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The metric-valued Lebesgue differentiation theorem in measure spaces and its applications

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Abstract

We prove a version of the Lebesgue differentiation theorem for mappings that are defined on a measure space and take values into a metric space, with respect to the differentiation basis induced by a von Neumann lifting. As a consequence, we obtain a lifting theorem for the space of sections of a measurable Banach bundle and a disintegration theorem for vector measures whose target is a Banach space with the Radon–Nikodým property.

Keywords Lebesgue differentiation theorem · von Neumann lifting · Measurable Banach bundle · Radon–Nikodým property · Disintegration of a measure

Mathematics Subject Classification 28A15 · 28A51 · 46G15 · 18F15 · 46G10 · 46B22 · 28A50

1 Introduction

In recent years, there have been impressive developments in the study of differential calculus over metric measure spaces. A breakthrough in the theory was Gigli’s work [18], where he proposed the language of L^p -normed L^∞ -modules, with the aim of building a tensor calculus on top of a robust functional-analytic framework. Similar notions – whose interest goes beyond the analysis on metric spaces – were previously considered, e.g., in Refs. [20, 24]. One of the main concepts introduced in Refs. [18] is the *cotangent module*, which can be regarded as the space of ‘abstract 1-

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forms'. Starting from this object, it was possible to develop an effective vector calculus even in very singular contexts, where a linear and/or smooth underlying structure is missing. However, differently from the classical Riemannian setting, the elements of the cotangent module do not have (a priori) a 'pointwise meaning'. The problem of providing a fiberwise description of normed modules was addressed in Refs. [11, 34], where it is shown that every separable normed module can be represented as the section space of a *measurable Banach bundle*. Previously, similar notions of a Banach bundle were investigated, e.g., in Refs. [21, 23]. Besides its theoretical interest, the relevance of a fiberwise characterisation of normed modules became evident with the recent paper [16], where it has been used to describe dual modules and pullback modules. We point out that the dual of the pullback of the cotangent module comes into play naturally, for instance, when studying the *differential of a map of bounded deformation* [18, Proposition 2.4.6] or the *velocity of a test plan* [18, Theorem 2.3.18], which are very important tools in analysis on nonsmooth spaces. As of now, no 'direct' description (i.e., without appealing to Banach bundles) of a dual normed module is available.

Motivated by this line of research, in the present paper we pursue the following plan:

- (i) Prove a *Lebesgue differentiation theorem* for the sections of a measurable Banach bundle. In this way, we are entitled to consider the *Lebesgue points* of such a section. At the level of generality we deal with in this paper (which covers, e.g., metric measure spaces that are not doubling), the differentiation bases we consider are those induced by a *von Neumann lifting* of the given measure, which exists under mild assumptions.
- (ii) Construct a (weak form of) *lifting* of the section space of a measurable Banach bundle. To this aim, the Lebesgue differentiation theorem in (i) has an essential role. This allows to select 'everywhere defined representatives' of sections in a consistent manner.
- (iii) Obtain a *Disintegration Theorem* for vector-valued measures of bounded variation whose target is a Banach space having the Radon–Nikodým property. The proof of the Disintegration Theorem builds upon the lifting of sections mentioned in (ii). Our interest in vector-valued measures mainly comes from the recent paper [7], where the theory of *local vector measures* has been introduced. The latter provides a unified framework for many key objects in nonsmooth analysis, such as metric currents, differentials of Sobolev functions, and distributional gradients of BV functions.

In addition to the new contributions mentioned above, we will provide a thorough (and partially revisited) account of well-established notions, such as differentiation bases and liftings, with two objectives in mind: to adapt previously known concepts to our setting, and to make the paper (almost) fully self-contained. Next we discuss in detail the contents of the paper.

Lifting theory

A fundamental result in Measure Theory is definitely the celebrated von Neumann–Maharam Theorem, which can be stated as follows: given a complete, finite measure space (X, Σ, μ) , there exists a selection $\ell: \Sigma \rightarrow \Sigma$ of μ -a.e. representatives which preserves the empty set, the whole space, finite intersections, and finite unions. An equivalent formulation can be given at the level of functions: there exists a linear and multiplicative operator $\ell: L^\infty(\mu) \rightarrow \mathcal{L}^\infty(\Sigma)$ that selects μ -a.e. representatives; in other words, $\ell(f) = f$ holds μ -a.e. for every $f \in L^\infty(\mu)$, and ℓ preserves constant functions, finite sums, and finite products. Here, $\mathcal{L}^\infty(\Sigma)$ stands for the space of all real-valued, Σ -measurable functions on X , while $L^\infty(\mu)$ denotes its quotient up to μ -a.e. equality. In either cases, the map ℓ is called a **von Neumann lifting**, or just a **lifting**, of the measure μ . The question about the existence of liftings (in the setting of the Euclidean space endowed with the Lebesgue measure) was originally raised by Haar in 1931 and soon after solved by von Neumann [46]. Afterwards, this result has been generalised by Maharam [36] and Ionescu Tulcea–Ionescu Tulcea [43] to more general measure spaces. On the one hand, the finiteness assumption on μ can be relaxed: the case of σ -finite measures can be easily reduced to the finite ones, but the result can be also proved in the larger class of strictly localisable measures. On the other hand, the role of the completeness assumption on μ is more significant: it is still an open problem whether all non-complete probability measures admit a lifting. As observed by Shelah [38], it is consistent with ZFC that no Borel lifting of the Lebesgue measure on $[0, 1]$ exists. However, under the continuum hypothesis, von Neumann and Stone proved in Ref. [47] that a Borel lifting of the Lebesgue measure on $[0, 1]$ exists, a result later generalised by Mokobodzki [37]. For a more detailed discussion about these topics, we refer to the books [5, 15, 39, 45].

One of the possible strategies which have been used to prove existence of liftings is the following: first, to provide a **lower density** $\phi: \Sigma \rightarrow \Sigma$ of the measure μ under consideration, i.e., a map ϕ verifying the lifting axioms with the exception of the finite unions preservation; then, to ‘enlarge’ ϕ to a lifting ℓ , i.e., to provide a lifting ℓ of μ satisfying $\phi(E) \subset \ell(E)$ for every set $E \in \Sigma$. In many interesting cases, an explicit lower density can be constructed. For instance, whenever a Lebesgue Density Theorem is in force, one obtains a lower density by just associating to each measurable set the family of its density points; this idea dates back to the original paper [46] by von Neumann. On an arbitrary measure space, to show existence of a lower density is a more delicate issue, which requires a transfinite induction argument to be employed (see [19]). Once a lower density is at disposal, a standard argument (using, e.g., Zorn Lemma [15, 341J] or the Ultrafilter Lemma [40]) yields the existence of a lifting. The completeness of μ plays a role exclusively in the passage from lower densities to liftings. It is also worth pointing out that, excluding the very exceptional framework of Stone spaces, no explicit lifting can be constructed, even when the lower density is somewhat canonical (e.g., if the Lebesgue Density Theorem holds with respect to some reasonable differentiation basis). A natural question, though, is whether an additional structure on the measure space reflects on the possibility to require further regularity of the lifting. In this regard, a significant example is that of **strong liftings** [41]: if X

is a complete, separable metric space and Σ is the completion of the Borel σ -algebra of X under μ , then one can find a lifting ℓ which is strong, meaning that $\ell(f)$ and f agree everywhere on the support of μ whenever $f : X \rightarrow \mathbb{R}$ is bounded continuous. The metrisability assumption cannot be dropped, as a counterexample built in [33] shows. We point out that strong liftings that are not Borel liftings exist [29].

Our presentation of the lifting theory mainly follows along the lines of [15] and [5]. In Sect. 2.1, we recall the definition of a lifting, the statement of von Neumann–Maharam Theorem, and how to pass from liftings of sets to liftings of functions (and vice versa). In Sect. 5.1, we introduce strong liftings. In Appendix A, we discuss lower densities, how to pass from a lower density to a lifting, and how the validity of the Lebesgue Density Theorem induces a lower density. The latter fact provides a relatively easy proof of the existence of von Neumann liftings on a vast class of measure spaces (see Theorem A.6) and of strong liftings on (complete and separable) metric measure spaces (see Corollary A.7).

Lebesgue differentiation theorems in measure spaces

The classical Lebesgue differentiation theorem, which dates back to [31], states that for \mathcal{L}^n -a.e. point $x \in \mathbb{R}^n$, the value of an integrable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at x coincides with the limit of its infinitesimal averages around x . In fact, a stronger statement holds: setting $\int_E f \, d\mathcal{L}^n := \mathcal{L}^n(E)^{-1} \int_E f \, d\mathcal{L}^n$, one has

$$\lim_{r \searrow 0} \int_{B_r(x)} |f(y) - f(x)| \, d\mathcal{L}^n(y) = 0, \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

Later on, this fundamental result has been generalised in many different ways. For instance, it is possible to consider integrable functions defined on domains different from the Euclidean ones, and to take averages with respect to other classes of ‘shrinking’ sets than balls. The Lebesgue differentiation theorem holds, exactly in the same form as above, on (asymptotically) doubling metric measure space [25], but often it can be formulated also in terms of other differentiation bases. By a **differentiation basis** on an arbitrary measure space (X, Σ, μ) we mean a family \mathcal{I} of measurable sets, having positive and finite measure, which are directed by downward inclusion. Namely, given a point $x \in X$ and two sets $I, I' \in \mathcal{I}$ containing x , there exists $J \in \mathcal{I}$ such that $J \subset I \cap I'$. The family of elements of \mathcal{I} containing x is typically denoted by \mathcal{I}_x . Differentiation bases come with a natural notion of convergence: we say that a map $\Phi : \mathcal{I} \rightarrow Y$, the target Y being a topological space, has **\mathcal{I} -limit** $y \in Y$ at $x \in X$ provided

$$\forall U \text{ neighbourhood of } y, \quad \exists I \in \mathcal{I}_x : \quad \forall J \in \mathcal{I}_x \text{ with } J \subset I, \quad \Phi(J) \in U.$$

For brevity, we will just write $\lim_{I \Rightarrow x} \Phi(I) = y$. With this terminology, a generalised form of Lebesgue Differentiation Theorem can be formulated as follows: for $f : X \rightarrow$

\mathbb{R} integrable,

$$\lim_{I \Rightarrow x} \int_I |f(y) - f(x)| d\mu(y) = 0, \quad \text{for } \mu\text{-a.e. } x \in X.$$

A well-studied problem in the literature is to understand for which spaces and differentiation bases the above version of Lebesgue Differentiation Theorem holds; for a thorough account of this theory, we refer to [8–10] and the references therein. In this paper, we will focus our attention on a special class of differentiation bases: those induced by a von Neumann lifting. Given a complete, finite measure space (X, Σ, μ) and a lifting ℓ of μ , we denote by \mathcal{I}^ℓ the family of all sets of the form $\ell(E)$, where $E \in \Sigma$ satisfies $\mu(E) > 0$. It readily follows from the lifting axioms that \mathcal{I}^ℓ defines a differentiation basis on (X, Σ, μ) . Furthermore, it is known that \mathcal{I}^ℓ verifies the Lebesgue Differentiation Theorem as stated above (see [30]). On the one hand, this type of differentiation basis has the advantage of being available in the great generality of measure spaces, without the need for an underlying topological or metric structure. On the other hand, the procedure of finding liftings (and thus a fortiori the associated differentiation bases) is highly non-constructive. In other words, albeit very interesting from a theoretical perspective, differentiation bases induced by a lifting are maybe not very useful from a computational point of view. Nonetheless, this issue is irrelevant for the applications that we will describe in the present paper, since liftings will be used only as an intermediate technical tool during the proofs.

Our main result is Theorem 3.4, which is a generalisation of the above-discussed Lebesgue Differentiation Theorem in measure spaces to more general targets than the real line, namely, to metric-valued maps. More specifically, we will prove that any measurable map $\varphi : X \rightarrow Y$, where (X, Σ, μ) is a complete measure space and (Y, d) is a separable metric space, satisfies

$$\lim_{I \Rightarrow x} \int_I d(\varphi(y), \varphi(x)) d\mu(y) = 0, \quad \text{for } \mu\text{-a.e. } x \in X,$$

where the limit is taken with respect to the differentiation basis \mathcal{I}^ℓ induced by an arbitrary lifting ℓ of μ . As an immediate corollary, we deduce that if \mathbb{B} is a Banach space and $v : X \rightarrow \mathbb{B}$ is a strongly measurable map, then it holds that

$$\lim_{I \Rightarrow x} \int_I v(y) d\mu(y) = v(x), \quad \text{for } \mu\text{-a.e. } x \in X,$$

where the integral is intended in the sense of Bochner. The proof argument we provide is rather elementary and much shorter than other ones we found in the literature (for the real-valued Lebesgue Differentiation Theorem in measure spaces), such as the one in [10]. Our strategy is to show that every measurable map $\varphi : X \rightarrow Y$ as above is ‘approximately continuous’ at almost every point of X with respect to \mathcal{I}^ℓ , in the sense expressed by Lemma 3.2. Once this property is established, Theorem 3.4 easily follows. This line of thought, based on an approximate continuity argument, is similar in spirit to the work done in Refs. [3, 32].

An immediate consequence of Theorem 3.4 is Corollary 3.6, which states the following. Consider a \mathbb{B} -valued measure Ω on (X, Σ) with bounded variation, where \mathbb{B} is a Banach space having the Radon–Nikodým property (RNP for brevity), and suppose that $\Omega \ll \mu$. Then the Radon–Nikodým derivative $\frac{d\Omega}{d\mu} : X \rightarrow \mathbb{B}$ can be expressed as a ‘genuine’ derivative as

$$\frac{d\Omega}{d\mu}(x) = \lim_{I \ni x} \frac{\Omega(I)}{\mu(I)} \in \mathbb{B}, \quad \text{for } \mu\text{-a.e. } x \in X,$$

where the limit is taken with respect to the differentiation basis \mathcal{I}^ℓ .

In Sects. 2.2, 2.3, and 2.5 we remind the basic notions concerning differentiation bases, Bochner integration, and the Radon–Nikodým property, respectively. In Sect. 3 we prove our version of Lebesgue differentiation theorem in measure spaces for metric-valued maps, as well as its direct consequences. In Appendix B we study more in details the concept of approximate continuity with respect to a differentiation basis, which is used in Lemma 3.2.

Banach bundles and liftings of sections

The main motivation behind our study of the metric-valued Lebesgue Differentiation Theorem and its applications comes from the theory of measurable Banach bundles and L^p -normed L^∞ -modules. Let us introduce the relevant terminology. Fix a measure space (X, Σ, μ) and an ambient Banach space \mathbb{B} . Following [11], we say that a multivalued map $\mathbf{E} : X \rightarrow \mathbb{B}$ is a **measurable Banach \mathbb{B} -bundle** over (X, Σ) provided it is weakly measurable (in an appropriate sense) and each $\mathbf{E}(x)$ is a linear subspace of \mathbb{B} . We denote by $\bar{\Gamma}_p(\mu; \mathbf{E})$ the space of all p -integrable sections of \mathbf{E} , while $\Gamma_p(\mu; \mathbf{E})$ stands for its quotient up to μ -a.e. equality. Our interest towards measurable Banach \mathbb{B} -bundles is, in turn, due to the fact that the space of sections $\Gamma_p(\mu; \mathbf{E})$ is an $L^p(\mu)$ -**normed $L^\infty(\mu)$ -module** in the sense of Gigli [17, 18]. One of the primary outcomes of [11] is that actually every separable $L^p(\mu)$ -normed $L^\infty(\mu)$ -module can be written as a space of sections of the form $\Gamma_p(\mu; \mathbf{E})$ for some Banach \mathbb{B} -bundle \mathbf{E} . This result is relevant if read in conjunction with the fact that, under mild assumptions, the cotangent module associated with a metric measure space is separable (cf. with [11, Appendix B]). In reality, the language of normed modules was introduced precisely with the aim of building a solid differential calculus for metric measure spaces, and cotangent modules provide a convenient concept of a covector field in this (possibly very singular) framework. Ultimately, the results we will discuss next say that it is possible to talk about the Lebesgue points of a covector field on a metric measure space, whose pointwise meaning is inherently encoded in the Banach bundle machinery. We also mention that the analysis of measurable Banach bundles covers the well-studied theory of Lebesgue–Bochner spaces, which correspond to considering those Banach bundles whose fibers are constant.

In Theorem 3.5 we introduce the precise representative map for the L^p -sections of a given measurable Banach \mathbb{B} -bundle \mathbf{E} over (X, Σ) : as a corollary of Theorem 3.4, one has that any given element $v \in \Gamma_p(\mu; \mathbf{E})$ admits a unique representative $\hat{v} \in \bar{\Gamma}_p(\mu; \mathbf{E})$

satisfying

$$\hat{v}(x) = \lim_{I \rightarrow x} \int_I v(y) \, d\mu(y) \in \mathbf{E}(x),$$

for any $x \in X$ such that the limit in the right-hand side exists, and $\hat{v}(x) := 0_{\mathbf{E}(x)}$ elsewhere. Any such point (thus, μ -a.e. point) is called a **Lebesgue point** of v . Observe that \hat{v} is independent of the chosen representative of v . Building upon Theorem 3.5, we obtain (see Theorem 4.4) a lifting map at the level of the sections of a Banach bundle, which generalises the von Neumann liftings of functions in a natural way. More precisely, we show that a von Neumann lifting ℓ of μ induces a linear choice of a representative $\ell: \Gamma_\infty(\mu; \mathbf{E}) \rightarrow \bar{\Gamma}_\infty(\mu; \mathbb{B}'')$ such that

$$\begin{aligned} \ell(f \cdot v)(x) &= \ell(f)(x) \ell(v)(x), \quad \forall v \in \Gamma_\infty(\mu; \mathbf{E}), f \in L^\infty(\mu), x \in X, \\ \|\ell(v)(x)\|_{\mathbb{B}''} &\leq \ell(\|v(\cdot)\|_{\mathbf{E}(\cdot)})(x), \quad \forall v \in \Gamma_\infty(\mu; \mathbf{E}), x \in X. \end{aligned}$$

The first condition expresses a compatibility with the lifting ℓ at the level of functions, whence no ambiguity should arise. About the second condition, one would probably expect it to be an identity rather than an inequality. However, with the exception of some special situations, this is not only unlikely but in fact truly impossible. Roughly speaking, the lifting of sections can verify $\|\ell(v)(\cdot)\|_{\mathbb{B}''} = \ell(\|v(\cdot)\|_{\mathbf{E}(\cdot)})$ only for those elements v that have compact (essential) image, as we will see in Proposition 4.9. In the case of Lebesgue–Bochner spaces, this kind of behaviour is consistent with the fact that a norm-preserving lifting $\ell: L^\infty(\mu; \mathbb{B}) \rightarrow \mathcal{L}^\infty(\mu; \mathbb{B})$ can exist if and only if \mathbb{B} is finite-dimensional (or μ is purely atomic). This surprising phenomenon was known, we will report it in Theorem C.2. Due to this reason, in the literature we only found notions of liftings for maps taking values into a locally compact space (see, e.g., [44, 45]). To the best of our knowledge, the weaker form of lifting we provide in Theorem 4.4 for arbitrary target (which contracts, but does not preserve, the norm) was not previously investigated. However, it has some interesting applications, such as a disintegration theorem for RNP-valued measures (see Sect. 5.2). Coming back to our lifting of sections of \mathbf{E} , we point out that its target is the larger space $\bar{\Gamma}_\infty(\mu; \mathbb{B}'')$, where \mathbb{B}'' stands for the bidual of \mathbb{B} . In some cases of interest (for instance, if \mathbb{B} has a predual, cf. with Remark 4.5), the target of ℓ can be actually required to be $\bar{\Gamma}_\infty(\mu; \mathbb{B})$, but we do believe this is not possible in full generality.

Let us spend a few words to describe the strategy we will adopt to achieve Theorem 4.4, which is partially inspired by von Neumann’s original proof of existence of liftings in the Euclidean space. Fix a complete, finite measure space (X, Σ, μ) , a lifting ℓ of μ , and a measurable Banach \mathbb{B} -bundle \mathbf{E} on (X, Σ) . Given any $x \in X$, the differentiation basis \mathcal{I}^ℓ induces a filter of ‘generalised neighbourhoods’ of x , which we can extend to an ultrafilter ω_x by applying the Ultrafilter Lemma. The key feature of the ultrafilter ω_x is that $\lim_{I \rightarrow x} \Phi(I) = y$ implies that the ultralimit of Φ with respect to ω_x coincides with y . The construction of ω_x will be committed to Lemma 4.2. At this point, the lifting map $\ell: \Gamma_\infty(\mu; \mathbf{E}) \rightarrow \bar{\Gamma}_\infty(\mu; \mathbb{B}'')$ can be defined as follows:

given any element $v \in \Gamma_\infty(\mu; \mathbf{E})$ and any point $x \in X$, we set

$$\ell(v)(x) := \omega_x\text{-}\lim \int_I v(y) \, d\mu(y),$$

where we are tacitly identifying \mathbb{B} with its canonical image into \mathbb{B}'' . The ultralimit is taken with respect to the weak* topology of \mathbb{B}'' , and its existence is ensured by Banach–Alaoglu Theorem; this explains why the lifting of sections takes values into \mathbb{B}'' . Theorem 3.4 plays a fundamental role here, as it ensures that $\ell(v)$ is a representative of v . The linearity of ℓ stems from general properties of ultralimits. The multiplicative identity $\ell(f \cdot v) = \ell(f)\ell(v)$ boils down to the almost tautological (yet interesting) fact that for any $f \in L^\infty(\mu)$ one has

$$\ell(f)(x) = \omega_x\text{-}\lim \int_I f(y) \, d\mu(y), \quad \text{for every point } x \in X,$$

which will be proved in Proposition 4.3. Finally, the validity of $\|\ell(v)(\cdot)\|_{\mathbb{B}''} \leq \ell(\|v(\cdot)\|_{\mathbf{E}(\cdot)})$ follows from the weak* lower semicontinuity of the norm $\|\cdot\|_{\mathbb{B}''}$.

In Sects. 2.4 and 2.6 we recall the theory of measurable Banach bundles and the most important concepts related to ultrafilters/ultralimits, respectively. In Sect. 4 we prove the existence theorem for liftings of sections, as well as all the preparatory results that we need to achieve it. Appendix C is dedicated to a slight revision of the previously-known lifting theory of Lebesgue–Bochner spaces, which constitutes a particular case of our study.

Disintegration of vector measures

One of the most striking applications of the theory of von Neumann liftings (or, more specifically, of strong liftings) is the Disintegration Theorem for measures. By suitably adapting a known strategy and employing Theorem 4.4, we obtain in Sect. 5 a rather general disintegration result for RNP-valued measures on Polish spaces, which states the following. Let $\varphi: X \rightarrow Y$ be a Borel map between Polish spaces, \mathbb{B} a Banach space having the Radon–Nikodým property, and Ω a \mathbb{B} -valued measure on X of bounded variation. Then there exists an (essentially unique) disintegration $\{\Omega_y\}_{y \in Y}$ of Ω along the map φ . Namely, Ω_y is a \mathbb{B} -valued measure on X of bounded variation with $\|\Omega_y\|_{\mathbb{B}}(X) = 1$ and concentrated on $\varphi^{-1}(\{y\})$ for $\varphi_\# \|\Omega\|_{\mathbb{B}}$ -a.e. $y \in Y$; moreover, the map $Y \ni y \mapsto \int f \, d\Omega_y \in \mathbb{B}$ is strongly measurable and satisfies

$$\int f \, d\Omega = \iint f \, d\Omega_y \, d(\varphi_\# \|\Omega\|_{\mathbb{B}})(y),$$

whenever $f \in C_b(X)$ is given. The existence of $\{\Omega_y\}_{y \in Y}$ will be proved in Theorem 5.9.

