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ON THE SCIENTIFIC WORK OF VICTOR ISAKOV

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1. **Introduction.** This special issue of Inverse Problems and Imaging with title "Analytical aspects of inverse problems in PDEs" is dedicated to the memory of Victor Isakov, a world leading expert in the field of inverse problems for partial differential equations (PDE). Over the course of his distinguished career, Isakov made numerous fundamental contributions to inverse scattering, the Calderón problem, inverse problems for elliptic, parabolic, and hyperbolic PDE, inverse problems of potential theory, as well as uniqueness in the Cauchy problem. He wrote two very influential research monographs [27] and [35].

Victor Isakov was born on November 4, 1947, in Stalinsk (later renamed Novokuznetsk), Russia. He received his PhD in 1973 from the (now Sobolev) Institute of Mathematics, Novosibirsk, Russia, where he was also appointed as a researcher from 1971 to 1983. He also served as a part time associate professor at the Novosibirsk State University from 1974 to 1982. In 1987 Victor Isakov moved to the US, after staying for two months as a visiting professor at the University of Florence,

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Italy. He held visiting positions at the Courant Institute of New York University, Cornell University, and University of Minnesota before being appointed a Professor at Wichita State University in 1988. In 2006 he was designated a Distinguished Professor by the Kansas Board of Regents, a position he held until his untimely death in 2021.

Apart from being a leading mathematician, Victor Isakov had wide cultural interests and was an outstanding pianist.

Let us now proceed to discuss some of the significant scientific results established by Victor Isakov.

2. **On the mathematics of Victor Isakov.**

2.1. **Applications of singular solutions to inverse problems.** In the study of inverse problems, besides complex geometric optics solutions [4], [69], singular solutions are also useful in many cases. The use of singular solutions in solving inverse problems was first introduced by Isakov [26]. In [26], Isakov considered the problem of determining a discontinuous conductivity coefficient from the knowledge of the Dirichlet-to-Neumann map. Let Ω be a C^2 bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Consider the conductivity equation

$$\operatorname{div}(a_j \nabla u_j) = 0 \quad \text{in} \quad \Omega, \tag{1}$$

where $a_j(x) = a_0(x) + \chi(\Omega_j) b_j(x)$ with $a_0 \in C^2(\bar{\Omega})$, $b_j \in C^2(\bar{\Omega}_j)$ satisfying $a_0 + b_j > 0$ on $\bar{\Omega}_j$ and $b_j \neq 0$ on $\partial\Omega_j$, $j = 1, 2$. Suppose that $\Omega_j \Subset \Omega$ is a Lipschitz domain and $\Omega \setminus \bar{\Omega}_j$ is connected. Let Γ_0 be a non-empty open subset of $\partial\Omega$. Associated to (1) and Γ_0 is the local Dirichlet-to-Neumann map $\Lambda_j^{\Gamma_0}$ given by

$$\Lambda_j^{\Gamma_0}(\varphi) = a_0 \frac{\partial u_j}{\partial \nu}\Big|_{\Gamma_0},$$

where u_j is the solution to (1) with Dirichlet data φ and $\operatorname{supp}(\varphi) \subset \Gamma_0$, and ν is the unit outer normal of $\partial\Omega$. Assume that a_0 is known.

Isakov in [26] showed that if $\Lambda_1^{\Gamma_0} = \Lambda_2^{\Gamma_0}$ then $\Omega_1 = \Omega_2$ and $b_1 = b_2$ on $\partial\Omega_1$. The result is proved by a contradiction argument. Assume that $\Omega_1 \neq \Omega_2$. Let $x_0 \in \partial\Omega_1 \setminus \bar{\Omega}_2$ and B be a ball centered at x_0 with $B \cap \bar{\Omega}_2 = \emptyset$. Using that $\Lambda_1^{\Gamma_0} = \Lambda_2^{\Gamma_0}$ and applying a Runge type approximation property, one can derive the following orthogonality relation,

$$\int_{\Omega_1} b_1 \nabla u_3 \cdot \nabla u_2 = \int_{\Omega_2} b_2 \nabla u_3 \cdot \nabla u_2, \tag{2}$$

