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# PARTIAL DATA INVERSE PROBLEMS AND SIMULTANEOUS RECOVERY OF BOUNDARY AND COEFFICIENTS FOR SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We study various partial data inverse boundary value problems for the semilinear elliptic equation $\Delta u + a(x, u) = 0$ in a domain in $\mathbb{R}^n$ by using the higher order linearization technique introduced in [LLLS19, FO19]. We show that the Dirichlet-to-Neumann map of the above equation determines the Taylor series of $a(x, z)$ at $z = 0$ under general assumptions on $a(x, z)$. The determination of the Taylor series can be done in parallel with the detection of an unknown cavity inside the domain or an unknown part of the boundary of the domain. The method relies on the solution of the linearized partial data Calderón problem [FKSU09], and implies the solution of partial data problems for certain semilinear equations $\Delta u + a(x, u) = 0$ also proved in [KU20].

The results that we prove are in contrast to the analogous inverse problems for the linear Schrödinger equation. There recovering an unknown cavity (or part of the boundary) and the potential simultaneously are long-standing open problems, and the solution to the Calderón problem with partial data is known only in special cases when $n \geq 3$.

**Keywords.** Calderón problem, inverse obstacle problem, Schiffer's problem, simultaneous recovery, partial data.

## Contents

## 1. INTRODUCTION

In this paper, we extend the recent studies [LLLS19, FO19] to various partial data inverse problems for the semilinear elliptic equation

$$\Delta u + a(x, u) = 0 \text{ in } \Omega \subset \mathbb{R}^n,$$

for $n \geq 2$. The proofs rely on *higher order linearization.* This method reduces inverse problems for semilinear elliptic equations to related problems for the Laplace equation, with artificial source terms produced by the nonlinear interaction, and then employs the exponential solutions introduced in [Cal80] to solve these problems. Hence, one can regard the nonlinearity as a tool to solve inverse problems for elliptic equations with certain nonlinearities.

As a matter of fact, many researchers have studied inverse problems for nonlinear elliptic equations. A classical method, introduced in [Isa93] in the parabolic case, is to show that the first linearization of the nonlinear DN map is actually the

DN map of the corresponding linearized equation, and then to adapt the theory of inverse problems for linear equations. For the semilinear equation $\Delta u + a(x, u) = 0$, the problem of recovering the potential $a(x, u)$ was studied in [IS94, IN95, Sun10, IY13a]. Further results are available for inverse problems for quasilinear elliptic equations [Sun96, SU97, KN02, LW07, MU], for the degenerate elliptic $p$-Laplace equation [SZ12, BHKS18], and for the fractional semilinear Schrödinger equation [LL19]. Certain inverse problems for quasilinear elliptic equations on Riemannian manifolds were considered in [LLS19]. We refer to the surveys [Sun05, Uhl09] for more details on inverse problems for nonlinear elliptic equations.

Inverse problems for hyperbolic equations with various nonlinearities have also been studied. Many of the results mentioned above rely on a solution to a related inverse problem for a linear equation, which is in contrast to the study of inverse problems for nonlinear hyperbolic equations. In fact, it has been realized that the nonlinearity can be beneficial in solving inverse problems for nonlinear hyperbolic equations.

By regarding the nonlinearity as a tool, some unsolved inverse problems for hyperbolic linear equations have been solved for their nonlinear analogues. Kurylev-Lassas-Uhlmann [KLU18] studied the scalar wave equation with a quadratic nonlinearity. In [LUW18], the authors studied inverse problems for general semilinear wave equations on Lorentzian manifolds, and in [LUW17] they studied similar problems for the Einstein-Maxwell equations. We also refer readers to [CLOP19, dHUW18, KLOU14, WZ19] and references therein for further results on inverse problems of nonlinear hyperbolic equations.

In this work we employ the method introduced independently in [LLLS19] and [FO19] which uses nonlinearity as a tool that helps in solving inverse problems for certain nonlinear elliptic equations. The method is based on *higher order linearizations* of the DN map, and essentially amounts to using sources with several parameters and obtaining new linearized equations after differentiating with respect to these parameters. The works [LLLS19, FO19] considered inverse problems with boundary measurements on the whole boundary, also on manifolds of certain type. In this article we will consider similar problems in Euclidean domains with different types of assumptions. We consider situations where the data is given on the full boundary or just on a part of the boundary. We also consider cases when the domain includes an unknown cavity or an unknown part of the boundary. Especially we will prove uniqueness in the partial data case. Moreover, just before this article was submitted to arXiv, the preprint [KU20] of Krupchyk and Uhlmann appeared on arXiv. The work [KU20] considers the partial data Calderón problem for certain semilinear equations and proves Corollary 1.1 below.

In this work, we do not pursue optimal regularity assumptions for our inverse problems. Instead, we want to demonstrate how the nonlinearity helps us in understanding related inverse problems.

Let us describe more precisely the semilinear equations studied in this article. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^\infty$ boundary $\partial\Omega$, where $n \geq 2$. Consider the following second order semilinear elliptic equation

$$(1.1) \qquad \begin{cases} \Delta u + a(x, u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

We will assume that the boundary data satisfies $f \in C^s(\partial\Omega)$ and $\|f\|_{C^s(\partial\Omega)} \leq \delta$, where $s > 2$ is not an integer and $\delta > 0$ is a sufficiently small number. Here $C^s(\partial\Omega)$ is the standard Hölder space on $\partial\Omega$ (the precise definition is given in Appendix A).

For the function $a = a(x, z)$, we assume that $a$ is $C^\infty$ in $\overline{\Omega} \times \mathbb{R}$ and satisfies one of the following conditions: Either $a = a(x, z)$ satisfies

$$(1.2) \qquad a(x, 0) = 0, \text{ and } 0 \text{ is not a Dirichlet eigenvalue of } \Delta + \partial_z a(x, 0) \text{ in } \Omega,$$

or $a = a(x, z)$ satisfies

$$(1.3) \qquad a(x, 0) = \partial_z a(x, 0) = 0.$$

Note that the condition (1.3) is stronger than (1.2). Nonlinearities satisfying (1.3) together with the condition $\partial_z^k a(x, 0) \neq 0$ for some $k \geq 2$ are called *genuinely nonlinear* in [LUW18] in the context of inverse problems of nonlinear hyperbolic equations. The benefit of assuming (1.3) is that the linearized equation will be just the Laplace equation.

For nonlinearities satisfying (1.2), it follows from [LLLS19, Proposition 2.1] (restated in Appendix A) that there are $C, \delta > 0$ such that the boundary value problem (1.1) is well-posed for small boundary data $f \in C^s(\partial\Omega)$, $\|f\|_{C^s(\partial\Omega)} < \delta$, and there is a unique solution $u$ of (1.1) satisfying $\|u\|_{C^s(\overline{\Omega})} < C\delta$. The solution $u$ is called the unique small solution. See Appendix A for a more detailed discussion. We define the corresponding *Dirichlet-to-Neumann* map (DN map) $\Lambda_a$ such that

$$(1.4) \qquad \begin{aligned} &\Lambda_a : \{f \in C^s(\partial\Omega) \,;\, \|f\|_{C^s(\partial\Omega)} < \delta\} \to C^{s-1}(\partial\Omega), \\ &\Lambda_a(f) = \partial_\nu u|_{\partial\Omega}, \end{aligned}$$

where $\partial_\nu$ is the normal derivative on the boundary $\partial\Omega$.

We have three main theorems. The first one considers a full data inverse problem and the latter two theorems consider cases where measurements are made only on subsets of a boundary.

The first theorem is a full data uniqueness result that follows from the method of [LLLS19, FO19] for Euclidean domains (in fact the theorem is contained in [FO19] when $n \geq 3$ also for Hölder continuous $a(x, z)$, and in the case $a(x, z) = q(x)z^m$ where $q \in C^\infty(\overline{\Omega})$, $m \in \mathbb{N}$ and $m \geq 2$, it is contained in [LLLS19, Theorem 1.2]). The result considers the Calderón problem for semilinear elliptic equations for a large class of nonlinear coefficients $a(x, z)$, and shows that the Taylor series of $a(x, z)$ at $z = 0$ can be recovered from the DN map. This theorem is not covered by earlier results on inverse problems for semilinear equations [IS94, IN95, Sun10, IY13a], which often assume a sign condition such as $\partial_u a(x, u) \leq 0$.

**Theorem 1.1** (Global uniqueness). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^\infty$ boundary $\partial\Omega$, where $n \geq 2$. Let $a_j(x, z)$ be $C^\infty$ functions in $x, z$ satisfying (1.2) for $j = 1, 2$. Let $\Lambda_{a_j} : C^s(\partial\Omega) \to C^{s-1}(\partial\Omega)$, $s > 2$, be the (full data) DN maps of*

$$\Delta u + a_j(x, u) = 0 \text{ in } \Omega,$$

*for $j = 1, 2$, and assume that*

$$\Lambda_{a_1}(f) = \Lambda_{a_2}(f)$$

*for any $f \in C^s(\partial\Omega)$ with $\|f\|_{C^s(\partial\Omega)} < \delta$, where $\delta > 0$ is a sufficiently small number. Then we have*

$$(1.5) \qquad \partial_z^k a_1(x, 0) = \partial_z^k a_2(x, 0) \text{ in } \Omega, \text{ for } k \geq 1.$$

Even though the above result is mostly contained in [LLLS19, FO19], it will be helpful for the partial data results to give a proof of Theorem 1.1 as well as a reconstruction algorithm to recover the coefficients $\partial_z^k a(x, 0)$ for all $k \geq 2$ in Section 2.

Next, we introduce an *inverse obstacle problem* for semilinear elliptic equations. Let $\Omega$ and $D$ be a bounded open sets with $C^\infty$ boundaries $\partial\Omega$ and $\partial D$ such that $D \subset\subset \Omega$. Assume that $\partial\Omega$ and $\Omega \setminus \overline{D}$ are connected. Let $a(x, z) \in C^\infty((\overline{\Omega} \setminus D) \times \mathbb{R})$

be a function satisfying (1.3) for $x \in \Omega \setminus \overline{D}$. Consider the following semilinear elliptic equation

(1.6)
$$\begin{cases} \Delta u + a(x,u) = 0 & \text{in } \Omega \setminus \overline{D}, \\ u = 0 & \text{on } \partial D, \\ u = f & \text{on } \partial \Omega. \end{cases}$$

For $s > 2$ and $s \notin \mathbb{N}$, let $f \in C^s(\partial\Omega)$ with $\|f\|_{C^s(\partial\Omega)} < \delta$, where $\delta > 0$ is a sufficiently small number. The condition (1.3) yields the well-posedness of (1.6) for small solutions by the results in Appendix A, and one can define the corresponding DN map $\Lambda_a^D$, with Neumann values measured only on $\partial\Omega$, by

$$\Lambda_a^D : \{f \in C^s(\partial\Omega)\,;\, \|f\|_{C^s(\partial\Omega)} < \delta\} \to C^{s-1}(\partial\Omega), \quad \Lambda_a^D : f \mapsto \partial_\nu u|_{\partial\Omega}.$$

The inverse obstacle problem is to determine the unknown cavity $D$ and the coefficient $a$ from the DN map $\Lambda_a^D$. Our second main result is as follows.