We now briefly comment on the heuristic behind the proof of the Disintegration Theorem. For the sake of exposition, suppose to know that a disintegration $\{\Omega_y\}_{y \in Y}$

of Ω along φ exists, and that X is compact. Fix a strong lifting ℓ of $\nu := \varphi_{\#}\|\Omega\|_{\mathbb{B}}$. Then for every $f \in C(X)$ we can formally compute

$$\iint f \, d\Omega_y \, d\nu(y) = \int f \, d\Omega = \varphi_{\#}(f\Omega)(Y) = \int \ell\left(\frac{d\varphi_{\#}(f\Omega)}{d\nu}\right)(y) \, d\nu(y),$$

where in the last term the lifting $\ell : L^{\infty}(\nu; \mathbb{B}) \rightarrow \mathcal{L}^{\infty}(\nu; \mathbb{B}'')$ provided by Theorem 4.4 appears. Technically speaking there is an inconsistency, as the element $\ell\left(\frac{d\varphi_{\#}(f\Omega)}{d\nu}\right)(y)$ belongs to \mathbb{B}'' . However, ignoring this issue for the moment, the previous formula is suggesting that

$$\int f \, d\Omega_y = \ell\left(\frac{d\varphi_{\#}(f\Omega)}{d\nu}\right)(y), \quad \text{for every } f \in C(X).$$

We then take this as a definition of Ω_y . Using Theorem 4.4, one can prove that Ω_y takes actually values in \mathbb{B} for ν -a.e. $y \in Y$ and gives a disintegration of Ω along φ . The fact of considering $\ell\left(\frac{d\varphi_{\#}(f\Omega)}{d\nu}\right)(y)$ rather than $\frac{d\varphi_{\#}(f\Omega)}{d\nu}(y)$ is fundamental. The problem is already at the level of the definition: given any $f \in C(X)$, the vector $\frac{d\varphi_{\#}(f\Omega)}{d\nu}(y) \in \mathbb{B}$ is defined for ν -a.e. $y \in Y$, but it is not clear (unless we invoke the lifting theory) how to invert the quantifiers. In other words, how to show that for ν -a.e. $y \in Y$ the vector $\frac{d\varphi_{\#}(f\Omega)}{d\nu}(y)$ is well-defined for every function $f \in C(X)$. We also point out that the inequality $\|\ell(\nu)(\cdot)\|_{\mathbb{B}'} \leq \ell(\|\nu(\cdot)\|_{\mathbb{B}})$ appearing in Theorem 4.4 is still sufficient to achieve the Disintegration Theorem, so that no finite-dimensionality on the target space \mathbb{B} is required.

About the axiom of choice

We conclude the Introduction by concisely commenting on the passages where the Axiom of Choice, or some of its weaker forms, will be used throughout:

- The existence of von Neumann liftings strongly relies upon the Axiom of Choice.
- In Lemma 4.2, we employ the Ultrafilter Lemma.
- During the proof of Theorem 4.4, we use Banach–Alaoglu Theorem.

It is evident that, when working at this level of generality, the usage of the Axiom of Choice is unavoidable. Nevertheless, we believe that in many cases of interest (e.g., on metric measure spaces or when working with separable Banach bundles) the use of liftings (and more generally of the Axiom of Choice) is not necessary, as observed for example in [11, Remark 4.14].

2 Preliminaries

Let us begin by fixing some general terminology, which will be used throughout the whole paper. Given a topological space (Y, τ) and a point $y \in Y$, we denote by $\mathcal{N}_Y(y) \subset \tau$ the family of all open neighbourhoods of y in (Y, τ) . Given a measure

space (X, Σ, μ) and an exponent $p \in [1, \infty)$, we denote by $\mathcal{L}^p(\mu)$ the space of all p -integrable functions $f: X \rightarrow \mathbb{R}$, i.e., of all measurable functions $f: X \rightarrow \mathbb{R}$ whose associated quantity $\|f\|_{\mathcal{L}^p(\mu)}$ is finite, where

$$\|f\|_{\mathcal{L}^p(\mu)} := \left(\int |f|^p d\mu \right)^{1/p}.$$

We denote by $\mathcal{L}^\infty(\Sigma)$ the space of all bounded, measurable functions $f: X \rightarrow \mathbb{R}$ and we define

$$\|f\|_{\mathcal{L}^\infty(\Sigma)} := \sup_{x \in X} |f(x)|.$$

It holds that $\mathcal{L}^p(\mu)$ is a complete seminormed space for every $p \in [1, \infty)$, while $\mathcal{L}^\infty(\Sigma)$ is a Banach space. For any $p \in [1, \infty)$, we introduce an equivalence relation \sim_μ on $\mathcal{L}^p(\mu)$: given any $f, g \in \mathcal{L}^p(\mu)$, we declare that $f \sim_\mu g$ if and only if $f = g$ holds μ -a.e. on X . We denote by $L^p(\mu)$ the quotient space $\mathcal{L}^p(\mu)/\sim_\mu$, which is called the **p -Lebesgue space**, and by $\pi_\mu: \mathcal{L}^p(\mu) \rightarrow L^p(\mu)$ the canonical projection map. Observe that $L^p(\mu)$ is a Banach space. Similarly, we introduce an equivalence relation \sim_μ on $\mathcal{L}^\infty(\Sigma)$ as above, we denote by $L^\infty(\mu)$ the quotient space $\mathcal{L}^\infty(\Sigma)/\sim_\mu$, and we call $\pi_\mu: \mathcal{L}^\infty(\Sigma) \rightarrow L^\infty(\mu)$ the canonical projection map. It holds that $L^\infty(\mu)$ is a Banach space if endowed with the quotient norm, i.e., with

$$\|f\|_{L^\infty(\mu)} := \inf \{ \|\bar{f}\|_{\mathcal{L}^\infty(\Sigma)} \mid \bar{f} \in \mathcal{L}^\infty(\Sigma), \pi_\mu(\bar{f}) = f \}, \quad \text{for all } f \in L^\infty(\mu).$$

Given a Banach space \mathbb{B} , we denote by \mathbb{B}' its continuous dual space and by $J_{\mathbb{B}}: \mathbb{B} \hookrightarrow \mathbb{B}'$ the **James' embedding** operator into the bidual space, which is defined as

$$J_{\mathbb{B}}(v)(\omega) := \omega(v), \quad \text{for every } v \in \mathbb{B} \text{ and } \omega \in \mathbb{B}'.$$

Recall that $J_{\mathbb{B}}$ is a linear isometric map and that it is surjective if and only if \mathbb{B} is reflexive.

2.1 Liftings in the sense of von Neumann

Let us recall the basic notions in the lifting theory of von Neumann. Our presentation follows along the one in Fremlin's book [15].

A **Boolean ring** is a ring $(\mathcal{R}, +, \cdot)$ such that $a^2 = a$ for every $a \in \mathcal{R}$. In particular, it holds that $a = -a$ and $ab = ba$ for every $a, b \in \mathcal{R}$. A **Boolean algebra** is a Boolean ring with a multiplicative identity $1_{\mathcal{R}}$. Given any two Boolean algebras \mathcal{R} and \mathcal{R}' , we say that a map $\phi: \mathcal{R} \rightarrow \mathcal{R}'$ is a **Boolean homomorphism** provided it is a ring homomorphism such that $\phi(1_{\mathcal{R}}) = 1_{\mathcal{R}'}$. Given a set $X \neq \emptyset$ and an algebra of sets $\Sigma \subset 2^X$, the triplet (Σ, Δ, \cap) is a Boolean algebra with zero \emptyset and identity X . We recall the notion of a von Neumann lifting:

Definition 2.1 (*von Neumann lifting*) Let (X, Σ, μ) be a measure space with $\mu \neq 0$. Then a map $\ell: \Sigma \rightarrow \Sigma$ is called a **von Neumann lifting** of μ if it satisfies the following properties:

(i) ℓ is a Boolean homomorphism, i.e., it holds $\ell(\emptyset) = \emptyset, \ell(X) = X$, and

$$\ell(E \cap F) = \ell(E) \cap \ell(F), \quad \ell(E \cup F) = \ell(E) \cup \ell(F), \quad \forall E, F \in \Sigma.$$

(ii) $\ell(E) = \ell(F)$ for every $E, F \in \Sigma$ such that $\mu(E \Delta F) = 0$.

(iii) $\mu(E \Delta \ell(E)) = 0$ for every $E \in \Sigma$.

It easily follows from the lifting axioms that each von Neumann lifting ℓ satisfies

$$\ell(\ell(E)) = \ell(E), \quad \text{for every } E \in \Sigma.$$

Liftings exist in high generality. For our purposes, the following existence result is sufficient:

Theorem 2.2 (von Neumann–Maharam) *Let (X, Σ, μ) be a complete, σ -finite measure space with $\mu \neq 0$. Then a von Neumann lifting ℓ of the measure μ exists.*

We refer, e.g., to [15, Section 341] for a proof of Theorem 2.2. However, under some additional (quite mild) assumptions, its proof can be simplified (see Theorem A.6).

Remark 2.3 Given any σ -finite measure space (X, Σ, μ) , it is clearly possible to construct a finite measure $\tilde{\mu}$ on (X, Σ) such that $\mu \ll \tilde{\mu} \ll \mu$, and thus accordingly $L^\infty(\tilde{\mu}) = L^\infty(\mu)$. Observe also that a given map $\ell: \Sigma \rightarrow \Sigma$ is a von Neumann lifting of μ if and only if it is a von Neumann lifting of $\tilde{\mu}$. Therefore, in several occasions (for instance, during the proof of Theorem 4.4) it will not be restrictive to assume that μ is finite.

Given a measurable space (X, Σ) and a set $E \in \Sigma$, we denote by $\mathbb{1}_E \in \mathcal{L}^\infty(\Sigma)$ the **characteristic function** of E , namely, we define $\mathbb{1}_E(x) := 1$ if $x \in E$ and $\mathbb{1}_E(x) := 0$ if $x \in X \setminus E$. The space of **simple functions** on (X, Σ) is then given by

$$S(\Sigma) := \left\{ \sum_{i=1}^n \lambda_i \mathbb{1}_{E_i} \mid n \in \mathbb{N}, (\lambda_i)_{i=1}^n \subset \mathbb{R}, (E_i)_{i=1}^n \subset \Sigma \text{ partition of } X \right\}. \quad (2.1)$$

Now fix a measure μ on (X, Σ) . For the sake of brevity, we denote

$$\mathbb{1}_E^\mu := \pi_\mu(\mathbb{1}_E) \in L^\infty(\mu), \quad \text{for every } E \in \Sigma.$$

Moreover, we denote by $S(\mu) \subset L^\infty(\mu)$ the image of $S(\Sigma)$ under π_μ , namely,

$$S(\mu) := \left\{ \sum_{i=1}^n \lambda_i \mathbb{1}_{E_i}^\mu \mid n \in \mathbb{N}, (\lambda_i)_{i=1}^n \subset \mathbb{R}, (E_i)_{i=1}^n \subset \Sigma \text{ partition of } X \right\}.$$

It holds that the spaces $S(\Sigma)$ and $S(\mu)$ are dense in $\mathcal{L}^\infty(\Sigma)$ and $L^\infty(\mu)$, respectively.

Theorem 2.4 (Lifting of functions) *Let (X, Σ, μ) be a measure space and ℓ a von Neumann lifting of μ . Then there exists a unique linear, continuous map $\ell: L^\infty(\mu) \rightarrow \mathcal{L}^\infty(\Sigma)$ such that*

$$\ell(\mathbb{1}_E^\mu) = \mathbb{1}_{\ell(E)}, \quad \text{for every } E \in \Sigma. \quad (2.2)$$

Moreover, the operator $\ell: L^\infty(\mu) \rightarrow \mathcal{L}^\infty(\Sigma)$ satisfies

$$\pi_\mu(\ell(f)) = f, \quad \text{for every } f \in L^\infty(\mu), \quad (2.3a)$$

$$\ell(fg) = \ell(f)\ell(g), \quad \text{for every } f, g \in L^\infty(\mu), \quad (2.3b)$$

$$\|\ell(f)\|_{\mathcal{L}^\infty(\Sigma)} = \|f\|_{L^\infty(\mu)}, \quad \text{for every } f \in L^\infty(\mu). \quad (2.3c)$$

Proof The required linearity and (2.2) force to define $\ell: S(\mu) \rightarrow \mathcal{S}(\Sigma)$ as

$$\ell\left(\sum_{i=1}^n \lambda_i \mathbb{1}_{E_i}^\mu\right) := \sum_{i=1}^n \lambda_i \mathbb{1}_{\ell(E_i)}, \quad \text{for every } \sum_{i=1}^n \lambda_i \mathbb{1}_{E_i}^\mu \in S(\mu). \quad (2.4)$$

Then the required continuity and the density of $S(\mu)$ in $L^\infty(\mu)$ imply that the map defined in (2.4) can be uniquely extended to a linear, continuous map $\ell: L^\infty(\mu) \rightarrow \mathcal{L}^\infty(\Sigma)$. Finally, the identities stated in (2.3a)–(2.3c) can be first checked on $S(\mu)$ directly from the definition (2.4), and then deduced on the whole $L^\infty(\mu)$ by an approximation argument. \square

Under the identification $\Sigma \ni E \mapsto \mathbb{1}_E \in \mathcal{L}^\infty(\Sigma)$, we know from (2.2) that the von Neumann lifting $\ell: \Sigma \rightarrow \Sigma$ agrees with $\ell \circ \pi_\mu: \mathcal{L}^\infty(\Sigma) \rightarrow \mathcal{L}^\infty(\Sigma)$. Therefore, to adopt the same symbol ℓ both for the von Neumann lifting $\ell: \Sigma \rightarrow \Sigma$ and for the induced lifting $\ell: L^\infty(\mu) \rightarrow \mathcal{L}^\infty(\Sigma)$ at the level of functions should cause no ambiguity. We also point out that in some references (see, e.g., the monograph [5]) von Neumann liftings are defined directly on the space of functions and a posteriori their action on sets is deduced via (2.2).

Remark 2.5 In general, in Theorem 2.4 the space $L^\infty(\mu)$ cannot be replaced by other reasonable function spaces. For instance, it is shown in Ref. [46] that liftings of $L^0(\mu)$ might not exist, where $L^0(\mu)$ stands for the space of measurable functions on X considered up to μ -a.e. equality. Concerning $L^p(\mu)$ spaces with $p \in [1, \infty)$, a first issue is given by the fact that they are typically not closed under multiplication. However, it is proved in Ref. [39] that not even linear liftings defined on $L^p(\mu)$ exist for a generic measure μ .

2.2 Differentiation bases

Below we recall the key concepts concerning the language of differentiation bases, which we learnt from [10]. As a matter of convenience, we slightly modify some definitions, to adapt them to our purposes. In this paper, we will be mostly concerned with differentiation bases induced by a lifting (cf. with Lemma 2.8).

Definition 2.6 (*Differentiation basis*) Let (X, Σ, μ) be a measure space. Then a family $\mathcal{I} \subset \Sigma$ is said to be a **differentiation basis** on (X, Σ, μ) provided the following properties hold:

- (i) $0 < \mu(I) < +\infty$ for every $I \in \mathcal{I}$.
- (ii) Calling $\mathcal{I}_x := \{I \in \mathcal{I} : x \in I\}$ for any $x \in X$, we have that

$$\hat{X} := \{x \in X \mid \mathcal{I}_x \neq \emptyset\} \in \Sigma, \quad \mu(X \setminus \hat{X}) = 0.$$

We say that \hat{X} is the **regular set** associated with \mathcal{I} .

- (iii) \mathcal{I} is **directed by downward inclusion**, meaning that if $x \in \hat{X}$ and $I, I' \in \mathcal{I}_x$, then there exists $J \in \mathcal{I}_x$ such that $J \subset I \cap I'$.

Moreover, given any $x \in \hat{X}$ and $I \in \mathcal{I}_x$, we define the family $\mathcal{U}_x^{\mathcal{I}}(I) \subset \mathcal{I}_x$ as

$$\mathcal{U}_x^{\mathcal{I}}(I) := \{J \in \mathcal{I}_x \mid J \subset I\}.$$

Each differentiation basis \mathcal{I} is naturally associated with the following concept of \mathcal{I} -limit.

Definition 2.7 (\mathcal{I} -limit) Let (X, Σ, μ) be a measure space and let \mathcal{I} be a differentiation basis on (X, Σ, μ) . Let (Y, τ) be a Hausdorff topological space and let $\Phi : \mathcal{I} \rightarrow Y$ be a given map. Fix $x \in \hat{X}$. Then we declare that the \mathcal{I} -**limit** of Φ at x exists and is equal to $y \in Y$, briefly

$$\lim_{I \Rightarrow x} \Phi(I) = y,$$

provided for any $U \in \mathcal{N}_Y(y)$ there exists $I \in \mathcal{I}_x$ such that $\Phi(J) \in U$ holds for every $J \in \mathcal{U}_x^{\mathcal{I}}(I)$.

Observe that if two maps $\Phi, \Psi : \mathcal{I} \rightarrow Y$ agree on $\mathcal{U}_x^{\mathcal{I}}(I)$ for some $x \in \hat{X}$ and $I \in \mathcal{I}_x$, then it holds $\lim_{I \Rightarrow x} \Phi(I) = \lim_{I \Rightarrow x} \Psi(I)$, in the sense that one such \mathcal{I} -limit exists if and only if the other one exists, and in that case they coincide. Consequently, one can define $\lim_{I \Rightarrow x} \Phi(I)$ as soon as Φ is defined just on $\mathcal{U}_x^{\mathcal{I}}(I)$ for some $I \in \mathcal{I}_x$, and not necessarily on the whole \mathcal{I} .

Hereafter, we will mostly focus our attention on those differentiation bases that are induced by a von Neumann lifting, in the sense that is explained by the following result.

Lemma 2.8 (*Differentiation basis induced by a lifting*) Let (X, Σ, μ) be a complete, σ -finite measure space. Let ℓ be a von Neumann lifting of μ . Then the family

$$\mathcal{I}^\ell := \{\ell(E) \mid E \in \Sigma, 0 < \mu(E) < +\infty\} \subset \Sigma$$

is a differentiation basis on (X, Σ, μ) .

Proof Items (i) and (iii) of Definition 2.6 easily follow from items (iii) and (i) of Definition 2.1, respectively. Concerning item (ii) of Definition 2.6, observe that the

σ -finiteness assumption on μ yields existence of a partition $(E_n)_{n \in \mathbb{N}} \subset \Sigma$ of X with $0 < \mu(E_n) < +\infty$ for all $n \in \mathbb{N}$. Hence, $\ell(E_n) \in \mathcal{I}_x^\ell$ for every $n \in \mathbb{N}$ and $x \in \ell(E_n)$, thus accordingly $\bigcup_n \ell(E_n) \subset \hat{X}$. Since μ is complete and $\mu(X \setminus \bigcup_n \ell(E_n)) = 0$, we conclude that $\hat{X} \in \Sigma$ and $\mu(X \setminus \hat{X}) = 0$. \square

Remark 2.9 It can be readily checked that if \hat{X} is the regular set associated with \mathcal{I}^ℓ , then

$$\mu(X) < +\infty \implies \hat{X} = X.$$

Indeed, in this case it holds that $x \in X = \ell(X)$, and thus $X \in \mathcal{I}_x^\ell$, for every $x \in X$.

2.3 Bochner integration

Here, we recall the basics of the integration theory in the sense of Bochner for maps that take values into a Banach space. The notion of Bochner integral, which was introduced in Refs. [4, 13], is a vector-valued generalisation of the Lebesgue integral, so in particular it is defined by means of an approximation argument using simple maps. Some classical references about the Bochner integral are [12, 14].

Let (X, Σ, μ) be a measure space and \mathbb{B} a Banach space. Then we denote by $\mathcal{S}(\mu; \mathbb{B})$ the family of all **\mathbb{B} -simple maps** on (X, Σ, μ) . Namely, we define

$$\mathcal{S}(\mu; \mathbb{B}) := \left\{ \sum_{i=1}^n \mathbb{1}_{E_i} v_i \mid \begin{array}{l} n \in \mathbb{N}, (E_i)_{i=1}^n \subset \Sigma \text{ pairwise disjoint sets,} \\ \mu(E_i) < \infty \text{ for all } i = 1, \dots, n, (v_i)_{i=1}^n \subset \mathbb{B} \end{array} \right\}. \tag{2.5}$$

Observe that $\mathcal{S}(\mu; \mathbb{B})$ is a linear space if endowed with the natural pointwise operations. Given a set $E \in \Sigma$, we define the map $\mathcal{S}(\mu; \mathbb{B}) \ni v \mapsto \int_E v \, d\mu \in \mathbb{B}$ as

$$\int_E v \, d\mu := \sum_{i=1}^n \mu(E \cap E_i) v_i \in \mathbb{B}, \quad \text{for every } v = \sum_{i=1}^n \mathbb{1}_{E_i} v_i \in \mathcal{S}(\mu; \mathbb{B}). \tag{2.6}$$

The resulting operator $\int_E \# \, d\mu$ is linear, and satisfies

$$\int_E \|v(\cdot)\|_{\mathbb{B}} \, d\mu = \sum_{i=1}^n \mu(E \cap E_i) \|v_i\|_{\mathbb{B}}, \quad \left\| \int_E v \, d\mu \right\|_{\mathbb{B}} \leq \int_E \|v(\cdot)\|_{\mathbb{B}} \, d\mu.$$

In the case $E = X$, we just write $\int v \, d\mu$ in place of $\int_X v \, d\mu$.

Remark 2.10 We point out that $\mathcal{S}(\Sigma)$ and $\mathcal{S}(\mu; \mathbb{R})$ do not coincide when μ is an infinite measure. However, denoting $\{f \neq 0\} := \{x \in X \mid f(x) \neq 0\}$ for brevity, we have that

$$\mathcal{S}(\mu; \mathbb{R}) = \{f \in \mathcal{S}(\Sigma) \mid \mu(\{f \neq 0\}) < +\infty\}.$$

On the one hand, for the elements of $\mathcal{S}(\Sigma)$ a finiteness assumption on the quantity $\mu(\{f \neq 0\})$ would destroy the density of $\mathcal{S}(\Sigma)$ in $\mathcal{L}^\infty(\Sigma)$. On the other hand, such finiteness assumption is needed for the elements of $\mathcal{S}(\mu; \mathbb{B})$ to give a meaning to the integral in (2.6).

Definition 2.11 (*Strong measurability*) Let (X, Σ, μ) be a σ -finite measure space, \mathbb{B} a Banach space. Then a map $v : X \rightarrow \mathbb{B}$ is said to be **strongly measurable** if it is μ -measurable and there exists a sequence $(v_n)_{n \in \mathbb{N}} \subset \mathcal{S}(\mu; \mathbb{B})$ of \mathbb{B} -simple maps on (X, Σ, μ) such that

$$\lim_{n \rightarrow \infty} \|v_n(x) - v(x)\|_{\mathbb{B}} = 0, \quad \text{for } \mu\text{-a.e. } x \in X.$$

As an immediate consequence of the definition, all \mathbb{B} -simple maps are strongly measurable. Hereafter, we will often consider strongly measurable maps up to μ -a.e. equality. Namely, we define an equivalence relation \sim_μ on the set of all strongly measurable maps from X to \mathbb{B} as follows: given any $v, w : X \rightarrow \mathbb{B}$ that are strongly measurable, we declare that

$$v \sim_\mu w \iff v(x) = w(x), \quad \text{for } \mu\text{-a.e. } x \in X. \tag{2.7}$$

We denote by $S(\mu; \mathbb{B})$ the quotient of $\mathcal{S}(\mu; \mathbb{B})$ with respect to such equivalence relation \sim_μ .

