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Title: Universal Infinitesimal Hilbertianity of Sub-Riemannian Manifolds

Please cite the original version:

Le Donne, E., Lučić, D., & Pasqualetto, E. (2023). Universal Infinitesimal Hilbertianity of Sub-Riemannian Manifolds. Potential Analysis, 59(1), 349-374. https://doi.org/10.1007/s11118-021-09971-8

Universal Infinitesimal Hilbertianity
of Sub-Riemannian Manifolds

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Abstract

We prove that sub-Riemannian manifolds are infinitesimally Hilbertian (i.e., the associated Sobolev space is Hilbert) when equipped with an arbitrary Radon measure. The result follows from an embedding of metric derivations into the space of square-integrable sections of the horizontal bundle, which we obtain on all weighted sub-Finsler manifolds. As an intermediate tool, of independent interest, we show that any sub-Finsler distance can be monotonically approximated from below by Finsler ones. All the results are obtained in the general setting of possibly rank-varying structures.

1 Introduction

General overview In the last two decades, weakly differentiable functions over metric measure spaces have been extensively studied and have played a fundamental role in the development of abstract calculus in the nonsmooth setting (see, e.g., [13, 14, 18]). The definition of Sobolev space we adopt in this paper is the one introduced in [10], which is

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equivalent to the notions proposed in [5, 8, 24]. At this level of generality, however, Sobolev calculus might not be fully satisfactory from a functional-analytic viewpoint. For instance, not only the Sobolev space can fail to be Hilbert (consider the Euclidean space endowed with the L^∞-norm and the Lebesgue measure), but it can be also non-reflexive (as shown in [2, Proposition 7.8]). In view of this, the class of *infinitesimally Hilbertian* metric measure spaces (i.e., whose associated Sobolev space is Hilbert) is particularly relevant. These spaces enjoy nice features, among which the strong density of boundedly-supported Lipschitz functions in the Sobolev space (as proven in [4]); we refer to the introduction of [21] for an account of the several benefits of working within this class of spaces.

A strictly related concept is that of *universally infinitesimally Hilbertian* metric space, that is to say, a metric space that is infinitesimally Hilbertian with respect to whichever Radon measure. The interest in this property is mainly motivated by the study of metric structures that are important from a geometric perspective, but do not carry any 'canonical' measure (such as sub-Riemannian manifolds that are not equiregular). The purpose of this paper is to prove the following claim:

> All sub-Riemannian manifolds are universally infinitesimally Hilbertian.

The goal will be achieved by building an isometric embedding of the 'analytic' space of *derivations* over any weighted sub-Finsler manifold (which provide us with a synthetic notion of vector field, linked to the Sobolev calculus) into the 'geometric' space of sections of the horizontal bundle. The abstract differential structure of the space under consideration and the behaviour of its (purely metric) tangent spaces are – a priori – unrelated, thus the role of the above-mentioned embedding result is to bridge this gap, showing that Sobolev functions are suitable to capture the fiberwise Hilbertianity of the horizontal bundle. As an intermediate tool, of independent interest, we prove that a sub-Finsler distance can be monotonically approximated from below by Finsler distances.

We would like to point out that the universal infinitesimal Hilbertianity is a non-trivial property already in the case of the Euclidean space $\mathbb{R}^n$: unless the measure under consideration is the Lebesgue measure $\mathcal{L}^n$ (or some measure $\mu \ll \mathcal{L}^n$ having smooth density), it is not clear how one can characterise the Sobolev space over $(\mathbb{R}^n, |\cdot|, \mu)$. For example, if the support of μ is totally disconnected, then the Sobolev space is trivial (in the sense that every 2-integrable function is Sobolev and has null weak gradient), thus it does not carry any information from the structure of the underlying space; cf. [5, Remark 4.12] for an instance of this phenomenon.

Outline of the paper We consider a *(generalised) sub-Finsler manifold* (M, E, σ, ψ). This means that M is a smooth connected manifold, while E is a smooth vector bundle over M equipped with a *continuous metric* $\sigma : E \to [0, +\infty)$ (as in Definition 3.7) and $\psi : E \to TM$ is a bundle morphism; moreover, a Hörmander-like condition is required to hold, cf. Definition 4.1. Whenever it holds that for every $x \in M$ the norm $\sigma|_{E_x}$ on the fiber E_x is induced by a scalar product that smoothly depends on x, we say that (M, E, σ, ψ) is a *(generalised) sub-Riemannian manifold*. This notion of sub-Riemannian manifold is the most general one that we have in the literature (see, e.g., [1]).

The *horizontal bundle* HM is obtained by 'patching together' the horizontal fibers $\mathcal{D}_x := \psi(E_x)$, which form a *continuous distribution* on M (in the sense of Theorem 3.2). We then define a *generalised metric* $\rho : TM \to [0, +\infty]$ over the tangent bundle (cf. Definition 3.4 for this term) as

$$\rho(x, v) = \|v\|_x := \inf \left\{ \sigma(u) \mid u \in E_x, \ (x, v) = \psi(u) \right\}, \quad \text{for every } (x, v) \in TM.$$

Observe that the finiteness domain of $\rho(x, \cdot)$ coincides with the horizontal fiber $\mathcal{D}_x$ for every $x \in M$. The space M can be made into a metric space by considering the *Carnot–Carathéodory distance*: given any two points $x, y \in M$, we define $\mathsf{d}_{CC}(x, y)$ as the length of the shortest path among all horizontal curves (i.e., tangent to HM) joining x and y. Here, the length of a horizontal curve is computed with respect to the generalised metric ρ. See Definition 4.3 for the details.

Let us now fix a non-negative Radon measure μ on (M, d_{CC}), say that μ is finite (for simplicity). We may consider two (completely different in nature) notions of vector field over $(M, \mathsf{d}_{CC}, \mu)$:

- The space $\mathrm{Der}^{2,2}(M; \mu)$ of L^2-*derivations* having divergence in L^2 (in the sense of [10]). These are linear functionals acting on Lipschitz functions and taking values into the space of Borel functions over M, that satisfy a suitable Leibniz rule and a locality property. The Sobolev space $W^{1,2}(M, \mathsf{d}_{CC}, \mu)$ is then obtained in duality with $\mathrm{Der}^{2,2}(M; \mu)$, as described in Definition 2.4. The whole Section 2 is devoted to the key results about L^2-derivations.
- The space $L^2(HM; \mu)$ of 2-integrable sections of the horizontal bundle; see Definition 4.7. Whenever M is a sub-Riemannian manifold, the elements of $L^2(HM; \mu)$ satisfy a pointwise parallelogram rule (thanks to geometric considerations, see Remark 4.8). Nevertheless, it is not clear – a priori – how to deduce from this information that the metric measure space $(M, \mathsf{d}_{CC}, \mu)$ is infinitesimally Hilbertian.

The main result of the paper aims at providing a relation between $\mathrm{Der}^{2,2}(M; \mu)$ and $L^2(HM; \mu)$: the former space is isometrically embeddable into the latter one. The precise statement is:

Theorem 1.1 (Embedding theorem) *Let* (M, E, σ, ψ) *be a sub-Finsler manifold with* d_{CC} *complete. Let* μ *be a finite, non-negative Borel measure on* (M, d_{CC})*. Then there exists a unique linear operator* $I \colon \mathrm{Der}^{2,2}(M; \mu) \to L^2(HM; \mu)$ *such that*

$$\mathsf{d}_H f(x)\big[I(\boldsymbol{b})(x)\big] = \boldsymbol{b}(f)(x) \quad \text{holds for } \mu\text{-a.e. } x \in M,$$

for every $\boldsymbol{b} \in \mathrm{Der}^{2,2}(M; \mu)$ *and* $f \in C_c^1(M) \cap LIP(M)$*. Moreover, the operator* I *satisfies*

$$\big\| I(\boldsymbol{b})(x) \big\|_x = |\boldsymbol{b}|(x) \quad \text{for } \mu\text{-a.e. } x \in M,$$

for every $\boldsymbol{b} \in \mathrm{Der}^{2,2}(M; \mu)$*.*

As a consequence, sub-Riemannian manifolds are universally infinitesimally Hilbertian:

Theorem 1.2 (Infinitesimal Hilbertianity of sub-Riemannian manifolds) *Let* (M, E, σ, ψ) *be a sub-Riemannian manifold with* d_{CC} *complete. Let* μ *be a non-negative Radon measure on* (M, d_{CC})*. Then the metric measure space* $(M, \mathsf{d}_{CC}, \mu)$ *is infinitesimally Hilbertian.*

The proof of the embedding result (Theorem 1.1) builds upon the following key ingredients:

a) The Carnot–Carathéodory distance d_{CC} can be written as pointwise limit of an increasing sequence of Finsler distances; cf. Theorem 5.1. This property follows from the results we develop in Section 3, where we show that the sub-Finsler metric ρ (or, more generally, any generalised metric as in Definition 3.4) can be approximated from

below by Finsler ones. This technical statement can be achieved by exploiting the lower semicontinuity of ρ, as done in Lemma 3.8.

b) The pointwise norm of a given derivation can be recovered by just considering its evaluation at smooth 1-Lipschitz functions. More precisely, we can find a sequence $(f_n)_n \subseteq C_c^1(\mathrm{M})$ of 1-Lipschitz functions (with respect to d_{CC}) such that the identity $|\boldsymbol{b}| = \sup_n \boldsymbol{b}(f_n)$ holds μ-a.e. for every $\boldsymbol{b} \in \mathrm{Der}^{2,2}(\mathrm{M}; \mu)$. This representation formula is obtained by combining item a) above with an approximation result for Finsler manifolds proven in [21].

c) Any derivation $\boldsymbol{b} \in \mathrm{Der}^{2,2}(\mathrm{M}; \mu)$ can be represented by a suitable measure $\boldsymbol{\pi}$ on the space of continuous curves in M, as granted by the metric version [22] of Smirnov's superposition principle for normal 1-currents; see Theorem 2.8. The presence of such a measure $\boldsymbol{\pi}$ is an essential tool in the construction of the embedding map $\mathrm{I}: \mathrm{Der}^{2,2}(\mathrm{M}; \mu) \to L^2(\mathrm{HM}; \mu)$, which preserves the pointwise norm of all vector fields as a consequence of item b).

Comparison with previous works The results of the present paper enrich the list of metric spaces that are known to be universally infinitesimally Hilbertian, which previously consisted of:

i) Euclidean spaces [15], see also [12],
ii) Riemannian manifolds [21],
iii) Carnot groups [21],
iv) Hilbert spaces [23],
v) locally CAT(κ)-spaces [11].

Let us now briefly comment on the main differences and analogies between the technique we exploit here and the previous approaches. To the best of our knowledge, the strategy proposed in [15, 21, 23] does not carry over to the framework of sub-Riemannian manifolds. In the classes of spaces i), ii), iii), iv), a fact which plays a fundamental role is that – thanks to a convolution argument – any Lipschitz function f (with respect to the relevant distance) can be approximated by smooth functions whose local Lipschitz constant is sufficiently close to that of f. In the context of sub-Riemannian manifolds, a relevant result in this direction is given by [17, Proposition 11.10], but still it does not seem to be sufficient for our purposes, as a more local estimate would be needed.

However, a different approach has been developed in [11] in order to overcome the lack of smoothness of the spaces in v). The proof in the sub-Riemannian case is inspired by the ideas introduced in [11]: indeed, the universal infinitesimal Hilbertianity of CAT spaces stems – similarly to what described above – from an embedding result, which in turn relies upon Smirnov's superposition principle and a representation formula for the pointwise norm of derivations. While the former is available on any metric measure space, the latter requires an ad hoc argument for the sub-Riemannian setting. This makes a significant difference with [11]: on CAT spaces, distance functions from given points are 1-Lipschitz and everywhere have some form of differentiability, thus they are suitable candidates for the representation formula; on sub-Riemannian manifolds, on the contrary, this is no longer true, whence we need to find an alternative way to show that there is plenty of smooth 1-Lipschitz functions that are μ-a.e. differentiable (where μ is an arbitrary measure). Most of the present paper is actually dedicated to addressing this last point. Once the representation formula is at disposal, the proof of the embedding result closely follows along the lines of [11, Theorem 6.2].

2 Derivations and Sobolev Calculus on Metric Measure Spaces

We recall here the notions of derivation and Sobolev space that have been proposed by S. Di Marino in [10]. For our purposes, a *metric measure space* is a triple (X, d, m), where (X, d) is a complete and separable metric space, while $m \geq 0$ is a locally finite Borel measure on (X, d).

We call $LIP(X)$ or $LIP^d(X)$ the space of real-valued Lipschitz functions on (X, d), while $LIP_{bs}(X)$ or $LIP^d_{bs}(X)$ stand for the set of elements of $LIP(X)$ with bounded support. The (global) Lipschitz constant of $f \in LIP(X)$ is denoted by $Lip(f)$ or $Lip^d(f)$, while the functions $lip(f) \colon X \to [0, +\infty)$ and $lip_a(f) \colon X \to [0, +\infty)$ are defined as

$$lip(f)(x) := \varlimsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)}, \quad lip_a(f)(x) := \inf_{r > 0} Lip\big(f|_{B_r(x)}\big)$$

whenever $x \in X$ is an accumulation point, and $lip(f)(x) = lip_a(f)(x) := 0$ elsewhere. We say that $lip(f)$ and $lip_a(f)$ are the *local Lipschitz constant* and the *asymptotic Lipschitz constant* of the function f, respectively. The vector space of all (equivalence classes up to m-a.e. equality of) real-valued Borel functions on X is denoted by $L^0(m)$.

A *derivation* on (X, d, m) is a linear map $\boldsymbol{b} \colon LIP_{bs}(X) \to L^0(m)$ with these two properties:

a) LEIBNIZ RULE. The identity $\boldsymbol{b}(fg) = f\,\boldsymbol{b}(g) + g\,\boldsymbol{b}(f)$ holds for every $f, g \in LIP_{bs}(X)$.

b) WEAK LOCALITY. There exists a function $G \in L^0(m)$ such that $|\boldsymbol{b}(f)| \leq G\, lip_a(f)$ is satisfied in the m-a.e. sense for every $f \in LIP_{bs}(X)$.

The *pointwise norm* $|\boldsymbol{b}| := \operatorname{ess\,sup}\big\{\boldsymbol{b}(f) \,\big|\, f \in LIP_{bs}(X),\, Lip(f) \leq 1\big\}$ is the minimal function (in the m-a.e. sense) that can be chosen as G in item b) above.

