METRIC EQUIVALENCES OF HEINTZE GROUPS AND APPLICATIONS TO CLASSIFICATIONS IN LOW DIMENSION

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ABSTRACT. We approach the quasi-isometric classification questions on Lie groups by considering low dimensional cases and isometries alongside quasi-isometries. First, we present some new results related to quasi-isometries between Heintze groups. Then we will see how these results together with the existing tools related to isometries can be applied to groups of dimension 4 and 5 in particular. Thus we take steps towards determining all the equivalence classes of groups up to isometry and quasi-isometry. We completely solve the classification up to isometry for simply connected solvable groups in dimension 4, and for the subclass of groups of polynomial growth in dimension 5.

CONTENTS

Date: December 12, 2021.
2010 *Mathematics Subject Classification.* 20F67, 53C23, 22E25, 17B70, 20F69, 30L10, 54E40.
Key words and phrases. isometries, quasi-isometries, homogeneous groups, Heintze groups.
V.K. and E.L.D. were partially supported by the European Research Council (ERC Starting Grant 713998 GeoMeG '*Geometry of Metric Groups*'). E.L.D. and S.N.G. were partially supported by the Academy of Finland (grant288501 '*Geometry of subRiemannian groups*' and by grant 322898 '*Sub-Riemannian Geometry via Metric-geometry and Lie- group Theory*'). S. N. G. has been supported by the Academy of Finland (grant 314172 '*Quantitative rectifiability in Euclidean and non-Euclidean spaces*', and grant 328846 '*Singular integrals, harmonic functions, and boundary regularity in Heisenberg groups*'). V.K. was also supported by Emil Aaltonen Foundation.

0. Introduction

This paper is a contribution to various metric classifications of Lie groups. The study of quasi-isometries between solvable groups is an active area of research [Pan89, Sha04, Sau06, dC18, EFW12, EFW13, Dym10, Xie14, Xie15, CPS17, Pia17, Pal20]. Distinguished examples of solvable groups are Heintze groups, i.e., those solvable simply connected Lie groups that admit left-invariant Riemannian structures with negative sectional curvatures [Hei74]. Every Heintze group G is a semidirect product of $\mathbb{R}$ and a nilpotent graded Lie group N. The parabolic visual boundary of G has a structure of homogeneous group. Namely, the boundary may be identified with N equipped with a distance that has dilation properties. Moreover, the existence of a quasi-isometry between two Heintze groups implies the existence of a biLipschitz map between the associated nilpotent groups and vice versa; this is a known result that we summarize in Proposition 1.5. See also [CPS17, p.6] for a good exposition and list of references.

The main aim of this article is twofold: First, we introduce quasi-isometry invariants that finally distinguish some low dimensional Heintze groups. Second, we study a finer metric classification. We say that two Lie groups G and H *can be made isometric* if there are left-invariant Riemannian metrics ρ_G and ρ_H so that (G, ρ_G) is isometric to (H, ρ_H). This is an equivalence relation among simply connected solvable groups, and we find the equivalence classes in low dimension: we consider all simply connected solvable Lie groups in dimension 4 and those with polynomial growth in dimension 5. For each equivalence class, there is a Riemannian manifold for which each element of the class acts isometrically and simply transitively. In our construction, such a Riemannian manifold is a Lie group, which we call the "real-shadow". In particular, we make a contribution to the conjecture that claims that every two Heintze groups are either not quasi-isometric or they can be made isometric.

0.1. **Quasi-isometries of Heintze groups.** First we present our results related to distinguishing Heintze groups up to quasi-isometry equivalence. We work on the level of parabolic visual boundaries, thus our objects of interest are *homogeneous groups*, by which we mean pairs (N, α) where N is a simply connected nilpotent Lie group and α is a derivation of N, such that $N \rtimes_\alpha \mathbb{R}$ defines a Heintze group. We may assume that $N \rtimes_\alpha \mathbb{R}$ is purely real, i.e., that all the eigenvalues of α are real numbers. For a homogeneous group (N, α), we always consider the biLipschitz class of distances that are homogeneous under the one-parameter subgroups of automorphisms induced by the derivation α. This class may be empty in some cases, see Remark 1.3.

Below we use the following notation: If (N, α) is a homogeneous group and $\bigoplus_{\lambda>0} V_\lambda^\alpha$ is the decomposition of the Lie algebra of N by the generalised eigenspaces of the derivation α, then for every $s > 0$ we denote by $(N, \alpha)^{(s)}$ the subgroup of N with the Lie algebra $\mathrm{LieSpan}(\bigoplus_{s\geq\lambda>0} V_\lambda^\alpha)$.

Theorem A. *Let (N_1, α) and (N_2, β) be purely real homogeneous groups that are biLipschitz equivalent via a map $F\colon N_1 \to N_2$.*

(i) Then N_1 and N_2 are quasi-isometric as Riemannian Lie groups.
(ii) For every $p \in N_1$ and every $s \geq 1$ we have $F(p(N_1, \alpha)^{(s)}) = F(p)(N_2, \beta)^{(s)}$ and the same holds for all the iterated normalisers of the subgroups $(N_1, \alpha)^{(s)}$ and $(N_2, \beta)^{(s)}$, respectively.

The proof of this result, which is inspired by the results of [CPS17], is presented in Section 2. We also present some examples to illustrate how this result helps to distinguish some particular pairs of low dimensional Heintze groups up to quasi-isometry. Notice that part (i) implies via [Pan89] that the Carnot groups associated to N_1 and N_2 as their asymptotic cones are isomorphic. In particular, the nilpotency steps of N_1 and N_2 agree.

0.2. **On the classification up to isometries.** To motivate and give some background, let us compare the state of the art of the classification up to isometry and quasi-isometry for two distinct subclasses of the class of solvable simply connected Lie groups: Heintze groups and solvable groups of polynomial growth (with nilpotent groups as main examples). These subclasses have some similarities when it comes to isometries and quasi-isometries. In both cases every group has "a representative with real roots" and those representatives are known to be distinguished by isometries, and are conjectured to be distinguished by quasi-isometries. More precisely, we have the following facts and folklore conjectures:

Proposition H1 (Alekseevskiĭ [Ale75])**.** *Every Heintze group can be made isometric to a purely real Heintze group.*

Proposition H2 (Gordon–Wilson [GW88])**.** *If two purely real Heintze groups can be made isometric, then they are isomorphic.*

Conjecture H3. *If two purely real Heintze groups are quasi-isometric then they are isomorphic.*

Proposition P1 (Breuillard [Bre14])**.** *Every simply connected solvable Lie group of polynomial growth can be made isometric to a nilpotent group.*

Proposition P2 (Wolf [Wol63])**.** *If two simply connected nilpotent Lie groups can be made isometric, then they are isomorphic.*

Conjecture P3. *If two simply connected nilpotent Lie groups are quasi-isometric then they are isomorphic.*

Recently, the articles [CKL+21] and [Jab19] clarified quite a bit this field, when it comes to isometries. Now we know that to every simply connected solvable Lie group it is possible to canonically associate a completely solvable (a.k.a. split-solvable or real triangulable) Lie group, so called "real-shadow" of the group, which is unique up to isomorphism. In particular, this construction satisfies the following theorem.

Fact 0.1 (Corollary 4.23 in [CKL+21])**.** *Let G and H be simply connected solvable Lie groups. Then G can be made isometric to H if and only if the real-shadows of G and H are isomorphic.*

This result, besides containing the information of propositions H1-2 and P1-2 above, implies that "can be made isometric" is an equivalence relation within the class of simply connected solvable Lie groups. Moreover, it implies that the isometric classification of such groups boils down to the algebraic problem of calculating their real-shadows. Remark that it is not known if Fact 0.1 holds when isometries are replaced by quasi-isometries. This is because the more general version, due to Y. Cornulier [dC18, Conjecture 19.113], of Conjecture H3 and Conjecture P3 is also open: whether two quasi-isometric completely solvable simply connected Lie groups are necessarily isomorphic or not.

Since Lie groups that can be made isometric are necessarily quasi-isometric, we are led to study the following problem: Which pairs of groups in the same quasi-isometry class can be made isometric? This problem is completely solved for groups of dimension 3 and it is surveyed in [FLD21]. One of the main contributions of the present article is to push towards a solution for simply connected groups of dimension 4. While we are not able to completely solve the quasi-isometry relations of 4-dimensional groups, we can solve the isometry relations: it is clear that Fact 0.1 is enough for that. However, Theorem B below is a more explicit result and can be proved with elementary methods. In its statement, we denote by $\alpha_0 = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{nil}}$ the *real part* of the derivation α: we shall recall the relevant decomposition more precisely in Proposition 1.12.

Theorem B. *Let H be a simply connected Lie group and α a derivation of H. Then the Lie group $H \rtimes_\alpha \mathbb{R}$ can be made isometric to the Lie group $H \rtimes_{\alpha_0} \mathbb{R}$, where α_0 is the real part of α.*

Theorem B is a corollary of a more general result, Theorem 3.1, which we will state and prove in Section 3. In the category of solvable groups, Theorem B is a special case of Fact 0.1, but it may also provide information about isometry questions of non-solvable semidirect products. Notice that there is no assumptions on the eigenvalues of α.

Theorem B has practical value also within the family of solvable Lie groups: In Section 4 we find all the pairs of Lie groups that can be made isometric within the family of 4-dimensional simply connected solvable Lie groups. In Section 5 we do the same within the family of 5-dimensional simply connected solvable Lie groups of polynomial growth. The method is described as follows. Since the algebraic classification of Lie groups is known within these families, we first indicate all the completely solvable ones: these are the groups that are isomorphic to their real-shadows. Then for each solvable group G that is not completely solvable, we find a completely solvable group to which it is isometric by finding a suitable decomposition of G as a semi-direct product $H \rtimes_\alpha \mathbb{R}$ where H is completely solvable. This happens to be always possible within the families we investigate. Now we know from Theorem B that such a group G can be made isometric to the completely solvable group $H \rtimes_{\alpha_0} \mathbb{R}$, while Fact 0.1 then guarantees that $H \rtimes_{\alpha_0} \mathbb{R}$ is the real-shadow of G, and any other solvable group G' that can be made isometric to G must also have $H \rtimes_{\alpha_0} \mathbb{R}$ as the real-shadow.

The result we get in dimension 4 is summarised in the theorem below.

