. If $x \in \mathbb{R}^n$ and $r > 0$, we let $B(x,r)$ denote the open ball $\{y \in \mathbb{R}^n \mid |y - x| < r\}$, and $S^{n-1}(x,r)$ is the sphere $\partial B(x,r)$. Furthermore, we let $Q(x,r)$ denote the open cube $\{y \in \mathbb{R}^n \mid |y_i - x_i| < r,\ i = 1,2,\ldots,n\}$. We shall employ the abbreviations $B(r) = B(0,r)$, $S^{n-1}(r) = S^{n-1}(0,r)$, $S^{n-1} = S^{n-1}(1)$, $Q(r) = Q(0,r)$ and $Q_{n-1}(r) = Q(r) \cap \{x \in \mathbb{R}^n \mid x_1 = 0\}$.

We let $GL(\mathbb{R}^n,\mathbb{R}^n)$ denote the space of all regular linear maps $A\colon \mathbb{R}^n \to \mathbb{R}^n$ with the sup-norm.

The Lebesgue measure of a set $A \subset \mathbb{R}^n$ will be written as $m_n(A) = m(A)$.

If $A \subset \mathbb{R}^n$, then $C(A)$ is the class of all continuous (real valued) functions on A. If U is an open set in $\mathbb{R}^n$, we let $C^k(U)$ denote the family of all k times continuously differentiable functions $u: U \to \mathbb{R}$, and $C_0^k(U)$ the family of all $u \in C^k(U)$ whose support spt u is a compact subset of U.

2.2. *Sobolev space* W_p^1 *and* ACL^p-*functions*. If A is a Lebesgue measurable subset of $\mathbb{R}^n$, then $L^p(A)$, $p \geq 1$, is the Banach space of all measurable functions $u: A \to \dot{\mathbb{R}} = \mathbb{R} \cup \{-\infty,\infty\}$ with the norm

$$\|u\|_p = \|u\|_{p,A} = \left(\int_A |u|^p \, dm \right)^{1/p} .$$

Given an open set G in $\mathbb{R}^n$, $W_p^1(G)$ is the Sobolev space of functions $u \in L^p(G)$ whose first distributional partial derivatives $D_i u$ belong to $L^p(G)$ with the norm

$$\|u\|_{1,p} = \|u\|_p + \sum_{i=1}^{n} \|D_i u\|_p .$$

We let $\nabla u = (D_1 u, D_2 u, \ldots, D_n u)$ denote the gradient of u. The space $W_{p,0}^1(G)$ is the closure of $C_0^\infty(G)$ in $W_p^1(G)$.

A mapping $f: G \to \mathbb{R}^m$ is said to be ACL if f is continuous and if for each open n-interval $Q \subset\subset G$, f is absolutely continuous on almost every line segment in $\overline{Q}$ parallel to the coordinate axes. An ACL-mapping has partial derivatives a.e. If these are locally L^p-integrable, $p \geq 1$, f is said to be ACL^p. It is well-known that f is ACL^p if and only if f is continuous and belongs to $W_p^1(D)$ for each open set $D \subset\subset G$. For basic properties of Sobolev spaces and ACL-functions see [M].

2.3. C-*functions*. Let G be an open set in $\mathbb{R}^n$. Functions $F: G \times \mathbb{R}^m \to \mathbb{R}$ satisfying the Carathéodory conditions

$$x \mapsto F(x,h) \text{ is measurable for all } h \in \mathbb{R}^m ,$$

$$h \mapsto F(x,h) \text{ is continuous for a.e. } x \in G ,$$

will be called *Carathéodory-functions*, abbreviated C-*functions*. The following *Scorza-Dragoni property*, see [ET, p. 235], is well-known.

2.4. Lemma. *A function* $F: G \times \mathbb{R}^m \to \mathbb{R}$ *is a* C-*function if and only if for each open set* $D \subset\subset G$ *and* $\varepsilon > 0$ *there is a compact set* $C \subset D$ *such that* $m(D \setminus C) < \varepsilon$ *and* $F|_{C \times \mathbb{R}^m}$ *is continuous.*

2.5. *Weak convergence.* If X is a topological vector space and if X' stands for its dual, then a sequence $x_n \in X$, $n = 1,2,\ldots$, is said to *converge weakly* to $x_0 \in X$, abbreviated $x_n \to x_0$ weakly (in X), if

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

for all $f \in X'$.

In particular, $u_n \to u$ weakly in $L^p(A)$ means that

$$\lim_{n\to\infty} \int_A u_n v \, dm = \int_A u_0 v \, dm$$

for each $v \in L^q(A)$, where $q = p/(p-1)$; if $p = 1$, then $q = \infty$.

The following elementary consequence of Hahn-Banach theorem is well-known, see e.g. [DS, Lemma II.3.27].

2.6. Lemma. *Let* X *be a normed space and* $x_n \in X$, $n = 0;1,2,\ldots$. *If* $x_n \to x_0$ *weakly, then*

$$\|x_0\| \leq \varliminf_{n\to\infty} \|x_n\| .$$

2.7. *Conformal mappings.* Let G and G' be domains in $\mathbb{R}^n$. A homeomorphism $f: G \to G'$ is *conformal* if $f \in C^1(G)$ and if

$$|f'(x)\,h| = |f'(x)|\,|h|$$

for every $x \in G$ and $h \in \mathbb{R}^n$. Here $|A|$ denotes the sup-norm of a linear map A. Alternatively, a C^1-homeomorphism f is conformal if and only if $|f'(x)|^n = |J(x,f)|$ for all $x \in G$, where $J(x,f)$ denotes the Jacobian determinant of f at x. We shall frequently use the following fact: *Suppose that* u *is an* ACL^p*-function in* G' *and that* $f: G \to G'$ *is a* C^1*-mapping. Then* $v = u \circ f$ *is* ACL^p *in* G *and*

$$\nabla v(x) = f'(x)^* \nabla u(f(x))$$

a.e. in G ; see e.g. [GLM, 6.10]. As usual, A^* is the adjoint of a linear map $A: \mathbb{R}^n \to \mathbb{R}^n$.

A function $f: G \to \mathbb{R}^n$ is *a similarity* if $f(x) = \lambda O(x) + h$, $x \in G$, for some orthogonal map $O: \mathbb{R}^n \to \mathbb{R}^n$ and for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $h \in \mathbb{R}^n$.

3. Variational kernels

We are mainly interested in variational integrals

$$I(u) = \int F(x, \nabla u(x))\, dx ,$$

where $F(x,h) \approx |h|^p$. Integrals of this type have been extensively studied, see e.g. [GLM, M, R3]. In this section we shall derive some alternative characterizations for kernels F of the elliptic and homogeneous type.

Suppose that G is an open set in $\mathbb{R}^n$ and that the kernel $F: G \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumptions, cf. [GLM]:

(a) For each open set $D \subset\subset G$ and $\varepsilon > 0$ there is a compact set $C \subset D$ with $m(D \setminus C) < \varepsilon$ and $F|_{C \times \mathbb{R}^n}$ is continuous.

(b) For a.e. $x \in G$ the function $h \to F(x,h)$ is convex.

(c) There are constants $1 \leq p < \infty$ and $0 < \alpha \leq \beta < \infty$ such that for a.e. $x \in G$

$$\alpha |h|^p \leq F(x,h) \leq \beta |h|^p$$

for all $h \in \mathbb{R}^n$.

(d) For a.e. $x \in G$, $F(x,\lambda h) = |\lambda|^p F(x,h)$ for all $\lambda \in \mathbb{R}$ and $h \in \mathbb{R}^n$.

The kernels F can also be characterized as follows.

3.1. Theorem. *A kernel* $F: G \times \mathbb{R}^n \to \mathbb{R}$ *satisfies (a)-(d) if and only if it satisfies the following two conditions:*

(i) *For all* $h \in \mathbb{R}^n$ *the function* $x \mapsto F(x,h)$ *is measurable in* G .

(ii) *There exist a constant* $p \in [1,\infty)$ *and a set* $D \subset G$ *such that* $m(G \setminus D) = 0$ *and*

$$\mathcal{P} = \{ h \mapsto F(x,h)^{1/p} \mid x \in D \}$$

is a uniformly equivalent family of norms in $\mathbb{R}^n$.