Definition 2.1 (The space $Der^{2,2}(X; m)$) Let (X, d, m) be a metric measure space. Then we denote by $Der^{2,2}(X; m)$ the space of all derivations $\boldsymbol{b}$ on (X, d, m) such that $|\boldsymbol{b}| \in L^2(m)$ and whose distributional divergence can be represented as a function in $L^2(m)$, i.e., there exists a (uniquely determined) function $div(\boldsymbol{b}) \in L^2(m)$ such that

$$\int \boldsymbol{b}(f)\, dm = -\int f\, div(\boldsymbol{b})\, dm \quad \text{for every } f \in LIP_{bs}(X).$$

The space $Der^{2,2}(X; m)$ is a module over the commutative ring $LIP_{bs}(X)$ and is a Banach space when endowed with the norm $Der^{2,2}(X; m) \ni \boldsymbol{b} \mapsto \|\boldsymbol{b}\|_{2,2} := \big(\int |\boldsymbol{b}|^2\, dm + \int div(\boldsymbol{b})^2\, dm\big)^{\frac{1}{2}}$.

Lemma 2.2 *Let (X, d, m) be a metric measure space and $\boldsymbol{b} \in Der^{2,2}(X; m)$. Let $(f_n)_n \subseteq LIP_{bs}(X)$ be a sequence with $\sup_n Lip(f_n) < +\infty$ that pointwise converges to some limit $f \in LIP_{bs}(X)$. Then*

$$\int \varphi\, \boldsymbol{b}(f_n)\, dm \longrightarrow \int \varphi\, \boldsymbol{b}(f)\, dm \quad \text{for every } \varphi \in LIP_{bs}(X).$$

Proof See item (1) of [11, Lemma 5.4]. ☐

It will be convenient to work with the following representation formula for the pointwise norm of the elements of $\mathrm{Der}^{2,2}(X; \mathfrak{m})$.

Proposition 2.3 *Let* $(X, d, \mathfrak{m})$ *be a metric measure space. Let* $\boldsymbol{b} \in \mathrm{Der}^{2,2}(X; \mathfrak{m})$ *be given. Fix a countable dense set* $(x_k)_k \subseteq X$. *For any* $j, k \in \mathbb{N}$, *let* $\eta_{jk} \colon X \to [0, 1 - 1/j]$ *be a boundedly-supported Lipschitz function such that* $\eta_{jk} = 1 - 1/j$ *on* $B_j(x_k)$ *and* $\mathrm{Lip}(\eta_{jk}) \leq 1/j^2$. *Then it holds that*

$$|\boldsymbol{b}| = \operatorname*{ess\,sup}_{j,k \in \mathbb{N}} \boldsymbol{b}\big((d(\cdot, x_k) \wedge j)\, \eta_{jk}\big) \quad \text{in the } \mathfrak{m}\text{-a.e. sense.} \tag{2.1}$$

Proof It follows from [11, Proposition 5.5]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By duality with $\mathrm{Der}^{2,2}(X; \mathfrak{m})$, it is possible to introduce a notion of Sobolev space $W^{1,2}(X, d, \mathfrak{m})$.

Definition 2.4 (Sobolev space) Let $(X, d, \mathfrak{m})$ be a metric measure space. Then we say that a function $f \in L^2(\mathfrak{m})$ belongs to the *Sobolev space* $W^{1,2}(X, d, \mathfrak{m})$ provided there exists a continuous morphism $L_f \colon \mathrm{Der}^{2,2}(X; \mathfrak{m}) \to L^1(\mathfrak{m})$ of $\mathrm{LIP}_{bs}(X)$-modules such that

$$\int L_f(\boldsymbol{b}) \, d\mathfrak{m} = -\int f \operatorname{div}(\boldsymbol{b}) \, d\mathfrak{m} \quad \text{for every } \boldsymbol{b} \in \mathrm{Der}^{2,2}(X; \mathfrak{m}).$$

The map L_f is uniquely determined. Furthermore, there exists a function $G \in L^2(\mathfrak{m})$ such that

$$\big|L_f(\boldsymbol{b})\big| \leq G\,|\boldsymbol{b}| \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } \boldsymbol{b} \in \mathrm{Der}^{2,2}(X; \mathfrak{m}).$$

The minimal such function G (in the $\mathfrak{m}$-a.e. sense) is called 2-*weak gradient* of f and is denoted by $|Df|$ or $|Df|_{\mathfrak{m}}$. Then $W^{1,2}(X, d, \mathfrak{m})$ is a Banach space when equipped with the norm

$$\|f\|_{W^{1,2}(X, d, \mathfrak{m})} := \left(\int |f|^2 \, d\mathfrak{m} + \int |Df|^2 \, d\mathfrak{m} \right)^{1/2} \quad \text{for every } f \in W^{1,2}(X, d, \mathfrak{m}).$$

Remark 2.5 Let $(X, d, \mathfrak{m})$ be a metric measure space. Consider the (not necessarily complete) norm $\mathrm{Der}^{2,2}(X; \mathfrak{m}) \ni \boldsymbol{b} \mapsto \|\boldsymbol{b}\|_2 := \left(\int |\boldsymbol{b}|^2 \, d\mathfrak{m} \right)^{1/2}$. Call $\mathbb{B}$ the dual space of $\big(\mathrm{Der}^{2,2}(X; \mathfrak{m}), \|\cdot\|_2\big)$. Given any function $f \in W^{1,2}(X, d, \mathfrak{m})$, we define the element $\mathscr{L}_f \in \mathbb{B}$ as

$$\mathscr{L}_f(\boldsymbol{b}) := \int L_f(\boldsymbol{b}) \, d\mathfrak{m} \quad \text{for every } \boldsymbol{b} \in \mathrm{Der}^{2,2}(X; \mathfrak{m}). \tag{2.2}$$

Then it holds the map $W^{1,2}(X, d, \mathfrak{m}) \ni f \mapsto \mathscr{L}_f \in \mathbb{B}$ is linear and $\|\mathscr{L}_f\|_{\mathbb{B}} = \big\||Df|\big\|_{L^2(\mathfrak{m})}$ is satisfied for every $f \in W^{1,2}(X, d, \mathfrak{m})$, as proven in [11, Proposition 5.10].

The following definition – which has been introduced in [14] – plays a key role in this paper.

Definition 2.6 (Infinitesimal Hilbertianity) We say that a metric measure space $(X, d, \mathfrak{m})$ is *infinitesimally Hilbertian* provided $W^{1,2}(X, d, \mathfrak{m})$ is a Hilbert space.

The following result provides a sufficient condition for the infinitesimal Hilbertianity to hold.

Proposition 2.7 *Let* $(X, d, \mathfrak{m})$ *be a metric measure space. Suppose that*

$$|\boldsymbol{b} + \boldsymbol{b}'|^2 + |\boldsymbol{b} - \boldsymbol{b}'|^2 = 2\,|\boldsymbol{b}|^2 + 2\,|\boldsymbol{b}'|^2 \quad \mathfrak{m}\text{-}a.e. \quad \textit{for every } \boldsymbol{b}, \boldsymbol{b}' \in Der^{2,2}(X; \mathfrak{m}). \quad (2.3)$$

Then $(X, d, \mathfrak{m})$ *is infinitesimally Hilbertian.*

Proof By integrating Eq. 2.3 we see that the norm $\|\cdot\|_2$ on $\mathrm{Der}^{2,2}(X; \mathfrak{m})$ (defined in Remark 2.5) satisfies the parallelogram rule, whence the dual space $\mathbb{B}$ of $\left(\mathrm{Der}^{2,2}(X; \mathfrak{m}), \|\cdot\|_2\right)$ is a Hilbert space. Therefore, we know from Remark 2.5 that for every $f, g \in W^{1,2}(X, d, \mathfrak{m})$ it holds that

$$
\begin{aligned}
\left\|D(f + g)\right\|_{L^2(\mathfrak{m})}^2 + \left\|D(f - g)\right\|_{L^2(\mathfrak{m})}^2 &= \|\mathscr{L}_{f+g}\|_{\mathbb{B}}^2 + \|\mathscr{L}_{f-g}\|_{\mathbb{B}}^2 = \|\mathscr{L}_f + \mathscr{L}_g\|_{\mathbb{B}}^2 + \|\mathscr{L}_f - \mathscr{L}_g\|_{\mathbb{B}}^2 \\
&= 2\,\|\mathscr{L}_f\|_{\mathbb{B}}^2 + 2\,\|\mathscr{L}_g\|_{\mathbb{B}}^2 = 2\,\left\|Df\right\|_{L^2(\mathfrak{m})}^2 + 2\,\left\|Dg\right\|_{L^2(\mathfrak{m})}^2,
\end{aligned}
$$

which proves that $W^{1,2}(X, d, \mathfrak{m})$ is a Hilbert space, as required. $\quad\square$

Finally, we conclude the subsection by reporting the following consequence of the metric version of Smirnov's superposition principle, which has been proven by E. Paolini and E. Stepanov in [22].

Theorem 2.8 (Superposition principle) *Let* $(X, d, \mathfrak{m})$ *be a metric measure space with* $\mathfrak{m}$ *finite. Let* $\boldsymbol{b} \in Der^{2,2}(X; \mathfrak{m})$. *Then there exists a finite, non-negative Borel measure* $\boldsymbol{\pi}$ *on* $C\big([0, 1], X\big)$, *concentrated on the set of non-constant Lipschitz curves on* X *having constant speed, such that*

$$\int g\,\boldsymbol{b}(f)\,\mathrm{d}\mathfrak{m} = \iint_0^1 g(\gamma_t)\,(f \circ \gamma)_t'\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma), \quad\quad (2.4a)$$

$$\int g\,|\boldsymbol{b}|\,\mathrm{d}\mathfrak{m} = \iint_0^1 g(\gamma_t)\,|\dot{\gamma}_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) \quad\quad (2.4b)$$

for every $f, g \in LIP_{bs}(X)$.

Proof Combine [11, Theorem 4.9] with [11, Lemma 6.1]. $\quad\square$

3 Monotone Approximation of Generalised Metrics

3.1 Set-up and Auxiliary Results

We begin with some classical definitions. A norm $\mathbf{n}$ defined on a finite-dimensional vector space V is said to be *smooth* provided it is of class C^∞ on $V \setminus \{0\}$. In addition, we say that $\mathbf{n}$ is *strongly convex* if the Hessian matrix of $\mathbf{n}^2$ at any vector $v \in V \setminus \{0\}$ is positive definite. With the notation $W \leq V$ we intend that W is a vector subspace of V.

By *smooth manifold* we shall always mean a connected differentiable manifold of class C^∞. Given a smooth manifold M and a smooth vector bundle (E, π) over M, we say that a function $F \colon E \to [0, +\infty)$ is a *Finsler metric* over E if it is continuous, it is smooth on the complement of the zero section, and $F|_{E_x}$ is a strongly convex norm on the fiber $E_x := \pi^{-1}(x)$ for every $x \in$ M.

By Finsler metric on M we mean a Finsler metric F over the tangent bundle TM. In this case, we also say that the couple (M, F) is a *Finsler manifold*. (In the literature, (M, F) is often referred to as a reversible Finsler manifold; cf., for instance, the monograph [7].)

Definition 3.1 (Generalised norm) Let V be a vector space. Then a function $\mathbf{n}\colon V \to [0, +\infty]$ is said to be a *generalised norm* if there exists a vector subspace $D(\mathbf{n}) \neq \{0\}$ of V such that $\mathbf{n}|_{D(\mathbf{n})}$ is a norm on $D(\mathbf{n})$ and $\mathbf{n}(v) = +\infty$ holds for every $v \in V \setminus D(\mathbf{n})$.

Theorem 3.2 (Definition of continuous distribution) *Let* M *be a smooth manifold and let* (E, π) *be a smooth vector bundle over* M. *Let* $\{V_x\}_{x \in M}$ *be a family of vector spaces such that* $V_x \leq E_x$ *for all* $x \in M$. *Then the following conditions are equivalent:*

i)   *Given* $\bar{x} \in M$ *and* $\bar{v} \in V_{\bar{x}}$, *there exists a continuous section* v *of* E, *defined on some neighbourhood* U *of* $\bar{x}$, *such that* $v(\bar{x}) = \bar{v}$ *and* $v(x) \in V_x$ *for every* $x \in U$.

ii) *Given* $\bar{x} \in M$, *there exist finitely many continuous sections* $v_1, \ldots, v_k$ *of* E, *defined on some neighbourhood* U *of* $\bar{x}$, *such that* $V_x = span\{v_1(x), \ldots, v_k(x)\}$ *for every* $x \in U$.

iii) *Given* $\bar{x} \in M$, *there exist a neighbourhood* U *of* $\bar{x}$, *a smooth vector bundle* $\tilde{E}$ *over* U, *and a continuous vector bundle morphism* $\psi\colon \tilde{E} \to E|_U$, *such that* $V_x = \psi(\tilde{E}_x)$ *for all* $x \in U$.

If the above conditions are satisfied, we say that $\{V_x\}_{x \in M}$ *is a continuous distribution (of possibly varying rank) over* M. *Moreover, we can assume that* k *and the rank of* $\tilde{E}$ *are at most* $d\,2^{2 \cdot 5^n - 1}$, *where* d *is the rank of* E *and* n *is the dimension of* M.

Proof The real novelty of the theorem is the implication i) $\implies$ ii).

i) $\implies$ ii) Suppose item i) holds. Given a point $\bar{x} \in M$, we can choose an open set $U' \subseteq \mathbb{R}^n$ containing 0 and a map $\varphi\colon U' \to M$ satisfying $\varphi(0) = \bar{x}$ that is a homeomorphism with its image. Possibly shrinking U', we can assume there exists a Finsler metric F over $E|_U$, where $U := \varphi(U')$. Fix any radius $\lambda > 0$ such that $\bar{B}_\lambda(0) \subseteq U'$ and call $K := \varphi(\bar{B}_\lambda(0)) \subseteq U$. For any $i = 1, \ldots, d$ we set $C_i := \{x \in K \ : \ \dim V_x \leq i\}$. In order to prove ii), it would be enough to find some finite families $\mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_d$ of continuous sections of $E|_K$ such that for any $i = 1, \ldots, d$ it holds that

$$V_x = \mathcal{F}_i(x) := span\{v(x) \mid v \in \mathcal{F}_i\} \qquad \text{for every } x \in C_i,$$
$$\mathcal{F}_i(x) \leq V_x \ \text{ and } \ \dim \mathcal{F}_i(x) \geq i \qquad \text{for every } x \in K \setminus C_i.$$

We build $\mathcal{F}_1, \ldots, \mathcal{F}_d$ via a recursive argument. Suppose to have already defined $\mathcal{F}_1, \ldots, \mathcal{F}_i$ for some $i < d$. Notice that item i) grants that the function $M \ni x \mapsto \dim V_x$ is lower semicontinuous, thus C_i is a compact set. For any $j \in \mathbb{N}$ we define the compact set $K_j \subseteq K$ as

$$K_j := \varphi\left(\left\{y \in \bar{B}_\lambda(0) \ \Big| \ \frac{\lambda}{2^{j+1}} \leq \mathrm{dist}(\varphi^{-1}(C_i), y) \leq \frac{\lambda}{2^{j-1}}\right\}\right).$$

Observe that $\bigcup_{j \in \mathbb{N}} K_j = K \setminus C_i$ and that $\mathring{K}_j \cap \mathring{K}_{j'} = \emptyset$ for all $j, j' \in \mathbb{N}$ such that $|j - j'|$ is even.