Theorem C. *Let G and H be simply connected solvable Lie groups of dimension 4. If G and H are both completely solvable, then they can be made isometric if and only if they are isomorphic. Instead, if at least one of them is not completely solvable, then they can be made isometric if and only if they belong to the same set of groups in the following list (the notation is w.r.t. the classification given by* [PSWZ76]*):*

(I) $\{\mathbb{R}^4,\ \mathbb{R} \times A_{3,6}\}$,
(II) $\{\mathbb{R} \times A_{3,1},\ A_{4,10}\}$,
(III$_\lambda$) $\{A_{4,5}^{\lambda,\lambda}\} \cup \{A_{4,6}^{a,b} : \lambda = \operatorname{sign}(ab)\min(|b/a|, |a/b|)\}$,
(IV) $\{A_{4,9}^{1}\} \cup \{A_{4,11}^{a} : a \in]0,\infty[\,\}$,
(V) $\{\mathbb{R} \times A_{3,3},\ A_{4,12}\} \cup \{\mathbb{R} \times A_{3,7}^{a} : a \in]0,\infty[\,\}$,
(VI) $\{\mathbb{R}^2 \times A_2\} \cup \{A_{4,6}^{a,0} : a \in \mathbb{R}\}$

Here (III$_\lambda$) *stands for distinct sets depending on parameter $\lambda \in \mathbb{R}\backslash\{0\}$. Hence the above list contains 5 sets (2 finite and 3 infinite) and one family of sets depending on a parameter.*

In Section 5 we find similar classification for simply connected solvable groups of polynomial growth in dimension 5. Table 3 in Section 5 summarises the results within this family.

Acknowledgements. We thank Pierre Pansu, Matias Carrasco Piaggio and Xiangdong Xie for constructive comments. In particular, Pansu helped us to strengthen an early version of Theorem A.(i).

1. Preliminaries

1.1. **Homogeneous groups.** We shall approach the quasi-isometric classification problems in Heintze groups by studying the biLipschitz maps on their boundary, and we now define precisely the terminology of this setting.

In this paper, we will always use the convention that if $N, H, G, \dots$ are Lie groups, then the fraktur letters $\mathfrak{n}, \mathfrak{h}, \mathfrak{g}, \dots$ denote their Lie algebras, and vice versa.

Definition 1.1. A pair (N, α) is a *homogeneous group* if N is a simply connected nilpotent Lie group and α is a derivation of N so that for each eigenvalue λ of α it holds $\operatorname{Re}(\lambda) > 0$. Further, we say that a homogeneous group (N, α) is

- *purely real*, if the eigenvalues of α are real numbers,

- *of Carnot type* if it is purely real, if α is diagonalisable over $\mathbb{R}$, and if the eigenspace corresponding to the smallest of the eigenvalues Lie-generates $\mathfrak{n}$.

Two homogeneous groups (N_1, α) and (N_2, β) are *isomorphic (as homogeneous groups)* if there is an isomorphism of Lie groups $F\colon N_1 \to N_2$ so that $\beta \circ F_* = F_* \circ \alpha$, where F_* is the Lie algebra isomorphism induced by F.

The data defining homogeneous groups exactly coincide with the data defining Heintze groups. The terms *purely real Heintze group* and *Heintze group of Carnot type* appear in the literature and correspond to the terms above, see for example [CPS17] and [dC18].

For purposes of classifications up to isometry or quasi-isometry, only the purely real homogeneous groups play a role due to the result of [Ale75] presented here as Proposition H1 in the introduction. Hence we will always assume that the derivation has real eigenvalues even in the cases when it would be not strictly necessary.

Next we discuss homogeneous distances on homogeneous groups.

Definition 1.2. Let (N, α) be a homogeneous group. A distance function ρ on the set N is said to be *homogeneous (for (N, α))*, if ρ is left-invariant, induces the manifold topology of N, and for all $\lambda > 0$ we have $\rho(\delta_\lambda x, \delta_\lambda y) = \lambda\rho(x, y)$ for all $x, y \in N$, where δ_λ is the automorphism of N with the differential $(\delta_\lambda)_* = \mathrm{e}^{\log(\lambda)\alpha}$. The triple (N, α, ρ) is called *a homogeneous metric group* if the distance function ρ is homogeneous for (N, α).

Remark 1.3. In [LDNG19, Theorem B] it is characterised when a purely real homogeneous group (N, α) admits a distance ρ making it a homogeneous metric group: denoting by ν the smallest eigenvalue of α, a distance exists if and only if $\nu \geq 1$ and the restriction of α to its generalised eigenspace of eigenvalue 1 is diagonalisable over $\mathbb{R}$. Consequently, if (N, α) is a homogeneous group, then for every $\lambda > 1/\nu$, the homogeneous group $(N, \lambda\alpha)$ admits a distance ρ making it a homogeneous metric group, and this may or may not be true for $\lambda = 1/\nu$.

Remark 1.4. Given a homogeneous group (N, α), all the distance functions that are homogeneous for (N, α) are biLipschitz equivalent via the identity map. More generally, it is straightforward to prove the following statement. Let ρ and ρ' be two distances metrising the same topological space M. Suppose there is a transitive group of homeomorphisms acting by isometries for both of the distances. Suppose there is $o \in M$ and a bijection $\delta\colon M \to M$, fixing the point o, and $\lambda \in]0, 1[$ with

$$\rho(\delta(x), \delta(y)) = \lambda\rho(x, y) \qquad \text{and} \qquad \rho'(\delta(x), \delta(y)) = \lambda\rho'(x, y)\,, \qquad \forall x, y \in M\,.$$

Then ρ and ρ' are biLipschitz equivalent via the identity map of M.

Due to Remark 1.4, when considering biLipschitz maps between two homogeneous groups, it is not necessary to specify the homogeneous distance functions, provided they exist, for which we refer to Remark 1.3. Whenever we assume that two homogeneous groups are biLipschitz equivalent we mean that on both of them some homogeneous distances exist for which the metric spaces are biLipschitz equivalent.

The following result summarises the known correspondence between the quasi-isometries of Heintze groups and the biLipschitz maps on their boundaries. For a good exposition and list of references, see [CPS17, p.6].

Proposition 1.5. *Let (N_1, α) and (N_2, β) be homogeneous groups. Then the Heintze groups $N_1 \rtimes_\alpha \mathbb{R}$ and $N_2 \rtimes_\beta \mathbb{R}$ are quasi-isometric if and only if there exists $\lambda_1, \lambda_2 > 0$ so that $(N_1, \lambda_1\alpha)$ and $(N_2, \lambda_2\beta)$ are biLipschitz equivalent.*

Proof. The two Heintze groups $N_1 \rtimes_\alpha \mathbb{R}$ and $N_2 \rtimes_\beta \mathbb{R}$ are quasi-isometric if and only if there are $\lambda_1, \lambda_2 > 0$ so that $(N_1, \lambda_1\alpha)$ and $(N_2, \lambda_2\beta)$ are quasisymmetric [Pau96, BS00, dC18]. The constants are needed to ensure the existence of homogeneous distances, rather than quasidistances, see Remark 1.3.

Notice that we do not claim that the boundary map induced by a quasi-isometry between two Heintze groups gives a quasisymmetry between the nilpotent groups. But, by [dC18, Lemma 6.D.1], one can modify the quasi-isometry to obtain another quasi-isometry whose boundary extension restricts to a map between the parabolic boundaries.

If $(N_1, \lambda_1\alpha)$ and $(N_2, \lambda_2\beta)$ are biLipschitz equivalent, then they are quasisymmetric. Vice versa, suppose that $(N_1, \lambda_1\alpha)$ and $(N_2, \lambda_2\beta)$ are quasisymmetric. Without changing their biLipschitz class, we may assume that α and β have only real eigenvalues, see [LDNG19, Theorem C]. Up to changing the constants, we may assume that the smallest of the eigenvalues of $\lambda_1\alpha$ and $\lambda_2\beta$ agree.

If $(N_1, \lambda_1\alpha)$ is of Carnot type, then by [Pan89, Pia17] we have that $(N_1, \lambda_1\alpha)$ and $(N_2, \lambda_2\beta)$ are isomorphic as homogeneous groups and thus biLipschitz equivalent. If $(N_1, \lambda_1\alpha)$ is not of Carnot type, then the quasisymmetry from $(N_1, \lambda_1\alpha)$ to $(N_2, \lambda_2\beta)$ is a biLipschitz map, by [Pia17, LDX16]. □

Next, we translate to our language Lemma 5.1 of [CPS17].

Proposition 1.6 ([CPS17])**.** *Let (N, α, ρ) be a purely real homogeneous metric group, and let λ_1 be the smallest of the eigenvalues of α. Then the Hausdorff dimension of any non-constant curve on N is at least λ_1, and the curve $t \mapsto \exp(tX)$ has Hausdorff dimension λ_1 if X is an eigenvector of α with eigenvalue λ_1.*

The next result is also a consequence of the work of [CPS17]. It tells us that whenever one is able to prove that some subgroups are preserved in the sense that all their left cosets are preserved, then the normalisers of these subgroups provide new invariants.

Proposition 1.7 ([CPS17])**.** *Let $F\colon (N_1, \alpha) \to (N_2, \beta)$ be a biLipschitz map between homogeneous groups, and suppose A_1 and A_2 are subgroups of N_1 and N_2, respectively. Let $\mathcal{N}(A_i)$ be the normaliser of A_i, for $i \in \{1, 2\}$. If for all $p \in N_1$ we have $F(pA_1) = F(p)A_2$, then it holds $F(p\mathcal{N}(A_1)) = F(p)(\mathcal{N}(A_2))$ for all $p \in N_1$.*

Proof. Fix $p, q \in N_1$. Then the following are equivalent statements

(i) $q \in p\mathcal{N}(A_1)$.
(ii) Hausdorff distance of qA_1 and pA_1 is finite.

(iii) Hausdorff distance of $F(qA_1) = F(q)A_2$ and $F(pA_1) = F(p)A_2$ is finite.
(iv) $F(q) \in F(p)\mathcal{N}(A_2)$.

Indeed, the equivalences (i)⇔(ii) and (iii)⇔(iv) are given by [CPS17, Lemma 3.2]. The equivalence (ii)⇔(iii) is a consequence of F being a biLipschitz map. □

We will need to understand quotients of homogeneous groups: for that the important lemma is the following straightforward consequence of the ideas of [LDR16] (see their results 2.8 and 2.10 in particular).

Lemma 1.8. *Suppose H is a normal subgroup of a homogeneous group (N, α). If $\mathfrak{h}$ is preserved under α, then the quotient N/H is a homogeneous group when equipped with the induced derivation $\hat{\alpha}$. Moreover, if ρ is a homogeneous distance on (N, α), then $\hat{\rho}$ given by*

$$\hat{\rho}(nH, mH) = \inf\{\rho(n, mh) : h \in H\}$$

is a homogeneous distance on $(N/H, \hat{\alpha})$ for which the projection $N \to N/H$ is a 1-Lipschitz map.

1.2. **Isometries of not necessarily solvable groups.** Above we said that two connected Lie groups G and H *can be made isometric* if there are left-invariant Riemannian metrics ρ_G and ρ_H so that (G, ρ_G) is isometric to (H, ρ_H). By [KLD17, Proposition 2.4] requiring the distances ρ_G and ρ_H to be Riemannian is not restrictive: We could suppose only that the distances are left-invariant and induce the respective manifold topologies. In any case, while quasi-isometries give a transitive relation between Lie groups, the relation by isometries is not transitive; we next wish to show an instructive example.