A family $\mathcal{P}$ of norms $f: \mathbb{R}^n \to \mathbb{R}$ is called *uniformly equivalent* if there exist constants $0 < \alpha \leq \beta < \infty$ such that $\alpha |h| \leq f(h) \leq \beta |h|$ for all $h \in \mathbb{R}^n$ and $f \in \mathcal{P}$.

Theorem 3.1 is an immediate consequence of the Scorza-Dragoni property (2.4) and the following lemma.

3.2. Lemma. *Suppose that* X *is a vector space and that* $p \in [1,\infty)$. *A non-negative function* $f: X \to \mathbb{R}$ *is a norm if and only if*

(i) f^p *is convex,*

(ii) $f(\lambda x)^p = |\lambda|^p f(x)^p$ *for all* $\lambda \in \mathbb{R}$ *and* $x \in X$,

(iii) $f(x)^p = 0$ *implies* $x = 0$.

Proof. Obviously, a norm $f: X \to \mathbb{R}$ satisfies (i)-(iii). For the converse, it suffices to show the triangle inequality. Fix $x, y \in X$. Set $f(x) = t$ and $f(y) = s$. We may assume that $ts \neq 0$. Write $u = t + s$. Now

$$\frac{x+y}{u} = \frac{t}{u}\left(\frac{x}{t}\right) + \frac{s}{u}\left(\frac{y}{s}\right) ,$$

whence the convexity of f^p together with (ii) yield

$$f\left(\frac{x+y}{u}\right)^p \leq \frac{t}{u} f\left(\frac{x}{t}\right)^p + \frac{s}{u} f\left(\frac{y}{s}\right)^p = \frac{t+s}{u} = 1 .$$

Thus the condition (ii) implies the triangle inequality.

3.3. Remark. A similar reasoning to above gives: *For a fixed* $p \in [1,\infty)$ *a non-negative function* $f: X \to \mathbb{R}$ *is a seminorm if and only* *if the conditions (i) and (ii) of Lemma 3.2 are satisfied.*

A norm $f: X \to \mathbb{R}$ is called *strict* if $f(x + y) = f(x) + f(y)$ implies $y = 0$ or $x = \lambda y$ for some $\lambda \geq 0$.

A kernel function $F: G \times \mathbb{R}^n \to \mathbb{R}$ is said to be *strictly convex* if for a.e. $x \in G$

$$F(x, ty + (1 - t)h) < tF(x,y) + (1 - t)F(x,h)$$

for $t \in (0,1)$ and $y, h \in \mathbb{R}^n$, $y \neq h$.

3.4. Theorem. *Suppose that* $F: G \times \mathbb{R}^n \to \mathbb{R}$ *satisfies (a)-(d) for some* $1 < p < \infty$. *Then* F *is strictly convex if and only if for a.e.* $x \in G$ *the norm* $h \mapsto F(x,h)^{1/p}$ *is strict.*

Proof. The sufficiency easily follows from the strict convexity of $h \mapsto |h|^p$. For the converse, let $f(h) = F(x,h)^{1/p}$ be a norm such that f^p is strictly convex. Fix $h, y \in \mathbb{R}^n$ with $f(h + y) = f(h) + f(y)$. We may assume that $h, y \neq 0$. As in the proof for Lemma 3.2 we obtain

$$1 = f\left(\frac{h + y}{u}\right)^p \leq \frac{t}{u} f\left(\frac{h}{t}\right)^p + \frac{s}{u} f\left(\frac{y}{s}\right)^p = 1 ,$$

where $f(h) = t$, $f(y) = s$ and $u = t + s$. Observe that $t, s \in (0,u)$. Thus the strict convexity of f^p yields

$$\frac{h}{t} = \frac{y}{s} .$$

This completes the proof.

3.5. *Inner product kernels.* If $\langle\ ,\ \rangle: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is an inner product in $\mathbb{R}^n$, there is a self-adjoint, positively definite linear mapping $T: \mathbb{R}^n \to \mathbb{R}^n$ with

$$(3.6) \qquad \langle h, y \rangle = Th \cdot y$$

for all $h, y \in \mathbb{R}^n$, and, conversely, by (3.6) each such mapping T

induces an inner product in $\mathbb{R}^n$. Thus a norm f in $\mathbb{R}^n$ is induced by an inner product if and only if there exists a self-adjoint T such that

$$f(h) = (Th \cdot h)^{1/2}$$

for all $h \in \mathbb{R}^n$.

If $\theta \colon G \to GL(\mathbb{R}^n, \mathbb{R}^n)$ is a measurable mapping such that for a.e. $x \in G$ the linear map $\theta(x)$ is self-adjoint and that there exist constants $0 < a \leq b < \infty$ with

$$a|h|^2 \leq \theta(x)\, h \cdot h \leq b|h|^2$$

for a.e. $x \in G$ and all $h \in \mathbb{R}^n$, then the kernels

$$F_\theta(x,h) = (\theta(x)\, h \cdot h)^{p/2}, \qquad 1 \leq p < \infty,$$

satisfy (a)-(d). Note that a kernel function F is of the type F_θ if and only if for a.e. $x \in G$ the norm $h \mapsto F(x,h)^{1/p}$ is induced by an inner product. Therefore we call kernels F_θ *inner product kernels*. These kernels have been studied by Yu. G. Reshetnyak [R2, R3] in the case $p = n$. They play an essential role in the theory of quasiregular mappings; see [R2, R3] and also [BI, GLM].

The inner product property is connected to the smoothness of F:

3.7. Theorem. *Let* $F \colon G \times \mathbb{R}^n \to \mathbb{R}$ *satisfy (a)-(d) for* $p \in [1,\infty)$. *Then* F *is an inner product kernel if and only if for a.e.* $x \in G$ *the function* $h \mapsto F(x,h)^{2/p}$ *belongs to* $C^2(\mathbb{R}^n)$.

Proof. For an inner product kernel F, the function $h \mapsto F(x,h)^{2/p}$ is a C^∞-function in $\mathbb{R}^n$ for a.e. $x \in G$. For the converse, it suffices to prove: If $f(h) = F(x,h)^{1/p}$ is a norm such that $g = f^2 \in C^2(\mathbb{R}^n)$, then the parallelogram law

$$g(h + y) + g(h - y) = 2g(h) + 2g(y)$$

holds for all $h, y \in \mathbb{R}^n$; see [Y, p. 39].

First note that the homogeneity of f implies

$$\partial_i \partial_j g(\lambda h) = \partial_i \partial_j g(h) , \qquad i , j = 1,2,\ldots,n ,$$

for all $\lambda \neq 0$ and $h \in \mathbb{R}^n$. Letting $\lambda \to 0$ we obtain

$$\partial_i \partial_j g(0) = \partial_i \partial_j g(h) , \qquad i , j = 1,2,\ldots,n ,$$

for all $h \in \mathbb{R}^n$. Now Taylor's formula yields

$$(3.8) \qquad g(h) = \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j g(0) \, h_i h_j$$

for all $h \in \mathbb{R}^n$.

To complete the proof let $h , y \in \mathbb{R}^n$. By (3.8)

$$\begin{aligned} &g(h + y) + g(h - y) \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j g(0)(h_i + y_i)(h_j + y_j) + \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j g(0)(h_i - y_i)(h_j - y_j) \\ &= \sum_{i,j=1}^{n} \partial_i \partial_j g(0)(h_i h_j + y_i y_j) = 2 g(h) + 2 g(y) \end{aligned}$$

as desired.

3.9. Remark. Theorem 3.7 yields for $p = 2$: *If* $h \mapsto F(x,h)$ *belongs to* $C^p(\mathbb{R}^n)$, *then* F *is an inner product kernel.* For $p \neq 2$ this is not true. For example, for $n \geq 2$, consider the kernels

$$F(x,h) = \sum_{i=1}^{n} |h_i|^p , \qquad 1 \leq p < \infty .$$

Observe that the norm

$$f(h) = \Big(\sum_{i=1}^{n} |h_i|^p \Big)^{1/p}$$

is induced by an inner product if and only if $p = 2$. However, the function $h \mapsto F(x,h)$ belongs to $C^p(\mathbb{R}^n)$ for $p = 2,3,4,\ldots$. Furthermore, $h \mapsto F(x,h)$ even belongs to $C^\infty(\mathbb{R}^n)$ for $p = 2k$, $k = 1,2,\ldots$.

