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REFINED INSTABILITY ESTIMATES FOR SOME INVERSE PROBLEMS

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Abstract. Many inverse problems are known to be ill-posed. The ill-posedness can be manifested by an instability estimate of exponential type, first derived by Mandache [29]. In this work, based on Mandache's idea, we refine the instability estimates for two inverse problems, including the inverse inclusion problem and the inverse scattering problem. Our aim is to derive explicitly the dependence of the instability estimates on key parameters.

The first result of this work is to show how the instability depends on the depth of the hidden inclusion and the conductivity of the background medium. This work can be regarded as a counterpart of the depth-dependent and conductivity-dependent stability estimate proved by Li, Wang, and Wang [28], or pure dependent stability estimate proved by Nagayasu, Uhlmann, and Wang [31]. We rigorously justify the intuition that the exponential instability becomes worse as the inclusion is hidden deeper inside a conductor or the conductivity is larger.

The second result is to justify the optimality of increasing stability in determining the near-field of a radiating solution of the Helmholtz equation from the far-field pattern. Isakov [16] showed that the stability of this inverse problem increases as the frequency increases in the sense that the stability estimate changes from a logarithmic type to a Hölder type. We prove in this work that the instability changes from an exponential type to a Hölder type as the frequency increases. This result is inspired by our recent work [25].

1. **Introduction.** Many inverse problems are known to be ill-posed. Even the uniqueness holds in most cases, the continuous dependence of the unknown on the measurements is very weak. For some inverse problems, two estimates have been proved to quantify this ill-posedness. For example, in Calderón's problem, a logarithmic stability estimate was proved by Alessandrini [1] and an exponential instability was derived by Mandache [29]. The estimate obtained in [29] guarantees that the logarithmic stability estimate in Calderón's problem is optimal. More refined

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stability estimates involving parameters of the equations, such as the frequency, the depth of the unknown, or the conductivity, etc. were derived for many cases, not just in inverse problems, but also in the unique continuation. Following Mandache's idea, exponential instability estimates containing the effect of the frequency in some inverse problems were proved in [42] (for the transport equation) and in [25] (for the Schrödinger equation). Inspired by the results in [25], in this paper, we derive exponential instability estimates emphasizing on the effect of the parameters for two inverse problems, the inverse inclusion and the inverse scattering problems. We will explain our main results in detail below.

In a recent article [23], Koch, Rüland, and Salo investigated the mechanisms that cause the instability for some linear and nonlinear inverse problems. The instability mechanisms were categorized by three smoothing properties – strong global smoothing, only weak global smoothing, and microlocal smoothing for the corresponding forward operators. They derived instability estimates in more general geometries and coefficients. Here we are interested in how instability estimates depend on some key parameters. We achieve this by refining Mandache's approach and, therefore, work in the situation of symmetric geometries and constant coefficients. In order to present the phenomena cleanly, we choose not to explore the possibility of extending the results to more general settings.

We dedicate this paper to the memory of Victor Isakov, who made numerous fundamental contributions in the development of inverse problems. His original research on the phenomenon of increasing stability gives us a better understanding of the ill-posedness in inverse problems. This paper is largely influenced by his results.

1.1. **Depth-dependent and conductivity-dependent instability of the electrical impedance tomography (EIT).** We first study the exponential instability of the EIT. Different from early woks [7, 8, 29], here we would like to refine the previous estimates in which one can understand the influence of other a priori factors of the conductivity in instability. Precisely, we consider the inverse inclusion problem with the information of boundary data. We now describe the problem in more detailed. Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary and $\gamma(x) > 0$ (with a sufficient regularity) represent the conductivity of Ω. Due to the conservation law, the electric potential u satisfies the conductivity equation

$$\nabla \cdot (\gamma(x)\nabla u) = 0 \quad \text{in } \Omega. \tag{1}$$

It is known that given any $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution u to (1) with $u|_{\partial\Omega} = f$. The boundary data is given in the form of the Dirichlet-to-Neumann map (DN-map):

$$\Lambda_\gamma(f) := \gamma\partial_\nu u\bigg|_{\partial\Omega}, \tag{2}$$

where ν is the unit exterior normal vector of $\partial\Omega$. The information of the conductivity is encoded in Λ_γ and the EIT is to determine γ from the knowledge of Λ_γ.

This inverse problem was proposed by Calderón [5] where he showed that the linearized DN-map at the constant conductivity is injective. The global uniqueness of the EIT was proved by Sylvester and Uhlmann [37] (for dimensions higher than two) and by Nachman [30] (for dimension two). The EIT is known to be ill-posed. A log-type stability estimate was first established by Alessandrini [1], while Mandache [29] confirmed that Alessandrini's result is optimal by showing that the problem of exponentially unstable. In several practical situations, the conductivity coefficient γ takes the following form:

$$\gamma(x) = \gamma_0(x) + \gamma_1(x)\chi_D,$$

where χ_D is the characteristic function of the domain D. Here, D represents an inclusion in Ω having a different conductivity γ_1. In [13], Isakov showed that, if $\gamma_0(x)$ is known, then both $\gamma_1(x)$ and D can be uniquely determined by the DN-map (2). A log-type stability estimate was obtained in [2] for this inverse inclusion problem, i.e. determination of D from Λ_γ.

We now consider the inverse inclusion problem with $\gamma_0(x) = 1$ and $\gamma_1(x) = \kappa \neq 1$, that is,

$$\nabla \cdot ((1 + (\kappa - 1)\chi_D)\nabla u) = 0 \quad \text{in } \Omega \tag{3}$$

and the DN-map Λ_D is defined by

$$\Lambda_D : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \quad \Lambda_D(u|_{\partial\Omega}) := \partial_\nu u|_{\partial\Omega}. \tag{4}$$

The exponential instability for the inverse inclusion problem described above was proved in [2]. However, the estimate obtained in [2] did not show that influence of the depth of D on the instability. In [31], they obtained a depth-dependent stability estimate by studying the linearized DN-map. Recently, the stability estimate of [31] was extended to the multi-layer medium in [28] where the effect of the conductivity of each layer on the stability was also discovered. To simplify the discussions, we consider the medium which has 3-layer structure (the ideas can be easily extended to multi-layer structure). Let Ω' be Lipschitz domains such that $\overline{D} \subset \Omega'$ and $\overline{\Omega'} \subset \Omega$. In this work, we study the inverse inclusion problem with

$$\gamma(x) = \kappa_1\chi_D + \kappa_2\chi_{\Omega'\setminus\overline{D}} + \chi_{\Omega\setminus\overline{\Omega'}},$$

where $\kappa_i > 0$ are different with $\kappa_i \neq 1$ $(i = 1, 2)$. We define the following operator:

$$L_D u := \nabla \cdot ((\kappa_1\chi_D + \kappa_2\chi_{\Omega'\setminus\overline{D}} + \chi_{\Omega\setminus\overline{\Omega'}})\nabla u) \quad \text{in } \Omega.$$

Likewise, we can define the DN-map Λ_D by (4).

One of the main theme of this work is to investigate how the depth of the inclusion D and the conductivity κ_2 affect the instability of the inverse problem. To formulate our problem precisely, we consider $\Omega = B_1$, $\Omega' = B_{\frac{3}{4}}$, and $D = B_r$ with $0 < r < \frac{1}{4}$. We introduce a smooth function

$$\psi : \partial D \to \mathbb{R}$$

and the perturbed boundary ∂D_s of the inclusion D_s is described by the image of

$$y = F_s(x) := x + s\psi(x)\nu_x(x), \quad x \in \partial D.$$

Now the linearized DN-map of Λ_{D_s} at $s = 0$, denoted by $\mathsf{d}\Lambda_{B_r}(\psi)$, is formally defined by

$$\mathsf{d}\Lambda_{B_r}(\psi) := \lim_{s\to 0} \frac{1}{s}(\Lambda_{D_s} - \Lambda_D). \tag{5}$$

Indeed, $\mathsf{d}\Lambda_{B_r}(\psi) : H^{\frac{1}{2}}(\partial B_1) \to H^{-\frac{1}{2}}(\partial B_1)$ is a bounded linear operator, see [28, Lemma 2.3]. A log-type stability estimate with $\mathsf{d}\Lambda_{B_r}$ including the effect of the depth r of the inclusion B_r and the conductivity κ_2 was proved in [28]. Precisely, under some apriori assumptions, the following estimate holds:

$$\|\psi\|_{L^2(\partial B_1)} \leq C(\kappa_2 + 1)\log(r^{-1})|\log\|\mathsf{d}\Lambda_{B_r}(\psi)\|_*|^{-1}, \tag{6}$$

where

$$\| \bullet \|_* = \| \bullet \|_{H^{\frac{1}{2}}(\partial B_1) \to H^{-\frac{1}{2}}(\partial B_1)}.$$

Estimate (6) clearly indicates that the stability becomes worse as the depth of the inclusion increases, i.e. r becomes smaller, or the conductivity κ_2 becomes larger. It was also showed in [31] that, given any $\epsilon > 0$, there exists no positive constant C' such that

$$\|\psi\|_{L^2(\partial B_r)} \le C' |\log \|\mathsf{d}\Lambda_{B_r}(\psi)\|_*|^{-1-\epsilon},$$

that is, the logarithmic stability (6) is optimal. The deterioration of the stability of reconstructing a deeply hidden inclusion by the DN-map was also observed numerically in [12, 39, 40].

By combining the ideas of [28] and [29], we proved the following depth-dependent and conductivity-dependent exponential instability for the linearized DN-map $\mathsf{d}\Lambda_{B_r}$:

Theorem 1.1. *Fixing any $0 < r < \frac{1}{4}$ and $\kappa_2 > 1 + \kappa_1$. There exists an absolute constant $0 < E < 1$ such that, given any $0 < \epsilon < E$, there exists a function $\psi \in \mathcal{C}^\infty(\partial B_r)$ with*

$$\|\psi\|_{L^\infty(\partial B_r)} \ge \epsilon$$

such that

$$\|\mathsf{d}\Lambda_{B_r}(\psi)\|_* \le C \frac{1}{\kappa_2 + 1} \exp(-|\log r|^{\frac{2}{3}} \epsilon^{-\frac{1}{3\alpha}}) \tag{7}$$

for some absolute constant C which is independent of $\kappa_1, \kappa_2, r, \epsilon$.

Estimate (7) corresponds to the statement that the depth-dependent and conductivity-dependent stability obtained in [28], as well as the depth-dependent stability obtained in [31], are optimal from the instability perspective. We want to point out that, since $\mathsf{d}\Lambda_{B_r}$ is a linear operator, a norm estimate was derived in [28, Corollary 1], precisely,

$$\|\mathsf{d}\Lambda_{B_r}(\psi)\|_* \le \frac{C|\kappa_1 - \kappa_2|}{|\kappa_1 + \kappa_2|} \frac{1}{\kappa_2 + 1} r^{\frac{1}{2}} \|\psi\|_{L^2(\partial B_r)} \tag{8}$$

for some constant C. The norm estimate (8) holds for *all* perturbations of the inclusion ψ. It gives us only an upper bound of the size of $\mathsf{d}\Lambda_{B_r}(\psi)$ in terms of ψ. The merit of (7) is that it provides a fact that the size of $\mathsf{d}\Lambda_{B_r}(\psi)$ could be much smaller (exponentially small) in terms of *some* perturbation ψ. The derivation of (7) is more delicate that that of (8).