Definition 2.12 (*Lebesgue–Bochner space*) Let (X, Σ, μ) be a σ -finite measure space, \mathbb{B} a Banach space. Let $p \in [1, \infty]$ be given. Then we say that a strongly measurable map $v : X \rightarrow \mathbb{B}$ belongs to $\mathcal{L}^p(\mu; \mathbb{B})$ if it holds that $\|v(\cdot)\|_{\mathbb{B}} \in \mathcal{L}^p(\mu)$. The **Lebesgue–Bochner space** $L^p(\mu; \mathbb{B})$ is defined as the quotient of $\mathcal{L}^p(\mu; \mathbb{B})$ with respect to the equivalence relation \sim_μ as in (2.7).

Each Lebesgue–Bochner space $L^p(\mu; \mathbb{B})$ is a Banach space with respect to the norm

$$\|v\|_{L^p(\mu; \mathbb{B})} := \|\|v(\cdot)\|_{\mathbb{B}}\|_{L^p(\mu)}, \quad \text{for every } v \in L^p(\mu; \mathbb{B}).$$

It is well-known (see, e.g., [27, Proposition 1.2.2]) that for any $v \in \mathcal{L}^1(\mu; \mathbb{B})$ there exists a sequence $(v_n)_{n \in \mathbb{N}} \subset \mathcal{S}(\mu; \mathbb{B})$ such that $\lim_n \|v_n(x) - v(x)\|_{\mathbb{B}} = 0$ holds for μ -a.e. point $x \in X$ and $\lim_n \int \|v_n(\cdot) - v(\cdot)\|_{\mathbb{B}} \, d\mu = 0$. Consequently, the following definition is meaningful.

Definition 2.13 (*Bochner integral*) Let (X, Σ, μ) be a σ -finite measure space, \mathbb{B} a Banach space. Let $v : X \rightarrow \mathbb{B}$ be strongly measurable. Then we say that the map v is **Bochner integrable** over a set $E \in \Sigma$ provided $\mathbb{1}_E \cdot v \in \mathcal{L}^1(\mu; \mathbb{B})$.

The **Bochner integral** of v over E is then defined as follows: given any sequence $(v_n)_{n \in \mathbb{N}} \subset \mathcal{S}(\mu; \mathbb{B})$ such that $\lim_n \|v_n(x) - v(x)\|_{\mathbb{B}} = 0$ for μ -a.e. point $x \in E$ and $\lim_n \int_E \|v_n(\cdot) - v(\cdot)\|_{\mathbb{B}} \, d\mu = 0$, we set

$$\int_E v \, d\mu := \lim_{n \rightarrow \infty} \int_E v_n \, d\mu \in \mathbb{B}, \tag{2.8}$$

where the limit is intended with respect to the strong topology of \mathbb{B} . In the case where $E = X$, we just write $\int v \, d\mu$ in place of $\int_X v \, d\mu$.

It is easy to show that (2.8) is well-posed, in the sense that the limit appearing in (2.8) exists and it does not depend on the specific choice of the approximating sequence $(v_n)_{n \in \mathbb{N}}$.

It is worth pointing out that if $v: X \rightarrow \mathbb{B}$ is Bochner integrable on $E \in \Sigma$, then it holds

$$\left\| \int_E v \, d\mu \right\|_{\mathbb{B}} \leq \int_E \|v(\cdot)\|_{\mathbb{B}} \, d\mu.$$

This follows from the corresponding statement for \mathbb{B} -simple maps and the density of $S(\mu; \mathbb{B})$ in $L^1(\mu; \mathbb{B})$, thus the integral operator

$$\int_E \# \, d\mu: L^1(\mu; \mathbb{B}) \rightarrow \mathbb{B}$$

is linear and continuous.

Remark 2.14 (Local Bochner integrability) Let (X, Σ, μ) be a σ -finite measure space, \mathcal{I} a differentiation basis on (X, Σ, μ) , and \mathbb{B} a Banach space. Then a given strongly measurable map $v: X \rightarrow \mathbb{B}$ is said to be **locally Bochner integrable with respect to \mathcal{I}** provided for μ -a.e. point $x \in \hat{X}$ there exists a set $I \in \mathcal{I}_x$ such that v is Bochner integrable over I .

In the case where $\mu(X) < +\infty$ and $\mathcal{I} = \mathcal{I}^\ell$ for some von Neumann lifting ℓ of μ , every μ -measurable map $v: X \rightarrow \mathbb{B}$ is locally Bochner integrable with respect to \mathcal{I}^ℓ . Indeed, calling

$$E_k := \{x \in X \mid k-1 \leq \|v(x)\|_{\mathbb{B}} < k\} \in \Sigma, \quad \text{for every } k \in \mathbb{N},$$

we have that $(E_k)_{k \in \mathbb{N}}$ is a partition of X and that $(\mathbb{1}_{E_k} \cdot v)_{k \in \mathbb{N}} \subset \mathcal{L}^\infty(\mu; \mathbb{B})$. In particular, the map v is Bochner integrable over each set $\ell(E_k) \in \mathcal{I}^\ell$. Since $\bigcup_k \ell(E_k)$ has full μ -measure in X , we deduce that v is locally Bochner integrable with respect to \mathcal{I}^ℓ . This observation explains why, differently from the classical Lebesgue differentiation theorem, in the statement of our main Theorem 3.4 no local integrability assumption will be explicitly required.

2.4 Measurable Banach bundles

In this section, we recall the notion of a measurable Banach bundle over a measurable space, which has been introduced in Ref. [11]. Before passing to the actual definition, we first need to fix some terminology. A **multivalued map** $\varphi: S \rightrightarrows T$ between two sets S and T is a mapping from S to the power set 2^T of T . Given a measurable space (X, Σ) and a topological space (Y, τ) , we say that a multivalued map $\varphi: X \rightrightarrows Y$ is **weakly measurable** provided it holds that

$$\{x \in X \mid \varphi(x) \cap U \neq \emptyset\} \in \Sigma, \quad \text{for every } U \subset Y \text{ open.}$$

Let us now introduce the notion of a measurable Banach bundle over a measurable space:

Definition 2.15 (*Banach bundle*) Let (X, Σ) be a measurable space and \mathbb{B} a Banach space. Then a given multivalued map $\mathbf{E}: X \rightarrow \mathbb{B}$ is said to be a **measurable Banach \mathbb{B} -bundle** (or just a **Banach \mathbb{B} -bundle**) over (X, Σ) provided it is weakly measurable and $\mathbf{E}(x)$ is a closed, linear subspace of \mathbb{B} for every $x \in X$.

In [11, Definition 4.1] the definition of Banach \mathbb{B} -bundle was formulated only for \mathbb{B} separable. Here, we allow for an arbitrary Banach space \mathbb{B} , so that also the case of Lebesgue–Bochner spaces is covered by this theory. In this regard, observe that the multivalued map associating to every $x \in X$ the whole space \mathbb{B} is, trivially, a Banach \mathbb{B} -bundle over (X, Σ) ; with a slight abuse of notation, we denote it by \mathbb{B} . A consistency check will be expressed by (2.9) below.

Let us remind the concept of a measurable section of a given measurable Banach bundle:

Definition 2.16 (*Section of a Banach bundle*) Let (X, Σ) be a measurable space. Let \mathbb{B} be a Banach space and \mathbf{E} a Banach \mathbb{B} -bundle over (X, Σ) . Then a map $v: X \rightarrow \mathbb{B}$ is said to be a **measurable section** of \mathbf{E} provided it is strongly measurable and $v(x) \in \mathbf{E}(x)$ for all $x \in X$.

Given a σ -finite measure space (X, Σ, μ) , a Banach space \mathbb{B} , a Banach \mathbb{B} -bundle \mathbf{E} over (X, Σ) , and an exponent $p \in [1, \infty]$, we define the space of **p -integrable sections** of \mathbf{E} as

$$\bar{\Gamma}_p(\mu; \mathbf{E}) := \{v \in \mathcal{L}^p(\mu; \mathbb{B}) \mid v \text{ is a measurable section of } \mathbf{E}\}.$$

Calling $\pi_\mu: \mathcal{L}^p(\mu; \mathbb{B}) \rightarrow L^p(\mu; \mathbb{B})$ the quotient map, the space of **L^p -sections** of \mathbf{E} is given by

$$\Gamma_p(\mu; \mathbf{E}) := \pi_\mu(\bar{\Gamma}_p(\mu; \mathbf{E})).$$

It holds that $\Gamma_p(\mu; \mathbf{E})$ is a closed, linear subspace of $L^p(\mu; \mathbb{B})$. Also, the very definitions yield

$$\bar{\Gamma}_p(\mu; \mathbb{B}) = \mathcal{L}^p(\mu; \mathbb{B}), \quad \Gamma_p(\mu; \mathbb{B}) = L^p(\mu; \mathbb{B}). \tag{2.9}$$

Remark 2.17 It is easy to check that the space $\Gamma_p(\mu; \mathbf{E})$ is actually an **$L^p(\mu)$ -normed $L^\infty(\mu)$ -module**, in the sense of Gigli (see [18, Definition 1.2.10] for the relevant definition).

As a matter of fact, the theory of normed modules (or, more specifically, of separable normed modules) was one of the main motivations behind the introduction of measurable Banach bundles in the sense of Definition 2.15. Separable normed modules play a key role in the setting of differential calculus over metric measure spaces, since under mild assumptions the *cotangent module* is separable (see [18, Proposition 2.2.5] and [11, Theorem B.1]).

When \mathbb{B} is separable, Banach \mathbb{B} -bundles are completely characterised by their sections. One of the main achievements of [11] is that every separable $L^p(\mu)$ -normed $L^\infty(\mu)$ -module (over a σ -finite measure space) can be written as the space of L^p -sections of some Banach \mathbb{B} -bundle, where \mathbb{B} is a fixed *universal* separable Banach space, i.e., a separable Banach space where every separable Banach space can be embedded linearly and isometrically. We recall that the existence of universal separable Banach spaces follows from the *Banach–Mazur Theorem* (cf. with [2, Proposition 7.5 of Chapter II]), which states that $C([0, 1])$ enjoys this property.

Conversely, in the case where \mathbb{B} is non-separable, we do not know if Banach \mathbb{B} -bundles are completely determined by their sections, the reason being that we do not have a measurable selection theorem for an arbitrary target \mathbb{B} . In other words, there might possibly exist two different Banach \mathbb{B} -bundles \mathbf{E} and \mathbf{F} such that $\Gamma_p(\mu; \mathbf{E})$ and $\Gamma_p(\mu; \mathbf{F})$ are isomorphic.

2.5 Radon–Nikodým property

Aim of this section is to recall the basics of the theory of vector-valued measures, as well as the strictly related concept of a Banach space having the Radon–Nikodým property, often abbreviated as RNP in the literature. In short, a Banach space \mathbb{B} has the RNP provided every \mathbb{B} -valued measure of bounded variation verifies the classical Radon–Nikodým Theorem. For a thorough account of the Radon–Nikodým property, we refer to [1, 12] and the references therein. Our brief presentation, which will cover only those few notions that are strictly needed for our purposes, is mostly taken from [27].

Let (X, Σ) be a measurable space and \mathbb{B} a Banach space. A mapping $\Omega: \Sigma \rightarrow \mathbb{B}$ is said to be a **\mathbb{B} -valued measure** on (X, Σ) provided the following property is satisfied: given any partition $(E_n)_{n \in \mathbb{N}} \subset \Sigma$ of $E \in \Sigma$, it holds that

$$\lim_{N \rightarrow \infty} \left\| \Omega(E) - \sum_{n=1}^N \Omega(E_n) \right\|_{\mathbb{B}} = 0.$$

We say that Ω is **bounded** (or **of bounded semivariation**) provided its range is a bounded subset of \mathbb{B} (cf. with [12, Proposition I.1.11]). Following [12, Definition I.1.12], it is possible to integrate bounded scalar-valued functions with respect to a bounded vector-valued measure:

Definition 2.18 (*Elementary Bartle integral*) Let (X, Σ) be a measurable space and \mathbb{B} a Banach space. Let Ω be a bounded \mathbb{B} -valued measure on (X, Σ) . Then we define the map

$$\mathcal{L}^\infty(\Sigma) \ni f \mapsto \int f \, d\Omega \in \mathbb{B},$$

called the **elementary Bartle integral**, as the unique linear continuous operator such that

$$\int f \, d\Omega = \sum_{i=1}^n \lambda_i \Omega(E_i), \quad \text{for every } f = \sum_{i=1}^n \lambda_i \mathbb{1}_{E_i} \in \mathcal{S}(\Sigma). \quad (2.10)$$

Moreover, given $f \in \mathcal{L}^\infty(\Sigma)$ and $E \in \Sigma$, we define $\int_E f \, d\Omega := \int \mathbb{1}_E f \, d\Omega \in \mathbb{B}$.

Observe that, indeed, the operator $\mathcal{S}(\Sigma) \ni f \mapsto \int f \, d\Omega \in \mathbb{B}$ defined in (2.10) verifies

$$\begin{aligned} \left| \omega\left(\int f \, d\Omega\right) \right| &= \left| \sum_{i=1}^n \lambda_i \omega(\Omega(E_i)) \right| \leq \sum_{i=1}^n |\lambda_i| |\omega(\Omega(E_i))| \\ &\leq \|f\|_{\mathcal{L}^\infty(\Sigma)} \left(\sum_{i \in P} \omega(\Omega(E_i)) - \sum_{i \in N} \omega(\Omega(E_i)) \right) \\ &= \|f\|_{\mathcal{L}^\infty(\Sigma)} \left(\omega(\Omega(E^+)) - \omega(\Omega(E^-)) \right) \\ &\leq \|f\|_{\mathcal{L}^\infty(\Sigma)} \left(\|\Omega(E^+)\|_{\mathbb{B}} + \|\Omega(E^-)\|_{\mathbb{B}} \right) \\ &\leq 2 \|f\|_{\mathcal{L}^\infty(\Sigma)} \sup_{E \in \Sigma} \|\Omega(E)\|_{\mathbb{B}}, \end{aligned}$$

for every $\omega \in \mathbb{B}'$ with $\|\omega\|_{\mathbb{B}'} \leq 1$ and $f = \sum_{i=1}^n \lambda_i \mathbb{1}_{E_i} \in \mathcal{S}(\Sigma)$, where we set

$$\begin{aligned} E^+ &:= \bigcup_{i \in P} E_i, \quad \text{where } P := \left\{ i = 1, \dots, n \mid \omega(\Omega(E_i)) \geq 0 \right\}, \\ E^- &:= \bigcup_{i \in N} E_i, \quad \text{where } N := \{1, \dots, n\} \setminus P. \end{aligned}$$

By passing to the supremum over all $\omega \in \mathbb{B}'$ with $\|\omega\|_{\mathbb{B}'} \leq 1$, we thus obtain that

$$\left\| \int f \, d\Omega \right\|_{\mathbb{B}} \leq 2 \|f\|_{\mathcal{L}^\infty(\Sigma)} \sup_{E \in \Sigma} \|\Omega(E)\|_{\mathbb{B}}, \quad \text{for every } f \in \mathcal{S}(\Sigma). \quad (2.11)$$

Moreover, standard verifications show that the map $\mathcal{S}(\Sigma) \ni f \mapsto \int f \, d\Omega \in \mathbb{B}$ is linear. Therefore, Definition 2.18 is well-posed and the inequality (2.11) holds for all $f \in \mathcal{L}^\infty(\Sigma)$.

The **variation** $\|\Omega\|_{\mathbb{B}} : \Sigma \rightarrow [0, +\infty]$ of a \mathbb{B} -valued measure Ω is defined at $E \in \Sigma$ as

$$\|\Omega\|_{\mathbb{B}}(E) := \sup \left\{ \sum_{i=1}^n \|\Omega(E_i)\|_{\mathbb{B}} \mid n \in \mathbb{N}, (E_i)_{i=1}^n \subset \Sigma \text{ partition of } E \right\}. \quad (2.12)$$

We say that Ω has **bounded variation** provided $\|\Omega\|_{\mathbb{B}}(X) < +\infty$. In this case, it holds that the variation $\|\Omega\|_{\mathbb{B}}$ is a finite measure on (X, Σ) and that Ω is bounded.

Observe that any (non-negative) finite measure μ on (X, Σ) has bounded variation and satisfies $\|\mu\|_{\mathbb{B}} = \mu$.

Remark 2.19 If $\Omega: \Sigma \rightarrow \mathbb{B}$ has bounded variation, its variation $\|\Omega\|_{\mathbb{B}}$ can be characterised as the least measure $\mu: \Sigma \rightarrow [0, +\infty)$ such that the inequality $\|\Omega(E)\|_{\mathbb{B}} \leq \mu(E)$ holds for every $E \in \Sigma$.

We say that $\Omega: \Sigma \rightarrow \mathbb{B}$ is **absolutely continuous** with respect to μ , briefly $\Omega \ll \mu$, if

$$\Omega(N) = 0_{\mathbb{B}}, \quad \text{for every } N \in \Sigma \text{ with } \mu(N) = 0.$$

It holds that $\Omega \ll \mu$ if and only if $\|\Omega\|_{\mathbb{B}} \ll \mu$.

The set of \mathbb{B} -valued measures having bounded variation is closed under various operations:

- (i) **RESTRICTION.** Let (X, Σ) be a measurable space, \mathbb{B} a Banach space, and Ω a bounded \mathbb{B} -valued measure on (X, Σ) . Then for any given $f \in \mathcal{L}^{\infty}(\Sigma)$ we define $f\Omega: \Sigma \rightarrow X$ as

$$(f\Omega)(E) := \int_E f \, d\Omega, \quad \text{for every } E \in \Sigma.$$

It holds that $f\Omega$ is a bounded \mathbb{B} -valued measure on (X, Σ) . Moreover, if Ω has bounded variation, then $f\Omega$ has bounded variation as well and it holds that

$$\|f\Omega\|_{\mathbb{B}} = |f| \|\Omega\|_{\mathbb{B}}, \quad \text{for every } f \in \mathcal{L}^{\infty}(\Sigma). \tag{2.13}$$

In particular, for any set $F \in \Sigma$ we can consider the **restriction** of Ω to F , which is given by $\Omega|_F := \mathbb{1}_F \Omega$ and satisfies $\|\Omega|_F\|_{\mathbb{B}} = \|\Omega\|_{\mathbb{B}}|_F$ when Ω has bounded variation.

- (ii) **PUSHFORWARD.** Let $(X, \Sigma_X), (Y, \Sigma_Y)$ be measurable spaces, \mathbb{B} a Banach space, Ω a \mathbb{B} -valued measure on (X, Σ_X) , and $\varphi: X \rightarrow Y$ a measurable map. Then we define the **pushforward** of Ω under φ as

$$\varphi_{\#}\Omega(E) := \Omega(\varphi^{-1}(E)) \in \mathbb{B}, \quad \text{for every } E \in \Sigma_Y.$$

It holds that $\varphi_{\#}\Omega: \Sigma_Y \rightarrow \mathbb{B}$ is a \mathbb{B} -valued measure on (Y, Σ_Y) . Moreover, if Ω has bounded variation, then $\varphi_{\#}\Omega$ has bounded variation as well and it holds that

$$\|\varphi_{\#}\Omega\|_{\mathbb{B}} \leq \varphi_{\#}\|\Omega\|_{\mathbb{B}}. \tag{2.14}$$

In Sect. 5 we will focus our attention on the following special class of Banach spaces:

Definition 2.20 (*Radon–Nikodým property*) Let \mathbb{B} be a Banach space. Let (X, Σ, μ) be a σ -finite measure space. Then we say that \mathbb{B} has the **Radon–Nikodým property with respect to** (X, Σ, μ) if the following property holds: given any \mathbb{B} -valued measure Ω on

(X, Σ) having bounded variation and satisfying $\Omega \ll \mu$, there exists $\frac{d\Omega}{d\mu} \in L^1(\mu; \mathbb{B})$ such that

$$\Omega(E) = \int_E \frac{d\Omega}{d\mu} d\mu, \quad \text{for every } E \in \Sigma. \tag{2.15}$$

The element $\frac{d\Omega}{d\mu}$, which is uniquely determined by (2.15), is called the **Radon–Nikodým derivative** of Ω with respect to μ . Moreover, we say that \mathbb{B} has the **Radon–Nikodým property** if it has the Radon–Nikodým property with respect to every σ -finite measure space.

A Banach space has the Radon–Nikodým property if and only if it has the Radon–Nikodým property with respect to $([0, 1], \mathcal{B}([0, 1]), \mathcal{L}^1|_{[0,1]})$, where $\mathcal{B}([0, 1])$ is the Borel σ -algebra of the interval $[0, 1]$ and \mathcal{L}^1 is the Lebesgue measure on \mathbb{R} ; cf. with [27, Theorem 1.3.26].

Remark 2.21 The Radon–Nikodým property is **separably determined**, in the sense that a Banach space \mathbb{B} has the RNP if and only if every closed, separable linear subspace of \mathbb{B} has the RNP. We refer, e.g., to [27, Theorem 1.3.18] for a proof of this fact. We also recall that the Banach space c_0 of all sequences $(\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ converging to 0, endowed with the supremum norm $\|(\alpha_n)_{n \in \mathbb{N}}\|_{c_0} := \sup_{n \in \mathbb{N}} |\alpha_n|$, does not have the RNP (according to [12, Example III.1.1]). Therefore, no linear subspace of a Banach space \mathbb{B} having the RNP is isometrically isomorphic to c_0 . In particular, given any compact, Hausdorff topological space (X, τ) , we know from [12, Theorem VI.2.15] that every linear continuous operator $T: C(X) \rightarrow \mathbb{B}$ is **weakly compact**, meaning that the weak closure of the image under T of the unit sphere of $C(X)$ is a weakly compact set. This fact will have a role during the proof of Theorem 5.9.

2.6 Ultrafilters and ultralimits

In the sequel, it will be often convenient to work with filter convergence and ultralimits, which we are going to remind for the reader’s usefulness. The material we will present is very standard, see for instance the book [6].

Definition 2.22 (*Filter and ultrafilter*) Let Λ be a non-empty set. Then a family $\mathcal{F} \subset 2^\Lambda$ of subsets of Λ is said to be a **filter** on Λ provided the following properties are satisfied:

- (i) $\emptyset \notin \mathcal{F}$.
- (ii) If $S, T \in \mathcal{F}$, then $S \cap T \in \mathcal{F}$.
- (iii) If $S \in \mathcal{F}$ and $T \in 2^\Lambda$ satisfy $S \subset T$, then $T \in \mathcal{F}$.

A filter ω on Λ which is maximal with respect to inclusion is said to be an **ultrafilter** on Λ .

It holds that a given filter \mathcal{F} on Λ is an ultrafilter if and only if for any $S \in 2^\Lambda$ one has that either $S \in \mathcal{F}$ or $\Lambda \setminus S \in \mathcal{F}$. An ultrafilter ω on Λ is said to be **principal** if it admits a least element, **non-principal** if not. In the case where ω is principal, its

least element is uniquely determined and is a singleton. When the set Λ is finite, every ultrafilter on Λ is principal.