Proposition 1.9. *Both the groups $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{R})$ can be made isometric to the group $\mathbb{S}^1 \times \mathrm{Aff}(\mathbb{R})^+$, but the groups $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{R})$ cannot be made isometric (to each other).*

The argument for the fact that the groups $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{R})$ cannot be made isometric is readily recorded in [FLD21, Proposition 2.11], but it goes back to Cornulier, and eventually to [Gor81, Theorem 2.2]. The first part of Proposition 1.9 may be deduced from [CKL+21, Theorem 3.24], but in this particular example the argument of [CKL+21] simplifies so much that we feel it is worth giving the following elementary proof.

Proof of the first part of Proposition 1.9. Let G denote either $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{PSL}(2, \mathbb{R})$, and let d be a left-invariant admissible distance on G. In either case, G has the Iwasawa decomposition $G = ANK$, where the factor AN forms a subgroup isomorphic to $\mathrm{Aff}(\mathbb{R})^+$ and K is instead isomorphic to $\mathbb{S}^1$. Our aim is to construct from d a new metric d' in G and find a metric d'' on $AN \times K$ so that (G, d') is isometric to $(AN \times K, d'')$.

We define, taking advantage of the compactness of K, a new distance function on G by the formula

$$d'(g, h) = \sup_{k \in K} d(gk, hk)\,.$$

It is trivial that d' satisfies the axioms of a distance function and that it is left-invariant. One may also see by a straightforward argument that any open d'-ball contains an open d-ball. Consequently, as $d'(g,h) \geq d(g,h)$ for all $g,h \in G$, then the distance d' induces the same topology as d.

We define d'' to be the pull-back distance on $AN \times K$ via the homeomorphism $\omega\colon (AN \times K) \to ANK$ given by $\omega(s,k) = sk^{-1}$. The resulting distance is left-invariant since for any fixed $k, k_1, k_2 \in K$ and $s, s_1, s_2 \in AN$ we have

$$\begin{aligned} d''((s,k)(s_1,k_1),(s,k)(s_2,k_2)) &= d'(ss_1(kk_1)^{-1}, ss_2(kk_2)^{-1}) \\ &= \sup_{k' \in K} d(ss_1k_1^{-1}k^{-1}k', ss_2k_2^{-1}k^{-1}k') \\ &= \sup_{k'' \in K} d(ss_1k_1^{-1}k'', ss_2k_2^{-1}k'') \\ &= \sup_{k'' \in K} d(s_1k_1^{-1}k'', s_2k_2^{-1}k'') \\ &= d'(s_1k_1^{-1}, s_2k_2^{-1}) = d''((s_1,k_1),(s_2,k_2)) \,. \end{aligned}$$

In conclusion, the map ω is an isometry between (G, d') and $(AN \times K, d'')$. □

The proof of the following fact is just slightly more involved, and the details are recorded in [CKL+21, Theorem 3.24]. The main difference is that one does not have a compact factor K in the Iwasawa decomposition, but instead there is a non-compact central group involved.

Proposition 1.10. *The universal cover of the group* $\mathrm{SL}(2,\mathbb{R})$ *can be made isometric to the group* $\mathbb{R} \times \mathrm{Aff}(\mathbb{R})^+$.

Even if the transitivity of isometry-relation is shown by Proposition 1.9 to be false in general, we are not aware of counterexamples in the class of simply connected Lie groups. Moreover, Fact 0.1 implies the transitivity among simply connected solvable Lie groups. Despite Fact 0.1 some questions remain unanswered, like the following.

Question 1.11. Is there a non-solvable simply connected group G and two solvable groups S_1, S_2, so that both S_1 and S_2 can be made isometric to G (with different metrics) and S_1 and S_2 have different real-shadow, i.e., they cannot be made isometric?

1.3. **Algebraic tools for isometries.** The aim of this section is to recall the results related to the real-shadow of a simply connected solvable Lie group, so that after proving Theorem B in Section 3, we are able to link it to Fact 0.1 and real-shadows. We will make the link explicit in Corollary 3.4.

We start by recalling a decomposition result which is necessary both for the construction of the real-shadow and also for the statement of Theorem B. The ingredients of its proof are recorded in [LDNG19, Section 2] while it might be considered well known.

Proposition 1.12. *Let α be a derivation on a Lie algebra $\mathfrak{g}$. Then there are derivations α_{sr}, α_{si} and α_{nil} on $\mathfrak{g}$ satisfying the following properties:*

(i) *The maps α, α_{sr}, α_{si} and α_{nil} all pairwise commute.*

(ii) $\alpha = \alpha_{\rm sr} + \alpha_{\rm si} + \alpha_{\rm nil}$.
(iii) The map $\alpha_{\rm nil}$ *is the nilpotent part of* α.
(iv) The maps $\alpha_{\rm sr}$ *and* $\alpha_{\rm si}$ *are semisimple.*
(v) The spectrum of $\alpha_{\rm sr}$ *is real, and the spectrum of* $\alpha_{\rm si}$ *is purely imaginary.*

If $\alpha = \mathrm{ad}_X$ for a vector X of a Lie algebra, we denote $\mathrm{ad}_{\rm s}(X) = \alpha_{\rm sr} + \alpha_{\rm si}$ and $\mathrm{ad}_{\rm si}(X) = \alpha_{\rm si}$; In the latter, "si" stands for semisimple and imaginary.

We recall some standard terminology: A Lie algebra $\mathfrak{g}$ is said to be *of type (R)* if all the eigenvalues of ad_X are purely imaginary for all $X \in \mathfrak{g}$. Instead, a Lie algebra is said to be *completely solvable*, if it is solvable and all these eigenvalues are real. The Lie algebra of a simply connected Lie group G is of type (R) if and only if G has polynomial growth [Jen73, Theorem 1.4], i.e., the Haar measure of the powers of neighbourhoods of identity grows bounded by a polynomial.

We recall here, using a slightly different viewpoint, the method of [CKL+21] to determine the real-shadow of a simply connected solvable Lie group. The arguments may be found inside the proof of Theorem 4.16 in [CKL+21].

Lemma 1.13. *Let* $\mathfrak{g}$ *be a solvable Lie algebra with nilradical* $\mathfrak{n}$. *Then there is a vector subspace* $\mathfrak{a} \subseteq \mathfrak{g}$ *so that*

(i) $\mathfrak{n} \oplus \mathfrak{a} = \mathfrak{g}$,
(ii) $\mathrm{ad}_{\rm s}(X)(Y) = 0$ *for all* $X, Y \in \mathfrak{a}$, *and*
(iii) $[\mathrm{ad}_{\rm s}(X), \mathrm{ad}_{\rm s}(Y)] = 0$ *for all* $X, Y \in \mathfrak{a}$.

Such a subspace $\mathfrak{a}$ is found by noticing that there is a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{c} + \mathfrak{n}$; then $\mathfrak{a}$ may be chosen inside $\mathfrak{c}$ to complement $\mathfrak{n}$.

The following statement gives naturally a very constructive definition of the real-shadow in the level of Lie algebras.

Proposition 1.14. *Let* $\mathfrak{g}$ *be a solvable Lie algebra. Choose a vector subspace* $\mathfrak{a} \subseteq \mathfrak{g}$ *with the properties of Lemma 1.13 and let* $\pi_{\mathfrak{a}}$ *denote the projection to* $\mathfrak{a}$ *along* $\mathfrak{n}$. *Define a map*

$$\varphi_{\mathfrak{a}} \colon \mathfrak{g} \to \mathrm{der}(\mathfrak{g}) \qquad \varphi_{\mathfrak{a}}(X) = -\mathrm{ad}_{\rm si}(\pi_{\mathfrak{a}}(X)).$$

Then

(i) $\varphi_{\mathfrak{a}}$ *is a homomorphism of Lie algebras, with Abelian image,*
(ii) the graph of $\varphi_{\mathfrak{a}}$, $\mathrm{Gr}(\varphi_{\mathfrak{a}}) = \{(X, \varphi_{\mathfrak{a}}(X)) \mid X \in \mathfrak{g}\}$, *is a completely solvable subalgebra of* $\mathfrak{g} \rtimes \mathrm{der}(\mathfrak{g})$,
(iii) if the vector space $\mathfrak{g}$ *is equipped with the operation defined by*

$$[X, Y]_{\mathbb{R}} = [X, Y] + \varphi_{\mathfrak{a}}(X)(Y) - \varphi_{\mathfrak{a}}(Y)(X)$$

then the map $X \mapsto (X, \varphi_{\mathfrak{a}}(X))$ *is a Lie algebra isomorphism from* $(\mathfrak{g}, [\cdot,\cdot]_{\mathbb{R}})$ *to* $\mathrm{Gr}(\varphi_{\mathfrak{a}})$.

Moreover, for every vector subspace $\mathfrak{a}' \subset \mathfrak{g}$ *as in Lemma 1.13 we have that* $\mathrm{Gr}(\varphi_{\mathfrak{a}})$ *is isomorphic to* $\mathrm{Gr}(\varphi_{\mathfrak{a}'})$.

Definition 1.15. Let $\mathfrak{g}$ be a solvable Lie algebra. Its *real-shadow* is the Lie algebra $\mathrm{Gr}(\varphi_{\mathfrak{a}})$ constructed as in Proposition 1.14.

The main result of [CKL+21] regarding this construction is that Fact 0.1 indeed holds for such a construction.

Remark 1.16. In many applications of low dimension, there is an Abelian subalgebra $\mathfrak{a}$ complementary to the nilradical $\mathfrak{n}$ of $\mathfrak{g}$. Then such $\mathfrak{a}$ trivially satisfies Lemma 1.13 and can be used to construct the real-shadow. Another remark is that if $\mathfrak{g}$ is of type (R), then $\mathrm{ad}_{\mathrm{si}}(X) = \mathrm{ad}_{\mathrm{s}}(X)$ for any $X \in \mathfrak{g}$, and consequently, the real-shadow of a Lie algebra of type (R) is its nilshadow as defined in [DtER03].

1.4. **Algebraic tools for quasi-isometries.** When considering the class of simply connected solvable Lie groups, the algebraic tools relevant for our study of groups of dimension 4 and 5 up to quasi-isometry are the following invariants:

(Inv-1) Carnot groups are quasi-isometrically distinct among themselves by Pansu's Theorem [Pan89]. More generally, [Pan89] implies that if two simply connected nilpotent Lie groups are quasi-isometric, their associated Carnot groups are isomorphic.
(Inv-2) For nilpotent groups, the Betti numbers (by [Sha04]) and more generally the Lie algebra cohomology rings (by [Sau06]) are quasi-isometry invariants.
(Inv-3) For the groups of polynomial growth, their degree of growth is quasi-isometry invariant. It is because this degree is the Hausdorff dimension of the asymptotic cone.
(Inv-4) Two simply connected solvable groups are quasi-isometric if and only if their real-shadows are quasi-isometric. This is because these groups are quasi-isometric to their real-shadows by Fact 0.1.
(Inv-5) Topological dimension of the asymptotic cone, called *cone dimension*, is a quasi-isometry invariant.

