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Norm-inflation results for purely BBM-type Boussinesq systems

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ABSTRACT

This article is concerned with the norm-inflation phenomena associated with a periodic initial-value *abcd*-Benjamin-Bona-Mahony type Boussinesq system. We show that the initial-value problem is ill-posed in the periodic Sobolev spaces $H_p^{-s}(0,2\pi) \times H_p^{-s}(0,2\pi)$ for all $s > 0$. Our proof is constructive, in the sense that we provide smooth initial data that generates solutions arbitrarily large in $H_p^{-s}(0,2\pi) \times H_p^{-s}(0,2\pi)$-norm for arbitrarily short time. This result is sharp since in [13] the well-posedness is proved to holding for all positive periodic Sobolev indexes of the form $H_p^{s}(0,2\pi) \times H_p^{s}(0,2\pi)$, including $s = 0$.

1. Introduction

The physical phenomena are usually modeled by equations involving differential operators. In the study of partial differential equations (PDEs), it is crucial to know if the equation or system are well-posed in the Hadamard's sense: existence, uniqueness, and continuous dependence of the solutions with respect to the initial data. The lack of the latter condition represents one of the main obstacles to tackle any further analysis for the underlying PDE. In particular, it would cause incorrect solutions or non meaningful solutions at all. As consequence one can not address, for instance, the numerical implementation of the solutions [8–10] or controllability properties of the PDE [3]. One way to prove ill-posedness is to evidence the lack of continuous dependence with respect to the initial data by showing that small initial data could generate arbitrarily large solutions. This phenomena is so-called norm-inflation phenomena by obvious reasons. The main purpose of this article is to study ill-posedness for a family of Boussinesq systems proposed by J. L. Bona, M. Chen and J.-C. Saut in [5,6]:

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$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = 0. \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0. \end{cases} \tag{1}$$

We prove that the system satisfies the norm-inflation phenomena for certain constant parameters $a, b, c, d \in \mathbb{R}$. Thus, the third Hadarmard's condition is violated. Here η and w are real-valued functions whose physical interpretation will be described in next lines. System (1) approximates the motion of small amplitude long waves on the surface of an ideal fluid under the force of gravity in situations where the motion is sensibly two-dimensional. In (1), the variable x is proportional to the distance in the direction of propagation, while t is proportional to elapsed time. The quantity $\eta(x,t) + h_0$ corresponds to the liquid's total depth at the point x and at time t, where h_0 is the undisturbed water depth. The variable $w(x,t)$ represents the horizontal velocity at the point $(x,y) = (x, \theta h_0)$, at time t, where y is the vertical coordinate, with $y = 0$ corresponding to the channel bottom or sea bed. Thus, w is the horizontal velocity field at the height θh_0, where θ is a fixed constant in the interval $[0,1]$. According to the choice of the constants a, b, c and d, we can distinguish several Boussinesq systems. In all these cases, the parameters must satisfy the following consistency conditions

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \qquad c + d = \frac{1}{2}(1 - \theta^2) \geq 0. \tag{2}$$

In particular, one always has

$$a + b + c + d = 1/3.$$

A detailed study on local well-posedness of system (1) on the real line was initially addressed in [5,6]. For results on ill/well-posedness in periodic domains, depending on the sign of the *abcd*-parameters, we refer the reader to [1] and [13].

In contrast with other classical wave models like Korteweg-de Vries (KdV) systems [11], Boussinesq system (1) does not assume the uni-directional propagation of shallow water waves but describing the bi-directional propagation of such waves. Its two-way propagation feature seems to have a wide range of applications in different physical and mathematical branches. Indeed, among other studies, system (1) has recently been addressed from the control theory point of view, see for example [2–4,12–14], and the references contained therein. In the articles above, the well-posedness property is necessary to prove their controllability and stability results by introducing appropriate dissipative mechanisms into the system.

In this work we focus on studying ill-posedness of the system (1)-(2) posed in a periodic domain. It will follows by showing that the system posses the norm-inflation phenomena. We restrict the *abcd*-parameters to the cases

$$a = 0, \quad c = 0, \quad b > 0 \quad \text{and} \quad d > 0. \tag{3}$$

The underlying system with such those restrictions is called purely BBM-type Boussinesq system, which in turn is an instance of weakly dispersive systems, see e.g. [6, Sections 2.1 and 2.2]. To be more precise, the system (1)–(3) becomes

$$\begin{cases} \eta_t + w_x - b\eta_{txx} + (\eta w)_x = 0 & \text{for } x \in (0, 2\pi), \ \ t > 0, \\ w_t + \eta_x - d w_{txx} + w w_x = 0 & \text{for } x \in (0, 2\pi), \ \ t > 0, \\ \eta(0,x) = \eta^0(x) & \text{for } x \in (0, 2\pi), \\ w(0,x) = w^0(x) & \text{for } x \in (0, 2\pi), \end{cases} \tag{4}$$

with periodic boundary conditions

$$\begin{cases} \eta(t,0) = \eta(t,2\pi); \ \ \eta_x(t,0) = \eta_x(t,2\pi) & \text{for } t > 0, \\ w(t,0) = w(t,2\pi); \ \ w_x(t,0) = w_x(t,2\pi) & \text{for } t > 0. \end{cases} \tag{5}$$

The main result of this work reads as follows.

Theorem 1.1. *Let $s > 0$ be given. There exist two sequences, one consisting of periodic initial data* $\begin{pmatrix} \eta_\nu^0 \\ w_\nu^0 \end{pmatrix}_{\nu\in\mathbb{N}} \in \left[C_p^\infty([0,2\pi])\right]^2$ *satisfying*

$$\left\| \begin{pmatrix} \eta_\nu^0 \\ w_\nu^0 \end{pmatrix} \right\|_{\dot{H}_p^{-s}(0,2\pi)\times \dot{H}_p^{-s}(0,2\pi)} \to 0 \quad as \quad \nu \to +\infty$$

and another one $(T_\nu)_{\nu\in\mathbb{N}}$ of positive times, tending to zero as $\nu \to \infty$, such that if $\begin{pmatrix} \eta_\nu \\ w_\nu \end{pmatrix}$ *is the solution of (4)-(5) coming up from the initial data* $\begin{pmatrix} \eta_\nu^0 \\ w_\nu^0 \end{pmatrix}$*, then*

$$\left\| \begin{pmatrix} \eta_\nu \\ w_\nu \end{pmatrix} \right\|_{\dot{H}_p^{-s}(0,2\pi)\times \dot{H}_p^{-s}(0,2\pi)} \to +\infty \quad as \quad \nu \to +\infty.$$

Here $\dot{H}_p^{-s}(0,2\pi)$ stands for the homogeneous version of $H_p^{-s}(0,2\pi)$, whose definition can be found in Section 2. The precise definition of the periodic Sobolev spaces we shall use throughout this article is given in Section 2. Furthermore, unless otherwise stated, we reserve the letter s to indicate a non-negative real number standing for the periodic Sobolev index in $H^s(0,2\pi)$. We remark that Theorem 1.1 is sharp since the system (3)-(5) is well-posed in $H^\beta(0,2\pi) \times H^\beta(0,2\pi)$ for $\beta \geq 0$ as showed in [13, Theorem 3.2]. The analogous of the latest result in the real line was proved in [6, Theorem 2.1]. Moreover, Theorem 1.1 adds one extra family to the ill-posedness result in [1, Theorem 5.2].

