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UNIVERSITY OF JYVÄSKYLÄ

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Author(s): Eriksson-Bique, Sylvester; Vähäkangas, Antti V.

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Sylvester Eriksson-Bique, Antti V. Vähäkangas
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SELF-IMPROVEMENT OF POINTWISE HARDY INEQUALITY

SYLVESTER ERIKSSON-BIQUE AND ANTTI V. VÄHÄKANGAS

Abstract. We prove the self-improvement of a pointwise p-Hardy inequality. The proof relies on maximal function techniques and a characterization of the inequality by curves.

1. Introduction

Let $X = (X, d, \mu)$ be a metric measure space and let $1 \leq p < \infty$. In this paper we are interested in the self-improvement properties of the *pointwise p-Hardy inequality*

$$|u(x)| \leq C_{\mathrm{H}}\, d(x, \Omega^c)(\mathcal{M}_{p,\kappa d(x,\Omega^c)} g(x))\,. \tag{1}$$

We say that an open set $\Omega \subsetneq X$ satisfies pointwise p-Hardy inequality, if there are constants C_{H} and κ such that inequality (1) holds for all $x \in \Omega$ whenever u is a Lipschitz function such that $u = 0$ in $\Omega^c = X \setminus \Omega$ and g is a bounded upper gradient of u; we refer to Section 2 for the definition of $\mathcal{M}_{p,\kappa d(x,\Omega^c)} g(x)$ and the standing assumptions on X. By Hölder's inequality, we see that increasing p will result in a weaker inequality (1). Self-improvement is concerned with the opposite, and far less intuitive, possibility of lowering the exponent p slightly. Our main result reads as follows. Let $1 \leq p_0 < p < \infty$ and assume that X supports a p_0-Poincaré inequality. Assume that Ω satisfies a pointwise p-Hardy inequality. Then there exists $q \in (p_0, p)$ such that Ω satisfies a pointwise q-Hardy inequality; we refer to Theorem 5.1. In this paper we provide a direct proof of this self-improvement result with transparent and quantitative bounds for the quantity $p - q > 0$ of self-improvement; see Remark 5.2.

The pointwise p-Hardy inequality was first independently studied by Hajłasz in [6] and by Kinnunen–Martio in [10]. Korte et al. proved in [11] that a pointwise p-Hardy inequality characterizes the so-called uniform p-fatness of the complement Ω^c; we note that uniform p-fatness is a uniform p-capacity density condition that appears often in potential theory and PDE's, see e.g. [7]. Consequently, our proof can be used to show the deep self-improvement property of uniform p-fatness. This result was first discovered in Euclidean spaces by Lewis [15] using potential theoretical arguments. Subsequently Mikkonen generalized Lewis' result to the Euclidean weighted setting in his PhD-thesis [17]. Mikkonen's approach, in turn, was adapted to metric spaces by Björn et al. in [2]. This adaptation relies on the impressive theory of differential structures on complete (or at least locally complete) metric spaces, established by Cheeger in [3].

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An alternative approach to the self-improvement of uniform p-fatness was recently provided by Lehrbäck et al. in [13]. Their proof builds upon a localization of the argument due to Koskela–Zhong [12] which, in turn, is concerned with the self-improvement of integral p-Hardy inequalities. Consequently, either one of the papers [2] or [13] together with the mentioned characterization in [11] can be used to provide an indirect proof of our main result. In comparison, our approach is more direct with the additional benefit of yielding transparent and quantitative bounds for the self-improvement. Our approach is new even in the classical setting of Euclidean space equipped with the Euclidean metric and the Lebesgue measure. For a survey on Hardy inequalities, and their connections to uniform p-fatness, we refer to [9] and references therein. See also [16].

The outline of this paper is as follows. Notation and maximal function techniques are presented in Section 2. The pointwise p-Hardy inequality is characterized by using curves in Section 3. The actual work for self-improvement via curves is done in Section 4 and our main results are stated and proved in Section 5. The main line of our proof is adapted from the paper [4] of the first author, where the self-improvement of a p-Poincaré inequality is proved with the aid of maximal functions and a characterization by curves; this result was originally obtained in [8] by a different method. Curiously, the present approach simultaneously explains the self-improvement property of both p-Poincaré inequality and pointwise p-Hardy inequality. We also remark that Lerner–Pérez [14] established self-improvement properties of Muckenhoupt weights by a similar approach to maximal functions. It is an open question, to what extent these ideas can be taken to unify proofs of various self-improvement phenomena that are ubiquitous in analysis and PDE.

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2. Notation and auxiliary results

Here, and throughout the paper, we assume that $X = (X, d, \mu)$ is a C_{QC}-quasiconvex proper metric measure space equipped with a metric d and a positive complete D-doubling Borel measure μ such that $\#X \geq 2$, $0 < \mu(B) < \infty$ and

$$\mu(2B) \leq D\,\mu(B) \tag{2}$$

for some $D > 1$ and for all balls $B = B(x, r) = \{y \in X : d(y, x) < r\}$. Here we use for $0 < \lambda < \infty$ the notation $\lambda B = B(x, \lambda r)$. The space X is separable under these assumptions, see [1, Proposition 1.6]. Moreover, the measure μ is regular and, in particular for every Borel set $E \subset X$ and every $\varepsilon > 0$, there exists an open set $V \supset E$ such that $\mu(E) \leq \mu(V) + \varepsilon$; we refer to [5, Theorem 7.8] for further details.

We denote by $\mathrm{Lip}(X)$ the space of Lipschitz functions on X. That is, we have $u \in \mathrm{Lip}(X)$ iff there exists a constant $\lambda > 0$ such that

$$|u(x) - u(y)| \leq \lambda d(x, y)\,, \qquad \text{for all} \;\; x, y \in X\,.$$

We let $\Omega \subset X$ be an open set. We denote by $\mathrm{Lip}_0(\Omega)$ the space of Lipschitz functions on X that vanish on $\Omega^c = X \setminus \Omega$. The set of continous functions on X is denoted by $C(X)$, and $C_0(\Omega) \subset C(X)$ consists of those continuous functions that

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vanish on Ω^c. We denote by $LC(X)$ the set of lower semicontinuous functions on X, and by $LC_0(\Omega)$ we denote the set of those functions in $LC(X)$ that vanish on Ω^c.

By a *curve* we mean a nonconstant, rectifiable, continuous mapping from a compact real interval to X. By $\Gamma(X)$ we denote the set of all curves in X. The length of a curve $\gamma \in \Gamma(x)$ is written as $\operatorname{len}(\gamma)$. We say that a curve $\gamma\colon [a,b] \to X$ *connects* $x \in X$ to $y \in X$ (or a point $x \in X$ to a set $E \subset X$), if $\gamma(a) = x$ and $\gamma(b) = y$ ($\gamma(b) \in E$, respectively). If $x, y \in X$, $E \subset X$ and $\nu \geq 1$ we denote by $\Gamma(X)^\nu_{x,y}$ the set of curves that connect x to y and whose lengths are at most $\nu d(x,y)$, and by $\Gamma(X)^\nu_{x,E}$ we denote the set of curves that connect x to E and whose lengths are at most $\nu d(x,E)$.