We now mention some other results related to our work. Besides considering the linearized map $\mathsf{d}\Lambda_{B_r}$, there are other ways to formulate the depth-dependence of the inverse problem, see, for example, [9, 10]. Let $0 \le \rho < 1$, $0 < r < 1$ and $\kappa > -1$. In [9, Theorems 2.6 and 4.4], Garde and Hyvönen proved that

$$\frac{1-\rho}{1+\rho} \le \frac{\|\Lambda_{1+\kappa\chi_{B_r(0)}} - \Lambda_1\|_{L^2(\partial B_1)\to L^2(\partial B_1)}}{\|\Lambda_{1+\kappa\chi_{B_R(z)}} - \Lambda_1\|_{L^2(\partial B_1)\to L^2(\partial B_1)}} \le \frac{1-\rho^2}{1+\rho^2}, \tag{9}$$

where Λ_γ is the DN-map given by (2),

$$z = \frac{\rho(1-r^2)}{1-\rho^2 r^2}\hat{x} \quad \text{and} \quad R = \frac{r(1-\rho^2)}{1-\rho^2 r^2},$$

provided that $|\hat{x}| = 1$, and the estimate (9) is optimal in the sense of

$$\inf_{0<r<1} \frac{\|\Lambda_{1+\kappa\chi_{B_r(0)}} - \Lambda_1\|_{L^2(\partial B_1)\to L^2(\partial B_1)}}{\|\Lambda_{1+\kappa\chi_{B_R(z)}} - \Lambda_1\|_{L^2(\partial B_1)\to L^2(\partial B_1)}} = \frac{1-\rho}{1+\rho},$$

$$\sup_{0<r<1} \frac{\|\Lambda_{1+\kappa\chi_{B_r(0)}} - \Lambda_1\|_{L^2(\partial B_1)\to L^2(\partial B_1)}}{\|\Lambda_{1+\kappa\chi_{B_R(z)}} - \Lambda_1\|_{L^2(\partial B_1)\to L^2(\partial B_1)}} = \frac{1-\rho^2}{1+\rho^2}.$$

Both the upper and lower bounds in (9) converge to zero as $\rho \to 1$, which indicates that the reconstruction of $B_R(z)$ is more stable when it close to the boundary.

The depth-dependent of the inverse problem can also be described in terms of the resolution limit. Let $K > 0$ and the function space

$$\Gamma_\Omega(\rho, q) = \left\{ \gamma \in L^\infty(\Omega) \;\middle|\; K^{-1} \le \gamma \le K, \gamma = 1 + (\gamma - 1)\chi_{B_\rho(q)} \right\},$$

where the point q is called the *center of the perturbation*. Fixing a constant $\epsilon_0 > 0$, and we say that two conductivities γ_1, γ_2 are ϵ_0*-indistinguishable* if

$$\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{\frac{1}{2}}(\partial\Omega)/H_0^{\frac{1}{2}}(\partial\Omega)\to H^{-\frac{1}{2}}(\partial\Omega)} \le \epsilon_0.$$

Accordingly, we can define the *resolution limit* at level ϵ_0 relative to the center $q \in \Omega$ by the number

$$\ell_q = \sup\left\{ \rho > 0 \;\middle|\; \gamma_1, \gamma_2 \in \Gamma_\Omega(\rho, q) \text{ are } \epsilon_0\text{-indistinguishable} \right\}.$$

Let $\Omega = B_1 \subset \mathbb{R}^2 \cong \mathbb{C}$, Alessandrini and Scapin [3, Theorems 3.3 and 3.8] derived that

$$\ell_q = \frac{1 + \ell_0^2 - \sqrt{1 + (4q^2 - 2)\ell_0^2 + \ell_0^4}}{2\ell_0} \quad \text{with} \quad \ell_0 = \sqrt{\frac{\sqrt{4 + \epsilon_0^2} - 2}{\epsilon_0 \frac{K-1}{K+1}}}$$

for all $q \in [0, 1)$, where ℓ_0 is the resolution limit at the center of the disk B_1. From this, we see that ℓ_q increases with respect to the depth $1 - q$. In other words, the resolution in the determination of the inclusion deteriorates as it is hidden deeper inside a conductor.

1.2. **Instability estimate for the determination of the near-field from the far-field.** We now study the instability phenomenon of determining the near-field of a radiating solution to the Helmholtz equation from the far-field pattern. The uniqueness follows easily from Rellich's lemma. Likewise, this inverse problem is also ill-posed. Nonetheless, it was proved by Isakov [16] that the stability of this inverse problem increases as the frequency increases. In this work, we want to verify this increasing stability phenomenon from the viewpoint of instability estimate and hence shows that the result obtained in [16] is optimal. The increasing stability phenomena were rigorously proved in other situations [11, 14, 15, 16, 17, 18, 19, 26, 27, 32, 35, 36], not only for inverse problems, but also for the unique continuation property.

Given any $f \in H^{\frac{1}{2}}(\partial B_1)$, there exists a unique $u \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{B_1})$ solving the following exterior problem:

$$\begin{cases} (\Delta + \kappa^2)u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B_1}, \\ u = f & \text{on } \partial B_1, \\ u \text{ satisfies Sommerfeld radiation condition} & \text{at } |x| \to \infty, \end{cases} \tag{10}$$

and the following estimate holds

$$\|u\|_{H^1(B_R\setminus\overline{B_1})} \le C(R, \kappa)\|f\|_{H^{\frac{1}{2}}(\partial B_1)}, \tag{11}$$

see, for example, [22, Theorem 1.1] (see also [33, Theorem 2.6.2] for refined inequality of (11) and [4, Theorem 3.3] for elastic waves). It is well-known that u satisfies the following asymptotic expansion [6, 22, 38]:

$$u(x) = \frac{e^{i\kappa r}}{r} u^{\infty}(\hat{x}) + O(r^{-2}) \quad \text{as } \; r = |x| \to \infty$$

uniformly for all $\hat{x} = x/|x| \in \mathcal{S}^2$, where $u^{\infty}(\hat{x})$ is called the far-field pattern. We use $u^{\infty}(f)$ to indicate the dependence of u^{∞} on f.

It follows from Rellich's lemma that $u^{\infty}(f)$ uniquely determine u in $\mathbb{R}^3 \setminus \overline{B_1}$ and therefore the boundary data f is uniquely also recovered, i.e., the mapping $f \to u^{\infty}(f)$ is injective. We now want to remark on the stability estimate of determining f from $u^{\infty}(f)$.

Let $\left\{ \; Y_n^m \;\middle|\; n \geq 0, |m| \leq n \; \right\}$ be the spherical harmonics, which forms a complete orthonormal basis in $L^2(\mathcal{S}^2)$. Therefore, we can write

$$u^{\infty} = \sum_{n \geq 0} \sum_{|m| \leq n} u_{nm}^{\infty} Y_n^m.$$

Define

$$\epsilon_1^2 := \sum_{n=0}^{\lfloor \sqrt{\kappa} \rfloor} \sum_{|m| \leq n} |u_{nm}^{\infty}|^2 \quad \text{and} \quad \epsilon_2^2 := \sum_{n=\lfloor \sqrt{\kappa} \rfloor + 1}^{\infty} \sum_{|m| \leq n} |u_{nm}^{\infty}|^2$$

Under some a priori assumptions, it was shown in [16, Theorem 1.1] that

$$\|f\|_{L^2(\partial B_1)}^2 \leq \frac{2e^2}{\pi} \epsilon_1^2 + \frac{2e^2}{\pi} \epsilon_2 + \frac{M_1}{\kappa + |\log \epsilon_2|}, \tag{12a}$$

$$\|f\|_{L^2(\partial B_1)}^2 \leq \frac{2e^2}{\pi} \epsilon_1^2 + \sqrt{\frac{2}{\pi \kappa}} e M_1 \epsilon_2^{\frac{1}{2}} + \frac{M_1^2}{\kappa + |\log \epsilon_2|} \tag{12b}$$

for some constant $M_1 > 0$. The estimates (12a) and (12b) indicate that the logarithmic part $(\kappa + |\log \epsilon_2|)^{-1}$ decreases as κ increases, and both estimates change from a logarithmic type to a Hölder type. In other words, Isakov's work [16] can be regarded as a quantitative version of Rellich's lemma. Moreover, using (11), one can see that Neumann data can be easily recovered from Dirichlet data, and the recovery process is stable.

In this work, we will study the counterpart of the increasing stability by investigating how the exponential instability is affected by the frequency. Inspired by the work [42] and our recent preprint [25], we prove the following theorem.

Theorem 1.2. *Fixing any frequency $\kappa > 0$, and let $\tilde{\kappa} := (\frac{\kappa}{2})^{\exp(\kappa)}$. There exists an absolute positive constant E such that for any $0 < \epsilon < E$, there exists a function $f \in \mathcal{C}^{\infty}(\partial B_1)$ satisfying*

$$\|f\|_{L^{\infty}(\partial B_1)} \geq \epsilon$$

and

$$\|u^{\infty}(f)\|_{H^{-\frac{5}{2}}(\mathcal{S}^2)} \leq C \left[\exp\left(- \frac{\max\{\tilde{\kappa}, 1\}}{3} \epsilon^{-\frac{1}{\alpha}} \right) + \min\{1, \tilde{\kappa}\} \epsilon^{\frac{1}{\alpha}} \right] \tag{13}$$

for some absolute constant C which is independent of κ and ϵ.

Before proceeding further, we would like to mention some interesting results obtained in [34] where the authors derived a stability of determining the scattered field from the far-field data and an instability estimate for the inverse scattering problem in the acoustic equation with a sound-soft obstacle. Similar to the spirit of our work, results in [34] also emphasize on the dependence of the wave number in the stability and instability estimates.

Estimate (13) shows that the instability changes from an exponential type to a Hölder type when κ increases, and vice versa. Such transition of instability was also established for an inverse problem in the stationary radiative transport equation in [42] and in the Schrödinger equation in [25]. In addition, this result shows that Isakov's result in [16] is optimal.

Our proof relies a well-known expression (43) of u in terms of spherical harmonics Y_n^m. The crucial step is the identity, which connects the Bessel function with Lommel polynomials, given in (45). This gives an explicit lower bound of the spherical Hankel functions, see Lemma 4.1. It is also interesting to mention that, using some refined properties of Bessel functions, John [21] constructed an example showing a logarithmic stability uniformly in κ in the continuation of solution to the Helmholtz equation from the unit disk into its complement in the plane.

1.3. **Organization of the paper.** We will follow the general procedure introduced in [29]. We first discuss the construction of an ϵ-discrete set in some function space in Section 2. Using this ϵ-discrete set, we will prove Theorem 1.1 and Theorem 1.2 in Section 3 and Section 4, respectively.