The scalar version of Theorem 1.1, in a periodic domain, was obtained by Bona and Dai in [7]. A similar result for the scalar case in the whole real line $\mathbb{R}$ was proved by Phantee in [15]. The proof of Theorem 1.1 closely follows the ideas from [7]. Bona and Dai constructed initial periodic data and proved that the corresponding solution blows up in $H_p^{-s}(0,2\pi)$-norm when time is sufficiently short. One of the main ingredients in their proof relies on the solutions' knowledge of the forward linear problem. In our case, thanks to the periodic framework and the spectral analysis carried out in [2] —to study controllability and stability issues— we also have the explicit expressions for the solutions to the linear counterpart of (4)-(5). This was done in [2] by combining tools from Fourier analysis with the well known Duhamel's principle. See Section 2 for details. In fact, the authors in [2] made a more refined spectral analysis to deduce the asymptotic behavior of the eigenvalues associated with (4)-(5), which is essential for proving their exact and approximate controllability results. On the other hand, apart from mentioned above, we show that a sequence of initial periodic data (constructed by hand) generates another sequence of solutions to (4)-(5), which in turn can be decomposed as the sum of three terms. The decomposition is closely linked to a Picard's iteration of second order applied to sequence of solutions to (4)-(5). The result then follows by showing that the first and third terms of the expansion remain bounded in $H_p^{-s}(0,2\pi) \times H_p^{-s}(0,2\pi)$-norm while the second one can be arbitrarily large and eventually goes to infinity when times goes to zero.

This paper is structured as follows. In Section 2, we state the well-posedness property of the linear version of (4)-(5). We also collect some useful results from the spectral analysis made in [2]. In Section 3, we analyze the Picard's iteration method to solutions to (4)-(5). As a consequence, we prove Theorem 1.1, the main result of this work. Finally, the Appendix is dedicated to describing computations needed for intermediate steps in proving Theorem 1.1.

2. Preliminaries

In this section we state some well-posedness results for both linear and nonlinear Boussinesq systems. A remarkable fact is the knowledge of the solution associated with the linear case, see Theorem 2.2.

2.1. Linear systems

We first analyze the linearized version of (4)-(5), that is, we consider the following linear system

$$\begin{cases} \eta_t + w_x - b\eta_{txx} = 0 & \text{for } x \in (0, 2\pi),\ \ t > 0, \\ w_t + \eta_x - dw_{txx} = 0 & \text{for } x \in (0, 2\pi),\ \ t > 0, \\ \eta(t,0) = \eta(t,2\pi);\ \ \eta_x(t,0) = \eta_x(t,2\pi) & \text{for } t > 0, \\ w(t,0) = w(t,2\pi);\ \ w_x(t,0) = w_x(t,2\pi) & \text{for } t > 0, \\ \eta(0,x) = \eta^0(x) & \text{for } x \in (0, 2\pi), \\ w(0,x) = w^0(x) & \text{for } x \in (0, 2\pi), \end{cases} \tag{6}$$

where $b > 0$ and $d > 0$ as in (3). Its well-posedness was derived from the spectral analysis done in [13] by using a Fourier approach. For the sake of completeness, we include such those results here. Firstly, we introduce a few notations.

Given any $v \in L^2(0, 2\pi)$ and $k \in \mathbb{Z}$, we denote by $\widehat{v}_k$ the k-Fourier coefficient of v, defined by

$$\widehat{v}_k = \frac{1}{2\pi} \int_0^{2\pi} v(x) e^{-ikx}\, \mathrm{d}x,$$

and, for any $m \in \mathbb{N}$, we define the space

$$H_p^m(0, 2\pi) = \left\{ v \in L^2(0, 2\pi) \;\middle|\; v = \sum_{k \in \mathbb{Z}} \widehat{v}_k e^{ikx},\ \sum_{k \in \mathbb{Z}} |\widehat{v}_k|^2 (1 + k^2)^m < \infty \right\},$$

which is a Hilbert space with respect to the inner product

$$(v, w)_m = \sum_{k \in \mathbb{Z}} \widehat{v}_k \overline{\widehat{w}_k} (1 + k^2)^m. \tag{7}$$

The norm associated with (7) is denoted by $||\ ||_m$. It can be seen that

$$H_p^m(0, 2\pi) = \left\{ v \in H^m(0, 2\pi) \;\middle|\; \frac{\partial^r v}{\partial x^r}(0) = \frac{\partial^r v}{\partial x^r}(2\pi),\ 0 \le r \le m - 1 \right\},$$

where $H^m(0, 2\pi)$ stands for the classical Sobolev space of index m in the interval $(0, 2\pi)$. We can extend the definition of $H_p^m(0, 2\pi)$ to the case $m = s \ge 0$, a non-negative real number, by setting

$$H_p^s(0, 2\pi) = \left\{ v = \sum_{k \in \mathbb{Z}} \widehat{v}_k e^{ikx} \in H^s(0, 2\pi) \;\middle|\; \sum_{k \in \mathbb{Z}} |\widehat{v}_k|^2 (1 + k^2)^s < \infty \right\}.$$

For any nonnegative real number s, $H_p^s(0, 2\pi)$ can also be seen as a Hilbert space with respect to the inner product defined by (7) with m replaced by s. In particular, for any $v \in H_p^s(0, 2\pi)$,

$$\|v\|_s^2 = \sum_{k \in \mathbb{Z}} |\widehat{v}_k|^2 (1 + k^2)^s.$$

For $s > 0$, we define the space $H_p^{-s}(0, 2\pi)$ as the topological dual of $H_p^s(0, 2\pi)$:

$$H_p^{-s}(0, 2\pi) = \left(H_p^s(0, 2\pi)\right)'.$$

Similarly, and for a given $\beta \in \mathbb{R}$, one can define the homogeneous version of $H_p^\beta(0, 2\pi)$ as

$$\dot{H}_p^\beta(0, 2\pi) = \left\{ v = \sum_{k\in\mathbb{Z}} \widehat{v}_k e^{ikx} \in H^\beta(0, 2\pi) \;\middle|\; \sum_{k\in\mathbb{Z}} |k|^{2\beta} |\widehat{v}_k|^2 < \infty \right\}$$

with norm

$$\|v\|^2_{\dot{H}_p^\beta} = \sum_{k\in\mathbb{Z}} |k|^{2\beta} |\widehat{v}_k|^2.$$

On the other hand, for $\alpha > 0$, let $\psi_\alpha(D_x)$ be the Fourier multiplier operator given in terms of the Fourier transform by

$$\widehat{\psi_\alpha(D_x)}u(k) = \frac{k}{1 + \alpha k^2}\widehat{u}(k), \quad D_x := -i\partial_x.$$

Given $\beta \in \mathbb{R}$, let us introduce the Hilbert space

$$V^\beta = H_p^\beta(0, 2\pi) \times H_p^\beta(0, 2\pi),$$

endowed with the inner product defined by

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = b\,(f_1, g_1)_\beta + d\,(f_2, g_2)_\beta\,,$$

where $(\cdot, \cdot)_\beta$ denotes the inner product given in (7) with m replaced by β. One can see that system (6) can be rewritten in the following vectorial form

$$i \begin{pmatrix} \eta \\ w \end{pmatrix}_t (t) + A \begin{pmatrix} \eta \\ w \end{pmatrix} (t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \eta \\ w \end{pmatrix} (0) = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix},$$

where A is a linear and compact operator in V^β, see for instance [13], defined by

$$A = \begin{pmatrix} 0 & \psi_b(D_x) \\ \psi_d(D_x) & 0 \end{pmatrix}. \tag{8}$$

Thus, if we assume that the initial data in (6) are given by

$$\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} = \sum_{k\in\mathbb{Z}} \begin{pmatrix} \widehat{\eta}_k^0 \\ \widehat{w}_k^0 \end{pmatrix} e^{ikx},$$

then the solution of (6) can be formally written as

$$\begin{pmatrix} \eta \\ w \end{pmatrix} (t, x) = \sum_{k\in\mathbb{Z}} \begin{pmatrix} \widehat{\eta}_k(t) \\ \widehat{w}_k(t) \end{pmatrix} e^{ikx},$$

where the pair $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ fulfill the following initial-value ODE with $T > 0$

$$\begin{cases} (1+bk^2)(\widehat{\eta}_k)_t + ik\widehat{w}_k = 0, & t \in (0,T), \\ (1+dk^2)(\widehat{w}_k)_t + ik\widehat{\eta}_k = 0, & t \in (0,T), \\ \widehat{\eta}_k(0) = \widehat{\eta}_k^0, \qquad \widehat{w}_k(0) = \widehat{w}_k^0. \end{cases} \tag{9}$$