Let $j \in \mathbb{N}$ be fixed. For any $x \in K_j$, we choose a vector $\bar{w}_x \in V_x$ such that $F(\bar{w}_x) = 1$ and $\dim(\mathcal{F}_i(x) + \mathbb{R}\,\bar{w}_x) \geq i + 1$. By item i), we can find a neighbourhood $W_x \subseteq U$ of x and a continuous section w_x of $E|_{W_x}$, such that $w_x(x) = \bar{w}_x$ and $w_x(z) \in V_z$ for all $z \in W_x$. Possibly shrinking W_x, we can further assume that $0 < F(w_x(z)) \leq 2$ and $\dim(\mathcal{F}_i(z) + \mathbb{R}\,w_x(z)) \geq i+1$ hold for every point $z \in W_x$. By compactness of K_j, we can thus find an open covering $W_1, \ldots, W_m \subseteq U$ of K_j and continuous sections $w_1, \ldots, w_m$ of $E|_{W_1}, \ldots, E|_{W_m}$, respectively, such that $0 < F(w_\iota(x)) \leq 2$ and $\dim(\mathcal{F}_i(x) + \mathbb{R}\,w_\iota(x)) \geq i + 1$ for every $\iota = 1, \ldots, m$ and $x \in W_\iota$. By Lebesgue's number lemma, there exists $r > 0$ such that any ball in $\mathbb{R}^n$ of radius r centered at $\varphi^{-1}(K_j)$ is entirely contained in one of the sets $\varphi^{-1}(W_1), \ldots, \varphi^{-1}(W_m)$. Choose a maximal r-separated subset S of $\varphi^{-1}(K_j)$, i.e., S is

maximal among all subsets satisfying $|p - q| \geq r$ for every $p, q \in S$ with $p \neq q$. Note that S is a finite set by compactness of $\varphi^{-1}(K_j)$. For any $p \in S$, call $G_p := \varphi(B_r(p))$ and pick $\iota(p) \in \{1, \ldots, m\}$ such that $G_p \subseteq W_{\iota(p)}$. By definition of S, it holds that $K_j \subseteq \bigcup_{p \in S} G_p$. Moreover, given any $p \in S$ we have that the balls $\{B_{r/2}(q) : q \in S \setminus \{p\}, |p - q| < 2r\}$ are pairwise disjoint and contained in $B_{5r/2}(p) \setminus B_{r/2}(p)$, whence accordingly $\#\{q \in S \setminus \{p\} : |p - q| < 2r\} \leq 5^n - 1$ for all $p \in S$. Therefore, we can take a partition $S = S_1 \cup \ldots \cup S_{5^n}$ with the property that $G_p \cap G_q = \emptyset$ whenever $\ell = 1, \ldots, 5^n$ and $p, q \in S_\ell$ satisfy $p \neq q$. Given $\ell = 1, \ldots, 5^n$ and $p \in S_\ell$, we can pick a continuous function $\psi_p : K \to [0, 1]$ satisfying $\psi_p = 0$ on $K_j \setminus G_p$ and $\psi_p > 0$ on $K_j \cap G_p$. For any multi-index $\alpha = (\alpha_2, \ldots, \alpha_{5^n}) \in \{-1, 1\}^{5^n - 1}$, we define

$$v_{j\alpha}(x) := \sum_{p \in S_1} \psi_p(x)\, w_{\iota(p)}(x) + \sum_{\ell=2}^{5^n} \alpha_\ell \sum_{p \in S_\ell} \psi_p(x)\, w_{\iota(p)}(x) \quad \text{for every } x \in K_j.$$

Notice that $F(v_{j\alpha}(x)) \leq 2 \cdot 5^n$ for every $\alpha \in \{-1, 1\}^{5^n - 1}$ and $x \in K_j$. For any $j \in \mathbb{N}$ we fix a continuous functions $\eta_j : K \to [0, 1]$ such that $\eta_j = 0$ on $K \setminus \mathring{K}_j$ and $\eta_j > 0$ on $\mathring{K}_j$. Then we set

$$\mathcal{F}'_{i+1} := \left\{ \sum_{j \text{ even}} \frac{\eta_j}{2^j} v_{j\alpha} \pm \sum_{j \text{ odd}} \frac{\eta_j}{2^j} v_{j\beta} \;\middle|\; \alpha, \beta \in \{-1, 1\}^{5^n - 1} \right\}.$$

Therefore, it follows from the construction that the family $\mathcal{F}_{i+1} := \mathcal{F}_i \cup \mathcal{F}'_{i+1}$ of continuous sections of $\mathrm{E}|_K$ satisfies $\dim \mathcal{F}_{i+1}(x) \geq i + 1$ for every $x \in K$, as required. Observe also that $\#\mathcal{F}'_i \leq 2^{2 \cdot 5^n - 1}$ for all $i = 1, \ldots, d$, thus the cardinality of $\mathcal{F} := \mathcal{F}_d$ does not exceed $d\, 2^{2 \cdot 5^n - 1}$. This proves ii).

ii) $\implies$ iii) Suppose item ii) holds. Given a point $\bar{x} \in \mathrm{M}$, pick a neighbourhood U of $\bar{x}$ and some continuous sections $v_1, \ldots, v_k$ of $\mathrm{E}|_U$ such that $V_x = \mathrm{span}\{v_1(x), \ldots, v_k(x)\}$ for every $x \in U$. Let us define $\tilde{E} := U \times \mathbb{R}^k$ and the continuous vector bundle morphism $\psi : \tilde{E} \to \mathrm{E}|_U$ as

$$\psi(x, \lambda) := \sum_{i=1}^{k} \lambda_i\, v_i(x) \quad \text{for every } x \in U \text{ and } \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k.$$

Therefore, we conclude that $\psi(\tilde{E}_x) = \mathrm{span}\{v_1(x), \ldots, v_k(x)\} = V_x$ for all $x \in U$, thus proving iii).

iii) $\implies$ i) Suppose item iii) holds. Fix $\bar{x} \in \mathrm{M}$ and $\bar{v} \in V_{\bar{x}}$. There exist a smooth vector bundle $\tilde{E}$ over some neighbourhood U' of $\bar{x}$ and a continuous vector bundle morphism $\psi : \tilde{E} \to \mathrm{E}|_{U'}$ such that $V_x = \psi(\tilde{E}_x)$ for all $x \in U'$. Choose any $\bar{w} \in \tilde{E}_{\bar{x}}$ for which $\psi(\bar{w}) = \bar{v}$. Then we can find a neighbourhood $U \subseteq U'$ of $\bar{x}$ and a continuous section w of $\tilde{E}|_U$ such that $w(\bar{x}) = \bar{w}$. Now let us define $v(x) := \psi(w(x))$ for every $x \in U$. Therefore, it holds that v is a continuous section of $\mathrm{E}|_U$ such that $v(\bar{x}) = \bar{v}$ and $v(x) \in V_x$ for all $x \in U$, thus proving i). $\qquad \square$

Remark 3.3 As already observed during the proof of Theorem 3.2, the function $\mathrm{M} \ni x \mapsto \dim V_x$ is lower semicontinuous whenever $\{V_x\}_{x \in \mathrm{M}}$ is a continuous distribution over M.

Definition 3.4 (Generalised metric) Let M be a smooth manifold, (E, π) a smooth vector bundle over M. Then a *generalised metric* over E is a lower semicontinuous function $\rho : \mathrm{E} \to [0, +\infty]$ having the following properties:

i) $\rho_x := \rho|_{E_x}$ is a generalised norm on E_x for every $x \in M$.
ii) The family $\{D(\rho_x)\}_{x \in M}$ is a continuous distribution.

In the case $E = TM$, we just say that ρ is a generalised metric on M.

Let us fix some notation: given any vector subspace $V \leq \mathbb{R}^d$, we denote by $V^\perp$ its orthogonal complement (with respect to the Euclidean norm). We denote by $\mathbb{S}^{d-1}$ the Euclidean unit sphere in $\mathbb{R}^d$, i.e., the set of all points $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ such that $x_1^2 + \ldots + x_d^2 = 1$, while $x \cdot y$ stands for the Euclidean scalar product between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$. Finally, given a metric space (X, d) and two compact non-empty sets $A, B \subseteq X$, we shall denote by $d_H(A, B)$ the Hausdorff distance between A and B, i.e.,

$$d_H(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \ \sup_{y \in B} \inf_{x \in A} d(x, y) \right\}. \tag{3.1}$$

Lemma 3.5 *Let* $\{0\} \neq V \leq \mathbb{R}^d$ *be given. Let* $\| \cdot \|$ *be a norm on* V. *Fix a constant* $\lambda > 0$ *and a norm* $\| \cdot \|'$ *on* $\mathbb{R}^d$ *such that* $\|v\|' < \|v\|$ *for all* $v \in V \setminus \{0\}$. *Then there exists a norm* $\mathbf{n}$ *on* $\mathbb{R}^d$ *such that the following properties are satisfied:*

$$\begin{aligned} \mathbf{n}(v) &= \|v\| \quad &\textit{for every } v \in V, \\ \mathbf{n}(v) &> \|v\|' \quad &\textit{for every } v \in \mathbb{R}^d \setminus \{0\}, \\ \mathbf{n}(v) &\geq \lambda \quad &\textit{for every } v \in V^\perp \cap \mathbb{S}^{d-1}. \end{aligned}$$

Proof Call $\lambda' := \lambda + \max \{ \|v\|' : v \in \mathbb{S}^{d-1} \}$ and $k := \dim V$. Fix any orthonormal basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$ (equipped with the Euclidean norm) such that $e_1, \ldots, e_k$ is a basis of V. Hence, we define the norm $\mathbf{n}$ on $\mathbb{R}^d$ as follows: given any $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, we set

$$\mathbf{n}(\alpha_1 e_1 + \ldots + \alpha_d e_d) := \|\alpha_1 e_1 + \ldots + \alpha_k e_k\| + \lambda' |(\alpha_{k+1}, \ldots, \alpha_d)|. \tag{3.2}$$

It directly follows from its very definition that the norm $\mathbf{n}$ satisfies $\mathbf{n}(v) = \|v\|$ for every $v \in V$ and $\mathbf{n}(v) = \lambda' > \lambda$ for every $v \in V^\perp \cap \mathbb{S}^{d-1}$. Finally, for any choice of $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ with $(\alpha_{k+1}, \ldots, \alpha_d) \neq 0$ we have that

$$\begin{aligned} \mathbf{n}(\alpha_1 e_1 + \ldots + \alpha_d e_d) &= \|\alpha_1 e_1 + \ldots + \alpha_k e_k\| + \lambda' |(\alpha_{k+1}, \ldots, \alpha_d)| \\ &\geq \|\alpha_1 e_1 + \ldots + \alpha_k e_k\|' + \lambda' |(\alpha_{k+1}, \ldots, \alpha_d)| \\ &> \|\alpha_1 e_1 + \ldots + \alpha_k e_k\|' + \left\| \frac{\alpha_{k+1} e_{k+1} + \ldots + \alpha_d e_d}{|(\alpha_{k+1}, \ldots, \alpha_d)|} \right\|' |(\alpha_{k+1}, \ldots, \alpha_d)| \\ &\geq \|\alpha_1 e_1 + \ldots + \alpha_d e_d\|', \end{aligned}$$

thus completing the proof of the statement. □

In the following results, we shall consider the trivial bundle $M \times \mathbb{R}^d$ over M. Given any $x \in M$, a vector subspace of the fiber of $M \times \mathbb{R}^d$ at x is of the form $\{x\} \times V$, for some vector subspace $V \leq \mathbb{R}^d$. For simplicity, we will always implicitly identify $\{x\} \times V$ with the vector space V itself.

Lemma 3.6 *Let* M *be a smooth manifold and* ρ *a generalised metric over* $M \times \mathbb{R}^d$. *Fix* $\bar{x} \in M$ *and any norm* $\| \cdot \|$ *on* $\mathbb{R}^d$. *Call* $V_x := D(\rho_x)$ *for every* $x \in M$ *and* $k := \dim V_{\bar{x}}$. *Then for any* $\varepsilon > 0$ *there exists a neighbourhood* U *of* $\bar{x}$ *such that*

$$d_H(V_{\bar{x}} \cap \mathbb{S}^{d-1}, V_x \cap \mathbb{S}^{d-1}) \leq \varepsilon \quad \textit{for every } x \in U \textit{ with } \dim V_x = k,$$

where the Hausdorff distance d_H *is computed with respect to the norm* $\|\cdot\|$.