As nilshadows were already treated above, we turn the attention here to the cone dimension. Cornulier proved in [dC08] that the cone dimension of a simply connected solvable Lie group agrees with the codimension of the exponential radical of the group. We turn this result into the following observation.

Proposition 1.17. *Let G be a simply connected completely solvable Lie group with Lie algebra $\mathfrak{g}$. Then the cone dimension of G equals the codimension of the subspace $\bigcap_{n\geq 1}\mathfrak{g}^n$ of $\mathfrak{g}$, where the subspace $\mathfrak{g}^n$ denotes the nth term in the lower central series of $\mathfrak{g}$.*

Proof. By [Osi02] (see also [dC08, Theorem 6.1]), the exponential radical R of G is a closed connected normal subgroup of G, the quotient group G/R has polynomial growth, and there is no closed connected normal subgroup R' so that the quotient G/R' would be of polynomial growth and have strictly larger dimension.

By [dC08, Theorem 1.1], the cone dimension of G, denoted by $\mathrm{conedim}(G)$, equals to the codimension of the exponential radical of G. By the above

$$\mathrm{conedim}(G) = \max\{\dim(\mathfrak{g}/\mathfrak{r}) : \mathfrak{r} \text{ ideal of } \mathfrak{g} \text{ and } \mathfrak{g}/\mathfrak{r} \text{ is of type (R)}\}$$

Moreover, since G is completely solvable, a quotient of $\mathfrak{g}$ is of type (R) if and only if it is nilpotent.

The terms of the lower central series are nested vector subspaces of $\mathfrak{g}$, and the condition $\mathfrak{g}^n = \mathfrak{g}^{n+1}$ for some n implies that $\mathfrak{g}^n = \mathfrak{g}^k$ for all $k \geq n$. Thus there is $N \in \mathbb{N}$ so that $\bigcap_{n\geq 1} \mathfrak{g}^n = \mathfrak{g}^N$. The quotient $\mathfrak{g}/\mathfrak{g}^N$ is nilpotent, and we will show its dimension is maximal. Let $\mathfrak{q}$ be an ideal of $\mathfrak{g}$ so that $\mathfrak{g}/\mathfrak{q}$ is nilpotent of step s. It is enough to show $\mathfrak{g}^N \subset \mathfrak{q}$. Assuming the contrary, we have a non-zero vector $X \in \mathfrak{g}^N \backslash \mathfrak{q}$. Because $\mathfrak{g}^N = \mathfrak{g}^k$ for all $k \geq N$, we may express X as a bracket of arbitrary length. More precisely X may be expressed as linear combination of a terms of the form $\mathrm{ad}_{X_1} \circ \cdots \circ \mathrm{ad}_{X_s}(X_{s+1})$ for some $X_i \in \mathfrak{g}$. It holds $X_i \notin \mathfrak{q}$ since $\mathfrak{q}$ is an ideal. Hence when X is considered as a non-zero element of the quotient $\mathfrak{g}/\mathfrak{q}$, it can be expressed as a bracket of length $s+1$, which contradicts the nilpotency step of the quotient. □

The above result implies that the cone dimension can be algorithmically calculated at the Lie algebra level.

2. On biLipschitz maps of homogeneous groups

It is conjectured that if two purely real Heintze groups are quasi-isometric, then they are isomorphic. Many quasi-isometry invariants are known, but still there are non-isomorphic pairs of purely real Heintze groups that are not distinguished by those invariants. In this section we present new quasi-isometry invariants for purely real Heintze groups: we prove Theorem A. Our analysis is based on the fact that two purely real Heintze groups are quasi-isometric if and only if their parabolic boundaries are biLipschitz equivalent, see Proposition 1.5.

In Section 2.1 we prove Theorem A.(i) stating that biLipschitz equivalent homogenous groups are quasi-isometric when equipped with Riemannian distances. One important consequence, see also Theorem 6.4 in appendix, is that the family of purely real Heintze groups with Abelian nilradical is closed under quasi-isometries among the family of purely real Heintze groups. In addition, the quasi-isometry relations within the family of purely real Heintze groups with Abelian nilradical are completely understood by the results of Xie [Xie14].

In Section 2.2, we prove that on the level of the boundary, the set of points reachable by curves of a given Hausdorff dimension can be algebraically computed and hence used as an invariant. This leads to Theorem A.(ii). Such a result will enable us to distinguish up to quasi-isometry some examples of low dimension that we discuss in Section 2.3.

We recall that in this paper, we will always use the convention that if $N, H, G, \ldots$ are Lie groups, then the fraktur letters $\mathfrak{n}, \mathfrak{h}, \mathfrak{g}, \ldots$ denote their Lie algebras, and vice versa.

2.1. **Homogeneous biLipschitz implies Riemannian quasi-isometric.**

In this section we prove Theorem A.(i). We follow a suggestion of Pansu for treating arbitrary homogeneous groups. However, in Section 6, we consider the case where one of the groups is Abelian. It is a less general setting, but the proof is direct and might be of independent interest.

Proof of Theorem A.(i). Given a metric space (M,d) and $\ell > 0$, we recall from [CKL+21] the definition of derived semi-intrinsic metric with parameter ℓ as

$$d_{[\ell]}(p,q) = \inf\Big\{\sum_{j=1}^{k} d(x_j,x_{j-1}) \mid x_0,\dots,x_k \in M,\ x_0=p,\ x_k=q,\ d(x_j,x_{j-1}) \le \ell\Big\}\,.$$

It follows immediately from the definition, that if a map $f\colon (M,d) \to (M',d')$ is an L-Lipschitz-map with $L \ge 1$, then $d'_{[\ell]}(f(p),f(q)) \le L d_{[\ell/L]}(p,q)$. By [CKL+21, Lemma 2.3], for a homogeneous metric group (N,d) and $\ell > 0$, the function $d_{[\ell]}$ is a proper quasi-geodesic distance function inducing the topology of N. Thus, $d_{[\ell]}$ is quasi-isometric to any left-invariant Riemannian distance on N.

We conclude that if $F\colon N_1 \to N_2$ is an L-biLipschitz map between homogeneous metric groups (N_1,α,d^α) and (N_2,β,d^β), then the derived semi-intrinsic metrics satisfy the following inequalities:

$$\frac{1}{L} d^\alpha_{[L\ell]}(x,y) \le d^\beta_{[\ell]}(F(x),F(y)) \le L d^\alpha_{[\ell/L]}(x,y)\,.$$

Therefore, if D_1 and D_2 are left-invariant Riemannian distances, then the map $F\colon (N_1,D_1) \to (N_2,D_2)$ is a quasi-isometry. □

2.2. **Reachability sets.** Considering homogeneous groups up to biLipschitz equivalence, one obvious invariant is the set of those points that can be reached by curves starting from the identity element and having Hausdorff dimension at most s, for some fixed $s \ge 1$ (notice that curves have Hausdorff dimension at least 1). When (N,α) is a homogeneous group, we denote

$$R(s) = \{\gamma(1) \mid \gamma \in \mathcal{C}^0([0,1],N),\ \gamma(0)=1_N,\ \mathcal{H}\text{-dim}(\gamma([0,1])) \le s\}.$$

As one might expect and as we shall now prove, such a set may be computed as the subgroup $(N,\alpha)^{(s)} < N$ corresponding to the subalgebra $\mathrm{LieSpan}(\bigoplus_{0<\lambda\le s} V_\lambda)$. Here $\mathfrak{n} = \bigoplus_{\lambda>0} V_\lambda$ is the decomposition of the Lie algebra by the generalised eigenspaces of the derivation α. The fact that $R(s) = (N,\alpha)^{(s)}$ makes this set into a practically usable invariant.

Theorem 2.1. *Let (N,α) be a purely real homogeneous group. Then $R(s) = (N,\alpha)^{(s)}$ for every $s \ge 1$.*

Proof. Fix $s \ge 1$. Using the Orbit Theorem, one may show (see [BL19, Proposition 2.26]) the following. Suppose W is subset of a Lie algebra $\mathfrak{g}$ so that W is invariant under scalar multiplication, i.e., $\mathbb{R}W = W$, and so that no proper subalgebra of $\mathfrak{g}$ contains W. Then $\bigcup_{k=1}^\infty (\exp(W))^k$ has non-empty interior in G, and since it is also a subgroup it holds $\bigcup_{k=1}^\infty (\exp(W))^k = G$. Applying this observation to $W = \bigcup_{0<\lambda\le s} V_\lambda$ and $G = (N,\alpha)^{(s)}$, we get that every element of $(N,\alpha)^{(s)}$ is a finite product of exponentials of vectors $X \in \bigcup_{0<\lambda\le s} V_\lambda$. Thus, to show that $R(s) \supset (N,\alpha)^{(s)}$, we only need to see that the flow lines $t \mapsto \exp(tX)$ have Hausdorff dimension at most s, for $X \in \bigcup_{0<\lambda\le s} V_\lambda$. By [CPS17, Lemma 5.2], we may assume that X is an eigenvector of α with eigenvalue $\lambda \le s$. Fix a homogeneous distance ρ, and set $L = \exp(\mathbb{R}X)$. Identifying L with $\mathbb{R}$, we get a distance to $\mathbb{R}$ that is homogeneous under the family of dilations induced by α. Hence by Remark 1.4, (L, ρ) is biLipschitz equivalent to $(\mathbb{R}, \|\cdot\|^{1/\lambda})$ and hence it has Hausdorff dimension λ.

To prove that $R(s) \subset (N, \alpha)^{(s)}$, denote $H_0 = (N, \alpha)^{(s)}$ and let then recursively H_k denote the normaliser of H_{k-1}. Consider the finite chain of subgroups $(N, \alpha)^{(s)} = H_0 < H_1 < \cdots < H_m = N$, where $m \geq 1$ is the first integer so that the repeated normaliser is the full space. Since nilpotent Lie algebras don't have non-trivial self-normalising subalgebras, such m exists. Fix a continuous curve $\gamma\colon [0,1] \to N$ with $\mathcal{H}$-dim$(\gamma([0,1])) \leq s$ and $\gamma(0) = 1_N$. We shall prove inductively that γ does not leave H_k for any $0 \leq k \leq m$.

The case $k = m$ of the induction is trivial. So we assume γ does not leave H_k for some $k \leq m$. Since H_{k-1} is normal in H_k, we may consider the quotient H_k/H_{k-1}. Observe that if a derivation α preserves a subalgebra $\mathfrak{q} < \mathfrak{n}$, then α necessarily preserves the normaliser of $\mathfrak{q}$. Therefore, since α preserves $(N, \alpha)^{(s)}$, then, by induction and Lemma 1.8, the quotient H_k/H_{k-1} is a homogeneous group. Moreover, the curve γ projects to the curve $\pi \circ \gamma$ of H_k/H_{k-1}, and Lemma 1.8 guarantees that the Hausdorff dimension of $\pi(\gamma([0,1]))$ is at most the Hausdorff dimension of $\gamma([0,1])$, so at most s.