**Theorem 1.2** (Simultaneous recovery: Unknown cavity and coefficients). *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with connected $C^\infty$ boundary $\partial\Omega$. Let $D_1, D_2 \subset\subset \Omega$ be nonempty open subsets with $C^\infty$ boundaries such that $\Omega \setminus \overline{D_j}$ are connected. For $j = 1, 2$, let*

$$a_j(x,z) \in C^\infty((\Omega \setminus \overline{D_j}) \times \mathbb{R})$$

*satisfy (1.3) and denote by $\Lambda_{a_j}^{D_j}$ the DN maps of the following Dirichlet problems*

$$\begin{cases} \Delta u_j + a_j(x,u_j) = 0 & \text{in } \Omega \setminus \overline{D_j}, \\ u_j = 0 & \text{on } \partial D_j, \\ u_j = f & \text{on } \partial\Omega \end{cases}$$

*defined for any $f \in C^s(\partial\Omega)$ with $\|f\|_{C^s(\partial\Omega)} < \delta$, where $\delta > 0$ is a sufficiently small number. Assume that*

$$\Lambda_{a_1}^{D_1}(f) = \Lambda_{a_2}^{D_2}(f), \ \text{whenever } \|f\|_{C^s(\partial\Omega)} < \delta.$$

*Then*

$$D := D_1 = D_2 \quad \text{and} \quad \partial_z^k a_1(x,0) = \partial_z^k a_2(x,0) \text{ in } \Omega \setminus \overline{D} \text{ for } k \geq 2.$$

The proof is based on higher order linearizations, and relies on the solution of the linearized Calderón problem with partial data given in [FKSU09]. We remark that the analogous simultaneously recovering problem stays open when the lower order coefficient $a(x,u) = q(x)u$. More specifically, when the elliptic equation $\Delta u + a(x,u) = 0$ becomes the (linear) Schrödinger equation $\Delta u + q(x)u = 0$, one does not know how to determine the obstacle $D \subset\subset \Omega$ and $q(x)$ by using the knowledge of the corresponding DN map.

The inverse problem of determining the obstacle $D$ from the DN map $\Lambda_{D,a}$ is usually regarded as the *obstacle problem*. The obstacle problem with a single measurement, i.e., determining the obstacle $D$ by a single Cauchy data $\{u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}\}$ is a long-standing problem in inverse scattering theory. This type problem is also known as *Schiffer's problem*, and the problem has been widely studied when the surrounding coefficients are known a priori. We refer the readers to [CK12, Isa06, LZ08] for introduction and discussion.

Many researchers have made significant progress in recent years on Schiffer's problem for the case with general polyhedral obstacles. For the uniqueness and stability results, see [AR05, CY03, LZ06, LZ07, Ron03, Ron08]. Under the assumption that $\partial D$ is nowhere analytic, Schiffer's problem was solved in [HNS13]. However, Schiffer's problem still remains open for the case with general obstacles (i.e., when obstacles have no geometrical assumptions). Furthermore, a nonlocal

type Schiffer's problem was solved by [CLL19]. We also want to point out that the simultaneous recovery of an obstacle and an unknown surrounding potential is also a long-standing problem in the literature. This problem is closely related to the *partial data* Calderón problem [KSU07, IUY10]. Unique recovery results in the literature are based on knowing the embedded obstacle to recover the unknown potential [IUY10], knowing the surrounding potential to recover the unknown obstacle [KL13, KP98, LZZ15, LZ10, O'D06], or using multiple spectral data to recover both the obstacle and potential [LL17].

Based on the connection of simultaneous recovery problems and the partial data Calderón problem, i.e., we do not know the DN map on the full boundary, we will next study a partial data problem for semilinear elliptic equations. In fact, we will consider the case where both the coefficients of the equation and a part of the boundary are unknown. In the study of partial data inverse problems for (linear) elliptic equations one usually assumes that the non-accessible part of the boundary is a priori known. This is not always a reasonable assumption in practical situations. For example, in medical imaging the body shape outside of the attached measurement device may not be precisely known.

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain with $C^\infty$ boundary $\partial\Omega$. Let $\Gamma \subset \partial\Omega$ be nonempty open set (the known part of the boundary), and assume that we do not know $\partial\Omega \setminus \Gamma$ a priori. We consider the following semilinear elliptic equation

$$(1.7) \qquad \begin{cases} \Delta u + a(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ u = f & \text{on } \Gamma, \end{cases}$$

where $a(x,u)$ is a smooth function fulfilling (1.3). For $s > 2$ and $s \notin \mathbb{N}$, let $f \in C_c^s(\Gamma)$ with $\|f\|_{C^s(\Gamma)} < \delta$, where $\delta > 0$ is any sufficiently small number. Here

$$C_c^s(\Gamma) := \{f \in C^s(\Gamma) : \operatorname{supp}(f) \subset \Gamma\}.$$

Then by the well-posedness of (1.7) for small solutions (see Appendix A again), one can define the corresponding DN map $\Lambda_a^{\Omega,\Gamma}$ with

$$\Lambda_a^{\Omega,\Gamma} : \{f \in C_c^s(\Gamma) ; \|f\|_{C^s(\Gamma)} < \delta\} \to C^{s-1}(\Gamma), \quad f \mapsto \partial_\nu u|_\Gamma.$$

The inverse problem is to determine unknown part of the boundary $\partial\Omega \setminus \Gamma$ and the coefficient $a$ from the DN map $\Lambda_a^{\Omega,\Gamma}$.

**Theorem 1.3** (Simultaneous recovery: Unknown boundary and coefficients)**.** *Let $\Omega_j \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $C^\infty$ boundary $\partial\Omega_j$ for $j = 1, 2$, and let $\Gamma$ be a nonempty open subset of both $\partial\Omega_1$ and $\partial\Omega_2$. Let $G$ be the connected component of $\Omega_1 \cap \Omega_2$ whose boundary contains $\Gamma$ and suppose that $G \neq \emptyset$. Let $a_j(x,z)$ be smooth functions satisfying (1.3). Let $\Lambda_{a_j}^{\Omega_j,\Gamma}$ be the DN maps of the following problems*

$$\begin{cases} \Delta u_j + a_j(x, u_j) = 0 & \text{in } \Omega_j, \\ u_j = 0 & \text{on } \partial\Omega_j \setminus \Gamma, \\ u_j = f & \text{on } \Gamma, \end{cases}$$

*for $j = 1, 2$. Assume that*

$$\Lambda_{a_1}^{\Omega_1,\Gamma}(f) = \Lambda_{a_2}^{\Omega_2,\Gamma}(f)$$

*for any $f \in C_c^s(\Gamma)$ with $\|f\|_{C^s(\Gamma)} < \delta$, for a sufficiently small number $\delta > 0$. Then we have*

$$\Omega_1 = \Omega_2 := \Omega \quad \text{and} \quad \partial_z^k a_1(x,0) = \partial_z^k a_2(x,0) \text{ in } \Omega \text{ for } k \geq 2.$$

The proof again relies on higher order linearizations and on the solution of the linearized Calderón problem with partial data [FKSU09]. By using Theorem 1.3, we immediately have the following result, which was first proved in the preprint [KU20] that appeared on arXiv just before this preprint was submitted.

**Corollary 1.1** (Partial data). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $C^\infty$ boundary $\partial\Omega$, and let $\Gamma \subset \Omega$ be a nonempty open subset. Let $a_j(x,z)$ be smooth functions satisfying* (1.3) *and let $\Lambda_{a_j}^{\Omega,\Gamma}$ be the partial data DN map for the Dirichlet problem*

$$\begin{cases} \Delta u_j + a_j(x, u_j) = 0 & in\ \Omega, \\ u_j = 0 & on\ \partial\Omega \setminus \Gamma, \\ u_j = f & on\ \Gamma, \end{cases}$$

*for $j = 1, 2$. Assume that*

$$\Lambda_{a_1}^{\Omega,\Gamma}(f) = \Lambda_{a_2}^{\Omega,\Gamma}(f)\ on\ \Gamma,$$

*for any $f \in C_c^s(\Gamma)$ with $\|f\|_{C^s(\Gamma)} < \delta$, for a sufficiently small number $\delta > 0$. Then*

$$\partial_z^k a_1(x, 0) = \partial_z^k a_2(x, 0)\ in\ \Omega\ for\ k \geq 2.$$

For the corresponding linear equation, i.e., $a(x, u) = q(x)u$, the partial data problem of determining $q$ from the DN map $\Lambda_q^{\Omega,\Gamma}(f)|_\Gamma$ for any $f$ supported in $\Gamma$, where $\Gamma$ is an arbitrary nonempty open subset of $\partial\Omega$, was solved in [IUY10] for $n = 2$ and $q_j \in C^{2,\alpha}$. For $n \geq 3$, the partial data problem stays open, but there are partial results [BU02, KSU07, Isa07, KS14a] when $\partial\Omega$ is assumed to be known. We refer to the surveys [IY13b, KS14b] for further references.

**Remark 1.2.** *If we assume that $a_j(x, z)$ are real analytic in $z$ for $j = 1, 2$, then one can completely recover the nonlinearity and show that $a_1(x, z) = a_2(x, z)$ in Theorems 1.1, 1.2 and 1.3. In particular this applies to equations of the type $\Delta u + q(x)u^m = 0$, where $m \geq 2$ is an integer.*

The paper is structured as follows. In Section 2 we prove Theorem 1.1. We also provide reconstruction algorithms for $\partial_z^k a(x, z)|_{z=0}$ for all $k \geq 2$. Theorem 1.2 and Theorem 1.3 will be proved in Section 3 and Section 4, respectively. Appendix A contains the proof of a topological lemma required in the arguments.

## 2. Proof of Theorem 1.1

We use higher order linearizations to prove Theorem 1.1. Before the proof we recall Calderón's exponential solutions ([Cal80]) to the equation $\Delta v = 0$ in $\mathbb{R}^n$, and the complex geometrical optics solutions (CGOs) that solve $\Delta v + qv = 0$ on a domain $\Omega$ in $\mathbb{R}^n$. These solutions will be used in the proof of Theorem 1.1. The exponential solutions of Calderón are of the form

(2.1)        $v_1(x) := \exp((\eta + i\xi) \cdot x), \quad v_2(x) := \exp((-\eta + i\xi) \cdot x),$

where $\eta$ and $\xi$ are any vectors in $\mathbb{R}^n$ that satisfy $\eta \perp \xi$ and $|\eta| = |\xi|$. The functions $v_1$ and $v_2$ solve the Laplace equation

$$\Delta v_1 = \Delta v_2 = 0 \text{ in } \mathbb{R}^n.$$

The linear span of the products $v_1 v_2 = \exp(2i\xi \cdot x)$, $\xi \in \mathbb{R}^n$, of Calderón's exponential solutions forms a dense set in $L^1(\Omega)$. In particular, if

$$\int_\Omega f v_1 v_2 \, dx = 0$$

holds for all Calderón's exponential solutions $v_1$ and $v_2$, then $f = 0$.