Then, we have the following result:

Lemma 2.1. *(see [13]) Let*

$$\lambda_k^{\pm} = \pm ik\sigma(k); \quad \sigma(k) = \frac{1}{\sqrt{(1+bk^2)(1+dk^2)}}, \quad (k \in \mathbb{Z} \setminus \{0\}). \tag{10}$$

The solution $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ of (9) is given by

$$\begin{cases} \widehat{\eta}_k(t) &=& \frac{1}{2}\left[\left(\widehat{\eta}_k^0 + \sqrt{\frac{1+dk^2}{1+bk^2}}\widehat{w}_k^0\right)e^{-\lambda_k^+ t} + \left(\widehat{\eta}_k^0 - \sqrt{\frac{1+dk^2}{1+bk^2}}\widehat{w}_k^0\right)e^{-\lambda_k^- t}\right], \\ \\ \widehat{w}_k(t) &=& \frac{1}{2}\left[\left(\sqrt{\frac{1+bk^2}{1+dk^2}}\widehat{\eta}_k^0 + \widehat{w}_k^0\right)e^{-\lambda_k^+ t} - \left(\sqrt{\frac{1+bk^2}{1+dk^2}}\widehat{\eta}_k^0 - \widehat{w}_k^0\right)e^{-\lambda_k^- t}\right], \end{cases} \tag{11}$$

if $k \neq 0$ and

$$\begin{cases} \widehat{\eta}_0(t) &=& \widehat{\eta}_0^0, \\ \widehat{w}_0(t) &=& \widehat{w}_0^0. \end{cases} \tag{12}$$

Thanks to Lemma 2.1, one can prove that the operator A generates an analytic group in V^β.

Theorem 2.1. *(see [13]) The family of linear operators $(S(t))_{t\geq 0}$ defined by*

$$S(t)\begin{pmatrix}\eta^0\\ w^0\end{pmatrix} = \sum_{k\in\mathbb{Z}}\begin{pmatrix}\widehat{\eta}_k(t)\\ \widehat{w}_k(t)\end{pmatrix}e^{ikx}, \qquad \left(\begin{pmatrix}\eta^0\\ w^0\end{pmatrix} \in V^\beta\right),$$

where the coefficients $\begin{pmatrix}\widehat{\eta}_k(t)\\ \widehat{w}_k(t)\end{pmatrix}$ are given by (11)-(12), is a group of isometries in V^β, for each $\beta \in \mathbb{R}$. Moreover, its infinitesimal generator is the operator $(D(A), A)$, where $D(A) = V^\beta$ and A is given by (8).

From Theorem 2.1 and standard techniques from semigroup theory, we also have the following global well-posedness result:

Theorem 2.2. *(see [13]) Let $T > 0$ and $\beta \in \mathbb{R}$. For each $\begin{pmatrix}\eta^0\\ w^0\end{pmatrix} \in V^\beta$ and $\begin{pmatrix}f\\ g\end{pmatrix} \in L^1(0,T;V^\beta)$, there exists a unique solution $(\eta, w) \in W^{1,1}([0,T];V^\beta)$ of the system*

$$\begin{pmatrix}\eta\\ w\end{pmatrix}_t(t) + A\begin{pmatrix}\eta\\ w\end{pmatrix}(t) = \begin{pmatrix}f\\ g\end{pmatrix}, \quad \begin{pmatrix}\eta\\ w\end{pmatrix}(0) = \begin{pmatrix}\eta^0\\ w^0\end{pmatrix},$$

which verifies the constant variation formula

$$\begin{pmatrix} \eta \\ w \end{pmatrix}(t) = S(t)\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} + \int_0^t S(t-s)\begin{pmatrix} f \\ g \end{pmatrix}(s)\, ds.$$

Moreover, if $\begin{pmatrix} f \\ g \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *it follows that* $\begin{pmatrix} \eta \\ w \end{pmatrix} \in \mathcal{C}^\omega(\mathbb{R}, V^\beta)$*, the class of analytic functions in* $t \in \mathbb{R}$ *with values in* V^β*.*

2.2. Nonlinear systems

Consider now the nonlinear Boussinesq system

$$\begin{cases} \eta_t + w_x - b\eta_{txx} + (\eta w)_x = f & \text{for } x \in (0, 2\pi),\ \ t > 0, \\ w_t + \eta_x - dw_{txx} + ww_x = g & \text{for } x \in (0, 2\pi),\ \ t > 0, \\ \eta(t,0) = \eta(t,2\pi);\ \eta_x(t,0) = \eta_x(t,2\pi) & \text{for } t > 0, \\ w(t,0) = w(t,2\pi);\ w_x(t,0) = w_x(t,2\pi) & \text{for } t > 0, \\ \eta(0,x) = \eta^0(x) & \text{for } x \in (0, 2\pi), \\ w(0,x) = w^0(x) & \text{for } x \in (0, 2\pi). \end{cases}$$

As in the linear case, we can write it as

$$i\begin{pmatrix} \eta \\ w \end{pmatrix}_t(t) = A\begin{pmatrix} \eta \\ w \end{pmatrix}(t) + \mathcal{N}\begin{pmatrix} \eta \\ w \end{pmatrix}(t) + i\begin{pmatrix} \psi_b(f) \\ \psi_d(g) \end{pmatrix}, \quad \begin{pmatrix} \eta \\ w \end{pmatrix}(0) = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \tag{13}$$

where $\mathcal{N} : V^s \to V^s$ is the nonlinear operator defined by

$$\mathcal{N}\begin{pmatrix} \eta \\ w \end{pmatrix} = \begin{pmatrix} \psi_b(D_x)(\eta w) \\ \psi_d(D_x)\left(\frac{w^2}{2}\right) \end{pmatrix}. \tag{14}$$

We have the well-posedness result for the nonlinear system for $s \geq 0$:

Theorem 2.3. *(see [13]) Assume that* $b > 0$ *and* $d > 0$*. Let* $T > 0$ *and* $s \geq 0$ *be given. Then, there exists a constant* $M > 0$*, depending on* T*, such that for any* $\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \in V^s$ *and any* $\begin{pmatrix} f \\ g \end{pmatrix} \in L^1(0,T;V^{s-2})$ *satisfying*

$$\left\| \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \right\|_{V^s} \leq M \quad \text{and} \quad \left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{L^1(0,T;V^{s-2})} \leq M,$$

the system (13)-(14) *admits a unique mild solution* $\begin{pmatrix} \eta \\ w \end{pmatrix} \in C([0,T];V^s)$*.*

The next section is then dedicated to proving ill-posedness for the nonlinear system in case of negative Sobolev indexes.

3. Picard's iteration method and norm-inflation result

Picard's iteration method is a useful tool to prove, for instance, existence of solutions for differential equations. The method requires a starting point and later one makes an iterative procedure. It generates a sequence of elements whose convergence (in a suitable Hilbert space) is the primary purpose of the method. If the sequence converges, then the limit is usually the desired solution to the underlying differential equation. We use a truncated version of this method up to second order.

