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Two-dimensional metric spheres from gluing hemispheres

Toni Ikonen

University of Jyvaskyla, Department of Mathematics and Statistics, Jyvaskyla, Finland

Correspondence
Toni Ikonen, University of Jyvaskyla, Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014, Jyvaskyla, Finland.
Email: toni.m.h.ikonen@jyu.fi

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Abstract

We study metric spheres (Z, d_Z) obtained by gluing two hemispheres of $\mathbb{S}^2$ along an orientation-preserving homeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$, where d_Z is the canonical distance that is locally isometric to $\mathbb{S}^2$ off the seam. We show that if (Z, d_Z) is quasiconformally equivalent to $\mathbb{S}^2$, in the geometric sense, then g is a welding homeomorphism with conformally removable welding curves. We also show that g is bi-Lipschitz if and only if (Z, d_Z) has a 1-quasiconformal parametrization whose Jacobian is comparable to the Jacobian of a quasiconformal mapping $h : \mathbb{S}^2 \to \mathbb{S}^2$. Furthermore, we show that if g^{-1} is absolutely continuous and g admits a homeomorphic extension with exponentially integrable distortion, then (Z, d_Z) is quasiconformally equivalent to $\mathbb{S}^2$.

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1 | INTRODUCTION

In this paper, we work in the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. We denote the equator $\mathbb{S}^2 \cap (\mathbb{R}^2 \times \{0\})$ by $\mathbb{S}^1$ and endow $\mathbb{S}^2$ with the length distance σ induced by the Euclidean distance of $\mathbb{R}^3$. The open southern and northern hemispheres are denoted by Z_1 and Z_2, respectively. Here $(0, 0, 1) \in Z_2$.

Consider an orientation-preserving homeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$, mapping the boundary of $\overline{Z}_1$ to the boundary of $\overline{Z}_2$. We identify each $z \in \mathbb{S}^1$ with its image $g(z) \in \mathbb{S}^1$. With this identification, we obtain a set Z and inclusion maps $\iota_1 : \overline{Z}_1 \to Z$ and $\iota_2 : \overline{Z}_2 \to Z$. We call $S_Z = \iota_1(\mathbb{S}^1) = \iota_2(\mathbb{S}^1)$ the *seam* of Z.

We construct a pseudodistance d_Z on Z, see Section 3, making the inclusion maps local isometries off the seam and 1-Lipschitz everywhere. We consider the quotient map $Q: Z \to \widetilde{Z}$ identifying points $x, y \in Z$ whenever $d_Z(x,y) = 0$, and endow $\widetilde{Z}$ with the associated quotient distance.

We are interested in this construction for the following reason: whenever the metric space $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$, there exist Riemann maps $\phi_1 : Z_1 \to \Omega_1$, $\phi_2 : Z_2 \to \Omega_2$ onto the complementary components of a Jordan curve C with $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$; with the Carathéodory theorem we can make sense of the composition $\phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$ [16]. Any such g is called a *welding homeomorphism* and C a *welding curve*. A long-standing problem is to understand which homeomorphisms g satisfy $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$ for some Riemann maps. We refer to the survey articles [18, 37] for further background information.

We also investigate the properties of $\widetilde{Z}$, given an arbitrary welding homeomorphism g. We show in Section 4 that the 1D Hausdorff measures on the seam $Q(S_Z)$ and on (the tangents of) C are closely connected, using results from classical complex analysis [16]. For example, our results show that a given subarc of the welding curve has tangents only in a set negligible to the 1D Hausdorff measure if and only if the quotient map Q collapses the corresponding part of the seam to a point.

We present in Sections 7.1 and 7.2 examples illustrating that for some homeomorphisms g, after removing a portion E' of the seam $Q(S_Z)$, one can find a 1-quasiconformal embedding $\psi: \widetilde{Z} \setminus E' \to \mathbb{S}^2$, but not necessarily a quasiconformal homeomorphism $\Psi: \widetilde{Z} \to \mathbb{S}^2$. A similar phenomenon was investigated in [17] and [7] in more detail.

We now state our first result.

Theorem 1.1. *Let $g: \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism. The following are quantitatively equivalent.*

(1) *g is L-bi-Lipschitz;*
(2) *there exists an L'-bi-Lipschitz homeomorphism $\Psi: \widetilde{Z} \to \mathbb{S}^2$;*
(3) *there exists $C' \geqslant 0$ such that for every $y \in Q(S_Z)$,*

$$\liminf_{r \to 0^+} \frac{\mathcal{H}^2_{\widetilde{Z}}(\overline{B}_{\widetilde{Z}}(y,r))}{\pi r^2} \leqslant C'.$$

In the implications "(1) ⇒ (2)" we may take $L' = L$, in "(2) ⇒ (3)" $C' = (L')^4$, and in "(3) ⇒ (1)" $L = \pi C'$.

We prove "(1) ⇒ (2)" by observing that if $g: \mathbb{S}^1 \to \mathbb{S}^1$ admits an L'-bi-Lipschitz extension $\phi: \overline{Z}_2 \to \overline{Z}_2$, the space $\widetilde{Z}$ has an L'-bi-Lipschitz parametrization. That we may take $L' = L$ in "(1) ⇒ (2)," follows by applying a known planar extension result [24] and stereographic projection.

The claim "(2) ⇒ (3)" is a straightforward consequence of the properties of Hausdorff measures. The implication "(3) ⇒ (1)" is proved by carefully analysing the behaviour of the inclusion mappings $\iota_i: \overline{Z}_i \to \widetilde{Z}$ at the equator $\mathbb{S}^1$. Notice that the ι_i are 1-Lipschitz everywhere and local isometries outside the equator. This implies $C' \geqslant 1$ in (3). Remark 5.9 shows two ways to improve the bi-Lipschitz constant $\pi C'$. The improvements imply that as $C' \to 1^+$ in (3), the bi-Lipschitz constant of g converges to one. In particular, (3) holds with $C' = 1$ if and only if g is an isometry.

Theorem 1.1 is closely related to the following result.

Theorem 1.2. *If an orientation-preserving homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ is L-bi-Lipschitz, there exists a 1-quasiconformal homeomorphism $\varphi\colon \mathbb{S}^2 \to \widetilde{Z}$ and a K-quasiconformal homeomorphism $h\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that the Jacobians satisfy*

$$C^{-1}J_h(x) \leqslant J_\varphi(x) \leqslant CJ_h(x) \quad \text{for } \mathcal{H}^2_{\mathbb{S}^2}\text{-almost everywhere, } x \in \mathbb{S}^2 \tag{1.1}$$

for $K = L^4$ and $C = L^2$. Conversely, if there exists K, C, and h for which Equation (1.1) holds, then g is $\pi(KC)^2$-bi-Lipschitz.

The Jacobians are defined in Section 2.3. We note that if $h\colon \mathbb{S}^2 \to \mathbb{S}^2$ is an orientation-preserving quasiconformal homeomorphism, the J_h coincides with the usual distributional Jacobian; see for example [2, Section 3.8].

If g is L-bi-Lipschitz, the existence of φ and h is a straightforward consequence of the implication "(1) ⇒ (2)" in Theorem 1.1. If h and φ exist, we first check that $\Psi = h \circ \varphi^{-1}$ is bi-Lipschitz, the study of the seam requiring a careful argument, and use the implications "(2) ⇒ (3) ⇒ (1)" from Theorem 1.1 to verify that g is bi-Lipschitz.

Theorem 1.2 is a special case of the *quasiconformal Jacobian problem*: which weights $\omega\colon \mathbb{S}^2 \to [0,\infty]$ are comparable to the Jacobians of quasiconformal homeomorphisms $h\colon \mathbb{S}^2 \to \mathbb{S}^2$; see [6, 10], and references therein for further reading.

Given that (1) and (3) are equivalent in Theorem 1.1, it is not entirely clear for which classes of homeomorphisms one can expect $\widetilde{Z}$ to be quasiconformally equivalent to $\mathbb{S}^2$, or what kind of geometric properties one can expect from such a $\widetilde{Z}$.

Question 1.3. Let $\widetilde{Z}$ be the metric space obtained from a homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$. When can we find a quasiconformal homeomorphism $\psi\colon \widetilde{Z} \to \mathbb{S}^2$? What kind of restrictions does this impose on g?

As an example, if g is a welding homeomorphism corresponding to the von Koch snowflake, then $d_Z(x,y) = 0$ for every pair of points in the seam, see Remark 4.2. Hence $\widetilde{Z}$ can fail to be quasiconformally equivalent, or homeomorphic, to $\mathbb{S}^2$ when g is in quasisymmetry. We show that a simple measure-theoretic assumption removes this obstruction.

Proposition 1.4. *Let $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a quasisymmetry whose inverse is absolutely continuous. Then $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$.*

The absolute continuity of g^{-1} is used in two ways. First, it guarantees that $\widetilde{Z} = (Z, d_Z)$. Second, if $\psi\colon \overline{Z}_2 \to \overline{Z}_2$ is a quasisymmetric extension of g, we show that the homeomorphism $H\colon \mathbb{S}^2 \to \widetilde{Z}$ satisfying $H|_{Z_1} = \iota_1$ and $H|_{Z_2} = \iota_2 \circ \psi|_{Z_2}$ is quasiconformal. A key step in the proof is showing the Sobolev regularity $H^{-1} \in N^{1,2}(\widetilde{Z}, \mathbb{S}^2)$; the absolute continuity of g^{-1} is applied here.

Proposition 1.4 is a special case of the following stronger result.

Theorem 1.5. *Let $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism whose inverse is absolutely continuous. If g extends to a homeomorphism $\psi\colon \overline{Z}_2 \to \overline{Z}_2$ for which $\psi|_{Z_2}$ has exponentially integrable distortion, then $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$.*

We now explain the main steps of the proof of Theorem 1.5. We first show that there exists a homeomorphism $H\colon \mathbb{S}^2 \to \widetilde{Z}$ with exponentially integrable distortion. We also have

$H^{-1} \in N^{1,2}(\widetilde{Z}, \mathbb{S}^2)$; see Remark 6.8. The exponential integrability of the distortion of H is used to verify the reciprocality condition of $\widetilde{Z}$, see Definition 2.5. Then [31, Theorem 1.4] shows that $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$. The key ingredients in the proof are the condenser estimates for mappings of exponentially distortion [27], applicable because $H^{-1} \in N^{1,2}(\widetilde{Z}, \mathbb{S}^2)$, and the Stoilow factorization theorem [2, Chapter 20]. There are some known criteria which guarantee that g admits an extension as in Theorem 1.5; see [25, 39].

In Section 7.1, we present an example of $g : \mathbb{S}^1 \to \mathbb{S}^1$ that is locally bi-Lipschitz outside a single point, but for which $\widetilde{Z}$ is not quasiconformally equivalent to $\mathbb{S}^2$. This illustrates that the absolute continuity of g^{-1} is not enough to guarantee that $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$. This fact is a consequence of the following result, partially answering Question 1.3.

Theorem 1.6. *Suppose that $g : \mathbb{S}^1 \to \mathbb{S}^1$ is an orientation-preserving homeomorphism for which there exists a quasiconformal homeomorphism $h : \mathbb{S}^2 \to \widetilde{Z}$. Then $\widetilde{Z} = (Z, d_Z)$ and there exists a 1-quasiconformal homeomorphism $\pi : \mathbb{S}^2 \to \widetilde{Z}$. Furthermore, g is a welding homeomorphism whose welding curves are conformally removable.*

The first step in the proof of Theorem 1.6 is showing that h can be assumed to be 1-quasiconformal. Then, up to an orientation-reversing Möbius transformation, $\phi_i = h^{-1} \circ \iota_i : Z_i \to \mathbb{S}^2$ are Riemann maps with welding curve $C = h^{-1}(Q(S_Z))$ and welding homeomorphism $\phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$. The equality $\widetilde{Z} = (Z, d_Z)$ and the conformal removability of C follow from a connection we show between the tangents of the welding curve C and the Hausdorff 1-measure on the seam $Q(S_Z)$; see Section 4. The equality $\widetilde{Z} = (Z, d_Z)$ implies $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$.

We recall that a compact proper subset $K \subset \mathbb{S}^2$ is *conformally removable* if every homeomorphism $M : \mathbb{S}^2 \to \mathbb{S}^2$ conformal in $\mathbb{S}^2 \setminus K$ is Möbius. The von Koch snowflake example illustrates that conformal removability of a welding curve C is not enough to guarantee even that $\widetilde{Z}$ is homeomorphic to $\mathbb{S}^2$. We refer the reader to [37] and [38] for further reading on conformal weldings and the connections to conformal removability. See [20] for some results in the context of Theorem 1.5.

The paper is structured as follows. In Section 2, we introduce our notations and some preliminary results. In Section 3, we analyse the distance d_Z induced by any given homeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$. When g is a welding homeomorphim, we establish in Section 4 a connection between the geometry of the seam S_Z and the tangents of the corresponding welding curves C. We also prove Theorem 1.6 in this section. In Section 5, we prove Theorems 1.1 and 1.2. Proposition 1.4 and Theorem 1.5 are proved in Section 6. In Section 7, we give some concluding remarks.

2 | PRELIMINARIES

2.1 | Notation

Let (Y, d_Y) be a metric space. We sometimes drop the subscript from d_Y when there is no chance for confusion. For all $Q \geqslant 0$, the *Q-dimensional Hausdorff measure*, or a *Hausdorff Q-measure*, is defined by

$$\mathcal{H}_Y^Q(B) = \frac{\alpha(Q)}{2^Q} \sup_{\delta > 0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} B_i)^Q : B \subset \bigcup_{i=1}^{\infty} B_i, \operatorname{diam} B_i < \delta \right\}$$

for all sets $B \subset Y$, where $\alpha(Q)$ is chosen so that $\mathcal{H}^n_{\mathbb{R}^n}$ coincides with the Lebesgue measure $\mathcal{L}^n$ for all positive integers.

The *length* of a path $\gamma\colon [a,b] \to Y$ is defined as

$$\ell_d(\gamma) = \sup \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i+1})),$$

the supremum taken over all finite partitions $a = t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n+1} = b$. A path is *rectifiable* if it has finite length.

The *metric speed* of a path $\gamma\colon [a,b] \to Y$ at the point $t \in [a,b]$ is defined as

$$v_\gamma(t) = \lim_{t \neq s \to t} \frac{d(\gamma(s), \gamma(t))}{|s-t|}$$

whenever this limit exists. The limit exists $\mathcal{L}^1$-almost everywhere for every rectifiable path [12, Theorem 2.1].

A rectifiable path $\gamma\colon [a,b] \to Y$ is *absolutely continuous* if for all $a \leqslant s \leqslant t \leqslant b$,

$$d(\gamma(t), \gamma(s)) \leqslant \int_s^t v_\gamma(u)\, d\mathcal{L}^1(u)$$

with $v_\gamma \in L^1([a,b])$ and $\mathcal{L}^1$ the Lebesgue measure on the real line. Equivalently, the rectifiable path γ is absolutely continuous if it maps sets of $\mathcal{L}^1$-measure zero to sets of $\mathcal{H}^1_Y$-measure zero [12, Section 3].

Let $\gamma\colon [a,b] \to X$ be an absolutely continuous path. Then the *(path) integral* of a Borel function $\rho\colon X \to [0,\infty]$ over γ is

$$\int_\gamma \rho\, ds = \int_a^b (\rho \circ \gamma) v_\gamma\, d\mathcal{L}^1. \tag{2.1}$$

If γ is rectifiable, then the path integral of ρ over γ is defined to be the path integral of ρ over the arc length parametrization γ_s of γ; see [19, Chapter 5] for further details.

Given a Borel set $A \subset Y$, the *length* of a path $\gamma\colon [a,b] \to Y$ in A is defined as $\int_Y \chi_A(y) \#(\gamma^{-1}(y))\, d\mathcal{H}^1_Y(y)$, where $\#(\gamma^{-1}(x))$ is the counting measure of $\gamma^{-1}(x)$. For $A = Y$, [15, Theorem 2.10.13] states

$$\ell(\gamma) = \int_Y \#(\gamma^{-1}(y))\, d\mathcal{H}^1_Y(y). \tag{2.2}$$

When γ is rectifiable, for every Borel function $\rho\colon Y \to [0,\infty]$,

$$\int_\gamma \rho\, ds = \int_Y \rho(y) \#(\gamma^{-1}(y))\, d\mathcal{H}^1_Y(y). \tag{2.3}$$

The equality (2.3) follows from [15, Theorem 2.10.13] via a standard approximation argument using simple functions.

2.2 | Metric Sobolev spaces

In this section we give an overview of Sobolev theory in the metric surface setting, and refer to [19] for a comprehensive introduction.

Let Γ be a family of paths in Y. A Borel function $\rho\colon Y \to [0,\infty]$ is *admissible* for Γ if the path integral $\int_\gamma \rho\, ds \geqslant 1$ for all rectifiable paths $\gamma \in \Gamma$. Given $1 \leqslant p < \infty$, the *p-modulus* of Γ is

$$\operatorname{mod}_p \Gamma = \inf \int_Y \rho^p \, d\mathcal{H}^2_Y,$$

where the infimum is taken over all admissible functions ρ. Observe that if Γ_1 and Γ_2 are path families and every path $\gamma_1 \in \Gamma_1$ contains a subpath $\gamma_2 \in \Gamma_2$, then $\operatorname{mod}_p \Gamma_1 \leqslant \operatorname{mod}_p \Gamma_2$. In particular, this holds if $\Gamma_1 \subset \Gamma_2$. When $p = 2$, and there is no chance for confusion, we omit the subscript from mod_2.