4. F-norm

A variational kernel F satisfying (a)-(d) induces a new norm, called an F-norm in $L^p(G)^n$. This leads to a simple proof for the lower semicontinuity of a variational integral, and we also study the uniform convexity of $L^p(G)^n$, $p \in (1,\infty)$, under the F-norm.

4.1. *F-norm.* Suppose that G is an open set in $\mathbb{R}^n$ and that $F: G \times \mathbb{R}^n \to \mathbb{R}$ satisfies (a)-(d) for some $p \in [1,\infty)$. We denote by $L^p(G)^n$ the cartesian product $L^p(G) \times \dots \times L^p(G)$ (n times) with the norm

$$\|u\|_{L^p} = \sum_{i=1}^{n} \|u_i\|_p ,$$

where $u = (u_1,\dots,u_n) \in L^p(G)^n$ and $\|u_i\|_p$, $i = 1,2,\dots,n$, is the usual $L^p(G)$-norm of u_i. We introduce another norm $\| \ \|_F$ in $L^p(G)^n$, called *an F-norm*,

$$\|u\|_F = \left(\int_G F(x,u(x))\, dx \right)^{1/p} .$$

The assumptions (a)-(d) together with Lemmas 2.4 and 3.2 immediately yield:

4.2. Lemma. *F-norm is a norm in* $L^p(G)^n$ *equivalent to the norm* $\| \ \|_{L^p}$.

4.3. Remarks. (1) If $p = 2$ and F is an inner product kernel, then $L^2(G)^n$ is a Hilbert space under the F-norm.

(2) If $A \subset G$ is a measurable set, we write

$$\|u\|_{F,A} = \left(\int_A F(x,u)\, dx \right)^{1/p}$$

for $u \in L^p(A)^n$. Obviously, $\|u\|_{F,A}$ is equivalent to the usual $L^p(A)^n$-norm

$$\|u\|_{L^p,A} = \sum_{i=1}^{n} \|u_i\|_{p,A} .$$

4.4. *Lower semicontinuity.* The concept of an F-norm can be used to give a simple proof for the lower semicontinuity of the variational integral

$$I(u) = \int_G F(x,\nabla u(x))\, dx$$

under the assumptions (a)-(d). There exist several proofs for this fundamental result in a more general case; see e.g. [D, M, R1, R3, S]. For a simple proof based on Banach-Saks' theorem see [L]. These proofs do not make use of the homogeneity assumption (d).

4.5. Theorem. *Let* I *and* G *be as above and let* u_i , $i = 0;1,2,\ldots$, *be a sequence of* W_p^1*-functions in* G *such that* $\nabla u_i \to \nabla u_0$ *weakly in* $L^p(G)^n$. *Then*

$$I(u_0) \le \varliminf_{i\to\infty} I(u_i) .$$

Proof. In $L^p(G)^n$ we use the F-norm. By Lemma 4.2, $\nabla u_i \to \nabla u_0$ weakly in $L^p(G)^n$. Thus Lemma 2.6 implies

$$\|\nabla u_0\|_F \le \varliminf_{i\to\infty} \|\nabla u_i\|_F ,$$

and hence

$$I(u_0) \le \varliminf_{i\to\infty} I(u_i)$$

as desired.

4.6. Remarks. (1) The above reasoning can also be used in a more general situation. Let $F\colon G \times \mathbb{R}^k \times \mathbb{R}^{nk} \to \mathbb{R}$ satisfy the conditions (a)-(d) modified in an obvious way (i.e., $F(x,\lambda u,\lambda v) = |\lambda|^p F(x,u,v)$). Then the following result holds. *Let* $u^{(i)} = (u_1^{(i)},\ldots,u_k^{(i)})$, $i = 0;1,2,\ldots$, *be a sequence in* $W_p^1(G)^k$. *If* $\nabla u_j^{(i)} \to \nabla u_j^{(0)}$ *and* $u_j^{(i)} \to u_j^{(0)}$ *weakly in* $L^p(G)$, $j = 1,2,\ldots,k$, *then*

$$I(u^{(0)}) \le \varliminf_{i\to\infty} I(u^{(i)}) ,$$

where

$$I(u) = \int_G F(x,u_1(x),\ldots,u_k(x),\nabla u_1(x),\ldots,\nabla u_k(x))\,dx\ .$$

(2) It is clear that in the above results we can take any measurable set $A \subset G$ instead of the open set G .

4.7. C o r o l l a r y. *Suppose that* $G \subset \mathbb{R}^n$ *is an open set and that* $F: G \times \mathbb{R}^n \to \mathbb{R}$ *satisfies (a)-(d). Furthermore, suppose that* $F_i: G \times \mathbb{R}^n \to \mathbb{R}$, $i = 1,2,\ldots$, *are such that*

(i) F_i *satisfies (a) for each* i .

(ii) *For a.e.* $x \in G$, $F_i(x,h) \geq 0$ *for all* $h \in \mathbb{R}^n$ *and* $i \in \mathbb{N}$.

(iii) *For each* $\varepsilon > 0$ *there is a compact set* $C \subset G$ *such that* $m(G \setminus C) < \varepsilon$ *and* $F_i \to F$ *uniformly in* $C \times \mathbb{R}^n$.

If u_m , $m = 0;1,2,\ldots$, *are* W^1_p*-functions in* G *such that* $\nabla u_m \to \nabla u_0$ *weakly in* $L^p(G)^n$, *then*

$$\varliminf_{i\to\infty} \int_G F_i(x,\nabla u_i(x))\,dx \geq \int_G F(x,\nabla u_0(x))\,dx\ .$$

Proof. Consider first a set $A \subset G$ with $m(A) < \infty$. Fix $\varepsilon > 0$ and choose compact sets $C_k \subset A$ with $m(A \setminus C_k) < \frac{1}{k}$ and $F_i \to F$ uniformly in $C_k \times \mathbb{R}^n$. Now there is an i_0 such that $i \geq i_0$ implies

$$F_i(x,h) \geq F(x,h) - \frac{\varepsilon}{m(A)}$$

for all $(x,h) \in C_k \times \mathbb{R}^n$. Theorem 4.5 (cf. 4.6 (2)) yields

$$\varliminf_{i\to\infty} \int_A F_i(x,\nabla u_i(x))\,dx \geq \varliminf_{i\to\infty} \int_{C_k} F_i(x,\nabla u_i(x))\,dx$$

$$\geq \varliminf_{i\to\infty} \int_{C_k} F(x,\nabla u_i(x))\,dx - \frac{m(C_k)}{m(A)}\varepsilon$$

$$\geq \int_{C_k} F(x,\nabla u_0(x))\,dx - \frac{m(C_k)}{m(A)}\varepsilon\ .$$

The last term tends to

$$\int_A F(x,\nabla u_0(x))\, dx - \varepsilon$$

as $k \to \infty$. Letting $\varepsilon \to 0$ we obtain

$$\varliminf_{i\to\infty} \int_A F_i(x,\nabla u_i(x))\, dx \geq \int_A F(x,\nabla u_0(x))\, dx .$$

To complete the proof, note that the absolute continuity of the integral allows us to choose for each $\varepsilon > 0$ a compact set $A_\varepsilon \subset G$ such that

$$\int_G F(x,\nabla u_0(x))\, dx \leq \int_{A_\varepsilon} F(x,\nabla u_0(x))\, dx + \varepsilon .$$

Hence

$$\int_G F(x,\nabla u_0(x))\, dx \leq \varliminf_{i\to\infty} \int_{A_\varepsilon} F_i(x,\nabla u_i(x))\, dx + \varepsilon$$

$$\leq \varliminf_{i\to\infty} \int_G F_i(x,\nabla u_i(x))\, dx + \varepsilon$$

for each $\varepsilon > 0$.