One can easily prove that the solution of (13)-(14), with the homogeneous source $\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, can be written in $(0,T)\times(0,2\pi)$ with $T>0$, as follows

$$\begin{pmatrix} \eta(t,x) \\ w(t,x) \end{pmatrix} = \begin{pmatrix} v(t,x) \\ u(t,x) \end{pmatrix} + \begin{pmatrix} \xi(t,x) \\ \varphi(t,x) \end{pmatrix} + \begin{pmatrix} y(t,x) \\ z(t,x) \end{pmatrix}, \tag{15}$$

where

$$\begin{pmatrix} v \\ u \end{pmatrix}(t) = S(t)\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix},$$
$$\begin{pmatrix} \xi \\ \varphi \end{pmatrix}(t) = \int_0^t S(t-\tau)\mathcal{N}\left(S(\tau)\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}\right)d\tau, \tag{16}$$

and

$$\begin{pmatrix} y \\ z \end{pmatrix}(t) = \int_0^t S(t-\tau)\widetilde{\mathcal{N}}\begin{pmatrix} y \\ z \end{pmatrix}(\tau)d\tau. \tag{17}$$

The operator $\widetilde{\mathcal{N}}$ is defined as

$$\widetilde{\mathcal{N}}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \psi_b(D_x)\left(v(\varphi+z)+u(\xi+y)+(\varphi+z)(\xi+y)\right) \\ \psi_d(D_x)\left(\frac{1}{2}\left(\varphi^2+z^2+2(u\varphi+uz+\varphi z)\right)\right) \end{pmatrix}.$$

As in [7], the functions sine and cosine will be involved in our construction. The explicit computations below shall be useful in our analysis.

Remark 3.1. For $k\in\mathbb{N}$, we have

- $S(t)\begin{pmatrix} \cos(kx) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(kx+k\sigma(k)t)+\cos(kx-k\sigma(k)t) \\ 0 \end{pmatrix},$
- $S(t)\begin{pmatrix} \sin(kx) \\ 0 \end{pmatrix} = \begin{pmatrix} \sin(kx+k\sigma(k)t)+\sin(kx-k\sigma(k)t) \\ 0 \end{pmatrix},$
- $S(t)\begin{pmatrix} 0 \\ \cos(kx) \end{pmatrix} = \begin{pmatrix} 0 \\ \cos(kx+k\sigma(k)t)+\cos(kx-k\sigma(k)t) \end{pmatrix},$
- $S(t)\begin{pmatrix} 0 \\ \sin(kx) \end{pmatrix} = \begin{pmatrix} 0 \\ \sin(kx+k\sigma(k)t)+\sin(kx-k\sigma(k)t) \end{pmatrix},$

where $(S(t))_{t\geq 0}$ is the group defined in Theorem 2.1, and $\sigma(k)$ is given by (10).

Remark 3.2. A straightforward computation combined with Remark 3.1 yield

$$\int_0^t S(t-\tau)\begin{pmatrix} \sin(kx-l\tau) \\ 0 \end{pmatrix}d\tau = \begin{pmatrix} M_1(t,x)+M_2(t,x) \\ 0 \end{pmatrix},$$

$$\int_0^t S(t-\tau)\begin{pmatrix}\sin(kx+l\tau)\\ 0\end{pmatrix}d\tau = \begin{pmatrix}M_3(t,x)+M_4(t,x)\\ 0\end{pmatrix},$$

for all $k \in \mathbb{N}$ and $l \in \mathbb{R}$, where

$$\begin{aligned}
M_1(t,x) &= \frac{1}{l+k\sigma(k)}\left(\cos(kx-lt)-\cos(kx+k\sigma(k)t)\right),\\
M_2(t,x) &= \frac{1}{l-k\sigma(k)}\left(\cos(kx-lt)-\cos(kx-k\sigma(k)t)\right),\\
M_3(t,x) &= \frac{1}{l-k\sigma(k)}\left(\cos(kx+k\sigma(k)t)-\cos(kx+lt)\right),\\
M_4(t,x) &= \frac{1}{l+k\sigma(k)}\left(\cos(kx-k\sigma(k)t)-\cos(kx+lt)\right).
\end{aligned}$$

Remark 3.3. Let $k \in \mathbb{N}$. For $\alpha > 0$, the Fourier multiplier operator $\psi_\alpha(D_x)$ satisfy

- $\psi_\alpha(D_x)(\sin(kx)) = -i\dfrac{k}{1+\alpha k^2}\cos(kx)$,
- $\psi_\alpha(D_x)(\cos(kx)) = i\dfrac{k}{1+\alpha k^2}\sin(kx)$,
- $\psi_\alpha(D_x)(C) = 0$, for all $C \in \mathbb{C}$.

Taking into account these remarks, we can now pass to prove the main result of this work.

3.1. Proof of Theorem 1.1

Let $s > 0$. The proof consists of verifying that the first and third terms in (15) are bounded in V^{-s} while the second one is arbitrarily large by considering suitable initial value data. To do that, we take $k_1 \in \mathbb{N}$ large enough and set $k_2 := k_1 + 1$. Let

$$\begin{pmatrix}\overline{\eta^0}\\ \overline{w^0}\end{pmatrix} = \begin{pmatrix}\sin(k_1x)+\sin(k_2x)\\ \sin(k_1x)+\sin(k_2x)\end{pmatrix}.$$

Throughout this proof, we consider the following family of 2π-periodic initial data

$$\begin{pmatrix}\eta^0\\ w^0\end{pmatrix} = k_1^\gamma\begin{pmatrix}\overline{\eta^0}\\ \overline{w^0}\end{pmatrix},$$

where $\gamma > 0$ will be determined later on. Note that those values depend on k_1. To abbreviate the notation, we just write $\overline{\eta^0}$, $\overline{w^0}$, η^0 and w^0 instead of $\overline{\eta^0_{k_1}}$, $\overline{w^0_{k_1}}$, $\eta^0_{k_1}$ and $w^0_{k_1}$. On the other hand, observe that

$$\int_0^{2\pi}\eta^0(x)dx = \int_0^{2\pi}w^0(x)dx = 0.$$

It allows us to deduce that the following subspace of V^s

$$\mathcal{V} = \left\{\begin{pmatrix}\eta\\ w\end{pmatrix} \in V^s : \ \widehat{\eta_0} = \widehat{w_0} = 0\right\}$$

is closed in V^s. Then, Theorem 2.1 ensures that $(S(t))_{t\in\mathbb{R}}$ is a group of isometries in V^{-s}. It immediately yields

$$\left\| S(t)\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \right\|_{V^{-s}} = \left\| \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \right\|_{V^{-s}} \sim k_1^{\gamma-s}. \tag{18}$$

Hence the first term in Picard's iteration, see (15), is uniformly bounded in V^{-s}, whenever $\gamma - s < 0$. To bound the second term, we require a more delicate analysis. For the reader's convenience, we leave the proof of the following estimate in the Appendix, see (25):

$$\left\| \begin{pmatrix} \xi \\ \varphi \end{pmatrix}(t,\cdot) \right\|_{V^{-s}} \sim k_1^{2\gamma} t. \tag{19}$$

To analyze the last term, see (17), we shall need the next result whose proof can be found in [7].

Lemma 3.1. *Let $F, G \in H_p^s(0,2\pi)$ with $s \geq 0$. There exists a constant $C > 0$, depending only on s, such that*

$$\|\psi_\alpha(D_x)(FG)\|_{H_p^s(0,2\pi)} \leq C\|F\|_{H_p^s(0,2\pi)}\|G\|_{H_p^s(0,2\pi)}, \tag{20}$$

for any $\alpha > 0$.