The complex geometrical optics solutions (CGOs) generalize Calderón's exponential solutions. For $n \geq 3$, they are of the form (see e.g. [SU87])

(2.2) $$V_1(x) = e^{\rho_1 \cdot x}(1 + r_1), \quad V_2(x) = e^{\rho_2 \cdot x}(1 + r_2),$$

where $\rho_1 = \eta + i(\xi + \zeta) \in \mathbb{C}^n$ and $\rho_2 = -\eta + i(\xi - \zeta) \in \mathbb{C}^n$. Here $\eta, \xi, \zeta \in \mathbb{R}^n$ satisfy

$$\eta \cdot \xi = \xi \cdot \zeta = \zeta \cdot \eta = 0, \text{ and } |\eta|^2 = |\xi|^2 + |\zeta|^2.$$

The idea is that $\xi$ is fixed but $|\eta|, |\zeta| \to \infty$. If $q \in L^\infty$, the CGO solutions $V_1$ and $V_2$ satisfy

$$(\Delta + q)V_1 = (\Delta + q)V_2 = 0 \text{ in } \Omega$$

and $\|r_j\|_{L^2(\Omega)} \leq \frac{C}{|\rho_j|}$ for some constant $C > 0$ depending on $q_j$, for $j = 1, 2$. Thus the product $V_1 V_2$ converges to $e^{2ix \cdot \xi}$ as $|\eta|, |\zeta| \to \infty$. For $n = 2$ one needs to use CGOs of the form $e^{i\tau\theta(x)} a(x)$, where $\tau > 0$ is large and the phase function $\theta$ is quadratic in $x$, instead of CGOs with linear phase functions of the form $\rho \cdot x$ described above. See [Buk08] for more details.

The products of pairs of CGOs form a complete set in $L^1(\Omega)$ by [SU87] for $n \geq 3$ and in $L^2(\Omega)$ by [Buk08, BTW19] for $n = 2$. Regarding this we record the following.

**Proposition 2.1** (Density of products of CGO solutions). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a $C^\infty$ boundary $\partial\Omega$ and let $q_1, q_2 \in C^\infty(\overline{\Omega})$. Let $f \in L^\infty(\Omega)$. Suppose that*

(2.3) $$\int_\Omega f V_1 V_2 \, dx = 0,$$

*for all $V_j$ solving $(-\Delta + q_j)V_j = 0$ in $\Omega$. Then $f \equiv 0$ in $\Omega$.*

*Proof.* The result for $n \geq 3$ follows from [SU87]. The argument for $n = 2$ is based on the stationary phase method and requires some care, but following closely the argument in [BTW19, Section 5] (where $q_1 - q_2$ is replaced by our function $f$ which is in $L^p(\Omega)$ for $1 \leq p \leq \infty$) gives the required result. $\square$

We refer to the survey [Uhl09] for more details and references on CGOs.

Before the proof, we need to discuss a minor issue: the equation $\Delta u + a(x, u) = 0$ involves real valued solutions ($a$ is defined on $\overline{\Omega} \times \mathbb{R}$), whereas exponential solutions and CGOs are complex valued. However, in the proof we can just use the real and imaginary parts of these solutions (which are solutions themselves, since the coefficients are real valued) by virtue of the following simple lemma.

**Lemma 2.2.** *Let $f \in L^\infty(\Omega)$, $v_1, v_2 \in L^2(\Omega)$, and $v_3, \ldots, v_m \in L^\infty(\Omega)$ be complex valued functions where $m \geq 2$. Then*

$$\int_\Omega f v_1 \cdots v_m \, dx = \sum_{j=1}^{2^m} \int_\Omega c_j f w_1^{(j)} \cdots w_m^{(j)} \, dx$$

*where $c_j \in \{\pm 1, \pm i\}$ and $w_1^{(j)} \in \{\mathrm{Re}(v_1), \mathrm{Im}(v_1)\}, \cdots, w_m^{(j)} \in \{\mathrm{Re}(v_m), \mathrm{Im}(v_m)\}$ for $1 \leq j \leq 2^m$.*

*Proof.* The result follows by writing

$$\int_\Omega f v_1 \cdots v_m \, dx = \int_\Omega f(\mathrm{Re}(v_1) + i\mathrm{Im}(v_1)) \cdots (\mathrm{Re}(v_m) + i\mathrm{Im}(v_m)) \, dx$$

and by multiplying out the right hand side. $\qquad\square$

Now, we can prove Theorem 1.1.

*Proof of Theorem 1.1.* We split the proof into two parts, where in the first part we assume that the linear terms of the operators $\Delta + a_j(x, z)$ vanish: $\partial_z a_j(x, 0) \equiv 0$, $j = 1, 2$. The proof in this case is based on Calderón's exponential solutions. In the second part we consider the case $\partial_z a_j(x, 0) \neq 0$ and use CGOs instead of Calderón's exponential solutions.

*Case 1.* $\partial_z a_j(x, 0) \equiv 0$.

The proof is by induction on the order of the order of differentiation $k \in \mathbb{N}$. By assumption, we have that

$$\partial_z a_1(x, 0) = 0 = \partial_z a_2(x, 0).$$

Let then $N \in \mathbb{N}$ and assume that

(2.4) $$\partial_z^k a_1(x, 0) = \partial_z^k a_2(x, 0) \text{ for all } k = 1, 2, \cdots, N.$$

The induction step is to show that (2.4) holds for $k = N + 1$.

For $\ell = 1, \ldots, N + 1$, let $\epsilon_\ell$ be small positive real numbers, and let $f_\ell \in C^s(\partial\Omega)$ be functions on the boundary. Let us denote $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{N+1})$ and let the function

$$u_j := u_j(x; \epsilon), \quad j = 1, 2,$$

be the unique small solution of the Dirichlet problem

(2.5) $$\begin{cases} \Delta u_j + a_j(x, u_j) = 0 & \text{in } \Omega, \\ u_j = \sum_{\ell=1}^{N+1} \epsilon_\ell f_\ell & \text{on } \partial\Omega. \end{cases}$$

The existence of the unique small solution (by redefining $\epsilon_\ell$ to be smaller if necessary) is given in Appendix A together with an explanation what unique small solutions mean. To prove the induction step, we will differentiate the equation (2.5) with respect to the $\epsilon_\ell$ parameters several times. The differentiation is justified by Proposition A.2.

We begin with the first order linearization as follows. Let us differentiate (2.5) with respect to $\epsilon_\ell$, so that

(2.6) $$\begin{cases} \Delta \left( \frac{\partial}{\partial \epsilon_\ell} u_j \right) + \partial_z a_j(x, u_j) \left( \frac{\partial}{\partial \epsilon_\ell} u_j \right) = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial \epsilon_\ell} u_j = f_\ell & \text{on } \partial\Omega. \end{cases}$$

Evaluating (2.6) at $\epsilon = 0$ shows that

$$\Delta v_j^{(\ell)} = 0 \text{ in } \Omega \text{ with } v_j^{(\ell)} = f_\ell \text{ on } \partial\Omega,$$

where

$$v_j^{(\ell)}(x) = \frac{\partial}{\partial \epsilon_\ell}\Big|_{\epsilon=0} u_j(x; \epsilon).$$

Here we have used $u_j(x; \epsilon)|_{\epsilon=0} \equiv 0$ so that $\partial_z a_j(x, u_j)|_{\epsilon=0} \equiv 0$ in $\Omega$. The functions $v_j^{(\ell)}$ are harmonic functions defined in $\Omega$ with boundary data $f_\ell|_{\partial\Omega}$. By uniqueness of the Dirichlet problem for the Laplace operator we have that

(2.7) $$v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)} \text{ in } \Omega \quad \text{for } \ell = 1, 2, \cdots, N + 1.$$

For illustrative purposes we show next how to prove that $\partial_z^2 a_1(x,0) = \partial_z^2 a_2(x,0)$, which corresponds to the special case $N = 1$. The second order linearization is given by differentiating (2.6) with respect to $\epsilon_k$ for arbitrary $k \neq \ell$ where $k, \ell \in \{1, 2, \cdots, N+1\}$. Doing so yields

(2.8)
$$\begin{cases} \Delta \left( \frac{\partial^2}{\partial \epsilon_k \partial \epsilon_\ell} u_j \right) + \partial_z a_j(x, u_j) \left( \frac{\partial^2}{\partial \epsilon_k \partial \epsilon_\ell} u_j \right) + \partial_z^2 a(x, u_j) \left( \frac{\partial u_j}{\partial \epsilon_k} \right) \left( \frac{\partial u_j}{\partial \epsilon_\ell} \right) = 0 & \text{in } \Omega, \\ \frac{\partial^2}{\partial \epsilon_k \partial \epsilon_\ell} u_j = 0 & \text{on } \partial\Omega. \end{cases}$$

By evaluating (2.8) at $\epsilon = 0$ we have that

(2.9)
$$\begin{cases} \Delta w_j^{(k\ell)} + \partial_z^2 a_j(x,0) v^{(k)} v^{(\ell)} = 0 & \text{in } \Omega, \\ w_j^{(k\ell)} = 0 & \text{on } \partial\Omega, \end{cases}$$

where we have denoted $w_j^{(k\ell)}(x) = \frac{\partial^2}{\partial \epsilon_k \partial \epsilon_\ell} u_j(x; \epsilon) \big|_{\epsilon=0}$ and used $u_j(x; \epsilon)|_{\epsilon=0} \equiv 0$ in $\Omega$ for $j = 1, 2$. By using the fact that $\Lambda_{a_1} \left( \sum_{\ell=1}^{N+1} \epsilon_\ell f_\ell \right) = \Lambda_{a_2} \left( \sum_{\ell=1}^{N+1} \epsilon_\ell f_\ell \right)$, we have that

(2.10)
$$\partial_\nu u_1|_{\partial\Omega} = \partial_\nu u_2|_{\partial\Omega}.$$

By applying $\partial_{\epsilon_k} \partial_{\epsilon_\ell}|_{\epsilon=0}$ to the equation (2.10) above shows that

$$\partial_\nu w_1^{(k\ell)} \Big|_{\partial\Omega} = \partial_\nu w_2^{(k\ell)} \Big|_{\partial\Omega}, \text{ for } k, \ell = 1, \ldots, N+1.$$

(We remind that this formal looking calculation is justified by [LLLS19, Proposition 2.1].) Hence, by integrating the equation (2.9) over $\Omega$ and by using integration by parts we obtain the equation

$$0 = \int_{\partial\Omega} \left( \partial_\nu w_1^{(k\ell)} - \partial_\nu w_2^{(k\ell)} \right) dS = \int_\Omega \Delta \left( w_1^{(k\ell)} - w_2^{(k\ell)} \right) dx$$