2. **Construction of an ϵ-discrete set.** Let $d \geq 1$. We now want to construct an ϵ-discrete set (a.k.a. ϵ-distinguishable set) for some neighborhood which is not too large. Here we recall that a set Z of a metric space (M, d) is called an ϵ-discrete set if $\mathsf{d}(z_1, z_2) \geq \epsilon$ for all $z_1 \neq z_2 \in Z$. For each $\epsilon > 0$, we define

$$\hat{\mathcal{N}}_\epsilon(B_{\frac{1}{2}}) := \left\{ \ \psi \in \mathcal{C}_c^\infty(\mathbb{R}^d) \ \middle| \ \psi \text{ is real-valued}, \operatorname{supp}(\psi) \subset B_{\frac{1}{2}}, \|\psi\|_{L^\infty(\mathbb{R}^d)} \leq \epsilon \ \right\}.$$

We now prove the following lemma.

Lemma 2.1. *Given any $\alpha > 0$, there exists $\mu = \mu(d, \alpha) > 0$ such that the following statement holds for any auxiliary parameter $\beta > 0$: Given any $0 < \epsilon < \mu\beta$, there is an ϵ-discrete set $\hat{Z}$ of $(\hat{\mathcal{N}}_\epsilon(B_{\frac{1}{2}}), \|\bullet\|_{L^\infty})$ with*

$$|\hat{Z}| \geq \exp\left[\frac{1}{2^{d+1}}\left(\frac{\mu\beta}{\epsilon}\right)^{\frac{d}{\alpha}}\right], \tag{14}$$

where $|\hat{Z}|$ denotes the cardinality of $\hat{Z}$.

Remark 1. When $d \geq 2$, Lemma 2.1 is a special case of [29, Lemma 2]. See also [24, Theorem XIV] for more abstract setting, or [8, Proposition 3.1], [25, Proposition 2.1], [42, Lemma 5.2].

Proof of Lemma 2.1. It remains to prove this theorem for $d = 1$. We fix $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R}^1)$ such that $\operatorname{supp}(\psi_0) \subset B_{\frac{1}{2}} = (-\frac{1}{2}, \frac{1}{2})$ and $\|\psi_0\|_{L^\infty(\mathbb{R}^1)} = 1$. We now define

$$\mu := \|\psi_0\|_{\mathcal{C}^\alpha(\mathbb{R}^1)}^{-1} \quad \text{and} \quad N = \left\lfloor \left(\frac{\mu\beta}{\epsilon}\right)^{\frac{1}{\alpha}} \right\rfloor.$$

Since $0 < \epsilon < \mu\beta$, then $\frac{\mu\beta}{\epsilon} > 1$. Hence,

$$N > \frac{1}{2}\left(\frac{\mu\beta}{\epsilon}\right)^{\frac{1}{\alpha}}. \tag{15}$$

We divide $B_{\frac{1}{2}} = (-\frac{1}{2}, \frac{1}{2})$ into N smaller intervals of length $1/N$. Let $y_1, \cdots, y_N$ be their centers. Defining

$$\hat{Z} := \left\{ \psi \,\middle|\, \psi(x) = \epsilon \sum_{j=1}^{N} \sigma_j \psi_0(N(x - y_j)), \quad \sigma_j \in \{0,1\} \right\}.$$

Note that each element $\psi \in \hat{Z}$ is smooth with $\|\psi\|_{L^\infty} \le \epsilon$ and $\hat{Z} \subset \hat{\mathcal{N}}_\epsilon(B_{\frac{1}{2}})$. Moreover, we see that

$$\|\psi_1 - \psi_2\|_{L^\infty} = \epsilon \quad \text{for all } \psi_1 \neq \psi_2 \in \hat{Z}.$$

Finally, we see that

$$|\hat{Z}| = 2^N = \exp(N \log 2) \ge \exp\left(\frac{N}{2}\right). \tag{16}$$

Combining (15) and (16), we obtain (14). □

Let $r > 0$ and let $\mathcal{P} : \partial B_r \to \mathbb{R}^d \cup \{\infty\}$ be the stereographic projection. Let us define

$$\mathcal{N}_\epsilon(\partial B_r, \mathcal{P}) := \left\{ \psi : \partial B_r \to \mathbb{R} \,\middle|\, \psi \circ \mathcal{P}^{-1} \in \hat{\mathcal{N}}_\epsilon(B_{\frac{1}{2}}) \right\},$$
$$Z := \left\{ \psi : \partial B_r \to \mathbb{R} \,\middle|\, \psi \circ \mathcal{P}^{-1} \in \hat{Z} \right\},$$

and

$$\mathcal{N}_\epsilon(\partial B_r) := \left\{ \psi : \partial B_r \to \mathbb{R} \,\middle|\, \psi \text{ is smooth with } \|\psi\|_{L^\infty(\partial B_r)} \le \epsilon \right\}.$$

It is clear that $Z \subset \mathcal{N}_\epsilon(\partial B_r, \mathcal{P}) \subset \mathcal{N}_\epsilon(\partial B_r)$ and $|Z| = |\hat{Z}|$. Hence, we can rephrase Lemma 2.1 as follows:

Proposition 1. *Given any $\alpha > 0$, there exists $\mu = \mu(d, \alpha) > 0$ such that the following statement holds for any auxiliary parameter $\beta > 0$: Given any $0 < \epsilon < \mu\beta$, there is an ϵ-discrete set Z of $(\mathcal{N}_\epsilon(\partial B_r), \|\bullet\|_{L^\infty(\partial B_r)})$ with*

$$|Z| \ge \exp\left[\frac{1}{2^{d+1}}\left(\frac{\mu\beta}{\epsilon}\right)^{\frac{d}{\alpha}}\right],$$

3. Proof of Theorem 1.1.

We prove Theorem 1.1 in this section.

3.1. General framework of matrix representation.

For each $\rho > 0$ and $\gamma \in \mathbb{R}$, we have

$$\|\psi\|_{L^2(\partial B_\rho)}^2 = \frac{\rho}{2\pi} \sum_{k \in \mathbb{Z}} \left| \int_0^{2\pi} \psi(\rho\cos\theta, \rho\sin\theta) e^{-ik\theta} \, d\theta \right|^2, \tag{17a}$$

$$\|\psi\|_{H^\gamma(\partial B_\rho)}^2 = \frac{\rho}{2\pi} \sum_{k \in \mathbb{Z}} (1+k^2)^\gamma \left| \int_0^{2\pi} \psi(\rho\cos\theta, \rho\sin\theta) e^{-ik\theta} \, d\theta \right|^2, \tag{17b}$$

see [31, (2.1)]. For each $n \in \mathbb{Z}$, we define $\phi_n : \partial B_1 \to \mathbb{C}$

$$\phi_n(\cos\theta, \sin\theta) := \frac{1}{\sqrt{2\pi}(1+n^2)^{\frac{1}{4}}} e^{in\theta}.$$

Using (17b) with $\gamma = \frac{1}{2}$ and $\rho = 1$ gives

$$\|\phi_n\|_{H^{\frac{1}{2}}(\partial B_1)}^2 = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1+k^2)^{\frac{1}{2}} \left| \int_0^{2\pi} \frac{1}{\sqrt{2\pi}(1+n^2)^{\frac{1}{4}}} e^{in\theta} e^{-ik\theta} \, d\theta \right|^2$$

$$
\begin{aligned}
&= \frac{1}{2\pi}(1+n^2)^{\frac{1}{2}} \left| \frac{\sqrt{2\pi}}{(1+n^2)^{\frac{1}{4}}} \right|^2 \\
&= \frac{1}{2\pi}(1+n^2)^{\frac{1}{2}} \frac{2\pi}{(2+n^2)^{\frac{1}{2}}} = 1.
\end{aligned}
$$

That is, $\left\{ \ \phi_n \ \middle| \ n \in \mathbb{Z} \ \right\}$ forms a complete orthonormal set in $H^{\frac{1}{2}}(\partial B_1)$.

Let $\mathcal{A} : H^{\frac{1}{2}}(\partial B_1) \to H^{-\frac{1}{2}}(\partial B_1)$ be any bounded linear operator. For any pair $(n,m) \in \mathbb{Z} \times \mathbb{Z}$, we define the complex number

$$
a_{nm} := \langle \mathcal{A}\phi_n, \phi_m \rangle,
$$

where $\langle \bullet, \bullet \rangle$ is the $H^{-\frac{1}{2}}(\partial B_1) \times H^{\frac{1}{2}}(\partial B_1)$ duality pair. We consider the Banach space X, which consists tensors (a_{nm}) with

$$
\|(a_{nm})\|_X := \frac{1}{4} \sup_{n,m\in\mathbb{Z}} (1+\max\{|n|,|m|\})^2 |a_{nm}| < \infty.
$$

We have the following proposition.

Proposition 2. *There exists an absolute constant $C_{\mathrm{abs}} > 0$ such that*

$$
\|\mathcal{A}\|_* \leq C_{\mathrm{abs}} \|(a_{nm})\|_X. \tag{18}
$$

In other words, tensor (a_{nm}) can be treated as the matrix representation of the bounded linear operator $\mathcal{A}$.

Proof. Using the Hilbert-Schmidt norm, we have

$$
\|\mathcal{A}\|_* \leq \left(\sum_{n,m\in\mathbb{Z}} |a_{nm}|^2 \right)^{\frac{1}{2}} \leq 4 \left(\sum_{n,m\in\mathbb{Z}} \frac{1}{(1+\max\{|n|,|m|\})^4} \right)^{\frac{1}{2}} \|(a_{nm})\|_X.
$$

We now compute

$$
\begin{aligned}
&\sum_{n,m\in\mathbb{Z}} \frac{1}{(1+\max\{|n|,|m|\})^4} \\
&\leq \left(\sum_{n\geq 0, m\geq 0} + \sum_{n\geq 0, m\leq 0} + \sum_{n\leq 0, m\geq 0} + \sum_{n\leq 0, m\leq 0} \right) \frac{1}{(1+\max\{|n|,|m|\})^4} \\
&= 4 \sum_{n\geq 0, m\geq 0} \frac{1}{(1+\max\{n,m\})^4} \\
&\leq 4 \left(\sum_{n\geq m\geq 0} + \sum_{m\geq n\geq 0} \right) \frac{1}{(1+\max\{n,m\})^4} \\
&= 8 \sum_{n\geq m\geq 0} \frac{1}{(1+n)^4} = 8 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{(1+n)^4} = 8 \sum_{n=0}^{\infty} \frac{1}{(1+n)^3} < \infty,
\end{aligned}
$$

which proves (2). ☐

3.2. **Estimating the matrix representation of the linearized DN-map.** The task here is to estimate $\mathsf{d}\Lambda_{B_r}^{nm}(\psi) := \langle \mathsf{d}\Lambda_{B_r}(\psi)(\phi_n), \phi_m \rangle$. Precisely, we want to prove the following proposition.