For $T \geq 0$ we set $X_T := \left[C([0,T];L^2)\right]^2$. From (17) and estimate (20), it follows that

$$\begin{aligned}
\left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T} &\lesssim \int_0^T \left(\|v\|_{L^2}\|\varphi+z\|_{L^2} + \|u\|_{L^2}\|\xi+y\|_{L^2} + \|\varphi+z\|_{L^2}\|\xi+y\|_{L^2}\right)(\tau)d\tau \\
&\quad + \int_0^T \Big(\|\varphi\|_{L^2}^2 + \|z\|_{L^2}^2 + 2\left(\|u\|_{L^2}\|\varphi\|_{L^2} + \|u\|_{L^2}\|z\|_{L^2} + \|z\|_{L^2}\|\varphi\|_{L^2}\right)\Big)(\tau)d\tau \\
&\lesssim 2T\left\| S(t)\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \right\|_{X_T}\left(\left\| \begin{pmatrix} \xi \\ \varphi \end{pmatrix} \right\|_{X_T} + \left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T} \right) + T\left(\left\| \begin{pmatrix} \xi \\ \varphi \end{pmatrix} \right\|_{X_T} + \left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T} \right)^2 \\
&\quad + T\left\| \begin{pmatrix} \xi \\ \varphi \end{pmatrix} \right\|_{X_T}^2 + T\left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T}^2 + 2T\left\| S(t)\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \right\|_{X_T}\left\| \begin{pmatrix} \xi \\ \varphi \end{pmatrix} \right\|_{X_T} \\
&\quad + 2T\left\| S(t)\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \right\|_{X_T}\left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T} + 2T\left\| \begin{pmatrix} \xi \\ \varphi \end{pmatrix} \right\|_{X_T}\left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T}.
\end{aligned}$$

Hence, taking into account (18)-(19), we deduce

$$\left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T} \lesssim \Lambda + \Pi\left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T} + 2T\left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T}^2, \tag{21}$$

where the implicit constant is independent of k_1 and

$$\Lambda = 4T^2k_1^{3\gamma} + 2T^3k_1^{4\gamma}; \quad \Pi = 4\left(Tk_1^{\gamma} + T^2k_1^{2\gamma}\right).$$

Inequality (21) involves quadratic and linear terms, so one can see it as a polynomial inequality of order two with variable

$$\mathcal{L}(T) := \left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T}.$$

Since $\begin{pmatrix} y \\ z \end{pmatrix} \in \left[C([0,T];L^2)\right]^2$, it follows that $\mathcal{L}$ is a continuous function on T, with $\mathcal{L}(0) = 0$. The task now is optimizing $\gamma < s$, and finding $k_0 \in \mathbb{N}$ (large enough), T_0 (small enough) so that $k_1^{\gamma - s}$ and $\mathcal{L}(T)$ remain uniformly bounded while $k_1^{2\gamma} t$ will be arbitrarily large for all $t \in [0, T_0]$ and for all $k_1 \geq k_0$ and $0 \leq T \leq T_0$. The next step consists of validating this argument.

Let $T_0 = k_1^{-\rho}$ with $\rho > 0$ to be chosen later. For $T \leq T_0$, we see that

$$\Lambda = \mathcal{O}\left(k_1^{3\gamma - 2\rho} + k_1^{4\gamma - 3\rho}\right) \text{ and } \Pi = \mathcal{O}\left(k_1^{\gamma - \rho} + k_1^{2\gamma - 2\rho}\right).$$

All the above exponents of k_1 are negative if, for instance, $\rho > 3\gamma/2$. We choose $\rho = \mu\gamma$ with $\mu > 3/2$. Now take $k_0 \in \mathbb{N}$ large enough so that $\Pi \ll \frac{1}{2}$ and both T_0 and Λ shall be very small for all $k_1 \geq k_0$. Moreover, without loss of generality, we can assume that the implicit constant in inequality (21) is just the constant one. Otherwise, we consider another suitable k_0 larger than the previous one. Therefore, in this circumstance, it follows that for all $0 \leq T \leq T_0$

$$\triangle := \sqrt{(1 - \Pi)^2 - 8T\Lambda} > 0$$

and hence the quadratic polynomial

$$p(\zeta) = 2T\zeta^2 + (\Pi - 1)\zeta + \Lambda$$

has two positive roots $0 < \zeta_1 < \zeta_2$, where

$$\zeta_1 = \frac{(1 - \Pi) - \triangle}{4T} \leq 4\Lambda.$$

Furthermore, $p(\zeta) < 0$, for $\zeta \in (\zeta_1, \zeta_2)$. From inequality (21), we obtain

$$p(\mathcal{L}(T)) \geq 0.$$

From the continuity of the real valued-function $\mathcal{L}$ on T, we deduce that

$$\left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_{X_T} = \mathcal{L}(T) - \mathcal{L}(0) \leq \zeta_1 \leq 4\Lambda, \quad \text{for all } T \in [0, T_0].$$

By induction, we can construct sequences as below:

- An increasing sequence of positive numbers $(k_\nu)_{\nu \in \mathbb{N}} = (k_1^\nu)_{\nu \in \mathbb{N}}$ with $\lim\limits_{\nu \to \infty} k_\nu = +\infty$.
- A decreasing sequence of positive numbers $(T_\nu)_{\nu \in \mathbb{N}} = (k_1^{-\nu(\mu\gamma)})_{\nu \in \mathbb{N}}$, representing the times, with $\lim\limits_{\nu \to \infty} T_\nu = 0$.

- A sequence of initial data (first terms in Picard's iteration) $\begin{pmatrix} \eta_\nu^0 \\ w_\nu^0 \end{pmatrix}_{\nu\in\mathbb{N}}$ with

$$\left\| \begin{pmatrix} \eta_\nu^0 \\ w_\nu^0 \end{pmatrix} \right\|_{V^{-s}} \sim k_1^{\nu(\gamma-s)}.$$

- A sequence of second terms in Picard's iteration $\begin{pmatrix} \xi_\nu \\ \varphi_\nu \end{pmatrix}_{\nu\in\mathbb{N}}$ with

$$\left\| \begin{pmatrix} \xi_\nu \\ \varphi_\nu \end{pmatrix}(t,\cdot) \right\|_{V^{-s}} \sim k_1^{\nu(2\gamma)} t, \quad \text{for all } t \geq 0.$$

- A sequence of third terms in Picard's iteration $\begin{pmatrix} y_\nu \\ z_\nu \end{pmatrix}_{\nu\in\mathbb{N}}$ with

$$\left\| \begin{pmatrix} y_\nu \\ z_\nu \end{pmatrix}(t,\cdot) \right\|_{V^{-s}} \sim k_1^{\nu((3-2\mu)\gamma)}, \quad \text{for all } t \in [0, k_1^{\nu(-\mu\gamma)}].$$

In this way, according to (15), the sequence of solutions $\begin{pmatrix} \eta_\nu \\ w_\nu \end{pmatrix}_{\nu\in\mathbb{N}}$ to (13)-(14) associated to the initial data $\begin{pmatrix} \eta_\nu^0 \\ w_\nu^0 \end{pmatrix}_{\nu\in\mathbb{N}}$ will be arbitrarily large in V^{-s}-norm if, for instance, $\gamma \in (0, s)$ and $\mu \in (\frac{3}{2}, 2)$. This is due to the second term $\begin{pmatrix} \xi_\nu \\ \varphi_\nu \end{pmatrix}$ is arbitrarily large at time T_ν in V^{-s}-norm. The proof is now completed. □

4. Appendix

This section is dedicated to proving Estimate (19) for the sequence of second terms in Picard's iteration. Recall by (16), the second term is given by

$$\int_0^t S(t-\tau)\mathcal{N}\left(S(\tau) \begin{pmatrix} \overline{\eta^0} \\ \overline{w^0} \end{pmatrix} \right) d\tau, \tag{22}$$

where the operator $\mathcal{N}$ is defined in (14). Using Remark 3.1, we deduce

$$S(t) \begin{pmatrix} \overline{\eta^0} \\ \overline{w^0} \end{pmatrix} =: \begin{pmatrix} \overline{\overline{\eta^0}} \\ \overline{\overline{w^0}} \end{pmatrix},$$

where

$$\begin{aligned} \overline{\overline{\eta^0}} = \overline{\overline{w^0}} &= \sin(k_1 x + k_1\sigma(k_1)t) + \sin(k_1 x - k_1\sigma(k_1)t) \\ &\quad + \sin(k_2 x + k_2\sigma(k_2)t) + \sin(k_2 x - k_2\sigma(k_2)t), \quad k_2 := k_1 + 1, k_1 \in \mathbb{N}. \end{aligned}$$