Proof Since $\{V_x\}_{x\in M}$ is a continuous distribution, we can find a neighbourhood U' of $\bar{x}$ and some continuous maps $v_1, \dots, v_{k'}: U' \to \mathbb{R}^d$ such that $V_x = \mathrm{span}\{v_1(x), \dots, v_{k'}(x)\}$ for all $x \in U'$. Up to relabelling, we can assume that $v_1(\bar{x}), \dots, v_k(\bar{x})$ constitute a basis of $V_{\bar{x}}$. Then there is a neighbourhood $U \subseteq U'$ of $\bar{x}$ such that $v_1(x), \dots, v_k(x)$ are linearly independent for all $x \in U$. Define $W_x := \mathrm{span}\{v_1(x), \dots, v_k(x)\}$ for every point $x \in U$. Let us apply a Gram–Schmidt orthogonalisation process to the vector fields $v_1, \dots, v_k$, with respect to the Euclidean norm $|\cdot|$:

$$w_1(x) := \frac{v_1(x)}{|v_1(x)|},$$

$$w_2(x) := \frac{v_2(x) - \big(v_2(x) \cdot w_1(x)\big)w_1(x)}{\big|v_2(x) - \big(v_2(x) \cdot w_1(x)\big)w_1(x)\big|},$$

$$\vdots$$

$$w_k(x) := \frac{v_k(x) - \sum_{i=1}^{k-1} \big(v_k(x) \cdot w_i(x)\big)w_i(x)}{\big|v_k(x) - \sum_{i=1}^{k-1} \big(v_k(x) \cdot w_i(x)\big)w_i(x)\big|}$$

for every $x \in U$. Therefore, the resulting continuous maps $w_1, \dots, w_k: U \to \mathbb{R}^d$ satisfy

$$\begin{aligned} w_i(x) \cdot w_j(x) &= \delta_{ij} && \text{for every } i, j = 1, \dots, k \text{ and } x \in U, \\ W_x &= \mathrm{span}\{w_1(x), \dots, w_k(x)\} && \text{for every } x \in U. \end{aligned}$$

Fix any $C > 0$ such that $\|v\| \le C\,|v|$ for all $v \in \mathbb{R}^d$. Possibly shrinking U, we can assume that

$$\big|w_i(\bar{x}) - w_i(x)\big| \le \frac{\varepsilon}{C\sqrt{k}} \quad \text{for every } i = 1, \dots, k \text{ and } x \in U. \tag{3.3}$$

Since $\big\{\sum_{i=1}^{k} q_i\, w_i(x) : q = (q_1, \dots, q_k) \in \mathbb{Q}^k \cap \mathbb{S}^{k-1}\big\}$ is dense in $W_x \cap \mathbb{S}^{d-1}$ for all $x \in U$, we have

$$\mathsf{d}_H(W_{\bar{x}} \cap \mathbb{S}^{d-1}, W_x \cap \mathbb{S}^{d-1}) \overset{(3.1)}{\le} \sup_{q \in \mathbb{Q}^k \cap \mathbb{S}^{k-1}} \left\| \sum_{i=1}^{k} q_i\big(w_i(\bar{x}) - w_i(x)\big) \right\|$$

$$\le C \sup_{q \in \mathbb{Q}^k \cap \mathbb{S}^{k-1}} \left(\sum_{i=1}^{k} q_i^2\right)^{1/2} \left(\sum_{i=1}^{k} |w_i(\bar{x}) - w_i(x)|^2\right)^{1/2} \overset{(3.3)}{\le} \varepsilon$$

for every $x \in U$. The statement follows by noticing that $W_x = V_x$ if $x \in U$ satisfies $\dim V_x = k$. $\qquad\square$

Definition 3.7 (Continuous metric) Let M be a smooth manifold. Let (E, π) be a smooth vector bundle over M. Then a *continuous metric* σ over E is a continuous function $\sigma: E \to [0, +\infty)$ such that $\sigma|_{E_x}$ is a norm on the fiber E_x for every point $x \in M$.

Lemma 3.8 *Let* M *be a smooth manifold,* ρ *a generalised metric over* $M \times \mathbb{R}^d$. *Call* $V_x := D(\rho_x)$ *for every* $x \in M$. *Fix* $\bar{x} \in M$ *and two constants* $\varepsilon, \lambda > 0$. *Let* σ *be a continuous metric over* $M \times \mathbb{R}^d$ *such that* $\sigma(x, v) < \rho(x, v)$ *for every* $x \in M$ *and* $v \in \mathbb{R}^d \setminus \{0\}$. *Then there exist a smooth, strongly convex norm* $\mathbf{n}$ *on* $\mathbb{R}^d$ *and a neighbourhood* U *of* $\bar{x}$ *such that the following properties hold:*

i) $\big|\mathbf{n}(v) - \rho(\bar{x}, v)\big| \le \varepsilon$ *for every* $v \in V_{\bar{x}} \cap \mathbb{S}^{d-1}$.

ii) $\sigma(x, v) < \mathbf{n}(v) < \rho(x, v)$ *for every* $x \in U$ *and* $v \in \mathbb{R}^d \setminus \{0\}$.

iii) $\dim V_{\bar{x}} = \min \{ \dim V_x \ : \ x \in U \}$ *and*

$$\mathbf{n}(v) \geq \lambda \quad \text{for every } x \in D \text{ and } v \in V_x^{\perp} \cap \mathbb{S}^{d-1},$$

where we set $D := \{ x \in U \ : \ \dim V_x = \dim V_{\bar{x}} \}$.

Proof We divide the proof into several steps:
STEP 1. Lemma 3.5 grants the existence of a norm $\mathbf{n}'$ on $\mathbb{R}^d$ such that

$$\begin{aligned} \mathbf{n}'(v) &= \rho(\bar{x}, v) & \text{for every } v \in V_{\bar{x}}, \\ \mathbf{n}'(v) &> \sigma(\bar{x}, v) & \text{for every } v \in \mathbb{R}^d \setminus \{0\}, \\ \mathbf{n}'(v) &\geq \lambda + 1 & \text{for every } v \in V_{\bar{x}}^{\perp} \cap \mathbb{S}^{d-1}. \end{aligned} \quad (3.4)$$

Given that $M \ni x \mapsto \dim V_x$ is lower semicontinuous, we can find a neighbourhood U' of $\bar{x}$ such that $\dim V_{\bar{x}}$ is the minimum of the function $U' \ni x \mapsto \dim V_x$.
STEP 2. The function $\mathbb{S}^{d-1} \ni v \mapsto \mathbf{n}'(v) - \sigma(\bar{x}, v) > 0$ is continuous by construction, thus there exists $\varepsilon' \in (0, \varepsilon)$ such that $\sigma(\bar{x}, v) < \mathbf{n}'(v) - \varepsilon'$ for all $v \in \mathbb{S}^{d-1}$. Choose any constant $\delta > 0$ such that $(\lambda + 1)(1 - \delta) > \lambda$ and $\delta < \varepsilon' / \max \{ \mathbf{n}'(v) \ : \ v \in \mathbb{S}^{d-1} \}$. Pick also $\varepsilon'' \in (0, \varepsilon')$ such that $\varepsilon'' < \delta \min \{ \mathbf{n}'(v) \ : \ v \in \mathbb{S}^{d-1} \}$. Then it holds that

$$\sigma(\bar{x}, v) < \mathbf{n}'(v) - \varepsilon' < \mathbf{n}'(v) - \varepsilon'' < \rho(\bar{x}, v) \quad \text{for every } v \in \mathbb{S}^{d-1}. \quad (3.5)$$

Being $(x, v) \mapsto \rho(x, v) - \mathbf{n}'(v) + \varepsilon''$ lower semicontinuous and $(x, v) \mapsto \mathbf{n}'(v) - \varepsilon' - \sigma(x, v)$ continuous, we deduce from Eq. 3.5 that there exists a neighbourhood $U'' \subseteq U'$ of $\bar{x}$ such that

$$\sigma(x, v) < \mathbf{n}'(v) - \varepsilon' < \mathbf{n}'(v) - \varepsilon'' < \rho(x, v) \quad \text{for every } x \in U'' \text{ and } v \in \mathbb{S}^{d-1}. \quad (3.6)$$

Let us define $\mathbf{n}'' := (1 - \delta) \mathbf{n}'$. Our choice of δ and ε'' yields $\mathbf{n}'(v) - \varepsilon' < (1 - \delta) \mathbf{n}'(v) < \mathbf{n}'(v) - \varepsilon''$ for every $v \in \mathbb{S}^{d-1}$, which together with Eq. 3.6 imply that

$$\sigma(x, v) < \mathbf{n}''(v) < \rho(x, v) \quad \text{for every } x \in U'' \text{ and } v \in \mathbb{S}^{d-1}. \quad (3.7)$$

Moreover, for any $v \in V_{\bar{x}} \cap \mathbb{S}^{d-1}$ it holds $\mathbf{n}''(v) > \mathbf{n}'(v) - \varepsilon' > \rho(\bar{x}, v) - \varepsilon$ by the first line of Eq. 3.4.
STEP 3. In light of Eq. 3.7, there exists a constant $\delta' > 0$ with $(\lambda + 1)(1 - \delta) - \delta' > \lambda$ such that

$$\begin{aligned} \sigma(x, v) < \mathbf{n}''(v) - \delta' < \mathbf{n}''(v) + \delta' < \rho(x, v) & \quad \text{for every } x \in U'' \text{ and } v \in \mathbb{S}^{d-1}, \\ \mathbf{n}''(v) - \delta' > \rho(\bar{x}, v) - \varepsilon & \quad \text{for every } v \in V_{\bar{x}} \cap \mathbb{S}^{d-1}. \end{aligned} \quad (3.8)$$

Choose any smooth norm $\| \cdot \|$ on $\mathbb{R}^d$ such that $\big| \|v\| - \mathbf{n}''(v) \big| \leq \delta'/2$ holds for all $v \in \mathbb{S}^{d-1}$, whose existence follows, e.g., from [16, Theorem 103]. Then let us finally define the sought norm $\mathbf{n}$ as $\mathbf{n}(v) := \|v\| + \delta'|v|/2$ for every $v \in \mathbb{R}^d$. Clearly, it is a smooth and strongly convex norm by construction. Moreover, it can be immediately checked that $\mathbf{n}$ satisfies

$$\big| \mathbf{n}(v) - \mathbf{n}''(v) \big| \leq \delta' \quad \text{for every } v \in \mathbb{S}^{d-1}. \quad (3.9)$$

Accordingly, by combining Eq. 3.8 with Eq. 3.9 we obtain that

$$\begin{aligned} \sigma(x, v) < \mathbf{n}(v) < \rho(x, v) & \quad \text{for every } x \in U'' \text{ and } v \in \mathbb{S}^{d-1}, \\ \mathbf{n}(v) > \rho(\bar{x}, v) - \varepsilon & \quad \text{for every } v \in V_{\bar{x}} \cap \mathbb{S}^{d-1}. \end{aligned} \quad (3.10)$$

STEP 4. Observe that Eq. 3.9 and the third line of Eq. 3.4 give

$$\mathbf{n}(v) \geq (1 - \delta) \mathbf{n}'(v) - \delta' \geq (\lambda + 1)(1 - \delta) - \delta' \quad \text{for every } v \in V_{\bar{x}}^{\perp} \cap \mathbb{S}^{d-1}. \quad (3.11)$$

Since $M \ni x \mapsto V_x$ is a continuous distribution and $(\lambda + 1)(1 - \delta) - \delta' > \lambda$, we deduce from Eq. 3.11 that for some neighbourhood $U \subseteq U''$ of $\bar{x}$ we have

$$\mathbf{n}(v) \geq \lambda \quad \text{for every } x \in U \text{ with } \dim V_x = \dim V_{\bar{x}} \text{ and } v \in V_x^\perp \cap \mathbb{S}^{d-1}.$$

Therefore, item iii) is verified (recall the last claim in STEP 1). Finally, we deduce from Eq. 3.10 that also items i) and ii) hold, thus concluding the proof of the statement. $\qquad\square$

3.2 The Approximation Result

Let M be a smooth manifold and let ρ be a generalised metric over $M \times \mathbb{R}^d$. Calling $V_x := D(\rho_x)$ for every $x \in M$, it holds that $M \ni x \mapsto \dim V_x$ is a lower semicontinuous function (recall Remark 3.3), thus for any $x \in M$ there exists $r_x > 0$ such that

$$\dim V_x = \min \{ \dim V_y \mid y \in B_{r_x}(x) \}.$$

Let us define

$$G_n := \{ x \in M \mid r_x \geq 1/n \} \quad \text{for every } n \in \mathbb{N}. \tag{3.12}$$

Observe that for any point $x \in M$ there exists $\bar{n} \in \mathbb{N}$ such that $x \in \bigcap_{n \geq \bar{n}} G_n$.

Proposition 3.9 *Let M be a smooth manifold. Let* d *be any distance on M that induces the manifold topology. Fix a generalised metric ρ over $M \times \mathbb{R}^d$. Then there exists a sequence $(F_n)_n$ of Finsler metrics over $M \times \mathbb{R}^d$ such that the following properties are satisfied:*

a) *$F_{n-1}(x, v) < F_n(x, v) < \rho(x, v)$ for every $n \in \mathbb{N}$, $x \in M$, and $v \in \mathbb{R}^d \setminus \{0\}$.*
b) *Given any $n \in \mathbb{N}$, it holds that*

$$F_n(x, v) \geq n \quad \text{for every } x \in G_n \text{ and } v \in V_x^\perp \cap \mathbb{S}^{d-1},$$

where $V_x := D(\rho_x)$ for all $x \in M$ and the set G_n is defined as in Eq. 3.12.
c) *For any $n \in \mathbb{N}$ there exists a countable set $S_n \subseteq M$ such that*

$$\left| F_n(z, v) - \rho(z, v) \right| \leq \frac{1}{n} \quad \text{for every } z \in S_n \text{ and } v \in V_z \cap \mathbb{S}^{d-1}.$$

d) *Given any $n \in \mathbb{N}$, $x \in G_n$, and $v \in \mathbb{R}^d$, there exists a point $z \in S_n \cap B_{1/n}(x)$ such that*

$$F_n(x, v) \geq F_n(z, v) \quad \text{and} \quad \mathsf{d}_H(V_z \cap \mathbb{S}^{d-1}, V_x \cap \mathbb{S}^{d-1}) < \frac{1}{n},$$

where the Hausdorff distance d_H is computed with respect to the norm $F_n(z, \cdot) + |\cdot|$.

Proof We recursively define the Finsler metrics $F_n \colon M \times \mathbb{R}^d \to [0, +\infty)$. Suppose to have already defined $F_0, \ldots, F_{n-1}$ for some $n \in \mathbb{N}$, where $F_0 := 0$. By using Lemma 3.6, Lemma 3.8, and the paracompactness of M, we can find a family $\{ (U_i^n, z_i^n, \mathbf{n}_i^n) : i \in \mathbb{N} \}$ such that:

i) $\{U_i^n\}_{i \in \mathbb{N}}$ is a locally finite, open covering of M, and $\mathrm{diam}(U_i^n) < 1/n$ for every $i \in \mathbb{N}$.
ii) $z_i^n \in U_i^n$ and $\dim V_{z_i^n} = \min\{\dim V_x : x \in U_i^n\}$ for every $i \in \mathbb{N}$.
iii) Given any $i \in \mathbb{N}$, we have that $\mathbf{n}_i^n$ is a smooth, strongly convex norm on $\mathbb{R}^d$ that satisfies

$$F_{n-1}(x, v) < \mathbf{n}_i^n(v) < \rho(x, v) \quad \text{for every } x \in U_i^n \text{ and } v \in \mathbb{R}^d \setminus \{0\}.$$

iv) $\left| \mathbf{n}_i^n(v) - \rho(z_i^n, v) \right| \leq 1/n$ for every $i \in \mathbb{N}$ and $v \in V_{z_i^n} \cap \mathbb{S}^{d-1}$.

v) Calling $D_i^n := \left\{ x \in U_i^n : \dim V_x = \dim V_{z_i^n} \right\}$ for all $i \in \mathbb{N}$, we have that $\mathbf{n}_i^n(v) \geq n$ for every $x \in D_i^n$ and $v \in V_x^\perp \cap \mathbb{S}^{d-1}$.

vi) Given any $i \in \mathbb{N}$, it holds that

$$\mathsf{d}_H(V_{z_i^n} \cap \mathbb{S}^{d-1}, V_x \cap \mathbb{S}^{d-1}) < \frac{1}{n} \quad \text{for every } x \in D_i^n,$$

where d_H is intended with respect to $\mathbf{n}_i^n + |\cdot|$.