4.8. *Uniform convexity.* The rest of this section deals with uniform convexity; especially, we study the uniform convexity of $L^p(G)^n$, $p \in (1,\infty)$, under the F-norm.

A normed space $(X, \|\cdot\|)$ is called *uniformly convex* if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ imply $\|\frac{1}{2}x + \frac{1}{2}y\| \leq 1 - \delta$. For alternative definitions see [K]. We use some of them in what follows.

We call a family $\mathcal{N}$ of norms $g\colon X \to \mathbb{R}$ *equiuniformly convex* if for each $\varepsilon > 0$ there is $\delta > 0$ such that for each $g \in \mathcal{N}$ the conditions $g(x) \leq 1$, $g(y) \leq 1$ and $g(x - y) \geq \varepsilon$ imply

$$g(\tfrac{1}{2}x + \tfrac{1}{2}y) \leq 1 - \delta .$$

Further, if $U \subset X$ is a convex set and if $\mathcal{F}$ is a family of continuous functions $f\colon U \to \mathbb{R}$, then $\mathcal{F}$ is called *equistrictly convex* if for each compact set $C \subset U$ and $\varepsilon > 0$ there is $\delta > 0$ such that $x, y \in C$ and $\|x - y\| \geq \varepsilon$ imply

$$f(\frac{1}{2}x + \frac{1}{2}y) \leq (1 - \delta) \frac{f(x) + f(y)}{2} .$$

Obviously, an equiuniformly convex family of norms is not equistrictly convex if $\dim X \geq 1$. The connection between the above-mentioned concepts is discussed in 4.13; see also 4.22.

The following lemma is a generalization of McShane's idea [K, pp. 360-362, McS].

4.9. Lemma. *Suppose that* X *is a normed space and that* N *is a family of norms in* X . *Let* $p \in (1,\infty)$. *Then* N *is equiuniformly convex if and only if for each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for every* $f \in N$ *the conditions* $f(x) \leq 1$, $f(y) \leq 1$ *and* $f(x-y) \geq \varepsilon$ *imply*

$$f(\frac{1}{2}x + \frac{1}{2}y)^p \leq (1 - \delta) \frac{f(x)^p + f(y)^p}{2} . \tag{4.10}$$

Proof. The equiuniform convexity of N immediately follows from (4.10). To prove the converse, it suffices to show that for each $\varepsilon > 0$ there is $\delta > 0$ such that for every $f \in N$ the conditions $f(x) = 1$, $f(y) \leq 1$ and $f(x-y) \geq \varepsilon$ imply (4.10).

Let us suppose that there are $\varepsilon > 0$ and sequences $x_n , y_n \in X$ and $f_n \in N$ such that $f_n(x_n) = 1$, $f_n(y_n) \leq 1$, $f_n(x_n - y_n) \geq \varepsilon$ and that

$$\lim_{n\to\infty} \frac{f_n(\frac{1}{2}x_n + \frac{1}{2}y_n)^p}{\frac{1}{2} + \frac{1}{2}f_n(y_n)^p} = 1 . \tag{4.11}$$

Then $\lim_{n\to\infty} f_n(y_n) = 1$, since otherwise there is $q < 1$ with $f_n(y_n) \leq q$ for some subsequence denoted again by $f_n(y_n)$. Then there exists $\rho < 1$ such that

$$f_n(\frac{1}{2}x_n + \frac{1}{2}y_n)^p \leq (\frac{1}{2}(1 + f_n(y_n)))^p \leq \frac{\rho}{2} (1 + f_n(y_n)^p) , \tag{4.12}$$

since the function

$$t \mapsto \frac{(1 + t)^p}{1 + t^p}$$

is strictly increasing in $[0 , 1]$; we may choose

$\rho = (1+q)^p\ (1+q^p)^{-1}\ 2^{1-p}$. Now (4.12) yields

$$\frac{f_n(\frac{1}{2}x_n + \frac{1}{2}y_n)^p}{\frac{1}{2} + \frac{1}{2}f_n(y_n)^p} \le \rho < 1 ,$$

which contradicts (4.11). Hence $\lim_{n\to\infty} f_n(y_n) = 1$, and (4.11) implies

$$\lim_{n\to\infty} f_n(\tfrac{1}{2}x_n + \tfrac{1}{2}y_n) = 1 ,$$

which contradicts the equiuniform convexity of $\mathcal{N}$. This completes the proof.

4.13. Corollary. *Suppose that* $p \in (1,\infty)$ *and that* $\mathcal{F}$ *is a family of continuous functions* $f\colon \mathbb{R}^n \to \mathbb{R}$ *such that*

(i) $f(\lambda x) = |\lambda|^p f(x)$ *for all* $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, *and*

(ii) *there exist constants* $0 < \alpha \le \beta < \infty$ *with*

$$\alpha |x|^p \le f(x) \le \beta |x|^p$$

for all $x \in \mathbb{R}^n$ *and* $f \in \mathcal{F}$.

Then $\mathcal{F}$ *is equistrictly convex if and only if* $\mathcal{F}^{1/p} = \{ f^{1/p} \mid f \in \mathcal{F} \}$ *is an equiuniformly convex family of norms.*

Proof. Suppose that $\mathcal{F}$ is equistrictly convex. Then each $f \in \mathcal{F}$ is strictly convex, cf. [RV, Section 71]. By Lemma 3.2, $\mathcal{F}^{1/p}$ is a family of norms in $\mathbb{R}^n$. Choose

$$C = \overline{B(0,\alpha^{-1/p})} .$$

Then $f^{-1}([0,1]) \subset C$ for each $f \in \mathcal{F}$; hence Lemma 4.9 implies the equiuniform convexity of $\mathcal{F}^{1/p}$. For the converse let $C \subset \mathbb{R}^n$ be a compact set, $C \neq \{0\}$. Then condition (ii) yields

$$0 < M = \sup \{ g(x) \mid x \in C ,\ g \in \mathcal{F}^{1/p} \} < \infty .$$

Hence $\frac{1}{M} f^{1/p}(C) \subset [0,1]$, and Lemma 4.9 implies the desired result.

The above connection between equistrictly convex and equiuniformly convex families is not true in infinite dimensional spaces; see 4.22.

4.14. *Uniformly convex kernels.* Suppose that G is an open set in $\mathbb{R}^n$ and that $F\colon G \times \mathbb{R}^n \to \mathbb{R}$ satisfies (a)-(d) for $p \in (1,\infty)$. The kernel F is *uniformly convex* if there is $G' \subset G$ such that $m(G \setminus G') = 0$ and that the family $\mathcal{F} = \{ h \mapsto F(x,h) \mid x \in G' \}$ is equi-strictly convex.

Corollary 4.13 yields:

4.15. Lemma. *Suppose that* $p \in (1,\infty)$. *The kernel* $F\colon G \times \mathbb{R}^n \to \mathbb{R}$ *is uniformly convex if and only if there is* $G' \subset G$ *such that* $m(G \setminus G') = 0$ *and* $N = \{ h \mapsto F(x,h)^{1/p} \mid x \in G' \}$ *is an equiuniformly convex family of norms.*

The next theorem says that all inner product kernels belong to this class.

4.16. Theorem. *Suppose that* G *is an open set in* $\mathbb{R}^n$ *and that* $F\colon G \times \mathbb{R}^n \to \mathbb{R}$ *is an inner product kernel with* $p \in (1,\infty)$. *Then* F *is uniformly convex.*

Proof. Fix $\varepsilon > 0$ and choose

$$\delta = \begin{cases} 1 - (1 - \varepsilon^2/4)^{1/2} & \text{if } \varepsilon \le 2 \\ 1 & \text{if } \varepsilon \ge 2 \,. \end{cases}$$

The parallelogram law yields that the desired inequality in 4.8 holds for an inner product norm $g(h) = F(x,h)^{1/p}$.