It allows us to deduce

$$\overline{\overline{\eta^0}}\,\overline{\overline{w^0}} = \sum_{j=1}^{4} I_j(t,x) + 2\sum_{j=5}^{10} I_j(t,x),$$

where

$$I_1(t,x) = \frac{1-\cos 2(k_1x + k_1\sigma(k_1)t)}{2}, \qquad I_2(t,x) = \frac{1-\cos 2(k_1x - k_1\sigma(k_1)t)}{2},$$
$$I_3(t,x) = \frac{1-\cos 2(k_2x + k_2\sigma(k_2)t)}{2}, \qquad I_4(t,x) = \frac{1-\cos 2(k_2x - k_2\sigma(k_2)t)}{2},$$
$$I_5(t,x) = \frac{\cos(2k_1\sigma(k_1)t) - \cos(2k_1x)}{2},$$
$$I_6(t,x) = \frac{\cos\left((k_1-k_2)x + (k_1\sigma(k_1) - k_2\sigma(k_2))t\right) - \cos\left((k_1+k_2)x + (k_1\sigma(k_1) + k_2\sigma(k_2))t\right)}{2},$$
$$I_7(t,x) = \frac{\cos\left((k_1-k_2)x + (k_1\sigma(k_1) + k_2\sigma(k_2))t\right) - \cos\left((k_1+k_2)x + (k_1\sigma(k_1) - k_2\sigma(k_2))t\right)}{2},$$
$$I_8(t,x) = \frac{\cos\left((k_1-k_2)x - (k_1\sigma(k_1) + k_2\sigma(k_2))t\right) - \cos\left((k_1+k_2)x - (k_1\sigma(k_1) - k_2\sigma(k_2))t\right)}{2},$$
$$I_9(t,x) = \frac{\cos\left((k_1-k_2)x - (k_1\sigma(k_1) - k_2\sigma(k_2))t\right) - \cos\left((k_1+k_2)x - (k_1\sigma(k_1) + k_2\sigma(k_2))t\right)}{2},$$
$$I_{10}(t,x) = \frac{\cos(2k_2\sigma(k_2)t) - \cos(2k_2x)}{2}.$$

Note that

$$\mathcal{N}\begin{pmatrix} \overline{\overline{\eta^0}} \\ \overline{\overline{w^0}} \end{pmatrix} = \begin{pmatrix} \psi_b(D_x)(\overline{\overline{\eta^0 w^0}}) \\ \frac{1}{2}\psi_d(D_x)(\overline{\overline{w^0}}^2) \end{pmatrix}.$$

Using several times Remark 3.3, we obtain

$$\psi_b(D_x)\left(\overline{\overline{\eta^0}}\,\overline{\overline{w^0}}\right) = \sum_{j=1}^{4} L_j(t,x) + 2\sum_{j=5}^{10} L_j(t,x),$$

where now L_j are known functions given by

$$L_1(t,x) = \psi_b(D_x)(I_1(t,x)) = -i\frac{k_1}{1+b(2k_1)^2}\sin\left((2k_1x + 2k_1\sigma(k_1)t)\right),$$
$$L_2(t,x) = \psi_b(D_x)(I_2(t,x)) = -i\frac{k_1}{1+b(2k_1)^2}\sin\left((2k_1x - 2k_1\sigma(k_1)t)\right),$$
$$L_3(t,x) = \psi_b(D_x)(I_3(t,x)) = -i\frac{k_2}{1+b(2k_2)^2}\sin\left((2k_2x + 2k_2\sigma(k_2)t)\right),$$
$$L_4(t,x) = \psi_b(D_x)(I_4(t,x)) = -i\frac{k_2}{1+b(2k_2)^2}\sin\left((2k_2x - 2k_2\sigma(k_2)t)\right),$$
$$L_5(t,x) = \psi_b(D_x)(I_5(t,x)) = -i\frac{k_1}{1+b(2k_1)^2}\sin\left(2k_1x\right),$$
$$L_6(t,x) = \psi_b(D_x)(I_6(t,x)) = \frac{i(k_1-k_2)}{2(1+b(k_1-k_2)^2)}\sin\left((k_1-k_2)x + (k_1\sigma(k_1) - k_2\sigma(k_2))t\right)$$
$$- \frac{i(k_1+k_2)}{2(1+b(k_1+k_2)^2)}\sin\left((k_1+k_2)x + (k_1\sigma(k_1) + k_2\sigma(k_2))t\right),$$

$$L_7(t,x) = \psi_b(D_x)(I_7(t,x)) = \frac{i(k_1-k_2)}{2(1+b(k_1-k_2)^2)} \sin\left((k_1-k_2)x + (k_1\sigma(k_1)+k_2\sigma(k_2))t\right) - \frac{i(k_1+k_2)}{2(1+b(k_1+k_2)^2)} \sin\left((k_1+k_2)x + (k_1\sigma(k_1)-k_2\sigma(k_2))t\right),$$

$$L_8(t,x) = \psi_b(D_x)(I_8(t,x)) = \frac{i(k_1-k_2)}{2(1+b(k_1-k_2)^2)} \sin\left((k_1-k_2)x - (k_1\sigma(k_1)+k_2\sigma(k_2))t\right) - \frac{i(k_1+k_2)}{2(1+b(k_1+k_2)^2)} \sin\left((k_1+k_2)x - (k_1\sigma(k_1)-k_2\sigma(k_2))t\right),$$

$$L_9(t,x) = \psi_b(D_x)(I_9(t,x)) = \frac{i(k_1-k_2)}{2(1+b(k_1-k_2)^2)} \sin\left((k_1-k_2)x - (k_1\sigma(k_1)-k_2\sigma(k_2))t\right) - \frac{i(k_1+k_2)}{2(1+b(k_1+k_2)^2)} \sin\left((k_1+k_2)x - (k_1\sigma(k_1)+k_2\sigma(k_2))t\right),$$

$$L_{10}(t,x) = \psi_b(D_x)(I_{10}(t,x)) = -i\frac{k_2}{1+b(2k_2)^2} \sin(2k_2x).$$

Due to the explicit expression of $\sigma(k)$, see Lemma 2.1, all the denominators involved in the following computations are never zero. From Remark 3.2 and the previous identities, it follows that

$$\begin{pmatrix} N_1(t,x) \\ 0 \end{pmatrix} = \int_0^t S(t-\tau)\begin{pmatrix} L_1(\tau,x) \\ 0 \end{pmatrix} d\tau = \begin{pmatrix} -i\frac{k_1}{1+b(2k_1)^2}\left(N_1^1(t,x)+N_1^2(t,x)\right) \\ 0 \end{pmatrix},$$

with

$$N_1^1(t,x) = \frac{\cos(2k_1x + 2k_1\sigma(2k_1)t) - \cos(2k_1x + 2k_1\sigma(k_1)t)}{k_1\sigma(k_1) - k_1\sigma(2k_1)},$$

$$N_1^2(t,x) = \frac{\cos(2k_1x - 2k_1\sigma(2k_1)t) - \cos(2k_1x + 2k_1\sigma(k_1)t)}{k_1\sigma(k_1) + k_1\sigma(2k_1)},$$

$$\begin{pmatrix} N_2(t,x) \\ 0 \end{pmatrix} = \int_0^t S(t-\tau)\begin{pmatrix} L_2(\tau,x) \\ 0 \end{pmatrix} d\tau = \begin{pmatrix} -i\frac{k_1}{1+b(2k_1)^2}\left(N_2^1(t,x)+N_2^2(t,x)\right) \\ 0 \end{pmatrix},$$

with

$$N_2^1(t,x) = \frac{\cos(2k_1x - 2k_1\sigma(k_1)t) - \cos(2k_1x + 2k_1\sigma(2k_1)t)}{k_1\sigma(k_1) + k_1\sigma(2k_1)},$$