Next, remark that all the generalised eigenspaces of α corresponding to eigenvalues less or equal to s are contained in the Lie algebra of $(N, \alpha)^{(s)}$, thus they are contained in H_{k-1}. This shows that all the eigenvalues of the derivation induced to H_k/H_{k-1} are strictly larger than s. Therefore, by Proposition 1.6, either $\pi \circ \gamma$ is constant or the Hausdorff dimension of $\pi(\gamma([0,1]))$ is strictly larger than s. Since the second case is ruled out, the curve $\pi \circ \gamma$ must be constant, i.e., $\gamma([0,1]) \subset H_{k-1}$ as the induction requires. We conclude that γ does not leave $H_0 = (N, \alpha)^{(s)}$ and hence $R(s) \subset (N, \alpha)^{(s)}$. ☐

Proof of Theorem A.(ii). As the set $R(s)$ is metrically defined, we get Theorem A.(ii) as immediate corollary of Theorem 2.1 when applying also Proposition 1.7. ☐

2.3. **Examples.** In this section we present some examples of pairs of Heintze groups trying to distinguish them up to quasi-isometry using the results that we proved.

- Ex 2.2 This is a pair of 7-dimensional Heintze groups with identical nilradical and derivations with identical diagonal form. Theorem A.(ii) distinguishes them.
- Ex 2.3 This is a pair of 5-dimensional Heintze groups with identical nilradical. This pair cannot be distinguished even with the new invariants we presented.
- Ex 2.4 This is a pair of 7-dimensional Heintze groups with different nilradical, but identical diagonal derivation. Theorem A.(i) distinguishes them.
- Ex 2.5 This is a pair of 10-dimensional Heintze groups with identical nilradical and derivations with identical diagonal form. Here the reachability sets don't distinguish the pair directly, but the normalisers can be used to distinguish them.
- Ex 2.6 This is a pair of 7-dimensional Heintze groups with different nilradical, but identical diagonal derivation. This pair cannot be distinguished even with the new invariants we presented.

In the next examples, we use the notation Heis for the standard Heisenberg group and Heis(5) for the 5-dimensional Heisenberg group. These are indexed by $A_{3,1}$ and $A_{5,4}$, respectively, in [PSWZ76], see also Section 4 and Section 5 later.

Example 2.2. Consider the Lie group $N = \mathrm{Heis} \times \mathbb{R}^3$ and two derivations on it

$$\alpha = \mathrm{diag}(1, 2, 3, 4, 5, 9) \qquad \text{and} \qquad \beta = \mathrm{diag}(4, 5, 9, 1, 2, 3).$$

Then the Heintze groups $N \rtimes_\alpha \mathbb{R}$ and $N \rtimes_\beta \mathbb{R}$ are not isomorphic: if they were, then α should be conjugate to β by an automorphism of $\mathfrak{n}$ (see for example [HKMT20, Proposition 4.7]). However, there is a unique linear endomorphism of $\mathfrak{n}$ that conjugates α to β and it is not an automorphism.

The invariant $R(2)$ distinguishes these homogeneous groups (N, α) and (N, β) by Theorem 2.1, as these sets have topological dimension 3 for (N, α) and 2 for (N, β).

Example 2.3. Consider the 4-dimensional Lie algebra $\mathrm{Heis} \times \mathbb{R}$ given by a basis $X_1, \ldots, X_4$ with the only non-trivial bracket being $[X_1, X_2] = X_3$. Consider, for every parameter $a > 1$, the two linear maps given by matrices

$$\alpha = \begin{bmatrix} a-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix} \qquad \text{and} \qquad \beta = \begin{bmatrix} a-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}.$$

Both these maps are derivations of $\mathrm{Heis} \times \mathbb{R}$ with strictly positive eigenvalues, hence they define two 5-dimensional Heintze groups $(\mathrm{Heis} \times \mathbb{R}) \rtimes_\alpha \mathbb{R}$ and $(\mathrm{Heis} \times \mathbb{R}) \rtimes_\beta \mathbb{R}$. These Heintze groups are non-isomorphic, as they are the groups $A^a_{5,20}$ and $A^{a,a}_{5,19}$ in the classification [PSWZ76]. We are not aware of any method of distinguishing these Heintze groups up to quasi-isometry. We remark that while the Jordan-forms of the derivations are different, the Jordan form is not proven to be invariant in this generality. We also remark that these groups are sublinearly biLipschitz equivalent, by a result of Cornulier [dC11, Theorem 1.2], see also [Pal20, Theorem 3.2].

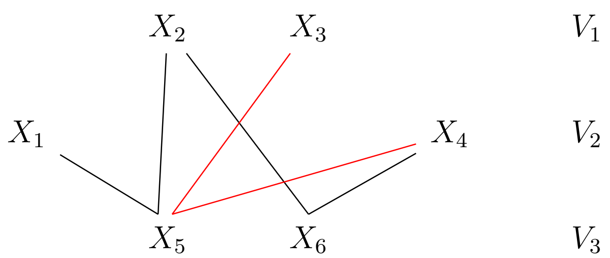

FIGURE 1. The graph representing schematically the bracket relations and positive gradings $V_1 \oplus V_2 \oplus V_3$ of Example 2.4.

Example 2.4. Consider the 6-dimensional vector space with a basis $X_1, \ldots, X_6$ with two different structures of Lie algebra: Let $\mathfrak{n}_1$ be the Lie algebra given by the non-trivial bracket relations

$$[X_1, X_2] = X_5 \qquad [X_2, X_4] = X_6$$

This is denoted by $L_{6,8} = L_{5,8} \times \mathbb{R}$ in the classification [dG07]. Let $\mathfrak{n}_2$ be instead given by

$$[X_1, X_2] = X_5, \qquad [X_2, X_4] = X_6, \qquad [X_3, X_4] = X_5.$$

This is denoted by $L_{6,22}(0)$ in the classification [dG07].

The linear map $\alpha = \mathrm{diag}(2, 1, 1, 2, 3, 3)$ in this basis is a derivation for both of these Lie algebra structures. For a schematic presentation, see Figure 1.

The homogeneous groups (N_1, α) and (N_2, α) cannot be biLipschitz-distinguished by Theorem A.(ii) but they can by Theorem A.(i) since the Lie algebras in question are stratifiable (even though homogeneous structures given are not of Carnot type).

Example 2.5. Consider the 10-dimensional Lie algebra $N = \mathrm{Heis}(5) \times \mathrm{Heis}(5)$ expressed as a vector space spanned by $X_1, \ldots, X_{10}$ with the non-trivial bracket relations

$$[X_1, X_2] = X_5, \qquad [X_3, X_4] = X_5, \qquad [X_6, X_7] = X_{10}, \qquad [X_8, X_9] = X_{10}.$$

Consider the two derivations given by matrices

$$\alpha = \mathrm{diag}(1, 7, 3, 5, 8, 2, 6, 4, 4, 8) \qquad \text{and} \qquad \beta = \mathrm{diag}(1, 7, 4, 4, 8, 2, 6, 3, 5, 8).$$

The resulting Heintze groups $G_\alpha = N \rtimes_\alpha \mathbb{R}$ and $G_\beta = N \rtimes_\beta \mathbb{R}$ are not isomorphic: If they were, α and β should be conjugate by an automorphism of N (again, [HKMT20, Proposition 4.7]), but this is impossible. Indeed, the conjugating automorphism is forced to map $X_3 \mapsto X_8$ and $X_4 \mapsto X_9$ while in the same time keeping the basis-vectors X_1 and X_2 fixed, which is not conceivable.

Distinguishing these spaces is a bit more involved and demonstrates the combined power of Theorem 2.1 and Proposition 1.7. We cannot distinguish them directly via the reachability sets of prescribed Hausdorff dimension. However, the following works.

Suppose $F\colon (N, \alpha) \to (N, \beta)$ is a biLipschitz map. Then F must map $(N, \alpha)^{(6)}$ to $(N, \beta)^{(6)}$. By Theorem 2.1, $(N, \alpha)^{(6)}$ and $(N, \beta)^{(6)}$ both agree with the subgroup spanned by all the other basis vectors except X_2. This subgroup is again a homogeneous group, and it is Lie isomorphic to $\mathbb{R} \times \mathrm{Heis} \times \mathrm{Heis}(5)$. The original biLipschitz map induces a biLipschitz map of this subgroup equipped with the two different homogeneous structures (the derivations), call these groups (N_0, α_0) and (N_0, β_0). For these two homogeneous groups, consider now the subgroups

$$(N_0, \alpha_0)^{(4)} = \langle X_1, X_3, X_6, X_8, X_9, X_{10} \rangle \quad \text{and} \quad (N_0, \beta_0)^{(4)} = \langle X_1, X_3, X_4, X_5, X_6, X_8 \rangle.$$

These are both isomorphic to the group $\mathbb{R}^3 \times \mathrm{Heis}$, so we did not yet distinguish the groups. However, the normalisers of these subgroups inside (N_0, α_0) and (N_0, β_0) are preserved by Lemma 1.7. These normalisers are

$$\mathcal{N}((N_0, \alpha_0)^{(4)}) = (N_0, \alpha_0)^{(4)} \oplus \langle X_5, X_7 \rangle \quad \text{and} \quad \mathcal{N}((N_0, \beta_0)^{(4)}) = (N_0, \beta_0)^{(4)} \oplus \langle X_{10} \rangle,$$

which have different topological dimension and this prevents the existence of a biLipschitz map.

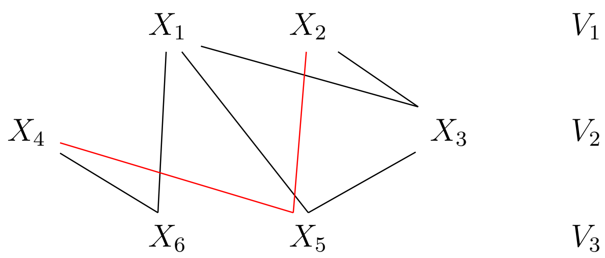

FIGURE 2. The graph representing schematically the bracket relations and positive gradings $V_1 \oplus V_2 \oplus V_3$ of Example 2.6.

Example 2.6. Consider the 6-dimensional vector space with a basis $X_1, \dots, X_6$ with two different structures of Lie algebra: Let $\mathfrak{n}_1$ be the Lie algebra given by the non-trivial bracket relations

$$[X_1, X_2] = X_3 \qquad [X_1, X_3] = X_5 \qquad [X_1, X_4] = X_6 \qquad [X_2, X_4] = X_5 \,.$$

This is denoted by $L_{6,23}$ in the classification [dG07]. Let $\mathfrak{n}_2$ be instead given by only the first three from above, i.e.,

$$[X_1, X_2] = X_3 \qquad [X_1, X_3] = X_5 \qquad [X_1, X_4] = X_6 \,.$$

This is denoted by $L_{6,25}$ in the classification [dG07].

