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Multi-scenario multi-objective robust optimization under deep uncertainty: A posteriori approach

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A B S T R A C T

This paper proposes a novel optimization approach for multi-scenario multi-objective robust decision making, as well as an alternative way for scenario discovery and identifying vulnerable scenarios even before any solution generation. To demonstrate and test the novel approach, we use the classic shallow lake problem. We compare the results obtained with the novel approach to those obtained with previously used approaches. We show that the novel approach guarantees the feasibility and robust efficiency of the produced solutions under all selected scenarios, while decreasing computation cost, addresses the scenario-dependency issues, and enables the decision-makers to explore the trade-off between optimality/feasibility in any selected scenario and robustness across a broader range of scenarios. We also find that the lake problem is ill-suited for reflecting trade-offs in robust performance over the set of scenarios and Pareto optimality in any specific scenario, highlighting the need for novel benchmark problems to properly evaluate novel approaches.

1. Introduction

Decision making in complex environmental problems typically involves several conflicting objectives to be considered simultaneously. There is no single optimal solution for these multi-objective problems because of the conflicting objectives. Instead, several so-called Pareto optimal solutions reflecting different trade-offs between the conflicting objectives can be found. In such multi-objective decision problems, decision support tools can help decision makers in balancing between conflicting objectives.

The task of decision makers in environmental problems is further complicated by the presence of uncertainty. To mitigate the potential negative consequences of uncertainty, it has been argued that decisions should have limited sensitivity to the consequences of uncertainty (so-called robust decision) and perform relatively well in a broad range of future states of the world or scenarios (Lempert et al., 2006). In environmental systems, the level of uncertainty is high and probabilities over the various alternative states of the world can only be approximately assessed. This kind of uncertainty is sometimes also known as deep uncertainty (Bankes 2002; Lempert et al., 2003; Kwakkel et al., 2010; Walker et al., 2013; Shavazipour and Stewart 2019). Uncertainties about future climate change and socio-economic conditions are two examples of deep uncertainty in environmental problems.

Therefore, decision makers in complex environmental problems are facing a multi-objective optimization problem to be solved in the presence of deep uncertainty, where the performance of a decision should be evaluated according to all objectives in all plausible scenarios (Shavazipour and Stewart 2019; Shavazipour et al., 2020; Stewart et al., 2013). This kind of decision problem is also known as a multi-scenario multi-objective decision making problem. Scenarios, in this paper, represent different plausible future realizations of the deep uncertainties (Maier et al., 2016). In practice, it is almost impossible to find a decision that is Pareto optimal (or even feasible) in all plausible scenarios. By a feasible solution in multi-scenario multi-objective optimization problems, we mean a solution that is feasible (i.e., satisfies all constraints) in all scenarios. Accordingly, decision makers seek robust solutions that are sufficiently good in a broad range of scenarios, i.e., robust satisficing. This introduces an additional trade-off between Pareto optimality (and feasibility) in any given scenario and robustness over a set of scenarios. In this paper, we refer to this as the trade-off between scenarios.

Recently, different approaches have been introduced for environmental multi-objective optimization problems under deep uncertainty, such as Many-Objective Robust Decision Making (MORDM) (Kasprzyk et al., 2013), multi-scenario MORDM (Watson and Kasprzyk 2017), and

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Multi-Objective Robust Optimization (MORO) (Hamarat et al., 2014; Kwakkel et al., 2015; Trindade et al., 2017). All these approaches are based on the Robust Decision Making (RDM) framework (Lempert et al., 2006; Groves and Lempert 2007). RDM is an iterative approach, where pre-specified policy alternatives/solutions are stress-tested over a wide range of scenarios in order to determine conditions under which each solution fails to perform adequately. Then, the policy alternatives are refined to find the most robust solution(s) in light of these failure conditions.

MORDM was introduced as an extension to RDM to help in generating a promising set of candidate solutions as input to the stress testing for decision problems involving multiple objectives. These solutions are produced using multi-objective optimization given a single reference scenario (i.e., only optimizing in the feasible region of a single scenario). This inadvertently introduces scenario dependency of the generated solutions (given that the Pareto approximation only includes solutions optimized in (and feasible for) that single scenario), which reduces both the robustness which can be attained during stress-testing (Eker and Kwakkel 2018; Giudici et al., 2020; Bartholomew and Kwakkel 2020), as well as the feasibility of the candidate solutions in other scenarios. To reduce this shortcoming, Watson and Kasprzyk (2017) proposed multi-scenario MORDM, which repeats the process of identifying candidate solutions prior to stress-testing for several scenarios. Expanding on this, Eker and Kwakkel (2018) introduced a more systematic scenario selection procedure that ensures high diversity among the scenarios which are used for the identification of candidate solutions. However, solutions generated with multi-scenario MORDM are still highly dependent on the selected scenarios. This is because multi-scenario MORDM does the search separately for few selected scenarios without checking the feasibility and performance of the solutions in the other scenarios during the optimization process. Therefore, there is no guarantee that the solutions generated are feasible in any other scenario.