If ρ is admissible for a path family $\Gamma \setminus \Gamma_0$, where $\operatorname{mod}_p \Gamma_0 = 0$, we say that ρ is *p-weakly admissible* for Γ. If a property holds for every path $\gamma \in \Gamma$ except in a subfamily of p-modulus zero, the property is said to hold *on p-almost every path* in Γ. We also refer to 2-almost every path as *almost every* path.

We recall the following lemma [19, Lemma 5.2.8].

Lemma 2.1. *Let $1 \leqslant p < \infty$. A family of nonconstant paths Γ satisfies $\operatorname{mod}_p \Gamma = 0$ if and only if there exists $\rho\colon Y \to [0,\infty]$, $\rho \in L^p(Y)$ with*

$$\infty = \int_\gamma \rho\, ds \quad \textit{for every } \gamma \in \Gamma.$$

Let $\psi\colon (Y, d_Y) \to (Z, d_Z)$ be a mapping between metric spaces Y and Z. A Borel function $\rho\colon Y \to [0,\infty]$ is an *upper gradient of* ψ if

$$d_Y(\psi(x), \psi(y)) \leqslant \int_\gamma \rho\, ds$$

for every rectifiable path $\gamma\colon [a,b] \to Y$ connecting x to y. The function ρ is a *p-weak upper gradient of* ψ if the same holds for p-almost every rectifiable path.

A p-weak upper gradient $\rho \in L^p_{\mathrm{loc}}(Y)$ of ψ is *minimal* if it satisfies $\rho \leqslant \widetilde{\rho}$ almost everywhere for all p-weak upper gradients $\widetilde{\rho} \in L^p_{\mathrm{loc}}(Y)$ of ψ. If ψ has a p-weak upper gradient $\rho \in L^p_{\mathrm{loc}}(Y)$, then ψ has a minimal p-weak upper gradient, which we denote by ρ_ψ. We refer to Section 6 of [19] and Section 3 of [36] for further details. Minimal 2-weak upper gradients are also referred to as *minimal weak upper gradients*.

Fix a point $z \in Z$, and let $d_z = d_Z(\cdot, z)$. The space $L^p(Y, Z)$ is defined as the collection of measurable maps $\psi\colon Y \to Z$ such that $d_z \circ \psi$ is in $L^p(Y)$. Moreover, $L^p_{\mathrm{loc}}(Y, Z)$ is defined as those measurable maps $\psi\colon Y \to Z$ for which, for all $y \in Y$, there is an open set $U \subset Y$ containing y such that $\psi|_U$ is in $L^p(U, Z)$.

The metric Sobolev space $N^{1,p}_{\mathrm{loc}}(Y, Z)$ consists of those maps $\psi\colon Y \to Z$ in $L^p_{\mathrm{loc}}(Y, Z)$ that have a minimal p-weak upper gradient $\rho_\psi \in L^p_{\mathrm{loc}}(Y)$.

For subsets $\emptyset \neq U \subset Y$, we say that $\psi \in N^{1,p}(U,Z)$ if $\psi|_U \in N^{1,p}_{\mathrm{loc}}(U,Z)$, $\rho_{\psi|U} \in L^p(U)$ and $\psi|_U \in L^p(U,Z)$. If $Z = \mathbb{R}$, we denote $N^{1,p}(U,Z) = N^{1,p}(U)$, and in the case $p = 2$,

$$E(\psi) := 2^{-1}\left\|\rho_\psi\right\|^2_{L^2(U)}.$$

We refer to $E(\psi)$ as the *Dirichlet energy* of ψ.

We repeatedly use the following technical lemma in later sections.

Lemma 2.2. *Let $\psi\colon Y \to Z$ be continuous, $\rho\colon Y \to [0,\infty]$ a Borel function and $\gamma\colon [0,1] \to Y$ absolutely continuous with $\int_\gamma \rho\, ds < \infty$.*

Suppose that $E \subset Y$ is compact, $\mathcal{H}^1_Z(\psi(E)) = 0$, and $\ell(\psi \circ \gamma|_I) \leqslant \int_{\gamma|_I} \rho\, ds$ for each closed interval $I \subset [0,1] \setminus \gamma^{-1}(E)$. Then $\ell(\psi \circ \gamma) \leqslant \int_\gamma \rho\, ds$.

Proof. First, for every closed interval $J \subset [0,1] \setminus \gamma^{-1}(E)$, $\psi \circ \gamma|_J$ is absolutely continuous with $v_{\psi\circ\gamma}(s) \leqslant (\rho \circ \gamma)(s) v_\gamma(s)$ for $\mathcal{L}^1$-almost every $s \in J$. This follows from [19, Proposition 6.3.2].

Second, consider the connected components $\{I_i\}_{i=1}^\infty$ of $[0,1] \setminus (\psi \circ \gamma)^{-1}(\psi(E))$. Notice that $I_i \subset [0,1] \setminus \gamma^{-1}(E)$ for every i.

Let $J_i = \overline{I_i}$. Then $v_{\psi\circ\gamma}(s) \leqslant (\rho \circ \gamma|_{J_i})(s)$ $\mathcal{L}^1$-almost everywhere on J_i (on I_i). This fact, the continuity of $\psi \circ \gamma$ and $\int_\gamma \rho\, ds < \infty$ imply

$$\ell(\psi \circ \gamma|_{J_i}) \leqslant \int_{I_i} (\rho \circ \gamma) v_\gamma\, ds < \infty.$$

By summing over i, we conclude

$$\sum_{i=1}^\infty \ell(\psi \circ \gamma|_{J_i}) \leqslant \int_{\bigcup_{i=1}^\infty I_i} (\rho \circ \gamma) v_\gamma\, ds \leqslant \int_\gamma \rho\, ds.$$

Given $\mathcal{H}^1_Z(\psi(E)) = 0$, (2.2) and (2.3) imply

$$\begin{aligned}\ell(\psi \circ \gamma) &= \int_{Z\setminus\psi(E)} \#((\psi \circ \gamma)^{-1}(x))\, d\mathcal{H}^1_Z(x)\\ &\leqslant \sum_{i=1}^\infty \int_{Z\setminus\psi(E)} \#((\psi \circ \gamma|_{J_i})^{-1}(x))\, d\mathcal{H}^1_Z(x)\\ &= \sum_{i=1}^\infty \ell(\psi \circ \gamma|_{J_i}) \leqslant \int_\gamma \rho\, ds.\end{aligned}$$

Hence $\ell(\psi \circ \gamma) \leqslant \int_\gamma \rho\, ds$. □

2.3 | Measure theory

Let Y be a Borel subset of a complete and separable metric space. A Borel measure μ on Y is *σ-finite* if there exists a Borel decomposition $\{B_i\}_{i=1}^\infty$ of Y for which $\mu(B_i) < \infty$ for every i.

A pair of σ-finite Borel measures μ and ν on Y are said to be *mutually singular* if there exists a Borel set $B \subset Y$ such that $\mu(B) = 0$ and $\nu(Y \setminus B) = 0$. The measure μ admits a *Lebesgue decomposition* (with respect to ν), where $\mu = f \cdot \nu + \mu^{\perp}$, with $\mu^{\perp}$ and ν mutually singular and f Borel measurable [9, Sections 3.1–3.2 in Volume I]. We say that μ and ν are *mutually absolutely continuous* if $\mu = f \cdot \nu$ with density $f > 0$ ν-almost everywhere.

Given a homeomorphism $\psi : Y \to Z$ and measures ν on Y and μ on Z, the measure $\psi^*\mu(B) = \mu(\psi(B))$ is called the *pullback measure*. Such a measure admits a decomposition $\psi^*\mu = f \cdot \nu + \mu^{\perp}$ with ν and $\mu^{\perp}$ mutually singular. If $\nu = \mathcal{H}^2_Y$ and $\mu = \mathcal{H}^2_Z$, the density f is called the *Jacobian* of ψ and denoted by J_ψ.

2.4 | Quasiconformal mappings

Here we define quasiconformal maps and recall some basic facts.

Definition 2.3. Let (Y, d_Y) and (Z, d_Z) be metric spaces with locally finite Hausdorff 2-measures. A homeomorphism $\psi : (Y, d_Y) \to (Z, d_Z)$ is *quasiconformal* if there exists $K \geqslant 1$ such that for all path families Γ in Y

$$K^{-1} \operatorname{mod} \Gamma \leqslant \operatorname{mod} \psi\Gamma \leqslant K \operatorname{mod} \Gamma, \tag{2.4}$$

where $\psi\Gamma = \{\psi \circ \gamma : \gamma \in \Gamma\}$. If Equation (2.4) holds with a constant $K \geqslant 1$, we say that ψ is K-*quasiconformal*.

A special case of [36, Theorem 1.1] yields the following.

Theorem 2.4. *Let Y and Z be locally compact separable metric spaces with locally finite Hausdorff 2-measure and $\psi : Y \to Z$ a homeomorphism. The following are equivalent for the same constant $K > 0$:*

(i) $\operatorname{mod} \Gamma \leqslant K \operatorname{mod} \psi\Gamma$ *for all path families Γ in Y.*
(ii) $\psi \in N^{1,2}_{\mathrm{loc}}(Y, Z)$ *and satisfies*

$$\rho^2_\psi(y) \leqslant K J_\psi(y)$$

for $\mathcal{H}^2_Y$-almost every $y \in Y$.

The *outer dilatation* of ψ is the smallest constant $K_O \geqslant 0$ for which the modulus inequality $\operatorname{mod} \Gamma \leqslant K_O \operatorname{mod} \psi\Gamma$ holds for all Γ in Y. The *inner dilatation* of ψ is the smallest constant $K_I \geqslant 0$ for which $\operatorname{mod} \psi\Gamma \leqslant K_I \operatorname{mod} \Gamma$ holds for all Γ in Y. The number $K(\psi) = \max\{K_I(\psi), K_O(\psi)\}$ is the *maximal dilatation* of ψ.

For a set $G \subset Y$ and disjoint sets $F_1, F_2 \subset G$, let $\Gamma(F_1, F_2; G)$ denote the family of paths with each path starting at F_1, ending at F_2 and whose images are contained in G. A *quadrilateral* is a set Q homeomorphic to $[0, 1]^2$ with boundary ∂Q consisting of four boundary arcs, overlapping only at the end points, labelled $\xi_1, \xi_2, \xi_3, \xi_4$ in cyclic order.

A *metric surface* is a separable metric space Y with locally finite Hausdorff 2-measure that is homeomorphic to a (connected) 2-manifold without boundary.

Definition 2.5. A metric surface Y is *reciprocal* if there exists a constant $\kappa \geqslant 1$ such that

$$\kappa^{-1} \leqslant \operatorname{mod} \Gamma(\xi_1, \xi_3; Q) \operatorname{mod} \Gamma(\xi_2, \xi_4; Q) \leqslant \kappa \tag{2.5}$$

for every quadrilateral $Q \subset Y$, and

$$\lim_{r \to 0^+} \operatorname{mod} \Gamma\left(\overline{B}_Y(y, r), Y \setminus B_Y(y, R); \overline{B}_Y(y, R)\right) = 0 \tag{2.6}$$

for all $y \in Y$ and $R > 0$ such that $Y \setminus B_Y(y, R) \neq \emptyset$.

We note that for every metric surface,

$$\kappa_0^{-1} \leqslant \operatorname{mod} \Gamma(\xi_1, \xi_3; Q) \operatorname{mod} \Gamma(\xi_2, \xi_4; Q), \tag{2.7}$$

with $\kappa_0 = (4/\pi)^2$ [14, 32].

We recall [31, Theorem 1.4] stating the following.

Theorem 2.6. *Let (Y, d_Y) be a metric surface homeomorphic to $\mathbb{R}^2$ or to $\mathbb{S}^2$. Then there exists a quasiconformal embedding $\psi : (Y, d_Y) \to \mathbb{S}^2$ if and only if Y is reciprocal.*

Similarly, Theorem 1.3 of [22] shows that if a metric surface (Y, d_Y) can be covered by quasiconformal images of domains $V \subset \mathbb{R}^2$, then (Y, d_Y) is quasiconformally equivalent to a Riemannian surface. In particular, we have the following.

Theorem 2.7. *Let (Y, d_Y) be a metric surface homeomorphic to $\mathbb{S}^2$. Then there exists a quasiconformal homeomorphism $\psi : (Y, d_Y) \to \mathbb{S}^2$ if and only if each point $y \in Y$ is contained in an open set U from which there exists a quasiconformal homeomorphism $\phi : U \to V \subset \mathbb{R}^2$.*

Since the duality lower bound (2.7) holds, Corollary 12.3 of [31] shows that any homeomorphism from a metric surface into a Euclidean domain having bounded outer dilatation has bounded inner dilatation. More precisely, we have the following.

Proposition 2.8. *Let Y be a metric surface, $U \subset Y$ a domain, and $\psi : U \to \Omega \subset \mathbb{R}^2$ a homeomorphism. If $K_O(\psi) < \infty$, then ψ is K-quasiconformal for $K = (2 \cdot \kappa_0) \cdot K_O(\psi)$.*

3 | HEMISPHERES

Recall that σ is the usual intrinsic length distance on the sphere $\mathbb{S}^2$, induced by the Euclidean distance of $\mathbb{R}^3$. We construct a (pseudo)distance d_Z on Z using a *predistance* $D : Z \times Z \to [0, \infty]$ defined in the following way, with the identification $S_Z \subset \overline{Z}_1$ for the seam,

$$D(x, y) = \begin{cases} \infty, & \text{if } (x, y) \in Z_1 \times Z_2 \cup Z_2 \times Z_1, \\ \min\{\sigma(x, y), \sigma(g(x), g(y))\}, & \text{if } x, y \in S_Z, \\ \sigma(x, y), & \text{otherwise.} \end{cases}$$

Then we denote $d_Z(x,y) = \inf \sum_{i=1}^n D(x_i, x_{i+1})$, the infimum taken over finite chains $(x_i)_{i=1}^{n+1}$ for which $x_1 = x$ and $x_{n+1} = y$. We obtain a metric space $\widetilde{Z}$ and a quotient map $Q\colon Z \to \widetilde{Z}$ by identifying $(x,y) \in Z \times Z$ whenever $d_Z(x,y) = 0$, and setting $d_{\widetilde{Z}}(x,y) = d_Z(Q^{-1}(x), Q^{-1}(y))$ for each $x, y \in \widetilde{Z}$.

In this section, we focus on analysing the distance d_Z on the seam S_Z. The main results of this section are Lemmas 3.2 and 3.3 and Proposition 3.6.

In the following two lemmas we abuse notation and identify $\iota_i(Z_i)$ with Z_i when convenient.

Lemma 3.1. *The following hold:*

1. *Let $x, y \in \mathbb{S}^1 \subset \overline{Z}_1$ and $(x_i)_{i=1}^{n+1}$ a chain with $x_1 = x$, $x_{n+1} = y$, and $x_i \in Z_1$ otherwise. Then $\sum_{i=1}^n D(x_i, x_{i+1}) \geqslant D(x,y)$.*
2. *Let $x, y \in \mathbb{S}^1 \subset \overline{Z}_1$ and $(x_i)_{i=1}^{n+1}$ a chain with $g(x_1) = g(x)$, $g(x_{n+1}) = g(y)$, and $x_i \in Z_2$ otherwise. Then $\sum_{i=1}^n D(x_i, x_{i+1}) \geqslant D(x,y)$.*

Proof. Given the chain from the claim (1), for every i, $D(x_i, x_{i+1}) = \sigma(x_i, x_{i+1})$. Thus, $\sum_{i=1}^n D(x_i, x_{i+1}) \geqslant \sigma(x_1, x_{n+1}) \geqslant D(x_1, x_{n+1})$. The corresponding inequalities hold for the chain from (2). □

Lemma 3.1 implies that when computing $d_Z(\iota_1(x), \iota_1(y))$ for $x, y \in \mathbb{S}^1$, it is sufficient to consider chains with intermediate points staying within the seam.

Lemma 3.2. *If $x, y \in Z_1$, then*

$$d_Z(\iota_1(x), \iota_1(y)) = \begin{cases} \sigma(x,y), & \text{or there exist } w, w' \in \mathbb{S}^1 \text{ with} \\ \sigma(x,w) + d_Z(\iota_1(w), \iota_1(w')) + \sigma(w', y) \leqslant \sigma(x,y). \end{cases} \tag{3.1}$$

The corresponding identity holds for points $x, y \in Z_2$.

Furthermore, if $x \in Z_1$ and $y \in Z_2$, there exist $w, w' \in \mathbb{S}^1$ such that

$$d_Z(\iota_1(x), \iota_2(y)) = \sigma(x,w) + d_Z(\iota_1(w), \iota_1(w')) + \sigma(g(w'), y). \tag{3.2}$$

Proof. We show Equation (3.1). Suppose that there exists a sequence $\epsilon_j \to 0^+$ and a sequence of chains $(x_{i,j})_{i=1}^{n_j+1}$ joining x to y with $d_Z(\iota_1(x), \iota_1(y)) \geqslant -\epsilon_j + \sum_{i=1}^{n_j} D(x_{i,j}, x_{i+1,j})$ so that every chain has an element in $\mathbb{S}^1$. If i_1 is the first index for which $x_{i,j} \in \mathbb{S}^1$ and i_2 the last one, then

$$\begin{aligned} \sum_{i=1}^{n_j} D(x_{i,j}, x_{i+1,j}) &\geqslant \sigma(x, x_{i_1,j}) + d_Z(\iota_1(x_{i_1,j}), \iota_1(x_{i_2,j})) + \sigma(x_{i_2,j}, y) \\ &\geqslant \inf\left\{\sigma(x,w) + d_Z(\iota_1(w), \iota_1(w')) + \sigma(w', y)\right\}, \end{aligned}$$

the infimum taken over every $w, w' \in \mathbb{S}^1$. Observe that the infimum is realized by some $w, w' \in \mathbb{S}^1$. Given such $w, w' \in \mathbb{S}^1$, we pass to the limit $j \to \infty$ and conclude

$$d_Z(\iota_1(x), \iota_1(y)) \geqslant \sigma(x,w) + d_Z(\iota_1(w), \iota_1(w')) + \sigma(w', y).$$

Since "$\leq$" holds for every pair $w, w' \in \mathbb{S}^1$, the lower equality in Equation (3.1) follows.