Proposition 3. *Given any $\epsilon > 0$ and $0 < r < \frac{1}{4}$. If $\kappa_2 > 1 + \kappa_1$, then there exists an absolute constant C such that*

$$
|\mathsf{d}\Lambda_{B_r}^{nm}(\psi)| \leq CR \frac{1}{\kappa_2 + 1} \ell^{\frac{1}{2}} r^{\ell-1} \tag{19}
$$

for all $\psi \in \mathcal{N}_R(\partial B_r)$*, where* $\ell = \max\{|n|,|m|\}$*.*

Remark 2. Observe that $\mathsf{d}\Lambda_{B_r}^{nm}(\psi) = 0$ when $n = 0$ or $m = 0$, since $\Lambda_{D_s}(1) = 0$ for all $s \geq 0$. Hence, we have

$$\begin{aligned}\|(\mathsf{d}\Lambda_{B_r}^{nm}(\psi))\|_X &= \frac{1}{4}\sup_{0\neq n,m\in\mathbb{Z}}(1+\max\{|n|,|m|\})^2|\mathsf{d}\Lambda_{B_r}^{nm}(\psi)| \\ &\leq \sup_{0\neq n,m\in\mathbb{Z}}\max\{|n|,|m|\}^2|\mathsf{d}\Lambda_{B_r}^{nm}(\psi)|. \end{aligned} \tag{20}$$

Given any function $g \in L^2(\partial B_r)$ and $k \in \mathbb{Z}$, we define the Fourier coefficient of g as

$$g_k := \int_0^{2\pi} g(r\cos\theta, r\sin\theta)e^{-ik\theta}\,d\theta.$$

It is easy to see that

$$|g_k| \leq 2\pi\|g\|_{L^\infty(\partial B_r)}. \tag{21}$$

For $f \in L^2(\partial B_1)$, we abuse the notation and define

$$f_k := \int_0^{2\pi} f(\cos\theta,\sin\theta)e^{-ik\theta}\,d\theta.$$

We need the following lemma, which is a special case of [28, (18)] (taking $R = 1$ in [28, (18)]).

Lemma 3.1. *For* $f \in H^{\frac{1}{2}}(\partial B_1)$*, we have*

$$\mathsf{d}\Lambda_{B_r}(\psi)(f)\Big|_{\partial B_1} = \sum_{a\in\mathbb{Z}}\lambda_a(f)e^{ia\theta},$$

where $\lambda_0(f) = 0$ *and for all* $a \in \mathbb{N}$

$$\lambda_{-a}(f) = \frac{\kappa_1-\kappa_2}{\pi^2}r^{-1}T_a\sum_{p=1}^{\infty}S_p\Big[(\kappa_1+\kappa_2)\psi_{-a+p}f_{-p} - (\kappa_2-\kappa_1)\psi_{-a-p}f_p\Big],$$

$$\lambda_{a}(f) = \frac{\kappa_1-\kappa_2}{\pi^2}r^{-1}T_a\sum_{p=1}^{\infty}S_p\Big[(\kappa_1+\kappa_2)\psi_{a-p}f_{p} - (\kappa_2-\kappa_1)\psi_{a+p}f_{-p}\Big],$$

where

$$T_a := \frac{-a}{(\kappa_2-\kappa_1)r^a + (\kappa_1+\kappa_2)r^{-a}}$$

$$\begin{aligned}S_p := 2p\Bigg(&\left(\frac{3}{4}\right)^{-p}\left[(\kappa_2-\kappa_1)(\kappa_2-1)r^p\left(\frac{3}{4}\right)^{-p} - (\kappa_1+\kappa_2)(\kappa_2+1)r^{-p}\left(\frac{3}{4}\right)^{p}\right] \\ &+\left(\frac{3}{4}\right)^{p}\left[-(\kappa_2-\kappa_1)(\kappa_2+1)r^p\left(\frac{3}{4}\right)^{-p} + (\kappa_1+\kappa_2)(\kappa_2-1)r^{-p}\left(\frac{3}{4}\right)^{p}\right]\Bigg)^{-1}.\end{aligned}$$

Remark 3. When $\kappa_2 = 1$ and $\kappa_1 = \kappa$, that is, the case of 2-layer medium, we have $S_p = T_p$, and hence Lemma 3.1 reduces to [31, Lemma 2.2] with $R = 1$.

The following inequalities can be found in the proof of [28, Lemma 2.3]:

$$|T_k| \leq \frac{2k}{\kappa_1+\kappa_2}r^k, \tag{22a}$$

$$|S_n| \leq \frac{1}{\min\{\frac{1}{2},c_0\}}\frac{2n}{(\kappa_1+\kappa_2)(\kappa_2+1)}r^n, \tag{22b}$$

where

$$c_0 := \inf_{\tau\in\mathbb{N}} \left|1 - \frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2} r^{2\tau} + \frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\frac{\kappa_2-1}{\kappa_2+1} r^{2\tau}\left(\frac{3}{4}\right)^{-2\tau} - \frac{\kappa_2-1}{\kappa_2+1}\left(\frac{3}{4}\right)^{2\tau}\right|.$$

Since $0 < r < \frac{1}{4}$ and $\kappa_2 \geq 1+\kappa_1$, it is easy to see that $c_0 \geq \frac{1}{5}$, and hence (22b) becomes

$$|S_n| \leq \frac{10n}{(\kappa_1+\kappa_2)(\kappa_2+1)} r^n. \tag{22c}$$

Now, we are ready to prove Proposition 3.

Proof of Proposition 3. Using (17b), we can estimate

$$\begin{aligned}
|\mathsf{d}\Lambda_{B_r}^{nm}(\psi)| &\leq \|\mathsf{d}\Lambda_{B_r}(\psi)(\phi_n)\|_{H^{-\frac{1}{2}}(\partial B_1)} \\
&= \left[\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}(1+k^2)^{-\frac{1}{2}}\left|\int_0^{2\pi}\left(\sum_{a\in\mathbb{Z}}\lambda_a(\phi_n)e^{ia\theta}\right)e^{-ik\theta}\,d\theta\right|^2\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}(1+k^2)^{-\frac{1}{2}}|2\pi\lambda_k(\phi_n)|^2\right]^{\frac{1}{2}} \\
&= \left[2\pi\sum_{k\in\mathbb{Z}}(1+k^2)^{-\frac{1}{2}}|\lambda_k(\phi_n)|^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{23}$$

Note that the Fourier coefficient $(\phi_n)_p$ of ϕ_n can be explicitly calculated:

$$\begin{aligned}
(\phi_n)_p &= \int_0^{2\pi}\phi_n(\cos\theta,\sin\theta)e^{-ip\theta}\,d\theta \\
&= \int_0^{2\pi}\frac{1}{\sqrt{2\pi}(1+n^2)^{\frac{1}{4}}}e^{in\theta}e^{-ip\theta}\,d\theta = \frac{\sqrt{2\pi}}{(1+n^2)^{\frac{1}{4}}}\delta_{np}.
\end{aligned} \tag{24}$$

Now we consider $n > 0$. For any $R > 0$ and $\psi \in \mathcal{N}_R(\partial B_r)$, we can see that for $k > 0$,

$$\begin{aligned}
&|\lambda_{-k}(\phi_n)| \\
&= \left|\frac{\kappa_1-\kappa_2}{\pi^2}r^{-1}T_k\sum_{p=1}^{\infty}S_p\left[(\kappa_1+\kappa_2)\psi_{-k+p}(\phi_n)_{-p} - (\kappa_2-\kappa_1)\psi_{-k-p}(\phi_n)_p\right]\right| \\
&= \left|\frac{\kappa_1-\kappa_2}{\pi^2}r^{-1}T_kS_n\left[(\kappa_2-\kappa_1)\psi_{-k-n}\frac{\sqrt{2\pi}}{(1+n^2)^{\frac{1}{4}}}\right]\right| \quad \text{(from (24))} \\
&= 2\pi\frac{|\kappa_1-\kappa_2|^2}{\pi^2}r^{-1}|T_k||S_n||\psi_{-k-n}|\frac{1}{(1+n^2)^{\frac{1}{4}}} \\
&\leq 20(2\pi)^{\frac{3}{2}}\frac{|\kappa_1-\kappa_2|^2}{\pi^2}Rr^{-1}\left(\frac{k}{\kappa_1+\kappa_2}r^k\right)\left(\frac{n}{(\kappa_1+\kappa_2)(\kappa_2+1)}r^n\right)\frac{1}{(1+n^2)^{\frac{1}{4}}} \\
&\qquad \text{(using (21),(22a), and (22c))} \\
&= \frac{20(2\pi)^{\frac{3}{2}}}{\pi^2}\left|\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right|^2 R\frac{1}{\kappa_2+1}r^{-1}\frac{knr^{k+n}}{(1+n^2)^{\frac{1}{4}}} \\
&\leq \frac{20(2\pi)^{\frac{3}{2}}}{\pi^2}R\frac{1}{\kappa_2+1}r^{-1}\frac{knr^{k+n}}{(1+n^2)^{\frac{1}{4}}}
\end{aligned} \tag{25a}$$

and

$$
\begin{aligned}
&|\lambda_k(\phi_n)| \\
&= \left| \frac{\kappa_1 - \kappa_2}{\pi^2} r^{-1} T_k \sum_{p=1}^{\infty} S_p \left[(\kappa_1 + \kappa_2)\psi_{k-p}(\phi_n)_p - (\kappa_2 - \kappa_1)\psi_{k+p}(\phi_n)_{-p} \right] \right| \\
&= \left| \frac{\kappa_1 - \kappa_2}{\pi^2} r^{-1} T_k S_n \left[(\kappa_2 + \kappa_1)\psi_{k-n} \frac{\sqrt{2\pi}}{(1+n^2)^{\frac{1}{4}}} \right] \right| \quad \text{(from (24))} \\
&= 2\pi \frac{|\kappa_1 - \kappa_2||\kappa_1 + \kappa_2|}{\pi^2} r^{-1} |T_k||S_n||\psi_{k-n}| \frac{1}{(1+n^2)^{\frac{1}{4}}} \\
&\leq 20(2\pi)^{\frac{3}{2}} \frac{|\kappa_1 - \kappa_2||\kappa_1 + \kappa_2|}{\pi^2} R r^{-1} \left(\frac{k}{\kappa_1 + \kappa_2} r^k \right) \left(\frac{n}{(\kappa_1 + \kappa_2)(\kappa_2 + 1)} r^n \right) \frac{1}{(1+n^2)^{\frac{1}{4}}} \\
&\qquad \text{(using (21),(22a), and (22c))} \\
&= \frac{20(2\pi)^{\frac{3}{2}}}{\pi^2} \left| \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right| R \frac{1}{\kappa_2 + 1} r^{-1} \frac{knr^{k+n}}{(1+n^2)^{\frac{1}{4}}} \\
&\leq \frac{20(2\pi)^{\frac{3}{2}}}{\pi^2} R \frac{1}{\kappa_2 + 1} r^{-1} \frac{knr^{k+n}}{(1+n^2)^{\frac{1}{4}}} \qquad \text{(25b)}
\end{aligned}
$$

From (25a) and (25b), if we define

$$
\tilde{C} := \frac{20(2\pi)^{\frac{3}{2}}}{\pi^2},
$$

we then have

$$
|\lambda_k(\phi_n)| \leq \tilde{C} R \frac{1}{\kappa_2 + 1} r^{-1} \frac{|k| n r^{|k|+n}}{(1+n^2)^{\frac{1}{4}}} \quad \text{for all } k \in \mathbb{Z}. \tag{26}
$$