Besides, gathering solutions generated by single-scenario optimizations cannot guarantee optimal robustness either. Of course, the feasibility and the performance of the solutions will be checked later in the robustness analysis. However, many of the solutions found in that way may have inferior performance (i.e., be dominated) in some scenarios or even be infeasible, which means wasting computational resources in finding poor solutions that will be eliminated later in the robustness analysis.

In contrast, MORO (Hamarat et al., 2014; Kwakkel et al., 2015; Trindade et al., 2017) only concentrates on robustness by optimizing the robustness measures as objective functions over a set of scenarios. However, this simulation-optimization approach is computationally demanding and (possibly) intractable, even for small sets of scenarios (Eker and Kwakkel 2018; Bartholomew and Kwakkel 2020; Giudici et al., 2020). Furthermore, utilizing different robustness measures results in different solutions. This highlights the meta-choice of selecting the most appropriate robustness measures, which might require a separate study (Giudici et al., 2020; Kwakkel et al., 2016; McPhail et al., 2018). Besides, the existing trade-offs between objectives in different scenarios cannot be explicitly verified in worst-case/min-max and aggregation-based robustness measures, which are often used in MORO. For instance, the overall robustness may be affected excessively because of poor performance in a few scenarios (Ben-Tal et al., 2017; Roos & den Hertog 2020; Shavazipour and Stewart 2019; Shavazipour et al., 2020). Bartholomew and Kwakkel (2020) compared MORDM, multi-scenario MORDM, and MORO, and confirmed that the more robustness is considered in the search for candidate solutions prior to stress-testing, the more robust the solutions will be. Nevertheless, there remains a trade-off between optimality (and feasibility) in any given scenario and robustness over the set of scenarios. This trade-off is sometimes also known as the price of robustness (Bertsimas and Sim 2004; Schöbel and Zhou-Kangas 2021). There is currently no approach for the search phase of MORDM that enables decision makers to explore this trade-off explicitly. Note that adding scenarios to a multi-objective optimization problem adds dimensions to the problem. Indeed, the resulting multi-scenario multi-objective optimization problem includes all objective-scenario combinations, in which the dimension of the space grows exponentially (e.g., in a problem with four objectives and five scenarios, the space becomes ($4\times 5 =$) 20-dimensional). This means that a solution may have an outstanding performance on one objective in one particular scenario, but its performance on other objectives may be poor, or the solution may even be infeasible in some scenarios. The previous variants of MORDM identify solutions in the objective space of a single-scenario problem. Then, they test the performance of these solutions on the uncertainty space constructed by an ensemble of random scenarios. This may imply losing robust solutions as well as the chance of exploring the mentioned trade-off.

In parallel to the continuous refinement of RDM via MORDM, multi-scenario MORDM and MORO, the concept of robustness in multi-objective optimization has been receiving theoretical attention as well. This has resulted in various novel theoretical concepts such as min-max robustness (Ehrgott et al., 2014), highly (Dranichak and Wiecek 2019), flimsy (Bitran 1980; Kuhn et al., 2016) and lightly robust efficiency (Ide and Schöbel 2016), regret robustness (Xidonas et al., 2017), and multi-scenario efficiency (Botte and Schöbel 2019; Shavazipour and Stewart 2019; Shavazipour et al., 2020). We refer the interested readers to Botte and Schöbel (2019), Ide and Schöbel (2016) and Schöbel and Zhou-Kangas (2021) for a review and comparison of different theoretical robustness concepts in multi-objective optimization. Although not all of these concepts and methods were primarily developed to deal with deep uncertainty, still, to some extent, they can be utilized for this purpose as a complement to the existing approaches in decision making under deep uncertainty (DMDU). Among these concepts, multi-scenario efficiency, defined particularly for a discrete uncertainty space (i.e., constructed with a finite number of scenarios), is similar to the concept of robustness utilized in the deep uncertainty literature.

The main difference between these two bodies of literature in how they use robustness concepts lies in where they are evaluating the robustness of a candidate solution. In mathematical optimization, robustness is often utilized as an a priori criterion or soft constraint in searching for candidate solutions leading to a particular set of solutions following that criterion/constraint (i.e., we are only looking for robust-efficient solutions). In contrast, in DMDU, the robustness of solutions is typically an attribute of a generated solution measured in an a posteriori manner (i.e., after the search phase). As a result, robustness in DMDU is used as an a posteriori measure for ranking already generated solutions. In mathematical optimization, all the robust-efficient solutions are compromise solutions distinguishable by different trade-offs between objective(s) in various scenarios. As the central common assumption, none of these two bodies of literature consider the probability of scenario occurrence in their definitions and models.