4.17. Remark. A strictly convex kernel is not necessarily uniformly convex, even though it is continuous. Fix $p \in (1,\infty)$ and let $G = B(0,1) \setminus \{0\} \subset \mathbb{R}^n$. Define $F\colon G \times \mathbb{R}^n \to \mathbb{R}$ by

$$F(x,h) = \Big(\sum_{i=1}^{n} |h_i|^{1/|x|} \Big)^{|x|\,p} .$$

Then F is continuous and a strictly convex kernel, see 3.4. However,

F is not uniformly convex. To see this, consider sequences $h_i = (a_i, a_i, \dots, a_i)$ and $y_i = (-a_i, a_i, a_i, \dots, a_i)$, where $a_i = n^{-1/i}$, $i = 2, 3, \dots$. Choose x_i such that $|x_i| = i^{-1}$. Then $F(x_i, h_i) = 1 = F(x_i, y_i)$ and

$$\lim_{i\to\infty} F(x_i, \tfrac{1}{2}h_i + \tfrac{1}{2}y_i) = 1 ,$$

but

$$\lim_{i\to\infty} F(x_i, h_i - y_i) = 2^p > 0 .$$

The next theorem characterizes the kernels F for which $L^p(G)^n$, $p \in (1,\infty)$, is uniformly convex under the F-norm. The sufficiency part generalizes an idea of McShane's [McS].

4.18. Theorem. *Suppose that* G *is an open set in* $\mathbb{R}^n$ *and that* $F\colon G \times \mathbb{R}^n \to \mathbb{R}$ *satisfies (a)–(d) for some* $p \in (1,\infty)$. *Then* $L^p(G)^n$ *is uniformly convex under the* F*-norm if and only if* F *is uniformly convex.*

Proof. Suppose first that F is uniformly convex. Fix $\varepsilon > 0$ and choose $f, g \in L^p(G)^n$ such that $\|f\|_F = 1$, $\|g\|_F = 1$ with $\|f - g\|_F \geq \varepsilon$. Write

$$A = \left\{ x \in G \mid F(x, f(x) - g(x)) > \frac{\varepsilon^p}{4}(F(x, f(x)) + F(x, g(x))) \right\} .$$

Then A is measurable, and by Lemmas 4.9 and 4.15 there is $\delta > 0$ such that

$$F(x, \tfrac{1}{2}f(x) + \tfrac{1}{2}g(x)) \leq (1 - \delta)\, \frac{F(x, f(x)) + F(x, g(x))}{2} \tag{4.19}$$

for a.e. $x \in A$. Since

$$\int_{G\setminus A} F(x, f(x) - g(x))\, dx$$

$$\leq \frac{\varepsilon^p}{4} \int_G (F(x, f(x)) + F(x, g(x)))\, dx \leq \frac{\varepsilon^p}{2} ,$$

we obtain

$$\int_A F(x\,,f(x)\,-\,g(x))\;dx \geq \varepsilon^p - \frac{\varepsilon^p}{2} = \frac{\varepsilon^p}{2}\,.$$

Now $\|f - g\|_{F,A} \geq \varepsilon\, 2^{-1/p}$, and hence $\max\,\{\|f\|^p_{F,A}\,,\,\|g\|^p_{F,A}\} \geq \varepsilon^p\, 2^{-(p+1)}$. Thus the condition (b) and (4.19) yield

$$\int_G \left(\tfrac{1}{2}F(x\,,f(x)) + \tfrac{1}{2}F(x\,,g(x)) - F(x\,,\tfrac{1}{2}f(x) + \tfrac{1}{2}g(x)\right)\,dx$$

$$\geq \int_A \left(\tfrac{1}{2}F(x\,,f(x)) + \tfrac{1}{2}F(x\,,g(x)) - F(x\,,\tfrac{1}{2}f(x) + \tfrac{1}{2}g(x))\right)\,dx$$

$$\geq \frac{\delta}{2}\int_A (F(x\,,f(x)) + F(x\,,g(x)))\,dx \geq \delta\,\varepsilon^p\, 2^{-(p+2)}\,.$$

Hence

$$\|\tfrac{1}{2}f + \tfrac{1}{2}g\|^p_F \leq \tfrac{1}{2}\,\|f\|^p_F + \tfrac{1}{2}\,\|g\|^p_F - \delta\,\varepsilon^p\, 2^{-(p+2)}$$

$$\leq 1 - \delta\,\varepsilon^p\, 2^{-(p+2)}\,,$$

which implies the desired result.

To prove the converse, suppose that there exist $\varepsilon > 0$, $A \subset G$ and sequences u_k, $v_k\colon G \to \mathbb{R}^n$, $k = 1,2,\dots$, such that $m(A) > 0$ and for all $x \in A$, $F(x\,,u_k(x)) \leq 1$, $F(x\,,v_k(x)) \leq 1$, $F(x\,,u_k(x) - v_k(x)) \geq 2\varepsilon$ and $F(x\,,\frac{1}{2}u_k(x) + \frac{1}{2}v_k(x)) > 1 - \frac{1}{k}$, cf. Lemma 4.15. By approximation we may suppose that $F(x\,,u_k(x)) < 1$ and $F(x\,,v_k(x)) < 1$ in A. Now the condition (a) gives a compact set $C \subset A$ such that $m(C) > 0$ and $F|_{C \times \mathbb{R}^n}$ is continuous. Fix $k \in \mathbb{N}$. The continuity of $F|_{C \times \mathbb{R}^n}$ allows us to choose for each $y \in C$ an open neighbourhood $N_{y,k}$ such that for every $x \in N_{y,k} \cap C$ $F(x\,,u_k(y)) < 1$, $F(x\,,v_k(y)) < 1$, $F(x\,,u_k(y) - v_k(y)) \geq \varepsilon$ and $F(x\,,\frac{1}{2}u_k(y) + \frac{1}{2}v_k(y)) > 1 - \frac{2}{k}$. Since C is compact and $m(C) > 0$, there is $y_0 \in C$ with $m(N_{y_0,k} \cap C) > 0$. Write $h_k = u_k(y_0)$, $y_k = v_k(y_0)$ and $B_k = N_{y_0,k} \cap C$. Set

$$f_k(x) = m(B_k)^{-1/p}\, h_k\, \chi_{B_k}$$

and

$$g_k(x) = m(B_k)^{-1/p}\, y_k\, \chi_{B_k} ,$$

where χ_{B_k} stands for the characteristic function of B_k. Then f_k and g_k are measurable,

$$\int_G F(x, f_k(x))\, dx < 1 \quad \text{and} \quad \int_G F(x, g_k(x))\, dx < 1 .$$

Thus $f_k, g_k \in L^p(G)^n$, $\|f_k\|_F \leq 1$, $\|g_k\|_F \leq 1$ and

$$\|f_k - g_k\|_F^p = \int_G F(x, f_k(x) - g_k(x))\, dx$$

$$= m(B_k)^{-1} \int_{B_k} F(x, h_k - y_k)\, dx \geq \varepsilon .$$

Furthermore,

$$\|\tfrac{1}{2} f_k + \tfrac{1}{2} g_k\|_F^p = m(B_k)^{-1} \int_{B_k} F(x, \tfrac{1}{2} h_k + \tfrac{1}{2} y_k)\, dx \geq 1 - \frac{2}{k} ,$$

which tends to 1 as $k \to \infty$. This contradicts the uniform convexity of $L^p(G)^n$ under the F-norm, and the proof is complete.

4.20. Corollary. *Suppose that* $F: G \times \mathbb{R}^n \to \mathbb{R}$ *is a uniformly convex kernel. If* $u_m \in W_p^1(G)$, $m = 0;1,2,\ldots$ *are such that* $u_m \to u_0$ *in* $L^p(G)$ *with*

$$\overline{\lim_{m\to\infty}} \int_G F(x, \nabla u_m(x))\, dx \leq \int_G F(x, \nabla u_0(x))\, dx , \tag{4.21}$$

then $u_m \to u_0$ *in* $W_p^1(G)$.

Proof. We use the F-norm in $L^p(G)^n$. Since $L^p(G)^n$ is uniformly convex, it suffices to show that $\nabla u_m \to \nabla u_0$ weakly in $L^p(G)^n$; cf. Theorem 4.5 and [HS, p. 233]. This is easily seen by standard reasoning. Note that (4.21) implies the boundedness of $\|D_j u_m\|_p$, $j = 1,2,\ldots,n$.