We say that a Borel function $g \geq 0$ on X is an *upper gradient* of a real-valued function u on X if, for any curve γ connecting any $x \in X$ to any $y \in X$, we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds\,. \tag{3}$$

We use the following familiar notation:

$$u_E = \fint_E u(y)\, d\mu(y) = \frac{1}{\mu(E)} \int_E u(y)\, d\mu(y)$$

is the integral average of $u \in L^1(E)$ over a measurable set $E \subset X$ with $0 < \mu(E) < \infty$. Moreover if $E \subset X$, then $\mathbf{1}_E$ denotes the characteristic function of E; that is, $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \in X \setminus E$. If $1 \leq p < \infty$ and $u\colon X \to \mathbb{R}$ is a μ-measurable function, then $u \in L^p_{\mathrm{loc}}(X)$ means that for each $x_0 \in X$ there exists $r > 0$ such that $u \in L^p(B(x_0, r))$, i.e., $\int_{B(x_0,r)} |u(y)|^p\, d\mu(y) < \infty$.

For $0 \leq r < \infty$ and $1 \leq p < \infty$, and every $f \in L^p_{\mathrm{loc}}(X)$, we define the r-restricted maximal function $\mathcal{M}_{p,r} f(x)$ at $x \in X$ by

$$\mathcal{M}_{p,r} f(x) := \begin{cases} |f(x)|\,, & r = 0\,, \\ \sup\limits_B \left(\fint_B |f(z)|^p \; d\mu(z) \right)^{\frac{1}{p}}\,, & r > 0\,, \end{cases}$$

where the supremum is taken over all balls $B = B(y,t)$ in X such that $x \in B$ and $0 < t < r$.

The definition of a pointwise p-Hardy inequality is as follows; recall that $\Omega^c = X \setminus \Omega$.

Definition 2.1. *Let $1 \leq p < \infty$. An open set $\emptyset \neq \Omega \subsetneq X$ is said to satisfy a* pointwise p-Hardy inequality *if there exists constants $C_{\mathrm{H}} > 0$ and $\kappa \geq 1$ such that for every Lipschitz function $u \in \mathrm{Lip}_0(\Omega)$, every bounded upper gradient g of u and every $x \in \Omega$, we have*

$$|u(x)| \leq C_{\mathrm{H}}\, d(x, \Omega^c) (\mathcal{M}_{p, \kappa d(x,\Omega^c)} g(x))\,. \tag{4}$$

Clearly by Hölder's inequality, a pointwise p-Hardy inequality implies a pointwise q-Hardy inequality for every $1 \leq p < q < \infty$.

The following p-Poincaré inequality has a corresponding property.

Definition 2.2. *Let $1 \leq p < \infty$. We say that X supports a p-Poincaré inequality, if there are constants $C_{\mathrm{PI}} > 0$ and $\lambda \geq 1$ such that for any ball B of radius $r > 0$*

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in X, any $u \in \mathrm{Lip}(X)$ and any bounded upper gradient g of u, we have

$$\fint_B |u(x) - u_B|\, d\mu(x) \le C_{\mathrm{PI}}\, r \left(\fint_{\lambda B} g(x)^p \, d\mu(x) \right)^{1/p} . \tag{5}$$

Here $u_B = \fint_B u \, d\mu$.

We remark that the p-Poincaré inequality has a self-improving property. More specifically, a p-Poincaré inequality for any $1 < p < \infty$ implies a p_0-Poincaré inequality for some $p_0 < p$; see [4] and [8]. For a self-contained exposition, we will explicitly assume such an improved Poincaré inequality. The following characterization from [4, Theorem 1.5] will be useful.

Lemma 2.3. *Let $1 \le p < \infty$. Then X supports a p-Poincaré inequality if and only if there are constants $C_{\mathrm{A}} > 0$, $\nu > C_{\mathrm{QC}}$ and $\kappa \ge 1$ such that, for any non-negative and bounded $g \in LC(X)$ and any $x, y \in X$, we have*

$$\inf_{\gamma \in \Gamma(X)^{\nu}_{x,y}} \int_\gamma g \, ds \le C_{\mathrm{A}}\, d(x,y) \left(\mathcal{M}_{p,\kappa d(x,y)} g(x) + \mathcal{M}_{p,\kappa d(x,y)} g(y) \right) . \tag{6}$$

We need a few auxiliary results involving maximal functions. We begin with the following scale invariant weak-type estimate that is originally from [4, Lemma 2.3].

Lemma 2.4. *Fix $1 \le q < \infty$ and $0 < r, s < \infty$. Let $f \in L^q_{\mathrm{loc}}(X)$, let $\Lambda > 0$, and define*

$$E_{q,s,\Lambda} = \{ x \in X \mid \mathcal{M}_{q,s} f(x) > \Lambda \} .$$

Then, for every $x \in X$,

$$\mathcal{M}_{1,r} \mathbf{1}_{E_{q,s,\Lambda}}(x) \le \frac{D^5 (\mathcal{M}_{q,r+3s} f(x))^q}{\Lambda^q} . \tag{7}$$

Proof. Fix $x \in X$ and $0 < t < r$. Let $B = B(y,t)$ be a ball in X such that $x \in B$. It suffices to prove that

$$\fint_B \mathbf{1}_{E_{q,s,\Lambda}} \, d\mu \le \frac{D^5 (\mathcal{M}_{q,r+3s} f(x))^q}{\Lambda^q} . \tag{8}$$

The proof of (8) is based upon a covering argument. For each $z \in E_{q,s,\Lambda} \cap B$ we fix a ball B_z of radius $0 < r_{B_z} < s$ such that $z \in B_z$ and

$$\left(\fint_{B_z} |f|^q \, d\mu \right)^{\frac{1}{q}} > \Lambda . \tag{9}$$

Suppose that $t < r_{B_z}$ for some $z \in E_{q,s,\Lambda} \cap B$. Then $x \in 3B_z$ and, therefore,

$$\fint_B \mathbf{1}_{E_{q,s,\Lambda}} \, d\mu \le 1 < \frac{\fint_{B_z} |f|^q \, d\mu}{\Lambda^q} \le \frac{D^2 \fint_{3B_z} |f|^q \, d\mu}{\Lambda^q} \le \frac{D^2 (\mathcal{M}_{q,3s} f(x))^q}{\Lambda^q} .$$

Since $\mathcal{M}_{q,3s} f(x) \le \mathcal{M}_{q,r+3s} f(x)$ and $D > 1$, we thus obtain inequality (8). Hence in the sequel, we can assume that $r_{B_z} \le t$ for all $z \in E_{q,s,\Lambda} \cap B$.

By using the $5r$-covering lemma [1, Lemma 1.7], we obtain a countable and disjoint family $\mathcal{B} \subset \{ B_z \mid z \in E_{q,s,\Lambda} \cap B \}$ of balls such that $E_{q,s,\Lambda} \cap B \subset \bigcup_{B' \in \mathcal{B}} 5B'$. Hence, by (9),

$$\fint_B \mathbf{1}_{E_{q,s,\Lambda}} \, d\mu \le \frac{1}{\mu(B)} \sum_{B' \in \mathcal{B}} \mu(5B') \le \frac{D^3}{\mu(B)} \sum_{B' \in \mathcal{B}} \mu(B') \le \frac{D^3}{\Lambda^q \mu(B)} \sum_{B' \in \mathcal{B}} \int_{B'} |f|^q \, d\mu .$$

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Since $r_{B'} \le \min\{s,t\}$, we have that $B' \subset B'' := B(y, t + 2\min\{s,t\})$ for every $B' \in \mathcal{B}$. Also, $B \subset B'' \subset 3B$, so $\mu(B'') \le D^2\mu(B)$. We can conclude that

$$\fint_B \mathbf{1}_{E_{q,s,\Lambda}}\, d\mu \le \frac{D^5}{\Lambda^q} \fint_{B''} |f|^q \, d\mu \le \frac{D^5 (\mathcal{M}_{q,t+3s} f(x))^q}{\Lambda^q} \, .$$

Since $\mathcal{M}_{q,t+3s} f(x) \le \mathcal{M}_{q,r+3s} f(x)$, we thus obtain inequality (8) also in this case. □

The following approximation lemma is originally from [4, Lemma 3.7]. For the convenience of the reader, we provide a proof. We remark that the regularity of the measure is needed in the proof. A Borel function $g\colon X \to [0,\infty)$ is *simple*, if $g = \sum_{j=1}^k a_j \mathbf{1}_{E_j}$ for some real $a_j > 0$ and Borel sets $E_j \subset X$, $j = 1, \ldots, k$.