Combining (23) and (26), we obtain

$$
\begin{aligned}
|\mathsf{d}\Lambda_{B_r}^{nm}(\psi)| &\leq \sqrt{2\pi} \tilde{C} R \frac{1}{\kappa_2 + 1} r^{-1} \frac{nr^n}{(1+n^2)^{\frac{1}{4}}} \left[\sum_{k \in \mathbb{Z}} (1+k^2)^{-\frac{1}{2}} k^2 r^{2|k|} \right]^{\frac{1}{2}} \\
&\leq \sqrt{2\pi} \tilde{C} R \frac{1}{\kappa_2 + 1} r^{-1} \frac{nr^n}{(1+n^2)^{\frac{1}{4}}} \left[\sum_{k \in \mathbb{Z}} (1+k^2)^{-\frac{1}{2}} k^2 \left(\frac{1}{4} \right)^{|k|} \right]^{\frac{1}{2}} \\
&= CR \frac{1}{\kappa_2 + 1} \frac{n}{(1+n^2)^{\frac{1}{4}}} r^{n-1} \qquad \text{(27)}
\end{aligned}
$$

with

$$
C = \sqrt{2\pi} \tilde{C} \left[\sum_{k \in \mathbb{Z}} (1+k^2)^{-\frac{1}{2}} k^2 \left(\frac{1}{4} \right)^{|k|} \right]^{\frac{1}{2}} < \infty.
$$

For $n < 0$, we can obtain an inequality similar to (27), precisely,

$$
|\mathsf{d}\Lambda_{B_r}^{nm}(\psi)| \leq CR \frac{1}{\kappa_2 + 1} \frac{|n|}{(1+n^2)^{\frac{1}{4}}} r^{|n|-1}.
$$

Since $\mathsf{d}\Lambda_{B_r}(\psi)$ is self-adjoint, i.e. $(\mathsf{d}\Lambda_{B_r}^{nm}(\psi))$ is symmetric, we thus conclude that

$$
|\mathsf{d}\Lambda_{B_r}^{nm}(\psi)| \leq CR \frac{1}{\kappa_2 + 1} \frac{\ell}{(1+\ell^2)^{\frac{1}{4}}} r^{\ell-1},
$$

which implies (19). □

3.3. **Construction of a δ-net.** Given any $\psi \in \mathcal{N}_R(\partial B_r)$, (19) implies that

$$\ell^2 |\mathsf{d}\Lambda^{nm}_{B_r}(\psi)| \le C_R \frac{1}{\kappa_2+1} \ell^{\frac{5}{2}} r^{\ell-1} \tag{28}$$

with $\ell = \max\{|n|,|m|\}$ and C_R depending on $R > 0$. Here, it suffices to take $C_R > 1$. Here, from (20), it follows that

$$\|(\mathsf{d}\Lambda^{nm}_{B_r}(\psi))\|_X \le C_R \frac{1}{\kappa_2+1} \sup_{\ell\ge 1} \ell^{\frac{5}{2}} r^{\ell-1} < \infty. \tag{29}$$

In other words, we have

$$(\mathsf{d}\Lambda^{nm}_{B_r}(\mathcal{N}_R(\partial B_r))) \subset X. \tag{30}$$

In view of (30), we want to construct a $\frac{\delta}{\kappa_2+1}$-net Y for $((\mathsf{d}\Lambda^{nm}_{B_r}(\mathcal{N}_R(\partial B_r))), \|\bullet\|_X)$, which is not too large. Precisely, we aim to derive the following proposition.

Proposition 4. *Let $0 < r < \frac{1}{4}$, $R > 0$, and $\kappa_2 > 1 + \kappa_1$. Given any $0 < \delta < 1$, there exists a $\frac{\delta}{\kappa_2+1}$-net Y for $((\mathsf{d}\Lambda^{nm}_{B_r}(\mathcal{N}_R(\partial B_r))), \|\bullet\|_X)$ such that*

$$\log|Y| \le C|\log r|^{-2} \log^3\left(\frac{\eta_R}{\delta}\right) + C\log\left(\frac{\eta_R}{\delta}\right), \tag{31}$$

where C is a general constant, and η_R is a constant depending only on R.

Remark 4. A set Y if a metric space (M,d) is called a δ-net for $Y_1 \subset M$ if for any $x \in Y_1$, there is a $y \in Y$ such that $d(x,y) \le \delta$.

Proof of Proposition 4. **Step 1: Initialization.** Let C_R be the constant given in Proposition 3. Given any $0 < \delta < 1$, let $\tau_0 > 1$ be the unique positive solution (not necessarily an integer) to

$$\tau_0^{\frac{5}{2}} r^{\tau_0 - 1} = \frac{\delta}{C_R}. \tag{32}$$

If we define $\ell_* = \lfloor \tau_0 \rfloor$ (note $1 \le \ell_* \le \tau_0$), then (32) implies

$$\frac{\delta}{C_R} \le \ell_*^{\frac{5}{2}} r^{\ell_*-1} \le \ell_*^{\frac{5}{2}} \left(\frac{1}{4}\right)^{\frac{\ell_*-1}{2}} r^{\frac{\ell_*-1}{2}} \le C' r^{\frac{\ell_*-1}{2}} \quad \text{with} \quad C' = \sup_{\tau\ge 1} \tau^{\frac{5}{2}} \left(\frac{1}{4}\right)^{\frac{\tau-1}{2}}. \tag{33}$$

Taking the logarithm both sides of (33) gives

$$\log\left(\frac{\delta}{C'C_R}\right) \le \log(r^{\frac{\ell_*-1}{2}}) = \frac{\ell_*-1}{2}\log r,$$

and thus

$$\log\left(\frac{C'C_R}{\delta}\right) = \log\left(-\frac{\delta}{C'C_R}\right) \ge -\frac{\ell_*-1}{2}\log r = \frac{\ell_*-1}{2}|\log r|,$$

which is equivalent to

$$\ell_* \le \frac{2}{|\log r|}\log\left(\frac{C'C_R}{\delta}\right) + 1. \tag{34}$$

Furthermore, we can observe that for any integer $\ell > \ell_*$, i.e. $\ell > \tau_0$, it holds

$$\ell^{\frac{5}{2}} r^{\ell-1} \le \frac{\delta}{C_R}. \tag{35}$$

Step 2: Construction of a set. For each pair $(n,m) \in \mathbb{Z}\times\mathbb{Z}$ with $0 < \ell = \max\{|n|,|m|\} \le \ell_*$, (28) implies

$$|\mathsf{d}\Lambda^{nm}_{B_r}(\psi)| \le C_R \frac{1}{\kappa_2+1} C'' \quad \text{with} \quad C'' = \sup_{\ell\ge 1} \ell^{\frac{1}{2}} \left(\frac{1}{4}\right)^{\ell-1}.$$

We set

$$\delta' := \frac{\delta}{\sqrt{2}\ell_*^2(\kappa_2+1)}, \quad Y' := \left\{ \; a = a_1 + ia_2 \in \delta'\mathbb{Z} + i\delta'\mathbb{Z} \;\middle|\; |a_1|, |a_2| \le \tfrac{C_R C''}{\ell_*^2(\kappa_2+1)} \; \right\},$$

and

$$Y := \left\{ \; (b_{nm}) \;\middle|\; \begin{array}{l} \text{if } \ell = \max\{|n|,|m|\} \le \ell_*, \text{ then } b_{nm} \in Y', \\ \text{otherwise, } b_{nm} = 0 \end{array} \; \right\}.$$

Step 3: Verifying that Y is a $\frac{\delta}{\kappa_2+1}$-net. Given any $\psi \in \mathcal{N}_R(\partial B_r)$, our goal is to construct a tensor $(b_{nm}) \in Y$ that is close to the tensor $(\mathsf{d}\Lambda_{B_r}^{nm}(\psi))$. If $0 < \ell = \max\{|n|,|m|\} \le \ell_*$, we choose $b_{nm} \in Y'$ as the closest element to $\mathsf{d}\Lambda_{B_r}^{nm}(\psi)$. Then, we have

$$\ell^2 |b_{nm} - \mathsf{d}\Lambda_{B_r}^{nm}(\psi)| \le \sqrt{2}\ell_*^2 \delta' = \frac{\delta}{\kappa_2+1}. \tag{36a}$$

Otherwise, if $\ell = \max\{|n|,|m|\} > \ell_*$, we choose $b_{nm} = 0$. For such choice of (b_{nm}), with the help of (28) and (35), we conclude that

$$\ell^2 |b_{nm} - \mathsf{d}\Lambda_{B_r}^{nm}(\psi)| = \ell^2 |\mathsf{d}\Lambda_{B_r}^{nm}(\psi)| \le C_R \frac{1}{\kappa_2+1} \ell^{\frac{5}{2}} r^{\ell-1} \le \frac{\delta}{\kappa_2+1}. \tag{36b}$$

Combining (20), (36a), and (36b), we conclude that

$$\|(b_{nm} - \mathsf{d}\Lambda_{B_r}^{nm}(\psi))\|_X \le \frac{\delta}{\kappa_2+1},$$

which shows that Y is a $\frac{\delta}{\kappa_2+1}$-net for $((\mathsf{d}\Lambda_{B_r}^{nm}(\mathcal{N}_R(\partial B_r))), \|\bullet\|_X)$.

Step 4: Calculating the cardinality of Y. We see that

$$|Y'| = \left(1 + 2\left\lfloor \frac{C_R C''}{\ell_*^2(\kappa_2+1)\delta'} \right\rfloor\right)^2 \le \left(1 + 2\sqrt{2}\frac{C_R C''}{\delta}\right)^2. \tag{37}$$

Let N_ℓ be the number of pairs $(n,m) \in (\mathbb{Z}\setminus\{0\}) \times (\mathbb{Z}\setminus\{0\})$ with $\max\{|n|,|m|\} = \ell$. We want to estimate N_ℓ. When $n = \pm\ell$, then m can be any no-zero integer between $-\ell$ and ℓ (i.e. there are 2ℓ choices). Switching the role of n and m, we hence obtain that $N_\ell \le 8\ell$. Consequently, we can estimate

$$N_* := \sum_{\ell=1}^{\ell_*} N_\ell \le \sum_{\ell=1}^{\ell_*} 8\ell = 4\ell_*(\ell_*+1) \quad \text{and} \quad |Y| = |Y'|^{N_*}. \tag{38}$$

Combining (34), (37), and (38), we obtain

$$\begin{aligned} \log|Y| = N_* \log|Y'| &\le 16\ell_*^2 \log\left(1 + 2\sqrt{2}\frac{C_R C''}{\delta}\right) \\ &\le 16\left[\frac{2}{|\log r|}\log\left(\frac{C' C_R}{\delta}\right) + 1\right]^2 \log\left(1 + 2\sqrt{2}\frac{C_R C''}{\delta}\right), \end{aligned}$$

which implies (31). □

Remark 5. Note that

$$\inf_{0<\delta<1} |\log r|^{-2} \log^3\left(\frac{\eta_R}{\delta}\right) = |\log r|^{-2}\log^3(\eta_R) < \left(\log\frac{1}{4}\right)^{-2} \log^3(\eta_R) := \mathring{E}_R.$$