4.22. Remark. Corollary 4.13 connects equistrictly convexity and equiuniform convexity in $\mathbb{R}^n$. Such a result is not true in infinite dimensional spaces, since the unit norm ball is never compact in an infinite dimensional normed space, cf. e.g. [Y, p. 85]. For example, consider the strictly convex function $I: L^p(G) \to \mathbb{R}$,

$$I(u) = \int_G F(x, u(x))\, dx ,$$

where F is as in 4.17. By Theorem 4.18, the F-norm $I^{1/p}$ is not uniformly convex though the singleton $\{I\}$ is equistrictly convex.

4.23. *An application to the Calculus of Variations.* For a bounded open set G in $\mathbb{R}^n$ fix $u_0 \in W^1_p(G)$. Write

$$\mathcal{F} = \{ u: G \to \dot{\mathbb{R}} \mid u - u_0 \in W^1_{p,0}(G) \} .$$

Function v in $\mathcal{F}$ is called *an extremal for the variational integral*

$$I(u) = \int_G F(x, \nabla u(x))\, dx$$

with boundary values u_0 if

$$I(v) = \inf \{ I(u) \mid u \in \mathcal{F} \} .$$

Suppose that $F: G \times \mathbb{R}^n \to \mathbb{R}$ is a uniformly convex kernel. Since $\nabla\mathcal{F} = \{ \nabla u \mid u \in \mathcal{F}\}$ is a convex and closed set in $L^p(G)^n$, it contains a unique element ∇v with minimal F-norm. This proves the existence and the uniqueness of the extremal with fixed boundary values u_0.

5. F-capacity and conformal mappings

For a given norm $f: \mathbb{R}^n \to \mathbb{R}$ and a constant $p \in (1,\infty)$ we write $F(h) = f(h)^p$, $h \in \mathbb{R}^n$. Suppose that G is a domain in $\mathbb{R}^n$ and that $C_0, C \subset \partial G$ are compact sets. The F-*capacity of* C_0 *and* C *relative to* G is the number

$$\operatorname{cap}_F(C_0,C;G) = \inf \left\{ \int_G F(\nabla u)\, dm \mid u \in W(C_0,C;G) \right\},$$

where

$$W(C_0,C;G)$$

$$= \{ u \in C(G \cup C_0 \cup C) \mid u \text{ is } ACL^p \text{ in } G ,\ u|_{C_0} \leq 0 \text{ and } u|_C \geq 1 \} .$$

We call an F-capacity *conformally invariant* (or *similarity-invariant*) if for each domain $G \subset \mathbb{R}^n$ and all compact sets $C_0 , C \subset \partial G$

$$\operatorname{cap}_F(C_0,C;G) = \operatorname{cap}_F(h(C_0), h(C) ; h(G))$$

for each conformal mapping (similarity) h defined in some open neighbourhood of $\overline{G}$. If f is the euclidean norm in $\mathbb{R}^n$, the F-capacity is the familiar p-capacity, which is known to be conformally invariant in the case $p = n$. This follows also from Theorem 6.3, where a more general situation is studied. In this section we shall prove the following theorem.

5.1. Theorem. *The F-capacity is similarity-invariant if and only if there exists* $\lambda > 0$ *such that* $F(h) = \lambda |h|^n$ *for all* $h \in \mathbb{R}^n$.

Theorem 5.1 immediately yields

5.2. Corollary. *The F-capacity is conformally invariant if and only if there exists* $\lambda > 0$ *such that* $F(h) = \lambda |h|^n$ *for all* $h \in \mathbb{R}^n$.

Since the conformal invariance of F-capacity, $F(h) = \lambda |h|^n$, is obtained from Theorem 6.3, we shall consider only the converse part. The proof is divided into four lemmas.

Write $g = f|_{S^{n-1}}: S^{n-1} \to \mathbb{R}$, $\alpha_0 = \min g$ and $\beta_0 = \max g$. Then $0 < \alpha_0 \leq \beta_0 < \infty$. Furthermore, $\alpha_0 = \beta_0$ if and only if for some $\lambda > 0$, $f(h) = \lambda |h|$ for all $h \in \mathbb{R}^n$. For $x_0 \in S^{n-1}$ we set

$$L_{x_0} = \{ t x_0 \mid t \in \mathbb{R} \} .$$

Let P_{x_0} be the projection of $\mathbb{R}^n$ onto L_{x_0} , i.e., $P_{x_0}(x) = (x \cdot x_0) x_0$.

5.3. Lemma. *Suppose that* $x_0 \in g^{-1}(\beta_0)$. *Then* $f(x) \geq f(P_{x_0}(x))$ *for all* $x \in \mathbb{R}^n$.

Proof. Write $y_0 = x_0/\beta_0$; then $f(y_0) = 1$. Fix $x \in \mathbb{R}^n$. We may assume that $f(x) = 1$ and that $P_{x_0}(x) = \mu y_0$, $\mu > 0$. Suppose that $\mu > 1$. Since

$$\frac{1}{\beta_0} < \frac{\mu}{\beta_0} = |\mu y_0| = |P_{x_0}(x)| = |x \cdot x_0| \ |x_0| \leq |x| ,$$

it follows by elementary geometry that there are $y \in S^{n-1}(\beta_0^{-1})$ and $t \in (0,1)$ such that

$$z_0 = t y + (1 - t) x \in \{ s y_0 \mid s \in (1,\mu) \} .$$

In particular, $f(z_0) > f(y_0) = 1$. On the other hand, $f(y) \leq \beta_0 |y| = 1$; hence the convexity of f yields

$$1 < f(z_0) \leq t f(y) + (1 - t) f(x) \leq 1 ,$$

which is impossible. Thus $\mu \leq 1$ and $f(x) \geq f(P_{x_0}(x))$, as required.

Fix $x_0 \in g^{-1}(\beta_0)$. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal mapping such that $Te_1 = x_0$. Write $G = TQ(1)$, $C_0 = T\{ x \in \partial Q(1) \mid x_1 = -1 \}$ and $C = T\{ x \in \partial Q(1) \mid x_1 = 1 \}$.

5.4. Lemma. $\operatorname{cap}_F(C_0,C;G) = \beta_0^p 2^{n-p}$.

Proof. If $u \in W(C_0,C;G)$, then Lemma 5.3 yields

$$F(\nabla u) \geq F(P_{x_0}(\nabla u)) = \beta_0^p \ |\nabla u \cdot x_0|^p \tag{5.5}$$

in G . Now Hölder's inequality implies for $x \in TQ_{n-1}(1)$

$$1 \leq \left(\int_{-1}^{1} |\nabla u(x + t x_0) \cdot x_0| \ dt \right)^p \leq 2^{p-1} \int_{-1}^{1} |\nabla u(x + t x_0) \cdot x_0|^p \ dt ,$$

and hence by (5.5) and by Fubini's theorem

$$\int_G F(\nabla u)\, dm \geq \beta_0^p \int_{TQ_{n-1}(1)} \int_{-1}^{1} |\nabla u(x + t x_0) \cdot x_0|^p \, dt \, dm_{n-1}$$

$$\geq \beta_0^p \, m_{n-1}(Q_{n-1}(1))\, 2^{1-p} = \beta_0^p \, 2^{n-p} .$$

Next set $v(x) = \frac{1}{2}(x_1 + 1)$. Then $u_0 = v \circ T^{-1}$ belongs to $W(C_0, C; G)$ and

$$\nabla u_0(x) = T \nabla v(T^{-1} x) = \frac{1}{2} x_0 \in L_{x_0}$$

in G. This yields

$$\int_G F(\nabla u_0)\, dm = \beta_0^p \int_G 2^{-p} \, dm = \beta_0^p \, 2^{n-p}$$

and thus

$$\mathrm{cap}_F(C_0, C; G) = \beta_0^p \, 2^{n-p} ,$$

as required.