where u_2 is any solution to (1) with $j = 2$ near $\bar{\Omega}_1 \cup \bar{\Omega}_2$ and u_3 solves $\operatorname{div}(a_3 \nabla u_3) = 0$ near $\bar{\Omega}_1 \cup \bar{\Omega}_2$, where a_3 is a C^1 extension of $a_0 + b_1$ in $B \cap \bar{\Omega}_1$. The key idea used by Isakov was to substitute $u_3(x) = K_3(y, x)$ and $u_2 = K_2(y, x)$ into (2), where K_3 and K_2 are Green's functions for the corresponding equations with singularity at a point y near x_0. Next it follows from (2) that

$$\begin{aligned} &\int_{B_0 \cap \Omega_1} b_1 \nabla K_3(y, \cdot) \cdot \nabla K_2(y, \cdot) \\ &= -\int_{B_0^c \cap \Omega_1} b_1 \nabla K_3(y, \cdot) \cdot \nabla K_2(y, \cdot) + \int_{\Omega_2} b_2 \nabla K_3(y, \cdot) \cdot \nabla K_2(y, \cdot), \end{aligned} \tag{3}$$

where $B_0 \subset B$. As $y \to x_0$, the left-hand side of (3) blows up, while the right-hand side of (3) remains bounded. This contradiction implies that $\Omega_1 = \Omega_2$. The proof of the fact that $b_1 = b_2$ on $\partial\Omega_1$ follows by a similar argument.

Isakov later applied the method to an inverse transmission scattering problem [28]. By simplifying some arguments in [28], Kirsch and Kress [41] extended Schiffer's result [57] to the case of sound-hard obstacles in which a sound-hard obstacle is uniquely determined by the far-field pattern at one fixed wave number. In [33] Isakov used the method of singular and complex geometric optics solutions to recover a transparent obstacle, transmission coefficients, as well as an electric potential in the Schrödinger equation inside the obstacle from partial boundary measurements as well as from scattering data at a fixed frequency, see also [47] for an extension of this result to the case of magnetic Schrödinger equations. We also mention that Alessandrini [2] used suitable singular solutions in the study of Calderón's problem. By modifying Isakov's method, Ikehata [20] proposed a reconstruction method of determining the discontinuity of the medium by the boundary measurements. Ikehata named the reconstruction method the probe method. Isakov's idea also played a key role in the development of a reconstruction method by Potthast [67]. Furthermore, the linear sampling method introduced by Colton and Kirsch [10] has some resemblance to Isakov's method.

2.2. **Complex geometric optics solutions and completeness of products of solutions.** In [29] Isakov developed an original, powerful, and flexible approach to the construction of complex geometric optics solutions for multiplicative perturbations of a large class of constant coefficient PDE. Specifically, let $P=\sum_{|\alpha|\le m} a_\alpha D^\alpha$ be a partial differential operator on $\mathbb{R}^n$ of order m with constant coefficients, $a_\alpha\in\mathbb{C}$, and let $P(\xi)=\sum_{|\alpha|\le m} a_\alpha \xi^\alpha$, $\xi\in\mathbb{R}^n$, be its full symbol. We also set $\widetilde{P}(\xi)=(\sum_{|\beta|\ge 0}|P^{(\beta)}(\xi)|^2)^{1/2}$, where $P^{(\beta)}(\xi)=\partial_\xi^\beta P(\xi)$. Assuming that

$$\sup_{\xi\in\mathbb{R}^n}\frac{1}{\widetilde{P}(\xi+\zeta)}\to 0, \text{ as } |\zeta|\to\infty,\ \zeta\in P^{-1}(0)\subset\mathbb{C}^n, \tag{4}$$

in [29] Isakov constructed solutions to the equation

$$(P+q)u=0 \quad \text{in} \quad \Omega, \tag{5}$$

that are of the form

$$u(x;\zeta)=e^{i\zeta\cdot x}(1+r(x;\zeta))\in L^2(\Omega). \tag{6}$$

Here $\Omega\subset\mathbb{R}^n$ is an open bounded set, $q\in L^\infty(\Omega)$, $\zeta\in\mathbb{C}^n$, $\zeta\cdot\zeta=0$, and $\|r\|_{L^2(\Omega)}\to 0$, as $|\zeta|\to\infty$. The proof proceeds as follows. In order for u given by (6) to satisfy (5), the remainder term r should solve

$$P(D+\zeta)r=-q(1+r). \tag{7}$$

To solve (7), a brilliant idea of Isakov is to employ a regular fundamental solution E_ζ of $P(D+\zeta)$, constructed by Hörmander [18], which gives a right inverse of $P(D+\zeta)$ on $L^2(\Omega)$ with the bound

$$\|E_\zeta\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\le C\sup_{\xi\in\mathbb{R}^n}\frac{1}{\tilde{P}(\xi+\zeta)}.$$