If no such sequence of $\epsilon_j \to 0^+$ exists, then there exists $\epsilon_0 > 0$ such that for every $\epsilon_0 > \epsilon > 0$, any chain joining x to y with $d_Z(\iota_1(x), \iota_2(y)) \geqslant -\epsilon + \sum_{i=1}^{n} D(x_i, x_{i+1})$ does not intersect $\mathbb{S}^1$. Hence $\sum_{i=1}^{n} D(x_i, x_{i+1}) \geqslant \sigma(x, y)$. So, either way, we obtain Equation (3.1). The claims for each $x, y \in Z_2$ and $(x, y) \in Z_1 \times Z_2$ are proved in a similar manner. □

For $i = 1, 2$, we denote $\widetilde{\iota}_i := Q \circ \iota_i : \overline{Z}_i \to \widetilde{Z}$. Lemma 3.2 implies that $\widetilde{\iota}_i$ is 1-Lipschitz everywhere and a local isometry in Z_i. We also establish that $\widetilde{\iota}_i$ is *monotone*, that is, the preimage of a point is a compact and connected set.

Lemma 3.3. *For $i = 1, 2$, the inclusion map $\widetilde{\iota}_i : \overline{Z}_i \to \widetilde{Z}$ is 1-Lipschitz everywhere and a local isometry on Z_i. Moreover, for every $z \in \widetilde{Z}$, the preimage $\widetilde{\iota}_i^{-1}(z)$ is compact and connected. It contains two or more points only if $\widetilde{\iota}_i^{-1}(z) \subset \mathbb{S}^1$.*

Before proving Lemma 3.3, we show two auxiliary results.

Lemma 3.4. *Let $x, y \in \mathbb{S}^1$ be distinct. Then there exists an arc $\gamma : [0, 1] \to \mathbb{S}^1$ joining x to y with $D(\iota_1(x), \iota_1(y)) = \min\{\ell(\gamma), \ell(g \circ \gamma)\}$. The arc satisfies*

$$D(\iota_1(x), \iota_1(y)) \geqslant \sup_{\{t_i\}_{i=1}^{n+1}} \sum_{i=1}^{n} D(\iota_1(\gamma(t_i)), \iota_1(\gamma(t_{i+1}))),$$

the supremum taken over finite partitions of $[0, 1]$. *In particular,* $D(\iota_1(x), \iota_1(y)) \geqslant \ell(\widetilde{\iota}_1(\gamma))$.

Proof. The existence of γ with $D(\iota_1(x), \iota_1(y)) = \min\{\ell(\gamma), \ell(g \circ \gamma)\}$ follows from the fact that σ is geodesic on $\mathbb{S}^1$. We identify $\iota_1(x)$ with x for every $x \in \mathbb{S}^1$ in the following computations.

The claim about the partitions is a consequence of the following observation and induction: If $0 \leqslant a < s < b \leqslant 1$, then

$$D(\gamma(a), \gamma(b)) \geqslant D(\gamma(a), \gamma(s)) + D(\gamma(s), \gamma(b)). \tag{3.3}$$

We first assume that $D(\gamma(a), \gamma(b)) = \sigma(\gamma(a), \gamma(b))$. Then γ is a length-minimizing geodesic joining $\gamma(a)$ and $\gamma(b)$. Consequently,

$$\sigma(\gamma(a), \gamma(b)) = \sigma(\gamma(a), \gamma(s)) + \sigma(\gamma(s), \gamma(b)).$$

Since $\sigma(c, d) \geqslant D(c, d)$ holds for every $c, d \in \mathbb{S}^1$, the inequality (3.3) holds in this case. In the remaining case, $g \circ \gamma$ is a length-minimizing geodesic joining $g(\gamma(a))$ and $g(\gamma(b))$ and

$$\sigma(g(\gamma(a)), g(\gamma(b))) = \sigma(g(\gamma(a)), g(\gamma(s))) + \sigma(g(\gamma(s)), g(\gamma(b))).$$

Since $\sigma(g(c), g(d)) \geqslant D(c, d)$ for every $c, d \in \mathbb{S}^1$, the inequality (3.3) holds also in this case.

The partition claim implies $D(x, y) \geqslant \sum_{i=1}^{n} d_{\widetilde{Z}}(\widetilde{\iota}_1(\gamma(t_i)), \widetilde{\iota}_1(\gamma(t_{i+1})))$ for every partition $\{t_i\}_{i=1}^{n+1}$ of $[0, 1]$. The inequality $D(x, y) \geqslant \ell(\widetilde{\iota}_1(\gamma))$ follows by taking the supremum over such partitions. □

Lemma 3.5. *Let $x, y \in \mathbb{S}^1$ be distinct. Then there exists an arc $\gamma : [0, 1] \to \mathbb{S}^1$ joining x to y such that $d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \ell(\widetilde{\iota}_1(\gamma))$.*

Proof. Let $\epsilon > 0$. The defining property of d_Z and Lemma 3.1 imply the existence of a chain $\{x_i\}_{i=1}^{n+1} \subset \mathbb{S}^1$ joining x to y for which

$$d_Z(\iota_1(x), \iota_1(y)) \geqslant -\epsilon + \sum_{i=1}^{n} D(\iota_1(x_i), \iota_1(x_{i+1})).$$

For each i, Lemma 3.4 yields the existence of an arc $\theta_i : [0,1] \to \mathbb{S}^1$ joining x_i to x_{i+1} with $D(\iota_1(x_i), \iota_1(x_{i+1})) \geqslant \ell(\widetilde{\iota}_1(\theta_i))$. Let θ denote the concatenation of these paths. Then $d_Z(\iota_1(x), \iota_1(y)) \geqslant -\epsilon + \ell(\widetilde{\iota}_1(\theta))$.

Let $\theta' : [0,1] \to \mathbb{S}^1$ be an arc joining x to y within the image of θ. Applying Equation (2.3) on $\widetilde{Z}$ with $\rho \equiv \chi_{\widetilde{Z}}$ implies that $\ell(\widetilde{\iota}_1(\theta)) \geqslant \ell(\widetilde{\iota}_1(\theta'))$. Such a θ' is one of the arcs joining x to y within $\mathbb{S}^1$.

Let $\epsilon_j \to 0^+$ and consider θ'_j as above for every such ϵ_j. Up to passing to a subsequence and relabeling, we may assume that every such θ'_j is the same arc θ'. Passing to the limit $j \to \infty$ establishes $d_Z(\iota_1(x), \iota_1(y)) \geqslant \ell(\widetilde{\iota}_1(\theta')) \geqslant d_Z(\iota_1(x), \iota_1(y))$. We set $\gamma = \theta'$ to conclude the proof. □

Proof of Lemma 3.3. The claimed 1-Lipschitz and local isometry properties of $\widetilde{\iota}_1$ follow from Lemma 3.2. The local isometry property implies that given $z \in \widetilde{Z}$, the preimage $\widetilde{\iota}_1^{-1}(z)$ has more than two points only if the preimage is a subset of $\mathbb{S}^1$.

Suppose the existence of a distinct pair $x, y \in \widetilde{\iota}_1^{-1}(z)$. Then $x, y \in \mathbb{S}^1$. Lemma 3.5 shows that there exists an arc γ joining x to y within $\mathbb{S}^1$ satisfying

$$0 = d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \ell(\widetilde{\iota}_1(\gamma)).$$

This implies $|\gamma| \subset \widetilde{\iota}_1^{-1}(z)$. Since x and y were arbitrary, we conclude that $\widetilde{\iota}_1^{-1}(z)$ is path connected. Consequently, $\widetilde{\iota}_1^{-1}(z)$ is a connected and compact subset of $\mathbb{S}^1$.

The properties of $\widetilde{\iota}_2$ follow from a symmetry in the argument. Hence the claim follows. □

Proposition 3.6. *Let* $g : (\mathbb{S}^1, \mathcal{H}^1_{\mathbb{S}^1}) \to (\mathbb{S}^1, \mathcal{H}^1_{\mathbb{S}^1})$ *be a homeomorphism with* $g^*\mathcal{H}^1_{\mathbb{S}^1} = v_g \mathcal{H}^1_{\mathbb{S}^1} + \mu^\perp$ *with* $\mathcal{H}^1_{\mathbb{S}^1}$ *and* $\mu^\perp$ *mutually singular. Then, for every Borel set* $B \subset \mathbb{S}^1$,

$$\mathcal{H}^1_{d_{\widetilde{Z}}}(\widetilde{\iota}_1(B)) = \int_B \min\{1, v_g\}\, d\mathcal{H}^1_{\mathbb{S}^1} = \int_{\widetilde{\iota}_1(B)} \#(\widetilde{\iota}_1^{-1}(z))\, d\mathcal{H}^1_{\widetilde{Z}}(z). \tag{3.4}$$

Moreover, for every $x, y \in \mathbb{S}^1$, *there exists an arc* $|\gamma| \subset \mathbb{S}^1$ *joining* x *to* y *for which*

$$d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \ell(\widetilde{\iota}_1(\gamma)). \tag{3.5}$$

Before proving Proposition 3.6, we first consider a Carathéodory construction on $\mathbb{S}^1$. First, fix a Borel set $B_0 \subset \mathbb{S}^1$ for which $\mathcal{H}^1_{\mathbb{S}^1}(B_0) = 0$ and $\mu^\perp(\mathbb{S}^1 \setminus B_0) = 0$. Set $\nu^{ABS}(B) := \int_B \min\{1, v_g\}\chi_{\mathbb{S}^1 \setminus B_0}\, d\mathcal{H}^1_{\mathbb{S}^1}$ for all Borel sets $B \subset \mathbb{S}^1$.

For every arc $\gamma : [0,1] \to \mathbb{S}^1$, we denote $\xi^{ABS}(|\gamma|) := \nu^{ABS}(|\gamma|)$ and $\xi(|\gamma|) := D(\gamma(0), \gamma(1))$. The set function ξ^{ABS} and the family of arcs $|\gamma| \subset \mathbb{S}^1$ yields Carathéodory premeasures ν^{ABS}_δ for each $\delta > 0$.

Lemma 3.7. *For every Borel set $B \subset \mathbb{S}^1$, we have $\nu^{ABS}(B) = \sup_{\delta>0} \nu_\delta^{ABS}(B) \geqslant \mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(B))$.*

Proof. The equality $\nu^{ABS}(B) = \sup_{\delta>0} \nu_\delta^{ABS}(B)$ follows from the fact that ν^{ABS} is a finite Borel regular Borel measure.

We denote $B_1 = \{v_g \geqslant 1\} \cup B_0$ and $B_2 = \mathbb{S}^1 \setminus B_1$. If $B \subset \mathbb{S}^1$ is Borel, we have

$$\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(B)) \leqslant \sum_{i=1}^{2} \mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(B \cap B_i)) \leqslant \mathcal{H}^1_{\mathbb{S}^1}(B \cap B_1) + \mathcal{H}^1_{\mathbb{S}^1}(g(B) \cap g(B_2))$$

since $\widetilde{\iota}_i$ is 1-Lipschitz for $i = 1, 2$. The right-hand side equals $\nu^{ABS}(B)$. Therefore $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(B)) \leqslant \nu^{ABS}(B)$ holds for all Borel sets. □

Lemma 3.8. *Let $x, y \in \mathbb{S}^1$ be distinct and $\gamma \colon [0,1] \to \mathbb{S}^1$ an arc joining x to y such that $d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \ell(\widetilde{\iota}_1(\gamma))$. Then $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(|\gamma|)) = d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \nu^{ABS}(|\gamma|)$.*

Proof. Let $\pi/2 > \delta_0 > 0$ be such that

$$D(\iota_1(a), \iota_1(b)) < \delta_0 \quad \text{implies} \quad \max\{\sigma(a,b), \sigma(g(a), g(b))\} < \pi/2.$$

Given such a pair $a, b \in \mathbb{S}^1$, the length-minimizing geodesic $\theta \colon [0,1] \to \mathbb{S}^1$ joining a to b satisfies $\xi(|\theta|) = \min\{\ell(\theta), \ell(g \circ \theta)\}$. Then $\xi(|\theta|) \geqslant \xi^{ABS}(|\theta|)$.

Let γ be as in the claim. Let $0 < \delta < \delta_0$ and $0 < \epsilon < \delta/2$. We consider a partition $\{t_i\}_{i=1}^{n+1}$ of $[0,1]$ such that $\sigma(\gamma(t_i), \gamma(t_{i+1})) < \delta/2$ for every i. Then there exists a chain $\{x_{i,j}\}_{j=1}^{n_i+1} \subset \mathbb{S}^1$ joining the ends of $\gamma|_{[t_i, t_{i+1}]}$ so that

$$d_Z(\iota_1 \circ \gamma(t_i), \iota_1 \circ \gamma(t_{i+1})) \geqslant -\frac{\epsilon}{n} + \sum_{j=1}^{n_i} D(\iota_1(x_{i,j}), \iota_1(x_{i,j+1})).$$

In particular, $D(\iota_1(x_{i,j}), \iota_1(x_{i,j+1})) < \delta < \delta_0$ for every j. Hence the length-minimizing geodesic $\gamma_{i,j}$ joining $x_{i,j}$ to $x_{i,j+1}$ satisfies the assumptions of Lemma 3.4. For every i, Lemma 3.4 implies that, up to further partitioning the paths $\gamma_{i,j}$ and relabeling, we may assume $\sigma(x_{i,j}, x_{i,j+1}) < \delta$ for every j. Given this property, we conclude $D(\iota_1(x_{i,j}), \iota_1(x_{i,j+1})) = \xi(|\gamma_{i,j}|) \geqslant \xi^{ABS}(|\gamma_{i,j}|)$ and

$$\ell(\widetilde{\iota}_1(\gamma)) = \sum_{i=1}^{n} d_Z(\iota_1 \circ \gamma(t_i), \iota_1 \circ \gamma(t_{i+1})) \geqslant -\epsilon + \nu_\delta^{ABS}\left(\bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} |\gamma_{i,j}| \right).$$

Since the concatenation θ_i of $\{\gamma_{i,j}\}_{j=1}^{n_i}$ is a path joining $\gamma(t_i)$ to $\gamma(t_{i+1})$, the concatenation θ of $\{\theta_i\}_{i=1}^{n}$ is a path joining x to y. Hence $\bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} |\gamma_{i,j}| = |\theta|$ contains $|\gamma|$ or $\mathbb{S}^1 \setminus |\gamma|$, and

$$\ell(\widetilde{\iota}_1(\gamma)) \geqslant -\epsilon + \min\left\{ \nu_\delta^{ABS}(|\gamma|), \nu_\delta^{ABS}(\mathbb{S}^1 \setminus |\gamma|) \right\}.$$

After passing to $\epsilon \to 0^+$ and then to $\delta \to 0^+$, we conclude

$$\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(\gamma)) = \ell(\widetilde{\iota}_1(\gamma)) \geqslant \min\left\{ \nu^{ABS}(|\gamma|), \nu^{ABS}(\mathbb{S}^1 \setminus |\gamma|) \right\}.$$

If we had $\nu^{ABS}(|\gamma|) > \nu^{ABS}(\mathbb{S}^1 \setminus |\gamma|)$, this would contradict Lemma 3.7 and the length-minimizing property of $\widetilde{\iota}_1(\gamma)$. Hence $\nu^{ABS}(|\gamma|) \leqslant \nu^{ABS}(\mathbb{S}^1 \setminus |\gamma|)$, and $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(|\gamma|)) = d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \nu^{ABS}(|\gamma|)$ follows from Lemma 3.7. □

Proof of Proposition 3.6. The existence of γ and equality in (3.5) already follows from Lemma 3.4.

We claim that (3.4) holds. To this end, we consider three arcs $\gamma_i : [0,1] \to \mathbb{S}^1$ overlapping only at their end points, whose images cover $\mathbb{S}^1$, with the arcs satisfying $\nu^{ABS}(|\gamma_i|) \leqslant \nu^{ABS}(\mathbb{S}^1 \setminus |\gamma_i|)$.

Lemmas 3.7 and 3.8 imply that $\widetilde{\iota}_1 \circ \gamma_i$ is a length-minimizing geodesic joining its end points and $\nu^{ABS}(|\gamma_i|) = \mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(|\gamma_i|))$. Lemma 3.7 implies that the metric speed of $\widetilde{\iota}_1|_{|\gamma_i|}$ is bounded from above by $\min\{1, v_g\}$. Hence the equality $\nu^{ABS}(|\gamma_i|) = \mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(|\gamma_i|))$ forces the metric speed of $\widetilde{\iota}_1$ to equal $\min\{1, v_g\}$ $\mathcal{H}^1_{\mathbb{S}^1}$-almost everywhere on $|\gamma_i|$ for $i = 1, 2, 3$. The equality (3.4) follows from the area formula (2.3) and the fact that $\#(\widetilde{\iota}_1^{-1}(x)) = 1$ $\mathcal{H}^1_{\widetilde{Z}}$-almost everywhere. The fact $\#(\widetilde{\iota}_1^{-1}(x)) = 1$ $\mathcal{H}^1_{\widetilde{Z}}$-almost everywhere follows from the monotonicity of $\widetilde{\iota}_1$ and the integrability of the multiplicity. The integrability of the multiplicity follows from (2.2). □

Remark 3.9. We consider a 2π-periodic doubling measure μ on $\mathbb{R}$ with $2\pi = \mu([0, 2\pi])$ such that for some Borel set $B \subset [0, 2\pi]$, $\mathcal{L}^1(B) = 0 = \mu([0, 2\pi] \setminus B)$, the existence of which is established by Ahlfors–Beurling [4, Section 7]. Then $\psi(x) = \int_0^x d\mu$ is a homeomorphism and there exists a quasisymmetry $g : \mathbb{S}^1 \to \mathbb{S}^1$ with $\theta \circ \psi = g \circ \theta$, where $\theta(t) = (\cos(t), \sin(t), 0)$. Then v_g in Equation (3.4) is identically zero. Consequently, $d_Z \equiv 0$ on the seam S_Z.