5.6. Lemma. *If* F*-capacity is similarity-invariant, then* $p = n$.

Proof. Let $h(x) = \lambda x$, $\lambda \neq 0$, be a dilation. Set $G' = h(G)$, $C_0' = h(C_0)$ and $C' = h(C)$. Now $u \in W(C_0, C; G)$ if and only if $u \circ h^{-1} \in W(C_0', C'; G')$. Fix $u \in W(C_0, C; G)$. Then

$$\nabla(u \circ h^{-1})(x) = \frac{1}{\lambda} \nabla u(h^{-1}(x))$$

in G' and thus

$$\int_{G'} F(\nabla(u \circ h^{-1}))\, dm = |\lambda|^{-p} \int_G |\lambda|^n F(\nabla u)\, dm$$

$$= |\lambda|^{n-p} \int_G F(\nabla u)\, dm .$$

So Lemma 5.4 yields for each $\lambda \neq 0$

$$\beta_0^p\, 2^{n-p} = \mathrm{cap}_F(C_0',C';G') = |\lambda|^{n-p}\, \mathrm{cap}_F(C_0,C;G)$$

$$= |\lambda|^{n-p}\, \beta_0^p\, 2^{n-p} \;;$$

hence $p = n$ as desired.

The next lemma completes the proof of Theorem 5.1.

5.7. Lemma. *If* $\alpha_0 < \beta_0$, *then F-capacity is not similarity-invariant.*

Proof. Fix $y_0 \in g^{-1}(\alpha_0)$. Let $O\colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal mapping with $O(x_0) = y_0$. Write $G' = O(G)$, $C_0' = O(C_0)$ and $C' = O(C)$. Suppose that u_0 is as in the proof of Lemma 5.4. Set $v_0 = u_0 \circ O^{-1}$. Then v_0 belongs to $W(C_0',C';G')$ and

$$\nabla v_0(x) = O\,\nabla u_0(O^{-1}(x)) = \tfrac{1}{2} y_0$$

in G' ; hence

$$\mathrm{cap}_F(C_0',C';G') \le \int_{G'} F(\nabla v_0)\, dm = \alpha_0^p \int_{G'} |\tfrac{1}{2} y_0|^p\, dm$$

$$= \alpha_0^p\, 2^{n-p} < \beta_0^p\, 2^{n-p} = \mathrm{cap}_F(C_0,C;G) \;,$$

which contradicts the similarity invariance.

6. Conformally invariant variational integrals

Suppose that G is an open set in $\mathbb{R}^n$ and that $F\colon G \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a C-function in $G \times (\mathbb{R} \times \mathbb{R}^n)$. Suppose, furthermore, that there exist constants $\alpha \in [0,\infty)$, $\beta \in (0,\infty)$ and $p \in [1,\infty)$ such that for a.e. $x \in G$

(6.1) $$0 \le F(x,u,h) \le \alpha\,|u|^p + \beta\,|h|^p$$

for all $(u,h) \in \mathbb{R} \times \mathbb{R}^n$. Setting $F(x,u,h) = \beta |h|^p$ for $x \notin G$ we may assume that $F: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

If $D \subset \mathbb{R}^n$ is an open set and if $u \in W_p^1(D)$, then the variational integral

$$I_F(u) = \int_D F(x, u(x), \nabla u(x))\, dx$$

exists and is finite. The variational integral I_F is *conformally invariant* (or *similarity-invariant*) if for each open set $D \subset\subset \mathbb{R}^n$ and for every conformal (similarity) $f: D \to \tilde{D} = f(D)$

$$I_F(u) = I_F(u \circ f)$$

i.e.,

$$\int_{\tilde{D}} F(y, u(y), \nabla u(y))\, dy = \int_D F(f(x), u \circ f(x), \nabla(u \circ f)(x))\, dx$$

whenever $u \in W_p^1(\tilde{D})$.

6.2. Remark. The growth restriction (6.1) is appropriate to guarantee $I_F(u)$ finite whenever $u \in W_p^1(D)$, see [Kr, p. 27].

The following theorem gives the structure of conformally invariant or similarity-invariant variational kernels and generalizes a result of M. Grüter's [Gr].

6.3. Theorem. *Suppose that a* C-*function* $F: \mathbb{R}^n \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}$ *satisfies (6.1). Then* I_F *is conformally invariant (or similarity-invariant) if and only if there is a* C-*function* $k: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *such that for a.e.* $x \in \mathbb{R}^n$

$$F(x,u,h) = k(x,u)\, |h|^n$$

for each $u \in \mathbb{R}$ *and* $h \in \mathbb{R}^n$.

Proof. To show the sufficiency, fix $D \subset \mathbb{R}^n$ and a conformal mapping $f: D \to f(D)$. Pick $u \in W_p^1(f(D))$. Using the change of variable, see [GLM, 6.8; V, p. 113; G, Theorem 6], and the conformality of f , we obtain

$$\int_{f(D)} F(y, u(y), \nabla u(y))\, dy = \int_D |J(x,f)|\, k(f(x), u \circ f(x))\, |\nabla u(f(x))|^n\, dx$$

$$= \int_D k(f(x), u(f(x)))\, |f'(x)|^n\, |\nabla u(f(x))|^n\, dx$$

$$= \int_D k(f(x), u(f(x)))\, |f'(x)^* \nabla u(f(x))|^n\, dx$$

$$= \int_D k(f(x), u(f(x)))\, |\nabla(u \circ f)(x)|^n\, dx$$

$$= \int_D F(f(x), u \circ f(x), \nabla(u \circ f)(x))\, dx ,$$

as desired.

To show the converse, we first prove some auxiliary results.

Suppose that G is an open set in $\mathbb{R}^n$ and that $f: G \to f(G) = G'$ is conformal. Suppose, furthermore, that

$$I_F(u) = I_F(u \circ f)$$

for each $u \in W_p^1(f(D))$ whenever D is a bounded open set in G .

6.4. Lemma. *For a.e.* $x \in G$

$$F(f(x), u, f'(x)^* h) = |J(x,f)|\, F(f(x), u, h)$$

for each $(u,h) \in \mathbb{R} \times \mathbb{R}^n$.

Proof. Since the problem is local, we may assume that f is defined in an open set containing $\overline{G}$ and that G is bounded. If $D \subset G$ is open, we obtain by a change of variable (see [GLM, 6.8; V, p. 113; G, Theorem 6])

$$\int_D F(f(x), u \circ f(x), \nabla(u \circ f)(x))\, dx = \int_{f(D)} F(y, u(y), \nabla u(y))\, dy$$

$$= \int_D |J(x,f)|\, F(f(x), u \circ f(x), \nabla u(f(x)))\, dx$$

for each $u \in W_p^1(f(D))$. Fix $c \in \mathbb{R}$ and $h \in \mathbb{R}^n$. Write $u(x) = h \cdot x + c$. Then $u \in W_p^1(f(D))$ for each open $D \subset G$ and $\nabla u(x) = h$. Choose M, N and $K \in \mathbb{R}$ such that

$$|f'(x)|^n < M , \qquad \alpha\,|u(f(x))|^p + \beta\,|h|^p < K$$

and

$$\alpha\,|u(f(x))|^p + \beta\,|f'(x)^* h|^p < N$$

for every $x \in G$. We show that for a.e. $x \in G$

(6.5) $$F(f(x), u(f(x)), f'(x)^* h) = |J(x,f)|\, F(f(x), u(f(x)), h) .$$

Fix $\delta > 0$ and write

$$U_\delta = \{ x \in G \mid F(f(x), u(f(x)), f'(x)^* h) > |J(x,f)|\, F(f(x), u(f(x)), h) + \delta \} .$$