Thanks to the assumption (4), we have $\|E_\zeta\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\to 0$, as $|\zeta|\to\infty$, $\zeta\in P^{-1}(0)\subset\mathbb{C}^n$. The solution $r\in L^2(\Omega)$ of (7) is then obtained as the unique fixed point of the contraction map $r\mapsto E_\zeta(-q(1+r))$ on $L^2(\Omega)$ for $|\zeta|$ large.

Another significant result of the work [29] is a proof of the completeness of the products of solutions to (5) with two different potentials q_1 and q_2, under an additional assumption on the operator P, which can be stated as follows: there exists an open non-empty set $U \subset \mathbb{R}^n$ such that for every $\xi \in U$ and every positive $R > 0$, there are $\zeta_1, \zeta_2 \in P^{-1}(0) \subset \mathbb{C}^n$ such that $\zeta_1 + \zeta_2 = \xi$, $|\zeta_j| > R$. The proof of this result is based on the construction of complex geometric optics solutions of the form (6) to the equations (5) with two different potentials q_1 and q_2, and two complex frequencies ζ_1 and ζ_2 as in the additional assumption above. As also shown in [29], the two conditions above are satisfied, in particular, when P is the biharmonic operator, the heat operator, as well as the ultrahyperbolic operator.

An account of the powerful method of [29] of constructing complex geometric optics solutions is given in the book [7]. This method turned out to be of crucial importance when establishing the Borg–Levinson theorem for elliptic operators of higher order with constant coefficients in [45]. It was also used, in particular, when solving an inverse problem of recovering a non-compactly supported potential in the Schrödinger equation in an unbounded waveguide from boundary measurements in [40] as well as when studying the inverse problem of determining a time dependent potential in the wave equation from partial boundary measurements in [44].

2.3. **Inverse problems for nonlinear equations.** Isakov began to study inverse boundary value problems for nonlinear equations in the early 90's. In [30], Isakov considered the problem of determining the semilinear term $a(x, u)$ in the parabolic equation $\partial_t u - \Delta u + a(x, u) = 0$ given all possible lateral boundary measurements (the lateral Dirichlet-to-Neumann map). A key observation in Isakov's approach was that the equality of the Dirichlet-to-Neumann maps corresponding to the semilinear parabolic equations implies the equality of the Dirichlet-to-Neumann maps for the linear parabolic equations obtained by the linearization technique. Precisely, let Λ_{a_j} be the lateral Dirichlet-to-Neumann map associated with the equation $\partial_t u_j - \Delta u_j + a_j(x, u_j) = 0$, $j = 1, 2$, and let $\Lambda^*_{a^*_j}$ be the lateral Dirichlet-to-Neumann map associated to the linearized equation

$$\partial_t v_j - \Delta v_j + a^*_j v_j = 0$$

with

$$a^*_j(x, t; g) = \frac{\partial a_j}{\partial u}(x, t; u_j(x, t; g)),$$

where $u_j(x, t; g)$ is the unique solution of the semilinear parabolic equation with the lateral Dirichlet condition g. Isakov then showed that if $\Lambda_{a_1} = \Lambda_{a_2}$ then $\Lambda^*_{a^*_1} = \Lambda^*_{a^*_2}$. Having established the equality of the Dirichlet-to-Neumann maps for the linear parabolic equation, the unique determination of $a(x, u)$ follows from the corresponding result for the linear equation. The idea of using the linearization technique in treating inverse problems for nonlinear equations is highly original.