Lemma 2.5. *Let $1 \le p < \infty$. Let $g\colon X \to [0,\infty)$ be a simple Borel function. Then, for each $x \in X$ and every $\varepsilon > 0$, there exists a non-negative and bounded $g_{x,\varepsilon} \in LC(X)$ such that $g(y) \le g_{x,\varepsilon}(y)$ for all $y \in X \setminus \{x\}$ and $\mathcal{M}_{p,r} g_{x,\varepsilon}(x) \le \mathcal{M}_{p,r} g(x) + \varepsilon$ if $r > 0$.*

Proof. We prove the claim, while assuming that $\operatorname{diam}(X) = \infty$. The case $\operatorname{diam}(X) < \infty$ is similar, and we omit the modifications. Fix $x \in X$ and $\varepsilon > 0$. In the first step, we prove an auxiliary statement for a Borel set $E \subset X$. Namely, we will show that there exists an open set $U \subset X$ such that $\mathbf{1}_E \le \mathbf{1}_U$ in $X \setminus \{x\}$ and

$$\mathcal{M}_{p,r}(\mathbf{1}_U - \mathbf{1}_E)(x) < \varepsilon \, , \qquad \text{if } r > 0 \, . \tag{10}$$

To prove this auxiliary statement, for each $m \in \mathbb{Z}$, we write

$$A_m = \{ y \in X \, : \, 2^{m-1} < d(x,y) < 2^{m+1} \} \, .$$

Observe that each $y \in X$ belongs to at most two annuli. We also have that $\mu(A_m) > 0$, since X is connected and unbounded. Hence, if $m \in \mathbb{Z}$ then by regularity of the measure μ, there is an open set $U_m \subset A_m$ such that

$$A_m \cap E \subset U_m \qquad \text{and} \qquad \mu(U_m \setminus E) = \mu(U_m \setminus (A_m \cap E)) < \frac{\varepsilon^p}{2D^4} \mu(A_m) \, . \tag{11}$$

Define $U = \bigcup_{m\in\mathbb{Z}} U_m$. Then

$$E \setminus \{x\} \subset \bigcup_{m\in\mathbb{Z}} (A_m \cap E) \subset \bigcup_{m\in\mathbb{Z}} U_m = U \, . \tag{12}$$

As a consequence, we then have $\mathbf{1}_E(y) \le \mathbf{1}_U(y)$ for every $y \in X \setminus \{x\}$. To prove (10), we let $r > 0$ and let $B(y,t) \subset X$ be a ball in X such that $x \in B(y,t)$ and $0 < t < r$. Then $\mathbf{1}_U - \mathbf{1}_E = \mathbf{1}_{U\setminus E}$ almost everywhere, and therefore by (11) we get

$$\begin{aligned}
\fint_{B(y,t)} |\mathbf{1}_U - \mathbf{1}_E|^p \, d\mu &= \fint_{B(y,t)} \mathbf{1}_{U\setminus E} \, d\mu \\
&\le \frac{1}{\mu(B(y,t))} \int_X \sum_{m=-\infty}^{\lceil \log_2(2t) \rceil} \mathbf{1}_{U_m \setminus E} \, d\mu \\
&= \frac{\varepsilon^p}{2D^4 \mu(B(y,t))} \sum_{m=-\infty}^{\lceil \log_2(2t) \rceil} \mu(A_m) \\
&\le \frac{\varepsilon^p}{D^4} \frac{\mu(B(x,8t))}{\mu(B(y,t))} \le \varepsilon^p \frac{\mu(B(x,t))}{\mu(B(y,2t))} \le \varepsilon^p \, .
\end{aligned}$$

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By raising this estimate to power $\frac{1}{p}$ and then taking supremum over all balls $B(y,t)$ as above, we obtain inequality (10).

We now turn to the proof of the actual lemma. Let $g = \sum_{j=1}^{k} a_j \mathbf{1}_{E_j}$ be such that $a_j > 0$ and $E_j \subset X$ is a Borel set for each $j = 1, \ldots, k$. By the auxiliary statement, for each $j = 1, \ldots, k$, there exists a non-negative and bounded $g_{x,\varepsilon,j} \in LC(X)$ such that $\mathbf{1}_{E_j} \le g_{x,\varepsilon,j}$ in $X \setminus \{x\}$ and

$$\mathcal{M}_{p,r}(g_{x,\varepsilon,j} - \mathbf{1}_{E_j})(x) \le \frac{\varepsilon}{k \max_j a_j} \,. \tag{13}$$

Now we define $g_{x,\varepsilon} = \sum_{j=1}^{k} a_j g_{x,\varepsilon,j}$. Then $g \le g_{x,\varepsilon}$ in $X \setminus \{x\}$. Moreover, by the subadditivity and positive homogeneity of the maximal function, and inequalities (13), we have

$$\begin{aligned}
\mathcal{M}_{p,r} g_{x,\varepsilon}(x) &= \mathcal{M}_{p,r}(g + g_{x,\varepsilon} - g)(x) \\
&\le \mathcal{M}_{p,r} g(x) + \mathcal{M}_{p,r}(g_{x,\varepsilon} - g)(x) \\
&\le \mathcal{M}_{p,r} g(x) + \sum_{j=1}^{k} a_j \mathcal{M}_{p,r}(g_{x,\varepsilon,j} - \mathbf{1}_{E_j})(x) \le \mathcal{M}_{p,r} g(x) + \varepsilon \,.
\end{aligned}$$

This concludes the proof. □

3. Characterization by curves

We translate the pointwise p-Hardy inequality to an equivalent problem of accessibility. This problem can be phrased as a problem of finding a single curve with a small integral. The standing assumptions concerning the space X are stated in Section 2.

Lemma 3.1. *Let $1 \le p < \infty$. Then an open set $\emptyset \ne \Omega \subsetneq X$ satisfies a pointwise p-Hardy inequality if, and only if, there are constants $C_\Gamma > 0$, $\nu > C_{\mathrm{QC}}$ and $\kappa \ge 1$ such that for each non-negative and bounded $g \in LC(X)$ and every $x \in \Omega$, we have*

$$\inf_{\gamma \in \Gamma(X)^{\nu}_{x,\Omega^c}} \int_\gamma g \, ds \le C_\Gamma \, d(x,\Omega^c) \left(\mathcal{M}_{p,\kappa d(x,\Omega^c)} g(x) \right) . \tag{14}$$

Proof. Throughout this proof, we tacitly assume that curves are parametrized by arc length. First suppose that an open set $\emptyset \ne \Omega \subsetneq X$ satisfies a pointwise p-Hardy inequality (4) with constants $C_{\mathrm{H}} > 0$ and $\kappa_\Gamma > 1$. Fix a non-negative and bounded function $g \in LC(X)$. Fix $x \in \Omega$ and let $\delta > 0$.