In a multi-scenario multi-objective decision making problem, ideally, candidate decisions are evaluated in terms of all objectives in all (or at least a representative set of selected) scenarios (Shavazipour and Stewart 2019; Shavazipour et al., 2020; Stewart et al., 2013). This kind of an assessment helps identifying solutions that are not only feasible in all (selected) scenarios but also robust efficient. That is, the used approach should guarantees that there exists no other solution which is not worse on all objectives in all selected scenarios and, is better on at least one objective in one scenario (Botte and Schöbel 2019; Shavazipour et al., 2021). Among the previously proposed methods developed to handle multiple objectives under deep uncertainty, only MORO can, to some extent, guarantee the robust efficiency of all generated solutions (without any extra filtering) in all (selected) scenarios (e.g., by considering all scenario-specific constraints within the optimization model).

The primary aim of this paper is to build a bridge between the literature on mathematical multi-objective optimization, which has a strong theoretical foundation, and the robust decision making literature which has shown successful real world applications. To the best of our knowledge, this is the very first step in this regard. By drawing on the theoretical developments in mathematical multi-objective optimization, we can address the issues of robust efficiency, feasibility, and the price of robustness which affect existing approaches for the search phase within the MORDM framework. In this paper, we propose a novel multi-scenario multi-objective robust optimization approach (called multi-scenario MORO, hereinafter) by incorporating uncertainties in the optimization phase and identify solutions that perform well in some (selected) scenarios. In this way, the performance of solutions in terms of all objectives in all selected scenarios are evaluated within a single optimization problem. As a result, the Pareto optimal solutions for considered scenarios can be found, which are not only feasible in all selected scenarios but also robust efficient, if any feasible solution is available.

In other words, we combine all single-scenario multi-objective optimization problems into a meta-optimization problem, and simultaneously consider the evaluation of the objective functions in multiple scenarios. Indeed, our objective functions include all the objective-scenario combinations (called meta-objective/meta-criteria (Stewart et al., 2013)) subject to constraints satisfaction in all considered scenarios. The proposed multi-scenario MORO has both a lower computation cost and increased robustness consideration during the search process, it generates less scenario dependent solutions. For our proposed approach, and likewise for other approaches for multi-scenario MORDM, selected scenarios should reflect the system vulnerabilities and/or the main decision maker's preferences (Giudici et al., 2020).

The classic shallow lake problem, first introduced by Carpenter et al. (1999), has been very often used to demonstrate, test, and compare methodological developments for decision making under deep uncertainty (Kwakkel 2017; Lempert and Collins 2007; Singh et al., 2015; Bartholomew and Kwakkel 2020; Eker and Kwakkel 2018; Quinn et al., 2017; Singh et al. 2015, 2015; Ward et al., 2015). It is a standard benchmark problem reflecting the required characteristics of real-world environmental problems such as tipping points affected by deeply uncertain parameters and multiple conflicting objectives. Therefore, we use the shallow lake problem to demonstrate our novel approach and compare it with existing approaches.

In brief, the main contributions of this paper are: (1) Proposing a multi-scenario MORO approach, which utilizes a different solution method from the mathematical multi-objective optimization literature to produce candidate solutions that reduce the computational cost and, also, can guarantee Pareto optimality; (2) Paving the way to explore the trade-offs between scenario-specific Pareto optimality and robustness by considering different numbers of scenarios in the optimization model; and (3) Introducing a novel way of scenario analysis to determine vulnerable scenarios using ideal points (best possible achievements on each objective in each scenario) and information about feasible regions in various scenarios. Since the proposed scenario analysis does not need any prior knowledge of solutions and their robustness, the decision makers can gain insight into the problem before solution determination.

The rest of the paper is organized as follows. Section 2 includes a brief description of multi-scenario multi-objective optimization problems and the solution method utilizing in this study, as well as the proposed multi-scenario MORO approach. The lake problem, as our case study, and the multi-scenario formulation of it are described in Section 3. In Section 4, we illustrate more details about how the proposed multi-scenario MORO can be applied and compare the results with the state-of-the-art methods in the literature. Finally, we discuss the feasibility, robustness, and computational costs of multi-scenario MORO regarding the different number of scenario considerations in Section 5, before concluding in Section 6.