The idea of linearization in [30] was later applied to inverse boundary value problems for semilinear elliptic equations [25], [23] and for quasilinear equations [70], [31]. The inverse boundary value problem for the Navier-Stokes equation studied in [61] was also based on the linearization technique, see also [59] and [38]. Subsequently, following [30], a second order linearization of the nonlinear Dirichlet–to–Neumann map has also been successfully exploited, see [3], [71], [70]. The work [43] developed the linearization idea further, by introducing higher order linearizations in the case of inverse problems for hyperbolic PDE. This allowed one to solve such inverse problems for nonlinear equations in situations where the corresponding inverse problems in the linear setting are open, see also [11], [60] and reference there for the use of higher order linearization in the hyperbolic setting. We refer to the works [16], [54] for the introduction of higher order linearization technique in the case of inverse problems for elliptic PDE and to the works [55], [52], [51], [6], [56] and references there for some further developments in the elliptic case.

2.4. **The Calderón problem with partial data.** Isakov made a fundamental contribution to the celebrated Calderón problem with partial data in his work [32]. When stating his result, let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with smooth boundary, and let $\Gamma \subset \partial\Omega$ be open nonempty with $\partial\Omega \setminus \Gamma$ being either part of a hyperplane or part of a sphere. Associated to Γ, we introduce the partial Cauchy data set for the Schrödinger equation,

$$C_q^\Gamma = \{(u|_\Gamma, \partial_\nu u|_\Gamma) : u \in H^1(\Omega) \text{ satisfying } (-\Delta + q)u = 0 \text{ in } \Omega,\ \operatorname{supp}(u|_{\partial\Omega}) \subset \Gamma\}.$$

Here $q \in L^\infty(\Omega)$ and $H^1(\Omega)$ is the standard Sobolev space on Ω. In [32] Isakov proved that the equality $C_{q_1}^\Gamma = C_{q_2}^\Gamma$ implies that $q_1 = q_2$ in Ω.

Let us proceed to explain the main ideas in the proof of this result in the case when $\Omega \subset \{x \in \mathbb{R}^n : x_n > 0\}$, $\partial\Omega \cap \{x_n = 0\} \neq \emptyset$, and $\Gamma = \partial\Omega \setminus (\partial\Omega \cap \{x_n = 0\})$. First, using that $C_{q_1}^\Gamma = C_{q_2}^\Gamma$, one can derive the following integral identity,

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0, \tag{8}$$

for all $u_j \in H^1(\Omega)$ satisfying $(-\Delta + q_j)u_j = 0$ in Ω and $\operatorname{supp}(u_j|_{\partial\Omega}) \subset \Gamma$, $j = 1, 2$. To construct solutions u_j which vanish on the hyperplane $x_n = 0$, the beautiful idea of Isakov is to use complex geometric optics solutions in combination with a reflection along this hyperplane. Specifically, let $\Omega^* = \{x^* = (x', -x_n) : x = (x', x_n) \in \Omega\}$ be the reflection of Ω with respect to the hyperplane $x_n = 0$, and let $\tilde{q}_j \in L^\infty(\Omega \cup \Omega^*)$ be the even extension of the potential q_j, $j = 1, 2$. Let

$$\tilde{u}_j = e^{\zeta_j \cdot x}(1 + \tilde{r}_j) \in H^1(\Omega \cup \Omega^*)$$

be a complex geometric optics solution to the Schrödinger equation $(-\Delta + \tilde{q}_j)\tilde{u}_j = 0$ in $\Omega \cup \Omega^*$, $j = 1, 2$. Here $\zeta_j \in \mathbb{C}^n$ is such that $\zeta_j \cdot \zeta_j = 0$, $|\zeta_j| \to \infty$, and the remainder term r_j satisfies $\|r_j\|_{L^2(\Omega \cup \Omega^*)} \leq C/|\zeta_j|$. The main point of [32] is to set

$$u_j(x) = \tilde{u}_j(x) - \tilde{u}_j(x^*) \in H^1(\Omega),$$

and observe that u_j satisfies $(-\Delta + q_j)u_j = 0$ in Ω and $u_j|_{x_n = 0} = 0$. Substituting the solutions u_j into the integral identity (8), we get