If $m(U_\delta) > 0$, pick an open set $U \subset G$ such that $U_\delta \subset U$ and $m(U \setminus U_\delta) < (\delta\, m(U_\delta))/MK$. Then

$$\int_{U\setminus U_\delta} |J(x,f)|\, F(f(x), u(f(x)), h)\, dx$$

$$= \int_{U\setminus U_\delta} |f'(x)|^n\, F(f(x), u(f(x)), h)\, dx$$

$$\le MK m(U \setminus U_\delta) < \delta\, m(U_\delta) .$$

Hence

$$\int_U F(f(x), u(f(x)), \nabla(u \circ f)(x))\, dx = \int_U F(f(x), u(f(x)), f'(x)^* h)\, dx$$

$$\geq \int_{U_\delta} F(f(x), u(f(x)), f'(x)^* h)\, dx$$

$$> \delta\, m(U_\delta) + \int_{U_\delta} |J(x,f)|\, F(f(x), u(f(x)), h)\, dx$$

$$> \int_U |J(x,f)|\, F(f(x), u(f(x)), h)\, dx = \int_{f(U)} F(y, u(y), \nabla u(y))\, dy ,$$

which contradicts the invariance of I_F . Hence $m(U_\delta) = 0$. Similarly, write

$$V_\delta = \{ x \in G \mid F(f(x), u(f(x)), f'(x)^* h) + \delta < |J(x,f)|\, F(f(x), u(f(x)), h) \} .$$

If $m(V_\delta) > 0$, pick an open set $V \subset G$ such that $V_\delta \subset V$ and $m(V \setminus V_\delta) < (\delta\, m(V_\delta))/N$. From the estimate

$$\int_{V \setminus V_\delta} F(f(x), u(f(x)), f'(x)^* h)\, dx$$

$$\leq \int_{V \setminus V_\delta} (\alpha |u(f(x))|^p + \beta |f'(x)^* h|^p)\, dx \leq N m(V \setminus V_\delta) < \delta\, m(V_\delta)$$

we obtain $m(V_\delta) = 0$ as above. Since $m(U_\delta) = 0 = m(V_\delta)$ for each $\delta > 0$, the equation (6.5) follows.

To complete the proof, write

$$G' = \{ x \in G \mid (u,h) \mapsto F(f(x), u, h) \ \text{is continuous} \} .$$

Then $m(G \setminus G') = 0$, and if we set

$$A_{c,h} = \{ x \in G' \mid F(f(x), f(x) \cdot h + c, f'(x)^* h) = |J(x,f)|\, F(f(x), f(x) \cdot h + c, h) \} ,$$

then $m(G \setminus A_{c,h}) = 0$ by (6.5). Hence $m(G \setminus A) = 0$, where

$$A = \bigcap_{\substack{c \in Q \\ h \in Q^n}} A_{c,h} .$$

Since

$$\begin{aligned} A = \{ x \in G' \mid & F(f(x), f(x) \cdot h + c, f'(x)^* h) \\ & = |J(x,f)| \, F(f(x), f(x) \cdot h + c, h) \\ & \text{for all } c \in Q \text{ and } h \in Q^n \} , \end{aligned}$$

the continuity of $(v,h) \mapsto F(f(x), v, h)$ for $x \in A$ and the density of $Q \times Q^n$ in $\mathbb{R} \times \mathbb{R}^n$ imply

$$\begin{aligned} A = \{ x \in G' \mid & F(f(x), f(x) \cdot h + c, f'(x)^* h) \\ & = |J(x,f)| \, F(f(x), f(x) \cdot h + c, h) \\ & \text{for all } c \in \mathbb{R} \text{ and } h \in \mathbb{R}^n \} . \end{aligned}$$

Since the function $c \mapsto f(x) \cdot h + c$ is surjective onto $\mathbb{R}$ for fixed $x \in A$ and $h \in \mathbb{R}^n$, we obtain

$$\begin{aligned} A = \{ x \in G' \mid & F(f(x), u, f'(x)^* h) = |J(x,f)| \, F(f(x), u, h) \\ & \text{for all } u \in \mathbb{R} \text{ and } h \in \mathbb{R}^n \} , \end{aligned}$$

and the proof is complete.

6.6. Corollary. *If* I_F *is similarity-invariant, then for a.e.* $x \in \mathbb{R}^n$

$$F(x,u,\lambda h) = |\lambda|^n F(x,u,h)$$

for every $\lambda \in \mathbb{R}$ *and* $(u,h) \in \mathbb{R} \times \mathbb{R}^n$.

Proof. Choose $f(x) = \lambda x$ in Lemma 6.4. Then for a.e. $x \in \mathbb{R}^n$

$$F(x,u,\lambda h) = |\lambda|^n F(x,u,h)$$

for each $(u,h) \in \mathbb{R} \times \mathbb{R}^n$. As in the proof for Lemma 6.4, the C-function property yields the desired result.

Proof for the necessity in 6.3. Fix $h_0 \in S^{n-1}$. For $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ set

$$k(x,u) = F(x,u,h_0) .$$

Pick $h \in S^{n-1}$ and choose an orthogonal mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(h_0) = h$. Then

$$f'(x)^* h = f^{-1}(h) = h_0$$

for each $x \in \mathbb{R}^n$. Hence Lemma 6.4 implies that for a.e. $x \in \mathbb{R}^n$

$$F(f(x), u, h_0) = F(f(x), u, h)$$

for all $u \in \mathbb{R}$. Observe that $|J(x,f)| = 1$. Now for a.e. $x \in \mathbb{R}^n$

$$k(x,u) = F(x,u,h_0) = F(x,u,h)$$

for all $u \in \mathbb{R}$. Write

$$G' = \{x \in \mathbb{R}^n \mid (u,h) \mapsto F(x,u,h) \text{ is continuous and } F(x,u,\lambda h) = |\lambda|^n F(x,u,h) \text{ for all } \lambda, u \in \mathbb{R} \text{ and } h \in \mathbb{R}^n\}$$

and

$$A_h = \{x \in G' \mid F(x,u,h) = k(x,u) \text{ for each } u \in \mathbb{R}\} .$$

Then $m(\mathbb{R}^n \setminus A_h) = 0$ for every $h \in S^{n-1}$, since $m(\mathbb{R}^n \setminus G') = 0$ by Lemma 6.4. Using similar reasoning to the proof of Lemma 6.4 we obtain $m(\mathbb{R}^n \setminus A) = 0$, where

$$A = \{ x \in G' \mid F(x,u,h) = k(x,u) \text{ for all } u \in \mathbb{R} \text{ and } h \in S^{n-1} \} .$$

Hence the homogeneity property of F in G' implies that

$$A = \{ x \in G' \mid F(x,u,h) = k(x,u)\ |h|^n \text{ for all } u \in \mathbb{R} \text{ and } h \in \mathbb{R}^n \} ,$$

which completes the proof.

6.7. Corollary. *Suppose that a C-function* $F\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *satisfies (6.1) for* $p = n$. *Then* I_F *is similarity-invariant if and only if there is a C-function* $k\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *such that for a.e.* $x \in \mathbb{R}^n$

$$F(x,u,h) = k(x,u)\ |h|^n$$

for each $u \in \mathbb{R}$ *and* $h \in \mathbb{R}^n$ *and that for each* $u \in \mathbb{R}$ *the function* $x \mapsto k(x,u)$ *belongs to* $L^\infty(\mathbb{R}^n)$.

Proof. The growth restriction (6.1) implies that for a.e. $x \in \mathbb{R}^n$

$$0 \le k(x,u) \le \frac{\alpha |u|^n}{|h|^n} + \beta$$

for every $u \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$, $h \neq 0$. Letting $|h| \to \infty$ we obtain the desired result.

6.8. Remark. The case $p \neq n$ in (6.1) is not interesting. In fact, if I_F is similarity-invariant and $p < n$, it is easily seen that $F = 0$. The same result is obtained if $p > n$ and $\alpha = 0$.

If for a.e. $x \in \mathbb{R}^n$, $F(x,u,h) = F(x,v,h)$ for every $u, v \in \mathbb{R}$ and $h \in \mathbb{R}^n$, we write $F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $F(x,h) = F(x,u,h)$ for short.

6.9. Corollary. *Suppose that a C-function* $F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *satisfies (6.1) for* $p = n$. *Then* I_F *is similarity-invariant if and only if there is* $k \in L^\infty(\mathbb{R}^n)$ *such that for a.e.* $x \in \mathbb{R}^n$

$$F(x,h) = k(x)\ |h|^n$$

for every $h \in \mathbb{R}^n$.

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