2. Methods

2.1. Multi-scenario multi-objective optimization

A multi-scenario multi-objective optimization problem (MSMOP), also called all-in-one or scenario-based multi-objective optimization problem with $k \geq 2$ objective functions and $s \geq 2$ scenarios can be formulated as follows (Shavazipour et al., 2021):

$$\begin{array}{ll} \text{minimize} & \{f_{1p}(\mathbf{x}), \dots, f_{kp}(\mathbf{x})\}, \quad p \in \Omega \\ \text{subject to} & \mathbf{x} \in S \subseteq \mathbb{R}^n, \end{array} \tag{1}$$

where the scenario space/set Ω is constructed by s plausible scenarios and each scenario includes k objective functions. Objective functions in scenario p ($p \in \{1, \dots, s\}$) are described by f_{ip} ($i = 1, \dots, k$), $\mathbf{x} = (x_1, \dots, x_n)^T$ is a vector of n decision variables in the feasible region S in the decision space $\mathbb{R}^n$ ($S \subseteq \mathbb{R}^n$) defined by constraint functions, $\mathbf{z}_p = (f_{1p}(\mathbf{x}), \dots, f_{kp}(\mathbf{x}))^T$ ($p \in \{1, \dots, s\}$) (called an *objective vector*) is the image of a decision vector $\mathbf{x}$ in the *objective space* $\mathbb{R}^k$ under the conditions of scenario p. A decision vector $\mathbf{x}^* \in S$ is called Pareto optimal (also called non-dominated) in scenario p if, under the conditions of scenario p, there does not exists another $\mathbf{x} \in S$ such that for all i, $f_{ip}(\mathbf{x}) \leq f_{ip}(\mathbf{x}^*)$ and $f_{jp}(\mathbf{x}) < f_{jp}(\mathbf{x}^*)$ for at least one index j. The image of the set of Pareto optimal decision vectors in the objective space is sometimes called a Pareto front. $\widehat{\mathbf{x}} \in S$ is *weakly* Pareto optimal in scenario p if, there does not exists another $\mathbf{x} \in S$ such that for all i, $f_{ip}(\mathbf{x}) < f_{ip}(\widehat{\mathbf{x}})$. A preferred solution refers to a Pareto optimal solution satisfying decision maker's preferences in terms of all ($k \times s$) meta-criteria. For any two objective vectors $\mathbf{z}'_p$, $\mathbf{z}''_p \in \mathbb{R}^k$, in scenario p, we say that $\mathbf{z}'_p$ *dominates* $\mathbf{z}''_p$ if and only if for all i, $z'_{ip} \leq z''_{ip}$ and $z'_{pj} < z''_{jp}$ for at least one index j.

The best and the worst possible values for individual objectives in the Pareto front are components of an *ideal point* $\mathbf{z}^{ideal} = (z_{11}^{ideal}, \dots, z_{ks}^{ideal})^T$ and a *nadir point* $\mathbf{z}^{\text{nadir}} = (z_{11}^{\text{nadir}}, \dots, z_{ks}^{\text{nad}})^T$, respectively. While ideal points can be simply calculated by solving a relevant single-scenario single-objective optimization problem, computing nadir points is difficult in practice. However, their estimations can either be provided by the decision maker or approximated, for instant, through a pay-off table (see, e.g., Miettinen (1999) and references therein). Also, $z_{ip}^{uto} = z_{ip}^{ideal} - \varepsilon$ ($i = 1, \dots, k; p = 1, \dots, s$) are components of an objective vector $\mathbf{z}_p^{uto} \in \mathbb{R}^n$, called a *Utopian objective vector* in scenario p, where $\varepsilon > 0$ is a relatively small scalar. It is strictly better than the ideal point.

2.2. Generating candidate solutions - achievement scalarizing function

Over the years, many different methods have been proposed to solve multi-objective optimization problems. The two most popular type of methods are 1) Multiple Criteria Decision Making (MCDM) (e.g., Chankong and Haimes (1983); Miettinen (1999)) and 2) Evolutionary Multi-objective Optimization (EMO) (e.g., Coello et al. (2007); Deb (2001)). A major advantage of EMO algorithms is that they generate a set of approximated Pareto optimal solutions in a single run of the algorithm. However, EMO algorithms tend to be inefficient when the number of objectives increases. In contrast, MCDM method guarantee Pareto optimality, have a strong theoretical foundation, and no limitation regarding the number of objectives. Many MCDM methods transform the original problem into a single-objective optimization problems (using a so-called scalarizing function) considering the decision maker's preferences (see, e.g. (Miettinen 1999), for more information on different MCDM methods and (Miettinen and Mäkelä 2002; Ruiz et al., 2009) for a comparison of various scalarizing functions).