4 | HARMONIC MEASURE AND WELDING HOMEOMORPHISMS

We consider a welding homeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$ and a welding circle C with complementary components Ω_1 and Ω_2, Riemann maps $\phi_i : Z_i \to \Omega_i$ for $i = 1, 2$, and $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$; in this section, we identify ϕ_i with its extension to a homeomorphism $\overline{Z}_i \to \overline{\Omega}_i$. With this identification understood, we consider the *harmonic measures* $\omega_i(E) := \mathcal{H}^1(\phi_i^{-1}(E) \cap \mathbb{S}^1)/(2\pi)$ for all Borel sets $E \subset \mathbb{S}^2$.

We define a homeomorphism $\pi : \mathbb{S}^2 \to (Z, d_Z)$ and a quotient map $\widetilde{\pi} : \mathbb{S}^2 \to \widetilde{Z}$ via the formulas

$$\pi(x) = \begin{cases} \iota_1 \circ \phi_1^{-1}(x), & \text{when } x \in \overline{\Omega_1}, \\ \iota_2 \circ \phi_2^{-1}(x), & \text{when } x \in \Omega_2 \end{cases} \quad \text{and} \quad \widetilde{\pi} = Q \circ \pi. \tag{4.1}$$

Recall that $Q : Z \to \widetilde{Z}$ is the quotient map identifying $x, y \in Z$ whenever $d_Z(x, y) = 0$. Lemma 3.3 implies that $\widetilde{\pi}$ is monotone and $\widetilde{\pi}^{-1}(x)$ contains two or more points only if x is a point of the seam $Q(S_Z)$, and in such a case $\widetilde{\pi}^{-1}(x) \subset C$.

For $\alpha = 1, 2$, we denote, for every Borel set $B \subset \mathbb{S}^2$,

$$\widetilde{\pi}^* \mathcal{H}^\alpha_{\widetilde{Z}}(B) := \int_{\widetilde{Z}} \#(B \cap \widetilde{\pi}^{-1}(x))\, d\mathcal{H}^\alpha_{\widetilde{Z}}(x) = \mathcal{H}^\alpha_{\widetilde{Z}}(\widetilde{\pi}(B)), \tag{4.2}$$

where the multiplicity can be ignored in the case $\alpha = 2$ since it equals one outside the negligible set $Q(S_Z)$. For $\alpha = 1$, the multiplicity is two or more only when it is ∞ and this happens in a set of negligible $\mathcal{H}^1_{\widetilde{Z}}$-measure. Either way, the multiplicity is negligible in Equation (4.2), so the second equality is justified.

Proposition 4.1. *Let g be a welding homeomorphism with a welding circle C and $I \subset C$ a subarc. Then $d_{\widetilde{Z}}(\widetilde{\pi}(x), \widetilde{\pi}(y)) = 0$ for all $x, y \in I$ if and only if $\omega_1|_I$ and $\omega_2|_I$ are mutually singular. If such an interval exists, then $\widetilde{Z}$ is not quasiconformally equivalent to $\mathbb{S}^2$.*

Remark 4.2. If g is a welding homeomorphism obtained from Remark 3.9 or any welding g corresponding to the von Koch snowflake [16, Example 4.3], Proposition 4.1 implies that $Q(S_Z)$ is a singleton. In particular, $\widetilde{Z}$ is not even homeomorphic to the sphere. For a given g, this happens if and only if $g^*\mathcal{H}^1_{\mathbb{S}^1}$ and $\mathcal{H}^1_{\mathbb{S}^1}$ are mutually singular.

A key step in the proof of the conformal removability in Theorem 1.6 is the following.

Proposition 4.3. *Let g be a welding homeomorphism and $\widetilde{\pi}$ as in Equation (4.1). Then $\widetilde{\pi}$ is continuous, monotone, and surjective. Moreover, for all path families Γ on $\mathbb{S}^2$, $\operatorname{mod}\Gamma \leqslant \operatorname{mod}\widetilde{\pi}\Gamma$. The metric space $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$ if and only if $\widetilde{\pi}$ is a homeomorphism for which* $\operatorname{mod}\Gamma = \operatorname{mod}\widetilde{\pi}\Gamma$ *for all path families.*

The proof of Proposition 4.3 requires some preparatory work. Given the curve C, we say that $x_0 \in C$ is a *tangent point* if there exists a homeomorphism $\gamma : (-\epsilon, \epsilon) \to C' \subset C$ with $\gamma(0) = x_0$, and a tangent vector $v_0 \in T_{x_0}\mathbb{S}^2$ with unit length such that for every smooth $f : \mathbb{S}^2 \to \mathbb{R}$, its differential df satisfies

$$df(v_0) = \lim_{t\to 0^+} \frac{f(\gamma(t)) - f(x_0)}{\sigma(\gamma(t), x_0)} \quad \text{and} \quad df(-v_0) = \lim_{t\to 0^-} \frac{f(\gamma(t)) - f(x_0)}{\sigma(\gamma(t), x_0)}.$$

If v_0 exists, the tangent vector v_0 is independent of the parametrization γ and C' up to multiplication by -1; see [16, Chapter II, Section 4]. The collection of *tangents points* of C is denoted by $\mathrm{Tn}(C)$. The key properties of $\mathrm{Tn}(C)$ are self-contained in the following statement.

Lemma 4.4. *The Borel set* $\mathrm{Tn}(C)$ *has σ-finite Hausdorff 1-measure. Moreover, on the set* $\mathrm{Tn}(C)$*, the measures ω_1, ω_2, and $\mathcal{H}^1_C$ are mutually absolutely continuous.*

Given any Borel set $E \subset C$ with $\omega_1(E) \cdot \omega_2(E) > 0$, the restrictions $\omega_1|_E$ and $\omega_2|_E$ are mutually singular on E if and only if $\mathcal{H}^1_C(\mathrm{Tn}(C) \cap E) = 0$.

Proof. The Borel measurability of $\mathrm{Tn}(C)$ follows from [16, Chapter II, Theorem 4.2] which connects the tangents of C and the angular derivatives of any given Riemann map $\phi_1' : Z_1 \to \Omega_1$, where $\partial\Omega_1 = C$. The fact that $\mathrm{Tn}(C)$ has σ-finite Hausdorff 1-measure follows from [16, Chapter VI, Theorem 4.2].

Theorem 6.3 of [16, Chapter VI] states that if a Borel set $E \subset C$ is such that $\omega_1(E) \cdot \omega_2(E) > 0$, then $\omega_1|_E$ and $\omega_2|_E$ are mutually singular on E if and only if $\mathcal{H}^1_C(\mathrm{Tn}(C) \cap E) = 0$.

The fact that on the set $\mathrm{Tn}(C)$ the measures ω_1, ω_2, and $\mathcal{H}^1_C$ are mutually absolutely continuous follows from [16, Chapter VI, Theorem 4.2 and the following discussion on p. 211]. □

Lemma 4.5. *The measures $\chi_C\widetilde{\pi}^*\mathcal{H}^1_{\widetilde{Z}}$, $\chi_{\mathrm{Tn}(C)}\omega_1$, $\chi_{\mathrm{Tn}(C)}\omega_2$ and $\chi_{\mathrm{Tn}(C)}\mathcal{H}^1_C$ are mutually absolutely continuous.*

More precisely, a given Borel set $B \subset \mathrm{Tn}(C)$ has positive 1D Hausdorff measure if and only if $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\pi}(B)) > 0$. Furthermore, if $B \subset C \setminus \mathrm{Tn}(C)$, then $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\pi}(B)) = 0$.

Proof. We write $g^*\mathcal{H}^1_{\mathbb{S}^1} = v_g \mathcal{H}^1_{\mathbb{S}^1} + 2\pi \cdot \mu^\perp$ with $\mathcal{H}^1_{\mathbb{S}^1}$ and $\mu^\perp$ being mutually singular. We recall from Proposition 3.6 that for every Borel set $B \subset C$,

$$\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\pi}(B)) = \int_{\phi_1^{-1}(B)} \min\left\{1, v_g\right\} d\mathcal{H}^1_{\mathbb{S}^1}. \tag{4.3}$$

We denote $h = v_g \circ \phi_1^{-1}$ and observe the equality $\omega_2 = h\omega_1 + (\phi_1)_*\mu^\perp$, where $(\phi_1)_*\mu^\perp(E) = \mu^\perp(\phi_1^{-1}(E))$ for each $E \subset \mathbb{S}^2$. Then Equation (4.3) is equivalent to

$$(2\pi)^{-1}\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\pi}(B)) = \int_B \min\{1, h\}\, d\omega_1. \tag{4.4}$$

Lemma 4.4 implies that the measures $\chi_{C\setminus \mathrm{Tn}(C)}\omega_1$ and $\chi_{C\setminus \mathrm{Tn}(C)}\omega_2$ are mutually singular. Consequently, $h = 0$ ω_1-almost everywhere in $C \setminus \mathrm{Tn}(C)$. In particular, if $B = C \setminus \mathrm{Tn}(C)$, the left-hand side equals zero in Equation (4.4).

Lemma 4.4 yields that the measures $\chi_{\mathrm{Tn}(C)}\omega_1$, $\chi_{\mathrm{Tn}(C)}\omega_2$ and $\chi_{\mathrm{Tn}(C)}\mathcal{H}^1_C$ are mutually absolutely continuous. Hence $\infty > h > 0$ ω_1-almost everywhere in $\mathrm{Tn}(C)$. This implies that the measure in Equation (4.4) is mutually absolutely continuous with the measures $\chi_{\mathrm{Tn}(C)}\omega_1$, $\chi_{\mathrm{Tn}(C)}\omega_2$ and $\chi_{\mathrm{Tn}(C)}\mathcal{H}^1_C$. The claim follows from the equalities (4.2) for $\alpha = 1$. □

Proof of Proposition 4.1. Consider a subarc $I \subset C$. Proposition 3.6 implies that $\widetilde{\pi}(I)$ has zero $\mathcal{H}^1_{\widetilde{Z}}$-measure if and only if for every $x, y \in I$, $d_{\widetilde{Z}}(\widetilde{\pi}(x), \widetilde{\pi}(y)) = 0$ if and only if $v_g = 0$ $\mathcal{H}^1_{\mathbb{S}^1}$-almost everywhere on $\phi_1^{-1}(I)$. Equivalently, $\omega_1|_I$ and $\omega_2|_I$ are mutually singular.

Lemma 3.3 shows that $\widetilde{Z} \neq (Z, d_Z)$ if and only if there exists a closed arc $I \subset \mathbb{S}^1$ such that $y = \widetilde{\iota}_1(I)$. Assume that such an I exists. Having fixed $x_0 \in Z_1$ and $0 < s < \sigma(x_0, \mathbb{S}^1)$, there exists $c = c(x_0, I, s)$ for which

$$\operatorname{mod} \Gamma(I, \overline{B}_{\mathbb{S}^2}(x_0, s); I \cup Z_1) \geqslant c > 0;$$

a positive lower bound can be shown, for example, by estimating the modulus of all geodesics joining I to $\overline{B}_{\mathbb{S}^2}(x_0, s)$ in $I \cup Z_1$.

When $R > 0$ is small enough, for every $R > r > 0$ and every path in $\Gamma(I, \overline{B}_{\mathbb{S}^2}(x_0, s)); I \cup Z_1)$, we find a subpath $\gamma' : [0,1] \to Z_1$ so that $\widetilde{\iota}_1 \circ \gamma'$ joins $\overline{B}_{\widetilde{Z}}(y, r)$ to $\widetilde{Z} \setminus B_{\widetilde{Z}}(y, R)$ within $\overline{B}_{\widetilde{Z}}(y, R)$. Since $\widetilde{\iota}_1$ is a local isometry off the seam, this implies

$$\liminf_{r \to 0^+} \operatorname{mod} \Gamma\left(\overline{B}_{\widetilde{Z}}(y, r), \widetilde{Z} \setminus B_{\widetilde{Z}}(y, R); \overline{B}_{\widetilde{Z}}(y, R)\right) \geqslant c.$$

Recalling Theorem 2.6, we see that $\widetilde{Z}$ is not quasiconformally equivalent to $\mathbb{S}^2$. □

Lemma 4.6. *For $i = 1, 2$, let $\rho_i : \Omega_i \to [0, \infty]$ denote the operator norm of the differential of $D(\phi_i^{-1})$. Then*

$$G = \chi_{\Omega_1}\rho_1 + \chi_{\Omega_2}\rho_2 + \infty \cdot \chi_{\mathrm{Tn}(C)} \in L^2(\mathbb{S}^2) \tag{4.5}$$

is a weak upper gradient of $\widetilde{\pi}$.

Proof. The L^2-integrability of G follows from the change of variables formulas of the Riemann maps ϕ_1 and ϕ_2 and the fact that $\mathrm{Tn}(C)$ has negligible area. Hence, as a consequence of Lemma 2.1, G is integrable along almost every absolutely continuous path $\gamma\colon [0,1] \to \mathbb{S}^2$. Given such a γ, we claim that

$$d_{\widetilde{Z}}(\widetilde{\pi}(\gamma(0)), \widetilde{\pi}(\gamma(1))) \leqslant \int_\gamma G\, ds, \tag{4.6}$$

implying that G is a weak upper gradient of $\widetilde{\pi}$.

Since G is integrable along γ, γ has negligible length in $\mathrm{Tn}(C)$. Then Equation (2.3) implies $\mathcal{H}^1_{\mathbb{S}^2}(\mathrm{Tn}(C) \cap |\gamma|) = 0$. We conclude $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\pi}(C) \cap |\widetilde{\pi} \circ \gamma|) = 0$ from Lemma 4.5. The assumptions of Lemma 2.2 are satisfied and the conclusion $\ell(\widetilde{\pi} \circ \gamma) \leqslant \int_\gamma G\, ds$ follows. The inequality (4.6) is a consequence. □

We define the *Jacobian* of $\widetilde{\pi}$ to be the density of $\widetilde{\pi}^*\mathcal{H}^2_{\widetilde{Z}}$, defined in Equation (4.2), with respect to $\mathcal{H}^2_{\mathbb{S}^2}$.

Lemma 4.7. *The mapping $\widetilde{\pi}$ satisfies Lusin's condition (N) and the Jacobian $J_{\widetilde{\pi}}$ coincides with G^2 $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere, with G being from Equation (4.5).*

Proof. The Lusin's condition (N) of $\widetilde{\pi}$ follows from the fact that $\widetilde{\pi}(C)$ has negligible $\mathcal{H}^2_{\widetilde{Z}}$-measure, the fact that $\iota_i\colon Z_i \to \widetilde{Z}_i$ is a local isometry, and as $\phi_i^{-1}\colon \Omega_i \to Z_i$ satisfies condition (N). Here $J_{\widetilde{\pi}} = 0$ $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere on C, so the equality $J_{\widetilde{\pi}} = G^2$ follows from the fact that ϕ_1 and ϕ_2 are Riemann maps. □

Proof of Proposition 4.3. The claimed topological properties of $\widetilde{\pi}$ were already verified at the beginning of this section. Lemmas 4.6 and 4.7 prove that $J_{\widetilde{\pi}} = G^2 \in L^1(\mathbb{S}^2)$ with G being a weak upper gradient of $\widetilde{\pi}$. This fact and the fact that the multiplicity of $\widetilde{\pi}$ is negligible for $\widetilde{\pi}^*\mathcal{H}^2_{\widetilde{Z}}$ imply $\operatorname{mod}\Gamma \leqslant \operatorname{mod}\widetilde{\pi}\Gamma$ for all path families Γ.

Lastly, we argue that a K-quasiconformal map $\psi\colon \widetilde{Z} \to \mathbb{S}^2$ exists (for some $K \geqslant 1$) if and only if $\widetilde{\pi}$ is a 1-quasiconformal homeomorphism. The "if"-direction is obvious.

In the "only if"-direction, the fact that $\widetilde{\pi}$ is a homeomorphism follows from Proposition 4.1. So $h = \psi \circ \widetilde{\pi}\colon \mathbb{S}^2 \to \mathbb{S}^2$ is a homeomorphism satisfying $\operatorname{mod}\Gamma \leqslant K \operatorname{mod} h\Gamma$ for all path families Γ. Theorem 2.4 and [2, Definition 3.1.1 and Theorem 3.7.7] prove that h is K-quasiconformal. Consequently, $\widetilde{\pi}$ is K'-quasiconformal for some $K' \leqslant K^2$. This self-improves to $K' = 1$ due to Lemma 4.8 below. This yields $\operatorname{mod}\widetilde{\pi}\Gamma = \operatorname{mod}\Gamma$ for all path families. □

Lemma 4.8. *Suppose that $\widetilde{\pi}\colon \mathbb{S}^2 \to \widetilde{Z}$ from Equation (4.1) is a homeomorphism. Then $\widetilde{\pi}\colon \mathbb{S}^2 \to \widetilde{Z}$ is 1-quasiconformal if and only if for every 1-Lipschitz $h\colon \mathbb{S}^2 \to \mathbb{R}$, $h \circ \widetilde{\pi}^{-1} \in N^{1,2}(\widetilde{Z})$.*

Proof. The "only if"-claim is clear, given Theorem 2.4 (ii). In the "if"-direction, fix a 1-Lipschitz $h\colon \mathbb{S}^2 \to \mathbb{R}$ for now.