Define a function $u\colon X \to [0,\infty)$ by setting

$$u(y) = \inf_\gamma \int_\gamma h \, ds \,, \qquad y \in X \,, \tag{15}$$

where $h = g + \mathcal{M}_{p,\kappa_\Gamma d(x,\Omega^c)} g(x) + \delta$, which is a non-negative bounded Borel function, and the infimum is taken over all curves γ in X connecting y to Ω^c. Let us remark that these curves are not subject to any distance conditions. Clearly, we have that $u = 0$ in Ω^c. Fix $y, w \in X$ and consider any curve σ connecting y to w. We claim that

$$|u(y) - u(w)| \le \int_\sigma h \, ds \,. \tag{16}$$

From this it follows, in particular, that h is an upper gradient of u. Moreover, since X is quasiconvex and h is bounded, it follows from (16) that $u \in \mathrm{Lip}_0(\Omega)$.

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In order to prove (16), we may assume that $u(y) > u(w)$. Fix $\varepsilon > 0$ and let γ be a curve in X such that connects w to Ω^c and satisfies inequality $u(w) \geq \int_\gamma h\,ds - \varepsilon$. Let $\sigma\gamma$ be the concatenation of σ and γ. Then

$$\begin{aligned}|u(y) - u(w)| &= u(y) - u(w)\\ &\leq \int_{\sigma\gamma} h\,ds - \int_\gamma h\,ds + \varepsilon = \int_\sigma h\,ds + \varepsilon\,.\end{aligned}$$

The desired inequality (16) follows by taking $\varepsilon \to 0_+$.

Now, applying the assumed pointwise p-Hardy inequality (2.1) to the function u and to its bounded upper gradient h yields

$$u(x) \leq C_\mathrm{H}\, d(x,\Omega^c)(\mathcal{M}_{p,\kappa_\Gamma d(x,\Omega^c)} h(x)) < \infty\,.$$

Since $u(x) \geq \delta d(x,\Omega^c) > 0$, by (15) there is a curve γ in X connecting x to Ω^c such that

$$\begin{aligned}\int_\gamma g\,ds + (\mathcal{M}_{p,\kappa_\Gamma d(x,\Omega^c)} g(x) + \delta)\,\mathrm{len}(\gamma) &= \int_\gamma h\,ds \leq 2u(x)\\ &\leq 2C_\mathrm{H}\, d(x,\Omega^c)(\mathcal{M}_{p,\kappa_\Gamma d(x,\Omega^c)} h(x))\\ &\leq 2C_\mathrm{H}\, d(x,\Omega^c)(2\mathcal{M}_{p,\kappa_\Gamma d(x,\Omega^c)} g(x) + \delta)\,.\end{aligned} \tag{17}$$

The last inequality follows from the sublinearity of maximal function. We can now conclude from (17) that $\mathrm{len}(\gamma) \leq 4C_\mathrm{H}\, d(x,\Omega^c)$. By taking $\delta \to 0_+$, we obtain from (17) that

$$\int_\gamma g\,ds \leq 4C_\mathrm{H}\, d(x,\Omega^c)(\mathcal{M}_{p,\kappa_\Gamma d(x,\Omega^c)} g(x))\,.$$

Thus, inequality (14) holds with

$$C_\Gamma = 4C_\mathrm{H}\,, \qquad \kappa = \kappa_\Gamma\,, \qquad \nu > \max\{C_\mathrm{QC}, 4C_\mathrm{H}\}\,.$$

For the converse implication, we assume that inequality (14) holds, for all non-negative and bounded $g \in LC(X)$, and for all $x \in \Omega$. We need to prove that Ω satisfies a pointwise p-Hardy inequality. To this end, we let $u \in \mathrm{Lip}_0(\Omega)$ and let g be a bounded upper gradient of u. We also fix $x \in \Omega$. Since g is not necessarily lower semicontinuous, some approximation is first needed so that we can get to apply (14) and thereby establish inequality (4).

Let $(g_N)_{N\in\mathbb{N}}$ be a pointwisely increasing sequence of non-negative simple Borel functions such that $\lim_{N\to\infty} g_N = g$ uniformly in X. Fix $\varepsilon > 0$. By the uniform convergence, there exists $N \in \mathbb{N}$ such that for all $\gamma \in \Gamma(X)^\nu_{x,\Omega^c}$ we have

$$\begin{aligned}\int_\gamma g\,ds &= \int_\gamma g_N\,ds + \int_\gamma (g - g_N)\,ds\\ &\leq \int_\gamma g_N\,ds + \sup_{y\in X}(g(y) - g_N(y))\,\mathrm{len}(\gamma)\\ &\leq \int_\gamma g_N\,ds + \sup_{y\in X}(g(y) - g_N(y))\nu d(x,\Omega^c) \leq \int_\gamma g_N\,ds + \varepsilon\,.\end{aligned} \tag{18}$$

Let $g_{N,x,\varepsilon} \in LC(X)$ be the non-negative bounded approximant of g_N given by Lemma 2.5. By inequality (14) and Lemma 2.5, there exists $\gamma_N \in \Gamma(X)^\nu_{x,\Omega^c}$ such

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that

$$\begin{aligned}\int_{\gamma_N} g_{N,x,\varepsilon}\,ds &\le C_\Gamma d(x,\Omega^c)\big(\mathcal{M}_{p,\kappa d(x,\Omega^c)} g_{N,x,\varepsilon}(x)\big)+\varepsilon\\ &\le\ C_\Gamma d(x,\Omega^c)\big(\mathcal{M}_{p,\kappa d(x,\Omega^c)} g_N(x)+\varepsilon\big)+\varepsilon\\ &\le C_\Gamma d(x,\Omega^c)\big(\mathcal{M}_{p,\kappa d(x,\Omega^c)} g(x)+\varepsilon\big)+\varepsilon\,.\end{aligned}$$

Without loss of generality, we may assume that $\gamma_N(t)=x$ only if $t=0$. On the other hand, by Lemma 2.5, we have $g_N\le g_{N,x,\varepsilon}$ in $X\setminus\{x\}$. Inequality (18), with $\gamma=\gamma_N$, implies that

$$\begin{aligned}\int_{\gamma_N} g\,ds \le \int_{\gamma_N} g_N\,ds+\varepsilon &\le \int_{\gamma_N} g_{N,x,\varepsilon}\,ds+\varepsilon\\ &\le C_\Gamma d(x,\Omega^c)\big(\mathcal{M}_{p,\kappa d(x,\Omega^c)} g(x)+\varepsilon\big)+2\varepsilon\,.\end{aligned}$$

Since g is an upper gradient of $u\in \operatorname{Lip}_0(\Omega)$, we get

$$\begin{aligned}|u(x)| &= |u(\gamma_N(0))-u(\gamma_N(\operatorname{len}(\gamma_N)))|\\ &\le \int_{\gamma_N} g\,ds \le C_\Gamma d(x,\Omega^c)\big(\mathcal{M}_{p,\kappa d(x,\Omega^c)} g(x)+\varepsilon\big)+2\varepsilon\,,\end{aligned}$$

and letting $\varepsilon\to 0$ gives the pointwise p-Hardy inequality (4) with $C_{\mathrm H}=C_\Gamma$ and $\kappa\ge 1$. □

While seemingly technical, the task of infimizing in (14) is often reduced to constructing an explicit curve, for which the upper bound holds. In particular, our proof for self-improvement of pointwise Hardy inequalities is based on establishing the existence of such a single curve for some exponent $q<p$.

Next we define a convenient α-function that condenses the pointwise p-Hardy inequality, or inequality (14) to be more specific, in a single function at the expense of abstraction. Indeed, the following definition looks complicated at first sight, but for our purposes the quantity $\alpha_{p,\Omega}$ is precisely the correct way to express the pointwise p-Hardy inequality.