$$N_1^2(t,x) = \frac{\cos(2k_1x2k_1\sigma(k_1)t) - \cos(2k_1x + 2k_1\sigma(2k_1)t)}{k_1\sigma(k_1) - k_1\sigma(2k_1)},$$

$$\begin{pmatrix} N_3(t,x) \\ 0 \end{pmatrix} = \int_0^t S(t-\tau)\begin{pmatrix} L_3(\tau,x) \\ 0 \end{pmatrix} d\tau = \begin{pmatrix} -i\frac{k_2}{1+b(2k_2)^2}\left(N_3^1(t,x)+N_3^2(t,x)\right) \\ 0 \end{pmatrix},$$

with

$$N_3^1(t,x) = \frac{\cos(2k_2x + 2k_2\sigma(2k_2)t) - \cos(2k_2x + 2k_2\sigma(k_2)t)}{k_2\sigma(k_2) - k_2\sigma(2k_2)},$$

$$N_3^2(t,x) = \frac{\cos(2k_2x - 2k_2\sigma(2k_2)t) - \cos(2k_2x + 2k_2\sigma(k_2)t)}{k_2\sigma(k_2) + k_2\sigma(2k_2)},$$

$$\begin{pmatrix} N_4(t,x) \\ 0 \end{pmatrix} = \int_0^t S(t-\tau) \begin{pmatrix} L_4(\tau,x) \\ 0 \end{pmatrix} d\tau = \begin{pmatrix} -i\frac{k_2}{1+b(2k_2)^2}\left(N_4^1(t,x)+N_4^2(t,x)\right) \\ 0 \end{pmatrix},$$

with

$$N_4^1(t,x) = \frac{\cos(2k_2x - 2k_2\sigma(k_1)t) - \cos(2k_2x + 2k_2\sigma(2k_2)t)}{k_1\sigma(k_2) + k_2\sigma(2k_2)},$$

$$N_4^2(t,x) = \frac{\cos(2k_2x2k_2\sigma(k_2)t) - \cos(2k_2x + 2k_2\sigma(2k_2)t)}{k_2\sigma(k_2) - k_2\sigma(2k_2)},$$

$$\begin{pmatrix} N_5(t,x) \\ 0 \end{pmatrix} = \int_0^t S(t-\tau) \begin{pmatrix} L_5(\tau,x) \\ 0 \end{pmatrix} d\tau = \begin{pmatrix} -i\frac{k_1}{1+b(2k_1)^2}\left(N_5^1(t,x)\right) \\ 0 \end{pmatrix},$$

with

$$N_5^1(t,x) = \frac{\cos(2k_1x - 2k_1\sigma(2k_1)t) - \cos(2k_1x + 2k_1\sigma(2k_1)t)}{2k_1\sigma(2k_1)},$$

$$\begin{aligned}\begin{pmatrix} N_6(t,x) \\ 0 \end{pmatrix} &= \int_0^t S(t-\tau) \begin{pmatrix} L_6(\tau,x) \\ 0 \end{pmatrix} d\tau \\ &= \begin{pmatrix} \frac{i(k_1-k_2)}{2(1+b(k_1-k_2)^2)}\left(N_6^1(t,x)+N_6^2(t,x)\right) - \frac{i(k_1+k_2)}{2(1+b(k_1+k_2)^2)}\left(N_6^3(t,x)+N_6^4(t,x)\right) \\ 0 \end{pmatrix},\end{aligned}$$

with

$$N_6^1(t,x) = \frac{\cos((k_1-k_2)x + (k_1-k_2)\sigma(k_1-k_2)t) - \cos((k_1-k_2)x + (k_1\sigma(k_1) - k_2\sigma(k_2))t)}{k_1\sigma(k_1) - k_2\sigma(k_2) - (k_1-k_2)\sigma(k_1-k_2)},$$

$$N_6^2(t,x) = \frac{\cos((k_1-k_2)x - (k_1-k_2)\sigma(k_1-k_2)t) - \cos((k_1-k_2)x + (k_1\sigma(k_1) - k_2\sigma(k_2))t)}{k_1\sigma(k_1) - k_2\sigma(k_2) + (k_1-k_2)\sigma(k_1-k_2)},$$

$$N_6^3(t,x) = \frac{\cos((k_1+k_2)x + (k_1+k_2)\sigma(k_1+k_2)t) - \cos((k_1+k_2)x + (k_1\sigma(k_1) + k_2\sigma(k_2))t)}{k_1\sigma(k_1) + k_2\sigma(k_2) - (k_1+k_2)\sigma(k_1+k_2)},$$

$$N_6^4(t,x) = \frac{\cos((k_1+k_2)x - (k_1+k_2)\sigma(k_1+k_2)t) - \cos((k_1+k_2)x + (k_1\sigma(k_1) + k_2\sigma(k_2))t)}{k_1\sigma(k_1) + k_2\sigma(k_2) + (k_1+k_2)\sigma(k_1+k_2)},$$

$$\begin{aligned}\begin{pmatrix} N_7(t,x) \\ 0 \end{pmatrix} &= \int_0^t S(t-\tau) \begin{pmatrix} L_7(\tau,x) \\ 0 \end{pmatrix} d\tau \\ &= \begin{pmatrix} \frac{i(k_1-k_2)}{2(1+b(k_1-k_2)^2)}\left(N_7^1(t,x)+N_7^2(t,x)\right) - \frac{i(k_1+k_2)}{2(1+b(k_1+k_2)^2)}\left(N_7^3(t,x)+N_7^4(t,x)\right) \\ 0 \end{pmatrix},\end{aligned}$$

with

$$N_7^1(t,x) = \frac{\cos((k_1-k_2)x + (k_1-k_2)\sigma(k_1-k_2)t) - \cos((k_1-k_2)x + (k_1\sigma(k_1) + k_2\sigma(k_2))t)}{k_1\sigma(k_1) + k_2\sigma(k_2) - (k_1-k_2)\sigma(k_1-k_2)},$$

$$N_7^2(t,x) = \frac{\cos((k_1-k_2)x - (k_1-k_2)\sigma(k_1-k_2)t) - \cos((k_1-k_2)x + (k_1\sigma(k_1) + k_2\sigma(k_2))t)}{k_1\sigma(k_1) + k_2\sigma(k_2) + (k_1-k_2)\sigma(k_1-k_2)},$$

$$N_7^3(t,x) = \frac{\cos((k_1+k_2)x + (k_1+k_2)\sigma(k_1+k_2)t) - \cos((k_1+k_2)x + (k_1\sigma(k_1) - k_2\sigma(k_2))t)}{k_1\sigma(k_1) - k_2\sigma(k_2) - (k_1+k_2)\sigma(k_1+k_2)},$$

$$N_7^4(t,x) = \frac{\cos\left((k_1+k_2)x - (k_1+k_2)\sigma(k_1+k_2)t\right) - \cos\left((k_1+k_2)x + (k_1\sigma(k_1) - k_2\sigma(k_2))t\right)}{k_1\sigma(k_1) - k_2\sigma(k_2) + (k_1+k_2)\sigma(k_1+k_2)},$$

$$\begin{aligned}\begin{pmatrix} N_8(t,x) \\ 0 \end{pmatrix} &= \int_0^t S(t-\tau) \begin{pmatrix} L_8(\tau,x) \\ 0 \end{pmatrix} d\tau \\ &= \begin{pmatrix} \frac{i(k_1-k_2)}{2(1+b(k_1-k_2)^2)} \left(N_8^1(t,x) + N_8^2(t,x)\right) - \frac{i(k_1+k_2)}{2(1+b(k_1+k_2)^2)} \left(N_8^3(t,x) + N_8^4(t,x)\right) \\ 0 \end{pmatrix},\end{aligned}$$