Consider the Borel function $G\colon \mathbb{S}^2 \to [0,\infty]$ defined on Lemma 4.6. Then $\rho = 1/G \circ \widetilde{\pi}^{-1}$ is such that ρ^2 is the Jacobian of $\widetilde{\pi}^{-1}$, as a consequence of Lemma 4.7. Hence $\rho \in L^2(\widetilde{Z})$.

Given that $h \circ \widetilde{\pi}^{-1} \in N^{1,2}(\widetilde{Z})$ and $\mathcal{H}^2_{\widetilde{Z}}(Q(S_Z)) = 0$, for almost every $\gamma\colon [0,1] \to \widetilde{Z}$, the composition $(h \circ \widetilde{\pi}^{-1}) \circ \gamma$ is absolutely continuous, γ has negligible length on the seam $Q(S_Z)$, and

$\int_\gamma \rho\, ds < \infty$. Indeed, the absolute continuity of $(h \circ \widetilde{\pi}^{-1}) \circ \gamma$ for almost every path follows from [19, Proposition 6.3.2]. The fact that almost every path has negligible length on $Q(S_Z)$ follows from Lemma 2.1 and the L^2-integrability of $\infty \cdot \chi_{Q(S_Z)}$. Similarly, the conclusion $\int_\gamma \rho\, ds < \infty$ follows from Lemma 2.1 and the L^2-integrability of ρ.

If we denote $E = (h \circ \widetilde{\pi}^{-1})(|\gamma| \cap Q(S_Z))$, the absolute continuity of $(h \circ \widetilde{\pi}^{-1}) \circ \gamma$ implies $\mathcal{H}^1_{\mathbb{R}}(E) = 0$. Then Lemma 2.2 yields $\ell((h \circ \widetilde{\pi}^{-1}) \circ \gamma) \leqslant \int_\gamma \rho\, ds$. We conclude that ρ is a weak upper gradient of $h \circ \widetilde{\pi}^{-1}$.

Since ρ is independent of h and h is an arbitrary 1-Lipschitz function, Theorem 7.1.20 [19] shows that ρ is a weak upper gradient of $\widetilde{\pi}^{-1}$. Since ρ^2 is the Jacobian of $\widetilde{\pi}^{-1}$, we conclude $K_O(\widetilde{\pi}^{-1}) = 1$. Recall $K_O(\widetilde{\pi}) = 1$ from Proposition 4.3. □

Remark 4.9. If the welding curve C happens to be rectifiable, the Hausdorff 1-measure on C and $\chi_{\mathrm{Tn}(C)}\mathcal{H}^1_C$ are mutually absolutely continuous [16, Chapter VI, Theorem 1.2 (F. and M. Riesz)]. With this fact at hand, Lemma 4.5 implies that $\widetilde{\pi}$ is a homeomorphism. Moreover, one can show that $h \circ \widetilde{\pi}^{-1} \in N^{1,2}(\widetilde{Z})$ for every 1-Lipschitz $h : \mathbb{S}^2 \to \mathbb{R}$. Hence $\widetilde{\pi}$ is 1-quasiconformal by Lemma 4.8.

Proof of Theorem 1.6. Suppose the existence of a quasiconformal homeomorphism $\psi : \widetilde{Z} \to \mathbb{S}^2$. Up to postcomposing ψ by an orientation-reversing Möbius transformation of $\mathbb{S}^2$, we may assume that $\widetilde{\phi}_i := \psi \circ \widetilde{\iota}_i|_{Z_i} : Z_i \to \mathbb{S}^2$ is orientation-preserving for $i = 1, 2$. Let $C = \psi(Q(S_Z))$.

The set $\mathbb{S}^2 \setminus C$ is the disjoint union of Jordan domains Ω_1 and Ω_2, where Ω_i is the image of $\widetilde{\phi}_i$ for $i = 1, 2$.

Next, since $\psi : \widetilde{Z} \to \mathbb{S}^2$ is a quasiconformal homeomorphism, ψ satisfies Lusin's Condition (N) [31, Section 17]. Consequently, C has zero 2D Hausdorff measure.

We consider the Beltrami differential $\mu = \chi_{\Omega_1}\mu_1 + \chi_{\Omega_2}\mu_2$, where μ_i is the Beltrami differential of $\widetilde{\phi}_i^{-1}$. If $h : \mathbb{S}^2 \to \mathbb{S}^2$ is a normalized solution to the Beltrami equation induced by μ [2, Measurable Riemann mapping theorem], the mapping $\widetilde{\psi} = h \circ \psi$ is 1-quasiconformal. Since C has zero measure, this is readily verified by hand or by applying [22, Theorem 4.12].

We have verified that $(Z, d_Z) = \widetilde{Z}$ and we may assume that $\psi : (Z, d_Z) \to \mathbb{S}^2$ is 1-quasiconformal with $\phi_i = \psi \circ \widetilde{\iota}_i|_{Z_i}$ being Riemann maps [2, Weyl's lemma]. Proposition 4.1 implies $(Z, d_Z) = \widetilde{Z}$. The definition of Z implies that $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$. Consequently, g is a welding homeomorphism.

In order to show the removability of $C := \psi(S_Z)$, we are given an orientation-preserving homeomorphism $M : \mathbb{S}^2 \to \mathbb{S}^2$ conformal in the complement of C. Then $\pi' := \psi^{-1} \circ M^{-1}$ defines a mapping as in Equation (4.1) for the curve $C' = M(C)$. Proposition 4.3 implies that π' is 1-quasiconformal. Consequently, $M^{-1} = \psi \circ \pi'$ is 1-quasiconformal, that is, a Möbius transformation. □

5 | MASS UPPER BOUND

In this section, we prove Theorems 1.1 and 1.2. We first consider the implication "(3)⇒(1)." Recall that we are given an orientation-preserving homeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$ and the canonical quotient map $Q : Z \to \widetilde{Z}$. We are assuming the existence of a constant $C > 0$ for which

$$\liminf_{r \to 0^+} \frac{\mathcal{H}^2_{\widetilde{Z}}(\overline{B}_{\widetilde{Z}}(y, r))}{\pi r^2} \leqslant C \quad \text{for every } y \in Q(S_Z). \tag{5.1}$$

In order to make transparent how the Lipschitz constant of g (respectively, g^{-1}) is related to C in Equation (5.1), we define $C_1, C_2 \geqslant 0$ to be the smallest constants for which

$$\liminf_{r\to 0^+} \frac{\mathcal{H}^2_{\widetilde{Z}}(\overline{\widetilde{\iota}_1(Z_1)} \cap \overline{B}_{\widetilde{Z}}(y,r))}{\pi r^2} \leqslant C_1 \quad \text{for every } y \in Q(S_Z) \tag{5.2}$$

$$\liminf_{r\to 0^+} \frac{\mathcal{H}^2_{\widetilde{Z}}(\overline{\widetilde{\iota}_2(Z_2)} \cap \overline{B}_{\widetilde{Z}}(y,r))}{\pi r^2} \leqslant C_2 \quad \text{for every } y \in Q(S_Z). \tag{5.3}$$

Recalling from Lemma 3.3 the fact that the inclusion maps are 1-Lipschitz and local isometries outside the seam, the limit infimum in Equations (5.2) and (5.3) are bounded from below by $1/2$. Since the seam is negligible, we also have $C_1, C_2 \geqslant 1/2$ and $\max\{C_1, C_2\} \leqslant C - 1/2$.

We show that the constant C_1 in Equation (5.2) and the Lipschitz constant L_1 of g^{-1} are connected via the following function

$$f(\epsilon) := \frac{\left(\sin|_{(0,\pi/2]}\right)^{-1}(\epsilon)}{\pi} + \frac{\sqrt{1-\epsilon^2}}{\pi\epsilon} \quad \text{for } 0 < \epsilon \leqslant 1. \tag{5.4}$$

Definition 5.1. For every $C \geqslant 1/2$, $L = L(C) \geqslant 1$ denotes the unique positive number such that for every $0 < \epsilon \leqslant L^{-1}$, $f(\epsilon) \geqslant C$. Equivalently, $L = 1/f^{-1}(C)$.

Remark 5.2. We note that for every $0 < \epsilon \leqslant 1$, we have $f(\epsilon) \geqslant (\pi\epsilon)^{-1}$. We use this fact during the proof of Theorem 1.1.

Proposition 5.3. *If Equation (5.2) holds with constant C_1 and $L_1 = L(C_1)$ is as in Definition 5.1, then g^{-1} is L_1-Lipschitz and $\widetilde{\iota}_1 : \overline{Z}_1 \to \widetilde{Z}$ satisfies for every $x, y \in \overline{Z}_1$, $\sigma(x,y) \geqslant d_Z(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) \geqslant \sigma(x,y)/L_1$.*

The symmetry in the argument yields the following result.

Proposition 5.4. *If Equation (5.3) holds with constant C_2 and $L_2 = L(C_2)$ is as in Definition 5.1, then g is L_2-Lipschitz and $\widetilde{\iota}_2 : Z_2 \to \widetilde{Z}$ satisfies for every $x, y \in \overline{Z}_2$, $\sigma(x,y) \geqslant d_Z(\widetilde{\iota}_2(x), \widetilde{\iota}_2(y)) \geqslant \sigma(x,y)/L_2$.*

We start the proof of Proposition 5.3. We consider the decomposition $g^*\mathcal{H}^1_{\mathbb{S}^1} = v_g \mathcal{H}^1_{\mathbb{S}^1} + \mu^\perp$ with $\mu^\perp$ and $\mathcal{H}^1_{\mathbb{S}^1}$ being singular. We fix a Borel representative of v_g. Let f be as in Equation (5.4). The following statement holds for every $\widetilde{Z}$.

Proposition 5.5. *Given $1 > \epsilon > 0$ and a $\mathcal{H}^1_{\mathbb{S}^1}$-density point $x_0 \in \mathbb{S}^1$ of $E := \{v_g \leqslant \epsilon\}$, we have*

$$f(\epsilon) \leqslant \liminf_{r\to 0^+} \frac{\mathcal{H}^2_{\widetilde{Z}}(\overline{\widetilde{\iota}_1(Z_1)} \cap \overline{B}_{\widetilde{Z}}(x_0,r))}{\pi r^2}. \tag{5.5}$$

Proof. For the duration of the proof, we fix normal coordinates $F : B(0,\pi/2) \to \mathbb{S}^2$ centred at x_0 in such a way that the preimage of $\mathbb{S}^1 \cap B(x_0, \pi/2)$ is $(-\pi/2, \pi/2) \times \{0\}$ [28, Section 5]. Recall that this means that F is an isometry along radial geodesics and the metric has the expansion

$g_{ij}(x) = \delta_{ij} + O(\|x\|_2^2)$ in these coordinates. In particular, as $r \to 0^+$, the bi-Lipschitz constant of $F|_{B(0,r)}$ is of the form $1 + O(r^2)$. We denote $\Gamma(s) := F(s, 0)$ for $|s| \leqslant \pi/2$.

We fix $0 < \eta < 1/\epsilon - 1$. Since x_0 is a density point of E, there exists $s_0 < \pi/2$ such that for every $0 < s \leqslant s_0$,

$$\mathcal{H}^1_{\mathbb{S}^1}(\Gamma([-s, s]) \setminus E) \leqslant \epsilon\eta s. \tag{5.6}$$

We fix $0 < r \leqslant \epsilon s_0$. Then, for every $0 < s < r/\epsilon$, Proposition 3.6 yields, for both $I = [0, s]$ and $I = [-s, 0]$,

$$\ell(E \cap (\widetilde{\iota}_1 \circ \Gamma|_I)) \leqslant \epsilon s. \tag{5.7}$$

Since $\widetilde{\iota}_1$ is 1-Lipschitz, according to Lemma 3.3, Equations (5.6) and (5.7) imply

$$\ell(\widetilde{\iota}_1 \circ \Gamma|_I) \leqslant s\epsilon + \epsilon\eta s = \epsilon(1 + \eta)s < s. \tag{5.8}$$

We denote for every $|s| < r/(\epsilon(1 + \eta))$, $\rho_s := r - \epsilon(1 + \eta)|s|$. For each $z \in Z_1 \cap B_{\mathbb{S}^2}(F(s, 0), \rho_s)$, the inequality (5.8) implies $\widetilde{\iota}_1(z) \in \overline{B}_{\widetilde{Z}}(\widetilde{\iota}_1(x_0), r)$.

We estimate $A_r := \mathcal{H}^2_{\widetilde{Z}}(\overline{\widetilde{\iota}_1(Z_1)} \cap \overline{B}_{\widetilde{Z}}(\widetilde{\iota}_1(x_0), r))$ as $r \to 0^+$. In estimating A_r, we use the fact that the seam $Q(S_Z)$ has negligible $\mathcal{H}^2_{\widetilde{Z}}$-measure and that $\widetilde{\iota}_1$ is a local isometry outside the seam. We claim that for each $0 < \theta < \pi/2$ the following holds:

$$A_r \geqslant (1 + O((r/\epsilon)^2))^{-2}\left(\theta r^2 + \cos(\theta)\frac{r^2}{(1 + O((r/\epsilon)^2))\epsilon(1 + \eta)}\right). \tag{5.9}$$

The term $(1 + O((r/\epsilon)^2))^{-2}$ comes from estimating the Jacobian of $\widetilde{\iota}_1 \circ F$. The first term in the brackets comes from the fact that F preserves the speed of radial geodesics, so

$$(\widetilde{\iota}_1 \circ F)\left(\left\{(s, t) : \sqrt{s^2 + t^2} < r, 0 < t\right\}\right) \subset \overline{\widetilde{\iota}_1(Z_1)} \cap \overline{B}_{\widetilde{Z}}(\widetilde{\iota}_1(x_0), r).$$

We use this inclusion in a circular sector $C_\theta(r)$ which has a total angle 2θ and an angle bisector $\{0\} \times \mathbb{R}$.

The second term in the brackets is twice the area of a suitable triangle. The factor of two comes from the symmetry of the estimate in Equation (5.8) with respect to the parameter $s = 0$. We consider a triangle $T_\theta(r) \subset \mathbb{R}^2$ foliated by line segments $\ell(s)$, where $0 \leqslant s < r/((1 + \eta)\epsilon)$, with $\ell(s)$ having the start point $(s, 0)$, tangent in the direction $(\sin(\theta), \cos(\theta))$, and has length $\rho_a/(1 + O((r/\epsilon)^2))$. The $\widetilde{\iota}_1 \circ F$ image of such a triangle $T_\theta(r)$ contributes to A_r. The inequality (5.9) follows.

We choose the angle θ to satisfy $\sin(\theta) = \epsilon(1 + \eta)$. We divide Equation (5.9) by πr^2, pass to the limit $r \to 0^+$, and then to $\eta \to 0^+$, and conclude

$$\liminf_{r \to 0^+} \frac{A_r}{\pi r^2} \geqslant \frac{\left(\sin|_{(0,\pi/2]}\right)^{-1}(\epsilon)}{\pi} + \frac{\sqrt{1 - \epsilon^2}}{\pi\epsilon} = f(\epsilon). \tag{5.10}$$

The inequality (5.5) is the same as Equation (5.10). □

Remark 5.6. Given $0 < \epsilon < 1$, the lower bound in Equation (5.10) is sharp. This can be shown by considering a bi-Lipschitz $g : \mathbb{S}^1 \to \mathbb{S}^1$ with metric speed $v_g \equiv \epsilon$ everywhere in an open neighbourhood of $x_0 \in \mathbb{S}^1$.

If the circular sector $C_\theta(r)$ and triangle $T_\theta(r)$ are defined as in the proof of the lower bound Equation (5.10), with $\eta = 0$, and $\theta = (\sin|_{(0,\pi/2]})^{-1}(\epsilon)$, we have

$$\liminf_{r\to 0^+} \frac{A_r}{\pi r^2} = \frac{\mathcal{H}^2_{\mathbb{R}^2}(C_\theta(1)) + 2\mathcal{H}^2_{\mathbb{R}^2}(T_\theta(1))}{\pi} = f(\epsilon).$$

This can be showed using Lemma 3.2 and Proposition 3.6. The key property of the angle θ is that the line on $\mathbb{R}^2$ containing $(r/\epsilon, 0)$ with tangent vector $(-\cos(\theta), \sin(\theta))$ intersects every ball $\overline{B}_{\mathbb{R}^2}((s,0),\rho_s)$ tangentially when $0 \leqslant s < 1/\epsilon$ and $\rho_s = 1 - \epsilon s$.

Proof of Proposition 5.3. Given Equation (5.2) and Proposition 5.5, we have $v_g(x) \geqslant L_1^{-1}$ for $\mathcal{H}^1_{\mathbb{S}^1}$-almost every $x \in \mathbb{S}^1$. This implies that g^{-1} is absolutely continuous and $v_{g^{-1}}(x) \leqslant L_1$ for $\mathcal{H}^1_{\mathbb{S}^1}$-almost every $x \in \mathbb{S}^1$. Therefore g^{-1} is L_1-Lipschitz.

The fact that $\widetilde{\iota}_1$ 1-Lipschitz follows from Lemma 3.3. Proposition 3.6 implies that

$$d_Z(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) \geqslant \sigma(x,y)/L_1, \quad \text{for every } x, y \in \mathbb{S}^1.$$

Given this inequality, the equality (3.1) in Lemma 3.2 implies the corresponding inequality for every pair $x, y \in \overline{Z}_1$. Hence $\widetilde{\iota}_1^{-1}$ is L_1-Lipschitz. □

Next, we verify a lemma about radial extensions of bi-Lipschitz maps, which we need during the proof of Theorem 1.1.