(2.11)
$$= \int_\Omega \left( \partial_z^2 a_2(x,0) - \partial_z^2 a_1(x,0) \right) v^{(k)} v^{(\ell)} dx$$

where $v^{(k)}$ and $v^{(\ell)}$ are defined in (2.7). (More generally, as in [LLLS19] we could as well have integrated against a third harmonic function $v^{(m)}$.) Therefore, by choosing $f_k$ and $f_\ell$ as the boundary values of the real or imaginary parts of Calderón's exponential solutions $v_1$ and $v_2$ in (2.1) (note that the real and imaginary parts of $v_1$ and $v_2$ are also harmonic), and by using Lemma 2.2, we obtain that

$$\int_\Omega \left( \partial_z^2 a_2(x,0) - \partial_z^2 a_1(x,0) \right) v_1 v_2 \, dx = 0.$$

It follows that the Fourier transform of the difference $\partial_z^2 a_1(x,0) - \partial_z^2 a_2(x,0)$ is zero. Thus $\partial_z^2 a_1(x,0) = \partial_z^2 a_2(x,0)$. We define

(2.12)
$$\partial_z^2 a(x,0) := \partial_z^2 a_1(x,0) = \partial_z^2 a_2(x,0).$$

We also note that by using (2.12), the equation (2.9) shows that the function $w_1^{(k\ell)} - w_2^{(k\ell)}$ solves

$$\Delta \left( w_1^{(k\ell)} - w_2^{(k\ell)} \right) = 0, \text{ with } w_1^{(k\ell)} - w_2^{(k\ell)} = 0 \text{ on } \partial\Omega.$$

Thus we have that

(2.13)
$$w^{(k\ell)} := w_1^{(k\ell)} = w_2^{(k\ell)} \text{ in } \Omega.$$

We have now shown how to prove the special case of how to go from $N = 1$ to $N = 2$. Let us return to the general case $N \in \mathbb{N}$. To prove the general case, we first show by induction within induction, call it subinduction, that

$$(2.14) \qquad \frac{\partial^k u_1(x;0)}{\partial \epsilon_{\ell_1} \cdots \partial \epsilon_{\ell_k}} = \frac{\partial^k u_2(x;0)}{\partial \epsilon_{\ell_1} \cdots \partial \epsilon_{\ell_k}} \text{ in } \Omega,$$

for all $k = 1, \ldots, N$. The claim holds for $k = 1$ by (2.7). Let us then assume that (2.14) holds for all $k \leq K < N$. The linearization of order $K + 1$ evaluated at $\epsilon = 0$ reads

$(2.15)$
$$\Delta \left( \frac{\partial^{K+1} u_j(x,0)}{\partial \epsilon_{\ell_1} \cdots \partial \epsilon_{\ell_{K+1}}} \right) + R_K(u_j, a_j, 0) + \partial_z^{K+1} a_j(x,0) \left( \Pi_{k=1}^{K+1} v^{(\ell_k)} \right) = 0 \text{ in } \Omega,$$

where $R_K(u_j, a_j, 0)$ is a polynomial of the functions $\partial_z^k a_j(x,0)$ and $\frac{\partial^k u_j(x;0)}{\partial \epsilon_{\ell_1} \cdots \partial \epsilon_{\ell_k}}$ for all $k \leq K$. By the induction assumptions (2.4) and (2.14) these functions agree for $j = 1, 2$. Thus it follows that

$$\begin{cases} \Delta \left( \partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_{K+1}}}^{K+1} u_1(x,0) - \partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_{K+1}}}^{K+1} u_2(x,0) \right) = 0 & \text{in } \Omega \\ \partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_{K+1}}}^{K+1} u_1(x,0) - \partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_{K+1}}}^{K+1} u_2(x,0) = 0 & \text{on } \partial\Omega. \end{cases}$$

(Above we have used the abbreviation $\partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_{K+1}}}^{K+1} u_j(x,0) = \frac{\partial^{K+1} u_j(x,0)}{\partial \epsilon_{\ell_1} \cdots \partial \epsilon_{\ell_{K+1}}}$ for $j = 1, 2$ and for $K \in \mathbb{N}$, which will also be used later in the proof). Thus by the uniqueness of solutions to the Laplace equation we have that (2.14) holds for $k = 1, \ldots, K + 1$, which concludes the induction step of the subinduction. Thus (2.14) holds for all $k = 1, \ldots, N$.

Let us then continue with the main induction argument of the proof. The linearization of order $N + 1$ at $\epsilon = 0$ yields the equation (2.15) with $N$ in place of $K$. By the subinduction, we have that $R_N(u_1, a_1, 0) = R_N(u_2, a_2, 0)$. By using this fact, it follows by subtracting the equations (2.15) with $j = 1$ and $j = 2$ from each other (with $K = N$) that

$$\int_\Omega \left( \partial_z^{N+1} a_1(x,0) - \partial_z^{N+1} a_2(x,0) \right) \left( \Pi_{k=1}^{N+1} v^{(\ell_k)} \right) dx = 0.$$

Here we used integration by parts and the assumption $\Lambda_{a_1} = \Lambda_{a_2}$. We choose two of the functions $v^{(\ell_k)}$ to be the real or imaginary parts of the exponential solutions (2.1), and the remaining $N - 1$ of them to be the constant function 1. Using Lemma 2.2 again, it follows that $\partial_z^{N+1} a_1(x,0) = \partial_z^{N+1} a_2(x,0)$ in $\Omega$ as desired. This concludes the main induction step.

*Case 2.* $\partial_z a_j(x,0) \not\equiv 0$.

The proof is similar to the Case 1, and therefore we keep exposition short. As said before, the main difference is that we use CGOs (2.2) instead of Calderón's exponential solutions (2.1). We consider $\epsilon_\ell$ to be small numbers, $\ell = 1, 2, \cdots, N+1$, and $\epsilon = (\epsilon_1, \cdots, \epsilon_{N+1})$ and $f_\ell \in C^s(\partial\Omega)$, for all $\ell = 1, 2, \cdots, N+1$. Let the function $u_j := u_j(x; \epsilon)$ be the unique small solution of

$$\begin{cases} \Delta u_j + a_j(x, u_j) = 0 & \text{in } \Omega, \\ u_j = \sum_{\ell=1}^{N+1} \epsilon_\ell f_\ell & \text{on } \partial\Omega, \end{cases}$$

for $j = 1, 2$. We begin with the first order linearization as follows, which at $\epsilon = 0$ yields:

(2.16)
$$\begin{cases} (\Delta + \partial_z a_j(x, 0)) \, v_j^{(\ell)} = 0 & \text{in } \Omega, \\ v_j^{(\ell)} = f_\ell & \text{on } \partial\Omega, \end{cases}$$

where

$$v_j^{(\ell)}(x) = \left. \frac{\partial}{\partial \epsilon_\ell} \right|_{\epsilon=0} u_j(x; \epsilon) \,.$$

The functions $v_j^\ell$ are the solutions of the Schrödinger equation with potential $\partial_z a_j(x, 0)$ in $\Omega$ with boundary data $f_\ell|_{\partial\Omega}$.

We show that $\partial_z a_1(x, 0) = \partial_z a_2(x, 0)$ for $x \in \Omega$. Since the DN maps $\Lambda_{a_1}$ and $\Lambda_{a_2}$ agree, we have by [LLLS19, Proposition 2.1] that the DN maps corresponding to the equation (2.16) are the same. Let us consider the function

$$g(x) := \partial_z a_1(x, 0) - \partial_z a_2(x, 0) \in L^\infty(\Omega),$$

then by Proposition 2.1, it is easy to see that $g = 0$ for $n \geq 3$. On the other hand, by using the boundary determination (for example, see [GT11, Appendix]), then one has $\partial_z a_1(x, 0) = \partial_z a_2(x, 0)$ for $x \in \partial\Omega$ for $n = 2$. Now, with this $g = 0$ on $\partial\Omega$ at hand, by using Proposition 2.1 again, it follows that $g = 0$ in $\Omega \subset \mathbb{R}^2$, which implies that

(2.17)
$$\partial_z a_1(x, 0) = \partial_z a_2(x, 0).$$

Moreover, by using (2.17) and the uniqueness of solutions to the Dirichlet problem (2.16), we have that

(2.18)
$$v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)} \text{ in } \Omega \text{ for } \ell = 1, 2, \cdots, N+1,$$

and we simply denote

$$q(x) := \partial_z a_1(x, 0) = \partial_z a_2(x, 0), \text{ for } x \in \Omega.$$

Here we used the assumption (1.2), which says that operators $\Delta + \partial_z a_j(x, 0)$ are injective on $H_0^1(\Omega)$, , $j = 1, 2$.

Since $\partial_z a_1(x, 0) = \partial_z a_2(x, 0)$, we have that the claim (1.5) of the theorem holds for $k = 1$. We proceed by induction on $k$. To do that, we assume that (1.5) holds for all $k = 1, \ldots, N$. Again, we do the $N = 1$ case separately to explain how the induction works. The second order linearization yields the equations for $j = 1, 2$:

(2.19)
$$\begin{cases} \Delta w_j^{(k\ell)} + q(x) w_j^{(k\ell)} + \partial_z^2 a_j(x, 0) v^{(k)} v^{(\ell)} = 0 & \text{in } \Omega, \\ w_j^{(k\ell)} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $w_j^{(k\ell)}(x) = \left. \frac{\partial^2}{\partial \epsilon_k \partial \epsilon_\ell} u_j(x; \epsilon) \right|_{\epsilon=0}$ and we used $u_j(x; \epsilon)|_{\epsilon=0} \equiv 0$ in $\Omega$. Since $\Lambda_{a_1} = \Lambda_{a_2}$, we have (as in Case 1) that

$$\left. \partial_\nu w_1^{(k\ell)} \right|_{\partial\Omega} = \left. \partial_\nu w_2^{(k\ell)} \right|_{\partial\Omega}, \text{ for } k, \ell \in 1, \ldots, N.$$

Fix $x_0 \in \Omega$. We claim that there exists a solution $v^{(0)} \in H^s(\Omega)$, where $s$ can be chosen arbitrarily large, of the Schrödinger equation

(2.20)
$$\Delta v^{(0)} + q(x) v^{(0)} = 0 \text{ in } \Omega$$

with

$$v^{(0)}(x_0) \neq 0.$$

By the Runge approximation property (see e.g. [LLS19, Proposition A.2]), it is enough to construct such a solution in some small neighborhood $U$ of $x_0$. Since $q$ is smooth, by a perturbation argument it is enough to construct a nonvanishing

solution of $\Delta w + q(x_0)w = 0$ near $x_0$. Writing $q(x_0) = \lambda^2$ for some complex number $\lambda$, it is enough to take $w = e^{i\lambda x_1}$. This completes the construction of $v^{(0)}$.