So far, all variants of MORDM have utilized EMO algorithms. To the best of our knowledge, MCDM methods have not yet been utilized within the MORDM framework. As also mentioned in Kasprzyk et al. (2013), the main reason is related to the use of a priori (importance) weights (as one form of preference information) because of some concerns about the accuracy of these weights before the decision maker observes a broader set of solutions and gains a better understanding about the non-convexity/continuity of the Pareto front and potential non-linear relations between variables/parameters. However, on the one hand, a priori methods are designed to be used in problems in which the decision maker has a good enough understanding of the problem, interdependencies of the objectives, and possible outcomes and is able (or wishes) to express his/her expertise a priori. On the other hand, there are also other types of MCDM methods: interactive and a posteriori methods (see, e.g. (Miettinen 1999),). Furthermore, when the number of objectives grows, EMO algorithms cannot efficiently approximate the Pareto front. Thus, they cannot be used to solve MSMOPs which usually have tens or hundreds of objectives. To overcome this issue, in this paper, we utilize one of the most popular scalarizing functions, i.e., an achievement scalarizing function (Wierzbicki 1986). More specifically, we use the following achievement scalarizing function, which includes an augmentation term to avoid weakly Pareto optimal solutions (Wierzbicki 1986),

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{i=1,\dots,k}[w_i(f_i(\mathbf{x}) - \bar{z}_i)] + \varepsilon \sum_{i=1}^{k} w_i(f_i(\mathbf{x}) - \bar{z}_i) \\ s.t. \quad & \mathbf{x} \in S \end{aligned} \tag{2}$$

where w_i $(i = 1, \dots, k)$ are the weights for normalization.

As preference information set by a decision maker, $\bar{z}_i (i = 1, \dots, k)$, known as *aspiration levels*, represent desirable objective function values. The vector of k aspiration levels is called a *reference point*. A reference point in the objective space can be feasible or infeasible. In any case, a scalarizing function like (2) can identify the closest Pareto optimal solution to the given reference point. Accordingly, utilizing different reference points tends to lead to different Pareto optimal solutions. However, sometimes the same Pareto optimal solution may be associated with multiple reference points. The multiplier ε is a small positive number, and $\varepsilon \sum_{i=1}^{k} w_i(f_i(\mathbf{x}) - \bar{z}_i)$ is an augmentation term ensuring Pareto optimality. Thus, the optimal solution to problem (2) is a Pareto optimal solution to the original multi-objective optimization problem (Wierzbicki 1986; Miettinen 1999).

When we have multiple scenarios, the above formulation has an additional dimension:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{i=1,\dots,k;p=1,\dots,s}[w_{ip}(f_{ip}(\mathbf{x}) - \bar{z}_{ip})] + \varepsilon \sum_{i=1}^{n} \sum_{p=1}^{s} [w_{ip}(f_{ip}(\mathbf{x}) - \bar{z}_{ip})] \\ s.t. \quad & \mathbf{x} \in S \end{aligned} \tag{3}$$

where w_{ip} represents the weight for objective i in scenario p and $\bar{z}_{ip}$ is the aspiration level for the ith objective function in the pth scenario.

Solving the scalarized problem (3) provides a single optimal solution which is Pareto optimal for the original problem (1). Different solutions can be generated by using different reference points. Therefore, in contrast to EMO methods, to produce a set of Pareto optimal solutions, one needs to repeatedly solve the optimization problem using different reference points. Nonetheless, considering the decision maker's preferences (reference points here) not only gives rise to generating solutions that lie in the areas of interest to the decision makers, but it can also confine the search area and thus reduce computational cost.

In general, a decision maker may provide preferences a *priori*, choose a preferred solution among the provided set of solutions in a *posteriori* method, or be iteratively involved using an *interactive* approach (Miettinen 1999). Achievement scalarizing functions can be applied in any of these three ways. In this paper, following all variants of RDM, we utilize scalarizing functions in an a posteriori method. Therefore, we need to predetermine several reference points to produce different Pareto optimal solutions. Amongst various techniques that have been developed for setting the reference points, we utilize the method introduced by Mueller-Gritschneder et al. (2009).

2.3. The proposed multi-scenario multi-objective robust optimization approach

(MO)RDM is an iterative approach for finding robust solution(s). It consists of four steps: (1) model specification; (2) solution identification; (3) computational exploration, i.e., re-evaluation of candidate solutions in a broad range of plausible scenarios; and (4) scenario discovery, in which vulnerable scenarios are identified and this information can be used to modify the model and/or generate new solutions. This process continues until the decision maker is satisfied with a (set of) solution(s) (Lempert et al., 2006; Kasprzyk et al., 2013).