Therefore, given any $0 < \epsilon < \mathring{E}_R^{-\alpha}$, there exists a unique $0 < \delta < 1$ such that

$$\epsilon^{-\frac{1}{\alpha}} = |\log r|^{-2}\log^3\left(\frac{\eta_R}{\delta}\right) \quad \left(\text{equivalently,} \quad \delta = \eta_R \exp(-|\log r|^{\frac{2}{3}} \epsilon^{-\frac{1}{3\alpha}}\right). \tag{39}$$

Therefore, (31) can be rewritten as follows:

$$\log|Y| \leq C(\epsilon^{-\frac{1}{\alpha}} + |\log r|^{\frac{2}{3}}\epsilon^{-\frac{1}{3\alpha}}). \tag{40}$$

3.4. **Proof of the main result.**

Proof of Theorem 1.1. Fixing any auxiliary parameters $R > 0$ and $\alpha > 0$. For each $0 < \epsilon < \min\{\mu\beta, \mathring{E}_R^{-\alpha}, R, 1\}$, we can construct an ϵ-discrete Z as in Proposition 1 with $d = 1$. Then, let δ be the number given in (39), and we can construct a $\frac{\delta}{\kappa_2+1}$-net Y as in Proposition 4 such that (40) holds. Since $0 < \epsilon < R$, then $Z \subset \mathcal{N}_R(\partial B_r)$. Therefore, Y is also a $\frac{\delta}{\kappa_2+1}$-net of $((\mathsf{d}\Lambda_{B_r}^{nm}(Z)), \|\bullet\|_X)$.

Now, we choose $\beta = \beta(\alpha, r, R)$ such that

$$\frac{1}{8}(\mu\beta)^{\frac{1}{\alpha}} > C|\log r|^{\frac{2}{3}} \quad \text{and} \quad \mu\beta > \epsilon.$$

Then it follows from (40), $0 < r < \frac{1}{4}$, and $0 < \epsilon < 1$ that

$$\log|Z| \geq \frac{1}{4}\left(\frac{\mu\beta}{\epsilon}\right)^{\frac{1}{\alpha}} > C(\epsilon^{-\frac{1}{\alpha}} + |\log r|^{\frac{2}{3}}\epsilon^{-\frac{1}{3\alpha}}) \geq \log|Y|.$$

Using the pigeonhole principle, there exist two different $\psi_1, \psi_2 \in Z$ such that

$$\|(\mathsf{d}\Lambda_{B_r}^{nm}(\psi_1) - y_{nm})\|_X \leq \frac{\delta}{\kappa_2+1} \quad \text{and} \quad \|(\mathsf{d}\Lambda_{B_r}^{nm}(\psi_2) - y_{nm})\|_X \leq \frac{\delta}{\kappa_2+1}$$

for some $(y_{nm}) \in Y$. Letting $\psi = \psi_1 - \psi_2$, we obtain that

$$\|(\mathsf{d}\Lambda_{B_r}^{nm}(\psi))\|_X \leq \frac{2\delta}{\kappa_2+1} = \frac{1}{\kappa_2+1}C_R\exp(-|\log r|^{\frac{2}{3}}\epsilon^{-\frac{1}{3\alpha}}),$$

which, with the help of Proposition 2, gives

$$\|\mathsf{d}\Lambda_{B_r}(\psi)\|_* \leq \frac{1}{\kappa_2+1}C_R\exp(-|\log r|^{\frac{2}{3}}\epsilon^{-\frac{1}{3\alpha}}). \tag{41}$$

Finally, since Z is a ϵ-discrete set, $\|\psi\|_{L^\infty(\partial B_r)} \geq \epsilon$ and the proof is complete. □

Remark 6. One may choose

$$\epsilon^{-\frac{1}{\alpha}} = \log^3\left(\frac{\eta_R}{\delta}\right) \quad \left(\text{equivalently,} \quad \delta = \eta_R\exp(-\epsilon^{-\frac{1}{3\alpha}})\right)$$

and take β sufficiently large such that

$$\frac{1}{8}(\mu\beta)^{\frac{1}{\alpha}} > C > C|\log r|^{-2} \quad \text{and} \quad \mu\beta > \epsilon.$$

Then it follows from (31) and $0 < \epsilon < 1$ that

$$\log|Z| \geq \frac{1}{4}\left(\frac{\mu\beta}{\epsilon}\right)^{\frac{1}{\alpha}} > C(|\log r|^{-2}\epsilon^{-\frac{1}{\alpha}} + \epsilon^{-\frac{1}{3\alpha}}) \geq \log|Y|.$$

Following the same argument as above, we then conclude that there exists ψ with $\|\psi\|_{L^\infty} \geq \epsilon$, but

$$\|\mathsf{d}\Lambda_{B_r}(\psi)\|_* \leq \frac{1}{\kappa_2+1}C_R\exp(-\epsilon^{-\frac{1}{3\alpha}}). \tag{42}$$

Comparing with (41), the estimate (42) is clearly not optimal.

4. **Proof of Theorem 1.2.**

In this section, we want to prove Theorem 1.2. We will follow the lines in the proof of Theorem 1.1.

4.1. **Spherical harmonics and series expansion.** For each $\gamma \in \mathbb{R}$, the Banach space $H^\gamma(\mathcal{S}^2)$ can be equipped with the following equivalent norm:

$$\|A\|_{H^\gamma(\mathcal{S}^2)}^2 = \sum_{n\geq 0}\sum_{|m|\leq n}(1+n)^{2\gamma}|a_{nm}|^2 \quad \text{where} \quad A = \sum_{n\geq 0}\sum_{|m|\leq n} a_{nm}Y_n^m.$$

The following proposition is simple but crucial in our work.

Proposition 5. *Let $s \in \mathbb{R}$, and we define the following Banach space:*

$$X_s := \left\{ (a_{nm}) \;\middle|\; \|(a_{nm})\|_{X_s} := \sup_{n\geq 0, |m|\leq n} (1+n)^{\frac{3}{2}-s}|a_{nm}| < \infty \right\}.$$

If $A = \sum_{n\geq 0}\sum_{|m|\leq n} a_{nm}Y_n^m$, then there exists an absolute constant $C_{\rm abs} > 0$ such that

$$\|A\|_{H^{-s}(\mathcal{S}^2)} \leq C_{\rm abs}\|(a_{nm})\|_{X_s}.$$

Proof. Using direct computations, we have

$$\begin{aligned}
\|A\|_{H^{-s}(\mathcal{S}^2)}^2 &= \sum_{n\geq 0}\sum_{|m|\leq n}(1+n)^{-2s}|a_{nm}|^2 = \sum_{n\geq 0}\sum_{|m|\leq n}\frac{1}{(1+n)^3}(1+n)^{3-2s}|a_{nm}|^2 \\
&\leq \sum_{n\geq 0}\sum_{|m|\leq n}\frac{1}{(1+n)^3}\|(a_{nm})\|_{X_s}^2 \\
&= \sum_{n\geq 0}\frac{2n+1}{(1+n)^3}\|(a_{nm})\|_{X_s}^2,
\end{aligned}$$

which proves the proposition with $C_{\rm abs}^2 = \sum_{n\geq 0}\frac{2n+1}{(1+n)^3} < \infty$. □

4.2. **Some elementary computations.** Recall from [6, Theorem 2.15 and 2.16] the following representation of u satisfying (10):

$$u(x) = \sum_{n\geq 0}\sum_{|m|\leq n}\left(\kappa i^{n+1}u_{nm}^\infty h_n^{(1)}(\kappa r)\right)Y_n^m(\hat{x}) \text{ where } u^\infty = \sum_{n\geq 0}\sum_{|m|\leq n}u_{nm}^\infty Y_n^m(\hat{x}), \tag{43}$$

In view of the boundary condition $u = f$ on ∂B_1, we have that

$$f_{nm} = \kappa i^{n+1}u_{nm}^\infty h_n^{(1)}(\kappa) \quad \text{with} \quad f(x) = \sum_{n\geq 0}\sum_{|m|\leq n}f_{nm}Y_n^m(\hat{x}), \tag{44}$$

where $h_n^{(1)}(t)$ is the spherical Hankel function. We can prove the following elementary lemma.

Lemma 4.1. *Let $\kappa > 0$ and define $\tilde{\kappa} := (\frac{\kappa}{2})^{\exp(\kappa)}$. Then there exists a constant $C > 0$, which is independent of n and κ, such that*

$$|h_n^{(1)}(\kappa)|^{-1} \leq \begin{cases} C\kappa 2^{-n} & \text{if } \kappa \leq \log(n), \\ C\kappa\tilde{\kappa} & \text{if } \kappa \geq \log(n), \end{cases}$$

for all $n = 0, 1, 2, \cdots$.

Proof. From [41, (4) (5), Sec. 9.61, p.297], it follows

$$|J_{n+\frac{1}{2}}(\kappa)|^2 + |J_{-n-\frac{1}{2}}(\kappa)|^2 = \frac{2}{\pi\kappa}(-1)^n R_{2n,\frac{1}{2}-n}(\kappa)$$

$$= \frac{2}{\pi\kappa} \sum_{m=0}^{n} \frac{(2\kappa)^{2m-2n}(2n-m)!(2n-2m)!}{((n-m)!)^2 m!} \tag{45}$$

where J_ν is the Bessel function of 1^{st} kind and $R_{n,\nu}$ is the Lommel polynomials (see also DLMF:10.49). From [41, (3), Sec. 3.61, p.74] or [20, p.142], we have

$$N_{n+\frac{1}{2}}(\kappa) = (-1)^{n+1} J_{-n-\frac{1}{2}}(\kappa),$$

where N_ν (some authors denote Y_ν, see e.g. [41]) is the Bessel function of 2^{nd} kind. Therefore, the Hankel function $H_\nu^{(1)}(\kappa) = J_\nu(\kappa) + iN_\nu(\kappa)$ satisfies

$$|H^{(1)}_{n+\frac{1}{2}}(\kappa)|^2 = |J_{n+\frac{1}{2}}(\kappa)|^2 + |N_{n+\frac{1}{2}}(\kappa)|^2 = |J_{n+\frac{1}{2}}(\kappa)|^2 + |J_{-n-\frac{1}{2}}(\kappa)|^2. \tag{46}$$

Combining (45) and (46) gives

$$|H^{(1)}_{n+\frac{1}{2}}(\kappa)|^2 = \frac{2}{\pi\kappa} \sum_{m=0}^{n} \frac{(2\kappa)^{2m-2n}(2n-m)!(2n-2m)!}{((n-m)!)^2 m!}.$$

By the relation

$$h_n^{(1)}(\kappa) = \left(\frac{\pi}{2\kappa}\right)^{\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}}(\kappa),$$