Now, multiplying (2.19) by $v^{(0)}$ and integrating by parts yields that

$$
\begin{aligned}
0 &= \int_{\partial\Omega} v^{(0)} \partial_\nu \left( w_1^{(k\ell)} - w_2^{(k\ell)} \right) dS \\
&= \int_\Omega v^{(0)} \Delta \left( w_1^{(k\ell)} - w_2^{(k\ell)} \right) dx + \int_\Omega \nabla v^{(0)} \cdot \nabla \left( w_1^{(k\ell)} - w_2^{(k\ell)} \right) dx \\
\text{(2.21)} \quad &= \int_\Omega q(x)(w_2^{(k\ell)} - w_1^{(k\ell)})v^{(0)} \, dx + \int_\Omega \left( \partial_z^2 a_2 - \partial_z^2 a_1 \right) v^{(k)} v^{(\ell)} v^{(0)} \, dx \\
&\quad - \int_\Omega \left( w_1^{(k\ell)} - w_2^{(k\ell)} \right) \Delta v^{(0)} \, dx \\
&= \int_\Omega \left( \partial_z^2 a_2(x,0) - \partial_z^2 a_1(x,0) \right) v^{(k)} v^{(\ell)} v^{(0)} \, dx.
\end{aligned}
$$

Here $v^{(k)}$ and $v^{(\ell)}$ are harmonic functions defined (2.18), which we now choose specifically to be real or imaginary parts of the CGOs (since $q$ is real valued, the real and imaginary parts of CGOs are also solutions of $\Delta v + qv = 0$). Then, by using Lemma 2.2 we can reduce to the case where $v^{(k)}$ and $v^{(\ell)}$ are the actual complex valued CGOs, and by applying the completeness of products of pairs of CGOs [Buk08, SU87] we obtain that

$$\partial_z^2 a_2(x,0)v^{(0)}(x) = \partial_z^2 a_1(x,0)v^{(0)}(x) \text{ for } x \in \Omega.$$

In particular, when $x = x_0$, we have $\partial_z^2 a_2(x_0,0) = \partial_z^2 a_1(x_0,0)$ since $v^{(0)}(x_0) \neq 0$. Since $x_0 \in \Omega$ was arbitrary, we have that

$$\text{(2.22)} \quad \partial_z^2 a(x,0) := \partial_z^2 a_1(x,0) = \partial_z^2 a_2(x,0) \text{ for } x \in \Omega.$$

This concludes the induction step in the special case of how to go from $N = 1$ to $N = 2$. We also have from (2.19) and (2.22) that $w_1^{(k\ell)} - w_2^{(k\ell)}$ solves

$$
\begin{cases}
\Delta \left( w_1^{(k\ell)} - w_2^{(k\ell)} \right) + q(x) \left( w_1^{(k\ell)} - w_2^{(k\ell)} \right) = 0 & \text{in } \Omega, \\
w_1^{(k\ell)} - w_2^{(k\ell)} = 0 & \text{on } \partial\Omega,
\end{cases}
$$

then the uniqueness of the solution to the Schrödinger equation yields that

$$w^{(k\ell)} := w_1^{(k\ell)} = w_2^{(k\ell)} \text{ in } \Omega.$$

Let us return to general case $N \in \mathbb{N}$. As in the Case 1, we first prove by the subinduction that

$$\partial_{\ell_1 \cdots \ell_k}^k u_1(x;0) = \partial_{\ell_1 \cdots \ell_k}^k u_2(x;0) \text{ in } \Omega,$$

for all $k \leq N$. Then the linearization for $j = 1, 2$ of order $N + 1$ shows that

(2.23)

$$(\Delta + q)\left( \partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_{N+1}}}^{N+1} u_j(x,0) \right) + R_N(u_j, a_j, 0) + \partial_z^{N+1} a_j(x,0) \left( \Pi_{k=1}^{N+1} v^{(\ell_k)} \right) = 0,$$

for $x \in \Omega$, and where $R_N(u_j, a_j, 0)$ is a polynomial of the functions $\partial_z^k a_j(x,0)$ and $\partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_k}}^k u_j(x;0)$ for $k \leq N$. By the subinduction we have that $R_N(u_1, a_1, 0) = R_N(u_2, a_2, 0)$.

Finally, by multiplying (2.23) by $v^{(0)}$ and repeating an integration by parts argument similar to that in (2.21) shows that we have the following integral identity

$$\int_\Omega \left( \partial_z^{N+1} a_1(x,0) - \partial_z^{N+1} a_2(x,0) \right) \left( \Pi_{k=1}^{N+1} v^{(\ell_k)} \right) v^{(0)} dx = 0.$$

With the help of Lemma 2.2 we can choose $v^{(\ell_1)}$ and $v^{(\ell_2)}$ to be the CGOs as before, and we choose the remaining $N - 1$ solutions as $v^{(\ell_3)} = \cdots = v^{(\ell_{N+1})} = v^{(0)}$, where

$v^{(0)}$ is the solution in (2.20). We conclude that $\partial_z^{N+1} a_1(x_0, 0) = \partial_z^{N+1} a_2(x_0, 0)$. Since $x_0 \in \Omega$ was arbitrary, we obtain that $\partial_z^{N+1} a_1(x, 0) = \partial_z^{N+1} a_2(x, 0)$ in $\Omega$. This concludes the proof. $\qquad\square$

**Remark 2.3.** *In the proof of Theorem 1.1, we have used the Runge approximation property to construct solutions to the Schrödinger equation that are nonzero at a given point $x_0$. An alternative method is to construct a nonvanishing solution of $\Delta v + q(x)v = 0$. This can be done by considering a complex geometrical optics solution*

$$v(x) = e^{\rho \cdot x}(1 + r) \ in \ \Omega,$$

*where $\rho \in \mathbb{C}^n$. Then $r$ solves*

$$e^{-\rho \cdot x}(\Delta + q)e^{\rho \cdot x} r = -q \ in \ \Omega,$$

*with the estimate (see [SU87, Theorem 1.1], the argument applies also in our case when $n \geq 2$)*

$$\|r\|_{H^s(\Omega)} \leq \frac{C}{|\rho|}\|q\|_{H^s(\Omega)},$$

*for $s > n/2$ and $|\rho|$ large enough. Then by the Sobolev embedding we have that*

$$\|r\|_{L^\infty(\Omega)} \leq \frac{1}{2},$$

*for $|\rho|$ large enough. This implies that $v(x)$ is nonvanishing in $\Omega$, and the solution $v^{(0)}$ in the proof of Theorem 1.1 could be replaced by $v$ here.*

*In the case $q(x) \leq 0$ in $\Omega$ (with the sign convention $\Delta = \sum_{k=1}^n \partial_{x_k}^2$), another alternative is to apply the maximum principle to construct a positive solution to the Schrödinger equation in $\Omega$.*

Furthermore, when the coefficient $a = a(x, z)$ of the operator $\Delta + a(x, \cdot)$ satisfies

$$\partial_z a(x, 0) \equiv 0$$

one has the following reconstruction result.

**Theorem 2.1** (Reconstruction)**.** *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^\infty$ boundary $\partial\Omega$. Let $\Lambda_a$ be the DN map of the equation*

$$\Delta u + a(x, u) = 0 \ in \ \Omega,$$

*and assume that $a(x, z) \in C^\infty(\overline{\Omega} \times \mathbb{R})$ satisfies (1.3). Then we can reconstruct $\partial_z^k a(x, 0)$ from the knowledge of $\Lambda_a$, for all $k \geq 2$.*

*Proof.* For $k = 2$, the reconstruction formula can be easily obtained by reviewing the argument between the equations (2.9) and (2.11). Formally we have

$$\widehat{\partial_z^2 a}(\cdot, 0)(-2\xi) = -\int_{\partial\Omega} \frac{\partial^2}{\partial\epsilon_1 \partial\epsilon_2}\bigg|_{\epsilon_1 = \epsilon_2 = 0} \Lambda_a \left(\epsilon_1 f_1 + \epsilon_2 f_2\right) dS,$$

which reconstructs the coefficient $\partial_z^2 a(x, 0)$. Here $\widehat{\partial_z^2 a}$ denotes the Fourier transformation of $a(x, z)$ in the $x$-variable, and $f_1$ and $f_2$ are the boundary values of the Calderón's exponential solutions (2.1). More precisely, we can take $f_1$ and $f_2$ to be the real or imaginary parts of the boundary values of the solutions (2.1), and we can then use a suitable combination as in Lemma 2.2 to recover $\widehat{\partial_z^2 a}(\cdot, 0)(-2\xi)$. Moreover, by using (2.9), one can solve the boundary value problem (2.9) uniquely to construct the function $w^{(kl)}$ given by (2.13).

The proof for general $k$ is by recursion, but let us show separately how to reconstruct $\partial_z^3 a(x,z)$ corresponding to $k = 3$. To reconstruct $\partial_z^3 a(x,0)$, we apply third order linearization for the equation

$$\begin{cases} \Delta u + a(x, u_j) = 0 & \text{in } \Omega, \\ u = \sum_{\ell=1}^{N+1} \epsilon_\ell f_\ell & \text{on } \partial\Omega, \end{cases}$$

at $\epsilon = (\epsilon_1, \ldots, \epsilon_{N+1}) = 0$, where $\epsilon_\ell$ are small and $f_\ell \in C^s(\partial\Omega)$. For $k = 3$, we can take $N$ to be 2. This shows that

$$\Delta w^{(ik\ell)} + \partial_z^2 a(x,0) \left( w^{(ik)} v^{(\ell)} + w^{(i\ell)} v^{(k)} + w^{(k\ell)} v^{(i)} \right)$$

$$(2.24) \qquad + \partial_z^3 a(x,0) \left( v^{(i)} v^{(k)} v^{(\ell)} \right) = 0 \qquad\qquad \text{in } \Omega,$$

holds, where $w^{(ik\ell)}(x) = \frac{\partial^3}{\partial\epsilon_i \partial\epsilon_k \partial\epsilon_\ell} u(x; 0)$. An integration by parts formula now yields that

$$\int_{\partial\Omega} \frac{\partial^3}{\partial\epsilon_i \partial\epsilon_k \partial\epsilon_\ell}\bigg|_{\epsilon_i = \epsilon_k = \epsilon_\ell = 0} \Lambda_a \left( \epsilon_i f_i + \epsilon_k f_k + \epsilon_\ell f_\ell \right) dS$$

$$+ \int_\Omega \partial_z^2 a(x,0) \left( w^{(ik)} v^{(\ell)} + w^{(i\ell)} v^{(k)} + w^{(k\ell)} v^{(i)} \right) dx$$

$$= - \int_\Omega \partial_z^3 a(x,0) v^{(i)} v^{(k)} v^{(\ell)} dx$$

Let $v^{(i)}$ and $v^{(k)}$ be real or imaginary parts of Calderón's exponential solutions (2.1) and let $v^{(\ell)} = 1$. By using Lemma 2.2 and the fact that we have already reconstructed $\partial_z^2 a(x,0)$ and $w^{(k\ell)}$, we see that we can reconstruct the Fourier transform of $\partial_z^3 a(x,0)$. Consequently, we know the all the coefficients of the equation (2.24) for $w^{(ik\ell)}$, thus we may solve (2.24) to reconstruct also $w^{(ik\ell)}$. (The boundary value for $w^{(ik\ell)}$ is 0.)