The linear map $\alpha = \mathrm{diag}(1, 1, 2, 2, 3, 3)$ in this basis is a derivation for both of these Lie algebra structures. For a schematic presentation, see Figure 2.

The homogeneous groups (N_1, α) and (N_2, α) cannot be biLipschitz-distinguished by any method we know: The Lie algebra $L_{6,25}$ is the associated Carnot algebra of $L_{6,23}$ and the simply connected nilpotent Lie groups associated are not distinguished by the known quasi-isometric invariants, see [dC18, p. 339]. This rules out the usage of Theorem A.(i).

The only non-trivial reachability set is the reachability set for Hausdorff dimension 1, and it is the same subspace $\langle X_1, X_2, X_3, X_5 \rangle$ for both. Its normaliser contains in addition X_6 in both cases, and not X_4, the next repeated normaliser being the full space. This rules out the usage of Theorem A.(ii).

3. On isometries of semi-direct products

In this section we focus in proving Theorem B, which is a consequence of the following more general result.

Theorem 3.1. *Let G and N be Lie groups and $A\colon G \to \mathrm{Aut}(N)$ a Lie group morphism. Suppose that there is a Lie group morphism $A_E\colon G \to \mathrm{Aut}(N)$ so that the set*

$$\left\{ A(g)^{-1} A_E(g) : g \in G \right\} \tag{1}$$

is contained in a compact subgroup of $\mathrm{Aut}(N)$. Then the Lie group $N \rtimes_A G$ can be made isometric to the Lie group $N \rtimes_{A_E} G$.

Proof. The groups $N \rtimes_A G$ and $N \rtimes_{A_E} G$ may be seen as acting by left-translations on the manifold $N \times G$. Hence, the statement is proven by constructing a Riemannian metric on the manifold $N \times G$ for which both these actions are by isometries. Denoting by 1 the element $(1_H, 1_G) \in N \times G$, it is enough to construct a scalar product ρ on the tangent space $T_1(N \times G)$ with the following property (J)

(J) whenever two elements $g_1 \in N \rtimes_A G$ and $g_2 \in N \rtimes_{A_E} G$ satisfy $g_1(1) = g_2(1)$, then the differential of $g_1^{-1} \circ g_2$ is an isometry of the scalar product ρ.

If ρ satisfies the property (J), then it can be transported to a Riemannian metric on $N \times G$ with the desired properties.

Let K be a compact subgroup of $\mathrm{Aut}(N)$ containing the set described in (1). Since K is compact, we may choose a scalar product ρ on $T_1(N \times G)$ that is invariant under the differentials of the maps $F \times \mathrm{Id}$ for $F \in K$. We will next see that ρ has the property (J), thus finishing the proof.

A point $(h, g) \in N \times G$ acts by left-translations with respect to the group law of $N \rtimes_A G$ on the manifold $N \times G$ as

$$L^A_{(h,g)}(m, \ell) = (hA(g)m, g\ell)$$

and similarly for $N \rtimes_{A_E} G$ by replacing A with A_E. We deduce that if $L^A_{(h,g)}(1) = L^{A_E}_{(h',g')}(1)$, then $(h, g) = (h', g')$. Therefore, to establish the property (J), it is enough to show that, for every (h, g), the differential of the map

$$Q_{(h,g)} := (L^A_{(h,g)})^{-1} \circ L^{A_E}_{(h,g)}$$

is an isometry for the scalar product ρ. By a computation, one now finds

$$Q_{(h,g)}(m, \ell) = L^A_{(A(g^{-1})h^{-1},g^{-1})} (hA_E(g)m, g\ell) = (A(g)^{-1}A_E(g)m, \ell).$$

This formula means that $Q_{(h,g)}$ is of the form $F \times \mathrm{Id}$ for $F \in K$, and by the construction of ρ, we are done. □

Corollary 3.2. *Let G and N be Lie groups and $A\colon G \to \mathrm{Aut}(N)$ a Lie group morphism. Suppose that $A(G)$ is connected and Abelian. Then there is a Lie group morphism $A_E\colon G \to \mathrm{Aut}(N)$ such that $A_E(G)$ is a vector space and $N \rtimes_A G$ can be made isometric to the Lie group $N \rtimes_{A_E} G$.*

Proof. Since $A(G)$ is connected and Abelian, its closure M in $\mathrm{Aut}(N)$ is also connected and Abelian. By standard results about the structure of Abelian Lie groups, M is the direct product of a torus K and a vector space E. Therefore $A = (A_K, A_E)$ with respect to the splitting $M = K \times E$, where $A_K\colon G \to K$ and $A_E\colon G \to E$ are Lie group morphisms. Moreover, the set $A_E(G)$ is a connected subgroup of E and therefore a vector space. Finally, for all $g \in G$, $A(g)^{-1}A_E(g) = (A_K(g)^{-1}, A_E(g)^{-1})(0, A_E(g)) = (A_K(g)^{-1}, 0) \in K$, so the hypothesis of Theorem 3.1 are satisfied. □

Theorem B, which we restate here for the reader's convinience, is a direct consequence of Theorem 3.1.

Corollary 3.3 (Theorem B). *Let H be a simply connected Lie group and α a derivation of H. Let $\alpha = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{si}} + \alpha_{\mathrm{nil}}$ be the decomposition to real, imaginary and nilpotent parts as in Proposition 1.12, and denote $\alpha_0 = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{nil}}$. Then the Lie group $H \rtimes_\alpha \mathbb{R}$ can be made isometric to the Lie group $H \rtimes_{\alpha_0} \mathbb{R}$.*

Proof. Notice that, since H is simply connected, the groups $\mathrm{Aut}(\mathfrak{h})$ and $\mathrm{Aut}(H)$ are canonically isomorphic. Thus, we can apply Theorem 3.1 to the maps $\mathbb{R} \to \mathrm{Aut}(H)$ given by $A(t) := e^{t\alpha}$, $A_E(t) := e^{t(\alpha_{\mathrm{sr}}+\alpha_{\mathrm{nil}})}$. Indeed, notice that $A(t)A_E(-t) = e^{t\alpha_{\mathrm{si}}}$ because $\alpha_{\mathrm{sr}} + \alpha_{\mathrm{nil}}$ and α_{si} commute by Proposition 1.12, and thus $\{A(t)A_E(-t) : t \in \mathbb{R}\} = \{e^{t\alpha_{\mathrm{si}}} : t \in \mathbb{R}\}$. Moreover, the closure in $\mathrm{Aut}(\mathfrak{h})$ of $\{e^{t\alpha_{\mathrm{si}}} : t \in \mathbb{R}\}$ is a compact group. □

While Theorem B may be applied outside the family of solvable groups, also within the family of solvable groups sometimes it might be practical to directly apply Theorem B to find isometries between two solvable groups when neither of them is completely solvable. Indeed, we remark that $H \rtimes_{\alpha_0} \mathbb{R}$ does not need to be completely solvable when H is not completely solvable. This approach would avoid the work to find their common real-shadow as in Fact 0.1.

We get the following corollary when combining Theorem B and Fact 0.1.

Corollary 3.4. *Let $\mathfrak{g}$ be a Lie algebra of the form $\mathfrak{g} = \mathfrak{h} \rtimes_\alpha \mathbb{R}$, where $\mathfrak{h}$ is completely solvable. Let $\alpha = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{si}} + \alpha_{\mathrm{nil}}$ be the decomposition of α as in Proposition 1.12. Then for $\alpha_0 = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{nil}}$ it holds that the Lie algebra $\mathfrak{h} \rtimes_{\alpha_0} \mathbb{R}$ is the real-shadow of $\mathfrak{g}$.*

4. Dimension 4

The aim of this section is to prove Theorem C. Namely, we find all pairs of solvable simply connected 4-dimensional Lie groups that can be made isometric. We start from dimension 4 because dimensions 3 and below are already solved (for a survey, see [FLD21]).

4.1. Solvable groups up to isometry.

The isomorphism classes of all simply connected solvable Lie groups are known in dimension 4. Thus determining within this family the pairs of non-isomorphic groups that can be made isometric reduces by Fact 0.1 to the determination of real-shadows of those solvable groups that are not completely solvable. Recall that by Fact 0.1 the relation "can be made isometric" is transitive, and hence *isometry (equivalence) classes* of groups are well defined objects.

The classification of simply connected Lie groups is equivalent to the classification of finite dimensional Lie algebras over $\mathbb{R}$. The list we shall use is given by Patera et al. [PSWZ76, Table I, p. 988], which in turn is based on the classification of Mubarakzjanov [Mub63a, Mub63b]. The list only contains the Lie algebras that are not direct products from lower dimension, and they are indexed from $A_{4,1}$ to $A_{4,12}$ with possible superscripts indicating one-parameter families. Table I in [PSWZ76] also contains the classification of 3D Lie algebras; in what follows we shall use those names from $A_{3,1}$ to $A_{3,9}$ together with $\mathbb{R}^n$ denoting the n-dimensional Abelian Lie algebra and A_2 denoting the unique non-Abelian 2D Lie algebra: the Lie group corresponding to A_2 was denoted by $\mathrm{Aff}^+(\mathbb{R})$ earlier.

In Table 1 we list all the simply connected completely solvable Lie groups of dimension 4. None of them can be made isometric to any other, and all the non-completely solvable groups (which in turn are listed in Table 2) can be made isometric to exactly one of these. In order to be able to divide the groups into families that seem to suit the purpose of classification up to isometry and quasi-isometry the best, in Table 1 we have relabelled the families in the left-most column, and we have written the unique completely solvable representative of the isometry class, in the notation of Patera et al., to the 2nd column. So the two left-most columns of Table 1 serve as a dictionary. We indicate the range of parameters immediately after the labels. Two concrete examples on how to read the table: our label (2) denotes the Lie algebra $\mathbb{R} \times A_{3,1}$ which is the direct product of the one-dimensional Abelian group and the Heisenberg group; instead, Lie algebra $(6, \frac{1}{2}, 1)$ denotes the Lie algebra $A_{4,5}^{1/2,1}$ of the classification of [PSWZ76].

The right-most column of Table 1 has a mark X if and only if the isometry class of this group consists of more than one isomorphism classes of simply connected solvable Lie groups. The third column is about quasi-isometric classification, and we come back to it in Section 4.3.

Table 2 lists all the remaining solvable Lie algebras of dimension 4. Namely, it lists those Lie algebras that are not completely solvable. Each of them has some completely solvable representative in its isometry class, namely the real-shadow. This real-shadow is indicated on the middle column, and our label for its isometry class is written in the right-most column. The computation of the real shadow is very simple after Corollary 3.4.