For the south pole $P_1 \in Z_1$, we consider the stereographic projection $P : \mathbb{S}^2 \setminus \{P_1\} \to \mathbb{R}^2 \times \{0\}$ fixing the equator and mapping the north pole $P_2 = (0,0,1)$ to the origin. We identify $\mathbb{R}^2 \times \{0\}$ with $\mathbb{R}^2$. We note that P^{-1} has the explicit definition

$$P^{-1}(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2} \right).$$

The Riemannian tensor of $\mathbb{S}^2$ in these coordinates is $I = (4/(1+r^2)^2)g_{\mathrm{E}}$, where r is the distance to the origin and g_E the Euclidean inner product. In polar coordinates, $g_{\mathrm{E}} = dr^2 + r^2 d\theta^2$. We see from the form of I that the bi-Lipschitz constants of $\widetilde{g} = P \circ g \circ (P|_{\mathbb{S}^1})^{-1}$ and $g : \mathbb{S}^1 \to \mathbb{S}^1$ coincide.

We represent the polar coordinates using the complex notation $re^{i\theta}$. We note that there exists a homeomorphism $\widetilde{G} : \mathbb{R} \to \mathbb{R}$ with $\widetilde{g}(e^{i\theta}) = e^{i\widetilde{G}(\theta)}$ for every $\theta \in \mathbb{R}$. For every $0 \leqslant r \leqslant 1$ and $\theta \in \mathbb{R}$, we set $\widetilde{\psi}(re^{i\theta}) := re^{i\widetilde{G}(\theta)}$ and refer to $\widetilde{\psi}$ as the *radial extension* of $\widetilde{g}$. We recall from [24, Theorem 2.2] that the bi-Lipschitz constants of $\widetilde{g}$ and $\widetilde{\psi}$ coincide. Let $\psi = P^{-1} \circ \widetilde{\psi} \circ P|_{Z_2} : Z_2 \to Z_2$.

We use the following fact during the proof of Lemma 5.8; see for example [11, 13].

Lemma 5.7. *For every $x, y \in \mathbb{S}^2$, $0 < \epsilon < 1$, and $0 < 4r < \sigma(x,y)$, the modulus of the family of paths joining $B_{\mathbb{S}^2}(x,r)$ to $B_{\mathbb{S}^2}(y,r)$ with length $(1+\epsilon)\sigma(x,y)$ is positive.*

Lemma 5.8. *The map $\psi : Z_2 \to Z_2$ is L-bi-Lipschitz if g is L-bi-Lipschitz.*

Proof. We refer the interested reader to [24, Section 2] for the proof of the fact that $\widetilde{\psi}$ is bi-Lipschitz if $\widetilde{g}$ (equivalently g) is bi-Lipschitz. We take this as given.

Since $\widetilde{\psi}$ is bi-Lipschitz, it has a differential at $\mathcal{L}^2$-almost every point in $\mathbb{D}$. Given this fact, the following computations are understood to hold at $\mathcal{L}^2$-almost every $(x, y) = re^{i\theta}$ in the unit disk.

The pullback $\widetilde{\psi}^* I$ is a diagonal matrix with respect to the basis $(dr, d\theta)$, with diagonal $4/(1 + r^2)^2$ and $4|\widetilde{G}'(\theta)|^2 r^2/(1+r^2)^2$. Hence the maximum of the operator norms of $D\widetilde{\psi} : (T\mathbb{D}, I) \to (T\mathbb{D}, I)$ and its inverse is equal to $L(re^{i\theta}) = \max\{|\widetilde{G}'(\theta)|, |\widetilde{G}'(\theta)|^{-1}\}$. Then, if L' denotes the essential supremum of $L(re^{i\theta})$, Lemma 5.7 implies that ψ is L'-bi-Lipschitz. On the other hand, L' is the bi-Lipschitz constant of g. □

Proof of Theorem 1.1. We first claim that "(1) ⇒ (2)." Lemma 5.8 provides us with an L-bi-Lipschitz $\psi : Z_2 \to Z_2$ extension of the given L-bi-Lipschitz g. We define $H(x) = \widetilde{\iota}_1(x)$ for each $x \in \overline{Z}_1$ and $H(x) = \widetilde{\iota}_2 \circ \psi(x)$ otherwise. Proposition 3.6 implies that H is L-bi-Lipschitz at the seam, and Lemma 3.2 implies that H is L-bi-Lipschitz everywhere.

Notice that if $H : \mathbb{S}^2 \to \widetilde{Z}$ is L'-bi-Lipschitz, we may choose $C = (L')^4$ as an upper bound for the 2D Hausdorff lower density. Hence "(2) ⇒ (3)" follows, quantitatively. Lastly, "(3) ⇒ (1)" follows from Propositions 5.3 and 5.4. In fact, given $C \geqslant 1$ for which the lower density bound of Equation (5.1) holds, g is L'-bi-Lipschitz for L' solving $C = f(1/L')$. Since $f(\epsilon) \geqslant 1/\pi\epsilon$ for every $0 < \epsilon \leqslant 1$, we have $C\pi \geqslant L'$. Hence g is $C\pi$-bi-Lipschitz. □

Remark 5.9. The estimates between the constants in "(3) ⇒ (1)" in Theorem 1.1 can be improved in two ways. First, the constants C_1 and C_2 in Equations (5.2) and (5.3) satisfy $\max\{C_1, C_2\} \leqslant C - 1/2$, so g is $(C - 1/2)\pi$-bi-Lipschitz.

The second improvement is obtained by using the constant $L' = L(C - 1/2)$ from Definition 5.1. Then g is L'-bi-Lipschitz, where $L' \leqslant (C - 1/2)\pi$.

These improvements imply that the bi-Lipschitz constant of g converges to 1 as $C \to 1^+$. These facts also improve Theorem 1.2 and the following result, Proposition 5.10.

Before proving Theorem 1.2, we investigate a related problem. To this end, suppose that we are given Riemann maps $\phi_i : Z_i \to \Omega_i$ with Ω_1 and Ω_2 denoting the complementary components of a welding curve C, and set $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$.

Proposition 5.10. *Let $K, C \geqslant 1$. The welding homeomorphism g is $\pi(KC)^2$-bi-Lipschitz if there exists a K-quasiconformal homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that for both $i = 1, 2$,*

$$C^{-1} J_h(x) \leqslant J_{\phi_i^{-1}}(x) \leqslant C J_h(x) \quad \text{for } \mathcal{H}^2_{\mathbb{S}^2}\text{-almost everywhere, } x \in \Omega_i. \tag{5.11}$$

Conversely, if g is L-bi-Lipschitz, then there exists L^4-quasiconformal homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that Equation (5.11) holds for $C = L^2$.

Proof. We first assume that $g : \mathbb{S}^1 \to \mathbb{S}^1$ is L-bi-Lipschitz. Then Theorem 1.1 provides us with an L-bi-Lipschitz homeomorphism $\Psi : \widetilde{Z} \to \mathbb{S}^2$. Proposition 4.3 and Equation (4.1) imply that $\widetilde{\pi} : \mathbb{S}^2 \to \widetilde{Z}$ defined via the formula

$$\widetilde{\pi}(x) = \begin{cases} \widetilde{\iota}_1 \circ \phi_1^{-1}(x), & x \in \overline{\Omega_1}, \\ \widetilde{\iota}_2 \circ \phi_2^{-1}(x), & x \in \Omega_2 \end{cases} \tag{5.12}$$

is a 1-quasiconformal homeomorphism. Therefore, $h := \Psi \circ \widetilde{\pi} : \mathbb{S}^2 \to \mathbb{S}^2$ is K-quasiconformal for $K = L^4$, and as Ψ is L-bi-Lipschitz, the Jacobians of h and $\widetilde{\pi}$ are comparable with comparison constant $C = L^2$.

Next, we are given a Jordan curve $\mathcal{C} \subset \mathbb{S}^2$ corresponding to a welding homeomorphism $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$, a K-quasiconformal homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$, and a constant $C \geqslant 1$ such that

$$C^{-1} J_h(x) \leqslant J_{\widetilde{\pi}}(x) \leqslant C J_h(x) \quad \mathcal{H}^2_{\mathbb{S}^2}\text{-almost everywhere, } x \in \mathbb{S}^2 \setminus \mathcal{C}. \tag{5.13}$$

For $i = 1, 2$, the composition $h \circ \phi_i$ is K-quasiconformal with Jacobian bounded from above C and below by C^{-1}, respectively; here we apply Equation (5.13). Theorem 2.4 (ii) and Hadamard's inequality imply that $C^{-1} \leqslant \rho^2_{h \circ \phi_i} \leqslant KC$ $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere in Z_i. Lemma 5.7 implies that the homeomorphism $h \circ \phi_i$ is locally L'-bi-Lipschitz for $L' = \sqrt{KC}$.

Since, for both $i = 1, 2$, $\overline{Z}_i$ is geodesic, it is immediate that $h \circ \phi_i : \overline{Z}_i \to \mathbb{S}^2$ is L'-Lipschitz. Since this holds for both $i = 1, 2$, the construction of d_Z implies that whenever $x, y \in \mathbb{S}^1$, $\sigma(h \circ \phi_1(x), h \circ \phi_1(y)) \leqslant L' d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y))$. Lemma 3.2 (3.1) establishes the same inequality for each $x, y \in \overline{Z}_1$. Hence the mapping $\widetilde{\pi}$ defined by the expression (5.12) is a homeomorphism and $\Psi := h \circ \widetilde{\pi}^{-1}$ is L'-Lipschitz on the southern hemisphere. A similar argument shows that Ψ is L'-Lipschitz on both of the hemispheres. Then Lemma 3.2 (3.2) implies that Ψ is L'-Lipschitz everywhere.

Since $\operatorname{mod} \Gamma \leqslant K \operatorname{mod} \Psi^{-1}\Gamma$ for all path families (recall Proposition 4.3), we have $\Psi^{-1} \in N^{1,2}(\mathbb{S}^2, \widetilde{Z})$. On the other hand, $\Psi(Q(S_Z))$ has negligible $\mathcal{H}^2_{\mathbb{S}^2}$-measure and Ψ^{-1} is locally L'-Lipschitz in the complement of that set. In particular, almost every absolutely continuous $\gamma : [0,1] \to \mathbb{S}^2$ has zero length in $\Psi(Q(S_Z))$ and $\Psi^{-1} \circ \gamma$ is absolutely continuous. As a consequence, $\mathcal{H}^1_{\widetilde{Z}}(Q(S_Z) \cap |\Psi^{-1} \circ \gamma|) = 0$.

Denoting $E = Q(S_Z) \cap |\Psi^{-1} \circ \gamma|$ and $\rho = L' \chi_{\mathbb{S}^2}$, we conclude from Lemma 2.2 that $\ell(\Psi^{-1} \circ \gamma) \leqslant \int_\gamma \rho \, ds \leqslant L' \ell(\gamma)$. Lemma 5.7 implies that Ψ^{-1} is L'-Lipschitz.

We have verified that Ψ is L'-bi-Lipschitz. By applying the implications "(2) ⇒ (3) ⇒ (1)" in Theorem 1.1, we conclude that g is L-bi-Lipschitz for $L = \pi(L')^4 = \pi(KC)^2$. □

Next, we prove Theorem 1.2. This essentially follows from Proposition 5.10.

Proof of Theorem 1.2. We claim that $g : \mathbb{S}^1 \to \mathbb{S}^1$ is bi-Lipschitz if and only if there exists a quasiconformal homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ and a 1-quasiconformal homeomorphism $\varphi : \mathbb{S}^2 \to \widetilde{Z}$ such that J_φ and J_h are comparable.

If such φ and h exist, we may assume that $\phi_i = \varphi^{-1} \circ \widetilde{\iota}_i|_{Z_i}$ is a Riemann map for both $i = 1, 2$. Then Proposition 5.10 shows that g is bi-Lipschitz.

Conversely, if g is bi-Lipschitz, Theorem 1.1 provides a bi-Lipschitz homeomorphism $\Psi : \widetilde{Z} \to \mathbb{S}^2$. Then Theorem 1.6 implies the existence of a 1-quasiconformal homeomorphism $\pi : \mathbb{S}^2 \to \widetilde{Z}$ such that $\phi_i = \pi^{-1} \circ \widetilde{\iota}_i|_{Z_i}$ is a Riemann map for $i = 1, 2$. We may also assume that $\Psi \circ \widetilde{\iota}_i|_{Z_i}$ is orientation-preserving for $i = 1, 2$, by post-composing Ψ with a suitable reflection, if need be. Defining $h = \Psi \circ \pi$ implies that the assumptions of Proposition 5.10 hold for g.

Since Theorem 1.1 and Proposition 5.10 are quantitative, so is Theorem 1.2. □

6 | MAPPINGS OF FINITE DISTORTION

In this section, we establish Proposition 1.4 and Theorem 1.5.

Definition 6.1. Let $\Omega, \Omega' \subset \mathbb{S}^2$ be open. A homeomorphism $\psi : \Omega \to \Omega'$ is a *mapping of finite distortion* if $\psi \in N^{1,1}(\Omega, \mathbb{S}^2)$; second, the determinant $J(D\psi)$ of the differential $D\psi$ is non-negative and integrable; lastly, there exists a function $1 \leqslant K'_\psi < \infty$ for which

$$|D\psi|_g^2 \leqslant K'_\psi J(D\psi) \quad \mathcal{H}^2_{\mathbb{S}^2}\text{-almost everywhere in } \Omega. \tag{6.1}$$

Here $|D\psi|_g$ refers to the operator norm of the differential $D\psi$. We let K_ψ denote a smallest Borel function which is bounded from below by χ_Ω and for which Equation (6.1) holds.

Definition 6.2. A smooth strictly increasing function $\mathcal{A} : [1, \infty) \to [0, \infty)$ is *admissible* if

(1) $\mathcal{A}(1) = 0$,
(2) $\int_1^\infty t^{-2}\mathcal{A}(t)\, d\mathcal{L}^1(t) = \infty$, and
(3) $t \mapsto t\mathcal{A}'(t)$ is increasing for large values t, and converges to ∞ as $t \to \infty$.

We obtain the same class of admissible $\mathcal{A}$ if we replace (2) with the condition

$$\int_1^\infty t^{-1}\mathcal{A}'(t)\, d\mathcal{L}^1(t) = \infty.$$

This follows from the fact that $\mathcal{A}(s)/s \leqslant 4\int_s^{2s} t^{-2}\mathcal{A}(t)\, d\mathcal{L}^1(t)$ whenever $s \geqslant 1$ and the integration by parts formula.

Definition 6.3. Let $\Omega, \Omega' \subset \mathbb{S}^2$ be open, and $\psi : \Omega \to \Omega'$ a homeomorphism. We say that ψ is *admissible* if ψ is a mapping of finite distortion and there exists an admissible $\mathcal{A}$ with

$$\int_\Omega e^{\mathcal{A}(K_\psi)}\, d\mathcal{H}^2_{\mathbb{S}^2} < \infty. \tag{6.2}$$

If $\mathcal{A}(t) = pt - p$ for some $p > 0$, we say that ψ has *exponentially integrable distortion*.

We recall some properties of such ψ. First, ψ satisfies Lusin's condition (N) [26, Theorem 1.1]. Second, $\psi^{-1} \in N^{1,2}(\Omega', \Omega)$ [27, Corollary 1.2]; this implies that ψ^{-1} satisfies Lusin's condition (N) [2, Theorem 3.3.7]. Third, the Jacobian $J(D\psi)$ appearing on the right-hand side of (6.1) coincides with the Jacobian J_ψ we defined in Section 2.2 [26].

In this section, we show the following theorem.

Theorem 6.4. *Suppose that $g : \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism, g^{-1} absolutely continuous, and there exists a homeomorphism $\psi : \overline{Z}_2 \to \overline{Z}_2$ extending g with $\psi|_{Z_2}$ admissible. Then $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$.*

Note that Theorem 1.5 is a consequence of Theorem 6.4 so it suffices to verify Theorem 6.4.

Definition 6.5. Given $x_0 \in \mathbb{S}^1$ and $\pi > R_0 > 0$, set $\widetilde{Q} := \overline{B}_{\mathbb{S}^2}(x_0, R_0) \subset \mathbb{S}^2$. We define $H(x) = \widetilde{\iota}_1(x)$ if $x \in \widetilde{Q} \cap \overline{Z}_1$ and $\widetilde{\iota}_2 \circ \psi(x)$ if $x \in \widetilde{Q} \cap Z_2$, and denote $\widetilde{R} = H(\widetilde{Q}) \subset \widetilde{Z}$.

Proposition 6.6. *If $\widetilde{R}$ and H are as in Definition 6.5, then H is a homeomorphism and there exists a 1-quasiconformal homeomorphism $f = (u, v)\colon \widetilde{R} \to [0,1] \times [0, M]$ for some $M > 0$.*

Proof of Theorem 6.4 assuming Proposition 6.6. We cover the seam in $\widetilde{Z}$ by the interiors of $\widetilde{R}$ as in Definition 6.5. This implies that $\widetilde{Z}$ can be covered by quasiconformal images of planar domains, and the quasiconformal equivalence of $\widetilde{Z}$ and $\mathbb{S}^2$ follows from Theorem 2.7. □

The following lemma is a key step in proving Proposition 6.6.

Lemma 6.7. *The H from Definition 6.5 is a homeomorphism, $H \in N^{1,1}(\widetilde{Q}, \widetilde{R})$ and $H^{-1} \in N^{1,2}(\widetilde{R}, \widetilde{Q})$. Furthermore, H satisfies Lusin's conditions (N) and (N^{-1}).*

Proof. The absolute continuity of g^{-1} implies for the Lebesgue decomposition $g^*\mathcal{H}^1 = v_g\mathcal{H}^1 + \mu^\perp$ that $\{v_g = 0\}$ has negligible $\mathcal{H}^1_{\mathbb{S}^1}$-measure in an open neighbourhood of $\mathbb{S}^1 \cap \widetilde{Q}$. Then Proposition 3.6 and Lemma 3.2 imply that H is a homeomorphism.