with

$$N_8^1(t,x) = \frac{\cos\left((k_1-k_2)x - (k_1\sigma(k_1) + k_2\sigma(k_2))t\right) - \cos\left((k_1-k_2)x + (k_1-k_2)\sigma(k_1-k_2)t\right)}{k_1\sigma(k_1) + k_2\sigma(k_2) + (k_1-k_2)\sigma(k_1-k_2)},$$

$$N_8^2(t,x) = \frac{\cos\left((k_1-k_2)x - (k_1\sigma(k_1) + k_2\sigma(k_2))t\right) - \cos\left((k_1-k_2)x - (k_1-k_2)\sigma(k_1-k_2)t\right)}{k_1\sigma(k_1) + k_2\sigma(k_2) - (k_1-k_2)\sigma(k_1-k_2)},$$

$$N_8^3(t,x) = \frac{\cos\left((k_1+k_2)x - (k_1\sigma(k_1) - k_2\sigma(k_2))t\right) - \cos\left((k_1+k_2)x + (k_1+k_2)\sigma(k_1+k_2)t\right)}{k_1\sigma(k_1) - k_2\sigma(k_2) + (k_1+k_2)\sigma(k_1+k_2)},$$

$$N_8^4(t,x) = \frac{\cos\left((k_1+k_2)x - (k_1\sigma(k_1) - k_2\sigma(k_2))t\right) - \cos\left((k_1+k_2)x - (k_1+k_2)\sigma(k_1+k_2)t\right)}{k_1\sigma(k_1) - k_2\sigma(k_2) - (k_1+k_2)\sigma(k_1+k_2)},$$

$$\begin{aligned}\begin{pmatrix} N_9(t,x) \\ 0 \end{pmatrix} &= \int_0^t S(t-\tau) \begin{pmatrix} L_9(\tau,x) \\ 0 \end{pmatrix} d\tau \\ &= \begin{pmatrix} \frac{i(k_1-k_2)}{2(1+b(k_1-k_2)^2)} \left(N_9^1(t,x) + N_9^2(t,x)\right) - \frac{i(k_1+k_2)}{2(1+b(k_1+k_2)^2)} \left(N_9^3(t,x) + N_9^4(t,x)\right) \\ 0 \end{pmatrix},\end{aligned}$$

with

$$N_9^1(t,x) = \frac{\cos\left((k_1-k_2)x - (k_1\sigma(k_1) - k_2\sigma(k_2))t\right) - \cos\left((k_1-k_2)x + (k_1-k_2)\sigma(k_1-k_2)t\right)}{k_1\sigma(k_1) - k_2\sigma(k_2) + (k_1-k_2)\sigma(k_1-k_2)},$$

$$N_9^2(t,x) = \frac{\cos\left((k_1-k_2)x - (k_1\sigma(k_1) - k_2\sigma(k_2))t\right) - \cos\left((k_1-k_2)x - (k_1-k_2)\sigma(k_1-k_2)t\right)}{k_1\sigma(k_1) - k_2\sigma(k_2) - (k_1-k_2)\sigma(k_1-k_2)},$$

$$N_9^3(t,x) = \frac{\cos\left((k_1+k_2)x - (k_1\sigma(k_1) + k_2\sigma(k_2))t\right) - \cos\left((k_1+k_2)x + (k_1+k_2)\sigma(k_1+k_2)t\right)}{k_1\sigma(k_1) + k_2\sigma(k_2) + (k_1+k_2)\sigma(k_1+k_2)},$$

$$N_9^4(t,x) = \frac{\cos\left((k_1+k_2)x - (k_1\sigma(k_1) + k_2\sigma(k_2))t\right) - \cos\left((k_1+k_2)x - (k_1+k_2)\sigma(k_1+k_2)t\right)}{k_1\sigma(k_1) + k_2\sigma(k_2) - (k_1+k_2)\sigma(k_1+k_2)},$$

$$\begin{pmatrix} N_{10}(t,x) \\ 0 \end{pmatrix} = \int_0^t S(t-\tau) \begin{pmatrix} L_{10}(\tau,x) \\ 0 \end{pmatrix} d\tau = \begin{pmatrix} -i\frac{k_2}{1+b(2k_2)^2} \left(N_{10}^1(t,x)\right) \\ 0 \end{pmatrix},$$

with

$$N_{10}^1(t,x) = \frac{\cos\left(2k_2x - 2k_2\sigma(2k_2)t\right) - \cos\left(2k_2x + 2k_2\sigma(2k_2)t\right)}{2k_2\sigma(2k_2)}.$$

Thanks to the real mean value theorem applied to the cosine function, we obtain

$$\left|\frac{\cos(kx - \omega_1 t) - \cos(kx - \omega_2 t)}{\omega_1 - \omega_2}\right| \le t, \quad \omega_1 \ne \omega_2,\ t \ge 0.$$

From all the above identities, we get

$$\begin{aligned}\int_0^t S(t-\tau)\begin{pmatrix}\psi_b\left(\overline{\overline{\eta^0}}\ \overline{\overline{w^0}}\right)\\ 0\end{pmatrix}(\tau,x)d\tau &= \int_0^t S(t-\tau)\begin{pmatrix}\sum_{l=1}^{10} L_l(\tau,x)\\ 0\end{pmatrix}d\tau\\ &= \begin{pmatrix}\sum_{l=1}^{10} N_l(t,x)\\ 0\end{pmatrix},\end{aligned} \tag{23}$$

where

$$N_l(t,x)\sim\begin{cases} ik_1^{-1}t, & 1\le l\le 5;\ l=10,\\ \frac{i}{1+b}t-ik_1^{-1}t, & 6\le l\le 9.\end{cases}$$

Proceeding in a similar way, we obtain

$$\int_0^t S(t-\tau)\begin{pmatrix}0\\ \frac{1}{2}\psi_d\left(\left(\overline{\overline{w^0}}\right)^2\right)\end{pmatrix}(\tau,x)d\tau = \begin{pmatrix}0\\ \sum_{l=1}^{10} R_l(t,x)\end{pmatrix}, \tag{24}$$

where

$$R_l(t,x)\sim\begin{cases} ik_1^{-1}t, & 1\le l\le 5;\ l=10,\\ \frac{i}{1+d}t-ik_1^{-1}t, & 6\le l\le 9.\end{cases}$$

Combining (22), (23) and (24), $\begin{pmatrix}\xi\\ \varphi\end{pmatrix}$ can be written as

$$\begin{aligned}&\begin{pmatrix}\xi\\ \varphi\end{pmatrix}(t,x) = k_1^{2\gamma}\int_0^t S(t-\tau)\mathcal{N}\left(S(\tau)\begin{pmatrix}\overline{\eta^0}\\ \overline{w^0}\end{pmatrix}\right)d\tau\\ &= k_1^{2\gamma}\left[\int_0^t S(t-\tau)\begin{pmatrix}\psi_b\left(\overline{\overline{\eta^0}}\ \overline{\overline{w^0}}\right)\\ 0\end{pmatrix}(\tau,x)d\tau + \int_0^t S(t-\tau)\begin{pmatrix}0\\ \frac{1}{2}\psi_d\left(\left(\overline{\overline{w^0}}\right)^2\right)\end{pmatrix}(\tau,x)d\tau\right]\\ &= \begin{pmatrix}k_1^{2\gamma}\sum_{l=1}^{10} N_l(t,x)\\ k_1^{2\gamma}\sum_{l=1}^{10} R_l(t,x)\end{pmatrix},\end{aligned}$$

where now

$$k_1^{2\gamma}\sum_{l=1}^{10} N_l(t,x)\sim ik_1^{2\gamma}t, \qquad k_1^{2\gamma}\sum_{l=1}^{10} R_l(t,x)\sim ik_1^{2\gamma}t.$$

Therefore, one gets the following estimate for the second term in Picard's iteration

$$\left\|\begin{pmatrix}\xi\\ \varphi\end{pmatrix}(t,\cdot)\right\|_{V^{-s}}\sim k_1^{2\gamma}t. \tag{25}$$

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