Definition 3.2. *Let $\emptyset\ne\Omega\subsetneq X$ be an open set. If $\tau\ge 0$, $\kappa,p\ge 1$ and $x\in\Omega$, we write*

$$\mathcal{E}^{\kappa,\tau}_{p,x,\Omega}=\{g\in LC(X)\mid \mathcal{M}_{p,\kappa d(x,\Omega^c)}g(x)\le\tau \text{ and } g(y)\in[0,1] \text{ for all } y\in X\}\,.$$

If also $\nu>C_{\mathrm{QC}}$, then we write

$$\alpha_{p,\Omega}(\nu,\kappa,\tau):=\sup_{x\in\Omega}\sup_{g\in\mathcal{E}^{\kappa,\tau}_{p,x,\Omega}}\frac{\inf_{\gamma\in\Gamma(X)^\nu_{x,\Omega^c}}\int_\gamma g\,ds}{d(x,\Omega^c)}\,. \tag{19}$$

Concerning definition (19), the parameter ν is related to the maximum length of the curves γ that are used so that $\operatorname{len}(\gamma)\le\nu d(x,\Omega^c)$. The parameters κ and τ measure the non-locality and size of the maximal function $\mathcal{M}_{p,\kappa d(x,\partial\Omega)}g(x)$, respectively.

The fundamental connection between inequality (14) and the α-function is established in the following lemma.

Lemma 3.3. *Let $\emptyset\ne\Omega\subsetneq X$ be an open set, and let $\kappa,p\ge 1$ and $\nu>C_{\mathrm{QC}}$. Let $g\in LC(X)$ be such that $g(y)\in[0,1]$ for every $y\in\Omega$. Then, for every $x\in\Omega$, we*

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have

$$\inf_{\gamma\in\Gamma(X)^{\nu}_{x,\Omega^c}} \int_\gamma g\,ds \le d(x,\Omega^c)\alpha_{p,\Omega}\left(\nu,\kappa,\left(\mathcal{M}_{p,\kappa d(x,\Omega^c)}g(x)\right)\right). \tag{20}$$

Proof. Fix $g \in LC(X)$ such that $g(y)\in[0,1]$ for all $y\in X$. Let $x\in\Omega$ and write

$$\tau = \mathcal{M}_{p,\kappa d(x,\Omega^c)}g(x) \ge 0\,.$$

Then $g\in\mathcal{E}^{\kappa,\tau}_{p,x,\Omega}$, and hence

$$\frac{\inf_{\gamma\in\Gamma(X)^{\nu}_{x,\Omega^c}}\int_\gamma g\,ds}{d(x,\Omega^c)} \le \sup_{h\in\mathcal{E}^{\kappa,\tau}_{p,x,\Omega}} \frac{\inf_{\gamma\in\Gamma(X)^{\nu}_{x,\Omega^c}}\int_\gamma h\,ds}{d(x,\Omega^c)} \le \alpha_{p,\Omega}(\nu,\kappa,\tau)$$

Where the last step follows, since $x\in\Omega$. □

In particular, from Lemma 3.3 we now obtain the following sufficient condition for the pointwise p-Hardy inequality in terms of a τ-linear upped bound for the α-function.

Lemma 3.4. *Let $1\le p<\infty$ and let $\emptyset\ne\Omega\subsetneq X$ be an open set. Suppose that there are constants $\nu>C_{\mathrm{QC}}$, $\kappa\ge 1$ and $C_\alpha>0$ such that, for any $\tau\ge 0$, we have*

$$\alpha_{p,\Omega}(\nu,\kappa,\tau)\le C_\alpha\tau\,.$$

Then Ω satisfies a pointwise p-Hardy inequality.

Proof. By Lemma 3.1, it suffices to find a constant $C_\Gamma>0$ such that inequality (14) holds for each non-negative bounded $g\in LC(X)$ and every $x\in\Omega$ — the remaining constants ν and κ are given in the assumptions of the present lemma. Fix such a function g and a point $x\in\Omega$. Since g is bounded and inequality (14) is invariant under multiplication of g with a strictly positive constant, we may further assume that $g(y)\in[0,1]$ for all $y\in X$.

Then the desired estimate (14), with $C_\Gamma=C_\alpha$, follows immediately from Lemma 3.3 and the assumptions. □

The converse of Lemma 3.4 is also true, as we will see in Section 4. Therein the following inequalities for the α-function become useful.

Lemma 3.5. *Let $\emptyset\ne\Omega\subsetneq X$ be an open set. Let $0\le\tau<\tau'$, $\kappa,p\ge 1$ and $\nu>C_{\mathrm{QC}}$. Then*

$$\alpha_{p,\Omega}(\nu,\kappa,\tau)\le\alpha_{p,\Omega}(\nu,\kappa,\tau')\,,\qquad \alpha_{p,\Omega}(\nu,\kappa,\tau)\le\nu\,,$$

and, for every $M\ge 1$,

$$\alpha_{p,\Omega}(\nu,\kappa,M\tau)\le M\alpha_{p,\Omega}(\nu,\kappa,\tau)\,.$$

Proof. These inequalities are clear from the definition (19). In this connection, it is important to observe that g is bounded by 1 and $\mathrm{len}(\gamma)\le\nu d(x,\Omega^c)$. □

4. Key theorem for self-improvement

In this section we formulate and prove our key Theorem 4.1. In the light of Lemma 3.4, Theorem 4.1 implies self-improvement of pointwise p-Hardy inequalities; see Theorem 5.1. This theorem also provides a converse of Lemma 3.4 for $p>1$; see Theorem 5.3.

Lemmata 2.3 and 3.1 give us the proper tools for the proof of Theorem 4.1. We assume that X supports a better p_0-Poincaré inequality for some $p_0<p$.

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This assumption allows us to focus on the new phenomena that arise especially in connection with the self-improvement of pointwise p-Hardy inequalities.

Theorem 4.1. *Let $1 \le p_0 < p < \infty$. Assume that X supports a p_0-Poincaré inequality. Let $\emptyset \neq \Omega \subsetneq X$ be an open set that satisfies a pointwise p-Hardy inequality. Then there exists an exponent $q \in (p_0, p)$ and constants $N > C_{\rm QC}$, $K \ge 1$ and $C_\alpha > 0$ such that*

$$\alpha_{q,\Omega}(N,K,\tau) \le C_\alpha \tau \tag{21}$$

whenever $\tau \ge 0$.

Proof. By Hölder's inequality, we can assume that $\max\{1, p/2\} \le p_0$. This assumption allows us to choose M below independent of p. This property, in turn, is beneficial in Remark 5.2, where a quantitative analysis is performed. Since Ω satisfies a pointwise p-Hardy inequality, by Lemma 3.1 it satisfies inequality (14) with constants $C_\Gamma > 0$, $\nu_\Gamma > C_{\rm QC}$ and $\kappa_\Gamma \ge 1$. Also, let $C_{\rm A} > 0$, $\nu_{\rm A} > C_{\rm QC}$ and $\kappa_{\rm A} \ge 1$ be the constants from inequality (6) in Lemma 2.3, for the exponent $p_0 < p$. Without loss of generality, we may assume that $\kappa_\Gamma = \kappa_{\rm A} =: \kappa$ and that $\nu_\Gamma = \nu_{\rm A} =: \nu$.