$$\begin{aligned}
&\int_\Omega (q_1 - q_2)\big(e^{(\zeta_1 + \zeta_2)\cdot x}(1 + \tilde{r}_1(x))(1 + \tilde{r}_2(x)) + e^{(\zeta_1^* + \zeta_2^*)\cdot x}(1 + \tilde{r}_1(x^*))(1 + \tilde{r}_2(x^*))\big)\, dx \\
&= \int_\Omega (q_1 - q_2)\big(e^{(\zeta_1 + \zeta_2^*)\cdot x}(1 + \tilde{r}_1(x))(1 + \tilde{r}_2(x^*)) + e^{(\zeta_1^* + \zeta_2)\cdot x}(1 + \tilde{r}_1(x^*))(1 + \tilde{r}_2(x))\big)\, dx.
\end{aligned} \tag{9}$$

Next, the idea is that for $\xi \in \mathbb{R}^n$ belonging to a suitable non-empty open set U, one can choose ζ_j so that $\zeta_j \cdot \zeta_j = 0$, $\zeta_1 + \zeta_2 = i\xi$, $\operatorname{Re}(\zeta_1 + \zeta_2^*) = \operatorname{Re}(\zeta_1^* + \zeta_2) = 0$, and $|\operatorname{Im}(\zeta_1 + \zeta_2^*)|, |\operatorname{Im}(\zeta_1^* + \zeta_2)| \to \infty$ as $|\zeta_j| \to \infty$. Letting $|\zeta_j| \to \infty$ and applying the Riemann–Lebesgue lemma in the right hand side of (9), we get

$$\int_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{ix\cdot\xi}\, dx = 0,$$

for $\xi \in U$. Therefore, $q_1 = q_2$ in Ω, showing the result.

In [19] logarithmic type stability estimates were obtained for Isakov's result [32], see also [53] and [64] for the increasing stability phenomenon. Isakov's reflection approach has been further employed for partial data inverse problems for other important PDE, in particular for Maxwell's systems in [13], [5], and magnetic Schrödinger equations in [42], [65]. Partial data results for the slab and half spaces geometries based on Isakov's reflection approach were obtained in [58], [42], [62], [12], and [66]. The work [48] unified the approaches [50] and [32] and extended both of them, see also [37]. We refer to [49] for a survey on partial data inverse problems.

2.5. **Increasing stability in inverse problems.** In his later career, Isakov was interested in the question of increasing stability in inverse problems. Most inverse problems are known to be ill-posed. The stability estimate is often an optimal log-type estimate, see for example [1], [63], [46]. However, numerical evidence [8] indicates that the stability increases with the wave number. Also, motivated by the result of John [39] where in general one can only expect a weak logarithmic stability in the Cauchy problem for the Helmholtz equation in the general geometry, Isakov [17] tried to provide a rigorous justification of the increasing stability phenomenon by considering the Cauchy problem for the Helmholtz equation under a convexity condition. The result of [17] shows that the stability of the continuation from the Cauchy data on the sub-domain of the boundary consists of two parts, a Lipschitz part and a Hölder part. The Hölder part decreases as the wave number increases. The argument used in [17] is elementary involving a suitable Carleman estimate. It is fair to say that Isakov initiated the investigation of increasing stability for inverse problems by carefully analyzing the effect of the wave number or the frequency on the stability estimate.

The result of [17] was later extended to the increasing stability for the Cauchy problem for the Helmholtz equation in the whole domain [68]. In this case, the stability estimate consists of two terms, a Lipschitz term and a logarithmic term. The interesting property is that the logarithmic part decreases as the wave number grows. This turns out to be a typical situation for some inverse boundary value problems, see [34], [24], [22], and for some inverse source problems, see [9], [21], [14], [15], [36]. The method used in the inverse source problems mentioned above is based on the observability bounds for the wave-type equations and an explicit bound in the analytic continuation.

2.6. **Other contributions.** It is impossible to cover all of Isakov's vast contributions to the field of inverse problems in this short editorial. For instance, the short review above does not address the fundamental work done by Isakov in the fields of the inverse problem of gravimetry, the inverse conductivity problem with a single measurement, Carleman estimates and applications to the unique continuation property and inverse problems, inverse parabolic problems, the inverse problem of option pricing, etc. For interested readers, we refer to Isakov's books [27], [35] and references therein for further details.

3. List of Victor Isakov's publications.

3.1. Research monographs.

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