An important example of filter is the following one: given a topological space (Y, τ) and a point $y \in Y$, the family of all (not necessarily open) neighbourhoods of y is a filter on Y . Conversely, $\mathcal{N}_Y(y)$ is typically not a filter, as there might well be neighbourhoods of y which are not open. In this paper, we mostly work with the filter induced by a differentiation basis:

Definition 2.23 (*Filter induced by a differentiation basis*) Let (X, Σ, μ) be a measure space and let \mathcal{I} be a differentiation basis on (X, Σ, μ) . Let us define $\tilde{\mathcal{I}} := \{X \setminus I : I \in \mathcal{I}\}$ and

$$\begin{aligned} \tilde{\mathcal{I}}_x &:= \{X \setminus I \mid I \in \mathcal{I}_x\}, \quad \text{for every } x \in \hat{X}, \\ \tilde{\mathcal{U}}_x^{\tilde{\mathcal{I}}}(I) &:= \{X \setminus J \mid J \in \mathcal{U}_x^{\mathcal{I}}(I)\}, \quad \text{for every } x \in \hat{X} \text{ and } I \in \mathcal{I}_x. \end{aligned}$$

Then for any regular point $x \in \hat{X}$ we define the filter $\mathcal{F}_x^{\tilde{\mathcal{I}}}$ on $\tilde{\mathcal{I}}_x$ as

$$\mathcal{F}_x^{\tilde{\mathcal{I}}} := \{S \subset \tilde{\mathcal{I}}_x \mid \tilde{\mathcal{U}}_x^{\tilde{\mathcal{I}}}(I) \subset S, \text{ for some } I \in \mathcal{I}_x\}.$$

We say that $\mathcal{F}_x^{\tilde{\mathcal{I}}}$ is the **filter at x induced by \mathcal{I}** . In the case where $\mathcal{I} = \mathcal{I}^\ell$ for some von Neumann lifting ℓ of μ , we write $\mathcal{U}_x^\ell(I)$ and $\tilde{\mathcal{U}}_x^\ell(I)$ in place of $\mathcal{U}_x^{\mathcal{I}^\ell}(I)$ and $\tilde{\mathcal{U}}_x^{\mathcal{I}^\ell}(I)$, respectively.

Filters on a topological space may be regarded as ‘locating schemes’, and as such they naturally come with the following notion of convergence:

Definition 2.24 (*Convergence of filters*) Let (Y, τ) be a Hausdorff topological space and \mathcal{F} a filter on Y . Let $y \in Y$ be given. Then we say that \mathcal{F} **converges to y** , briefly $\mathcal{F} \rightarrow y$, provided it holds that $\mathcal{N}_Y(y) \subset \mathcal{F}$.

Given two non-empty sets Λ and Ξ , a map $\phi: \Lambda \rightarrow \Xi$, and a filter \mathcal{F} on Λ , we define

$$\phi_*\mathcal{F} := \{T \in 2^\Xi \mid \phi(S) \subset T, \text{ for some } S \in \mathcal{F}\}.$$

The family $\phi_*\mathcal{F}$, which is a filter on Ξ , is called the **pushforward filter of \mathcal{F} under ϕ** .

The following result shows that \mathcal{I} -limits can be expressed in terms of filter convergence.

Lemma 2.25 *Let (X, Σ, μ) be a measure space and \mathcal{I} a differentiation basis on (X, Σ, μ) . Let (Y, τ) be a Hausdorff topological space and $\Phi: \mathcal{I} \rightarrow Y$ a map. Fix $x \in \hat{X}$ and $y \in Y$. Then*

$$\lim_{I \Rightarrow x} \Phi(I) = y \iff \tilde{\Phi}_*\mathcal{F}_x^{\tilde{\mathcal{I}}} \rightarrow y,$$

where the map $\tilde{\Phi}: \tilde{\mathcal{I}} \rightarrow Y$ is defined as $\tilde{\Phi}(X \setminus I) := \Phi(I)$ for every $I \in \mathcal{I}$.

Proof The statement follows from the observation that, given any $U \in \mathcal{N}_Y(y)$ and $I \in \mathcal{I}_x$, it holds that $\Phi(J) \in U$ for every $J \in \mathcal{U}_x^{\mathcal{I}}(I)$ if and only if $\tilde{\Phi}(\tilde{\mathcal{U}}_x^{\mathcal{I}}(I)) = \Phi(\mathcal{U}_x^{\mathcal{I}}(I)) \subset U$. □

Definition 2.26 (Ultralimit) Let $\Lambda \neq \emptyset$ be a set and ω an ultrafilter on Λ . Let (Y, τ) be a Hausdorff topological space and $\Phi: \Lambda \rightarrow Y$ a given map. Then we declare that the **ultralimit** of the map Φ with respect to ω exists and is equal to $y \in Y$, briefly

$$\omega\text{-}\lim_i \Phi(i) = y,$$

provided for any $U \in \mathcal{N}_Y(y)$ it holds that $\{i \in \Lambda : \Phi(i) \in U\} \in \omega$.

Remark 2.27 It is worth pointing out that the notion of ultralimit can be reformulated in terms of filter convergence. Namely, it holds $\omega\text{-}\lim_i \Phi(i) = y$ if and only if $\Phi_*\omega \rightarrow y$.

Let us recall a few basic calculus rules concerning ultralimits:

- (i) When the ultralimit $\omega\text{-}\lim_i \Phi(i)$ exists, it is uniquely determined (by the Hausdorff assumption on Y). If ω is principal and its least element is given by $\{i_0\}$, then

$$\exists \omega\text{-}\lim_i \Phi(i) = \Phi(i_0).$$

Conversely, if ω is non-principal, then the ultralimit $\omega\text{-}\lim_i \Phi(i)$ might not exist. A sufficient condition for its existence is given by the compactness of (Y, τ) .

- (ii) Let $f: Y \rightarrow \mathbb{R}$ be a lower semicontinuous function. Suppose that both $\omega\text{-}\lim_i \Phi(i)$ and $\omega\text{-}\lim_i f(\Phi(i))$ exist. Then it holds that

$$f\left(\omega\text{-}\lim_i \Phi(i)\right) \leq \omega\text{-}\lim_i f(\Phi(i)). \tag{2.16}$$

- (iii) Suppose (Y, d) is a separated topological vector space. Let $\Phi, \Psi: \Lambda \rightarrow Y$ be two given maps and $\lambda \in \mathbb{R}$. Suppose that $\omega\text{-}\lim_i \Phi(i)$ and $\omega\text{-}\lim_i \Psi(i)$ exist. Then it holds that

$$\exists \omega\text{-}\lim_i (\Phi(i) + \Psi(i)) = \left(\omega\text{-}\lim_i \Phi(i)\right) + \left(\omega\text{-}\lim_i \Psi(i)\right), \tag{2.17a}$$

$$\exists \omega\text{-}\lim_i (\lambda \Phi(i)) = \lambda\left(\omega\text{-}\lim_i \Phi(i)\right). \tag{2.17b}$$

The Ultrafilter Lemma – which we will enunciate in the next result – is particularly relevant for us. In ZF, the Ultrafilter Lemma is strictly weaker than the Axiom of Choice, but strictly stronger than the Hahn–Banach Extension Theorem.

Theorem 2.28 (Ultrafilter lemma) *Let $\Lambda \neq \emptyset$ be a set and \mathcal{F} a filter on Λ . Then there exists an ultrafilter ω on Λ containing \mathcal{F} .*

3 Metric-valued Lebesgue differentiation theorem in measure spaces

In this section, we prove a version of Lebesgue Differentiation Theorem for metric-valued maps defined on a measure space (Theorem 3.4), as well as some of its direct consequences.

We first introduce some relevant terminology. Let (X, Σ, μ) be a σ -finite measure space and (Y, τ) a Hausdorff topological space. Then we say that a measurable map $\varphi: X \rightarrow Y$ is **essentially separably valued** if there exists $N \in \Sigma$ such that $\mu(N) = 0$ and $\varphi(X \setminus N)$ is a separable subset of Y .

Remark 3.1 Given any Banach space \mathbb{B} , it is well-known that a μ -measurable map $v: X \rightarrow \mathbb{B}$ is essentially separably valued if and only if it is strongly measurable.

Theorem 3.4 will easily follow from the ensuing key result, which states that every essentially separably valued, measurable map from a complete measure space to a metric space is ‘approximately continuous’ at almost every point with respect to the differentiation basis induced by a lifting. In fact, this notion of approximate continuity can be expressed in terms of a suitable topology, called the *density topology*. Since this characterisation is interesting but not strictly needed for our purposes, we postpone the relative discussion to Appendix B.

Lemma 3.2 *Let (X, Σ, μ) be a complete, σ -finite measure space and let (Y, d) be a metric space. Let ℓ be a von Neumann lifting of μ . Let $\varphi: X \rightarrow Y$ be an essentially separably valued, measurable map. Then for μ -a.e. point $x \in X$ the following property holds:*

$$\forall \varepsilon > 0, \exists I \in \mathcal{I}_x^\ell : d(\varphi(z), \varphi(x)) < \varepsilon, \text{ for } \mu\text{-a.e. } z \in I. \tag{3.1}$$

Proof Pick a set $N' \in \Sigma$ with $\mu(N') = 0$ and a separable set $Y' \subset Y$ such that $\varphi(X \setminus N') \subset Y'$. Fix a dense sequence $(y_n)_{n \in \mathbb{N}}$ in Y' . Let us define

$$A_n^k := \{x \in X \setminus N' \mid d(\varphi(x), y_n) < 1/k\}, \quad B_n^k := \ell(A_n^k), \quad \text{for all } n, k \in \mathbb{N}.$$

Given that $\mu(A_n^k \Delta B_n^k) = 0$ for every $n, k \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} A_n^k = X \setminus N'$ for every $k \in \mathbb{N}$, we have that $N := X \setminus \bigcap_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} A_n^k \cap B_n^k \in \Sigma$ is μ -negligible. We claim that each point x belonging to the full μ -measure set $\hat{X} \setminus N$ satisfies (3.1). To prove it, fix $x \in \hat{X} \setminus N$ and $\varepsilon > 0$. Choose any $k \in \mathbb{N}$ for which $1/k \leq \varepsilon/2$. Hence, we can find $n \in \mathbb{N}$ such that $x \in A_n^k \cap B_n^k$. Being $x \in \hat{X}$, there exists some $I' \in \mathcal{I}_x^\ell$. Let us now define $I := I' \cap B_n^k \in \mathcal{I}_x^\ell$. Observe that

$$d(\varphi(z), \varphi(x)) \leq d(\varphi(z), y_n) + d(y_n, \varphi(x)) < \frac{1}{k} + \frac{1}{k} \leq \varepsilon,$$

for all $z \in A_n^k$ and thus for μ -a.e. $z \in I$. This proves (3.1) and accordingly the statement. □

Remark 3.3 Albeit not needed for our purposes, we point out that Lemma 3.2 holds in greater generality. More precisely, the target can be chosen to be a *uniformisable*,

Hausdorff topological space (Y, τ) . In this case, the statement reads as: for μ -a.e. $x \in X$, it holds that

$$\forall \text{entourage } U \text{ of } Y, \quad \exists I \in \mathcal{I}_x^\ell : (\varphi(z), \varphi(x)) \in U, \quad \text{for } \mu\text{-a.e. } z \in I,$$

as one can check by suitably adapting the proof of Lemma 3.2.

Having Lemma 3.2 at disposal, it is almost immediate to obtain the following result:

Theorem 3.4 (Metric-valued Lebesgue differentiation theorem) *Let (X, Σ, μ) be a complete, σ -finite measure space and let (Y, d) be a metric space. Let ℓ be a von Neumann lifting of μ . Let $\varphi : X \rightarrow Y$ be an essentially separably valued, measurable map. Then it holds that*

$$\lim_{I \Rightarrow x} \int_I d(\varphi(\cdot), \varphi(x)) \, d\mu = 0, \quad \text{for } \mu\text{-a.e. } x \in X. \tag{3.2}$$

In particular, if \mathbb{B} is a Banach space and $v : X \rightarrow \mathbb{B}$ is a strongly measurable map, then

$$v(x) = \lim_{I \Rightarrow x} \int_I v \, d\mu, \quad \text{for } \mu\text{-a.e. } x \in X. \tag{3.3}$$

The Bochner integral in the right-hand side of (3.3) is well-defined thanks to Remark 2.14.

Proof To prove (3.2) amounts to showing that for μ -a.e. $x \in X$ the following property holds:

$$\forall \varepsilon > 0 \quad \exists I \in \mathcal{I}_x^\ell : \int_J d(\varphi(\cdot), \varphi(x)) \, d\mu \leq \varepsilon, \quad \text{for every } J \in \mathcal{U}_x^\ell(I). \tag{3.4}$$

This readily stems from Lemma 3.2: given a point $x \in X$ satisfying (3.1) and any $\varepsilon > 0$, there exists $I \in \mathcal{I}_x^\ell$ such that $d(\varphi(z), \varphi(x)) < \varepsilon$ for μ -a.e. $z \in I$. In particular, for any $J \in \mathcal{U}_x^\ell(I)$ we may estimate $\int_J d(\varphi(\cdot), \varphi(x)) \, d\mu \leq \varepsilon$, whence the claim (3.4) follows.

We pass to the verification of (3.3). In view of Lemma 3.2 and (3.2), we know that for μ -a.e. $x \in X$ there exists $I' \in \mathcal{I}_x^\ell$ such that v is bounded (thus, Bochner integrable) on I' and

$$\lim_{I \Rightarrow x} \int_I \|v(\cdot) - v(x)\|_{\mathbb{B}} \, d\mu = 0. \tag{3.5}$$

Observe also that for every $I \in \mathcal{U}_x^\ell(I')$ it holds that $\|\int_I v \, d\mu - v(x)\|_{\mathbb{B}} \leq \int_I \|v(\cdot) - v(x)\|_{\mathbb{B}} \, d\mu$. By letting $I \Rightarrow x$ and using (3.5) in the previous estimate, we obtain (3.3), as required. \square

Theorem 3.4 ensures that, once a von Neumann lifting of the reference measure is fixed, the L^p -sections of a given Banach bundle admit a distinguished representative:

Theorem 3.5 (Lebesgue points and precise representatives) *Let (X, Σ, μ) be a complete, σ -finite measure space and ℓ a von Neumann lifting of μ . Let \mathbb{B} be a Banach space and \mathbf{E} a Banach \mathbb{B} -bundle over (X, Σ) . Let $p \in [1, \infty]$ and $v \in \Gamma_p(\mu; \mathbf{E})$ be given. Then μ -a.e. $x \in \hat{X}$ is a **Lebesgue point** of v , meaning that the following \mathcal{I}^ℓ -limit exists and belongs to $\mathbf{E}(x)$:*

$$\hat{v}(x) := \lim_{I \rightarrow x} \int_I v \, d\mu \in \mathbf{E}(x). \tag{3.6}$$

Moreover, setting $\hat{v}(x) := 0_{\mathbf{E}(x)}$ for every non-Lebesgue point $x \in X$ of v , it holds that the resulting map $\hat{v}: X \rightarrow \mathbb{B}$ belongs to $\bar{\Gamma}_p(\mu; \mathbf{E})$ and satisfies $\pi_\mu(\hat{v}) = v$. We will say that the operator $\hat{\#}: \Gamma_p(\mu; \mathbf{E}) \rightarrow \bar{\Gamma}_p(\mu; \mathbf{E})$ is the **precise representative map** associated with ℓ .

Proof The claim is a direct consequence of Theorem 3.4. Notice that the precise representative map $\hat{\#}: \Gamma_p(\mu; \mathbf{E}) \rightarrow \bar{\Gamma}_p(\mu; \mathbf{E})$ is well-posed, since two μ -a.e. equivalent elements of $\bar{\Gamma}_p(\mu; \mathbf{E})$ must share the same precise representative, as one can immediately deduce from (3.6). □

Another immediate consequence of Theorem 3.4 is the following: on Banach spaces having the RNP, the so-called Radon–Nikodým derivative truly behaves as a derivative with respect to the differentiation basis induced by some von Neumann lifting. More precisely:

Corollary 3.6 (Differentiation of \mathbb{B} -valued measures) *Let (X, Σ, μ) be a complete, σ -finite measure space. Let \mathbb{B} be a Banach space having the Radon–Nikodým property. Let Ω be any \mathbb{B} -valued measure on (X, Σ) having bounded variation and satisfying $\Omega \ll \mu$. Then it holds*

$$\frac{d\Omega}{d\mu}(x) = \lim_{I \rightarrow x} \frac{\Omega(I)}{\mu(I)} \in \mathbb{B}, \quad \text{for } \mu\text{-a.e. } x \in X. \tag{3.7}$$

Proof Fix a representative $v \in \mathcal{L}^1(\mu; \mathbb{B})$ of $\frac{d\Omega}{d\mu} \in L^1(\mu; \mathbb{B})$. Then Theorem 3.4 yields

$$v(x) = \lim_{I \rightarrow x} \int_I v \, d\mu = \lim_{I \rightarrow x} \int_I \frac{d\Omega}{d\mu} \, d\mu = \lim_{I \rightarrow x} \frac{\Omega(I)}{\mu(I)}, \quad \text{for } \mu\text{-a.e. } x \in X,$$

whence the claimed identity (3.7) follows. □

Corollary 3.6 can be used to provide an immediate proof of the following well-known fact, which will be used in Sect. 5.2. Under the assumptions of Corollary 3.6, we claim that

$$\left\| \frac{d\Omega}{d\mu}(\cdot) \right\|_{\mathbb{B}} \leq \frac{d\|\Omega\|_{\mathbb{B}}}{d\mu}, \quad \text{holds } \mu\text{-a.e. on } X. \tag{3.8}$$

Indeed, for μ -a.e. $x \in X$ we may estimate

$$\begin{aligned} \left\| \frac{d\Omega}{d\mu}(x) \right\|_{\mathbb{B}} &\stackrel{(3.7)}{=} \left\| \lim_{I \rightarrow x} \frac{\Omega(I)}{\mu(I)} \right\|_{\mathbb{B}} \stackrel{*}{=} \lim_{I \rightarrow x} \left\| \frac{\Omega(I)}{\mu(I)} \right\|_{\mathbb{B}} \leq \lim_{I \rightarrow x} \frac{\|\Omega\|_{\mathbb{B}}(I)}{\mu(I)} \\ &\stackrel{(3.7)}{=} \frac{d\|\Omega\|_{\mathbb{B}}}{d\mu}(x), \end{aligned}$$

where the starred identity follows from the continuity of $\|\cdot\|_{\mathbb{B}} : \mathbb{B} \rightarrow \mathbb{R}$. This yields (3.8).

4 Liftings of sections

Aim of this section is to obtain a lifting at the level of sections of a measurable Banach bundle (see Theorem 4.4) by applying the Lebesgue Differentiation Theorem 3.4.

Let (X, Σ, μ) be a complete, finite measure space and let ℓ be a von Neumann lifting of μ . In the technical Lemma 4.2, we will consider the set $\mathcal{A}(\ell) \subset X = \hat{X}$, which is defined as

$$\mathcal{A}(\ell) := \left\{ x \in X \mid (\mathcal{I}_x^\ell, \subset) \text{ has a least element } I_x \right\}.$$

The measurability of the set $\mathcal{A}(\ell)$ is a consequence of Lemma 4.1 below, which provides an equivalent characterisation of $\mathcal{A}(\ell)$ in terms of the atoms of the measure μ . Before passing to the actual statement, let us recall some basic terminology and properties about atoms. Given a σ -finite measure space (X, Σ, μ) , we say that a set $A \in \Sigma$ with $\mu(A) > 0$ is an **atom** of μ if

$$B \in \Sigma, B \subset A \implies \text{either } \mu(B) = 0 \text{ or } \mu(A \setminus B) = 0.$$

The σ -finiteness assumption on μ ensures that any atom of μ has finite μ -measure. Observe also that if two given sets $A, A' \in \Sigma$ satisfy $\mu(A \Delta A') = 0$, then A is an atom of μ if and only if A' is an atom of μ . Therefore, since the measure μ is σ -finite, there can exist at most countably many atoms of μ which are not mutually μ -a.e. equivalent.

Lemma 4.1 *Let (X, Σ, μ) be a complete, finite measure space. Let ℓ be a von Neumann lifting of μ . Then it holds that*

$$\mathcal{A}(\ell) = \bigcup_{A \text{ atom of } \mu} \ell(A). \tag{4.1}$$

In particular, it holds that $\mathcal{A}(\ell) \in \Sigma$.

Proof To prove the inclusion \subset in (4.1), let $x \in \mathcal{A}(\ell)$ be fixed. We aim to show that the least element I_x of $(\mathcal{I}_x^\ell, \subset)$ is an atom of μ . We argue by contradiction: suppose there exists $B \in \Sigma$ such that $B \subset I_x$ and $\mu(B), \mu(I_x \setminus B) > 0$. Since $I_x = \ell(I_x)$ is

the disjoint union of $\ell(B)$ and $\ell(I_x \setminus B)$, we can assume (possibly interchanging B and $I_x \setminus B$) that $x \in \ell(B)$, so that $\ell(B) \in \mathcal{I}_x^\ell$. Given that $\ell(B) \subset I_x$ and $\ell(B) \neq I_x$, we contradict the minimality of I_x . Then $\ell(I_x) = I_x$ is an atom of μ , thus its element x belongs to the right-hand side of (4.1).

Conversely, let us prove the inclusion \supset in (4.1). Fix an atom A of μ and a point $x \in \ell(A)$. We aim to show that $\ell(A)$ coincides with the minimal element I_x of $(\mathcal{I}_x^\ell, \subset)$, whence it would follow that $x \in \mathcal{A}(\ell)$. To prove it, pick any $I \in \mathcal{I}_x^\ell$. Since $A \setminus I \subset A$ and A is an atom of μ , we have that either $\mu(A \cap I) = 0$ or $\mu(A \setminus I) = 0$. This implies that either $\ell(A) \cap I = \ell(A \cap I) = \emptyset$ or $\ell(A) \setminus I = \ell(A \setminus I) = \emptyset$. Since $x \in \ell(A) \cap I$, it necessarily holds that $\ell(A) \setminus I = \emptyset$, which is equivalent to $\ell(A) \subset I$. Thanks to the arbitrariness of $I \in \mathcal{I}_x^\ell$, we conclude that $\ell(A) = I_x$.

Finally, it follows from the previous discussion concerning atoms that in the family \mathcal{I}^ℓ there can be at most countably many atoms, thus the set appearing in the right-hand side of (4.1) (if non-empty) can be written as a countable union of atoms and in particular it is measurable. Therefore, (4.1) ensures that $\mathcal{A}(\ell) \in \Sigma$, thus proving the last part of the statement. \square

Let us now construct a suitable family $\{\omega_x\}_{x \in X}$ of ultrafilters induced by a von Neumann lifting ℓ . These ultrafilters ω_x will be a key ingredient during the proof of Theorem 4.4.

Lemma 4.2 *Let (X, Σ, μ) be a complete, finite measure space. Let ℓ be a von Neumann lifting of μ . Let $x \in X$ be given. Then there exists an ultrafilter ω_x on the set $\tilde{\mathcal{I}}_x^\ell$ such that*

$$\tilde{\mathcal{U}}_x^\ell(I) \in \omega_x, \quad \text{for every } I \in \mathcal{I}_x^\ell. \tag{4.2}$$

In particular, if (Y, τ) is a Hausdorff topological space and $\Phi: \mathcal{I}_x^\ell \rightarrow Y$ is any given map, then

$$\lim_{I \rightarrow x} \Phi(I) = y, \quad \text{for some } y \in Y \implies \omega_x\text{-}\lim_{X \setminus I} \Phi(I) = y. \tag{4.3}$$

Moreover, the following properties are satisfied:

- (i) *If $x \in \mathcal{A}(\ell)$, then ω_x is the principal ultrafilter having $\{X \setminus I_x\}$ as its least element.*
- (ii) *If $x \in X \setminus \mathcal{A}(\ell)$, then ω_x is a non-principal ultrafilter.*

Proof Given any $x \in X = \hat{X}$, let us consider the filter $\mathcal{F}_x^\ell := \mathcal{F}_x^{\mathcal{I}^\ell}$ on $\tilde{\mathcal{I}}_x^\ell$ induced by \mathcal{I}^ℓ as in Definition 2.23. Using the Ultrafilter Lemma (Theorem 2.28), we obtain an ultrafilter ω_x on $\tilde{\mathcal{I}}_x^\ell$ containing \mathcal{F}_x^ℓ . Note that $\tilde{\mathcal{U}}_x^\ell(I) \in \mathcal{F}_x^\ell \subset \omega_x$ for every $I \in \mathcal{I}_x^\ell$ and thus (4.2) is satisfied.