Choose a partition of unity $\{\varphi_i^n : i \in \mathbb{N}\} \subseteq C^\infty(M)$ subordinated to $\{U_i^n : i \in \mathbb{N}\}$ such that $\varphi_i^n(z_i^n) = 1$ for every $i \in \mathbb{N}$. Let us define

$$F_n(x, v) := \sum_{i \in \mathbb{N}} \varphi_i^n(x)\, \mathbf{n}_i^n(v) \quad \text{for every } x \in M \text{ and } v \in \mathbb{R}^d.$$

Since each norm $\mathbf{n}_i^n$ is smooth and strongly convex, it can be readily checked that F_n is a Finsler metric over $M \times \mathbb{R}^d$. Let us then conclude by verifying that F_n satisfies the desired properties:

a) It follows from iii) that $F_{n-1}(x, v) < F_n(x, v) < \rho(x, v)$ for all $x \in M$ and $v \in \mathbb{R}^d \setminus \{0\}$.

b) Fix any point $x \in G_n$. We claim that for any $i \in \mathbb{N}$ with $x \in U_i^n$, it holds that $x \in D_i^n$. Indeed, we know that $r_x \geq 1/n$ by definition of G_n, whence it holds $U_i^n \subseteq B_{r_x}(x)$ by i) and accordingly $\dim V_x = \dim V_{z_i^n}$ by ii). This shows that $x \in D_i^n$, thus proving the above claim.

Fix $v \in V_x^\perp \cap \mathbb{S}^{d-1}$. Therefore, we deduce from the previous claim and v) that

$$\sum_{i \in \mathbb{N}} \varphi_i^n(x) = \sum_{\substack{i \in \mathbb{N}: \\ x \in D_i^n}} \varphi_i^n(x) = 1 \quad \text{and} \quad F_n(x, v) = \sum_{\substack{i \in \mathbb{N}: \\ x \in D_i^n}} \varphi_i^n(x)\, \mathbf{n}_i^n(v) \geq n.$$

Define $S_n := \{z_i^n : i \in \mathbb{N}\}$. Notice that $F_n(z_i^n, \cdot) = \mathbf{n}_i^n$ for any $i \in \mathbb{N}$, whence iv) gives c).

Fix $n \in \mathbb{N}$, $x \in G_n$, and $v \in \mathbb{R}^d$. Since the family $\{i \in \mathbb{N} : x \in U_i^n\}$ is finite, we can find $j \in \mathbb{N}$ such that $x \in U_j^n$ and $F_n(z_j^n, v) = \min\left\{ F_n(z_i^n, v) : i \in \mathbb{N}, x \in U_i^n \right\}$. Consequently,

$$F_n(x, v) = \sum_{\substack{i \in \mathbb{N}: \\ x \in U_i^n}} \varphi_i^n(x)\, \mathbf{n}_i^n(v) = \sum_{\substack{i \in \mathbb{N}: \\ x \in U_i^n}} \varphi_i^n(x)\, F_n(z_i^n, v) \geq F_n(z_j^n, v).$$

Moreover, as in the proof of item b) we deduce that $x \in D_j^n$. Therefore, we know from item vi) that $\mathsf{d}_H(V_{z_j^n} \cap \mathbb{S}^{d-1}, V_x \cap \mathbb{S}^{d-1}) < 1/n$, where d_H is taken with respect to $\mathbf{n}_j^n + |\cdot| = F_n(z_j^n, \cdot) + |\cdot|$. Notice also that $z_j^n \in B_{1/n}(x)$, as $\mathrm{diam}(U_j^n) < 1/n$ by i). This gives the statement. $\qquad\square$

Lemma 3.10 *Let* M, ρ, *and* $(F_n)_n$ *be as in Proposition 3.9. Then it holds that* $F_n(x, v) \nearrow \rho(x, v)$ *for every* $x \in M$ *and* $v \in \mathbb{R}^d$.

Proof It clearly suffices to prove that $F_n(x, v) \nearrow \rho(x, v)$ for any fixed $x \in M$ and $v \in \mathbb{S}^{d-1}$.

CASE 1. Assume $v \in V_x := D(\rho_x)$. We argue by contradiction: suppose there is $t > 0$ such that

$$F_n(x, v) \leq \rho(x, v) - t \quad \text{for every } n \in \mathbb{N}. \tag{3.13}$$

Fix any distance d on M that induces the manifold topology. Since the function ρ is lower semicontinuous by definition, we can find $r > 0$ such that

$$\rho(y, w) \geq \rho(x, v) - \frac{t}{2} \quad \text{for every } (y, w) \in \mathrm{M} \times \mathbb{R}^d \text{ with } \mathsf{d}(x, y), |v - w| < r. \quad (3.14)$$

Choose any $\bar{n} \in \mathbb{N}$ such that $1/\bar{n} < \min\{r, t/4\}$ and $x \in \bigcap_{n \geq \bar{n}} G_n$, with G_n defined as in Eq. 3.12. Fix any $n \geq \bar{n}$. Item d) of Proposition 3.9 grants the existence of a point $z \in S_n \cap B_{1/n}(x)$ and a vector $w_z \in V_z \cap \mathbb{S}^{d-1}$ such that $F_n(x, v) \geq F_n(z, v)$ and $F_n(z, v - w_z) + |v - w_z| \leq 1/n$. Moreover, item c) of Proposition 3.9 ensures that $F_n(z, w_z) \geq \rho(z, w_z) - 1/n$. All in all, we conclude that

$$F_n(x, v) \geq F_n(z, v) \geq F_n(z, w_z) - \frac{1}{n} \geq \rho(z, w_z) - \frac{2}{n} \overset{(3.14)}{\geq} \rho(x, v) - \frac{t}{2} - \frac{2}{n} > \rho(x, v) - t,$$

which is in contradiction with Eq. 3.13. This proves that $F_n(x, v) \nearrow \rho(x, v)$ in the case $v \in V_x$.

CASE 2. Assume $v \notin V_x$. Choose those elements $v' \in V_x$, $w \in V_x^\perp \cap \mathbb{S}^{d-1}$, and $\beta > 0$ for which $v = v' + \beta w$. Fix any $\bar{n} \in \mathbb{N}$ with $x \in \bigcap_{n \geq \bar{n}} G_n$. Then item b) of Proposition 3.9 yields $F_n(x, w) \geq n$ for all $n \geq \bar{n}$. Taking into account item a) of Proposition 3.9, this gives

$$F_n(x, v) \geq \beta F_n(x, w) - F_n(x, v') \geq \beta n - \rho(x, v') \overset{n}{\longrightarrow} +\infty = \rho(x, v).$$

Therefore, the statement is proven. $\qquad\square$

Theorem 3.11 (Approximation of generalised metrics) *Let M be a smooth manifold. Let ρ be a generalised metric on M. Then there exists a sequence $(F_n)_n$ of Finsler metrics on M such that*

$$F_n(x, v) \nearrow \rho(x, v) \quad \text{for every } x \in \mathrm{M} \text{ and } v \in \mathrm{T}_x\mathrm{M}. \quad (3.15)$$

Proof Let us denote $d := \dim \mathrm{M}$. Since M is paracompact (and second countable), we can find a locally finite, open covering $(\mathrm{M}_i)_{i \in \mathbb{N}}$ of M such that the tangent bundle TM admits a local trivialisation $\psi_i \colon \mathrm{TM}_i \to \mathrm{M}_i \times \mathbb{R}^d$ for every $i \in \mathbb{N}$. Fix any partition of unity $(\varphi_i)_i \subseteq C^\infty(\mathrm{M})$ subordinated to $(\mathrm{M}_i)_i$. Given any $i \in \mathbb{N}$, we can apply Proposition 3.9 and Lemma 3.10 to obtain a sequence $(\tilde{F}_n^i)_n$ of Finsler metrics over $\mathrm{M}_i \times \mathbb{R}^d$ such that $\tilde{F}_n^i(x, \tilde{v}) \nearrow (\rho \circ \psi_i^{-1})(x, \tilde{v})$ as $n \to \infty$ for every $x \in \mathrm{M}_i$ and $\tilde{v} \in \mathbb{R}^d$. Therefore, let us define $F_n \colon \mathrm{TM} \to [0, +\infty)$ as

$$F_n(x, v) := \sum_{i \in \mathbb{N}} \varphi_i(x) \, (\tilde{F}_n^i \circ \psi_i)(x, v) \quad \text{for every } x \in \mathrm{M} \text{ and } v \in \mathrm{T}_x\mathrm{M}.$$

It can be readily checked that $(F_n)_n$ is a sequence of Finsler metrics on M satisfying Eq. 3.15. $\qquad\square$

Remark 3.12 Under some additional assumptions, we can actually improve the statement of Theorem 3.11: if we further suppose that $\rho_x|_{D(\rho_x)}$ is a Hilbert norm for every $x \in \mathrm{M}$, then there exists a sequence $(g_n)_n$ of Riemannian metrics on M such that

$$\sqrt{(g_n)_x(v, v)} \nearrow \rho(x, v) \quad \text{for every } x \in \mathrm{M} \text{ and } v \in \mathrm{T}_x\mathrm{M}.$$

This fact can be proven by slightly modifying (actually, simplifying) the arguments we discussed in the present section. More precisely, it is sufficient to notice that in this case the norm n defined in Eq. 3.2 is induced by a scalar product, and to omit STEP 3 from the

proof of Lemma 3.8 (just defining $\mathbf{n} := \mathbf{n}''$). Another proof of this result can be found in [19, Proof of Corollary 1.5].

4 Sub-Finsler Manifolds

4.1 Definitions and Main Properties

In this subsection we recall the notion of sub-Finsler manifold and its main properties. The following material is taken from [20, Section 3.1], see also [6].

Given a smooth manifold M, we denote by $\mathrm{Vec}(M)$ the space of all smooth vector fields on M. Moreover, we define the map $\mathsf{Der} \colon C([0, 1], M) \times [0, 1] \to \mathrm{TM}$ as

$$\mathsf{Der}(\gamma, t) := \begin{cases} (\gamma_t, \dot{\gamma}_t) & \text{if } \dot{\gamma}_t \in \mathrm{T}_{\gamma_t} M \text{ exists,} \\ (\gamma_t, 0) & \text{otherwise.} \end{cases} \tag{4.1}$$

It is well-known that Der is a Borel map. For any $v, w \in \mathrm{Vec}(M)$, we denote by $[v, w] \in \mathrm{Vec}(M)$ the Lie brackets of v and w. Given any subset $\mathcal{F}$ of $\mathrm{Vec}(M)$, we define the space $\mathrm{Lie}(\mathcal{F}) \subseteq \mathrm{Vec}(M)$ as the Lie algebra generated by the family $\mathcal{F}$, i.e.,

$$\mathrm{Lie}(\mathcal{F}) := \mathrm{span}\Big\{[v_1, \ldots, [v_{j-1}, v_j] \ldots] \ \Big| \ j \in \mathbb{N}, \ v_1, \ldots, v_j \in \mathcal{F}\Big\}.$$

We set $\mathrm{Lie}_x(\mathcal{F}) := \big\{v(x) \ : \ v \in \mathrm{Lie}(\mathcal{F})\big\} \leq \mathrm{T}_x M$ for every $x \in M$. We say that the family $\mathcal{F}$ satisfies the *Hörmander condition* provided $\mathrm{Lie}_x(\mathcal{F}) = \mathrm{T}_x M$ holds for every $x \in M$.

Definition 4.1 (Sub-Finsler manifold) Let M be a smooth manifold. Then a triple $(\mathrm{E}, \sigma, \psi)$ is said to be a *sub-Finsler structure* on M provided the following properties hold:

i)   E is a smooth vector bundle over M,
ii) σ is a continuous metric over E,
iii) $\psi \colon \mathrm{E} \to \mathrm{TM}$ is a morphism of smooth vector bundles such that the family $\mathcal{D}$ of *smooth horizontal vector fields* on M, which is defined as

$$\mathcal{D} := \big\{\psi \circ u \mid u \text{ is a smooth section of E}\big\} \subseteq \mathrm{Vec}(M),$$

satisfies the Hörmander condition.

The quadruple (M, E, σ, ψ) is said to be a *generalised sub-Finsler manifold* (or just a *sub-Finsler manifold*, for brevity). If $(\mathrm{E}_x, \sigma|_{\mathrm{E}_x})$ is a Hilbert space for every $x \in M$ and the family of squared norms $(\sigma|_{\mathrm{E}_x})^2$ smoothly depends on x, then (M, E, σ, ψ) is called a *generalised sub-Riemannian manifold* (or just a *sub-Riemannian manifold*).

The family $\mathcal{D}$ of smooth horizontal vector fields is a finitely-generated module over $C^\infty(M)$. The continuous distribution $\{\mathcal{D}_x\}_{x \in M}$ associated with (M, E, σ, ψ) is defined as

$$\mathcal{D}_x := \big\{v(x) \mid v \in \mathcal{D}\big\} \leq \mathrm{T}_x M \quad \text{for every } x \in M.$$

We say that $r(x) := \dim \mathcal{D}_x \leq \dim M$ is the *rank* of the sub-Finsler structure $(\mathrm{E}, \sigma, \psi)$ at $x \in M$.

Given any point $x \in M$ and any vector $v \in \mathcal{D}_x$, we define the quantity $\|v\|_x$ as

$$\|v\|_x := \inf\big\{\sigma(u) \mid u \in \mathrm{E}_x, \ (x, v) = \psi(u)\big\}. \tag{4.2}$$

Therefore, it holds that $\|\cdot\|_x$ is a norm on $\mathcal{D}_x$. Furthermore, if (E, σ, ψ) is a sub-Riemannian structure on M, then each norm $\|\cdot\|_x$ is induced by some scalar product $\langle \cdot, \cdot \rangle_x$.

Definition 4.2 (Horizontal curve) Let (M, E, σ, ψ) be a sub-Finsler manifold. Let $\gamma \colon [0, 1] \to M$ be a continuous curve such that for any $\bar{t} \in [0, 1]$ there exist $\delta > 0$ and a chart (U, ϕ) of M such that $\gamma(I) \subseteq U$ and $\phi \circ \gamma|_I \colon I \to \mathbb{R}^{\dim M}$ is Lipschitz, where we set $I := (\bar{t} - \delta, \bar{t} + \delta) \cap [0, 1]$. Then the curve γ is said to be *horizontal* provided there is an L^∞-section u of the pullback bundle γ^*E – i.e., a map $[0, 1] \ni t \mapsto u(t) \in E_{\gamma_t}$ that is measurable and essentially bounded – such that

$$(\gamma_t, \dot{\gamma}_t) = \psi\big(u(t)\big) \quad \text{holds for a.e. } t \in [0, 1].$$

The *sub-Finsler length* of the curve γ is defined as $\ell_{CC}(\gamma) := \int_0^1 \|\dot{\gamma}_t\|_{\gamma_t} \, dt$.