It suffices to prove that there exists $k \in \mathbb{N}$, $K, S \in [1,\infty)$, $N \in (C_{\rm QC}, \infty)$, $M > 1$ and $\delta \in (0,1)$ such that, for each $q \in (p_0, p)$ and every $\tau > 0$, we have

$$\alpha_{q,\Omega}(N,K,\tau) \le S\tau + \delta \max_{i=1,\dots,k} \left(M^{-iq/p} \alpha_{q,\Omega}(N,K,M^i\tau) \right). \tag{22}$$

Indeed, from this inequality and Lemma 3.5, we get

$$\alpha_{q,\Omega}(N,K,\tau) \le S\tau + \delta M^{k\frac{p-q}{p}} \alpha_{q,\Omega}(N,K,\tau) \qquad \text{for all } q \in (p_0,p) \text{ and } \tau > 0\,.$$

In order to absorb the last term on the right to the left, we need $\delta M^{k\frac{p-q}{p}} < 1$. This can be ensured by choosing $q \in (p_0, p)$ so close to p that

$$0 < p - q < \frac{p \ln(\frac{1}{\delta})}{k \ln(M)}\,.$$

With this choice of q we find for all $\tau > 0$ that

$$\alpha_{q,\Omega}(N,K,\tau) \le \left(\frac{S}{1 - \delta M^{k\frac{p-q}{p}}} \right) \tau =: C_\alpha \tau\,.$$

Then, this inequality holds also for $\tau = 0$, which is seen by using monotonicity property of the α-function, see Lemma 3.5. Thus, the desired inequality (21) follows from (22). Hence, we are left with proving inequality (22).

At this stage, we fix the auxiliary parameters

$$K = 4\kappa\,, \quad N = 3\nu\,, \quad M = 4\,, \quad \delta = \frac{1}{4}\,.$$

We also fix $k \in \mathbb{N}$ so large that $C_\Gamma^p \frac{2^p D^5}{k^{p-1}} < \delta^p$, that is, $k > (2^p \delta^{-p} C_\Gamma^p D^5)^{\frac{1}{p-1}}$. The last auxiliary parameter is defined to be $S = 1 + M^k \nu + 3C_{\rm A} M^k$. We also let $q \in (p_0, p)$ and $\tau > 0$. Now, the overall strategy is as follows: we will construct, for any $x \in \Omega$ and any $g \in \mathcal{E}^{K,\tau}_{q,x,\Omega}$, a curve $\gamma \in \Gamma(X)^N_{x,\Omega^c}$ such that

$$\int_\gamma g\, ds \le S\tau d(x,\Omega^c) + \delta \max_{i=1,\dots,k} \left(M^{-iq/p} \alpha_{q,\Omega}(N,K,M^i\tau) \right) d(x,\Omega^c)\,. \tag{23}$$

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Dividing both sides of this estimate by $d(x,\Omega^c)$, and then taking the supremum over x and g as above, proves inequality (22).

Let us fix $x \in \Omega$ and $g \in \mathcal{E}^{K,\tau}_{q,x,\Omega}$. For each $i \geq 1$, we write

$$E_i := \{z \in \Omega \mid \mathcal{M}_{q,\kappa d(x,\Omega^c)} g(z) > M^i \tau\},$$

and define a bounded function $h\colon X \to [0,\infty)$ by setting

$$h = \frac{1}{k}\sum_{i=1}^{k} \mathbf{1}_{E_i} M^{iq/p}.$$

Since $E_j \supset E_i$ if $j \leq i$ and $p/2 \leq p_0 < q < p$, we have

$$h^p \leq \frac{1}{k^p}\sum_{j=1}^{k}\bigg(\sum_{i=1}^{j} M^{iq/p}\bigg)^p \mathbf{1}_{E_j} \leq \frac{2^p}{k^p}\sum_{j=1}^{k}\mathbf{1}_{E_j} M^{jq}.$$

In the final estimate, we also use the equation $M = 4$ to obtain the factor 2^p. Observe that $\mathbf{1}_{E_i} \in LC_0(\Omega)$ since E_i is open, for each $i = 1, \ldots, k$. Hence, we have $h \in LC_0(\Omega) \subset LC(X)$. By sublinearity and monotonicity of the maximal function, Lemma 2.4, and the assumption that $g \in \mathcal{E}^{K,\tau}_{q,x,\Omega}$, where $K = 4\kappa$, we obtain

$$\begin{aligned}\big(\mathcal{M}_{p,\kappa d(x,\Omega^c)} h(x)\big)^p &\leq \frac{2^p}{k^p}\sum_{j=1}^{k}(\mathcal{M}_{1,\kappa d(x,\Omega^c)}\mathbf{1}_{E_j}(x))M^{jq} \\ &\leq \frac{2^p D^5}{k^p}\sum_{j=1}^{k}\frac{(\mathcal{M}_{q,4\kappa d(x,\Omega^c)} g(x))^q}{M^{jq}\tau^q} M^{jq} \leq \frac{2^p D^5}{k^{p-1}}.\end{aligned} \tag{24}$$

Then, by the choice of k and estimate (24), we obtain that $C_\Gamma \mathcal{M}_{p,\kappa d(x,\Omega^c)} h(x) < \delta$, and therefore from Lemma 3.1 with exponent p we obtain a curve $\gamma_0 \in \Gamma(X)^\nu_{x,\Omega^c}$, which is parametrized by arc length, such that

$$\int_{\gamma_0} \frac{1}{k}\sum_{i=1}^{k}\mathbf{1}_{E_i} M^{iq/p}\, ds = \int_{\gamma_0} h\, ds \leq \delta d(x,\Omega^c), \tag{25}$$

and

$$\operatorname{len}(\gamma_0) \leq \nu d(x,\Omega^c). \tag{26}$$

Clearly, without loss of generality, we may also assume that $\gamma_0([0,\operatorname{len}(\gamma_0))) \subset \Omega$.

By inequality (25), there exists $i_0 \in \{1,\ldots,k\}$ such that

$$\int_{\gamma_0} \mathbf{1}_{E_{i_0}}\, ds \leq \delta M^{-i_0 q/p} d(x,\Omega^c). \tag{27}$$

Let $O = \gamma_0^{-1}(E_{i_0})$ and denote $T = [0,\operatorname{len}(\gamma_0)] \setminus O$. By the lower semicontinuity of g and the definition of E_{i_0} we have, for all $t \in T \setminus \{\operatorname{len}(\gamma_0)\}$,

$$g(\gamma_0(t)) \leq \mathcal{M}_{q,\kappa d(x,\Omega^c)} g(\gamma_0(t)) \leq M^{i_0}\tau. \tag{28}$$

Since E_{i_0} is open in X, the set O is relatively open in $[0,\operatorname{len}(\gamma_0)]$. Observe that $0 \notin O$ since $g \in \mathcal{E}^{K,\tau}_{q,x,\Omega}$ and $K > \kappa$. Likewise $\operatorname{len}(\gamma_0) \notin O$ since $\gamma_0(\operatorname{len}(\gamma_0)) \in \Omega^c$. There are now essentially two different cases to be handled; the remaining cases of corresponding finite unions are treated in a similar way. Namely, the two cases are:

$$O = \bigcup_{i\in\mathbb{N}}(a_i,b_i) \quad \text{or} \quad O = (a_0,b_0) \cup \bigcup_{i\in\mathbb{N}}(a_i,b_i), \tag{29}$$