4.2. **Dropping the assumption of solvability.** We do not have many tools to treat non-solvable simply connected Lie groups. However, in dimension 4 there are no Levi decompositions other than the direct products (see [Mac99, p. 301]), and hence the only two non-solvable Lie groups are $\mathbb{R} \times \mathbb{S}^3$ and $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$, and we can say something about these. We are thus interested if either of these two groups can be made isometric to some solvable groups, or if they can be made isometric to each other. For topological reasons, $\mathbb{R} \times \mathbb{S}^3$ cannot be made isometric to any other simply connected 4-dimensional group: it is the only group not homeomorphic to $\mathbb{R}^4$. The case of the group $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ however is more involved, and we will see next what we can say about it.

We know from Proposition 1.10 that $\widetilde{\mathrm{SL}}(2)$ can be made isometric to $\mathbb{R} \times A_2$, hence the groups $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ and $\mathbb{R}^2 \times A_2$ can be made isometric. Consequently, $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ must have cone dimension 3, which is the cone dimension of $\mathbb{R}^2 \times A_2$ as one may see from Proposition 1.17. Thus, checking the cone dimensions of solvable groups from Table 1, only the question remains whether or not $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ can be made isometric also to the group $A_{4,3}$ or to some groups in the family $A_{4,6}^{a,0}$ for $a \in \mathbb{R}$. Notice for example that the fact that $A_{4,3}$ and $\mathbb{R}^2 \times A_2$ cannot be made isometric does not rule out that $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ and $A_{4,3}$ can be made isometric, because $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ is not solvable. Similarly, if $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ can be made isometric to $A_{4,6}^{a,0}$ for some $a \in \mathbb{R}$, it does not

Our labelling	[PSWZ76]	**QI-type**	$*$
(1)	$\mathbb{R}^4$	poly growth	X
(2)	$\mathbb{R} \times A_{3,1}$	poly growth	X
(3)	$A_{4,1}$	poly growth	
$(4, a)\ \ a \in]0, \infty[$	$A_{4,2}^a$	Heintze	
(5)	$A_{4,4}$	Heintze	
$(6, a, b)\ \ a, b \in]0, 1],\ b > a$	$A_{4,5}^{a,b}$	Heintze	
$(7, a)\ \ a \in]0, 1]$	$A_{4,5}^{a,a}$	Heintze	X
(8)	$A_{4,7}$	Heintze	
$(9, a)\ \ a \in]0, 1[$	$A_{4,9}^a$	Heintze	
(10)	$A_{4,9}^1$	Heintze	X
$(11, a)\ \ a \in]-\infty, 0[$	$A_{4,2}^a$	conedim 1	
$(12, a, b)\ \ a, b \in]-1, 1[\setminus\{0\},\ b > a,\ a < 0$	$A_{4,5}^{a,b}$	conedim 1	
$(13, a)\ \ a \in [-1, 0[$	$A_{4,5}^{a,a}$	conedim 1	X
$(14, a)\ \ a \in]-1, 0[$	$A_{4,9}^a$	conedim 1	
(15)	$A_{4,8}$	conedim 1	
(16)	$A_{4,9}^0$	conedim 2	
(17)	$\mathbb{R} \times A_{3,2}$	conedim 2	
(18)	$\mathbb{R} \times A_{3,3}$	conedim 2	X
(19)	$A_2 \times A_2$	conedim 2	
$(20, a)\ \ a \in]-1, 1[\setminus\{0\}$	$\mathbb{R} \times A_{3,5}^a$	conedim 2	
(21)	$\mathbb{R} \times A_{3,4}$	conedim 2	
(22)	$A_{4,3}$	conedim 3	
(23)	$\mathbb{R}^2 \times A_2$	conedim 3	X

Table 1. Completely solvable Lie algebras of dimension 4.

imply anything for $A_{4,6}^{a',0}$ with $a' \neq a$. One might wish to compare this phenomenon to [CKL$^+$21, Theorem 4.21].

4.3. **Quasi-isometric classification of 4-dimensional groups.** We don't have a complete quasi-isometric classification of simply connected 4-dimensional Lie groups. In this section we show what is known about it. Recall that quasi-isometry equivalence classes are necessarily unions of the isometry classes: we just established these

Lie algebra	**real-shadow**	**isometry class**
$\mathbb{R} \times A_{3,6}$	$\mathbb{R}^4$	(1)
$A_{4,10}$	$\mathbb{R} \times A_{3,1}$	(2)
$A_{4,6}^{a,b}$ $a, b \in]0, \infty[,\ a \leq b$	$A_{4,5}^{a/b,a/b}$	$(7, a/b)$
$A_{4,6}^{a,b}$ $a, b \in]0, \infty[,\ a > b$	$A_{4,5}^{b/a,b/a}$	$(7, b/a)$
$A_{4,11}^{a}$ $a \in]0, \infty[$	$A_{4,9}^{1}$	(10)
$A_{4,6}^{-a,b}$ $a, b \in]0, \infty[,\ a \leq b$	$A_{4,5}^{-a/b,-a/b}$	$(13, -a/b)$
$A_{4,6}^{a,-b}$ $a, b \in]0, \infty[,\ b < a$	$A_{4,5}^{-b/a,-b/a}$	$(13, -b/a)$
$\mathbb{R} \times A_{3,7}^{a}$ $a \in]0, \infty[$	$\mathbb{R} \times A_{3,3}$	(18)
$A_{4,12}$	$\mathbb{R} \times A_{3,3}$	(18)
$A_{4,6}^{a,0}$	$\mathbb{R}^2 \times A_2$	(23)

TABLE 2. Solvable but not completely solvable Lie algebras of dimension 4, and their real-shadows.

isometry classes for simply connected solvable groups. Hence it is enough to consider the groups in Table 1 and the two non-solvable groups $\mathbb{R} \times \mathbb{S}^3$ and $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$.

Recall that the degree of polynomial growth is a quasi-isometric invariant. The degree of polynomial growth for $\mathbb{R} \times \mathbb{S}^3$ is 1, so it cannot be quasi-isometric either to any group in Table 1 or the group $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$. Consequently, the quasi-isometry class of $\mathbb{R} \times \mathbb{S}^3$ within the family of simply connected 4-dimensional groups, is a singleton.

About the group $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$, the only thing that we are able to say is that since it can be made isometric to $\mathbb{R}^2 \times A_2$, then it must have cone dimension 3.

For all completely solvable groups that are not Heintze groups and do not have polynomial growth, we have calculated, using Proposition 1.17, their cone dimensions and marked them to the third column titled "QI-type" of Table 1. The cone dimensions are quasi-isometry invariants by [dC11].

Recall that while Heintze groups have cone dimension 1, they are quasi-isometrically distinct from the non-Heintze groups of cone dimension 1 since in dimension 4 only the Heintze groups are Gromov hyperbolic by [dCT11] (see also [dC18, p. 277]).

The quasi-isometric classification of 4-dimensional purely real Heintze groups can be done by case-by-case study. However, a direct argument follows from Theorem A.(i), Proposition 1.5 and the results of Xie [Xie14], Carrasco Piaggio and Sequeira [CPS17, Theorem 1.3]: The purely real Heintze groups in Table 1 split into two categories

nilradical $\mathbb{R}^3$	$(4, a)$	(5)	$(6, a, b)$
nilradical Heis	(8)	$(9, a)$	(10)

Those with nilradical $\mathbb{R}^3$ are quasi-isometrically distinct from each other by [Xie14]. Those with nilradical Heis are quasi-isometrically distinct from each other by [CPS17, Theorem 1.3]. All the quasi-isometry relations between these two classes are excluded by Theorem A.(i). Thus, the quasi-isometry classes, isometry classes, and isomorphism classes all agree for purely real Heintze groups of dimension 4.

For the groups of polynomial growth, our classes (1), (2) and (3) are known to be quasi-isometry equivalence classes, because the completely solvable representatives (in this case, nilpotent representatives) are Carnot groups and quasi-isometric classification of Carnot groups is solved by Pansu [Pan89].

As a conclusion, we may present the following proposition.

Proposition 4.1. *Let $\mathcal{G}$ be the family of the isomorphism classes of 4-dimensional simply connected solvable groups that either have polynomial growth or are Heintze groups. Then two elements $G, H \in \mathcal{G}$ are quasi-isometric if and only if they can be made isometric. If the groups G and H are completely solvable, then they are quasi-isometric if and only if they are isomorphic.*

We stress that there are 4-dimensional simply connected solvable groups that neither have polynomial growth nor are Heintze groups. All groups that we labeled from (11) to (23) in Table 1 are outside the class $\mathcal{G}$. Therefore, the above Proposition 4.1 does not give a complete quasi-isometric classification of 4-dimensional groups.

5. Dimension 5

The classification of real solvable Lie algebras is known in dimensions five also, see [PSWZ76]. However, due to the multitude of isomorphism classes, we rather restrict our attention to the groups of polynomial growth.

The first task is to determine a list of all simply connected solvable Lie groups of polynomial growth in dimension 5. We are not aware of a reference where this is done, so we have to do it by ourselves using the classification of real solvable Lie algebras presented in Patera et al. [PSWZ76, p. 989]. Notice that one can pretty quickly find all the candidates for groups of polynomial growth by excluding the Lie algebras with a bracket relation of the type $[e_i, e_j] = \lambda e_j$ for $\lambda \neq 0$: This is an obstruction of being polynomial growth, since all the eigenvalues of all the adjoint maps should be purely imaginary (see Section 1.3). The candidates so found are possible to check by hand if they have polynomial growth or not.

Taking into account the direct products, the full list of solvable simply connected Lie groups of polynomial growth is presented in Table 3. In the first 2 columns, we have recalled a dictionary between classifications presented in Patera et al. and that by de Graaf [dG07] for nilpotent Lie algebras. For nilpotent algebras that are not Carnot algebras, we have indicated their associated Carnot algebras in the 4th column. For non-nilpotent Lie algebras, we have indicated their nilshadow in the 3rd column.