(see e.g. DLMF:10.47), we have

$$|h_n^{(1)}(\kappa)|^2 = \frac{\pi}{2\kappa}|H^{(1)}_{n+\frac{1}{2}}(\kappa)|^2 = \frac{1}{\kappa^2} \sum_{m=0}^{n} \frac{(2\kappa)^{2m-2n}(2n-m)!(2n-2m)!}{((n-m)!)^2 m!}$$

and hence,

$$|h_n^{(1)}(\kappa)|^2 \geq \frac{1}{\kappa^2} \frac{(2\kappa)^{2m-2n}(2n-m)!(2n-2m)!}{((n-m)!)^2 m!}\bigg|_{m=0} = \left(\frac{1}{\kappa}\frac{(2n)!}{(2\kappa)^n n!}\right)^2. \tag{47}$$

Note that

$$\frac{(2n)!}{n!} = \frac{2^{2n}}{\sqrt{\pi}}\Gamma\left(\frac{2n+1}{2}\right).$$

Consequently, we can rewrite the inequality (47) as

$$|h_n^{(1)}(\kappa)| \geq \frac{1}{\kappa}\frac{(2n)!}{(2\kappa)^n n!} = \frac{1}{\kappa}\frac{1}{(2\kappa)^n}\frac{2^{2n}}{\sqrt{\pi}}\Gamma\left(\frac{2n+1}{2}\right) = \frac{1}{\sqrt{\pi}}\frac{1}{\kappa}\left(\frac{2}{\kappa}\right)^n \Gamma\left(\frac{2n+1}{2}\right),$$

that is,

$$|h_n^{(1)}(\kappa)|^{-1} \leq \sqrt{\pi}\kappa\left(\frac{\kappa}{2}\right)^n \frac{1}{\Gamma(\frac{2n+1}{2})}. \tag{48}$$

If $0 < \kappa \leq \log(n)$, since

$$\lim_{n\to\infty} 2^n \left[\left(\frac{\log(n)}{2}\right)^n \frac{1}{\Gamma(\frac{2n+1}{2})}\right] = 0,$$

(48) implies

$$|h_n^{(1)}(\kappa)|^{-1} \leq \sqrt{\pi}\kappa\left(\frac{\log(n)}{2}\right)^n \frac{1}{\Gamma(\frac{2n+1}{2})} \leq C\kappa 2^{-n}$$

for some absolute constant $C > 0$. Otherwise, if $\kappa \geq \log(n)$, (48) gives

$$|h_n^{(1)}(\kappa)|^{-1} \leq \sqrt{\pi}\kappa\left(\frac{\kappa}{2}\right)^n \left[\sup_{r>\frac{1}{2}} \frac{1}{\Gamma(r)}\right] \leq C\kappa\left(\frac{\kappa}{2}\right)^{\exp(\kappa)} = \sqrt{\pi}\kappa\tilde{\kappa},$$

which is our desired lemma. □

In view of Lemma 4.1, we can express (44) as $u_{nm}^{\infty} = u_{nm}^{\infty,(1)} + u_{nm}^{\infty,(2)}$, where

$$u_{nm}^{\infty,(1)} = \frac{1}{\kappa i^{n+1}} (h_n^{(1)}(\kappa))^{-1} \chi_{\kappa \leq \log(n)} f_{nm}, \tag{49a}$$

$$u_{nm}^{\infty,(2)} = \frac{1}{\kappa i^{n+1}} (h_n^{(1)}(\kappa))^{-1} \chi_{\kappa > \log(n)} f_{nm}. \tag{49b}$$

We then estimate $u_{nm}^{\infty,(1)}$ and $u_{nm}^{\infty,(2)}$.

Proposition 6. *Let $R > 0$ and define $B_R^{\infty} := \left\{ f : \partial B_1 \to \mathbb{R} \,\middle|\, |f| \leq R \right\} \subset L^2(\mathcal{S}^2)$. Then there exists a constant C_R, depending only on R, such that*

$$|u_{nm}^{\infty,(1)}| \leq C_R 2^{-n} \leq C_R, \tag{50a}$$

$$|u_{nm}^{\infty,(2)}| \leq C_R \tilde{\kappa}, \tag{50b}$$

where $\tilde{\kappa}$ is defined in Lemma 4.1.

Proof. Let $f = \sum_{n\geq 0} \sum_{|m|\leq n} f_{nm} Y_n^m \in B_R^{\infty}$. For each $n' \geq 0$ and $|m'| \leq n'$, we see that

$$|f_{n'm'}| \leq \left(\sum_{n\geq 0} \sum_{|m|\leq n} |f_{nm}|^2 \right)^{\frac{1}{2}} = \|f\|_{L^2(\partial B_1)} \leq |\partial B_1|^{\frac{1}{2}} R. \tag{51}$$

Combining (49a) with (51), we obtain

$$|u_{nm}^{\infty,(1)}| = \frac{1}{\kappa} |h_n^{(1)}(\kappa)|^{-1} \chi_{\kappa \leq \log(n)} |f_{nm}| \leq \frac{|\partial B_1|^{\frac{1}{2}} R}{\kappa} |h_n^{(1)}(\kappa)|^{-1} \chi_{\kappa \leq \log(n)}. \tag{52}$$

By Lemma 4.1, (50a) follows directly from (52). Similarly, using (49b) and (51), we have

$$|u_{nm}^{\infty,(2)}| = \frac{1}{\kappa} |h_n^{(1)}(\kappa)|^{-1} \chi_{\kappa > \log(n)} |f_{nm}| \leq \frac{|\partial B_1|^{\frac{1}{2}} R}{\kappa} |h_n^{(1)}(\kappa)|^{-1} \chi_{\kappa > \log(n)}. \tag{53}$$

Then (50b) is an easy consequence of (53) with the help of Lemma 4.1. □

4.3. **Construction of a net.** If $s > \frac{3}{2}$, from (50a) and (50b), we have

$$\|(u_{nm}^{\infty}(f))\|_{X_s} \leq C_R \sup_{n\geq 0} \left\{ (1+n)^{\frac{3}{2}-s} (2^{-n} + \tilde{\kappa}) \right\} < \infty$$

for all $f \in B_R^{\infty}$. In other words, $(u_{nm}^{\infty}(B_R^{\infty})) \subset X_s$. Now, we want to construct a δ-net Y of $((u_{nm}^{\infty}(B_R^{\infty})), \|\bullet\|_{X_s})$ which is not too large. Precisely, we want to establish the following lemma.

Lemma 4.2. *Let $s > \frac{3}{2}$ and C_R be the constant given in Proposition 6. If $0 < \delta < C_R$, then there exists a δ-net Y of $((u_{nm}^{\infty}(B_R^{\infty})), \|\bullet\|_{X_s})$ such that*

$$\log |Y| \leq \eta_{s,R} \left[\log \left(1 + \frac{C_R}{\delta} \right) + \frac{C_R \tilde{\kappa}}{\delta} + \left(\frac{C_R \tilde{\kappa}}{\delta} \right)^{\frac{2}{2s-3}} \right]^2 \tag{54}$$

for some constant $\eta_{s,R}$, which depending only on s and R.

Proof of Lemma 4.2. **Step 1: Initialization.** Let ℓ_1 and ℓ_2 be the solution of

$$(1+\ell_1)^{\frac{3}{2}-s} 2^{-\ell_1} = \frac{\delta}{2C_R} \quad \text{and} \quad (1+\ell_2)^{\frac{3}{2}-s} \tilde{\kappa} = \frac{\delta}{2C_R}, \tag{55}$$

respectively. Let n_* be the smallest non-negative integer such that

$$(1+\ell)^{\frac{3}{2}-s} (2^{-\ell_1} + \tilde{\kappa}) \leq \frac{\delta}{C_R} \quad \text{for all } \ell \geq n_*. \tag{56}$$

Since $s > \frac{3}{2}$, we observe that

$$\begin{aligned}&(1+(\ell_1+\ell_2))^{\frac{3}{2}-s}(2^{-(\ell_1+\ell_2)}+\tilde{\kappa})\\&=(1+(\ell_1+\ell_2))^{\frac{3}{2}-s}2^{-(\ell_1+\ell_2)}+(1+(\ell_1+\ell_2))^{\frac{3}{2}-s}\tilde{\kappa}\\&\le(1+\ell_1)^{\frac{3}{2}-s}2^{-\ell_1}+(1+\ell_2)^{\frac{3}{2}-s}\tilde{\kappa}\\&\le\frac{\delta}{2C_R}+\frac{\delta}{2C_R}=\frac{\delta}{C_R}\end{aligned}$$

and, therefore,

$$n_* \le \ell_1+\ell_2. \tag{57}$$

Note that $(\frac{3}{2}-s)\log(1+\ell_1) < 0$. We can see that

$$-\ell_1\log 2 \ge \left(\frac{3}{2}-s\right)\log(1+\ell_1)-\ell_1\log 2=\log\left[(1+\ell_1)^{\frac{3}{2}-s}2^{-\ell_1}\right]=\log\left[\frac{\delta}{2C_R}\right],$$

hence,

$$\ell_1 \le -\frac{1}{\log 2}\log\left[\frac{\delta}{2C_R}\right]=\frac{1}{\log 2}\log\left[\frac{2C_R}{\delta}\right]. \tag{58a}$$

On the other hand, from the definition of ℓ_2, it follows

$$\ell_2+1=\left(\frac{\delta}{2C_R\tilde{\kappa}}\right)^{\frac{2}{3-2s}}=\left(\frac{2C_R\tilde{\kappa}}{\delta}\right)^{\frac{2}{2s-3}}. \tag{58b}$$

Combining (57), (58a), and (58b) implies

$$n_*+1 \le \frac{1}{\log 2}\log\left[\frac{2C_R}{\delta}\right]+\left(\frac{2C_R\tilde{\kappa}}{\delta}\right)^{\frac{2}{2s-3}}. \tag{59}$$

Step 2: Construction of sets. Define $\delta' = \frac{\delta}{2\sqrt{2}}$ and the sets

$$\begin{aligned}Y_1' &:= \left\{\ a=a_1+ia_2\in\delta'\mathbb{Z}+i\delta'\mathbb{Z}\ \middle|\ |a_1|,|a_2|\le C_R\ \right\},\\ Y_2' &:= \left\{\ a=a_1+ia_2\in\delta'\mathbb{Z}+i\delta'\mathbb{Z}\ \middle|\ |a_1|,|a_2|\le C_R\tilde{\kappa}\ \right\},\end{aligned}$$

as well as

$$\begin{aligned}Y_1 &:= \left\{\ (b_{nm})\ \middle|\ \begin{array}{l}\text{if } 0\le n\le n_*, \text{ then } b_{nm}\in Y_1'\\ \text{otherwise, } b_{nm}=0\end{array}\ \right\},\\ Y_2 &:= \left\{\ (c_{nm})\ \middle|\ \begin{array}{l}\text{if } 0\le n\le n_*, \text{ then } c_{nm}\in Y_2'\\ \text{otherwise, } c_{nm}=0\end{array}\ \right\},\end{aligned}$$

and $Y = Y_1 + Y_2$.