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where the interval (a_0, b_0) has a certain special property, to be shortly explained, that distinguishes it from the intervals (a_i, b_i), $i \in \mathbb{N} = \{1, 2, \ldots\}$. The second case takes place, if there exists $0 < t_0 < \mathrm{len}(\gamma_0)$ such that $\gamma_0(t) \in E_{i_0}$ for every $t_0 < t < \mathrm{len}(\gamma_0)$. In both cases the intervals (called 'gaps') are pairwise disjoint and $a_i < b_i < \mathrm{len}(\gamma_0)$ for each $i \in \mathbb{N}$, and in the second case $a_0 < b_0 = \mathrm{len}(\gamma_0)$. Moreover, in both cases $\gamma_0(a_i), \gamma_0(b_i) \in \Omega \setminus E_{i_0}$ for each $i \in \mathbb{N}$, and in the second case $\gamma_0(a_0) \in \Omega \setminus E_{i_0}$. We remark that in the second case $\gamma_0(b_0) \notin \Omega \setminus E_{i_0}$, and this special property of the 'final gap' (a_0, b_0) distinguishes it from the remaining gaps. Write $d_i := d(\gamma_0(a_i), \gamma_0(b_i))$ for each i. Then, by inequality (27), we have

(30)
$$\sum_i d_i \le \sum_i \mathrm{len}(\gamma_0|_{[a_i,b_i]}) = \sum_i \int_{\gamma_0|_{[a_i,b_i]}} \mathbf{1}_{E_{i_0}}\, ds \le \int_{\gamma_0} \mathbf{1}_{E_{i_0}}\, ds \le \delta M^{-i_0 q/p} d(x, \Omega^c)\,.$$

There are now two cases to be treated in a case study.

Let us first consider the case $O = \bigcup_{i\in\mathbb{N}} (a_i, b_i)$. Fix $i \in \mathbb{N}$. Since $\gamma_0(a_i), \gamma_0(b_i) \in \Omega \setminus E_{i_0}$, there holds

$$\mathcal{M}_{q,\kappa d(x,\Omega^c)} g(\gamma_0(a_i)) \le M^{i_0}\tau \quad \text{and} \quad \mathcal{M}_{q,\kappa d(x,\Omega^c)} g(\gamma_0(b_i)) \le M^{i_0}\tau\,. \tag{31}$$

Lemma 2.3 applied to the p_0-Poincaré inequality, and to the two points $\gamma_0(a_i)$ and $\gamma_0(b_i)$, provides us with a curve $\gamma^i\colon [a_i, b_i] \to X$ such that $\gamma^i(a_i) = \gamma_0(a_i)$, $\gamma^i(b_i) = \gamma_0(b_i)$,

$$\mathrm{len}(\gamma^i) \le \nu d(\gamma_0(a_i), \gamma_0(b_i)) = \nu d_i\,, \tag{32}$$

and, by using also the fact that $p_0 < q$ and Hölder's inequality,

(33)
$$\begin{aligned}&\int_{\gamma^i} g\, ds \\ &\le C_{\mathrm{A}} d(\gamma_0(a_i), \gamma_0(b_i)) \left(\mathcal{M}_{q,\kappa d(\gamma_0(a_i),\gamma_0(b_i))} g(\gamma_0(a_i)) + \mathcal{M}_{q,\kappa d(\gamma_0(a_i),\gamma_0(b_i))} g(\gamma_0(b_i))\right) \\ &\qquad + \underbrace{C_{\mathrm{A}} d(\gamma_0(a_i), \gamma_0(b_i)) M^{i_0}\tau}_{>0}\,.\end{aligned}$$

We observe that $\kappa d(\gamma_0(a_i), \gamma_0(b_i)) \le \kappa d(x, \Omega^c)$, which follows from (30) since

$$d(\gamma_0(a_i), \gamma_0(b_i)) = d_i \le \sum_i d_i \le d(x, \Omega^c)\,.$$

This estimate together with (31) and (33) yields

$$\int_{\gamma^i} g\, ds \le 3C_{\mathrm{A}} d(\gamma_0(a_i), \gamma_0(b_i)) M^{i_0}\tau = 3C_{\mathrm{A}} M^{i_0}\tau d_i\,. \tag{34}$$

Let us now define a curve $\gamma\colon [0, \mathrm{len}(\gamma_0)] \to X$ by setting $\gamma(t) = \gamma_0(t)$ if $t \in T$ and $\gamma(t) = \gamma^i(t)$ if $t \in (a_i, b_i)$ for some $i \in \mathbb{N}$ that is uniquely determined by t. Then, by the length estimates (26) and (32), followed by inequality (30), we obtain that

$$\begin{aligned}\mathrm{len}(\gamma) &\le \mathrm{len}(\gamma_0) + \sum_{i\in\mathbb{N}} \mathrm{len}(\gamma^i) \\ &\le \nu d(x, \Omega^c) + \nu \sum_{i\in\mathbb{N}} d_i \le 2\nu d(x, \Omega^c) \le N d(x, \Omega^c)\,.\end{aligned}$$

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From this it follows that $\gamma \in \Gamma(X)^N_{x,\Omega^c}$; we remark that the required continuity and connecting properties of γ are straightforward establish, and we omit the details. Also, by inequalities (26), (28), (30) and (34), we have

$$\begin{aligned}\int_\gamma g\, ds &= \int_T g(\gamma_0(t))\, dt + \sum_{i\in\mathbb{N}} \int_{\gamma^i} g\, ds \\ &\le M^{i_0}\tau\nu d(x,\Omega^c) + 3C_{\mathrm{A}} M^{i_0}\tau\delta M^{-i_0 q/p} d(x,\Omega^c) \\ &\le (M^{i_0}\nu + 3C_{\mathrm{A}} M^{i_0})\tau d(x,\Omega^c) \le S\tau d(x,\Omega^c)\,.\end{aligned}$$

In the present case, we have now constructed a curve γ such that inequality (23) holds, even without the absorption term. We are done in the first case of (29).

Next we consider the slightly more complicated case $O = (a_0, b_0) \cup \bigcup_{i\in\mathbb{N}}(a_i, b_i)$, in which there is also a final gap (a_0, b_0) such that $b_0 = \mathrm{len}(\gamma_0)$ and $\gamma_0(b_0) \in \Omega^c$. As in the previous case, for each $i \in \mathbb{N}$, we can first construct curves $\gamma^i \colon [a_i, b_i] \to X$ such that

$$\mathrm{len}(\gamma^i) \le \nu d(\gamma_0(a_i), \gamma_0(b_i)) = \nu d_i\,, \tag{35}$$

and

$$\int_{\gamma^i} g\, ds \le 3C_{\mathrm{A}} d(\gamma_0(a_i), \gamma_0(b_i)) M^{i_0}\tau = 3C_{\mathrm{A}} M^{i_0}\tau d_i\,. \tag{36}$$

For $i = 0$ we have to be more careful, since $\gamma_0(b_0) \notin \Omega \setminus E_{i_0}$. We now proceed as follows. By using (30) and the equality $K\delta = \kappa$, we first observe that

$$Kd(\gamma_0(a_0), \Omega^c) \le Kd(\gamma_0(a_0), \gamma_0(b_0)) = Kd_0 \le K\delta d(x,\Omega^c) = \kappa d(x,\Omega^c)\,.$$

On the other hand, we still have that $\gamma_0(a_0) \in \Omega \setminus E_{i_0}$, and thus

$$\mathcal{M}_{q,Kd(\gamma_0(a_0),\Omega^c)} g(\gamma_0(a_0)) \le \mathcal{M}_{q,\kappa d(x,\Omega^c)} g(\gamma_0(a_0)) \le M^{i_0}\tau\,.$$