From the algebraic classification given in Table 3 one may directly deduce the classification up to isometries and quasi-isometries (up to one open case we will mention soon) using the list of invariants we recorded in beginning of Section 1.4. Indeed, recalling invariant (Inv-4) and Remark 1.16, every group is isometric to its nilshadow,

Patera et al.	**de Graaf**	**nilshadow**	G_∞
$\mathbb{R}^5$			
$\mathbb{R}^2 \times A_{3,1}$			
$\mathbb{R} \times A_{4,1}$			
$A_{5,1}$	$L_{5,8}$		
$A_{5,2}$	$L_{5,7}$		
$A_{5,3}$	$L_{5,9}$		
$A_{5,4}$	$L_{5,4}$		
$A_{5,5}$	$L_{5,5}$		$\mathbb{R} \times A_{4,1}$
$A_{5,6}$	$L_{5,6}$		$A_{5,2}$
$\mathbb{R}^2 \times A_{3,6}$		$\mathbb{R}^5$	
$\mathbb{R} \times A_{4,10}$		$\mathbb{R}^2 \times A_{3,1}$	
$A_{5,17}^{s,0,0}$ $s \neq 0$		$\mathbb{R}^5$	
$A_{5,14}^{0}$		$\mathbb{R}^2 \times A_{3,1}$	
$A_{5,26}^{0,\varepsilon}$ $\varepsilon = \pm 1$		$A_{5,4}$	
$A_{5,18}^{0}$		$A_{5,1}$	

TABLE 3. Solvable Lie algebras of type (R) in dimension 5.

and in dimension 4 it happens that the nilshadows are always Carnot groups (Carnot groups are those with empty field both in "nilshadow" and in "G_∞"). Moreover, the nilshadows happen to be those Carnot groups that are not associated Carnot groups of some nilpotent non-Carnot groups. Hence the classification up to isometry and quasi-isometry is ready for the groups of polynomial growth and those Carnot groups that appear as their nilshadows. Only problem that remains after applying (Inv-1) is if $A_{5,5}$ or $A_{5,6}$ are quasi-isometric to their associated Carnot groups, recall that by [Wol63] isometries between non-isomorphic nilpotent groups cannot exist. The invariant (Inv-2) tells that $A_{5,5}$ is not quasi-isometric to its associated Carnot group $\mathbb{R} \times A_{4,1}$ (see [dC18, Section 19.7]), but the possible quasi-isometry relation between $A_{5,6}$ and $A_{5,2}$ remains unanswered by this analysis.

In conclusion, as was the case for the family of simply connected solvable Lie groups of dimension 4, we are unable to completely classify simply connected Lie groups of polynomial growth in dimension 5. However, here it is only one pair of groups whose possible quasi-isometry relation remains open: whether or not the Lie group $A_{5,6}$ is quasi-isometric to its associated Carnot group $A_{5,2}$. This question cannot be answered by the community for now.

6. Appendix: A direct proof in Abelian case

In this section we prove Theorem 6.4. It is a less general statement than Theorem A.(i), but the proof is completely different in spirit and might have independent interest and possibilities to generalise. The proof is highly inspired by the results of [CPS17].

The following definition appeared implicitly in [CPS17], but we prefer to have a name for it.

Definition 6.1. The *characteristic subalgebra* for a purely real homogeneous group (N, α) is the subalgebra $\mathfrak{h}_\alpha$ of $\mathfrak{n}$ constructed as follows. Consider a basis of $\mathfrak{n}$ where α is in Jordan form, and let λ_1 denote the smallest of the eigenvalues of α. Let V_{λ_1} be the subspace corresponding to the Jordan-blocks of α of eigenvalue λ_1 (i.e., the generalised eigenspace of eigenvalue λ_1). Let $\hat{V}_1 \subset V_{\lambda_1}$ be the sum of the subspaces corresponding to the Jordan blocks in V_{λ_1} of maximal size. Next, let $\mathcal{V}_1$ consist of eigenvectors of eigenvalue λ_1 inside $\hat{V}_1$, and finally define $\mathfrak{h}_\alpha = \mathrm{LieSpan}(\mathcal{V}_1)$. We further denote by H_α the subgroup of N with Lie algebra $\mathfrak{h}_\alpha$ and call it the *characteristic subgroup* of (N, α).

Remark 6.2. Equivalently, the characteristic subalgebra is defined as follows: Let $k \in \mathbb{N}$ be the unique integer such that $(\alpha|_{V_{\lambda_1}} - \lambda_1 \mathrm{Id})^k \neq 0$ and $(\alpha|_{V_{\lambda_1}} - \lambda_1 \mathrm{Id})^{k+1} = 0$. Then $\mathcal{V}_1 = \mathrm{Im}(\alpha|_{V_{\lambda_1}} - \lambda_1 \mathrm{Id})^k$ and $\mathfrak{h}_\alpha = \mathrm{LieSpan}(\mathcal{V}_1)$.

In the following, we list some facts related to characteristic subalgebras and subgroups.

Proposition 6.3. *(i)* $\mathfrak{h}_\alpha = \mathfrak{n}$ *if and only if* (N, α) *is of Carnot type.*

(ii) $\mathfrak{h}_\alpha$ *is preserved under* α.

(iii) Suppose $F\colon (N_1, \alpha) \to (N_2, \beta)$ *is a biLipschitz map between two purely real homogeneous groups, and suppose* $F(1_{N_1}) = 1_{N_2}$. *Then* $F(H_\alpha) = H_\beta$.

Proof. The part (i) follows by observing that both of the claims are equivalent to the condition $V_{\lambda_1} = \mathcal{V}_1$.

The part (ii) is proven by a straightforward induction on the length of a bracket in $\mathfrak{h}_\alpha$, using that $\mathcal{V}_1$ is preserved under α by construction.

The part (iii) is proven in [Pia17], see also [CPS17, p. 6] □

Theorem 6.4. *Let* (N_1, α) *and* (N_2, β) *be purely real homogeneous groups that are biLipschitz equivalent. If* N_1 *is Abelian, so is* N_2. *Consequently* (N_1, α) *and* (N_2, β) *are isomorphic as homogeneous groups, by* [Xie14].

Proof. We prove the claim inductively on the topological dimension of the groups in question. The case $n = 1$ (and also $n = 2$) is true due to the lack of non-Abelian nilpotent groups. So assume the claim holds for groups of dimension k and less and

$$\dim(N_1) = \dim(N_2) = k + 1 . \tag{2}$$

Let $F\colon N_1 \to N_2$ be a biLipschitz map, which after post-composing with a left-translation we may assume to satisfy $F(1_{N_1}) = 1_{N_2}$. Thus, when H_α and H_β denote the respective characteristic subgroups, by Proposition 6.3.(iii) it holds

$$F(H_\alpha) = H_\beta \,. \tag{3}$$

If $H_\alpha = N_1$, then by Proposition 6.3.(i) the homogeneous group (N_1, α) is of Carnot type, and as a consequence of [Pia17, Theorem 1.9] the homogeneous group (N_2, β) is also of Carnot type. In this case, by Pansu's Theorem [Pan89], (N_1, α) and (N_2, β) are isomorphic as homogeneous groups. We are left to consider the case

$$H_\alpha \subsetneq N_1 \,. \tag{4}$$

From (2) and (4) we have $\dim(H_\alpha) \le k$. Thus the induction assumption and (3) gives that H_β is Abelian because H_α is Abelian.

Moreover we claim that H_β is normal in N_2. Indeed, the normaliser of H_α is N_1 since N_1 is Abelian, hence by Proposition 1.7 the normaliser of H_β is N_2.

Next, we claim H_β is central in N_2. By the definition of the characteristic subalgebra we have $\mathfrak{h}_\beta = \mathrm{LieSpan}(\mathcal{V}_1)$, where $\mathcal{V}_1 \subset V_{\lambda_1}$, as in Definition 6.1. We know now that $\mathfrak{h}_\beta$ is an Abelian ideal, so $\mathcal{V}_1$ is Abelian and $\mathfrak{h}_\beta \subset V_{\lambda_1}$. Using the grading given by the generalised eigenspaces V_λ of β (see [Bou75, p. 16 Prop. 12]) and the fact that $\mathfrak{h}_\beta$ is an ideal of $\mathfrak{n}_2$ we get for all $H \in \mathfrak{h}_\beta$ and $X \in \mathfrak{n}_2$ that

$$[X, H] \in \mathfrak{h}_\beta \cap \bigoplus_{\lambda > \lambda_1} V_\lambda = \{0\} \,.$$

Hence $\mathfrak{h}_\beta$ is central in $\mathfrak{n}_2$.

Take W to be a complementary subspace to $\mathcal{V}_1$ inside V_{λ_1}. Define

$$\mathfrak{s}_\beta = W \oplus \bigoplus_{\lambda > \lambda_1} V_\lambda \,,$$

which is an ideal because it contains $[\mathfrak{n}_2, \mathfrak{n}_2]$. The subspaces $\mathfrak{s}_\beta$ and $\mathfrak{h}_\beta$ are in direct sum and they are both ideals, so the Lie algebra $\mathfrak{n}_2$ is the direct product of these two subalgebras: $\mathfrak{n}_2 = \mathfrak{s}_\beta \times \mathfrak{h}_\beta$. On the N_1 side, the same construction works but it is simpler because N_1 is Abelian. Anyway, we may decompose $\mathfrak{n}_1 = \mathfrak{h}_\alpha \times \mathfrak{s}_\alpha$, where $\mathfrak{s}_\alpha$ is an arbitrary complementary subspace to $\mathfrak{h}_\alpha$.

By Proposition 6.3.(ii) and the concrete formula for a homogeneous distance on the quotient given in Lemma 1.8, we have that the quotient groups N_1/H_α and N_2/H_β are biLipschitz equivalent purely real homogeneous groups. Their dimension is at most k, since the characteristic subgroups have at least dimension 1. Hence by induction, N_2/H_β is Abelian since N_1/H_α is Abelian. By the structure of direct products, N_2/H_β and S_β are isomorphic as Lie groups, hence S_β is Abelian. Since N_2 is a direct sum of two Abelian normal subgroups H_β and S_β, then N_2 is Abelian.

For the final statement, [Xie14, Theorem 1.1] tells that the Jordan forms of α and β are proportional. On the other hand, since the homogeneous groups are biLipschitz equivalent, the smallest of the eigenvalues of α and β must agree since by Proposition 1.6 we have that the common smallest eigenvalue λ_1 is the minimal Hausdorff dimension of curves. Therefore the Jordan forms of α and β agree and this is enough to give an isomorphism of homogeneous groups in the Abelian case. □

Remark 6.5. The proof above does not give a new proof of the main result of [Xie14], since it may happen that the complementary subspace W cannot be chosen to be preserved under the derivation β. Therefore, while N_2/H_β has a structure of a homogeneous group induced by β, the subgroup S^β is not preserved under β and does not inherit a structure of a homogeneous group.

Remark 6.6. Theorem 6.4 may also be proven from Theorem A.(ii) by an argument that we will next sketch, thereby giving a third proof for Theorem 6.4. A homogeneous group (N, α) is non-Abelian if and only if for some $s > 0$ the reachability set $(N, \alpha)^s$ is strictly larger than the subgroup corresponding to $\bigoplus_{0<\lambda\leq s} V_\lambda$. Suppose that homogeneous groups (N_1, α) and (N_2, β) are biLipschitz equivalent. On the one hand, the characteristic polynomials of α and β agree by [CPS17], and hence the dimensions of the generalised eigenspaces of the same eigenvalues agree. On the other hand, $(N_1, \alpha)^s$ and $(N_2, \beta)^s$ have the same dimension for every s by Theorem A.(ii). We conclude that, if N_1 is Abelian, then also N_2 is Abelian.

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