There exist three different approaches for identifying policy alternatives in the second step of MORDM: single-scenario (Kasprzyk et al., 2013), multiple single-scenario (Eker and Kwakkel 2018; Watson and Kasprzyk 2017), and robust optimization (Hamarat et al., 2014; Kwakkel et al., 2015; Trindade et al., 2017). However, they have shortcomings, e.g., scenario dependency (in the first two variants) and inability to reflect the trade-offs between scenarios (in all three variants). To overcome these weaknesses, we propose a novel multi-scenario multi-objective robust optimization approach that simultaneously considers multiple objectives in multiple scenarios (not an indirect aggregated value over a set of scenarios); i.e., the proposed multi-scenario MORO approach performs the search in a combined multi-scenario multi-objective space. In this way, all the generated solutions are robust-efficient in all (selected) scenarios, which increases robustness and reduces scenario dependency. Indeed, we propose to use the multi-scenario multi-objective optimization approach (model (1)) to generate solutions in the second step (search phase) of the (MO)RDM.

Yet, the proposed multi-scenario MORO involves four iterative steps portraying in Fig. 1 and detailed as follows:

1. **Model specification:** Determining the components of a decision making problem, such as the decisions to be made, decision variables, certain and uncertain parameters and relations between them, how to measure performance like objective functions in an optimization problem, problem constraints, etc.
2. **Solution determination:** This step, which is the main contribution of this study, divides into three sub-steps, as also shown in Fig. 1.
   (a) *Scenario selection:* Similar to the previous variants of multi-scenario MORDM, we need to select a set of scenarios to be considered within the optimization. This study follows the state-of-the-art scenario selection method proposed by Eker and Kwakkel (2018), although any other approaches can be utilized. The decision maker can set the number of scenarios to be considered in the optimization problem based on preferences, computation cost, complexity, or other considerations.
   (b) *Multi-scenario multi-objective optimization problem formulation:* In this step, to identify the candidate solutions, we formulate (using the information specified in the previous steps) and solve a multi-scenario multi-objective optimization problem that simultaneously considers multiple objectives and multiple scenarios within a single optimization problem of the form (1). By changing the number of scenarios considered in this problem, the decision maker can explore the trade-offs between all objectives in all selected scenarios. The higher the number of scenarios considered within the optimization problem, the more robust the identified solutions will be. However, increasing the number of scenarios considered within the problem can reduce the chance of feasibility and/or optimality in any given scenario.
   (c) *Solution process:* Since the total number of objective-scenario combinations (meta-objectives) utilizing in MSMOP, is often significantly high, we utilize the scalarizing function (3) to generate Pareto optimal solutions by solving it multiple times, by any appropriate single-objective solver, incorporating different reference points.

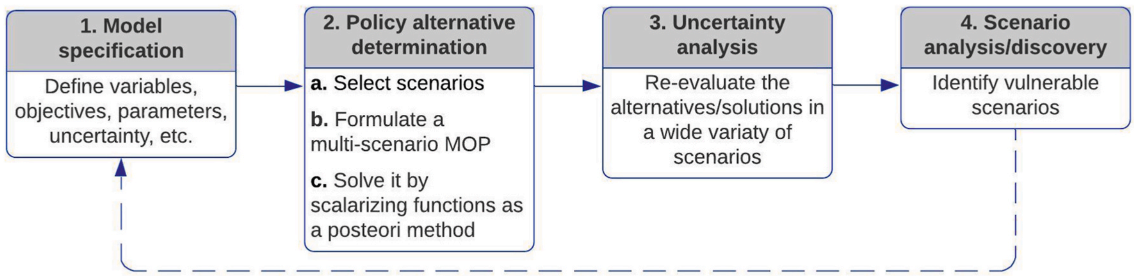

Fig. 1. Schematic of the proposed multi-scenario multi-objective robust optimization approach.

3. **Uncertainty/Robustness analysis:** In this study, following the previous variants of MORDM, the solutions identified in step 2 are re-evaluated over a wider range of plausible scenarios to assess their robustness and investigate the impacts of deep uncertainty on the objective functions.
4. **Scenario discovery/analysis:** Scenario discovery methods aim at identifying sub-spaces/subsets in the uncertainty space/set Ω where candidate solutions perform poorly. Different algorithms have been developed for this purpose in the literature (e.g., Bryant and Lempert 2010; Dalal et al., 2013; Kwakkel and Jaxa-Rozen 2016). However, as an alternative to available scenario discovery methods, we propose a novel method to determine vulnerable scenarios using ideal points and information about feasible regions in various scenarios. To this end, first, we calculate the ideal points for all objective functions in all new randomly generated scenarios in the previous step. Note that these randomly generated scenarios may not be the same as the scenarios considered in the optimization problem. Thus we need to calculate the best possible values for each objective (hereinafter called ideal values) in each randomly generated scenario. These ideal values are computed by solving the associated single-scenario single-objective optimization problem described in Section 2.1. Comparing the differences between the ideal values in various scenarios, will help us identify vulnerable scenarios and the combinations of the deeply uncertain parameters causing the poor performances in these vulnerable scenarios. This novel way of scenario analysis/discovery will be illustrated in more detail through the case study in Section 4.3.