We pass to the verification of (4.3). Define $\tilde{\Phi}(X \setminus I) := \Phi(I)$ for every $I \in \mathcal{I}_x^\ell$. Then the claimed implication (4.3) can be proven by observing that, given any $y \in Y$,

it holds that

$$\begin{aligned} \lim_{I \Rightarrow x} \Phi(I) = y &\stackrel{(a)}{\iff} \tilde{\Phi}_* \mathcal{F}_x^\ell \rightarrow y \stackrel{(b)}{\implies} \tilde{\Phi}_* \omega_x \rightarrow y \\ &\stackrel{(c)}{\iff} \omega_x\text{-}\lim_{X \setminus I} \Phi(I) = y, \end{aligned}$$

where (a) follows from Lemma 2.25, (b) from the inclusion $\mathcal{F}_x^\ell \subset \omega_x$, and (c) from Remark 2.27.

To prove i), it suffices to note that $\{X \setminus I_x\} = \tilde{\mathcal{U}}_x^\ell(I_x) \in \mathcal{F}_x^\ell \subset \omega_x$. To prove ii), we show that for any $I \in \mathcal{I}_x^\ell$ there exists $S \in \omega_x$ such that $X \setminus I \notin S$. To this aim, let $I \in \mathcal{I}_x^\ell$ be fixed. Given that $x \notin \mathcal{A}(\ell)$, we can find some element $J \in \mathcal{I}_x^\ell$ that is strictly contained in I , so that $X \setminus I \notin \tilde{\mathcal{U}}_x^\ell(J) \in \omega_x$, thus reaching the sought conclusion. \square

The fact that the ultrafilters ω_x are induced by the differentiation basis \mathcal{I}^ℓ entails a strong compatibility with the lifting ℓ itself, as it is evident from the following result, which will have an essential role in one step of the proof of Theorem 4.4 (namely, to achieve (4.5d)).

Proposition 4.3 *Let (X, Σ, μ) be a complete, finite measure space. Let ℓ be a von Neumann lifting of μ . Let us fix any family $\{\omega_x\}_{x \in X}$ of ultrafilters as in Lemma 4.2. Then for every function $f \in L^\infty(\mu)$ it holds that*

$$\ell(f)(x) = \omega_x\text{-}\lim_{X \setminus I} \int_I f \, d\mu, \quad \text{for every } x \in X. \tag{4.4}$$

Proof First of all, let us prove that the identity (4.4) holds for every simple function $f \in S(\mu)$. We can write $f = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}^\mu$, where $(\lambda_i)_{i=1}^n \subset \mathbb{R}$ and $(A_i)_{i=1}^n \subset \Sigma$ is a partition of X made of positive μ -measure sets. Given any $i = 1, \dots, n$ and $x \in \ell(A_i)$, we have that $\ell(A_i) \in \mathcal{I}_x^\ell$ and $\int_I f \, d\mu = \lambda_i$ for every $I \in \mathcal{U}_x^\ell(\ell(A_i))$, thus accordingly $\omega_x\text{-}\lim_{X \setminus I} \int_I f \, d\mu = \lambda_i = \ell(f)(x)$. Given that $\bigcup_{i=1}^n \ell(A_i) = \ell(\bigcup_{i=1}^n A_i) = X$, the identity (4.4) is proven for every $f \in S(\mu)$.

Let us pass to the verification of the general statement. Let $f \in L^\infty(\mu)$ and $\varepsilon > 0$ be fixed. Then we can find $g_\varepsilon, h_\varepsilon \in S(\mu)$ such that $f - \varepsilon \leq g_\varepsilon \leq f \leq h_\varepsilon \leq f + \varepsilon$ everywhere on X . By taking first the average integrals over $I \in \mathcal{I}_x^\ell$ and then the ultralimits, for any $x \in X$ we get

$$\begin{aligned} \ell(f)(x) - \varepsilon &= \ell(f - \varepsilon)(x) \leq \ell(g_\varepsilon)(x) = \omega_x\text{-}\lim_{X \setminus I} \int_I g_\varepsilon \, d\mu \leq \omega_x\text{-}\lim_{X \setminus I} \int_I f \, d\mu \\ &\leq \omega_x\text{-}\lim_{X \setminus I} \int_I h_\varepsilon \, d\mu = \ell(h_\varepsilon)(x) \leq \ell(f + \varepsilon)(x) = \ell(f)(x) + \varepsilon, \end{aligned}$$

where we used (4.4) for simple functions. By letting $\varepsilon \searrow 0$, we finally obtain (4.4) for f . \square

We are now in a position to construct the lifting of sections of a given Banach \mathbb{B} -bundle:

Theorem 4.4 (Lifting of sections) *Let (X, Σ, μ) be a complete, σ -finite measure space and ℓ a von Neumann lifting of μ . Let \mathbb{B} be a Banach space and \mathbf{E} a Banach \mathbb{B} -bundle over (X, Σ) . Then there exists a map $\ell: \Gamma_\infty(\mu; \mathbf{E}) \rightarrow \bar{\Gamma}_\infty(\mu; \mathbb{B}'')$ such that the following properties hold:*

$$\ell(v)(x) = J_{\mathbb{B}}(\hat{v}(x)), \quad \forall v \in \Gamma_\infty(\mu; \mathbf{E}) \text{ and } \mu\text{-a.e. } x \in X, \tag{4.5a}$$

$$\ell(v + w)(x) = \ell(v)(x) + \ell(w)(x), \quad \forall v, w \in \Gamma_\infty(\mu; \mathbf{E}), x \in X, \tag{4.5b}$$

$$\ell(f \cdot v)(x) = \ell(f)(x) \ell(v)(x), \quad \forall v \in \Gamma_\infty(\mu; \mathbf{E}), f \in L^\infty(\mu), x \in X, \tag{4.5c}$$

$$\|\ell(v)(x)\|_{\mathbb{B}''} \leq \ell(\|v(\cdot)\|_{\mathbf{E}(\cdot)})(x), \quad \forall v \in \Gamma_\infty(\mu; \mathbf{E}), x \in X, \tag{4.5d}$$

where $\hat{\#}: \Gamma_\infty(\mu; \mathbf{E}) \rightarrow \bar{\Gamma}_\infty(\mu; \mathbf{E})$ is the precise representative map provided by Theorem 3.5. In particular, the operator $\ell: \Gamma_\infty(\mu; \mathbf{E}) \rightarrow \bar{\Gamma}_\infty(\mu; \mathbb{B}'')$ is linear and continuous. Moreover, in the case where \mathbf{E} coincides with the constant bundle \mathbb{B} , we can additionally require that

$$\ell(\mathbb{1}_X^\mu \bar{v})(x) = J_{\mathbb{B}}(\bar{v}), \quad \text{for every } \bar{v} \in \mathbb{B} \text{ and } x \in X. \tag{4.6}$$

Proof Thanks to Remark 2.3, we can assume without loss of generality that μ is finite. Fix any family $\{\omega_x\}_{x \in X}$ of ultrafilters as in Lemma 4.2. Now consider an element $v \in \Gamma_\infty(\mu; \mathbf{E})$. Given any $x \in X$, we have that the family $\{ \int_I J_{\mathbb{B}} \circ \hat{v} \, d\mu : I \in \mathcal{I}_x^\ell \}$ is norm-bounded in \mathbb{B}'' . In view of the fact that the weak* topology of \mathbb{B}'' is Hausdorff, as well as of the compactness of its restriction to any closed ball in \mathbb{B}'' (which is guaranteed by Banach–Alaoglu Theorem), the following definition is well-posed:

$$\ell(v)(x) := \omega_x\text{-}\lim_{X \setminus I} \int_I J_{\mathbb{B}} \circ \hat{v} \, d\mu \in \mathbb{B}'', \quad \text{for every } x \in X, \tag{4.7}$$

where the ultralimit is taken with respect to the weak* topology of \mathbb{B}'' . Let us verify that the resulting map ℓ takes values into the space $\bar{\Gamma}_\infty(\mu; \mathbb{B}'')$ and fulfils the required properties.

Proof of (4.5a) Fix any $v \in \Gamma_\infty(\mu; \mathbf{E})$. We know from Theorem 3.5 that for μ -a.e. $x \in X$ the strong \mathcal{I}^ℓ -limit $\lim_{I \Rightarrow x} \int_I \hat{v} \, d\mu \in \mathbb{B}$ exists and coincides with $\hat{v}(x)$. In particular, it holds

$$(J_{\mathbb{B}} \circ \hat{v})(x) = J_{\mathbb{B}}\left(\lim_{I \Rightarrow x} \int_I \hat{v} \, d\mu\right) = \lim_{I \Rightarrow x} \int_I J_{\mathbb{B}} \circ \hat{v} \, d\mu$$

with respect to the strong topology of \mathbb{B}'' , so also with respect to the weak* topology of \mathbb{B}'' . By recalling (4.3), we obtain (4.5a) and thus in particular $\ell(v) \in \bar{\Gamma}_\infty(\mu; \mathbb{B}'')$ for all $v \in \Gamma_\infty(\mu; \mathbf{E})$.

Proof of (4.5b) Let $v, w \in \Gamma_\infty(\mu; \mathbf{E})$ be fixed. Then (2.17a) and Theorem 3.5 give

$$\begin{aligned} \ell(v)(x) + \ell(w)(x) &= \omega_x\text{-}\lim_{X \setminus I} \int_I J_{\mathbb{B}} \circ \hat{v} \, d\mu + \omega_x\text{-}\lim_{X \setminus I} \int_I J_{\mathbb{B}} \circ \hat{w} \, d\mu \\ &= \omega_x\text{-}\lim_{X \setminus I} \int_I J_{\mathbb{B}} \circ (\hat{v} + \hat{w}) \, d\mu = \ell(v + w)(x), \end{aligned}$$

for every $x \in X$, whence (4.5b) follows.

Proof of (4.5d) Recall that the bidual norm $\mathbb{B}'' \ni L \mapsto \|L\|_{\mathbb{B}''} \in \mathbb{R}$ is a weakly* lower semicontinuous function. This can be easily seen by noticing that (by definition) the norm $\|\cdot\|_{\mathbb{B}''}$ can be expressed as

$$\|L\|_{\mathbb{B}''} = \sup \{L(\omega) \mid \omega \in \mathbb{B}', \|\omega\|_{\mathbb{B}'} = 1\}$$

and that $\mathbb{B}'' \ni L \mapsto L(\omega) \in \mathbb{R}$ is weakly* continuous for every $\omega \in \mathbb{B}'$. Hence, for every $v \in \Gamma_\infty(\mu; \mathbf{E})$ and $x \in X$ one has

$$\begin{aligned} \|\ell(v)(x)\|_{\mathbb{B}''} &= \left\| \omega_x\text{-}\lim_{X \setminus I} \int_I J_{\mathbb{B}} \circ \hat{v} \, d\mu \right\|_{\mathbb{B}''} \stackrel{(2.16)}{\leq} \omega_x\text{-}\lim_{X \setminus I} \left\| \int_I J_{\mathbb{B}} \circ \hat{v} \, d\mu \right\|_{\mathbb{B}''} \\ &\leq \omega_x\text{-}\lim_{X \setminus I} \int_I \|(J_{\mathbb{B}} \circ \hat{v})(\cdot)\|_{\mathbb{B}''} \, d\mu = \omega_x\text{-}\lim_{X \setminus I} \int_I \|v(\cdot)\|_{\mathbb{B}} \, d\mu \\ &\stackrel{(4.4)}{=} \ell(\|v(\cdot)\|_{\mathbb{B}})(x). \end{aligned}$$

Proof of (4.5c) As an intermediate step, we are first going to prove that:

$$\ell(\lambda v)(x) = \lambda \ell(v)(x), \quad \forall v \in \Gamma_\infty(\mu; \mathbf{E}), \lambda \in \mathbb{R}, x \in X, \tag{4.8a}$$

$$\ell(\mathbb{1}_A^\mu \cdot v)(x) = \mathbb{1}_{\ell(A)}(x) \ell(v)(x), \quad \forall v \in \Gamma_\infty(\mu; \mathbf{E}), A \in \Sigma, x \in X. \tag{4.8b}$$

The identity (4.8a) follows from (2.17b). To verify (4.8b), we distinguish two cases. In the case where $x \in \ell(A)$, we have that $\ell(A) \in \mathcal{I}_x^\ell$ and thus accordingly it holds that

$$\begin{aligned} \ell(\mathbb{1}_A^\mu \cdot v)(x) &= \omega_x\text{-}\lim_{X \setminus I} \int_I J_{\mathbb{B}} \circ (\mathbb{1}_{\ell(A)} \cdot \hat{v}) \, d\mu = \omega_x\text{-}\lim_{X \setminus I} \int_{I \cap \ell(A)} J_{\mathbb{B}} \circ \hat{v} \, d\mu \\ &= \ell(v)(x). \end{aligned}$$

Conversely, if $x \in X \setminus \ell(A)$, then $\ell(X \setminus A) \in \mathcal{I}_x^\ell$ and consequently

$$\begin{aligned} \ell(\mathbb{1}_A^\mu \cdot v)(x) &= \omega_x\text{-}\lim_{X \setminus I} \int_I J_{\mathbb{B}} \circ (\mathbb{1}_{\ell(A)} \cdot \hat{v}) \, d\mu \\ &= \omega_x\text{-}\lim_{X \setminus I} \int_{I \cap \ell(X \setminus A)} J_{\mathbb{B}} \circ (\mathbb{1}_{\ell(A)} \cdot \hat{v}) \, d\mu = 0. \end{aligned}$$

All in all, we have proven that also (4.8b) is verified. Observe that (4.5b), (4.8a), and (4.8b) imply that (4.5c) holds for every $f \in S(\mu)$. Now fix any function $f \in L^\infty(\mu)$ and pick a sequence $(f_n)_{n \in \mathbb{N}} \subset S(\mu)$ that converges to f in $L^\infty(\mu)$. Therefore, for every $x \in X$ one has

$$\begin{aligned} \|\ell(f \cdot v)(x) - \ell(f_n \cdot v)(x)\|_{\mathbb{B}''} &= \|\ell((f - f_n) \cdot v)(x)\|_{\mathbb{B}''} \\ &\stackrel{(4.5d)}{\leq} \ell(\|((f - f_n) \cdot v)(\cdot)\|_{\mathbb{B}})(x) \\ &= \ell(|f - f_n|)(x) \ell(\|v(\cdot)\|_{\mathbb{B}})(x) \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, where we used the continuity of the map $\ell : L^\infty(\mu) \rightarrow \mathcal{L}^\infty(\Sigma)$. Since $\lim_n \ell(f_n)(x) = \ell(f)(x)$ holds as well, by letting $n \rightarrow \infty$ in the identity $\ell(f_n \cdot v)(x) = \ell(f_n)(x) \ell(v)(x)$ we finally obtain (4.5c).

Proof of (4.6) Assume that \mathbf{E} is the constant bundle \mathbb{B} . Then (4.7) immediately implies that $\ell(\mathbb{1}_X \bar{v})(x) = J_{\mathbb{B}}(\bar{v})$ holds for every $\bar{v} \in \mathbb{B}$ and $x \in X$, thus proving (4.6). \square

Remark 4.5 Note that in the case where the Banach space \mathbb{B} has a predual, the conclusion of Theorem 4.4 can be improved to: *There exists a map $\ell : \Gamma_\infty(\mu; \mathbf{E}) \rightarrow \bar{\Gamma}_\infty(\mu; \mathbb{B})$ such that (4.5b)–(4.5d) hold, and $\ell(v)(x) = \hat{v}(x)$ for every $v \in \Gamma_\infty(\mu; \mathbf{E})$ and μ -a.e. $x \in X$.*

Some comments on Theorem 4.4 are in order. As we will see next, the inequality in (4.5d) cannot, in general, be improved to an equality. In the last part of this section, we will introduce a subfamily of $\Gamma_\infty(\mu; \mathbf{E})$ where one can require the lifting to be norm-preserving. This has to do with the compactness of the essential range of those mappings one would like to lift, see Definition 4.7 and Proposition 4.9. This kind of issue was previously known in the setting of Lebesgue–Bochner spaces, as we will discuss in Appendix C, see in particular Theorem C.2. As far as we know, the weaker form of 1-Lipschitz lifting we introduced in Theorem 4.4 was not considered earlier. An application of Theorem 4.4 will be given in Sect. 5.2.

Remark 4.6 (Comparison with liftings of normed modules) As recalled in Remark 2.17, the space of sections $\Gamma_\infty(\mu; \mathbf{E})$ is an $L^\infty(\mu)$ -normed $L^\infty(\mu)$ -module. One of the main achievements of [11] is a lifting result for $L^\infty(\mu)$ -normed $L^\infty(\mu)$ -modules, see [11, Theorem 3.5]. Therefore, we actually have at disposal two different notions of liftings for $\mathcal{M} := \Gamma_\infty(\mu; \mathbf{E})$. Following the approach in [11], we obtain an $\mathcal{L}^\infty(\Sigma)$ -normed $\mathcal{L}^\infty(\Sigma)$ -module $\tilde{\mathcal{M}}$ together with a lifting map $\ell : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ that preserves the pointwise norm, meaning that $|\ell(v)| = \ell(|v|)$ for all $v \in \mathcal{M}$. Keeping in mind that in general the notion of lifting provided by Theorem 4.4 cannot preserve the norm (cf. with Theorem C.2 and Remark C.3), this is telling that (with the exception of very special cases) the fibers $\{\tilde{\mathcal{M}}_x\}_{x \in X}$ of the lifted module $\tilde{\mathcal{M}}$ built in [11, Section 3.3] cannot be embedded in a measurable way in any given Banach space. In particular, this means that the lifting of $\Gamma_\infty(\mu; \mathbf{E})$ as a normed module is somehow unrelated to the fibers of \mathbf{E} .

Let (X, Σ, μ) be a measure space, (Y, d) a metric space. Then we say that a μ -measurable map $\varphi: X \rightarrow Y$ is **essentially compactly valued** provided there exist $N \in \Sigma$ with $\mu(N) = 0$ and a compact set $K \subset Y$ such that $\varphi(X \setminus N) \subset K$.

Definition 4.7 Let (X, Σ, μ) be a finite measure space, \mathbb{B} a Banach space. Let \mathbf{E} be a Banach \mathbb{B} -bundle over (X, Σ) . Then we denote by $\text{ECV}(\mu; \mathbf{E})$ the set of all elements $v \in \Gamma_\infty(\mu; \mathbf{E})$ such that some (thus, any) representative $\bar{v} \in \bar{\Gamma}_\infty(\mu; \mathbf{E})$ of v is essentially compactly valued.

We point out that $\text{ECV}(\mu; \mathbf{E})$ is a linear subspace of $\Gamma_\infty(\mu; \mathbf{E})$. More generally, $\text{ECV}(\mu; \mathbf{E})$ is an $L^\infty(\mu)$ -submodule of $\Gamma_\infty(\mu; \mathbf{E})$. This can be either easily checked from the definition or immediately deduced from the following result:

Lemma 4.8 *Let (X, Σ, μ) be a finite measure space and \mathbb{B} a Banach space. Let \mathbf{E} be a Banach \mathbb{B} -bundle over (X, Σ) . Then it holds that*

$$\text{ECV}(\mu; \mathbf{E}) = \Gamma_\infty(\mu; \mathbf{E}) \cap \text{cl}_{\Gamma_\infty(\mu; \mathbb{B})}(S(\mu; \mathbb{B})). \tag{4.9}$$

Proof To prove \subset , let $v \in \text{ECV}(\mu; \mathbf{E})$ be fixed. Pick a representative $\bar{v} \in \bar{\Gamma}_\infty(\mu; \mathbf{E})$ of v . Fix $N \in \Sigma$ with $\mu(N) = 0$ such that the set $\bar{v}(X \setminus N)$ is totally bounded. Given any $\varepsilon > 0$, we can thus find finitely many vectors $v_1, \dots, v_n \in \mathbb{B}$ such that $\bar{v}(X \setminus N) \subset \bigcup_{i=1}^n B_\varepsilon(v_i)$. Then we define

$$E'_i := \{x \in X : \|\bar{v}(x) - v_i\|_{\mathbb{B}} < \varepsilon\} \in \Sigma, \quad \text{for every } i = 1, \dots, n.$$

We also set $E_1 := E'_1 \cup N$ and $E_i := E'_i \setminus \bigcup_{j < i} E'_j$ for every $i = 2, \dots, n$. The resulting family $(E_i)_{i=1}^n \subset \Sigma$ is a partition of the set X and the element $w := \sum_{i=1}^n \mathbb{1}_{E_i}^\mu \cdot v_i \in S(\mu; \mathbb{B})$ satisfies $\|v - w\|_{\Gamma_\infty(\mu; \mathbb{B})} \leq \varepsilon$ by construction. Thanks to the arbitrariness of ε , we conclude that $v \in \text{cl}_{\Gamma_\infty(\mu; \mathbb{B})}(S(\mu; \mathbb{B}))$, proving \subset in (4.9).