Step 3: Verifying Y is a δ-net. Our goal is to construct

$$\begin{cases}(b_{nm})\in Y_1, & \text{an approximation of } (u_{nm}^{\infty,(1)}(f)),\\ (c_{nm})\in Y_2, & \text{an approximation of } (u_{nm}^{\infty,(2)}(f)).\end{cases}$$

- If $n \le n_*$, we take $b'_{nm} \in Y_1'$ (resp. $c'_{nm} \in Y_1'$) be the closest element to $u_{nm}^{\infty,(1)}(f)$ (resp. $u_{nm}^{\infty,(2)}(f)$). Hence, we have

$$|b'_{nm}-u_{nm}^{\infty,(1)}(f)|\le\sqrt{2}\delta' \quad \left(\text{resp.}\quad |c'_{nm}-u_{nm}^{\infty,(2)}(f)|\le\sqrt{2}\delta'\right).$$

Note that $(1+n)^{\frac{3}{2}-s} \le 1$ and thus

$$(1+n)^{\frac{3}{2}-s}\bigg(|b'_{nm} - u_{nm}^{\infty,(1)}(f)| + |c'_{nm} - u_{nm}^{\infty,(2)}(f)|\bigg) \le 2\sqrt{2}\delta' = \delta. \tag{60a}$$

- Otherwise, if $n > n_*$, we simply choose $b'_{nm} = c'_{nm} = 0$. We have

$$\begin{aligned}
&(1+n)^{\frac{3}{2}-s}\bigg(|b'_{nm} - u_{nm}^{\infty,(1)}(f)| + |c'_{nm} - u_{nm}^{\infty,(2)}(f)|\bigg) \\
&= (1+n)^{\frac{3}{2}-s}\bigg(|u_{nm}^{\infty,(1)}(f)| + |u_{nm}^{\infty,(2)}(f)|\bigg) \\
&\le C_R(1+n)^{\frac{3}{2}-s}(2^{-n} + \tilde{\kappa}) \quad \text{(using Proposition 6)} \\
&\le \delta \quad \text{(using (56))}.
\end{aligned} \tag{60b}$$

Combining (60a) and (60b), we conclude that Y is a δ-net of $((u_{nm}^{\infty}(B_R^{\infty})), \|\bullet\|_{X_s})$.

Step 4: Estimating the size of Y. We know that

$$|Y_1'| = \left(1 + \left\lfloor \frac{2C_R}{\delta'} \right\rfloor\right)^2 \le \left(1 + \frac{4\sqrt{2}C_R}{\delta}\right)^2, \tag{61a}$$

$$|Y_2'| = \left(1 + \left\lfloor \frac{2C_R\tilde{\kappa}}{\delta'} \right\rfloor\right)^2 \le \left(1 + \frac{4\sqrt{2}C_R\tilde{\kappa}}{\delta}\right)^2, \tag{61b}$$

and

$$|Y| = |Y_1||Y_2| = |Y_1'|^{n_*+1}|Y_2'|^{n_*+1}. \tag{62}$$

Therefore, combining (59), (61a), and (61b), we can compute

$$\begin{aligned}
\log|Y| &= (n_*+1)\bigg(\log|Y_1| + \log|Y_2|\bigg) \\
&\le \left[\frac{1}{\log 2}\log\left[\frac{2C_R}{\delta}\right] + \left(\frac{2C_R\tilde{\kappa}}{\delta}\right)^{\frac{2}{2s-3}}\right] \times \\
&\quad \times \left[2\log\left(1 + \frac{4\sqrt{2}C_R}{\delta}\right) + 2\log\left(1 + \frac{4\sqrt{2}C_R\tilde{\kappa}}{\delta}\right)\right].
\end{aligned}$$

Since $\log(1+t) \le t$ for all $t \ge 0$, we have that

$$\begin{aligned}
\log|Y| \le 2&\left[\frac{1}{\log 2}\log\left[\frac{2C_R}{\delta}\right] + \left(\frac{2C_R\tilde{\kappa}}{\delta}\right)^{\frac{2}{2s-3}}\right] \\
&\times \left[\log\left(1 + \frac{4\sqrt{2}C_R}{\delta}\right) + \frac{4\sqrt{2}C_R\tilde{\kappa}}{\delta}\right]
\end{aligned}$$

and (54) is proved. □

Choosing $s = \frac{5}{2}$ in Lemma 4.2, we immediately obtain the following corollary.

Corollary 1. *For any $0 < \delta < C_R$, there exists a δ-net Y of $((u_{nm}^{\infty}(B_R^{\infty})), \|\bullet\|_{X_{\frac{5}{2}}})$ such that*

$$\log|Y| \le \eta_R\left[\log\left(1 + \frac{C_R}{\delta}\right) + \frac{2C_R\tilde{\kappa}}{\delta}\right]^2 \tag{63}$$

for some constant η_R, depending only on R.

Remark 7. If $\kappa \geq 2$, then $\tilde{\kappa} \geq 1$. In this case, we see that

$$\inf_{0<\delta<C_R} \frac{1}{\tilde{\kappa}}\left[\log\left(1+\frac{C_R}{\delta}\right)+\frac{2C_R\tilde{\kappa}}{\delta}\right] = \frac{\log 2}{\tilde{\kappa}}+2 \leq 2+\log 2.$$

Therefore, given any ϵ satisfying

$$2+\log 2 < \epsilon^{-\frac{1}{\alpha}} < \infty \quad \left(\text{equivalently,} \quad \epsilon \in (0,(2+\log 2)^{-\alpha})\right), \tag{64}$$

there exists a unique $\delta \in (0, C_R)$ such that

$$\epsilon^{-\frac{1}{\alpha}} = \frac{1}{\tilde{\kappa}}\left[\log\left(1+\frac{C_R}{\delta}\right)+\frac{2C_R\tilde{\kappa}}{\delta}\right]. \tag{65}$$

Otherwise, if $0 < \kappa < 2$, then $0 < \tilde{\kappa} \leq 1$. In this case, we note that

$$\inf_{0<\delta<C_R} \left[\log\left(1+\frac{C_R}{\delta}\right)+\frac{2C_R\tilde{\kappa}}{\delta}\right] = \log 2 + 2\tilde{\kappa} \leq 2+\log 2.$$

Similarly, given any ϵ satisfying (64), there exists a unique $\delta \in (0, C_R)$ such that

$$\epsilon^{-\frac{1}{\alpha}} = \log\left(1+\frac{C_R}{\delta}\right)+\frac{2C_R\tilde{\kappa}}{\delta}. \tag{66}$$

Putting together (65) and (66) implies that given any ϵ satisfying (64), there exists a unique $\delta \in (0, C_R)$ such that

$$\epsilon^{-\frac{1}{\alpha}} = \frac{1}{\max\{\tilde{\kappa},1\}}\left[\log\left(1+\frac{C_R}{\delta}\right)+\frac{2C_R\tilde{\kappa}}{\delta}\right]. \tag{67}$$

In view of (63) and (67), we then conclude that

$$\log|Y| \leq \eta_R \max\{\tilde{\kappa},1\}^2 \epsilon^{-\frac{2}{\alpha}}. \tag{68}$$

4.4. **Proof of the main result.**

Proof of Theorem 1.2. As above, we take $s=\frac{5}{2}$ and fix any auxiliary parameters $R>0$ and $\alpha>0$. For each $0<\epsilon<\min\{(2+\log 2)^{-\alpha}, R, \mu\beta\}$, let Z be an ϵ-discrete set constructed in Proposition 1 with $d=2$ and $r=1$. Let δ be given in (67). Next, we construct a δ-net Y described in Corollary 1 and (68) holds. Clearly, Y is also a δ-net for $((u_{nm}^{\infty}(Z)), \|\bullet\|_{X_{\frac{5}{2}}})$.

We now choose $\beta=\beta(\alpha,R,\kappa)$, which is independent of ϵ, such that

$$\mu\beta \geq R \quad \text{and} \quad |Z| \geq \exp\left[\frac{1}{8}\left(\frac{\mu\beta}{\epsilon}\right)^{\frac{2}{\alpha}}\right] > \exp\left[\eta_R \max\{\tilde{\kappa},1\}^2\epsilon^{-\frac{2}{\alpha}}\right] \geq |Y|.$$

Therefore, using pigeonhole principle, we can choose two different smooth functions $f_1, f_2 \in Z$ such that

$$\|(u_{nm}^{\infty}(f_1)-y_{nm})\|_{X_{\frac{5}{2}}} \leq \delta \quad \text{and} \quad \|(u_{nm}^{\infty}(f_2)-y_{nm})\|_{X_{\frac{5}{2}}} \leq \delta.$$

Letting $f=f_1-f_2$ and using Proposition 5, we obtain

$$\|u^{\infty}(f)\|_{H^{-\frac{5}{2}}(\mathcal{S}^2)} \leq C_{\rm abs}\|(u_{nm}^{\infty}(f))\|_{X_{\frac{5}{2}}} \leq 2C_{\rm abs}\delta \quad \text{and} \quad \|f\|_{L^{\infty}(\partial B_1)} \geq \epsilon. \tag{69}$$

To finish the proof, we discuss two cases.

- **Case 1.** If $\frac{C_R\tilde{\kappa}}{\delta} \leq \log(1+\frac{C_R}{\delta})$, then (67) implies
$$\begin{aligned}\epsilon^{-\frac{1}{\alpha}} &= \frac{1}{\max\{\tilde{\kappa},1\}}\left[\log\left(1+\frac{C_R}{\delta}\right)+\frac{2C_R\tilde{\kappa}}{\delta}\right] \\ &\leq \frac{3}{\max\{\tilde{\kappa},1\}}\log\left(1+\frac{C_R}{\delta}\right) \leq \frac{3}{\max\{\tilde{\kappa},1\}}\log\left(\frac{2C_R}{\delta}\right),\end{aligned}$$
which gives
$$\delta \leq 2C_R \exp\left(-\frac{\max\{\tilde{\kappa},1\}}{3}\epsilon^{-\frac{1}{\alpha}}\right). \tag{70}$$
- **Case 2.** If $\frac{C_R\tilde{\kappa}}{\delta} \geq \log(1+\frac{C_R}{\delta})$, then (67) implies
$$\begin{aligned}\epsilon^{-\frac{1}{\alpha}} &= \frac{1}{\max\{\tilde{\kappa},1\}}\left[\log\left(1+\frac{C_R}{\delta}\right)+\frac{2C_R\tilde{\kappa}}{\delta}\right] \\ &\leq \frac{1}{\max\{\tilde{\kappa},1\}}\frac{3C_R\tilde{\kappa}}{\delta} = \frac{3C_R\min\{1,\tilde{\kappa}\}}{\delta},\end{aligned}$$
that is,
$$\delta \leq 3\min\{1,\tilde{\kappa}\}C_R\epsilon^{\frac{1}{\alpha}}. \tag{71}$$

Combining (70) and (71), we obtain
$$\delta \leq 2C_R \exp\left(-\frac{\max\{\tilde{\kappa},1\}}{3}\epsilon^{-\frac{1}{\alpha}}\right) + 3\min\{1,\tilde{\kappa}\}C_R\epsilon^{\frac{1}{\alpha}}. \tag{72}$$
Finally, substituting (72) into (69), the proof is completed. □

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