From this it follows that $g \in \mathcal{E}^{K,M^{i_0}\tau}_{q,\gamma_0(a_0),\Omega}$. By definition (19) of $\alpha_{q,\Omega}(N,K,M^{i_0}\tau)$, we obtain a curve $\gamma^0 \colon [a_0, b_0] \to X$ connecting $\gamma_0(a_0) \in \Omega$ to Ω^c such that

$$\mathrm{len}(\gamma^0) \le Nd(\gamma_0(a_0), \Omega^c) \le Nd(\gamma(a_0), \gamma(b_0)) = Nd_0 \tag{37}$$

and

$$\begin{aligned}\int_{\gamma^0} g\, ds &\le d(\gamma_0(a_0), \Omega^c)\alpha_{q,\Omega}(N,K,M^{i_0}\tau) + \underbrace{\tau d(x,\Omega^c)}_{>0} \\ &\le d_0 \alpha_{q,\Omega}(N,K,M^{i_0}\tau) + \tau d(x,\Omega^c)\,.\end{aligned} \tag{38}$$

Now we define γ as in the first case but using also the final gap (a_0, b_0) by setting $\gamma(t) = \gamma^0(t)$ for every $t \in (a_0, b_0]$. Then by (26), (30), (37), and our choice of N and δ, we obtain

$$\mathrm{len}(\gamma) \le \mathrm{len}(\gamma_0) + \mathrm{len}(\gamma^0) + \sum_{i\in\mathbb{N}} \mathrm{len}(\gamma^i) \le (\nu + \delta N + \nu) d(x,\Omega^c) \le Nd(x,\Omega^c)\,.$$

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Thus, we find that $\gamma \in \Gamma(X)^N_{x,\Omega^c}$. Finally, by inequalities (26), (28), (30), (36), and (38) we have

$$\begin{aligned}\int_\gamma g\,ds &= \int_T g(\gamma_0(t))\,dt + \sum_{i\in\mathbb{N}} \int_{\gamma^i} g\,ds + \int_{\gamma^0} g\,ds \\ &\le M^{i_0}\tau\nu d(x,\Omega^c) + 3C_{\mathrm{A}}M^{i_0}\tau d(x,\Omega^c) + d_0\alpha_{q,\Omega}(N,K,M^{i_0}\tau) + \tau d(x,\Omega^c) \\ &\le S\tau d(x,\Omega^c) + \delta M^{-i_0 q/p}\alpha_{q,\Omega}(N,K,M^{i_0}\tau)d(x,\Omega^c)\,.\end{aligned}$$

Recall that $i_0 \in \{1,\dots,k\}$. Hence, the desired estimate (23) for γ follows and thus the proof is complete. □

5. Main results

As a consequence of Theorem 4.1 and Lemma 3.4, we obtain the following theorem. It is the main result of the present paper.

Theorem 5.1. *Let $1 \le p_0 < p < \infty$. Assume that X supports a p_0-Poincaré inequality (5). Let $\emptyset \ne \Omega \subsetneq X$ be an open set that satisfies a pointwise p-Hardy inequality (4). Then there exists an exponent $q \in (p_0,p)$ such that Ω satisfies a pointwise q-Hardy inequality.*

Remark 5.2. The conclusion of Theorem 5.1 reads as follows: there exists $q \in (p_0,p)$ such that Ω satisfies a pointwise q-Hardy inequality. We can establish a more quantitative result. Indeed, by examining the proof of Theorem 4.1, we see that it runs through if p, p_0 and q satisfy the following inequalities

$$\max\{1,p/2\} \le p_0 < q < p \qquad \text{and} \qquad \delta M^{k\frac{p-q}{p}} < 1\,,$$

where $M = 4$, $\delta = \frac{1}{4}$ and $\mathbb{N} \ni k > (2^p\delta^{-p}C_\Gamma^pD^5)^{\frac{1}{p-1}}$. Here $C_\Gamma > 0$ is the constant appearing in inequality (14). This inequality characterizes the pointwise p-Hardy inequality. Thus, we can choose

$$k := \lceil (8C_\Gamma)^{\frac{p}{p-1}}D^{\frac{5}{p-1}} + 1\rceil > (8C_\Gamma)^{\frac{p}{p-1}}D^{\frac{5}{p-1}} = (2^p\delta^{-p}C_\Gamma^pD^5)^{\frac{1}{p-1}}\,.$$

Then $\delta M^{k\frac{p-q}{p}} < 1 \Leftrightarrow 4^{k\frac{p-q}{p}} < 4 \Leftrightarrow p-q < \frac{p}{k}$. On the other hand, by examining the proof of Lemma 3.1, we have $C_\Gamma = 4C_{\mathrm{H}}$, where $C_{\mathrm{H}} > 0$ is the constant in the assumed pointwise p-Hardy inequality (4). All in all, we find that if the assumptions of Theorem 5.1 hold,

$$\max\{1,p/2\} \le p_0 < q < p \qquad \text{and} \qquad p-q < \frac{p}{\lceil (32C_{\mathrm{H}})^{\frac{p}{p-1}}D^{\frac{5}{p-1}} + 1\rceil}\,,$$

then Ω satisfies a pointwise q-Hardy inequality. Rather similar quantitative bounds for the self-improvement of p-Poincaré inequalities can be found in [4].

Theorem 5.3. *Let $1 \le p_0 < p < \infty$. Assume that X supports a p_0-Poincaré inequality. Let $\emptyset \ne \Omega \subsetneq X$ be an open set. Then the following conditions are equivalent:*

(A) *The open set Ω satisfies a pointwise p-Hardy inequality;*
(B) *There are constants $\nu > C_{\mathrm{QC}}$, $\kappa \ge 1$ and $C_\alpha > 0$ such that, for any $\tau \ge 0$, we have*

$$\alpha_{p,\Omega}(\nu,\kappa,\tau) \le C_\alpha\tau\,.$$

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(C) *There are constants $C_\Gamma > 0$, $\nu > C_{\mathrm{QC}}$ and $\kappa \geq 1$ such that for each non-negative and bounded $g \in LC_0(\Omega)$ and every $x \in \Omega$, we have*

$$\inf_{\gamma\in\Gamma(X)^{\nu}_{x,\Omega^c}} \int_\gamma g\, ds \leq C_\Gamma\, d(x,\Omega^c) \left(\mathcal{M}_{p,\kappa d(x,\Omega^c)} g(x)\right).$$

Proof. The implication from (A) to (B) follows from Theorem 4.1 and the pointwise estimate $\alpha_{p,\Omega} \leq \alpha_{q,\Omega}$ that trivially is valid if $p \geq q$. The converse follows from Theorem 3.4. The implication from (A) to (C) is a consequence of Lemma 3.1. On the other hand, by inspecting the proof of Theorem 4.1, we find that condition (C) implies (A). In particular, the test function h that is constructed in the proof actually belongs to $LC_0(\Omega)$. □

Remark 5.4. By combining Theorem 5.1 and Theorem 5.3 one obtains self-improvement of further inequalities (B) and (C) in Theorem 5.3; these inequalities are both equivalent with the pointwise p-Hardy inequality. We remark that inequality (C) differs from the characterizing condition appearing in Lemma 3.1 in that the test functions g in (C) are required to vanish outside Ω. The self-improvement results for the conditions (B) and (C) are naturally also subject to a better p_0-Poincaré inequality; we omit the explicit formulations.

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(S.E.-B.) Department of mathematics, UCLA, 520 Portola Plaza, Los Angeles CA 90095, USA

E-mail address: syerikss@math.ucla.edu

(A.V.V.) University of Jyvaskyla, Department of Mathematics and Statistics, P.O. Box 35, FI-40014 University of Jyvaskyla, Finland

E-mail address: antti.vahakangas@iki.fi

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