Conversely, fix $v \in \Gamma_\infty(\mu; \mathbf{E}) \cap \text{cl}_{\Gamma_\infty(\mu; \mathbb{B})}(S(\mu; \mathbb{B}))$ and a representative $\bar{v} \in \bar{\Gamma}_\infty(\mu; \mathbf{E})$ of v . Given any $k \in \mathbb{N}$, we can find $n_k \in \mathbb{N}$, pairwise disjoint sets $(E_i^k)_{i=1}^{n_k} \subset \Sigma$, and $(v_i^k)_{i=1}^{n_k} \subset \mathbb{B}$ such that $N_k := X \setminus \bigcup_{i=1}^{n_k} E_i^k \in \Sigma$ is μ -null and $\|\bar{v}(x) - v_i^k\|_{\mathbb{B}} \leq 1/k$ for every $i = 1, \dots, n_k$ and $x \in E_i^k$. We claim that $\bar{v}(X \setminus N)$ is totally bounded, where $N \in \Sigma$ stands for the μ -null set $\bigcup_{k \in \mathbb{N}} N_k$. Indeed, given any $\varepsilon > 0$ and chosen $k \in \mathbb{N}$ so that $1/k < \varepsilon$, we have that the set $\hat{v}(X \setminus N)$ is contained in $\bigcup_{i=1}^{n_k} B_\varepsilon(v_i^k)$, thus showing that $\hat{v}(X \setminus N)$ is totally bounded. Therefore, also the inclusion \supset in (4.9) is proven. \square

Proposition 4.9 *Let (X, Σ, μ) be a complete, σ -finite measure space, ℓ a von Neumann lifting of μ . Let \mathbb{B} be a Banach space, \mathbf{E} a Banach \mathbb{B} -bundle over (X, Σ) . Then there exists a map $\ell: \Gamma_\infty(\mu; \mathbf{E}) \rightarrow \bar{\Gamma}_\infty(\mu; \mathbb{B}')$ as in Theorem 4.4 that for every $v \in \text{ECV}(\mu; \mathbf{E})$ also satisfies*

$$\ell(v)(x) \in J_{\mathbb{B}}(\mathbb{B}), \quad \|\ell(v)(x)\|_{\mathbb{B}'} = \ell(\|v(\cdot)\|_{\mathbb{B}})(x), \quad \text{for every } x \in X. \tag{4.10}$$

Proof Thanks to Remark 2.3, we can assume without loss of generality that the measure μ is finite. Moreover, we can apply Theorem 4.4 to the constant bundle \mathbb{B} , thus

obtaining a lifting map $\tilde{\ell}: \Gamma_\infty(\mu; \mathbb{B}) \rightarrow \tilde{\Gamma}_\infty(\mu; \mathbb{B}')$, whose restriction $\ell := \tilde{\ell}|_{\Gamma_\infty(\mu; \mathbf{E})}$ clearly verifies (4.5a)–(4.5d). Moreover, thanks to (4.6) we can also require that

$$\tilde{\ell}(\mathbb{1}_X^\mu \bar{v})(x) = J_{\mathbb{B}}(\bar{v}), \quad \text{for every } \bar{v} \in \mathbb{B} \text{ and } x \in X. \tag{4.11}$$

Since both mappings

$$\begin{aligned} \Gamma_\infty(\mu; \mathbf{E}) \ni v &\longmapsto \|\tilde{\ell}(v)(\cdot)\|_{\mathbb{B}'} \in \mathcal{L}^\infty(\Sigma), \\ \Gamma_\infty(\mu; \mathbf{E}) \ni v &\longmapsto \ell(\|v(\cdot)\|_{\mathbb{B}}) \in \mathcal{L}^\infty(\Sigma) \end{aligned}$$

are continuous, we know from Lemma 4.8 that to prove (4.10) it suffices to show that

$$\|\tilde{\ell}(v)(x)\|_{\mathbb{B}'} = \ell(\|v(\cdot)\|_{\mathbb{B}})(x), \quad \text{for every } v \in S(\mu; \mathbb{B}) \text{ and } x \in X. \tag{4.12}$$

To this aim, fix any $v = \sum_{i=1}^n \mathbb{1}_{E_i}^\mu \bar{v}_i \in S(\mu; \mathbb{B})$. Then for every $x \in X$ we can compute

$$\begin{aligned} \|\tilde{\ell}(v)(x)\|_{\mathbb{B}'} &\stackrel{(4.5b)}{=} \left\| \sum_{i=1}^n \tilde{\ell}(\mathbb{1}_{E_i}^\mu \bar{v}_i)(x) \right\|_{\mathbb{B}'} \stackrel{(4.5c)}{=} \left\| \sum_{i=1}^n \mathbb{1}_{\ell(E_i)}(x) \tilde{\ell}(\mathbb{1}_X^\mu \bar{v}_i)(x) \right\|_{\mathbb{B}'} \\ &= \sum_{i=1}^n \mathbb{1}_{\ell(E_i)}(x) \|\tilde{\ell}(\mathbb{1}_X^\mu \bar{v}_i)(x)\|_{\mathbb{B}'} \stackrel{(4.11)}{=} \sum_{i=1}^n \mathbb{1}_{\ell(E_i)}(x) \|\bar{v}_i\|_{\mathbb{B}} \\ &= \ell\left(\sum_{i=1}^n \|\bar{v}_i\|_{\mathbb{B}} \mathbb{1}_{E_i}^\mu\right)(x) = \ell(\|v(\cdot)\|_{\mathbb{B}})(x), \end{aligned}$$

whence (4.12) follows. Therefore, the statement is achieved. □

5 Disintegration of vector-valued measures

In this section, we apply Theorem 4.4 to obtain a disintegration result (Theorem 5.9) for vector measures defined on a Polish space, whose target is a Banach space which has the Radon–Nikodým property. An important feature of Polish spaces, which is a key ingredient in the proof of Theorem 5.9, is the existence of strong liftings (cf. with Definition 5.1).

5.1 Strong liftings

A topological space (X, τ) is said to be a **Polish space** if there exists a complete, separable distance d on X which induces the topology τ . We will also say that τ is a **Polish topology** on X . We denote by $\mathcal{B}(X)$ the Borel σ -algebra of (X, τ) . Moreover, the **support** of a given σ -finite Borel measure μ on X is the closed set $\text{spt}(\mu) \subset X$ defined as

$$\text{spt}(\mu) := X \setminus \bigcup \{U \in \tau \mid \mu(U) = 0\}.$$

We denote by $C(X)$ the space of all continuous functions $f: X \rightarrow \mathbb{R}$, while $C_b(X)$ stands for the space of those elements of $C(X)$ that are bounded. Observe that $C_b(X) \subset \mathcal{L}^\infty(\mathcal{B}(X))$. Moreover, the linear space $C_b(X)$ becomes a Banach space if endowed with the supremum norm $\|f\|_{C_b(X)} := \sup_X |f|$. If X is compact, then the Banach space $C(X) = C_b(X)$ is separable.

Definition 5.1 (*Strong lifting*) Let (X, Σ, μ) be a complete, σ -finite measure space. Let τ be a Polish topology on X with $\tau \subset \Sigma$. Then we say that a given von Neumann lifting ℓ of μ is a **strong lifting** of μ provided it holds that

$$U \cap \text{spt}(\mu) \subset \ell(U), \quad \text{for every } U \in \tau. \tag{5.1}$$

One can readily check that (5.1) holds if and only if $\ell(\pi_\mu(f)) = f$ on $\text{spt}(\mu)$ for every $f \in C_b(X)$, where the map $\ell: L^\infty(\mu) \rightarrow \mathcal{L}^\infty(\Sigma)$ is the one given by Theorem 2.4.

Strong liftings exist on all Polish spaces, as proved for example in [41]. We will provide an alternative proof of the existence of strong liftings on Polish spaces in Corollary A.7.

Theorem 5.2 (Existence of strong liftings) *Let (X, Σ, μ) be a complete, σ -finite measure space. Let τ be a Polish topology on X with $\tau \subset \Sigma$. Then there exists a strong lifting ℓ of μ .*

The property of being a strong lifting carries over to the lifting of sections, as we are going to discuss in the following result. By $C_b(X; \mathbb{B})$ we will mean the Banach space of all continuous maps $v: X \rightarrow \mathbb{B}$ satisfying $\|v\|_{C_b(X; \mathbb{B})} := \sup_{x \in X} \|v(x)\|_{\mathbb{B}} < +\infty$. Observe that $C_b(X; \mathbb{B}) \subset \mathcal{L}^\infty(\mu; \mathbb{B})$ and that $\|v\|_{\mathcal{L}^\infty(\mu; \mathbb{B})} \leq \|v\|_{C_b(X; \mathbb{B})}$ for every $v \in C_b(X; \mathbb{B})$.

Proposition 5.3 *Let (X, Σ, μ) be a complete, σ -finite measure space and τ a Polish topology on X with $\tau \subset \Sigma$. Let ℓ be a strong lifting of μ and \mathbb{B} a Banach space. Then any lifting operator $\ell: L^\infty(\mu; \mathbb{B}) \rightarrow \mathcal{L}^\infty(\mu; \mathbb{B}')$ as in Theorem 4.4 satisfies the following property:*

$$\ell(\pi_\mu(v))(x) = J_{\mathbb{B}}(v(x)), \quad \text{for every } v \in C_b(X; \mathbb{B}) \text{ and } x \in \text{spt}(\mu).$$

Proof Let $v \in C_b(X; \mathbb{B})$ and $x \in \text{spt}(\mu)$ be fixed. Set $\bar{v} := v(x)$ for brevity. Since $\|v(\cdot) - \bar{v}\|_{\mathbb{B}}$ is a continuous function, one has that

$$\begin{aligned} & \|\ell(\pi_\mu(v))(x) - J_{\mathbb{B}}(\bar{v})\|_{\mathbb{B}'} \stackrel{(4.6)}{=} \|\ell(\pi_\mu(v))(x) - \ell(\mathbb{1}_X^\mu \bar{v})(x)\|_{\mathbb{B}'} \\ &= \|\ell(\pi_\mu(v) - \mathbb{1}_X^\mu \bar{v})(x)\|_{\mathbb{B}'} \stackrel{(4.5d)}{\leq} \ell(\|\pi_\mu(v)(\cdot) - \bar{v}\|_{\mathbb{B}})(x) \\ &= \ell(\pi_\mu(\|v(\cdot) - \bar{v}\|_{\mathbb{B}}))(x) = \|v(x) - \bar{v}\|_{\mathbb{B}} = 0. \end{aligned}$$

This shows the validity of the statement. □

5.2 Disintegration of RNP-valued measures of bounded variation

We begin with the relevant definition of the disintegration of a vector-valued measure of bounded variation.

Definition 5.4 (*Disintegration of a vector-valued measure*) Let (X, Σ_X) and (Y, Σ_Y) be two measurable spaces. Let $\varphi: X \rightarrow Y$ be a measurable map. Let \mathbb{B} be a Banach space and Ω a \mathbb{B} -valued measure on (X, Σ_X) having bounded variation. Then we say that a family $\{\Omega_y\}_{y \in Y}$ is a **disintegration of Ω along φ** provided the following conditions are verified:

- (i) Each Ω_y is a \mathbb{B} -valued measure on (X, Σ_X) with bounded variation. Moreover, $\|\Omega_y\|_{\mathbb{B}}(X) = 1$, $\{y\} \in \Sigma_Y$, and $\|\Omega_y\|_{\mathbb{B}}(X \setminus \varphi^{-1}(\{y\})) = 0$ hold for $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$ -a.e. $y \in Y$.
- (ii) Given any function $f \in \mathcal{L}^{\infty}(\Sigma_X)$, the mapping $Y \ni y \mapsto \int f \, d\Omega_y \in \mathbb{B}$ is strongly measurable, where the domain (Y, Σ_Y) is endowed with the pushforward measure $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$.
- (iii) It holds that

$$\int f \, d\Omega = \iint f \, d\Omega_y \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y), \quad \text{for all } f \in \mathcal{L}^{\infty}(\Sigma_X). \quad (5.2)$$

Observe that $\|\int f \, d\Omega_y\|_{\mathbb{B}} \leq \|f\|_{\mathcal{L}^{\infty}(\Sigma_X)}$ for $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$ -a.e. $y \in Y$, thus the map $Y \ni y \mapsto \|\int f \, d\Omega_y\|_{\mathbb{B}}$ belongs to $\mathcal{L}^1(\varphi_{\#}\|\Omega\|_{\mathbb{B}})$ and accordingly the right-hand side of (5.2) is well-posed.

Remark 5.5 By virtue of the Monotone Class Theorem, the requirements ii) and iii) in Definition 5.4 can be replaced by the following two conditions:

- (ii') Given any set $E \in \Sigma_X$, the mapping $Y \ni y \mapsto \Omega_y(E) \in \mathbb{B}$ is strongly measurable, where the domain (Y, Σ_Y) is endowed with the pushforward measure $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$.
- (iii') It holds that

$$\Omega(E) = \int \Omega_y(E) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y), \quad \text{for every } E \in \Sigma_X. \quad (5.3)$$

Moreover, in the case where X is a Polish space and $\Sigma_X = \mathcal{B}(X)$, one can equivalently require that the mapping $Y \ni y \mapsto \int f \, d\Omega_y \in \mathbb{B}$ is strongly measurable for every $f \in C_b(X)$ and that the defining identity (5.2) is satisfied for every $f \in C_b(X)$.

Whenever they exist, disintegrations are (essentially) unique; see Lemma 5.7 below. Before passing to the verification of this statement, let us recall a useful technical fact:

Remark 5.6 Let (X, τ) be a Polish space. Let \mathbb{B} be a Banach space and Ω a \mathbb{B} -valued measure on $(X, \mathcal{B}(X))$ having bounded variation. Fix any countable, dense subset D

of X . Define \mathcal{U} as the countable family of all finite unions of open balls in X having centre in D and rational radius. Then we claim that for any $E \in \mathcal{B}(X)$ there exists $(U_n)_{n \in \mathbb{N}} \subset \mathcal{U}$ such that

$$\|\Omega(U_n) - \Omega(E)\|_{\mathbb{B}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Indeed, for any Borel set $E \subset X$ we may estimate

$$\begin{aligned} \inf_{U \in \mathcal{U}} \|\Omega(U) - \Omega(E)\|_{\mathbb{B}} &= \inf_{U \in \mathcal{U}} \|\Omega(U \setminus E) - \Omega(E \setminus U)\|_{\mathbb{B}} \\ &\leq \inf_{U \in \mathcal{U}} (\|\Omega(U \setminus E)\|_{\mathbb{B}} + \|\Omega(E \setminus U)\|_{\mathbb{B}}) \\ &\leq \inf_{U \in \mathcal{U}} (\|\Omega\|_{\mathbb{B}}(U \setminus E) + \|\Omega\|_{\mathbb{B}}(E \setminus U)) \\ &= \inf_{U \in \mathcal{U}} \|\Omega\|_{\mathbb{B}}(U \Delta E) \stackrel{\star}{=} 0, \end{aligned}$$

where the starred identity follows from the regularity of the measure $\|\Omega\|_{\mathbb{B}}$.

Lemma 5.7 (Uniqueness of the disintegration) *Let $(X, \tau_X), (Y, \tau_Y)$ be Polish spaces and let $\varphi: X \rightarrow Y$ be a Borel measurable map. Let \mathbb{B} be a Banach space and let Ω be a \mathbb{B} -valued measure on $(X, \mathcal{B}(X))$ having bounded variation. Suppose that $\{\Omega_y\}_{y \in Y}$ and $\{\Omega'_y\}_{y \in Y}$ are disintegrations of Ω along φ . Then it holds that $\Omega_y = \Omega'_y$ for $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$ -a.e. $y \in Y$.*

Proof We argue by contradiction: suppose there exists a Borel set $P_1 \subset Y$ such that $\Omega_y \neq \Omega'_y$ for every $y \in P_1$ and $\varphi_{\#}\|\Omega\|_{\mathbb{B}}(P_1) > 0$. Letting \mathcal{U} be as in Remark 5.6, we can find $U \in \mathcal{U}$ and $P_2 \subset P_1$ Borel such that $\Omega_y(U) \neq \Omega'_y(U)$ for every $y \in P_2$ and $\varphi_{\#}\|\Omega\|_{\mathbb{B}}(P_2) > 0$. Hence, we can find $\lambda > 0$ and a Borel set $P_3 \subset P_2$ such that $\|\Omega_y(U) - \Omega'_y(U)\|_{\mathbb{B}} \geq \lambda$ for every $y \in P_3$ and $\varphi_{\#}\|\Omega\|_{\mathbb{B}}(P_3) > 0$. For brevity, we set $v(y) := \Omega_y(U)$ and $v'(y) := \Omega'_y(U)$ for every $y \in P_3$. Since v and v' are strongly measurable, we can find a closed, separable subset Z of \mathbb{B} and a Borel set $N \subset P_3$ such that $\varphi_{\#}\|\Omega\|_{\mathbb{B}}(N) = 0$ and $v(y), v'(y) \in Z$ for every $y \in P_3 \setminus N$. Then consider the finite Borel measure $\mu := (v, v')_{\#}(\varphi_{\#}\|\Omega\|_{\mathbb{B}}|_{P_3 \setminus N})$ on $Z \times Z$. Given that $\text{spt}(\mu)$ is not contained in the diagonal $\{(z, z) : z \in Z\}$, we can find distinct points $z, z' \in Z$ such that $(z, z') \in \text{spt}(\mu)$. Choose $r > 0$ so that $\|z - z'\|_{\mathbb{B}} > 2r$ and $r < \lambda/2$. Call B (resp. B') the closed ball in Z of centre z (resp. z') and radius r . Then $B \cap B' = \emptyset$ and $\mu(B \times B') > 0$, thus

$$P := \{y \in P_3 \setminus N \mid v(y) \in B, v'(y) \in B'\}$$

is a Borel set having positive $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$ -measure. Define $E := U \cap \varphi^{-1}(P)$. Therefore, one has that

$$\begin{aligned} & \left\| \frac{\Omega(E)}{\varphi_{\#}\|\Omega\|_{\mathbb{B}}(P)} - z \right\|_{\mathbb{B}} \\ &= \left\| \frac{1}{\varphi_{\#}\|\Omega\|_{\mathbb{B}}(P)} \int \Omega_y(U \cap \varphi^{-1}(P)) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y) - z \right\|_{\mathbb{B}} \\ &= \left\| \frac{1}{\varphi_{\#}\|\Omega\|_{\mathbb{B}}(P)} \int \Omega_y(U \cap \varphi^{-1}(\{y\} \cap P)) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y) - z \right\|_{\mathbb{B}} \\ &= \left\| \int_P \Omega_y(U) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y) - z \right\|_{\mathbb{B}} = \left\| \int_P (v(y) - z) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y) \right\|_{\mathbb{B}} \\ &\leq \int_P \|v(y) - z\|_{\mathbb{B}} \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y) \leq r, \end{aligned}$$

which implies that $\Omega(E)/(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(P) \in B$. The same computations, with (z', Ω'_y) in place of (z, Ω_y) , show that also $\Omega(E)/(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(P) \in B'$, thus leading to a contradiction. \square

This section ends with an existence result, see Theorem 5.9. First of all, an auxiliary fact:

Remark 5.8 We point out that disintegrations can be obtained by ‘patching together’ local disintegrations, as we are going to describe. Let $(X_n)_{n \in \mathbb{N}} \subset \Sigma_X$ be a family of pairwise disjoint subsets of X with $\|\Omega\|_{\mathbb{B}}(X \setminus \bigcup_n X_n) = 0$. Suppose to have a disintegration $\{\Omega_y^n\}_{y \in Y}$ of $\Omega|_{X_n}$ along $\varphi|_{X_n}$ for every $n \in \mathbb{N}$. Fix a measurable version $\alpha_n : Y \rightarrow [0, +\infty)$ of $\frac{d\varphi_{\#}\|\Omega|_{X_n}\|_{\mathbb{B}}}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}}$. Set

$$\Omega_y(E) := \sum_{n \in \mathbb{N}} \alpha_n(y) \Omega_y^n(E) \in \mathbb{B}, \quad \text{for every } y \in Y \text{ and } E \in \Sigma_X.$$

It is straightforward to check that $\{\Omega_y\}_{y \in Y}$ is a disintegration of Ω along φ . For instance, for every $E \in \Sigma_X$ we have that

$$\begin{aligned} \Omega(E) &= \sum_{n \in \mathbb{N}} \Omega|_{X_n}(E) = \sum_{n \in \mathbb{N}} \int \Omega_y^n(E) \, d(\varphi_{\#}\|\Omega|_{X_n}\|_{\mathbb{B}})(y) \\ &= \sum_{n \in \mathbb{N}} \int \alpha_n(y) \Omega_y^n(E) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y) = \int \Omega_y(E) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y), \end{aligned}$$

where the last identity can be justified using the Dominated Convergence Theorem.

In the case of non-negative measures, a proof (via strong liftings) of the Disintegration Theorem can be found in Refs. [26, 42]. In the next result, we adapt their strategy (using Theorem 4.4) to prove existence of the disintegration for RNP-valued measures of bounded variation.

Theorem 5.9 (Existence of disintegration for vector-valued measures) *Let (X, τ_X) , (Y, τ_Y) be Polish spaces and $\varphi: X \rightarrow Y$ a Borel measurable map. Let \mathbb{B} be a Banach space having the Radon–Nikodým property and Ω a \mathbb{B} -valued measure on $(X, \mathcal{B}(X))$ having bounded variation. Then there exists a disintegration $\{\Omega_y\}_{y \in Y}$ of Ω along φ .*

Proof Thanks to the inner regularity of $\|\Omega\|_{\mathbb{B}}$ and to Lusin’s Theorem, we can find a countable family $(K_n)_{n \in \mathbb{N}}$ of pairwise disjoint compact sets in X such that $\varphi|_{K_n}: K_n \rightarrow Y$ is continuous for every $n \in \mathbb{N}$ and $\|\Omega\|_{\mathbb{B}}(X \setminus \bigcup_n K_n) = 0$. In view of Remark 5.8, to achieve the statement it is sufficient to disintegrate each $\Omega|_{K_n}$ along $\varphi|_{K_n}$. Therefore, we may assume without loss of generality that X is compact and φ is continuous.

Step 1: Construction of Ω_y . Denote by Σ be the completion of $\mathcal{B}(Y)$ under $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$. Fix a strong lifting ℓ of $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$, which exists by Theorem 5.2. For any $f \in C(X)$, it holds that

$$\|\varphi_{\#}(f\Omega)\|_{\mathbb{B}} \stackrel{(2.14)}{\leq} \varphi_{\#}\|f\Omega\|_{\mathbb{B}} \stackrel{(2.13)}{=} \varphi_{\#}(\|f\|\|\Omega\|_{\mathbb{B}}) \leq \|f\|_{C(X)}\varphi_{\#}\|\Omega\|_{\mathbb{B}}. \tag{5.4}$$

Hence, we can consider the Radon–Nikodým derivative

$$\frac{d\varphi_{\#}(f\Omega)}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}} \in L^{\infty}(\varphi_{\#}\|\Omega\|_{\mathbb{B}}; \mathbb{B}).$$

Consider the lifting $\ell: L^{\infty}(\varphi_{\#}\|\Omega\|_{\mathbb{B}}; \mathbb{B}) \rightarrow \mathcal{L}^{\infty}(\varphi_{\#}\|\Omega\|_{\mathbb{B}}; \mathbb{B}'')$ given by Theorem 4.4. For $y \in Y$, we set

$$T_y(f) := \ell\left(\frac{d\varphi_{\#}(f\Omega)}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}}\right)(y) \in \mathbb{B}'', \quad \text{for every } f \in C(X).$$

The resulting map $T_y: C(X) \rightarrow \mathbb{B}''$ is linear by construction. Also, for any function $f \in C(X)$ one has that

$$\begin{aligned} \|T_y(f)\|_{\mathbb{B}''} &= \left\| \ell\left(\frac{d\varphi_{\#}(f\Omega)}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}}\right)(y) \right\|_{\mathbb{B}''} \stackrel{(4.5d)}{\leq} \ell\left(\left\| \frac{d\varphi_{\#}(f\Omega)}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}}(\cdot) \right\|_{\mathbb{B}}\right)(y) \\ &\stackrel{(3.8)}{\leq} \ell\left(\frac{d\|\varphi_{\#}(f\Omega)\|_{\mathbb{B}}}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}}\right)(y) \stackrel{(5.4)}{\leq} \|f\|_{C(X)}. \end{aligned} \tag{5.5}$$

Being $(C(X), \|\cdot\|_{C(X)})$ separable, we can pick a countable dense subset \mathcal{C} of $C(X)$. Thanks to (4.5a), we can find $N \in \Sigma$ with $\varphi_{\#}\|\Omega\|_{\mathbb{B}}(N) = 0$ such that $T_y(f) \in \mathbb{B}$ for every $y \in Y \setminus N$ and $f \in \mathcal{C}$, where we identify \mathbb{B} with its image $J_{\mathbb{B}}(\mathbb{B}) \subset \mathbb{B}''$. Since each T_y is linear and continuous, we deduce that $T_y(f) \in \mathbb{B}$ for every $y \in Y \setminus N$ and $f \in C(X)$. Now fix $y \in Y \setminus N$. Given that the linear continuous operator $T_y: C(X) \rightarrow \mathbb{B}$ is weakly compact by Remark 2.21, we know from [12, Theorems VI.2.1 and VI.2.5] (see also [14, Theorem VI.7.3]) that there exists a bounded \mathbb{B} -valued measure Ω_y on $(X, \mathcal{B}(X))$ such that $\int f \, d\Omega_y = T_y(f)$ for all $f \in C(X)$.