2.4. Robustness measures and trade-offs analysis

Following recent studies and for comparison purposes, in this paper, we use the *mean/standard deviation* (Hamarat et al., 2014) and the *domain criterion* (Starr 1963) to measure the robustness for each objective and avoid objectives aggregation to compare the robustness trade-offs between the objectives. The mean/standard deviation measure is used to compare the results with Eker and Kwakkel (2018), while the domain criterion measure is utilized mostly for scenario analyzes and relevant discussions. It is also used in comparison with the results of Quinn et al. (2017). These two robustness measures, which are used in robustness analyses and scenario discovery (not in optimization), are briefly described in this section.

2.4.1. Mean/standard deviation (signal-to-noise ratio)

The mean/standard deviation measure for solution j in objective function i, representing by R_{ij}, can be formulated as follows (Eker and Kwakkel 2018):

$$R_{ij} = \begin{cases} \dfrac{\mu\left(f^*_{ijp}+1\right)}{\sigma\left(f^*_{ijp}+1\right)}; & \text{if } f_i \text{ is to be maximized}; p = 1, \dots, s, \\ \mu\left(f^*_{ijp}+1\right) \times \sigma\left(f^*_{ijp}+1\right); & \text{if } f_i \text{ is to be minimized}; p = 1, \dots, s. \end{cases} \tag{4}$$

where $\mu(\cdot)$ is the mean over the set of scenarios for the ith objective function in the case of implementing the solution j, and σ is the standard deviation.

2.4.2. Domain criterion

The domain criterion is a satisficing robustness measure, introduced by Starr (1963), which directly applies the decision maker's preferences on minimum acceptable values for each objective function. This measure mirrors the fraction/percentage of all considered scenarios in which the minimum acceptable thresholds are met (i.e., the percentage/number of scenarios in which the solutions are meeting the criterion.). The robustness value lies between 0 and 1, where 1 shows that the given criterion is met in all scenarios for the relevant candidate solution, and 0 means that the given criterion is not met in any scenario.

3. Case study - The shallow lake problem

To demonstrate the proposed multi-scenario MORO, we use the shallow lake problem (Carpenter et al., 1999), which is often used to demonstrate and benchmark methodological developments for decision making under deep uncertainty (Bartholomew and Kwakkel 2020; Eker and Kwakkel 2018; Hadka et al., 2015; Lempert and Collins 2007; Quinn et al., 2017; Singh et al., 2015; Ward et al., 2015). In this stylized decision making problem, a city is located next to a shallow lake. Anthropogenic pollution produced by the city goes into the lake. If a eutrophication threshold is passed, the lake irreversible transitions to a eutrophic state. The decision problem is to decide on an annual pollution control strategy that gives rise to the highest economic benefits without passing a critical threshold. To complicate the decision problem, next to the controllable anthropogenic inflow, there is also uncontrollable natural inflow. Phosphorus pollution levels can be calculated by the following dimensionless differential equation (Quinn et al., 2017):

$$P_t = (1-b)P_{t-1} + \frac{P^q_{t-1}}{1+P^q_{t-1}} + x_{t-1} + \zeta_{t-1}, \tag{5}$$

where P represents the phosphorus level in the lake, x describes the phosphorus/anthropogenic pollution input, $\zeta \sim logn(\mu, \sigma^2)$ refers to the natural pollution input, t indicates the time period, and b and q are the parameters of the lake model which control the rate at which pollution is lost from the lake and recycled from the sediment.

3.1. Objective functions

Following the literature on the shallow lake problem (Bartholomew and Kwakkel 2020; Eker and Kwakkel 2018; Hadka et al., 2015; Quinn et al., 2017; Ward et al., 2015), we consider four conflicting objectives as follows:

3.1.1. Economic utility (to be maximized)

The first objective function is the economic utility by releasing anthropogenic phosphorus pollution into the lake. Following Quinn et al. (2017) and Ward et al. (2015), the economic utility is computed as the average of the discounted benefit in N simulations of T years of random natural inflows. In the first objective function

$$f_1(\mathbf{x}) = \frac{1}{N}\sum_{n=1}^{N}\sum_{t=0}^{T-1}\alpha x_{tn}\delta^t, \tag{6}$$

α is an economic constant (fixed at 0.04), δ is a discount rate, and x_{tn} is the decision variable describing the phosphorus pollution level that can be released in year t for the nth random natural inflows simulation/realization.