Definition 4.3 (Carnot–Carathéodory distance) Let (M, E, σ, ψ) be a sub-Finsler manifold. Fix any $x, y \in M$. Then we define the *Carnot–Carathéodory distance* between x and y as

$$\mathsf{d}_{CC}(x, y) := \inf \Big\{ \ell_{CC}(\gamma) \, \Big| \, \gamma \text{ is a horizontal curve in M such that } \gamma_0 = x \text{ and } \gamma_1 = y \Big\}. \tag{4.3}$$

Theorem 4.4 (Chow–Rashevskii) *Let (M, E, σ, ψ) be a sub-Finsler manifold. Then d_{CC} is a distance on M that induces the manifold topology.*

The metric space (M, d_{CC}) is complete if and only if $\bar{B}_r(x)$ is compact for all $x \in M$ and $r > 0$.

Proposition 4.5 *Let (M, E, σ, ψ) be a sub-Finsler manifold. Let $\gamma \colon [0, 1] \to M$ be a curve in M. Then γ is d_{CC}-Lipschitz if and only if it is horizontal. Moreover, in such case it holds that*

$$\|\dot{\gamma}_t\|_{\gamma_t} = \lim_{h \to 0} \frac{\mathsf{d}_{CC}(\gamma_{t+h}, \gamma_t)}{|h|} \quad \text{for a.e. } t \in [0, 1].$$

4.2 Structure of the Horizontal Bundle

Let (M, E, σ, ψ) be a sub-Finsler manifold, whose associated distribution is denoted by $\{\mathcal{D}_x\}_{x \in M}$. Then we define the *horizontal bundle* HM as

$$HM := \bigsqcup_{x \in M} \mathcal{D}_x.$$

Moreover, we define the function $\rho \colon TM \to [0, +\infty]$ as

$$\rho(x, v) := \begin{cases} \|v\|_x & \text{if } (x, v) \in HM, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.4}$$

Lemma 4.6 *Let (M, E, σ, ψ) be a sub-Finsler manifold. Then the function ρ defined in Eq. 4.4 is a generalised metric on M. In particular, the horizontal bundle HM is a Borel subset of TM.*

Proof First of all, observe that $x \mapsto D(\rho_x) = \mathcal{D}_x$ is a continuous distribution by Theorem 3.2. With this said, we only have to prove that the function ρ is lower semicontinuous. To

this aim, let us fix a sequence $\{(x_n, v_n)\}_{n \in \mathbb{N} \cup \{\infty\}} \subseteq \mathrm{TM}$ such that $(x_n, v_n) \to (x_\infty, v_\infty)$. We claim that

$$\rho(x_\infty, v_\infty) \leq \varliminf_{n \to \infty} \rho(x_n, v_n). \tag{4.5}$$

Without loss of generality, we can assume that $\varliminf_n \rho(x_n, v_n) < +\infty$. For any $n \in \mathbb{N}$ choose an element $u_n \in \mathrm{E}_{x_n}$ such that $\psi(u_n) = (x_n, v_n)$ and $\sigma(u_n) \leq \rho(x_n, v_n) + 1/n$. Moreover, pick a subsequence $(n_k)_k$ with $\lim_k \rho(x_{n_k}, v_{n_k}) = \varliminf_n \rho(x_n, v_n)$. Therefore, it holds that $\{\sigma(u_{n_k})\}_{k \in \mathbb{N}}$ is bounded, thus there exists $u_\infty \in \mathrm{E}_{x_\infty}$ such that (possibly passing to a not relabelled subsequence) we have $u_{n_k} \to u_\infty$. Note that $\psi(u_\infty) = \lim_k \psi(u_{n_k}) = \lim_k (x_{n_k}, v_{n_k}) = (x_\infty, v_\infty)$ by continuity of ψ. Consequently, by using the continuity of σ and the definition of ρ_{x_∞} we conclude that

$$\rho(x_\infty, v_\infty) \leq \sigma(u_\infty) = \lim_{k \to \infty} \sigma(u_{n_k}) \leq \lim_{k \to \infty} \rho(x_{n_k}, v_{n_k}) + \frac{1}{n_k} = \varliminf_{n \to \infty} \rho(x_n, v_n),$$

which proves the claim Eq. 4.5. Hence, the statement is finally achieved. $\square$

A vector field $v \colon \mathrm{M} \to \mathrm{TM}$ is said to be a *section* of HM provided $v(x) \in \mathcal{D}_x$ for every $x \in \mathrm{M}$. We say that a section v of HM is *Borel* provided it is Borel measurable as a map from M to TM.

It immediately follows from Lemma 4.6 that

$$\mathrm{M} \ni x \longmapsto \big\| v(x) \big\|_x \in \mathbb{R} \text{ is a Borel function,} \quad \text{for every Borel section } v \text{ of HM.}$$

The space of Borel sections of HM is a vector space with respect to the usual pointwise operations.

Definition 4.7 (The space $L^2(\mathrm{HM}; \mu)$) Let $(\mathrm{M}, \mathrm{E}, \sigma, \psi)$ be a sub-Finsler manifold. Let μ be a non-negative Borel measure on $(\mathrm{M}, \mathsf{d}_{CC})$. Then we define the space $L^2(\mathrm{HM}; \mu)$ as the set of (equivalence classes up to μ-a.e. equality of) all Borel sections v of the horizontal bundle HM such that $\mathrm{M} \ni x \mapsto \big\| v(x) \big\|_x \in \mathbb{R}$ belongs to $L^2(\mu)$. The space $L^2(\mathrm{HM}; \mu)$ is an $L^\infty(\mu)$-module with respect to the natural pointwise operations, thus in particular it is a vector space.

Remark 4.8 (Pointwise parallelogram identity) Let $(\mathrm{M}, \mathrm{E}, \sigma, \psi)$ be a sub-Riemannian manifold. Given that each space $(\mathcal{D}_x, \|\cdot\|_x)$ is Hilbert, we readily deduce that

$$\big\| v(x) + w(x) \big\|_x^2 + \big\| v(x) - w(x) \big\|_x^2 = 2 \big\| v(x) \big\|_x^2 + 2 \big\| w(x) \big\|_x^2 \quad \text{holds for } \mu\text{-a.e. } x \in \mathrm{M},$$

for every $v, w \in L^2(\mathrm{HM}; \mu)$.

Given any smooth function $f \in C^\infty(\mathrm{M})$, we denote by $\mathrm{d}f$ its differential, which is a smooth section of the cotangent bundle $\mathrm{T}^*\mathrm{M}$. Then the *horizontal differential* $\mathrm{d}_\mathrm{H} f$ of f is defined as

$$\mathrm{d}_\mathrm{H} f(x) := \mathrm{d}_x f|_{\mathcal{D}_x} \in \mathcal{D}_x^* \quad \text{for every } x \in \mathrm{M}. \tag{4.6}$$

Lemma 4.9 *Let $(\mathrm{M}, \mathrm{E}, \sigma, \psi)$ be a given sub-Finsler manifold. Then there exists a countable family of functions $\mathscr{C} \subseteq C_c^\infty(\mathrm{M})$ such that $\{\mathrm{d}_\mathrm{H} f(x) : f \in \mathscr{C}\}$ is dense in $\mathcal{D}_x^*$ for every $x \in \mathrm{M}$.*

Proof Call $n := \dim \mathrm{M}$. By Lindelöf lemma we know that there exists an open covering $(\Omega_j)_{j \in \mathbb{N}}$ of M with the following property: for every $j \in \mathbb{N}$ there exist some functions $f_1^j, \ldots, f_n^j \in C_c^\infty(\mathrm{M})$ such that $\mathrm{d}_x f_1^j, \ldots, \mathrm{d}_x f_n^j$ is a basis of $\mathrm{T}_x^*\mathrm{M}$ for every $x \in \Omega_j$.

Consequently, $d_H f_1^j(x), \ldots, d_H f_n^j(x)$ generate $\mathcal{D}_x^*$ for every $x \in \Omega_j$. Calling V_j the $\mathbb{Q}$-linear subspace of $C_c^\infty(M)$ generated by $f_1^j, \ldots, f_n^j$, one clearly has that $\left\{ d_H f(x) : f \in V_j \right\}$ is dense in $\mathcal{D}_x^*$ for every $x \in \Omega_j$. Therefore, the countable family of functions $\mathscr{C} := \bigcup_{j \in \mathbb{N}} V_j$ fulfills the required properties. $\qquad \square$

Lemma 4.10 *Let* (M, E, σ, ψ) *be a sub-Finsler manifold. Let* $f \in C_c^1(M)$. *Then* $f \in LIP(M)$ *and*

$$\left\| d_H f(x) \right\|_x^* \leq \mathrm{lip}(f)(x) \quad \text{for every } x \in M. \tag{4.7}$$

Proof Lipschitzianity of f can be proven by arguing, e.g., as in [1, Lemma 3.16]. To show Eq. 4.7, let $v \in \mathcal{D}_x$ and $\varepsilon > 0$ be fixed. We know from Eq. 4.2 that there exists $u \in E_x$ such that $(x, v) = \psi(u)$ and $\sigma(u) \leq \|v\|_x + \varepsilon$. Choose a smooth section η of E such that $\eta(x) = u$. Since $\psi \circ \eta$ is a smooth vector field on M, there exists a smooth solution $\gamma : [0, \delta'] \to M$ to the ODE

$$\begin{cases} \dot{\gamma}_t = (\psi \circ \eta)(\gamma_t) & \text{for every } t \in [0, \delta'], \\ \gamma_0 = x. \end{cases}$$

Being $\eta \circ \gamma$ continuous, we can find $\delta \in (0, \delta')$ such that $\sigma(\eta(\gamma_t)) \leq \sigma(u) + \varepsilon$ for every $t \in [0, \delta]$. Moreover, again by Eq. 4.2 we have that $\|\dot{\gamma}_t\|_{\gamma_t} \leq \sigma(\eta(\gamma_t))$ for all $t \in [0, \delta]$. Combining the previous estimates we get that $\|\dot{\gamma}_t\|_{\gamma_t} \leq \|v\|_x + 2\varepsilon$ for every $t \in [0, \delta]$. Therefore, we conclude that

$$d_H f(x)[v] = d_x f(\dot{\gamma}_0) = \lim_{t \searrow 0} \frac{f(\gamma_t) - f(x)}{t} \leq \mathrm{lip}(f)(x) \lim_{t \searrow 0} \frac{d_{CC}(\gamma_t, \gamma_0)}{t}$$

$$\overset{(4.3)}{\leq} \mathrm{lip}(f)(x) \lim_{t \searrow 0} \frac{1}{t} \int_0^t \|\dot{\gamma}_s\|_{\gamma_s} \, ds \leq \mathrm{lip}(f)(x) \left(\|v\|_x + 2\varepsilon \right).$$

Letting $\varepsilon \searrow 0$ we see that $d_H f(x)[v] \leq \mathrm{lip}(f)(x) \|v\|_x$ for all $v \in \mathcal{D}_x$, whence Eq. 4.7 follows. $\qquad \square$

5 Main Result: Infinitesimal Hilbertianity of Sub-Riemannian Manifolds

5.1 Derivations on Weighted Sub-Finsler Manifolds

The aim of this subsection is to provide an alternative to the representation formula Eq. 2.1 for the pointwise norm of a derivation (with divergence) over a weighted sub-Finsler manifold M. We would like to express the pointwise norm of a derivation $\boldsymbol{b}$ as the essential supremum of the functions $\boldsymbol{b}(f)$, where f varies in a countable family of 1-Lipschitz smooth functions. Given that the distance functions $d_{CC}(\cdot, \bar{x})$ from fixed points $\bar{x} \in M$ are not smooth (thus in particular not almost everywhere differentiable with respect to an arbitrary measure on M), a new representation formula is needed.

The following result states that any Carnot–Carathéodory distance can be (monotonically) approximated by distances associated to suitable Finsler metrics. A word on notation: given a Finsler metric F on a manifold M, we denote by d_F the induced distance on M.

Theorem 5.1 *Let* (M, E, σ, ψ) *be a sub-Finsler manifold. Then there exists a sequence* $(F_n)_n$ *of Finsler metrics on* M *such that* $d_{F_n}(x, y) \nearrow d_{CC}(x, y)$ *holds for every* $x, y \in M$.

Proof Define ρ as in Eq. 4.4 and consider a sequence $(F_n)_n$ of approximating Finsler metrics as in Theorem 3.11. Let $x, y \in M$ be fixed. Let $\gamma : [0, 1] \to M$ be a curve joining x and y that is Lipschitz when read in charts (i.e., as in Definition 4.2). Calling ℓ_{F_n} the length

functional associated to F_n, it holds that $\ell_{F_n}(\gamma) \leq \ell_{F_{n+1}}(\gamma) \leq \ell_{CC}(\gamma)$ for every $n \in \mathbb{N}$, thus by taking the infimum over γ we deduce that $\mathsf{d}_{F_n}(x, y) \leq \mathsf{d}_{F_{n+1}}(x, y) \leq \mathsf{d}_{CC}(x, y)$. Given any $n \in \mathbb{N}$, by definition of d_{F_n} we find a constant-speed Lipschitz curve $\gamma^n \colon [0, 1] \to \mathrm{M}$, where the target is endowed with d_{F_n}, such that

$$\ell_{F_n}(\gamma^n) \leq \mathsf{d}_{F_n}(x, y) + \frac{1}{n}. \tag{5.1}$$

Fix $n \in \mathbb{N}$. The above considerations yield

$$\ell_{F_n}(\gamma^i) \leq \ell_{F_i}(\gamma^i) \overset{(5.1)}{\leq} \mathsf{d}_{F_i}(x, y) + \frac{1}{i} \leq \mathsf{d}_{CC}(x, y) + 1 \quad \text{for every } i \geq n.$$

This shows that $(\gamma^i)_{i \geq n}$ is an equiLipschitz family of curves (with respect to d_{F_n}). By combining Arzelà–Ascoli theorem with a diagonalisation argument, we thus obtain a curve $\gamma \colon [0, 1] \to \mathrm{M}$, which is Lipschitz with respect to each distance d_{F_n}, such that (up to a not relabelled subsequence)

$$\lim_{i \to \infty} \sup_{t \in [0,1]} \mathsf{d}_{F_n}(\gamma_t^i, \gamma_t) = 0 \quad \text{for every } n \in \mathbb{N}. \tag{5.2}$$

Since ℓ_{F_n} is lower semicontinuous under uniform convergence of curves, we deduce from Eq. 5.2 that

$$\ell_{F_n}(\gamma) \leq \varliminf_{i \to \infty} \ell_{F_n}(\gamma^i) \leq \varliminf_{i \to \infty} \ell_{F_i}(\gamma^i) \overset{(5.3)}{\leq} \lim_{i \to \infty} \mathsf{d}_{F_i}(x, y) \quad \text{for every } n \in \mathbb{N}. \tag{5.3}$$

Therefore, by using the monotone convergence theorem we obtain that

$$\int_0^1 \rho(\gamma_t, \dot\gamma_t) \, dt = \lim_{n \to \infty} \int_0^1 F_n(\gamma_t, \dot\gamma_t) \, dt = \lim_{n \to \infty} \ell_{F_n}(\gamma) \overset{(5.3)}{\leq} \lim_{i \to \infty} \mathsf{d}_{F_i}(x, y) \leq \mathsf{d}_{CC}(x, y) < +\infty,$$

which implies that the curve γ is horizontal and satisfies $\ell_{CC}(\gamma) = \mathsf{d}_{CC}(x, y) = \lim_n \mathsf{d}_{F_n}(x, y)$. $\qquad\square$

Although not strictly needed for our purposes, let us point out an immediate well-known consequence (already proven in [19]) of the previous theorem.