To reconstruct $\partial_z^k a_j(x,0)$ for any $k \in \mathbb{N}$, one proceeds recursively. Let us assume that we have reconstructed $\partial_z^k a(x,0)$ and $w^{(\ell_1 \cdots \ell_k)}$ for all $k \leq N$. The linearization of order $N + 1$ then yields that (cf. (2.15))

$$(2.25) \quad \begin{cases} \Delta w^{(\ell_1 \cdots \ell_{N+1})} + R_N(u, a, 0) + \partial_z^{N+1} a(x,0) \left( \Pi_{k=1}^{N+1} v^{(\ell_k)} \right) = 0 & \text{in } \Omega, \\ w^{(\ell_1 \cdots \ell_{N+1})} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $w^{(\ell_1 \cdots \ell_{N+1})}(x) = \partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_{N+1}}}^{N+1} u(x,0)$ and where $R_N(u, a, 0)$ is a polynomial of the functions $\partial_z^k a(x,0)$ and $\partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_k}}^k u(x,0)$ for $k \leq N$. By the recursion assumption we have thus already recovered $R_N(u, a, 0)$. Finally, integrating the equation (2.25) over $\Omega$ shows that

$$\int_{\partial\Omega} \frac{\partial^{N+1}}{\partial\epsilon_{\ell_1} \cdots \partial\epsilon_{\ell_{N+1}}}\bigg|_{\epsilon=0} \Lambda_a \left( \sum_{k=1}^{N+1} \epsilon_{\ell_k} f_{\ell_k} \right) dS + \int_\Omega R_N(u, a, 0)\, dx$$

$$= - \int_\Omega \partial_z^L a(x,0) \left( \Pi_{\ell=1}^L v^{(i_\ell)} \right) dx.$$

We choose $v^{(\ell_1)}$, $v^{(\ell_2)}$ to be real or imaginary parts of exponential solutions (2.1) and $v^{(\ell_3)} = \cdots = v^{(\ell_{N+1})} = 1$ in $\Omega$. Using Lemma 2.2 again, this recovers $\partial_z^{N+1} a(x,0)$. To end the reconstruction argument, we insert the now reconstructed $\partial_z^{N+1} a(x,0)$ into (2.25) and solve the equation for $w^{(\ell_1 \cdots \ell_{N+1})}$ with zero Dirichlet boundary value.                                                                                    $\square$

## 3. Simultaneous recovery of cavity and coefficients

We prove Theorem 1.2 by first recovering the cavity $D$ from the first linearization of the equation
$$\Delta u(x) + a(x, u) = 0.$$
After that the function $a = a(x, z)$ is recovered by higher order linearization.

*Proof of Theorem 1.2.* Let $f = \sum_{\ell=1}^{N+1} \epsilon_\ell f_\ell$, where $\epsilon_\ell$ are sufficiently small numbers and let $f_\ell \in C^s(\partial\Omega)$ for all $\ell = 1, 2, \cdots, N + 1$. We denote $\epsilon = (\epsilon_1, \ldots, \epsilon_{N+1})$ and let $u_j(x) = u_j(x; \epsilon)$ be the solution of

$$(3.1) \qquad \begin{cases} \Delta u_j + a_j(x, u_j) = 0 & \text{in } \Omega \setminus \overline{D_j}, \\ u_j = 0 & \text{on } \partial D_j, \\ u_j = f & \text{on } \partial\Omega \end{cases}$$

with $f = \sum_{\ell=1}^{N+1} \epsilon_\ell f_\ell$, $j = 1, 2$.

*Step 1. Recovering the cavity.*

Let us differentiate (3.1) with respect to $\epsilon_\ell$, for $\ell = 1, \cdots, N + 1$. We obtain

$$(3.2) \qquad \begin{cases} \Delta \left( \frac{\partial}{\partial \epsilon_\ell} u_j \right) + \partial_z a_j(x, u_j) \left( \frac{\partial}{\partial \epsilon_\ell} u_j \right) = 0 & \text{in } \Omega \setminus \overline{D_j}, \\ \frac{\partial}{\partial \epsilon_\ell} u_j = 0 & \text{on } \partial D_j, \\ \frac{\partial}{\partial \epsilon_\ell} u_j = f_\ell & \text{on } \partial\Omega, \end{cases}$$

for all $\ell = 1, 2, \cdots, N + 1$ and $j = 1, 2$. Note that by (1.3), the function $u_j(x; 0)$ solves (3.1) with zero Dirichlet condition $\partial\Omega$ and $\partial D_j$. Thus we have $u_j(x; 0) \equiv 0$ in $\Omega \setminus \overline{D_j}$, for $j = 1, 2$. By letting $\epsilon = 0$ and by denoting $v_j^{(\ell)}(x) := \frac{\partial}{\partial \epsilon_\ell}\big|_{\epsilon=0} u_j$, the equation (3.2) becomes

$$(3.3) \qquad \begin{cases} \Delta v_j^{(\ell)} = 0 & \text{in } \Omega \setminus \overline{D_j}, \\ v_j^{(\ell)} = 0 & \text{on } \partial D_j, \\ v_j^{(\ell)} = f_\ell & \text{on } \partial\Omega. \end{cases}$$

We show that $D_1 = D_2$. This follows by a standard argument (see for instance [BV99, ABRV00]), but we include a proof for completeness. Let $G$ be the connected component of $\Omega \setminus (\overline{D_1 \cup D_2})$ whose boundary contains $\partial\Omega$ and let $\widetilde{v}^{(\ell)} := v_1^{(\ell)} - v_2^{(\ell)}$. Then $\widetilde{v}^{(\ell)}$ solves

$$\begin{cases} \Delta \widetilde{v}^{(\ell)} = 0 & \text{in } G, \\ \widetilde{v}^{(\ell)} = \partial_\nu \widetilde{v}^{(\ell)} = 0 & \text{on } \partial\Omega \end{cases}$$

since $\Lambda_{a_1}^{D_1}(f) = \Lambda_{a_2}^{D_2}(f)$ on $\partial\Omega$ for small $f$. By the unique continuation principle for harmonic functions, one has that $\widetilde{v}^{(\ell)} = 0$ in $G$. Thus

$$(3.4) \qquad v_1^{(\ell)} = v_2^{(\ell)} \text{ in } G,$$

for $\ell = 1, \ldots, N + 1$. In order to prove the uniqueness of the cavity, $D_1 = D_2$, one needs only to consider the case $\ell = 1$ of the problem (3.3). However, we need to consider all $\ell = 1, \cdots, N + 1$ to recover the coefficient $a(x, z)$.

We now argue by contradiction and assume that $D_1 \neq D_2$. Note that $G \neq \emptyset$, which satisfies the assumptions of Lemma A.3, then the next step is to apply Lemma A.3 in the appendix with the choices $\Omega_1 = \Omega \setminus \overline{D_1}$, $\Omega_2 = \Omega \setminus \overline{D_2}$ and $\Gamma = \partial\Omega$ (note that $\Omega_j$ and $\Gamma$ are connected by our assumptions). It follows, after interchanging $D_1$ and $D_2$ if necessary, that there exists a point $x_1$ such that

$$x_1 \in \partial G \cap (\Omega \setminus \overline{D_1}) \cap \partial D_2.$$

Since $x_1 \in \partial D_2$, we have $v_2^{(\ell)}(x_1) = 0$. By (3.4) and continuity, we also have that $v_1^{(\ell)}(x_1) = 0$. The point $x_1$ is an interior point of the open set $\Omega \setminus \overline{D_1}$. Let us fix one of the boundary values $f_\ell$ to be non-negative and not identically 0. Now, since $v_1^{(\ell)}(x_1) = 0$, the maximum principle implies that $v_1^{(\ell)} \equiv 0$ in the connected open set $\Omega \setminus \overline{D_1}$. This is in contradiction with the assumption that $v_1^{(\ell)} = f_\ell$ on $\partial \Omega$ is non-vanishing (since $v_1^{(\ell)}$ is continuous up to boundary). This shows that $D := D_1 = D_2$. Moreover, we have by (3.4) that

$$(3.5) \qquad v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)} \text{ in } \Omega \setminus \overline{D},$$

for all $\ell = 1, \cdots, N + 1$ as desired.

*Step 2. Recovering the coefficient.*

In order to prove the claim

$$(3.6) \qquad \partial_z^k a_1(x, 0) = \partial_z^k a_2(x, 0), \quad k \in \mathbb{N}$$

of the theorem, we proceed by induction similar to the proof of Theorem 1.1. The equation (3.6) is true for $k = 1$ by assumption. Let us make the induction assumption that (3.6) holds for $k \leq N$, and assume also that

$$(3.7) \qquad \partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_k}}^k u_1(x, 0) = \partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_k}}^k u_2(x, 0) \text{ for all } k \leq N,$$

then we want to show that (3.6) holds for $k = N + 1$. This equation holds for $k = 1$ by (3.5).

By differentiating $N + 1$ times the equation (3.1) with respect to the parameters $\epsilon_{\ell_1}, \ldots, \epsilon_{\ell_{N+1}}$ for $j = 1, 2$, and by subtracting the results from each other shows that in $\Omega \setminus \overline{D}$ one has

$$(3.8)$$
$$\Delta \partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_{N+1}}}^{N+1} (u_1(x, 0) - u_2(x, 0)) + \partial_z^{N+1} (a_1(x, 0) - a_2(x, 0)) \left( \Pi_{k=1}^{N+1} v^{(\ell_k)} \right) = 0$$

Here we used (3.6) for $k \leq N$ and (3.7) to deduce that the terms with derivatives of order $\leq N$ vanish in the subtraction. We also have $\partial_{\epsilon_{\ell_1} \cdots \epsilon_{\ell_{N+1}}}^{N+1} (u_1(x, 0) - u_2(x, 0)) = 0$ on $\partial\Omega \cup \partial D$.

We know the DN map only on $\partial\Omega$, but not on $\partial D$. Therefore integrating (3.8) and using integration by parts would produce an unknown integral over $\partial D$. To compensate for the lack of knowledge on $\partial D$, we proceed as follows. Let $v^{(0)}$ be the solution of

$$(3.9) \qquad \begin{cases} \Delta v^{(0)} = 0 & \text{in } \Omega \setminus \overline{D}, \\ v^{(0)} = 0 & \text{on } \partial D, \\ v^{(0)} = 1 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle and by the fact that $\Omega \setminus \overline{D}$ is connected, we have that $v^{(0)} > 0$ in $\Omega \setminus \overline{D}$. Multiplying the equation (3.8) by the function $v^{(0)}$, and then

integrating the resulting equation yields

$$
\begin{aligned}
0 &= \int_{\partial(\Omega\setminus\overline{D})} v^{(0)} \partial_\nu \left( w_2^{(\ell_1\cdots\ell_{N+1})} - w_1^{(\ell_1\cdots\ell_{N+1})} \right) dS \\
&= \int_{\Omega\setminus\overline{D}} v^{(0)} \Delta \left( w_2^{(\ell_1\cdots\ell_{N+1})} - w_1^{(\ell_1\cdots\ell_{N+1})} \right) dx \\
&\quad + \int_{\Omega\setminus\overline{D}} \nabla v^{(0)} \cdot \nabla \left( w_2^{(\ell_1\cdots\ell_{N+1})} - w_1^{(\ell_1\cdots\ell_{N+1})} \right) dx \\
&= \int_{\Omega\setminus\overline{D}} \partial_z^{N+1} (a_1(x,0) - a_2(x,0)) \left( \Pi_{k=1}^{N+1} v^{(\ell_k)} \right) v^{(0)} \, dx \\
&\quad + \int_{\partial(\Omega\setminus\overline{D})} \partial_\nu v^{(0)} \left( w_2^{(\ell_1\cdots\ell_{N+1})} - w_1^{(\ell_1\cdots\ell_{N+1})} \right) dS \\
&= \int_{\Omega\setminus\overline{D}} \partial_z^{N+1} (a_1(x,0) - a_2(x,0)) \left( \Pi_{k=1}^{N+1} v^{(\ell_k)} \right) v^{(0)} \, dx,
\end{aligned}
$$

(3.10)

where we denoted $w_j^{(\ell_1\cdots\ell_{N+1})}(x) = \partial_{\epsilon_{\ell_1}\cdots\epsilon_{\ell_{N+1}}}^{N+1} u_j(x,0)$ for $j = 1, 2$. In the first equality we used $v^{(0)} = 0$ on $\partial D$ and the assumption that the DN maps agree on $\partial\Omega$ so that $\partial_\nu w_1^{(\ell_1\cdots\ell_{N+1})} = \partial_\nu w_2^{(\ell_1\cdots\ell_{N+1})}$ on $\partial\Omega$. In the second to last equality we used the fact that $v^{(0)}$ is harmonic.