We recall from Lemma 3.3 the fact that the inclusion maps $\widetilde{\iota}_1|_{Z_1}\colon Z_1 \to \widetilde{Z}$ and $\widetilde{\iota}_2|_{Z_2}\colon Z_2 \to \widetilde{Z}$ are 1-Lipschitz local isometries. This implies that H and its inverse are absolutely continuous in measure; the seam has negligible Hausdorff 2-measure.

In the following proof, we write $\widetilde{\rho}_i$ for functions defined on $\widetilde{Q} \cap Z_i \subset \mathbb{S}^2$ and $\rho_i = (\widetilde{\rho}_i \circ \widetilde{\iota}_i^{-1})$ on $\widetilde{R} \cap \widetilde{\iota}_i(Z_i) \subset \widetilde{Z}$ for $i = 1, 2$.

Since $\psi^{-1} \in N^{1,2}(\widetilde{Q} \cap Z_2, \mathbb{S}^2)$, for $i = 1, 2$, there exists an upper gradient $\widetilde{\rho}_i \in L^2(\widetilde{Q} \cap Z_i)$ of $H^{-1} \circ \widetilde{\iota}_i|_{Z_i \cap \widetilde{Q}}$ for $i = 1, 2$. We fix such functions and denote $\rho := \chi_{\widetilde{R} \cap \widetilde{\iota}_1(Z_1)}\rho_1 + \chi_{\widetilde{R} \cap \iota_2(Z_2)}\rho_2 \in L^2(\widetilde{R})$.

Let Γ_0 denote the collection of non-constant paths on $\widetilde{R} \subset \widetilde{Z}$ which have positive length in the seam $Q(S_Z)$ or along which ρ fails to be integrable. Since $\rho + \infty \cdot \chi_{Q(S_Z)}$ is L^2-integrable, Lemma 2.1 yields $\operatorname{mod}\Gamma_0 = 0$.

Consider next an absolutely continuous path $\gamma\colon [0,1] \to \widetilde{R}$ in the complement of Γ_0. Then $\theta = H^{-1} \circ \gamma$ is such that $\mathcal{H}^1_{\mathbb{S}^2}(|\theta| \cap \mathbb{S}^1) = 0$. Indeed, since γ has zero length in the seam, the area formula (2.3) implies $\mathcal{H}^1_{\widetilde{Z}}(|\gamma| \cap Q(S_Z)) = 0$. This implies $\mathcal{H}^1_{\mathbb{S}^1}(|\theta| \cap \mathbb{S}^1) = 0$ due to Proposition 3.6 and the absolute continuity of $g|^{-1}_{\mathbb{S}^1 \cap \widetilde{Q}}$. Since $\mathcal{H}^1_{\mathbb{S}^1}(|\theta| \cap \mathbb{S}^1) = 0$, the assumptions of Lemma 2.2 are satisfied. Hence

$$\ell(\theta) \leqslant \int_\gamma \rho\, ds < \infty.$$

This implies that H^{-1} has an L^2-integrable weak gradient, so $H^{-1} \in N^{1,2}(\widetilde{R}, \widetilde{Q})$.

Lastly, we claim that $H \in N^{1,1}(\widetilde{Q}, \widetilde{R})$. To this end, we observe that $H|_{\widetilde{Q} \cap Z_i}$ has an upper gradient $\widetilde{\rho}_i \in L^1(\widetilde{Q} \cap Z_i)$, and denote $\widetilde{\rho} = \sum_{i=1}^2 \chi_{\widetilde{Q} \cap Z_i}\widetilde{\rho}_i \in L^1(\widetilde{Q})$. Now $\widetilde{\rho}$ is integrable along 1-almost every absolutely continuous path $\gamma\colon [0,1] \to \widetilde{Q}$ and 1-almost every such path has zero length in $\mathbb{S}^1$. Having fixed a path γ with these properties, Proposition 3.6 implies that $\theta = H \circ \gamma$ has zero length in the seam. The inequality $\ell(\theta) \leqslant \int_\gamma \rho\, ds$ follows from Lemma 2.2. This yields that $H \in N^{1,1}(\widetilde{Q}, \widetilde{R})$. □

Remark 6.8. The Sobolev regularity $H^{-1} \in N^{1,2}(\widetilde{Q}, \widetilde{R})$ is crucial in the following. Typically, the Sobolev regularity of the inverse of a Sobolev homeomorphism is a subtle issue in the metric surface setting.

To highlight the issue, we recall [23, Example 6.1]. There an example of a metric surface X was constructed for which there exists a 1-Lipschitz homeomorphism $H\colon \mathbb{R}^2 \to X$ with $\operatorname{mod}\Gamma \leqslant \operatorname{mod} H\Gamma$ for all path families, but $H^{-1} \notin N^{1,2}(X,\mathbb{R}^2)$. In fact, H is a local isometry outside a Cantor set $E \subset \mathbb{R}\times\{0\}$ of positive $\mathcal{L}^1$-measure and $H(E)$ has negligible $\mathcal{H}^1_X$-measure. The key point is that X is not reciprocal; recall Definition 2.5.

We define the following auxiliary function for later use:

$$P(t) := \begin{cases} t^2, & 0 \leqslant t < 1, \\ \dfrac{t^2}{\mathcal{A}^{-1}(\log t^2)}, & t \geqslant 1. \end{cases}$$

We note that for every $a \in [0,\infty)$,

$$P(a) \leqslant e^{\mathcal{A}(K_H)} + \frac{a^2}{K_H} \quad \text{for } \mathcal{H}^2_{\mathbb{S}^2}\text{-almost everywhere in } \widetilde{Q}. \tag{6.3}$$

This follows by first observing that $a^2 < e^{\mathcal{A}(K_H)}$ implies $P(a) \leqslant e^{\mathcal{A}(K_H)}$ and otherwise $P(a) \leqslant \frac{a^2}{K_H}$.

Also, for any measurable function $\widetilde{\rho}\colon \widetilde{Q} \to [0,\infty]$,

$$\int_{\widetilde{Q}} P(\widetilde{\rho})\, d\mathcal{H}^2_{\mathbb{S}^2} < \infty \quad \text{implies} \quad \int_{\widetilde{Q}} \widetilde{\rho}\, d\mathcal{H}^2_{\mathbb{S}^2} < \infty. \tag{6.4}$$

The implication in Equation (6.4) follows since $\mathcal{A}'(t)t$ is increasing for large t and converges to infinity as $t \to \infty$. Consequently, there exists $t_1 \geqslant 1$ for which the derivative of $h(t) = e^{\mathcal{A}(t)}/t^2$ is bounded from below by $h(t)/t$ for every $t \geqslant t_1$. This implies the existence of $t_0 \geqslant 1$ such that $h(t) \geqslant 1$ for every $t \geqslant t_0$. This is equivalent to saying that $P(t) \geqslant t$ for every $t \geqslant t_0$. This yields Equation (6.4).

We set $K_\psi(x) = |D\psi|_g^2/J(D(\psi))(x)$ and $K_{\psi^{-1}}(x) = |D(\psi^{-1})|_g^2/J(D(\psi^{-1}))$. Observe that $K_\psi = K_{\psi^{-1}} \circ \psi$ $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere.

We set $K_H(x) = 1$ if $x \in \widetilde{Q} \cap \overline{Z}_1$ and $K_H(x) = K_\psi(x)$ in $x \in \widetilde{Q} \cap Z_2$. Then

$$\int_{\widetilde{Q}} e^{\mathcal{A}(K_H)}\, d\mathcal{H}^2_{\mathbb{S}^2} < \infty. \tag{6.5}$$

Also, $K_{H^{-1}} := \rho^2_{H^{-1}}/J_{H^{-1}}$ satisfies $K_H = K_{H^{-1}} \circ H$ $\mathcal{H}^2_{\widetilde{Z}}$-almost everywhere, since, outside a $\mathcal{H}^2_{\widetilde{Z}}$-negligible set, either the number is one, or $\rho^2_{H^{-1}} \circ \widetilde{\iota}_2 = |D(\psi^{-1})|_g^2$, $J_{H^{-1}} \circ \widetilde{\iota}_2 = J(D(\psi^{-1}))$, and $K_\psi = K_{\psi^{-1}} \circ \psi$.

For every $z \in \widetilde{Q}$ and every pair $0 < r < r_0$, we denote $\Gamma(z,r,r_0) := \Gamma(\overline{B}_{\mathbb{S}^2}(z,r), \widetilde{Q} \setminus B_{\mathbb{S}^2}(z,r_0); \widetilde{Q})$.

Lemma 6.9. *For every $z \in \widetilde{Q}$ and $0 < r < r_0$ with $\widetilde{Q} \setminus B_{\mathbb{S}^2}(z,r_0) \neq \emptyset$,*

$$\operatorname{mod} H\Gamma(z,r,r_0) \leqslant \inf\left\{ \int_{\widetilde{Q}} \widetilde{\rho}^2 K_H\, d\mathcal{H}^2_{\mathbb{S}^2} : \ \widetilde{\rho} \text{ is admissible for } \Gamma(z,r,r_0) \right\}. \tag{6.6}$$

Proof. Fix an admissible function $\widetilde{\rho}$ for $\Gamma(z,r,r_0)$. Then for almost every $\gamma \in H\Gamma(z,r,r_0)$, $H^{-1}\circ\gamma$ is absolutely continuous, and

$$1 \leqslant \int_{H^{-1}\circ\gamma} \widetilde{\rho}\, ds \leqslant \int_{\gamma} (\widetilde{\rho}\circ H^{-1})\rho_{H^{-1}}\, ds.$$

In particular, $\rho = (\widetilde{\rho}\circ H^{-1})\rho_{H^{-1}}$ is weakly admissible for $H\Gamma(z,r,r_0)$. Consequently,

$$\operatorname{mod} H\Gamma(z,r,r_0) \leqslant \int_{\widetilde{R}} \rho^2\, d\mathcal{H}^2_{\widetilde{Z}}.$$

The change of variables formula for H and the fact that the seam $Q(S_Z)$ is $\mathcal{H}^2_{\widetilde{Z}}$-negligible establish the claim, after taking the infimum over such $\widetilde{\rho}$. □

Having observed Lemma 6.9 and Equation (6.5), the capacitary estimate [27, Theorem 5.3] implies that keeping r_0 fixed in Equation (6.6), we obtain $\operatorname{mod} H\Gamma(z,r,r_0) \to 0$ as $r \to 0^+$. A key point is that $\mathcal{A}$ in Equation (6.5) is admissible. Since H is a homeomorphism, this implies that Equation (2.6) holds for every $y \in \operatorname{int}(\widetilde{R}) \subset \widetilde{Z}$. By repeating the argument with a slightly larger $\widetilde{Q}$, we conclude the following.

Lemma 6.10. *The identity (2.6) holds for every $y \in \widetilde{R} \subset \widetilde{Z}$.*

Fix a decomposition $\widetilde{\xi}_1, \widetilde{\xi}_2, \widetilde{\xi}_3, \widetilde{\xi}_4$ of $\partial\widetilde{Q}$ of four arcs overlapping only at their end points, labelled in cyclic order consistently with the orientation of $\mathbb{S}^2$. For each i, we denote $\xi_i = H(\widetilde{\xi}_i)$.

Given the validity of Equation (2.6) for each $y \in \widetilde{R}$ and the universal lower bound in Equation (2.7), [31, Proposition 9.1] yields the existence of a homeomorphism $f = (u,v)\colon \widetilde{R} \to [0,1]\times[0,M]$ with the following properties:

(a) $u \in N^{1,2}(\widetilde{R})$ with $2E(u) =: M$ [31, Section 4];
(b) $u^{-1}(0) = \xi_1, u^{-1}(1) = \xi_3, v^{-1}(0) = \xi_2$, and $v^{-1}(M) = \xi_4$ [31, Theorem 5.1 and Proposition 7.3];
(c) The minimal weak upper gradient ρ_u is weakly admissible for the path family $\Gamma(\xi_1,\xi_3;\widetilde{R})$ and is a minimizer, that is, $M = \operatorname{mod}\Gamma(\xi_1,\xi_3;\widetilde{R})$ [31, Section 4-5];
(d) For every Borel set $E \subset \widetilde{R}$, $\mathcal{L}^2(f(E)) = \int_E \rho_u^2\, d\mathcal{H}^2_{\widetilde{Z}}$. In particular, the Jacobian of f coincides with ρ_u^2 [31, Proposition 8.2].

The third point implies that if $u' \in N^{1,2}(\widetilde{R})$ has the same boundary values as u in $\xi_1 \cup \xi_3$, the Dirichlet energies satisfy $E(u) \leqslant E(u')$. Given this, we say that u is an *energy minimizer* for $\Gamma(\xi_1,\xi_3;\widetilde{R})$.

During the proof of Proposition 6.11, the Beltrami differential of H is defined to be zero in $\operatorname{int}(\widetilde{Q}) \cap \overline{Z_1}$, and coincide with the one of ψ in $\operatorname{int}(\widetilde{Q}) \cap Z_2$.

Proposition 6.11. *The map $f = (u,v)\colon \widetilde{R} \to [0,1]\times[0,M]$ is a 1-quasiconformal homeomorphism.*

The proof of Proposition 6.11 is split into several lemmas.

Lemma 6.12. *Let* $0 < a < b < 1$ *and* $0 < c < d < M$ *for which*

$$Q^0 = \left\{ x \in \widetilde{R} : \ f(x) \in [a,b] \times [c,d] \right\} \subset \mathrm{int}(\widetilde{R}) \setminus Q(S_Z).$$

Then $f|_{\mathrm{int}(Q^0)}$ *is a 1-quasiconformal homeomorphism.*

Proof. For the duration of the proof, we denote

$$\xi_1^0 = f^{-1}(\{a\} \times [c,d]), \quad \xi_2^0 = f^{-1}([a,b] \times \{c\}),$$
$$\xi_3^0 = f^{-1}(\{b\} \times [c,d]), \quad \xi_4^0 = f^{-1}([a,b] \times \{d\}).$$

There exists a Jordan domain $V \subset \mathrm{int}(Q) \cap Z_i$, for some $i = 1, 2$, such that $\widetilde{\iota}_i(\overline{V}) = Q^0$. Equation (57) [31, Lemma 10.2] states that

$$\mathrm{mod}\, \Gamma(\xi_1^0, \xi_3^0; Q^0) = \frac{d-c}{b-a}.$$

Since $\widetilde{\iota}_i$ is 1-Lipschitz and a local isometry in $\overline{V}$, we have for every quadrilateral $Q' \subset Q^0$,

$$\mathrm{mod}\, \Gamma(\xi_1', \xi_3'; Q')\, \mathrm{mod}\, \Gamma(\xi_2', \xi_4'; Q') = 1. \tag{6.7}$$

In particular, we have

$$\mathrm{mod}\, \Gamma(\xi_1^0, \xi_3^0; Q^0)\, \mathrm{mod}\, \Gamma(\xi_2^0, \xi_4^0; Q^0) = 1. \tag{6.8}$$

We wish to apply [31, Proposition 11.1]. There Rajala assumes that Equation (2.5) holds for some $\kappa \geqslant 1$ and concludes that $2000 \cdot \sqrt{\kappa} \rho_u$ is a weak upper gradient of f. We do not assume this. However, a quick inspection of the proof shows that given any open set $\Omega \subset \mathrm{int}(Q^0)$, the Property (6.7) implies that $2000 \cdot \chi_{\mathrm{int}(Q^0)} \cdot \rho_u$ is a weak upper gradient of $f|_{\mathrm{int}(Q^0)}$ in Ω. By exhausting $\mathrm{int}(Q^0)$ by such open sets, we conclude that $f|_{\mathrm{int}(Q^0)} \in N^{1,2}(\mathrm{int}(Q^0); \mathbb{R}^2)$.

Since $u \in N^{1,2}(\widetilde{R})$ is a continuous energy minimizer, the composition $u \circ \iota_i|_V$ is harmonic [2, Weyl's lemma]. The Riemann mapping theorem, the Sobolev regularity of $f|_{\mathrm{int}(Q^0)}$, the boundary values of the components of $f|_{Q^0}$, and Equation (6.8) imply that $f \circ \iota_i|_V$ is a Riemann map. In particular, $f|_{\mathrm{int}(Q^0)}$ is a 1-quasiconformal homeomorphism. □

Lemma 6.13. *The composition* $\widetilde{f} = f \circ H : \widetilde{Q} \to [0,1] \times [0,M]$ *is an element of* $N^{1,1}(\mathrm{int}(\widetilde{Q}), \mathbb{R}^2)$. *Moreover, the Beltrami differential of* $\widetilde{f}$ *coincides with the one of* H *and Equation (6.5) holds for* $K_{\widetilde{f}}$ *in place of* K_H.

Proof. Given Lemma 6.12, the Beltrami differential of $\widetilde{f}$ and H coincide $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere in $\mathrm{int}(\widetilde{Q}) \setminus \mathbb{S}^1$, that is, $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere in $\mathrm{int}(\widetilde{Q})$. The result also implies that the pointwise distortions of $\widetilde{f}$ and H coincide $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere in $\mathrm{int}(\widetilde{Q})$.