Corollary 5.2 *Let* $(\mathrm{M}, \mathrm{E}, \sigma, \psi)$ *be a sub-Riemannian manifold. Then there exists a sequence* $(g_n)_n$ *of Riemannian metrics on* M *such that* $\mathsf{d}_{g_n}(x, y) \nearrow \mathsf{d}_{CC}(x, y)$ *holds for every* $x, y \in \mathrm{M}$.

Proof It follows from Theorem 5.1 by taking Remark 3.12 into account. $\qquad\square$

We shall also need the ensuing approximation result for real-valued Lipschitz functions that are defined on a Finsler manifold.

Lemma 5.3 *Let* (M, F) *be a Finsler manifold. Let* $f \in LIP_{bs}(\mathrm{M})$ *be given. Then there exists a sequence* $(f_n)_n \subseteq C_{bs}^1(\mathrm{M})$ *with* $\sup_n Lip(f_n) \leq Lip(f)$ *such that* $f_n \to f$ *uniformly on* M.

Proof We call $C := \max_{\mathrm{M}} |f|$. We know (for instance, from [21, Theorem 2.6]) that for any $n \in \mathbb{N}$ there exists a function $g_n \in C_{bs}^1(\mathrm{M})$ such that $Lip(g_n) \leq Lip(f) + 1/n$ and

$|g_n - f| \leq 1/n$ on M. Set $c_n := \mathrm{Lip}(f)/\big(\mathrm{Lip}(f) + 1/n\big)$ and $f_n := c_n\, g_n$. Therefore $f_n \in C^1_{bs}(\mathrm{M})$, $\mathrm{Lip}(f_n) \leq \mathrm{Lip}(f)$, and

$$|f_n - f| \leq |f_n - g_n| + |g_n - f| = (1 - c_n)|g_n| + |g_n - f| \leq \frac{|f| + 1/n}{n\,\mathrm{Lip}(f) + 1} + \frac{1}{n} \leq \frac{C + 1}{n\,\mathrm{Lip}(f) + 1} + \frac{1}{n},$$

thus accordingly $f_n \to f$ uniformly on M, as required. $\qquad\square$

For a complete study of limits of sub-Finsler structures in the Lipschitz category we point out the recent preprint [6].

We are now in a position to prove a representation formula for the pointwise norm of derivations on weighted sub-Finsler manifolds, by combining the above two results with Proposition 2.3.

Theorem 5.4 *Let* $(\mathrm{M}, \mathrm{E}, \sigma, \psi)$ *be a sub-Finsler manifold such that* d_{CC} *is complete. Let* $\mu \geq 0$ *be a finite Borel measure on* $(\mathrm{M}, \mathrm{d}_{CC})$. *Then there exists a countable family* $\mathscr{F} \subseteq C^1_c(\mathrm{M}) \cap LIP(\mathrm{M})$ *such that* $\mathrm{Lip}(f) \leq 1$ *for every* $f \in \mathscr{F}$ *and*

$$|\boldsymbol{b}| = \operatorname*{ess\,sup}_{f \in \mathscr{F}} \boldsymbol{b}(f) \quad \mu\text{-a.e.} \quad \text{for every } \boldsymbol{b} \in Der^{2,2}(\mathrm{M}; \mu).$$

Proof Fix a dense sequence $(x_k)_k \subseteq \mathrm{M}$. Theorem 5.1 grants the existence of a sequence $(F_i)_i$ of Finsler metrics on M such that $\mathrm{d}_{F_i} \nearrow \mathrm{d}_{CC}$ pointwise on $\mathrm{M} \times \mathrm{M}$. Choose a family $\{\eta_{jk}\}_{j,k \in \mathbb{N}}$ of cut-off functions with these properties: given $j, k \in \mathbb{N}$, we have that $\eta_{jk} \colon \mathrm{M} \to [0, 1 - 1/j]$ is a boundedly-supported Lipschitz function (with respect to d_{F_1}) such that $\eta_{jk} = 1 - 1/j$ on $B^{\mathrm{d}_{CC}}_j(x_k)$ and $\mathrm{Lip}^{\mathrm{d}_{F_1}}(\eta_{jk}) \leq 1/j^2$. Observe that for any $i, j, k \in \mathbb{N}$ it holds that

$$\mathrm{Lip}^{\mathrm{d}_{F_i}}\big((\mathrm{d}_{F_i}(\cdot, x_k) \wedge j)\,\eta_{jk}\big) \leq \mathrm{Lip}^{\mathrm{d}_{F_i}}\big(\mathrm{d}_{F_i}(\cdot, x_k) \wedge j\big)\max_{\mathrm{M}}|\eta_{jk}| + \mathrm{Lip}^{\mathrm{d}_{F_i}}(\eta_{jk})\max_{\mathrm{M}}\big|\mathrm{d}_{F_i}(\cdot, x_k) \wedge j\big|$$

$$\leq 1 - \frac{1}{j} + j\,\mathrm{Lip}^{\mathrm{d}_{F_1}}(\eta_{jk}) \leq 1.$$

Therefore, Lemma 5.3 guarantees the existence of a function $f_{ijk} \in C^1_c(\mathrm{M}) \cap \mathrm{LIP}^{\mathrm{d}_{F_i}}(\mathrm{M})$ that satisfies $\mathrm{Lip}^{\mathrm{d}_{F_i}}(f_{ijk}) \leq 1$ and

$$\big|(\mathrm{d}_{F_i}(x, x_k) \wedge j)\,\eta_{jk}(x) - f_{ijk}(x)\big| \leq \frac{1}{i} \quad \text{for every } x \in \mathrm{M}.$$

Define $\mathscr{F} := \{f_{ijk} \; : \; i, j, k \in \mathbb{N}\}$. Note that $\mathscr{F} \subseteq C^1_c(\mathrm{M}) \cap \mathrm{LIP}^{\mathrm{d}_{CC}}(\mathrm{M})$ and $\sup_{f \in \mathscr{F}} \mathrm{Lip}^{\mathrm{d}_{CC}}(f) \leq 1$, as $\mathrm{Lip}^{\mathrm{d}_{CC}}(f_{ijk}) \leq \mathrm{Lip}^{\mathrm{d}_{F_i}}(f_{ijk}) \leq 1$ for all $i, j, k \in \mathbb{N}$. Now let us call

$$\mathsf{n}(\boldsymbol{b}) := \operatorname*{ess\,sup}_{f \in \mathscr{F}} \boldsymbol{b}(f) \quad \text{for every } \boldsymbol{b} \in \mathrm{Der}^{2,2}(\mathrm{M}; \mu). \tag{5.4}$$

Since $\boldsymbol{b}(f) \leq |\boldsymbol{b}|\,\mathrm{Lip}^{\mathrm{d}_{CC}}(f) \leq |\boldsymbol{b}|$ holds μ-a.e. for all $f \in \mathscr{F}$, we deduce that $\mathsf{n}(\boldsymbol{b}) \leq |\boldsymbol{b}|$ holds μ-a.e. as well. To prove the converse inequality, fix $\varepsilon > 0$. Proposition 2.3 grants the existence of a Borel partition $(A_{jk})_{j,k}$ of M such that $\sum_{j,k} \chi_{A_{jk}} \boldsymbol{b}\big((\mathrm{d}_{CC}(\cdot, x_k) \wedge j)\,\eta_{jk}\big) \geq |\boldsymbol{b}| - \varepsilon$ in the μ-a.e. sense. Fix $j, k \in \mathbb{N}$ and choose a sequence $(\varphi_n)_n \subseteq \mathrm{LIP}^{\mathrm{d}_{CC}}_{bs}(\mathrm{M})$ with $\varphi_n \geq 0$ converging to $\chi_{A_{jk}}$ strongly in $L^2(\mu)$. Given that $\lim_i f_{ijk}(x) = (\mathrm{d}_{CC}(x, x_k) \wedge j)\,\eta_{jk}(x)$ for every $x \in \mathrm{M}$, Lemma 2.2 yields

$$\int \varphi_n\, \boldsymbol{b}\big((\mathrm{d}_{CC}(\cdot, x_k) \wedge j)\,\eta_{jk}\big)\,\mathrm{d}\mu = \lim_{i \to \infty} \int \varphi_n\, \boldsymbol{b}(f_{ijk})\,\mathrm{d}\mu \overset{(5.4)}{\leq} \int \varphi_n\, \mathsf{n}(\boldsymbol{b})\,\mathrm{d}\mu \quad \text{for every } n \in \mathbb{N},$$

thus by letting $n \to \infty$ we deduce that

$$\int_{A_{jk}} |\boldsymbol{b}| \, \mathrm{d}\mu - \varepsilon \, \mu(A_{jk}) \leq \int_{A_{jk}} \boldsymbol{b}\big((\mathrm{d}_{\mathrm{CC}}(\cdot, x_k) \wedge j) \, \eta_{jk}\big) \, \mathrm{d}\mu \leq \int_{A_{jk}} \mathsf{n}(\boldsymbol{b}) \, \mathrm{d}\mu.$$

By summing over $j, k \in \mathbb{N}$ we get that $\int |\boldsymbol{b}| \, \mathrm{d}\mu - \varepsilon \, \mu(\mathrm{M}) \leq \int \mathsf{n}(\boldsymbol{b}) \, \mathrm{d}\mu$. By letting $\varepsilon \searrow 0$ we finally conclude that $\int |\boldsymbol{b}| \, \mathrm{d}\mu \leq \int \mathsf{n}(\boldsymbol{b}) \, \mathrm{d}\mu$, which forces the μ-a.e. equality $|\boldsymbol{b}| = \mathsf{n}(\boldsymbol{b})$, as desired. $\qquad\square$

5.2 Embedding Theorem and its Consequences

This subsection is devoted to our main result, namely Theorem 1.1, which states that the space of derivations $\mathrm{Der}^{2,2}(\mathrm{M}; \mu)$ associated with a weighted sub-Finsler manifold M can be isometrically embedded into the space $L^2(\mathrm{HM}; \mu)$ of all 'geometric' 2-integrable sections of the horizontal bundle HM. For the reader's convenience, we also recall here the statement.

Theorem 5.5 (Embedding theorem) *Let* $(\mathrm{M}, \mathrm{E}, \sigma, \psi)$ *be a sub-Finsler manifold with* d_{CC} *complete. Let* μ *be a finite, non-negative Borel measure on* $(\mathrm{M}, \mathrm{d}_{\mathrm{CC}})$. *Then there exists a unique linear operator* $\mathrm{I} \colon \mathrm{Der}^{2,2}(\mathrm{M}; \mu) \to L^2(\mathrm{HM}; \mu)$ *such that*

$$\mathrm{d}_{\mathrm{H}} f(x)\big[\mathrm{I}(\boldsymbol{b})(x)\big] = \boldsymbol{b}(f)(x) \quad \textit{holds for } \mu\textit{-a.e. } x \in \mathrm{M}, \tag{5.5}$$

for every $\boldsymbol{b} \in \mathrm{Der}^{2,2}(\mathrm{M}; \mu)$ *and* $f \in C_c^1(\mathrm{M}) \cap \mathrm{LIP}(\mathrm{M})$. *Moreover, the operator* I *satisfies*

$$\big\|\mathrm{I}(\boldsymbol{b})(x)\big\|_x = |\boldsymbol{b}|(x) \quad \textit{for } \mu\textit{-a.e. } x \in \mathrm{M}, \tag{5.6}$$

for every $\boldsymbol{b} \in \mathrm{Der}^{2,2}(\mathrm{M}; \mu)$.

Proof We divide the proof into several steps:

BOREL REGULARITY. We aim to prove that any section $\mathrm{I}(\boldsymbol{b})$ of HM satisfying Eq. 5.5 is (equivalent to) a Borel section. Given any $\bar{x} \in \mathrm{M}$, we can find an open neighbourhood Ω of $\bar{x}$ and some functions $f_1, \ldots, f_n \in C_c^\infty(\mathrm{M})$ such that $\mathrm{d}_x f_1, \ldots, \mathrm{d}_x f_n$ is a basis of $\mathrm{T}_x^* \mathrm{M}$ for all $x \in \Omega$. Since each function $\Omega \ni x \mapsto \mathrm{d}_x f_i \big[\mathrm{I}(\boldsymbol{b})(x)\big]$ is equivalent to a Borel function by Eqs. 4.6 and 5.5, we deduce that $\mathrm{I}(\boldsymbol{b})$ is equivalent to a Borel section of HM on Ω. By Lindelöf lemma we can cover M with countably many sets Ω with this property, whence $\mathrm{I}(\boldsymbol{b})$ is equivalent to a Borel section of HM.

INTEGRABILITY. The property Eq. 5.6 ensures that each Borel section $\mathrm{I}(\boldsymbol{b})$ belongs to $L^2(\mathrm{HM}; \mu)$, since $|\boldsymbol{b}| \in L^2(\mu)$ by assumption.

UNIQUENESS. Fix $\boldsymbol{b} \in \mathrm{Der}^{2,2}(\mathrm{M}; \mu)$ and pick $\mathscr{C} \subseteq C_c^1(\mathrm{M}) \cap \mathrm{LIP}(\mathrm{M})$ as in Lemma 4.9. Then by writing Eq. 5.5 for every function $f \in \mathscr{C}$ we deduce that the element $\mathrm{I}(\boldsymbol{b})(x) \in \mathcal{D}_x$ is uniquely determined for μ-a.e. $x \in \mathrm{M}$, thus the operator I is unique.