Now, let us choose the boundary values as $f_3 = f_4 = \cdots = f_{N+1} = 1$ on $\partial\Omega$. With these boundary values the corresponding functions $v^{(\ell_k)}$ are harmonic functions in $\Omega\setminus\overline{D}$ with $v^{(\ell_k)} = 1$ on $\partial\Omega$ and $v^{(\ell_k)} = 0$ on $\partial D$. By the maximum principle, we have $0 < v^{(\ell_k)} < 1$ in $\Omega\setminus\overline{D}$ for $3 \le k \le N+1$. By [FKSU09, Theorem 1.1] we can find special complex valued harmonic functions in $\Omega\setminus\overline{D}$ whose boundary values vanish on $\partial D$ so that the products of pairs of these harmonic functions form a complete subset in $L^1(\Omega)$. We use real and imaginary parts of these special harmonic functions as $v^{(\ell_1)}$ and $v^{(\ell_2)}$. From the integral identity (3.10) and Lemma 2.2, we conclude that $\left( \partial_z^{N+1} a_1(x,0) - \partial_z^{N+1} a_2(x,0) \right) v^{(0)} v^{(\ell_3)} v^{(\ell_4)} \cdots v^{(\ell_{N+1})} = 0$ in $\Omega\setminus\overline{D}$. Since $v^{(\ell_k)}$ and $v^{(0)}$ are positive in $\Omega\setminus\overline{D}$ for $3 \le k \le N+1$, this implies $\partial_z^{N+1} a_1(x,0) = \partial_z^{N+1} a_2(x,0)$ in $\Omega\setminus\overline{D}$ as desired. $\qquad\square$

## 4. Simultaneous recovery of boundary and coefficients

We prove Theorem 1.3 by a similar method that we proved the Theorem 1.2.

*Proof of Theorem 1.3.* We consider boundary data of the form $f = \sum_{\ell=1}^{N+1} \epsilon_\ell f_\ell$, where $\epsilon_\ell$ are small numbers and $f_\ell \in C_c^s(\Gamma)$ for all $\ell = 1, \ldots, N+1$. Denote $\epsilon = (\epsilon_1, \cdots, \epsilon_{N+1})$. Let $u_j(x) = u_j(x; \epsilon)$, $j = 1, 2$, be the solution of

$$
\text{(4.1)} \qquad
\begin{cases}
\Delta u_j + a_j(x,u) = 0 & \text{in } \Omega_j, \\
u_j = 0 & \text{on } \partial\Omega_j \setminus \Gamma, \\
u_j = f & \text{on } \Gamma.
\end{cases}
$$

Note that by decreasing $\Gamma$ is necessary, we can assume that $\Gamma$ is connected.

*Step 1. Reconstruction of the boundary.*

By differentiating (4.1) with respect to $\epsilon_\ell$ for $\ell \in \mathbb{N}$, we obtain

$$
\begin{cases}
\Delta \left( \frac{\partial}{\partial\epsilon_\ell} u_j \right) + \partial_z a(x, u_j) \left( \frac{\partial}{\partial\epsilon_\ell} u_j \right) = 0 & \text{in } \Omega_j, \\
\frac{\partial}{\partial\epsilon_\ell} u_j = 0 & \text{on } \partial\Omega_j \setminus \Gamma, \\
\frac{\partial}{\partial\epsilon_\ell} u_j = f_\ell & \text{on } \Gamma,
\end{cases}
$$

for $j = 1, 2$. By letting $\epsilon = 0$ and using $u_j(x; 0) = 0$, we have that $v_j^{(\ell)} := \frac{\partial}{\partial \epsilon_\ell} u_j \big|_{\epsilon=0}$ solves:

$$\begin{cases} \Delta v_j^{(\ell)} = 0 & \text{in } \Omega_j, \\ v_j^{(\ell)} = 0 & \text{on } \partial \Omega_j \setminus \Gamma, \\ v_j^{(\ell)} = f_\ell & \text{on } \Gamma, \end{cases}$$

for $j = 1, 2$ and $\ell = 1, \cdots, N + 1$. Let $G$ be the connected component of $\Omega_1 \cap \Omega_2$ whose boundary contains the set $\Gamma$. Let $\widetilde{v}^{(\ell)} := v_1^{(\ell)} - v_2^{(\ell)}$ in the domain $G$. Then, using that $\Lambda_{a_1}^{\Omega_1, \Gamma}(f) = \Lambda_{a_2}^{\Omega_2, \Gamma}(f)$ on $\Gamma$ for small $f \in C_c^s(\Gamma)$, the function $\widetilde{v}^{(\ell)}$ solves

$$\begin{cases} \Delta \widetilde{v}^{(\ell)} = 0 & \text{in } G, \\ \widetilde{v}^{(\ell)} = \partial_\nu \widetilde{v}^{(\ell)} = 0 & \text{on } \Gamma. \end{cases}$$

Then by the unique continuation principle for harmonic functions, we have that $\widetilde{v}^{(\ell)} = 0$ in $G$. In other words, $v_1^{(\ell)} = v_2^{(\ell)}$ in $G$ for all $\ell = 1, \cdots, N + 1$. We remark that as in Section 3, one only needs one harmonic function $v^{(1)}$ to recover the unknown boundary. For the coefficients, we still need many harmonic functions. Let us choose on the functions $f_\ell \in C_c^s(\Gamma)$ to be non-negative and not identically zero.

If $\Omega_1 \neq \Omega_2$ and the connected component $G \neq \emptyset$, we can use Lemma A.3 in the appendix to conclude that (possibly after interchanging $\Omega_1$ and $\Omega_2$) there is a point $x_1$ with

$$x_1 \in \partial G \cap \Omega_1 \cap (\partial \Omega_2 \setminus \Gamma).$$

Since $x_1 \in \partial \Omega_2 \setminus \Gamma$, it follows that $v_2^{(\ell)}(x_1) = 0$. As $x_1$ is an interior point of the connected open set $\Omega_1$ and the boundary value of $v_2^{(\ell)}$ is non-negative, the maximum principle implies that $v_2^{(\ell)} \equiv 0$ in $\Omega_1$. This is in contradiction with the assumption that $f_\ell$ is not identically zero. This shows that $\Omega_1 = \Omega_2$. Furthermore, by denoting $\Omega := \Omega_1 = \Omega_2$, we have that $v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)}$ in $\Omega$ for $\ell = 1, \cdots, N + 1$.

*Step 2. Reconstruction of the coefficient.*

The reconstruction of the Taylor series of $a(x, z)$ at $z = 0$ is similar to Step 2 in the proof of Theorem 1.2. First one shows by higher order linearization and by induction that the equation (3.8) holds in $\Omega$. After that one constructs a harmonic function that vanishes on $\partial \Omega \setminus \Gamma$ and which is positive on $\Gamma$. This is similar to the construction of $v^{(0)}$ in (3.9). The maximum principle shows that the constructed harmonic function is positive in $\Omega$. Integrating by parts as in (3.10) and using [FKSU09, Theorem 1.1] finishes the proof.  $\square$

## Appendix A.

In the appendix, we provide some results we used earlier in the text. We first recall the well-posedness of semilinear elliptic equations from [LLLS19]. This is given in terms of Hölder spaces, which we also define next.

If $s = k + \alpha$, $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0, 1)$, and if $D \subset \mathbb{R}^n$ is a closed set, we define the Hölder space $C^s(D) = C^{k,\alpha}(D)$ as the set of all $h : D \to \mathbb{R}$ such that $\|h\|_{C^{k,\alpha}(D)}$ is finite, where

$$\|h\|_{C^{k,\alpha}(D)} := \sum_{|\beta| \leq k} \|\partial^\beta h\|_{L^\infty(D)} + \sup_{x \neq y, \ x,y \in D} \sum_{|\beta| = k} \frac{|\partial^\beta h(x) - \partial^\beta h(y)|}{|x - y|^\alpha},$$

and $\beta = (\beta_1, \cdots, \beta_n)$ is a multi-index with $\beta_i \in \mathbb{N} \cup \{0\}$ and $|\beta| = \beta_1 + \cdots + \beta_n$. In particular, this defines the space $C^s(\overline{\Omega})$ when $\Omega \subset \mathbb{R}^n$ is a bounded open set

with $C^\infty$ boundary. Next we define $C^s(\partial\Omega)$. Since $\partial\Omega$ is a compact manifold, it can be covered by finitely many open sets $(U_j)_{j=1}^N$ so that for each $j$ there is a diffeomorphism $\varphi_j : \overline{U}_j \to D_j$ where $D_j$ is a closed set in $\mathbb{R}^{n-1}$. Choosing a partition of unity $(\chi_j)_{j=1}^N$ subordinate to the cover $(U_j)$, we may define

$$\|f\|_{C^s(\partial\Omega)} := \sum_{j=1}^N \|(\chi_j f) \circ \varphi_j^{-1}\|_{C^s(D_j)}.$$

Choosing a different partition of unity leads to an equivalent norm.