Step 2: Ω_y has bounded variation. Given that

$$C(X) \ni f \mapsto \ell \left(\frac{d\varphi_{\#}(f \|\Omega\|_{\mathbb{B}})}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}} \right) (y) \in \mathbb{R}$$

is a linear 1-Lipschitz operator (i.e., it is an element of $C(X)'$ with norm at most 1), using the Riesz–Markov–Kakutani Representation Theorem we obtain a Borel measure $\mu_y \geq 0$ on X such that $\mu_y(X) \leq 1$ and

$$\int f \, d\mu_y = \ell \left(\frac{d\varphi_{\#}(f \|\Omega\|_{\mathbb{B}})}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}} \right) (y) \quad \text{for every } f \in C(X).$$

The operator

$$S_y : \pi_{\mu_y}(C(X)) \rightarrow \mathbb{B}, \quad S_y(\pi_{\mu_y}(f)) := T_y(f), \quad \text{for every } f \in C(X),$$

is well-defined, is linear, and satisfies $\|S_y(\pi_{\mu_y}(f))\|_{\mathbb{B}} \leq \int |f| \, d\mu_y$ by (5.4) and (5.5). Hence, since $\pi_{\mu_y}(C(X))$ is dense in $L^1(\mu_y)$, we can uniquely extend S_y to a linear and continuous operator $S_y : L^1(\mu_y) \rightarrow \mathbb{B}$. Using [12, Theorem VI.3.3 and Corollary VI.3.7], we deduce that the \mathbb{B} -valued measure Ω_y has bounded variation. We also claim that

$$\|\Omega_y\|_{\mathbb{B}}(E) \leq \mu_y(E), \quad \text{for every } E \in \mathcal{B}(X). \tag{5.6}$$

In view of Remark 2.19, to prove (5.6) amounts to checking that $\|\Omega_y(E)\|_{\mathbb{B}} \leq \mu_y(E)$ holds for all $E \in \mathcal{B}(X)$. To this aim, pick a sequence $(f_n)_{n \in \mathbb{N}} \subset C(X)$ such that $f_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim_n \int |f_n - \mathbb{1}_E| \, d(\|\Omega_y\|_{\mathbb{B}} + \mu_y) = 0$. Since $\|\int f_n \, d\Omega_y\|_{\mathbb{B}} \leq \int f_n \, d\mu_y$ for every $n \in \mathbb{N}$ and

$$\left\| \int f_n \, d\Omega_y - \Omega_y(E) \right\|_{\mathbb{B}} \leq \int |f_n - \mathbb{1}_E| \, d\|\Omega_y\|_{\mathbb{B}} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

we conclude that $\|\Omega_y(E)\|_{\mathbb{B}} = \lim_n \|\int f_n \, d\Omega_y\|_{\mathbb{B}} \leq \lim_n \int f_n \, d\mu_y = \mu_y(E)$, as desired.

Step 3: Ω_y is concentrated on $\varphi^{-1}(\{y\})$. Next we show that

$$\|\Omega_y\|_{\mathbb{B}}(X \setminus \varphi^{-1}(\{y\})) = 0 \quad \text{for } \varphi_{\#}\|\Omega\|_{\mathbb{B}}\text{-a.e. } y \in Y;$$

the fact that the von Neumann lifting ℓ is strong comes into play only now. Fix any point $y \in \text{spt}(\varphi_{\#}\|\Omega\|_{\mathbb{B}}) \setminus N$ and a complete, separable distance d_Y on Y that induces the topology τ_Y . Define $\bar{g}_y := \max\{1 - d_Y(y, \cdot), 0\} \in C_b(Y)$ and $g_y := \pi_{\varphi_{\#}\|\Omega\|_{\mathbb{B}}}(\bar{g}_y) \in L^\infty(\varphi_{\#}\|\Omega\|_{\mathbb{B}})$. Notice that $\bar{g}_y < 1$ on $Y \setminus \{y\}$ and $\ell(g_y)(y) = 1$. Then,

since $\bar{g}_y \circ \varphi \in C(X)$, we can compute

$$\begin{aligned} \int (\bar{g}_y \circ \varphi) f \, d\mu_y &= \ell\left(\frac{d\varphi_{\#}((\bar{g}_y \circ \varphi) f \|\Omega\|_{\mathbb{B}})}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}}\right)(y) = \ell\left(\frac{d\varphi_{\#}(f \|\Omega\|_{\mathbb{B}})}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}}\right) g_y(y) \\ &= \ell(g_y)(y) \int f \, d\mu_y = \int f \, d\mu_y, \quad \text{for every } f \in C(X). \end{aligned}$$

Hence, $\bar{g}_y \circ \varphi = 1$ holds μ_y -a.e., which forces the validity of

$$\|\Omega_y\|_{\mathbb{B}}(X \setminus \varphi^{-1}(\{y\})) \leq \mu_y(X \setminus \varphi^{-1}(\{y\})) = 0.$$

Step 4: $\{\Omega_y\}_{y \in Y}$ is the disintegration of Ω . To conclude, it remains to prove that $\{\Omega_y\}_{y \in Y}$ is the disintegration of Ω along φ . Notice that the map $Y \ni y \mapsto \int f \, d\Omega_y \in \mathbb{B}$ is strongly measurable for every $f \in C(X)$ by construction. Moreover, for any $f \in C(X)$ we have that

$$\begin{aligned} \int f \, d\Omega &= (f\Omega)(X) = \varphi_{\#}(f\Omega)(Y) = \int \frac{d\varphi_{\#}(f\Omega)}{d\varphi_{\#}\|\Omega\|_{\mathbb{B}}} \, d\varphi_{\#}\|\Omega\|_{\mathbb{B}} \\ &= \int T_y(f) \, d\varphi_{\#}\|\Omega\|_{\mathbb{B}} = \iint f \, d\Omega_y \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y), \end{aligned} \tag{5.7}$$

whence (5.2) follows. Finally, we claim that $\|\Omega_y\|_{\mathbb{B}}(X) = 1$ for $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$ -a.e. point $y \in Y$. Indeed, we have that

$$\begin{aligned} \sum_{i=1}^n \|\Omega(E_i)\|_{\mathbb{B}} &= \sum_{i=1}^n \left\| \int \Omega_y(E_i) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y) \right\|_{\mathbb{B}} \\ &\leq \sum_{i=1}^n \int \|\Omega_y\|_{\mathbb{B}}(E_i) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y) \\ &= \int \|\Omega_y\|_{\mathbb{B}}(X) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y) \leq \varphi_{\#}\|\Omega\|_{\mathbb{B}}(Y) = \|\Omega\|_{\mathbb{B}}(X), \end{aligned}$$

holds whenever $(E_i)_{i=1}^n \subset \mathcal{B}(X)$ is a partition of X ; the starred inequality follows from the fact that $\|\Omega_y\|_{\mathbb{B}}(X) \leq \mu_y(X) \leq 1$ for $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$ -a.e. $y \in Y$. By passing to the supremum over $(E_i)_{i=1}^n$, we thus obtain that

$$\|\Omega\|_{\mathbb{B}}(X) = \int \|\Omega_y\|_{\mathbb{B}}(X) \, d(\varphi_{\#}\|\Omega\|_{\mathbb{B}})(y),$$

which in turn forces the identity $\|\Omega_y\|_{\mathbb{B}}(X) = 1$ for $\varphi_{\#}\|\Omega\|_{\mathbb{B}}$ -a.e. $y \in Y$, as claimed. The proof is complete. □

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Appendix A: From Lebesgue differentiation theorem to liftings

In this appendix, we show how to get the existence of von Neumann liftings in a quite vast class of measure spaces, containing for instance all Polish spaces. First, we recall the concept of a lower density (Definition A.1) and how it induces a lifting (Lemma A.2). Then we show that each differentiation basis verifying the Lebesgue Differentiation Theorem induces a lower density (Theorem A.3). Finally, we provide (under rather mild assumptions) a quick way to obtain a lower density verifying the Lebesgue differentiation theorem and thus to prove the existence of von Neumann liftings (Theorem A.6), which become automatically strong liftings in the case of measures over a Polish space (Corollary A.7).

We begin with the notion of a lower density, whose definition is taken from [15, 341C]:

Definition A.1 (*Lower density*) Let (X, Σ, μ) be a measure space. Then by a **lower density** of μ we mean a map $\phi: \Sigma \rightarrow \Sigma$ satisfying the following properties:

- (i) $\phi(\emptyset) = \emptyset$, $\phi(X) = X$, and $\phi(E \cap F) = \phi(E) \cap \phi(F)$ for every $E, F \in \Sigma$.
- (ii) $\phi(E) = \phi(F)$ for every $E, F \in \Sigma$ such that $\mu(E \Delta F) = 0$.
- (iii) $\mu(E \Delta \phi(E)) = 0$ for every $E \in \Sigma$.

Observe that each von Neumann lifting is a lower density. Moreover, a lower density of μ is a von Neumann lifting of μ if and only if

$$\phi(E \cup F) = \phi(E) \cup \phi(F) \quad \text{for every } E, F \in \Sigma.$$

The following key result states that by suitably ‘enlarging’ the sets appearing in the range of a lower density of a complete measure μ , one can obtain a von Neumann lifting of μ . We sketch the proof of this fact, expanding the argument in [39, page 1136].

Lemma A.2 (From lower density to lifting) *Let (X, Σ, μ) be a complete measure space and let ϕ be a lower density of μ . Then there exists a von Neumann lifting ℓ of μ such that*

$$\phi(E) \subset \ell(E) \subset X \setminus \phi(X \setminus E), \quad \text{for every } E \in \Sigma. \quad (\text{A.1})$$

Sketch of proof Given any $x \in X$, it is easy to check that \mathcal{F}_x is a filter on X , where we set

$$\mathcal{F}_x := \{F \in 2^X \mid \exists E \in \Sigma : x \in \phi(E), E \subset F\}.$$

By applying the Ultrafilter Lemma, we obtain an ultrafilter ω_x on X containing \mathcal{F}_x . Then we define the map $\ell: \Sigma \rightarrow \Sigma$ as $\ell(E) := \{x \in X : E \in \omega_x\}$ for every $E \in \Sigma$. For any $E \in \Sigma$, we have that $\phi(E) \in \mathcal{F}_x \subset \omega_x$ holds for every $x \in \phi(E)$. Moreover, if $x \in \ell(E)$, then $E \in \omega_x$, so that $X \setminus E \notin \omega_x$ and accordingly $X \setminus E \notin \mathcal{F}_x$, which implies that $x \notin \phi(X \setminus E)$. This shows that $\phi(E) \subset \ell(E) \subset X \setminus \phi(X \setminus E)$, whence (A.1) follows. Let us check that ℓ is a von Neumann lifting of μ . If $N \in \Sigma$ satisfies $\mu(N) = 0$, then $\mu((X \setminus N) \Delta X) = 0$ and thus $\phi(X \setminus N) = \phi(X) = X$, so that the second inclusion in (A.1) implies that $\ell(N) = \emptyset$. Moreover, given any $E \in \Sigma$, we have that $\mu(E \Delta \phi(E)) = 0 = \mu(E \Delta (X \setminus \phi(X \setminus E)))$, thus (A.1) and the completeness of μ ensure that $\ell(E) \in \Sigma$ and that $\mu(E \Delta \ell(E)) = 0$. Finally, it is easy to show that ℓ is a Boolean homomorphism as a consequence of the fact that ω_x is an ultrafilter. \square

The following well-known result states that every differentiation basis for which the Lebesgue differentiation theorem holds induces a lower density, just taking the density points.

Theorem A.3 (From Lebesgue differentiation theorem to lifting) *Let us consider a complete, finite measure space (X, Σ, μ) . Let \mathcal{I} a differentiation basis on (X, Σ, μ) such that*

$$D_{\mathcal{I}}(E) \in \Sigma, \quad \mu(E \Delta D_{\mathcal{I}}(E)) = 0, \quad \text{for every } E \in \Sigma, \tag{A.2}$$

where the set $D_{\mathcal{I}}(E) \subset X$ of **density points** of E with respect to \mathcal{I} is defined as

$$D_{\mathcal{I}}(E) := \left\{ x \in \hat{X} \mid \lim_{I \rightarrow x} \frac{\mu(E \cap I)}{\mu(I)} = 1 \right\}. \tag{A.3}$$

Then there exists a von Neumann lifting ℓ of μ such that

$$D_{\mathcal{I}}(E) \subset \ell(E) \subset X \setminus D_{\mathcal{I}}(X \setminus E), \quad \text{for every } E \in \Sigma.$$

Proof By virtue of Lemma A.2, to prove the statement it suffices to show that $D_{\mathcal{I}}: \Sigma \rightarrow \Sigma$ is a lower density of μ . Item iii) of Definition A.1 is exactly (A.2). Moreover, we immediately deduce from the definition (A.3) that $D_{\mathcal{I}}(\emptyset) = \emptyset$, that $D_{\mathcal{I}}(X) = X$, and that item ii) of Definition A.1 holds. Finally, to prove item i) of Definition A.1, it is only left to check

$$D_{\mathcal{I}}(E \cap F) = D_{\mathcal{I}}(E) \cap D_{\mathcal{I}}(F) \quad \text{for every } E, F \in \Sigma.$$

On the one hand, since $E \cap F \subset E$ it is immediate to see that $D_{\mathcal{I}}(E \cap F) \subset D_{\mathcal{I}}(E)$, and similarly $D_{\mathcal{I}}(E \cap F) \subset D_{\mathcal{I}}(F)$, thus the inclusion $D_{\mathcal{I}}(E \cap F) \subset D_{\mathcal{I}}(E) \cap D_{\mathcal{I}}(F)$

is proven. On the other hand, for any $I \in \mathcal{I}$ we have

$$\begin{aligned} \frac{\mu(E \cap F \cap I)}{\mu(I)} &= \frac{\mu(E \cap I)}{\mu(I)} + \frac{\mu(F \cap I)}{\mu(I)} - \frac{\mu((E \cup F) \cap I)}{\mu(I)} \\ &\geq \frac{\mu(E \cap I)}{\mu(I)} + \frac{\mu(F \cap I)}{\mu(I)} - 1, \end{aligned}$$

so that by first taking $x \in D_{\mathcal{I}}(E) \cap D_{\mathcal{I}}(F)$ and then letting $I \ni x$ we conclude that $x \in D_{\mathcal{I}}(E \cap F)$. \square

We point out that, in fact, the assumption in Theorem A.3 is a priori weaker than the statement of the Lebesgue differentiation theorem. Namely, we are only requiring that the differentiation basis \mathcal{I} verifies the statement of the Lebesgue Density Theorem.

Remark A.4 In general, the map $D_{\mathcal{I}}: \Sigma \rightarrow \Sigma$ defined in (A.3) is not a von Neumann lifting. For example, consider the real line \mathbb{R} together with the Lebesgue measure as a measure space, and the family \mathcal{I} of bounded non-trivial intervals as a differentiation basis. Then $D_{\mathcal{I}}$ does not preserve unions: 0 is in $D_{\mathcal{I}}([-1, 1])$, but neither in $D_{\mathcal{I}}([-1, 0])$ nor in $D_{\mathcal{I}}([0, 1])$.

Let us now focus on those measure spaces (X, Σ, μ) that are **separable**, which means that for some (thus, any) exponent $p \in [1, \infty)$ the p -Lebesgue space $L^p(\mu)$ is separable. It follows, e.g., from Remark 5.6 that $(X, \mathcal{B}(X), \mu)$ is separable whenever (X, τ) is a Polish space and μ is a σ -finite Borel measure on X . Next we state a corollary of a result concerning separable measure spaces, recently obtained by Brena and Gigli in [7, Theorem A]. Its proof is based on Doob’s Martingale Convergence Theorem and relies only on the Axiom of Countable Choice.

Theorem A.5 *Let (X, Σ, μ) be a complete, separable, finite measure space. Then there exists a sequence $\mathcal{P} = (\mathcal{P}_k)_{k \in \mathbb{N}}$ of partitions $\mathcal{P}_k = (P_{k,n})_{n \in I_k} \subset \Sigma$ of X , with $I_k \subset \mathbb{N}$, such that:*

- (i) $\mu(P_{k,n}) > 0$ holds for every $k \in \mathbb{N}$ and $n \in I_k$.
- (ii) Each family \mathcal{P}_{k+1} is a refinement of \mathcal{P}_k , meaning that for any $k \in \mathbb{N}$ and $n \in I_{k+1}$ there exists (a unique) index $m \in I_k$ such that $P_{k+1,n} \subset P_{k,m}$.
- (iii) For any $x \in X$, let $(n_k(x))_{k \in \mathbb{N}} \subset \mathbb{N}$ be the unique sequence such that $x \in P_{k,n_k(x)}^x := P_{k,n_k(x)}$ for every $k \in \mathbb{N}$. To any $E \in \Sigma$ we associate the set $\mathcal{D}_{\mathcal{P}}(E) \in \Sigma$, which is given by

$$\mathcal{D}_{\mathcal{P}}(E) := \left\{ x \in X \mid \lim_{k \rightarrow \infty} \frac{\mu(E \cap P_k^x)}{\mu(P_k^x)} = 1 \right\}.$$

Then it holds that $\mu(E \Delta \mathcal{D}_{\mathcal{P}}(E)) = 0$ for every $E \in \Sigma$.

Moreover, if (X, τ) is a Polish space and Σ is the completion of $\mathcal{B}(X)$ under μ , then the above statement holds for any sequence $\mathcal{P} = (\mathcal{P}_k)_{k \in \mathbb{N}}$ of partitions $\mathcal{P}_k = (P_{k,n})_{n \in I_k} \subset \mathcal{B}(X)$ of X verifying the following conditions: each family \mathcal{P}_{k+1} is a

refinement of \mathcal{P}_k and it holds that

$$\begin{aligned} \mu(P_{k,n}) &> 0, \quad \forall k \in \mathbb{N}, n \in I_k, \\ \lim_{k \rightarrow \infty} \sup_{n \in I_k} \text{diam}(P_{k,n} \cap \text{spt}(\mu)) &= 0, \end{aligned} \tag{A.4}$$

where diameters are intended with respect to some complete, separable distance d on X that induces the topology τ .

Observe that, given any (X, Σ, μ) and \mathcal{P} as in Theorem A.5, the family of sets

$$\mathcal{I}(\mathcal{P}) := \{P_{k,n} \mid k \in \mathbb{N}, n \in I_k\} \subset \Sigma$$

is a differentiation basis on (X, Σ, μ) and it holds that $D_{\mathcal{I}(\mathcal{P})}(E) = \mathcal{D}_{\mathcal{P}}(E)$ for every $E \in \Sigma$. Given that (X, Σ, μ) and $\mathcal{I}(\mathcal{P})$ verify the hypotheses of Theorem A.3, it thus follows that:

Theorem A.6 (von Neumann liftings on separable measure spaces) *Let us consider a complete, separable, finite measure space (X, Σ, μ) . Let \mathcal{P} be any sequence of partitions of the space X as in Theorem A.5. Then there exists a von Neumann lifting $\ell_{\mathcal{P}}$ of μ such that*

$$\mathcal{D}_{\mathcal{P}}(E) \subset \ell_{\mathcal{P}}(E) \subset X \setminus \mathcal{D}_{\mathcal{P}}(X \setminus E), \quad \text{for every } E \in \Sigma. \tag{A.5}$$

Finally, the same argument yields existence of strong liftings on an arbitrary Polish space:

Corollary A.7 (Existence of strong liftings) *Let (X, Σ, μ) be a complete, finite measure space. Assume that (X, τ) is a Polish space and that Σ is the completion of $\mathcal{B}(X)$ under μ . Let $\mathcal{P} = (\mathcal{P}_k)_{k \in \mathbb{N}}$ be any sequence of partitions $\mathcal{P}_k = (P_{k,n})_{n \in I_k} \subset \mathcal{B}(X)$ of X as in Theorem A.5 and satisfying (A.4). Then there exists a strong lifting $\ell_{\mathcal{P}}$ of μ such that (A.5) is verified.*

Proof Theorem A.6 yields a von Neumann lifting $\ell_{\mathcal{P}}$ of μ satisfying (A.5). Choose a complete, separable distance d on X that induces the topology τ . Let $U \in \tau \setminus \{\emptyset\}$ and $x \in U$ be fixed. Then there exists $r > 0$ such that the open d -ball of centre x and radius r is contained in U . Thanks to (A.4), we can also choose $\bar{k} \in \mathbb{N}$ so that $\text{diam}(P_k^x \cap \text{spt}(\mu)) < r/2$ for every $k \geq \bar{k}$. Hence, we have that $P_k^x \cap \text{spt}(\mu) \subset U$ for every $k \geq \bar{k}$, thus accordingly

$$\frac{\mu(U \cap P_k^x)}{\mu(P_k^x)} = \frac{\mu(U \cap \text{spt}(\mu) \cap P_k^x)}{\mu(P_k^x)} = \frac{\mu(\text{spt}(\mu) \cap P_k^x)}{\mu(P_k^x)} = 1, \quad \forall k \geq \bar{k}.$$

This implies that $x \in \mathcal{D}_{\mathcal{P}}(U) \subset \ell_{\mathcal{P}}(U)$. Thanks to the arbitrariness of $x \in U$, we have shown that $U \subset \ell_{\mathcal{P}}(U)$. Therefore, $\ell_{\mathcal{P}}$ is a strong lifting of μ . □

Appendix B: Approximate continuity with respect to a differentiation basis

In Sect. 3, the main Theorem 3.4 was obtained as a corollary of Lemma 3.2, where we proved that ‘measurable maps are almost everywhere approximately continuous with respect to a differentiation basis’, in a sense which we did not make precise. In this appendix, we expand the discussion concerning this point. In Definition B.1 we recall the notion of density topology, which is a topology induced by a von Neumann lifting. In Proposition B.3 we provide different characterisations of the notion of approximate continuity mentioned above. We learnt of this kind of approach from [3].

Let us first recall the concept of density topology associated with a lower density (see [35]):

Definition B.1 (*Density topology*) Let (X, Σ, μ) be a complete, σ -finite measure space and let ϕ be a lower density of μ . Then we define the family of sets $\mathcal{T}_\phi \subset \Sigma$ as

$$\mathcal{T}_\phi := \{E \in \Sigma \mid E \subset \ell(E)\}.$$

We say that \mathcal{T}_ϕ is the **density topology** on X associated with ϕ .

Let us verify that \mathcal{T}_ϕ is actually a topology on X . Trivially, one has that $\emptyset, X \in \mathcal{T}_\phi$. Given any $E, F \in \mathcal{T}_\phi$, it holds $E \cap F \subset \phi(E) \cap \phi(F) = \phi(E \cap F)$ and thus $E \cap F \in \mathcal{T}_\phi$. To show that \mathcal{T}_ϕ is closed under arbitrary unions, fix a (possibly uncountable) family $\{E_i\}_{i \in I} \subset \mathcal{T}_\phi$. Call $E := \bigcup_{i \in I} E_i$ for brevity. Note that, a priori, we do not know whether $E \in \Sigma$. However, we can find a countable family $C \subset I$ such that $\mu(E \Delta \bigcup_{i \in C} E_i) = 0$, thus accordingly the completeness of μ and the measurability of $\bigcup_{i \in C} E_i$ ensure that $E \in \Sigma$. Finally, for any $i \in I$ we have $E_i \subset \phi(E_i) \subset \phi(E)$, whence it follows that $E \subset \phi(E)$. This shows $\bigcup_{i \in I} E_i \in \mathcal{T}_\phi$.

Remark B.2 (Strong liftings and density topology) Let (X, Σ, μ) be a complete, σ -finite measure space such that $\text{spt}(\mu) = X$. Let τ be a Polish topology on X with $\tau \subset \Sigma$. Let ℓ be any strong lifting of μ . Then it holds that τ is contained in the density topology \mathcal{T}_ℓ .