Next, we show that $\widetilde{u} = u \circ H \in N^{1,1}(\widetilde{Q})$. We recall that $H \in N^{1,1}(\widetilde{Q}, \widetilde{R})$. Moreover, if $\rho_0 \in L^2(\widetilde{R})$ is an upper gradient of u, the function $\rho = (\rho_0 \circ H)\rho_H$ is a 1-weak upper gradient of $\widetilde{u}$ with

$$\int_{\widetilde{Q}} P(\rho)\, d\mathcal{H}^2_Z \leqslant \int_{\widetilde{Q}} e^{\mathcal{A}(K_H)}\, d\mathcal{H}^2_Z + \|\rho_0\|^2_{L^2(Q)} < \infty,$$

where we apply Equation (6.3) and the distortion inequality $\rho_H^2 \leqslant K_H J_H$. The $L^1(\widetilde{Q})$-integrability of ρ follows from Equation (6.4), so $\widetilde{u} \in N^{1,1}(\widetilde{Q})$.

Let $\widetilde{v} = v \circ H$. Lemma 6.12 implies that $\rho = (\rho_0 \circ H)\rho_H \in L^1(\widetilde{Q})$ is a 1-weak upper gradient of $\widetilde{v}$ in every open $U \subset \operatorname{int}(\widetilde{Q}) \setminus \mathbb{S}^1$. Therefore, $\widetilde{v} \in N^{1,1}(\operatorname{int}(\widetilde{Q}) \setminus \mathbb{S}^1)$. Given the continuity of $\widetilde{v}$, we actually have $\widetilde{v} \in N^{1,1}(\operatorname{int}(\widetilde{Q}))$. This is seen by verifying the ACL (absolute continuity on lines) property for $\widetilde{v}|_{\operatorname{int}(\widetilde{Q})}$ on charts covering $\mathbb{S}^1 \cap \operatorname{int}(\widetilde{Q})$. The ACL property on charts follows from a minor modification of the proof in [35, Theorem 35.1] showing that closed sets with σ-finite Hausdorff 1-measure are quasiconformally removable. This implies that ρ is a 1-weak upper gradient of $\widetilde{v}$ on $\operatorname{int}(\widetilde{Q})$. The claim follows from this. □

Lemma 6.14. *Let u' denote the energy minimizer for $\Gamma(\xi_2, \xi_4; Q)$. Then $v = Mu'$.*

Proof. Similarly to f and $\widetilde{f}$, let $f' = (u', v')$ and $\widetilde{f}'$ denote the homeomorphisms obtained from the energy minimizer u' for $\Gamma(\xi_2, \xi_4; \widetilde{R})$. Let R' denote the image of f' and R the image of f.

Lemma 6.13 shows that the Beltrami differentials of $\widetilde{f}$ and $\widetilde{f}'$ coincide with one another $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere and their distortion satisfies Equation (6.5) for an admissible $\mathcal{A}$. Then the Stoilow factorization theorem [2, Theorems 20.5.1, 20.5.2] implies that $\varphi = \widetilde{f}' \circ \widetilde{f}^{-1}$ is conformal; note also that $\varphi = f' \circ f^{-1}$.

Since φ is conformal, the energy minimizer π_1 for $\Gamma(f'(\xi_2), f'(\xi_4); R')$ is such that $\pi_1 \circ \varphi$ is the energy minimizer for $\Gamma(f(\xi_2), f(\xi_4); R)$. On the other hand, here π_1 is the projection to the x-axis and $\pi_1 \circ \varphi$ is M^{-1} times the projection to the y-axis. Since $\varphi = f' \circ f^{-1}$, the equality $u' = \pi_1 \circ \varphi \circ f = M^{-1} v$ follows. □

Proof of Proposition 6.11. Lemma 6.14 implies that $f = (u, v) \in N^{1,2}(\widetilde{R}, \mathbb{R}^2)$. Furthermore, Lemma 6.12 implies $\rho_f^2 = J_f \in L^1(\widetilde{R})$. Hence $\operatorname{mod} \Gamma \leqslant \operatorname{mod} f\Gamma$ for every path family in $\widetilde{R}$. This improves to K-quasiconformality for some $K \geqslant 1$ due to Proposition 2.8. As $f(Q(S_Z) \cap \widetilde{R})$ is negligible due to the change of variables formula for f, and as f^{-1} is 1-quasiconformal outside $f(S_Z \cap \widetilde{R})$, we immediately obtain $\operatorname{mod} \Gamma \leqslant \operatorname{mod} f^{-1}\Gamma$ for every path family in $f(\widetilde{R})$. Thus f is 1-quasiconformal. □

Proof of Proposition 6.6. This is proved by Proposition 6.11. □

Remark 6.15. Notice that if Lemma 6.10 holds for a given homeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$ having an admissible extension, even without assuming the absolute continuity of g^{-1}, the rest of the proof of Proposition 6.11 (and Proposition 6.6) go through the same way.

Proof of Proposition 1.4. Given a quasisymmetry $g : \mathbb{S}^1 \to \mathbb{S}^1$, its *Beurling–Ahlfors* extension $\psi : \overline{Z}_2 \to \overline{Z}_2$ is a quasisymmetry and $\psi|_{Z_2}$ is K-quasiconformal for some $K \geqslant 1$ [4]. Thus, if g^{-1} is absolutely continuous, g satisfies the assumptions of Theorem 1.5. Alternatively, if H is as in Definition 6.5, Lemma 6.9 implies that H^{-1} has outer dilatation $K_O(H^{-1}) \leqslant K$. Proposition 2.8 implies that H is quasiconformal; this self-improves to K-quasiconformality. Clearly H extends to a K-quasiconformal homeomorphism $H : \mathbb{S}^2 \to \widetilde{Z}$. □

7 | CONCLUDING REMARKS

7.1 | A point of positive capacity

For a general orientation-preserving homeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$, the $\widetilde{Z}$ can have points of positive capacity (in the sense that Equation (2.6) can fail) even if g is locally bi-Lipschitz in the complement of a single point. For example, having fixed arbitrary $1 < \alpha < \beta$, we consider the homeomorphism $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x^{\alpha}, & x \geqslant 0, \\ -(-x)^{\beta}, & x < 0. \end{cases} \tag{7.1}$$

We construct a homeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$ by restricting h to the interval $[-1, 1]$, extending the restriction to $\mathbb{R}$ periodically, and by considering the covering map $\theta(t) = (\cos(\pi t), \sin(\pi t), 0)$, and a homeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$ satisfying $g \circ \theta = \theta \circ h^{-1}$. Then g^{-1} is an L-Lipschitz homeomorphism for some $L \geqslant 1$, and one can check directly from the definition of d_Z that the inclusion map $\widetilde{\iota}_1 : \overline{Z}_1 \to \widetilde{Z}$ is L-bi-Lipschitz onto its image.

Let $x_0 \in \widetilde{Z}$ denote the point corresponding to $(1, 0, 0)$. By using the techniques from Section 5, we can show that $\widetilde{Z} \setminus \{x_0\}$ can be covered by bi-Lipschitz images of planar domains. Then [22, Theorem 1.3] implies that $\widetilde{Z} \setminus \{x_0\}$ is 1-quasiconformally equivalent to a Riemannian surface (that is homeomorphic to a planar domain). Such a Riemannian surface can be conformally embedded into $\mathbb{S}^2$ [1, Section III.4]. Hence there exists a 1-quasiconformal embedding $\psi : \widetilde{Z} \setminus \{x_0\} \to \mathbb{S}^2$.

We claim that the complement of the image of ψ is a non-trivial continuum (which is equivalent to the failure of Equation (2.6) at x_0). Indeed, otherwise ψ would extend to a 1-quasiconformal homeomorphism and g would be a welding homeomorphism, as a consequence of Theorem 1.6. This would contradict both [29, Example 1] and [34, Theorem 3], where both of these result show that g is not a welding homeomorphism.

In contrast, if we set $\alpha = \beta \geqslant 1$ in Equation (7.1), the homeomorphism g is a quasisymmetry, so $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$, as a consequence of Proposition 1.4.

7.2 | Points of positive capacity

We construct another example for which points of positive capacity occur. To this end, consider a Cantor set $E \subset [0, 1]$ and

$$h(x) = \begin{cases} (\mathcal{L}^1([0,1] \setminus E))^{-1} \displaystyle\int_0^x \chi_{\mathbb{R} \setminus E}(y)\, d\mathcal{L}^1(y), & 0 \leqslant x \leqslant 1, \\ x, & \text{otherwise}. \end{cases} \tag{7.2}$$

Then $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz homeomorphism coinciding with the identity map outside $(0, 1)$.

Next, consider the Möbius transformation $\theta_1(z) = (z - i)/(z + i)$ from the upper half-space $\overline{\mathbb{H}}$ onto the Euclidean unit disk $\overline{\mathbb{D}}$. Let $\theta_2(x, y) = (2x/(1 + x^2 + y^2), 2y/(1 + x^2 + y^2), (1 - x^2 - y^2)/(1 + x^2 + y^2))$. Then $\theta := \theta_2 \circ \theta_1 : \overline{\mathbb{H}} \to \overline{Z}_2$ defines a 1-quasiconformal homeomorphism, given that θ_2^{-1} is a (n orientation-reversing) stereographic projection.

There exists a unique homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ satisfying $g \circ \theta = \theta \circ h^{-1}$. We see from Equation (7.2) that g^{-1} is L-Lipschitz and $\widetilde{\iota}_1$ is L-bi-Lipschitz with a constant L depending only on $\mathcal{L}^1(E)$. In particular, $\widetilde{Z} = (Z, d_Z)$.

We denote $E' = \widetilde{\iota}_2(\theta(E)) \subset \widetilde{Z}$, and apply [22, Theorem 1.3] as in Section 7.1, and find a 1-quasiconformal embedding $\psi\colon \widetilde{Z} \setminus E' \to \mathbb{S}^2$.

Consider on $\mathbb{R}^2$ the distance d_E obtained as follows: For each absolutely continuous $\gamma\colon [0,1] \to \mathbb{R}^2$, denote $\ell_E(\gamma) := \int_\gamma \chi_{\mathbb{R}^2\setminus E}\, ds$. We set $d_E(x,y) = \inf \ell_E(\gamma)$, the infimum taken over absolutely continuous paths joining x to y.

We denote $X = (\mathbb{R}^2, d_E)$. The change of distance map $H\colon \mathbb{R}^2 \to X$ is a 1-Lipschitz homeomorphism that is a local isometry on $\mathbb{R}^2 \setminus E$. Moreover, if $\theta\colon [0,1] \to \mathbb{R}^2$ is absolutely continuous, the metric speeds satisfy

$$v_{H\circ\theta} = \left(\chi_{\mathbb{R}^2\setminus E} \circ \theta\right) \cdot v_\theta \quad \mathcal{L}^1\text{-almost everywhere.} \tag{7.3}$$

The composition $G = \widetilde{\iota}_2 \circ \theta \circ (H|_{[-1,2]\times[0,1]})^{-1}$ is a 1-quasiconformal homeomorphism. This follows from Lemma 2.2, the equalities $\mathcal{H}^1_{\widetilde{Z}}(E') = 0 = \mathcal{H}^1_X(H(E))$, together with Proposition 3.6 and Equation (7.3).

We consider a Cantor set E obtained from [23, Example 6.1]. The key property of E is the following: there exists a path family Γ on $[0,1]^2$, each path joining (0,0) to (1,0), such that $\operatorname{mod} H\Gamma \geqslant (4\pi)^{-1}$ and $\operatorname{mod}\Gamma = 0$. Given that G is 1-quasiconformal, the points $\widetilde{\iota}_2(\theta(x))$, where $x = (0,0), (1,0)$, fail Equation (2.6). Consequently, $\widetilde{Z}$ is not quasiconformally equivalent to $\mathbb{S}^2$, and the embedding ψ does not have a quasiconformal extension $\Psi\colon \widetilde{Z} \to \mathbb{S}^2$.

Question 7.1. Are there Cantor sets E with $\mathcal{L}^1(E) > 0$ such that a quasiconformal embedding $\psi\colon \widetilde{Z} \setminus E' \to \mathbb{S}^2$ extends to a quasiconformal homeomorphism $\Psi\colon \widetilde{Z} \to \mathbb{S}^2$?

Given a compact set $F \subset Y$ with $Y = \mathbb{R}^2$ or $Y = \mathbb{S}^2$, we say that F has *zero absolute area* if every 1-quasiconformal embedding $f\colon Y \setminus F \to \mathbb{S}^2$ satisfies $\mathcal{H}^2_{\mathbb{S}^2}(\mathbb{S}^2 \setminus f(Y\setminus F)) = 0$.

We expect that the quasiconformal extension Ψ exists if and only if the set $F = \mathbb{S}^2 \setminus \psi(\widetilde{Z} \setminus E')$ has zero absolute area; the "only if"-direction follows by applying the techniques used in Section 4, by noting that the composition $(f \circ \psi)^{-1}$ has a continuous, monotone, and surjective extension $\widetilde{\pi}$ with $\operatorname{mod}\Gamma \leqslant \operatorname{mod}\widetilde{\pi}\Gamma$ for all path families. We expect that the "if"-direction follows from [23, Theorems 1.3 and 1.4, together with Lemma 5.1].

If E in Question 7.1 has zero absolute area, [23, Theorem 1.3] implies that the change of distance map H is a 1-quasiconformal homeomorphism. Given that the G above is 1-quasiconformal, one readily verifies that $\widetilde{\iota}_2$ is a 1-quasiconformal homeomorphism onto its image. We ask the following.

Question 7.2. Let E, g, and ψ be as in Question 7.1. If $\widetilde{\iota}_2\colon \overline{Z}_2 \to \widetilde{Z}$ is a 1-quasiconformal parametrization of its image, does $\psi\colon \widetilde{Z} \setminus E' \to \mathbb{S}^2$ extend to a quasiconformal homeomorphism $\Psi\colon \widetilde{Z} \to \mathbb{S}^2$? In particular, if E has zero absolute area, does $F = \mathbb{S}^2 \setminus \psi(\widetilde{Z} \setminus E')$ have zero absolute area?

As a related note, it is clear, for example, by [21, Theorem 1.1 and Proposition 1.2], that the inclusion map $\widetilde{\iota}_2$ is a 1-quasiconformal homeomorphism if and only if there exists a quasiconformal homeomorphism $h : \widetilde{\iota}_2(\overline{Z}_2) \to \overline{\mathbb{D}}$, where $\overline{\mathbb{D}}$ is the closed Euclidean unit disk.

7.3 | Welding homeomorphisms

We consider a welding homeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$ with welding curve $C \subset \mathbb{S}^2$. Consider the monotone mapping $\widetilde{\pi} : \mathbb{S}^2 \to \widetilde{Z}$ obtained from Equation (4.1).

Question 7.3. If $\widetilde{\pi}$ is a homeomorphism, is it a 1-quasiconformal homeomorphism?

We showed in Proposition 4.1 that if $\widetilde{\pi}$ is not a homeomorphism, then $\widetilde{Z}$ is not quasiconformally equivalent to $\mathbb{S}^2$; the collapsing creates points of positive capacity—by which we mean that Equation (2.6) fails—in $\widetilde{Z}$. Question 7.3 asks if the collapsing is the only obstruction for quasiconformal uniformization. Lemma 4.8 reduces the question to understanding when $\widetilde{\pi}^{-1} \in N^{1,2}(\widetilde{Z}, \mathbb{S}^2)$.

7.4 | Quasisymmetries

Observe that the assumptions of Proposition 1.4 are satisfied by every quasisymmetry $g : \mathbb{S}^1 \to \mathbb{S}^1$ that is *strongly quasisymmetric* [3, 5 8, 33]: for every $\epsilon > 0$ there exists $\delta > 0$ such that for every subarc $I \subset \mathbb{S}^1$ and Borel set $E \subset I$,

$$\mathcal{H}^1_{\mathbb{S}^1}(E) \leqslant \delta \mathcal{H}^1_{\mathbb{S}^1}(I) \quad \text{implies} \quad \mathcal{H}^1_{\mathbb{S}^1}(g(E)) \leqslant \epsilon \mathcal{H}^1_{\mathbb{S}^1}(g(I)).$$

The welding curves corresponding to strongly quasisymmetric homeomorphisms are special cases of the *asymptotically conformal* quasicircles; see [30]. One might ask whether or not $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$ whenever g is a welding homeomorphism corresponding to such a curve. Corollary 4 of [30] provides us with an example of asymptotically conformal quasicircle C which has an uncountable number of tangent points, with the tangent points dense in C, but they also have zero 1D Hausdorff measure.

Lemma 7.4. *There exists a quasisymmetric $g : \mathbb{S}^1 \to \mathbb{S}^1$ with asymptotically conformal welding curve C such that $\widetilde{Z}$ is not homeomorphic to $\mathbb{S}^2$.*

Lemma 7.4 follows from Proposition 4.1, Lemma 4.5, and the cited example.

Question 7.5. Is the answer to Question 7.3 yes if we also assume that $g : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetry?

To answer Question 7.5 negatively, one needs to construct a quasisymmetry $\psi : \overline{Z}_2 \to \overline{Z}_2$, with $g = \psi|_{\mathbb{S}^1}$, for which the measures $g^*\mathcal{H}^1_{\mathbb{S}^1}$ and $\mathcal{H}^1_{\mathbb{S}^1}$ are not mutually singular in any subarc $I \subset \mathbb{S}^1$, yet the corresponding $\widetilde{Z}$ is not quasiconformally equivalent to $\widetilde{Z}$. Equivalently, one only needs to show that the homeomorphism $H : \mathbb{S}^2 \to \widetilde{Z}$, coinciding with $\widetilde{\iota}_1$ in Z_1 and with $\widetilde{\iota}_2 \circ \psi$ in Z_2, is not quasiconformal. By arguing as in the proof of Lemma 4.8, one sees that H is quasiconformal if and only if $H^{-1} \in N^{1,2}(\widetilde{Z}, \mathbb{S}^2)$.

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