**Proposition A.1** (Well-posedness [LLLS19]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^\infty$ boundary $\partial\Omega$, and let $Q$ be the semilinear elliptic operator*

$$Q(u) := \Delta u + a(x,u),$$

*where $a \in C^\infty(\overline{\Omega} \times \mathbb{R})$ satisfies the following two conditions:*

(A.1) $$a(x,0) = 0,$$

(A.2) $$\text{The map } v \mapsto \Delta v + \partial_u a(\,\cdot\,,0)v \text{ is injective on } H_0^1(\Omega).$$

*Let $s > 2$ with $s \notin \mathbb{Z}$. There exist $\delta, C > 0$ such that for any $f$ in the set*

$$U_\delta := \{f \in C^s(\partial\Omega)\,;\, \|f\|_{C^s(\partial\Omega)} < \delta\},$$

*there is a solution $u = u_f$ of*

(A.3) $$\begin{cases} \Delta u + a(x,u) = 0 & in~\Omega, \\ u = f & on~\partial\Omega, \end{cases}$$

*which satisfies*

$$\|u\|_{C^s(\overline{\Omega})} \le C\|f\|_{C^s(\partial\Omega)}.$$

*The solution $u_f$ is unique within the class $\left\{ w \in C^s(\overline{\Omega})\,;\, \|w\|_{C^s(\overline{\Omega})} \le C\delta \right\}$, and if $f \in C^\infty(\partial\Omega)$, then $u_f \in C^\infty(\overline{\Omega})$. Moreover, there are $C^\infty$ maps*

$$\begin{aligned} S &: U_\delta \to C^s(\overline{\Omega}), & f &\mapsto u_f, \\ \Lambda &: U_\delta \to C^{s-1}(\partial\Omega), & f &\mapsto \partial_\nu u_f|_{\partial\Omega}. \end{aligned}$$

The proof of the above proposition can be found in [LLLS19, Section 2] and is based on the use of Banach fixed point theorem. The proposition shows that we have a solution to (A.3) if the Dirichlet data has $C^s$ norm less than some fixed number $\delta > 0$. The solution is unique if it has $C^s$ norm less than $C\delta$, where $C > 0$ is some fixed number. In this case we say that the problem (A.3) has unique small solutions.

The next proposition can be found in [LLLS19, Section 2]. By introducing the Dirichlet data $f = \epsilon_1 f_1 + \cdots \epsilon_m f_m$, one can also define the solution $u = u(x;\epsilon)$ with $\epsilon = (\epsilon_1, \cdots, \epsilon_m)$. We can justify the formal calculation that we may differentiate the equation

(A.4) $$\begin{cases} \Delta u + a(x,u) = 0 & in~\Omega, \\ u = \epsilon_1 f_1 + \cdots + \epsilon_m f_m & on~\partial\Omega, \end{cases}$$

in the $\epsilon_j$ variables to have equations corresponding to first and $m$th linearizations,

$$\Delta v^{(\ell)} + \partial_u a(x,0)v^{(\ell)} = 0 \quad \text{and} \quad \Delta w = -\left.\partial_{\epsilon_1 \cdots \epsilon_m}^m\right|_{\epsilon=0}(a(x,u)),$$

where $v^{(\ell)} = \left.\partial_{\epsilon_\ell}\right|_{\epsilon=0} u(x;\epsilon)$ and $w = \left.\partial_{\epsilon_1 \cdots \epsilon_m}^m\right|_{\epsilon=0} u(x;\epsilon)$. The normal derivative of $w$ is the $m$th linearization of the DN map of (A.4). In the proposition, we write

$$(D^k f)_x(y_1, \ldots, y_k)$$

to denote the $k$th derivative at $x$ of a mapping $f$ between Banach spaces, considered as a symmetric $k$-linear form acting on $(y_1, \ldots, y_k)$. We refer to [Hor85, Section 1.1], where the notation $f^{(k)}(x; y_1, \ldots, y_k)$ is used instead of $(D^k f)_x(y_1, \ldots, y_k)$.

**Proposition A.2** (Smoothness of the DN map and integral identity [LLLS19])**.** *Let $a \in C^\infty(\overline{\Omega} \times \mathbb{R})$, and let $\Lambda_a$ be the DN map for the semilinear equation*

(A.5) $$\Delta u + a(x, u) = 0 \ in \ \Omega,$$

*where*

$$a(x, 0) = 0.$$

*For $f \in C^s(\partial\Omega)$, let $v_f$ be the solution of the Laplace equation*

(A.6) $$\Delta v_f = 0 \ in \ \Omega, \qquad v_f|_{\partial\Omega} = f.$$

*The first linearization $(D\Lambda_a)_0$ of $\Lambda_a$ at $f = 0$ is the DN map of the Laplace equation:*

$$(D\Lambda_a)_0 : C^s(\partial\Omega) \to C^{s-1}(\partial\Omega), \quad f \mapsto \partial_\nu v_f|_{\partial\Omega}.$$

*For $m \in \mathbb{N}$ with $m \geq 2$, the $m$-th linearization $(D^m \Lambda_a)_0$ of $\Lambda_a$ at $f = 0$ can be characterized by the following identity: for any $f_1, \ldots, f_{m+1} \in C^s(\partial\Omega)$ one has*

(A.7) $$\int_{\partial\Omega} (D^m\Lambda_a)_0(f_1, \ldots, f_m) f_{m+1} \, dS = - \int_\Omega \partial^m_{\epsilon_1 \cdots \epsilon_m}\big|_{\epsilon=0} (a(x, u)) \, dx.$$

Finally, we give for completeness a standard lemma [BV99] that was used for recovering an unknown cavity or an unknown part of the boundary.

**Lemma A.3.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be bounded connected open sets with $C^\infty$ boundaries, and assume that $\Gamma$ is a nonempty connected open subset of $\partial\Omega_1 \cap \partial\Omega_2$. Let $G$ be the connected component of $\Omega_1 \cap \Omega_2$ whose boundary contains $\Gamma$. Suppose that $G \neq \emptyset$, if*

$$\Omega_1 \neq \Omega_2,$$

*then, after interchanging $\Omega_1$ and $\Omega_2$ if necessary, one has*

$$\partial G \cap \Omega_1 \cap (\partial\Omega_2 \setminus \Gamma) \neq \emptyset.$$

*Proof.* Without loss of generality, we may assume that $\Omega_1 \setminus \Omega_2 \neq \emptyset$. We claim that we then have the inclusion relation

(A.8) $$\partial(\Omega_1 \setminus G) \subset \{\partial G \cap (\partial\Omega_2 \setminus \Gamma)\} \cup (\partial\Omega_1 \setminus \Gamma).$$

First, we prove (A.8). Using the fact that $\partial E = \overline{E} \cap \left(\overline{\mathbb{R}^n \setminus E}\right)$ for any $E \subset \mathbb{R}^n$, and using that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$, one has

$$\begin{aligned}
\partial(\Omega_1 \setminus G) &= \left(\overline{\Omega_1 \setminus G}\right) \cap \left(\overline{\mathbb{R}^n \setminus (\Omega_1 \setminus G)}\right) \\
&= \left(\overline{\Omega_1 \setminus G}\right) \cap \left(\overline{(\mathbb{R}^n \setminus \Omega_1) \cup (\mathbb{R}^n \cap G)}\right) \\
&= \left(\overline{\Omega_1 \setminus G} \cap \overline{\mathbb{R}^n \setminus \Omega_1}\right) \cup \left(\overline{\Omega_1 \setminus G} \cap \overline{G}\right) \\
&\subset (\partial\Omega_1 \setminus \Gamma) \cup (\partial G \setminus \Gamma).
\end{aligned}$$

Here we used that $(\overline{\Omega_1 \setminus G}) \cap \Gamma = \emptyset$. Next, one has $\overline{G} \cap (\Omega_1 \cap \Omega_2) \subset G$ (since any component of $\Omega_1 \cap \Omega_2$ that meets $\overline{G}$ must be equal to $G$), and thus we have

$$\partial G = \overline{G} \setminus G \subset \overline{G} \setminus (\Omega_1 \cap \Omega_2) \subset \left(\overline{\Omega_1 \cap \Omega_2}\right) \setminus (\Omega_1 \cap \Omega_2) = \partial(\Omega_1 \cap \Omega_2) \subset \partial\Omega_1 \cup \partial\Omega_2.$$

It follows that $\partial G \setminus \Gamma = \{(\partial\Omega_1 \cup \partial\Omega_2) \cap \partial G\} \setminus \Gamma$. Combining the above facts, we have proved (A.8).

Next, by the above inclusion relation $(A.8)$, it is easy to see that

$$\partial(\Omega_1 \setminus G) \cap \Omega_1 \subset \{(\partial G \cap (\partial \Omega_2 \setminus \Gamma)) \cup (\partial \Omega_1 \setminus \Gamma)\} \cap \Omega_1$$
$$(A.9) \qquad = \{(\partial G \cap (\partial \Omega_2 \setminus \Gamma) \cap \Omega_1\} \cup \{(\partial \Omega_1 \setminus \Gamma) \cap \Omega_1\}$$
$$= \partial G \cap \Omega_1 \cap (\partial \Omega_2 \setminus \Gamma),$$

where we have used that $\Omega_1$ is a bounded open set such that $(\partial \Omega_1 \setminus \Gamma) \cap \Omega_1 = \emptyset$.

We will now show that $\partial G \cap \Omega_1 \cap (\partial \Omega_2 \setminus \Gamma) \neq \emptyset$. Suppose that this is not true, i.e., $\partial G \cap \Omega_1 \cap (\partial \Omega_2 \setminus \Gamma) = \emptyset$, then $(A.9)$ implies that

$$(A.10) \qquad \partial(\Omega_1 \setminus G) \cap \Omega_1 = \emptyset.$$

Note that the following facts hold:

$$(A.11) \qquad (\Omega_1 \setminus G) \cap \Omega_1 \neq \emptyset,$$
$$(A.12) \qquad \{\mathbb{R}^n \setminus (\Omega_1 \setminus G)\} \cap \Omega_1 \neq \emptyset.$$

These facts are proved as follows. For $(A.11)$, we have $(\Omega_1 \setminus G) \cap \Omega_1 = \Omega_1 \setminus G$. If $\Omega_1 \setminus G = \emptyset$, we have $\Omega_1 \subset G$. However, by using the definition of $G$, we have that $G \subset \Omega_1 \cap \Omega_2 \subset \Omega_1$, which implies that $\Omega_1 = \Omega_1 \cap \Omega_2$. This violates our assumption that $\Omega_1 \setminus \Omega_2 \neq \emptyset$. Thus we must have $\Omega_1 \setminus G \neq \emptyset$. Similarly, for $(A.12)$, we can also obtain that

$$\{\mathbb{R}^n \setminus (\Omega_1 \setminus G)\} \cap \Omega_1 = \{(\mathbb{R}^n \setminus \Omega_1) \cup G\} \cap \Omega_1 = G \cap \Omega_1 \neq \emptyset.$$

Finally, writing $V = \text{int}(\Omega_1 \setminus G)$ and using $(A.10)$–$(A.12)$, we obtain that

$$V \cap \Omega_1 \neq \emptyset,$$
$$(\mathbb{R}^n \setminus \overline{V}) \cap \Omega_1 \neq \emptyset.$$

Using $(A.10)$ again in the form $\partial V \cap \Omega_1 = \emptyset$, we may decompose $\Omega_1$ as

$$\Omega_1 = (V \cap \Omega_1) \cup \left\{(\mathbb{R}^n \setminus \overline{V}) \cap \Omega_1\right\}.$$

Since $V$ is open, this implies that $\Omega_1$ can be written as the union of two nonempty disjoint open sets. This contradicts the assumption that $\Omega_1 \subset \mathbb{R}^n$ is a connected set. Therefore, $\partial G \cap \Omega_1 \cap (\partial \Omega_2 \setminus \Gamma)$ must be a nonempty set, which completes the proof of Lemma A.3. $\qquad \square$

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