LINEARITY. Let $\boldsymbol{b}, \boldsymbol{b}' \in \mathrm{Der}^{2,2}(\mathrm{M}; \mu)$ and $\lambda, \lambda' \in \mathbb{R}$ be given. Then Eq. 5.5 ensures that

$$\begin{aligned}
\mathrm{d}_{\mathrm{H}} f(x)\big[\lambda \, \mathrm{I}(\boldsymbol{b})(x) + \lambda' \, \mathrm{I}(\boldsymbol{b}')(x)\big] &= \lambda \, \mathrm{d}_{\mathrm{H}} f(x)\big[\mathrm{I}(\boldsymbol{b})(x)\big] + \lambda' \, \mathrm{d}_{\mathrm{H}} f(x)\big[\mathrm{I}(\boldsymbol{b}')(x)\big] \\
&= \lambda \, \boldsymbol{b}(f)(x) + \lambda' \, \boldsymbol{b}'(f)(x) = (\lambda \, \boldsymbol{b} + \lambda' \, \boldsymbol{b}')(f)(x)
\end{aligned}$$

for every $f \in C_c^1(\mathrm{M}) \cap \mathrm{LIP}(\mathrm{M})$ and μ-a.e. $x \in \mathrm{M}$. By uniqueness, we conclude that I is linear.

EXISTENCE. Let $\boldsymbol{b} \in \mathrm{Der}^{2,2}(\mathrm{M}; \mu)$ be fixed. Consider its associated measure $\boldsymbol{\pi}$ as in Theorem 2.8. Define the measure $\hat{\boldsymbol{\pi}} := \boldsymbol{\pi} \otimes \mathcal{L}_1$ on $C\big([0, 1], \mathrm{M}\big) \times [0, 1]$, where $\mathcal{L}_1$ stands

for the restriction of the Lebesgue measure $\mathcal{L}^1$ to the interval $[0, 1]$. The evaluation map $e: C([0, 1], M) \times [0, 1] \to M$, i.e.,

$$e(\gamma, t) := \gamma_t \quad \text{for every } \gamma \in C([0, 1], M) \text{ and } t \in [0, 1],$$

is continuous. Therefore, it makes sense to consider the finite Borel measure $\nu := e_*\hat{\boldsymbol{\pi}}$ on M. An application of the disintegration theorem [3, Theorem 5.3.1] provides us with a weakly measurable family $\{\hat{\boldsymbol{\pi}}_x\}_{x \in M}$ of Borel probability measures on $C([0, 1], M) \times [0, 1]$ such that

$$\hat{\boldsymbol{\pi}}_x \text{ is concentrated on } e^{-1}(\{x\}) \qquad \text{for } \nu\text{-a.e. } x \in M, \qquad (5.7a)$$

$$\int \Psi(\gamma, t) \, d\hat{\boldsymbol{\pi}}(\gamma, t) = \int \left(\int \Psi(\gamma, t) \, d\hat{\boldsymbol{\pi}}_x(\gamma, t) \right) d\nu(x) \qquad \text{for every } \Psi \in L^1(\hat{\boldsymbol{\pi}}). \quad (5.7b)$$

Since $\boldsymbol{\pi}$-a.e. curve γ is horizontal by Proposition 4.5, we have that $\dot{\gamma}_t \in \mathcal{D}_{\gamma_t}$ holds for $\hat{\boldsymbol{\pi}}$-a.e. (γ, t) by Fubini theorem. Consider the Borel map $\mathsf{Der}: C([0, 1], M) \times [0, 1] \to TM$ defined in Eq. 4.1. Then we know from Eq. 5.7a that for ν-a.e. $x \in M$ the measure $\mathfrak{n}_x := \mathsf{Der}_*\hat{\boldsymbol{\pi}}_x$ can be viewed as a Borel probability measure on $\mathcal{D}_x$. Therefore, for any function $g \in \mathrm{LIP}_{bs}(M)$ we have that

$$\int g \, |\boldsymbol{b}| \, d\mu \overset{(2.4b)}{=} \iint_0^1 g(\gamma_t) \, |\dot{\gamma}_t| \, dt \, d\boldsymbol{\pi}(\gamma) = \int g(e(\gamma, t)) \, \rho(\mathsf{Der}(\gamma, t)) \, d\hat{\boldsymbol{\pi}}(\gamma, t)$$

$$\overset{(5.7b)}{=} \int \left(\int g(e(\gamma, t)) \, \rho(\mathsf{Der}(\gamma, t)) \, d\hat{\boldsymbol{\pi}}_x(\gamma, t) \right) d\nu(x)$$

$$= \int g(x) \left(\int_{\mathcal{D}_x} \|v\|_x \, d\mathfrak{n}_x(v) \right) d\nu(x), \qquad (5.8)$$

where the function ρ is defined as in Eq. 4.4. Let us set $\Phi(x) := \int_{\mathcal{D}_x} \|v\|_x \, d\mathfrak{n}_x(v)$ for ν-a.e. $x \in M$. The measurability of Φ is granted by the fact that $\{\hat{\boldsymbol{\pi}}_x\}_{x \in M}$ is a weakly measurable family of measures. Moreover, by the arbitrariness of $g \in \mathrm{LIP}_{bs}(M)$ we deduce from Eq. 5.8 that $|\boldsymbol{b}|\mu = \Phi\nu$. In particular $\Phi \in L^1(\nu)$, which implies that

$$\int_{\mathcal{D}_x} \|v\|_x \, d\mathfrak{n}_x(v) < +\infty \qquad \text{for } \nu\text{-a.e. } x \in M. \qquad (5.9)$$

Given that $\boldsymbol{\pi}$ is concentrated on non-constant Lipschitz curves having constant speed, we also have that $\dot{\gamma}_t \neq 0$ for $\hat{\boldsymbol{\pi}}$-a.e. (γ, t), or equivalently that $\rho \circ \mathsf{Der} > 0$ in the $\hat{\boldsymbol{\pi}}$-a.e. sense. Hence Eq. 5.8 ensures that $\Phi(x) = \int \rho \circ \mathsf{Der} \, d\hat{\boldsymbol{\pi}}_x > 0$ holds for ν-a.e. point $x \in M$, which together with the identity $|\boldsymbol{b}|\mu = \Phi\nu$ imply that $\nu \ll \mu$. The Bochner integral $\int_{\mathcal{D}_x} v \, d\mathfrak{n}_x(v)$ is well-posed for ν-a.e. point $x \in M$ by Eq. 5.9, therefore it makes sense to define

$$\mathrm{I}(\boldsymbol{b})(x) := \frac{d\nu}{d\mu}(x) \int_{\mathcal{D}_x} v \, d\mathfrak{n}_x(v) \in \mathcal{D}_x \qquad \text{for } \mu\text{-a.e. } x \in M,$$

where $\frac{d\nu}{d\mu}$ stands for the Radon–Nikodým derivative of ν with respect to μ. Now fix $g \in \mathrm{LIP}_{bs}(M)$ and $f \in C_c^1(M) \cap \mathrm{LIP}(M)$. We call $df: TM \to \mathbb{R}$ the smooth map $(x, v) \mapsto$

$d_x f[v]$. Therefore

$$\int g \, \boldsymbol{b}(f) \, d\mu \overset{(2.4a)}{=} \iint_0^1 g(\gamma_t) \, (f \circ \gamma)_t' \, dt \, d\boldsymbol{\pi}(\gamma) = \int g(e(\gamma, t)) \, df(\mathsf{Der}(\gamma, t)) \, d\hat{\boldsymbol{\pi}}(\gamma, t)$$

$$\overset{(5.7b)}{=} \int \left(\int g(e(\gamma, t)) \, df(\mathsf{Der}(\gamma, t)) \, d\hat{\boldsymbol{\pi}}_x(\gamma, t) \right) d\nu(x)$$

$$= \int g(x) \left(\int_{\mathcal{D}_x} d_x f[v] \, d\mathfrak{n}_x(v) \right) d\nu(x).$$

Since $g \in \mathrm{LIP}_{bs}(\mathrm{M})$ is arbitrary, we deduce that $\boldsymbol{b}(f)(x) = \frac{d\nu}{d\mu}(x) \int_{\mathcal{D}_x} d_x f[v] \, d\mathfrak{n}_x(v)$ holds for μ-a.e. point $x \in \mathrm{M}$. Being the map $d_x f|_{\mathcal{D}_x} : \mathcal{D}_x \to \mathbb{R}$ linear and continuous, we conclude that

$$\boldsymbol{b}(f)(x) = \frac{d\nu}{d\mu}(x) \int_{\mathcal{D}_x} d_x f[v] \, d\mathfrak{n}_x(v) = d_x f\left[\frac{d\nu}{d\mu}(x) \int_{\mathcal{D}_x} v \, d\mathfrak{n}_x(v) \right] = d_x f\big[\mathrm{I}(\boldsymbol{b})(x)\big]$$

$$= d_{\mathrm{H}} f(x)\big[\mathrm{I}(\boldsymbol{b})(x)\big]$$

is satisfied for μ-a.e. $x \in \mathrm{M}$, thus proving Eq. 5.5.

ISOMETRY. Let $\boldsymbol{b} \in \mathsf{Der}^{2,2}(\mathrm{M}; \mu)$ be fixed. We deduce from the μ-a.e. identity $|\boldsymbol{b}| = \Phi \frac{d\nu}{d\mu}$ that

$$\big\| \mathrm{I}(\boldsymbol{b})(x) \big\|_x = \frac{d\nu}{d\mu}(x) \left\| \int_{\mathcal{D}_x} v \, d\mathfrak{n}_x(v) \right\|_x \le \frac{d\nu}{d\mu}(x) \int_{\mathcal{D}_x} \|v\|_x \, d\mathfrak{n}_x(v) = |\boldsymbol{b}|(x) \quad \text{for } \mu\text{-a.e. } x \in \mathrm{M}.$$

In order to prove the converse inequality, pick a countable family $\mathscr{F} \subseteq C_c^1(\mathrm{M}) \cap \mathrm{LIP}(\mathrm{M})$ as in Theorem 5.4. Therefore, for any $f \in \mathscr{F}$ it holds that

$$\boldsymbol{b}(f)(x) \overset{(5.5)}{=} d_{\mathrm{H}} f(x)\big[\mathrm{I}(\boldsymbol{b})(x)\big] \le \big\| d_{\mathrm{H}} f(x) \big\|_x^* \big\| \mathrm{I}(\boldsymbol{b})(x) \big\|_x \overset{(4.7)}{\le} \mathrm{Lip}(f) \big\| \mathrm{I}(\boldsymbol{b})(x) \big\|_x \le \big\| \mathrm{I}(\boldsymbol{b})(x) \big\|_x$$

for μ-a.e. $x \in \mathrm{M}$, whence $|\boldsymbol{b}|(x) = \big(\mathrm{ess\,sup}_{f \in \mathscr{F}} \, \boldsymbol{b}(f)\big)(x) \le \big\| \mathrm{I}(\boldsymbol{b})(x) \big\|_x$ holds for μ-a.e. point $x \in \mathrm{M}$. This completes the proof of Eq. 5.6. $\qquad\square$

Finally, we conclude by expounding how to deduce from Theorem 5.5 that all sub-Riemannian manifolds are universally infinitesimally Hilbertian. This is the content of the following result, which has been already stated in Theorem 1.2.

Theorem 5.6 (Infinitesimal Hilbertianity of sub-Riemannian manifolds) *Let* $(\mathrm{M}, \mathrm{E}, \sigma, \psi)$ *be a sub-Riemannian manifold with* d_{CC} *complete. Let* μ *be a non-negative Radon measure on* (M, d_{CC}). *Then the metric measure space* $(\mathrm{M}, d_{CC}, \mu)$ *is infinitesimally Hilbertian.*

Proof Let $\bar{x} \in \mathrm{spt}(\mu)$ be fixed. We define $B_n := \bar{B}_n(\bar{x})$ and $\mu_n := \mu|_{B_n}$ for every $n \in \mathbb{N}$. We know from [14, Proposition 2.6] and [9, Theorem 7.2.5] that for any $n \in \mathbb{N}$ it holds that

$$f \in W^{1,2}(\mathrm{M}, d_{CC}, \mu) \implies f \in W^{1,2}(\mathrm{M}, d_{CC}, \mu_n) \text{ and } |Df|_{\mu_n} = |Df|_\mu \ \mu_n\text{-a.e.}. \tag{5.10}$$

This ensures that, in order to prove that $(\mathrm{M}, d_{CC}, \mu)$ is infinitesimally Hilbertian, it is enough to show that $W^{1,2}(\mathrm{M}, d_{CC}, \mu_n)$ is a Hilbert space for every $n \in \mathbb{N}$. Given that μ_n is a finite measure, we can apply Theorem 5.5 and Remark 4.8 to deduce that

$$|\boldsymbol{b} + \boldsymbol{b}'|^2 + |\boldsymbol{b} - \boldsymbol{b}'|^2 = 2|\boldsymbol{b}|^2 + 2|\boldsymbol{b}'|^2 \quad \mu\text{-a.e.} \quad \text{for every } \boldsymbol{b}, \boldsymbol{b}' \in \mathsf{Der}^{2,2}(\mathrm{M}; \mu).$$

Hence $(\mathrm{M}, d_{CC}, \mu_n)$ is infinitesimally Hilbertian by Proposition 2.7. The statement is achieved. $\qquad\square$

Remark 5.7 Given a sub-Finsler manifold (M, E, σ, ψ) equipped with a non-negative Radon measure μ, it is not necessarily true that $W^{1,2}(M, d_{CC}, \mu)$ is a Hilbert space. Nevertheless, we can still deduce from Theorem 5.5 that $W^{1,2}(M, d_{CC}, \mu)$ is reflexive, as we are going to explain.

First of all, it can be readily checked that $L^2(HM; \mu)$ is a reflexive Banach space if endowed with the norm $L^2(HM; \mu) \ni v \mapsto \left(\int \|v(x)\|_x^2 \, d\mu(x) \right)^{1/2}$. Calling $\mathbb{B}$ the dual of $\left(\text{Der}^{2,2}(M; \mu), \|\cdot\|_2 \right)$, where the norm $\|\cdot\|_2$ is defined as in Remark 2.5, we deduce from Theorem 5.5 that $\mathbb{B}$ is a reflexive Banach space. Consequently, the product space $L^2(\mu) \times \mathbb{B}$ is reflexive as well. Define $\mathscr{L}_f \in \mathbb{B}$ for every $f \in W^{1,2}(M, d_{CC}, \mu)$ as in Eq. 2.2. Observe that Remark 2.5 grants that the linear operator

$$W^{1,2}(M, d_{CC}, \mu) \longrightarrow L^2(\mu) \times \mathbb{B},$$
$$f \longmapsto (f, \mathscr{L}_f)$$

is an isometry. Therefore, we can finally conclude that $W^{1,2}(M, d_{CC}, \mu)$ is reflexive, as claimed.

We also point out that, by slightly adapting the arguments we carried out in the paper, it is possible to deal with any exponent $p \in (1, \infty)$ and to show that $W^{1,p}(M, d_{CC}, \mu)$ is reflexive.

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