Proposition B.3 Let (X, Σ, μ) be a complete, σ -finite measure space and (Y, τ) a topological space. Let ℓ be a von Neumann lifting of μ . Let $\varphi: X \rightarrow Y$ be a measurable map. Let $x \in \hat{X}$ be a given point. Then the following conditions are equivalent:

- (i) For every $U \in \mathcal{N}_Y(\varphi(x))$ there exists $I \in \mathcal{I}_x^\ell$ such that $\varphi(z) \in U$ for μ -a.e. $z \in I$.
- (ii) It holds that $x \in \ell(\varphi^{-1}(U))$ for every $U \in \mathcal{N}_Y(\varphi(x))$.
- (iii) The mapping $\varphi: (X, \mathcal{T}_\ell) \rightarrow (Y, \tau)$ is continuous at x .

When the above hold, we say that φ is **approximately continuous** at x with respect to \mathcal{I}^ℓ .

Proof (i) \Rightarrow (ii). Fix $U \in \mathcal{N}_Y(\varphi(x))$ and pick any $I \in \mathcal{I}_x^\ell$ such that $\varphi(z) \in U$ for μ -a.e. $z \in I$. Then the set $E := I \cap \varphi^{-1}(U) \in \Sigma$ satisfies $\mu(E \Delta I) = 0$ and thus $x \in I = \ell(I) = \ell(E) \subset \ell(\varphi^{-1}(U))$.

(ii) \Rightarrow (iii). Given any neighbourhood $U \in \mathcal{N}_Y(\varphi(x))$, we define

$$E := \varphi^{-1}(U) \cap \ell(\varphi^{-1}(U)) \in \Sigma.$$

Observe that $\ell(E) = \ell(\varphi^{-1}(U)) \supset E$, which shows that $E \in \mathcal{T}_\ell$. Notice that, by definition, we have $\varphi(z) \in U$ for every $z \in E$. Given that $x \in \varphi^{-1}(U) \subset E$ (which says that E is a neighbourhood of x in the topology \mathcal{T}_ℓ) and thanks to the arbitrariness of U , we can conclude that φ is \mathcal{T}_ℓ -continuous at x .

(iii) \Rightarrow (i). Let $U \in \mathcal{N}_Y(\varphi(x))$ be given. Then the \mathcal{T}_ℓ -continuity assumption on φ at x yields the existence of a set $E \in \mathcal{T}_\ell$ such that $x \in E$ and $\varphi(E) \subset U$. Now fix an arbitrary set $I' \in \mathcal{I}_x^\ell$ and define $I := I' \cap \ell(E)$. Given that $x \in E \subset \ell(E)$, we know that $I \in \mathcal{I}_x^\ell$. Moreover, the fact that $\varphi(z) \in U$ for every $z \in E$ and $\mu(E \Delta \ell(E)) = 0$ imply that $\varphi(z) \in U$ for μ -a.e. $z \in I$. \square

Observe that Lemma 3.2 can be rephrased as follows. Every essentially separably valued, measurable map $\varphi : X \rightarrow Y$, from a complete, σ -finite measure space (X, Σ, μ) with lifting ℓ to a metric space (Y, d) , is approximately continuous at μ -a.e. point of X with respect to \mathcal{I}^ℓ .

Appendix C: More on liftings of Lebesgue–Bochner spaces

As we already pointed out, the theory of measurable Banach bundles covers the case of Lebesgue–Bochner spaces, which have been thoroughly studied in the literature, also from the point of view of lifting theory. We will report here some results in this direction, taken from the paper [22], which show that the lifting of sections we obtained in Theorem 4.4 is the best one can hope for. More precisely, it will be clear from Theorem C.2 that the inequality in (4.5d) cannot, in general, be improved to an equality.

Let (X, Σ, μ) be a σ -finite measure space. Then μ is said to be **atomic** provided it has at least one atom, while it is said to be **non-atomic** when no atom of μ exists. Moreover, we say that μ is **purely atomic** provided $\mu|_E$ is atomic for every $E \in \Sigma$ such that $\mu(E) > 0$.

Lemma C.1 *Let (X, Σ, μ) be a complete, σ -finite measure space. Suppose that μ is not purely atomic. Let ℓ be a von Neumann lifting of μ . Then there exist a point $\bar{x} \in X$ and a family $\{E_n\}_{n \in \mathbb{N}} \subset \Sigma$ of pairwise disjoint sets such that $0 < \mu(E_n) < \infty$ for every $n \in \mathbb{N}$ and*

$$\mu\left(X \setminus \bigcup_{n \in \mathbb{N}} E_n\right) = 0, \quad \bar{x} \in X \setminus \bigcup_{n \in \mathbb{N}} \ell(E_n). \tag{C.1}$$

Proof Fix a set $F \in \Sigma$ with $0 < \mu(F) < \infty$ such that $\mu|_F$ is non-atomic, whose existence stems from the assumption that μ is not purely atomic. Being μ a σ -finite measure, we can find an at most countable partition $(G_i)_{i \in I} \subset \Sigma$ of $X \setminus F$ into sets having finite, positive μ -measure. Now fix any point $\bar{x} \in \ell(F)$ and notice that $\bar{x} \notin$

$\bigcup_{i \in I} \ell(G_i)$. Since $\mu|_F$ is non-atomic, we know that it has full range (it follows, e.g., from Lyapunov Convexity Theorem, see [12]), meaning that for any $\lambda \in (0, \mu(F))$ there exists $F' \in \Sigma$ such that $F' \subset F$ and $\mu(F') = \lambda$. We can thus proceed with the construction via the following recursive argument. First, take a measurable set $F_1 \subset F$ such that $\mu(F_1) = \mu(F)/2$. Given that $\bar{x} \in \ell(F) = \ell(F_1) \cup \ell(F \setminus F_1)$, we may assume (up to replacing F_1 by $F \setminus F_1$) that $\bar{x} \notin \ell(F_1)$. Next, we can argue in the very same way starting from $F \setminus F_1$, thus getting a set $F_2 \subset F \setminus F_1$ such that $\mu(F_2) = \mu(F)/4$ and $\bar{x} \notin \ell(F_2)$, and so on. All in all, with this procedure we obtain a sequence $(F_j)_{j \in \mathbb{N}}$ of pairwise disjoint measurable subsets of F such that $\mu(F_j) = \mu(F)/2^j$ holds for every $j \in \mathbb{N}$ and $\bar{x} \notin \bigcup_{j \in \mathbb{N}} \ell(F_j)$. Hence, the family $\{E_n\}_{n \in \mathbb{N}} := \{G_i\}_{i \in I} \cup \{F_j\}_{j \in \mathbb{N}}$ satisfies (C.1). \square

The previous lemma is useful to achieve the next result, taken from [22, Section 4.1]; we report its proof for the reader’s usefulness. It is convenient, for the moment, to consider only measures that are not purely atomic. The case of purely atomic measures is easier to handle and will be discussed in the last part of this appendix (see Remark C.4).

Theorem C.2 (Lifting of Lebesgue–Bochner spaces) *Let (X, Σ, μ) be a complete, σ -finite measure space. Suppose that μ is not purely atomic. Let \mathbb{B} be a Banach space and ℓ a von Neumann lifting of μ . Then the following two conditions are equivalent:*

(i) *There exists an operator $\ell : L^\infty(\mu; \mathbb{B}) \rightarrow \mathcal{L}^\infty(\mu; \mathbb{B})$ satisfying*

$$\begin{aligned} \ell(\mathbb{1}_X^\mu \bar{v})(x) &= \bar{v}, \quad \forall \bar{v} \in \mathbb{B}, x \in X, \\ \ell(v)(x) &= \hat{v}(x), \quad \forall v \in L^\infty(\mu; \mathbb{B}) \text{ and } \mu\text{-a.e. } x \in X, \\ \ell(v + w)(x) &= \ell(v)(x) + \ell(w)(x), \quad \forall v, w \in L^\infty(\mu; \mathbb{B}), x \in X, \\ \ell(f \cdot v)(x) &= \ell(f)(x) \ell(v)(x), \quad \forall v \in L^\infty(\mu; \mathbb{B}), f \in L^\infty(\mu), x \in X, \\ \|\ell(v)(x)\|_{\mathbb{B}} &= \ell(\|v(\cdot)\|_{\mathbb{B}})(x), \quad \forall v \in L^\infty(\mu; \mathbb{B}), x \in X, \end{aligned}$$

where $\hat{\#} : L^\infty(\mu; \mathbb{B}) \rightarrow \mathcal{L}^\infty(\mu; \mathbb{B})$ is the precise representative given by Theorem 3.5.

(ii) *The Banach space \mathbb{B} is finite-dimensional.*

Proof The implication (ii) \Rightarrow (i) stems from Proposition 4.9 and the observation that $L^\infty(\mu; \mathbb{B})$ and $\text{ECV}(\mu; \mathbb{B})$ coincide when \mathbb{B} is finite-dimensional. Conversely, to prove the validity of the converse implication (i) \Rightarrow (ii), we argue by contradiction: suppose that (i) holds and \mathbb{B} is infinite-dimensional. Then we can find a sequence $(\bar{v}_n)_{n \in \mathbb{N}} \subset \mathbb{B}$ such that $\|\bar{v}_n\|_{\mathbb{B}} = 1$ for every $n \in \mathbb{N}$ and $\|\bar{v}_n - \bar{v}_m\|_{\mathbb{B}} \geq 1/2$ for every $n, m \in \mathbb{N}$ with $n \neq m$. Recalling Lemma C.1, we can find $\bar{x} \in X$ and a sequence of pairwise disjoint sets $(E_n)_{n \in \mathbb{N}} \subset \Sigma$ with $\bar{x} \notin \bigcup_n E_n$ such that $\ell(E_n) = E_n$ for all $n \in \mathbb{N}$ and $\mu(X \setminus \bigcup_n E_n) = 0$. Define $v := \sum_{n \in \mathbb{N}} \mathbb{1}_{E_n}^\mu \bar{v}_n \in L^\infty(\mu; \mathbb{B})$. Calling $\bar{v} := \ell(v)(\bar{x}) \in \mathbb{B}$, we claim that $\bar{v} = \bar{v}_{\bar{n}}$ for some $\bar{n} \in \mathbb{N}$. If this were not the case, then there would exist $\varepsilon > 0$ such that $\|\bar{v} - \bar{v}_n\|_{\mathbb{B}} \geq \varepsilon$ for every $n \in \mathbb{N}$; here, we are using the fact that $\{\bar{v}_n : n \in \mathbb{N}\}$ is a discrete set. This is not possible, since it would

imply that

$$\begin{aligned} 0 &= \|\ell(v)(\bar{x}) - \bar{v}\|_{\mathbb{B}} = \|\ell(v)(\bar{x}) - \ell(\mathbb{1}_X^\mu \bar{v})(\bar{x})\|_{\mathbb{B}} = \|\ell(v - \mathbb{1}_X^\mu \bar{v})(\bar{x})\|_{\mathbb{B}} \\ &= \ell(\|v(\cdot) - \bar{v}\|_{\mathbb{B}})(\bar{x}) = \ell\left(\sum_{n \in \mathbb{N}} \|\bar{v}_n - \bar{v}\|_{\mathbb{B}} \mathbb{1}_{E_n}^\mu\right)(\bar{x}) \geq \ell(\varepsilon \mathbb{1}_{\bigcup_n E_n}^\mu)(\bar{x}) \\ &= \varepsilon \ell(\mathbb{1}_X^\mu)(\bar{x}) = \varepsilon. \end{aligned}$$

Let $\bar{n} \in \mathbb{N}$ be such that $\bar{v} = \bar{v}_{\bar{n}}$. Note that $\bar{x} \in X \setminus E_{\bar{n}} = X \setminus \ell(E_{\bar{n}}) = \ell(X \setminus E_{\bar{n}})$ and thus

$$\begin{aligned} 0 &= \|\ell(v)(\bar{x}) - \bar{v}\|_{\mathbb{B}} = \|\mathbb{1}_{\ell(X \setminus E_{\bar{n}})}(\bar{x}) \ell(v)(\bar{x}) - \bar{v}_{\bar{n}}\|_{\mathbb{B}} \\ &= \|\ell(\mathbb{1}_{X \setminus E_{\bar{n}}}^\mu \cdot v - \mathbb{1}_{X \setminus E_{\bar{n}}}^\mu \bar{v}_{\bar{n}})(\bar{x})\|_{\mathbb{B}} = \ell(\mathbb{1}_{X \setminus E_{\bar{n}}}^\mu \|v(\cdot) - \bar{v}_{\bar{n}}\|_{\mathbb{B}})(\bar{x}) \\ &= \ell\left(\sum_{n \neq \bar{n}} \|\bar{v}_n - \bar{v}_{\bar{n}}\|_{\mathbb{B}} \mathbb{1}_{E_n}^\mu\right)(\bar{x}) \geq \ell\left(\frac{1}{2} \mathbb{1}_{X \setminus E_{\bar{n}}}^\mu\right)(\bar{x}) = \frac{1}{2}, \end{aligned}$$

which leads to a contradiction. Therefore, we have proven that \mathbb{B} is finite-dimensional. \square

Remark C.3 By inspecting the proof of Theorem C.2, it is evident that a stronger claim holds: if \mathbb{B} is infinite-dimensional, then it is not possible to find a lifting $L^\infty(\mu; \mathbb{B}) \rightarrow \mathcal{L}^\infty(\mu; \mathbb{V})$ for any Banach space \mathbb{V} where \mathbb{B} can be embedded linearly and isometrically.

We point out that all measurable maps are *almost everywhere constant on atoms*: given a metric space (Y, d) , a strongly measurable map $\varphi: X \rightarrow Y$, and an atom A of μ , it holds that

$$\mu(\{x \in A \mid \varphi(x) \neq y\}) = 0, \quad \text{for a (unique) element } y \in Y. \tag{C.2}$$

To prove it, fix $N \in \Sigma$ and a separable Borel set $\tilde{Y} \subset Y$ such that $\mu(N) = 0$ and $\varphi(X \setminus N) \subset \tilde{Y}$. Thanks to Lindelöf Lemma, for any $n \in \mathbb{N}$ we can find a Borel partition $(E_n^k)_{k \in \mathbb{N}}$ of \tilde{Y} such that the diameter of each set E_n^k does not exceed $1/n$. Without loss of generality, we can also assume that $(E_{n+1}^k)_k$ is a refinement of $(E_n^k)_k$, i.e., for any $k \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$ for which $E_{n+1}^k \subset E_n^\ell$. Since A is an atom of μ , we can find $(k_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ such that $\mu(A \setminus \varphi^{-1}(E_n)) = 0$, where $E_n := E_{n+1}^{k_n}$. Notice that $E := \bigcap_{n \in \mathbb{N}} E_n$ has null diameter, thus in particular it is either empty or a singleton. Given that $\mu(A \setminus \varphi^{-1}(E)) = 0$, we deduce that $E \neq \emptyset$, so that $E = \{y\}$ for some $y \in Y$ and thus $\varphi(x) = y$ for μ -a.e. $x \in A$. Consequently, the claim (C.2) is proven.

Remark C.4 (Lifting of purely atomic measures) Let (X, Σ, μ) be a σ -finite measure space, with μ purely atomic. As proven, e.g., in [28, Theorem 2.2], one can find an at most countable partition $(A_i)_{i \in I}$ of X made of atoms of μ . Observe that for any $E \in \Sigma$ there exists a unique index set $I_E \subset I$ such that $\mu(E \Delta \bigcup_{i \in I_E} A_i) = 0$. Then

we define the map $\ell: \Sigma \rightarrow \Sigma$ as

$$\ell(E) := \bigcup_{i \in I_E} A_i, \quad \text{for every } E \in \Sigma.$$

It is straightforward to verify that ℓ is a von Neumann lifting of μ .

Now fix an arbitrary Banach space \mathbb{B} . Recalling (C.2), we see that for any $v \in L^\infty(\mu; \mathbb{B})$ there is a unique $(\bar{v}_i)_{i \in I} \subset \mathbb{B}$ such that $\sup_{i \in I} \|\bar{v}_i\|_{\mathbb{B}} < +\infty$ and $v = \sum_{i \in I} \mathbb{1}_{A_i}^\mu \bar{v}_i$, thus we set

$$\ell(v) := \sum_{i \in I} \mathbb{1}_{A_i} \bar{v}_i \in \mathcal{L}^\infty(\mu; \mathbb{B}).$$

It is then easy to check that the resulting operator $\ell: L^\infty(\mu; \mathbb{B}) \rightarrow \mathcal{L}^\infty(\mu; \mathbb{B})$ verifies all the properties listed in item i) of Theorem C.2.

References

1. Benyamini, Y., Lindenstrauss, J.: Geometric Nonlinear Functional Analysis. Vol. 1, vol. 48. American Mathematical Society Colloquium Publications, American Mathematical Society, Providence (2000)
2. Bessaga, C., Pełczyński, A.: Selected Topics in Infinite-dimensional Topology. Polska Akademia Nauk. Instytut Matematyczny, Monografie matematyczne, Warsaw (1975)
3. Bliedtner, J., Loeb, P.A.: The optimal differentiation basis and liftings of L^∞ . Trans. Am. Math. Soc. **352**, 4693–4710 (2000)
4. Bochner, S.: Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind. Fundam. Math. **20**, 262–276 (1933)
5. Bogachev, V.I.: Measure Theory, vol. II. Springer, Berlin (2007)
6. Bourbaki, N.: Topologie générale. Hermann, Paris (1965)
7. Brena, C., Gigli, N.: Local vector measures. Preprint. [arXiv:2206.14864](https://arxiv.org/abs/2206.14864) (2022)
8. Bruckner, A.M.: Differentiation of integrals. Am. Math. Mon. **78**, 53 (1971)
9. Bruckner, A.M., Bruckner, J.B., Thomson, B.S.: Real Analysis. Prentice-Hall, Upper Saddle River (1997)
10. Bruckner, A.M., Bruckner, J.B., Thomson, B.S.: Real Analysis, 2nd edn. ClassicalRealAnalysis.com, Vancouver (2008)
11. Di Marino, S., Lučić, D., Pasqualetto, E.: Representation theorems for normed modules. Preprint. [arXiv:2109.03509](https://arxiv.org/abs/2109.03509) (2021)
12. Diestel, J., Uhl, J.J., Jr.: Vector Measures. American Mathematical Society, Providence (1977)
13. Dunford, N.: Integration in general analysis. Trans. Am. Math. Soc. **37**, 441–453 (1935)
14. Dunford, N., Schwartz, J.: Linear Operators. (I): General Theory. Interscience Publishers, Pure and Applied Mathematics, New York (1958)
15. Fremlin, D.: Measure Theory: Measure Algebras. Measure Theory, vol. 3. Torres Fremlin, Colchester (2011)
16. Gigli, N., Lučić, D., Pasqualetto, E.: Duals and pullbacks of normed modules. Preprint. [arXiv:2207.04972](https://arxiv.org/abs/2207.04972) (2022)
17. Gigli, N.: Lecture notes on differential calculus on $RC D$ spaces. Publ. Res. Inst. Math. Sci. **54**, 855–918 (2018)
18. Gigli, N.: Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below. Mem. Am. Math. Soc. **251**, 161 (2018)
19. Graf, S., von Weizsäcker, H.: On the Existence of Lower Densities in Noncomplete Measure Spaces. Measure Theory, pp. 155–158. Springer, Berlin (1976)
20. Guo, T.X.: Recent progress in random metric theory and its applications to conditional risk measures. Sci. China Math. **54**, 633–660 (2011)

21. Gutman, A.E.: Banach bundles in the theory of lattice-normed spaces. I. Continuous Banach bundles. *Sib. Adv. Math.* **3**, 1–55 (1993)
22. Gutman, A.E.: Banach bundles in the theory of lattice-normed spaces. II. Measurable Banach bundles. *Sib. Adv. Math.* **3**, 8–40 (1993)
23. Gutman, A.E., Koptev, A.V.: Dual Banach Bundles. *Nonstandard Analysis and Vector Lattices, Mathematical Applications*, vol. 525, pp. 105–159. Kluwer Academic Publishers, Dordrecht (2000)
24. Haydon, R., Levy, M., Raynaud, Y.: Randomly Normed Spaces. *Travaux en Cours [Works in Progress]*, vol. 41. Hermann, Paris (1991)
25. Heinonen, J.: *Lectures on Analysis on Metric Spaces*. Universitext. Springer, New York (2001)
26. Hoffmann-Jørgensen, J.: Existence of conditional probabilities. *Math. Scand.* **28**, 257–264 (1971)
27. Hytönen, T., Neerven, J., Veraar, M., Weis, L.: *Analysis in Banach Spaces: Volume I: Martingales and Littlewood-Paley Theory*. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge/A Series of Modern Surveys in Mathematics*, Springer (2016)
28. Johnson, R.A.: Atomic and nonatomic measures. *Proc. Am. Math. Soc.* **25**, 650–655 (1970)
29. Johnson, R.A.: Strong liftings that are not Borel liftings. *Proc. Am. Math. Soc.* **80**, 234–236 (1980)
30. Kölzow, D.: *Differentiation von Maßen*. *Lecture Notes in Mathematics*. Springer, Berlin (1968)
31. Lebesgue, H.: Sur l'intégration des fonctions discontinues. *Ann. Sci. Éc. Norm. Supér.* **3**(27), 361–450 (1910)
32. Loeb, P.A., Talvila, E.: Lusin's theorem and Bochner integration. *Sci. Math. Jpn.* **10**, 55–62 (2004)
33. Losert, V.: A measure space without the strong lifting property. *Math. Ann.* **239**, 119–128 (1979)
34. Lučić, D., Pasqualetto, E.: The Serre–Swan theorem for normed modules. *Rend. Circ. Mat. Palermo* **2**(68), 385–404 (2019)
35. Lukeš, J., Malý, J., Zajíček, L.: *Fine Topological Methods in Real Analysis and Potential Theory*. Vol. 1189 of *Lecture Notes in Mathematics*. Springer, Berlin (1986)
36. Maharam, D.: On a theorem of von Neumann. *Proc. Am. Math. Soc.* **9**, 987–994 (1958)
37. Mokobodzki, G.: Relèvement borélien compatible avec une classe d'ensembles négligeables. *Application à la désintégration des mesures*. *Séminaire de probabilités de Strasbourg* **9**, 437–442 (1975)
38. Shelah, S.: Lifting problem of the measure algebra. *Isr. J. Math.* **45**, 90–96 (1983)
39. Strauss, W., Macheras, N.D., Musiał, K.: Liftings. *Handbook of Measure Theory: In two volumes* (edited by E. Pap), Volume II, Part 7. Elsevier Science, Amsterdam (2002)
40. Traynor, T.: An elementary proof of the lifting theorem. *Pac. J. Math.* **53**, 267–272 (1974)
41. Tulcea, A.I., Tulcea, C.I.: On the existence of a lifting commuting with the left translations of an arbitrary locally compact group. In: *Proceedings of the Fifth Berkeley Symposium of Mathematical Statistics and Probability* (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1, pp. 63–97. University of California Press (1967)
42. Tulcea, C.I.: On the lifting property and disintegration of measures. *Bull. Am. Math. Soc.* **71**, 829–842 (1965)
43. Tulcea, A.I., Tulcea, C.I.: On the lifting property (I). *J. Math. Anal. Appl.* **3**, 537–546 (1961)
44. Tulcea, A.I., Tulcea, C.I.: Liftings for abstract valued functions and separable stochastic processes. *Probab. Theory Relat. Fields* **13**, 114–118 (1969)
45. Tulcea, A.I., Tulcea, C.I.: *Topics in the Theory of Lifting*. Vol. 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, Berlin (1969)
46. von Neumann, J.: Algebraische Repräsentanten der Funktionen “bis auf eine Menge von Maße Null”. *J. Reine Angew. Math.* **165**, 109–115 (1931)
47. von Neumann, J., Stone, M.H.: The determination of representative elements in the residual classes of a Boolean algebra. *Fundam. Math.* **25**, 353–378 (1935)