3.1.2. Phosphorus pollution (to be minimized)

Minimizing the maximum average phosphorus level is considered as an environmental objective for water quality targets. This objective function is naturally conflicting with the economic objective. If $P_{tn}(\mathbf{x})$ represents the concentration of the phosphorus pollution in year t for the nth random natural inflows simulation, the second objective function is

$$f_2(\mathbf{x}) = max_{t\in\{1,\dots,T\}}\left\{\frac{1}{N}\sum_{n=1}^{N}P_{tn}\right\} \tag{7}$$

3.1.3. Inertia (to be maximized)

To avoid extremely rapid declines in phosphorus pollution in one year, which needs a massive amount of investments in infrastructure and to control the maintain decision inertia, the decision maker can set an annual reduction limit (I_{limit}) on phosphorus pollution. Therefore, in the third objective function

$$f_3(\mathbf{x}) = \frac{1}{N(T-1)}\sum_{n=1}^{N}\sum_{t=1}^{T-1}\varphi_{tn}, \quad \text{where} \quad \varphi_{tn} = \begin{cases} 1, & x_{t-1,n} - x_{tn} < I_{limit} \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

the inertia of a decision is maximized. Inertia is defined as the average fraction of (T-1) planning years over N random natural inflows simulations, where inter-annual pollution declines are lower than I_{limit}% of the maximum possible reduction. In this paper, following (Eker and Kwakkel 2018; Quinn et al., 2017; Ward et al., 2015), I_{limit} is set as 0.02 (i.e. 20% of the maximum possible reduction).

3.1.4. Reliability (to be maximized)

The last objective function

$$f_4(\mathbf{x}) = \frac{1}{NT}\sum_{n=1}^{N}\sum_{t=1}^{T}\theta_{tn}, \quad \text{where} \quad \theta_{tn} = \begin{cases} 1, & P_{tn} < P_{crit} \\ 0, & P_{tn} \geq P_{crit} \end{cases} \tag{9}$$

also called an average reliability of a decision, reflects the decision maker's desire in abstaining from the eutrophication of the lake which occurs if the concentration of the phosphorus in the lake passes a critical threshold (P_{crit}). If the phosphorus level in the lake lies below P_{crit} in a given period, the reliability index θ_{tn} is 1 and 0 otherwise. Thus, maximizing the reliability means maximizing the periods (out of T and across N simulations) in which the phosphorus level in the lake stays below the critical threshold P_{crit}.

3.2. Uncertainties and scenario selection

Two different degrees/levels of uncertainty (see, e.g., Shavazipour and Stewart (2019) or Kwakkel and Walker (2010) for the definitions of various degrees or levels of uncertainty), i.e., mild (also called stochastic) and deep, are present in the shallow lake problem. The mild uncertainty in natural pollution inflow (ζ) is handled by the average values of random samples generating by a log-normal distribution. The mean (μ) and standard deviation (σ) of the log-normal distribution, the discount factor (δ), the natural recycling rate (q), and the loss rate (b) are the deeply uncertain parameters in this problem. Following previous work (Bartholomew and Kwakkel 2020; Eker and Kwakkel 2018; Quinn et al., 2017), these five deeply uncertain parameters and their ranges are shown in Table 1. A combination of the values of these five deeply uncertain parameters, sampled from their ranges, generates a scenario from the scenario space Ω. For scenario selection, we followed Eker and Kwakkel (2018) in selecting four more scenarios in addition to the baseline scenario and, thus, consider the same scenarios (by utilizing the same values for deeply uncertain parameters constructing those four scenarios). For more information about how these four scenarios were selected, see Eker and Kwakkel (2018, Section 4.1, pages 205–207). The last five columns of Table 1 denote these five selected scenarios.

3.3. Multi-scenario inter-temporal open-loop control formulation for the lake problem

Different variants of the shallow lake problem have been proposed in the literature. The widely known ones are inter-temporal open-loop control (Eker and Kwakkel 2018; Hadka et al., 2015; Quinn et al., 2017; Singh et al., 2015; Ward et al., 2015), direct policy search (Quinn et al., 2017), and planned adaptive direct policy search (Bartholomew and Kwakkel 2020). In this paper, we use the often used inter-temporal open loop control version in a multi-scenario manner which includes T decision variables.

The optimization formulation for this multi-scenario inter-temporal open-loop control version of the lake problem is:

$$\begin{array}{lll} \text{minimize} & \{-f_{1p}(\mathbf{x}), f_{2p}(\mathbf{x}), -f_{3p}(\mathbf{x}), -f_{4p}(\mathbf{x})\} & p \in \Omega \\ \text{s.t.} & 0.0001 \leq x_t \leq 0.1, & \text{for all} t, \end{array} \tag{10}$$

where $\mathbf{x} = (x_0, x_1, \dots, x_{T-1})$ is a vector of decision variables, T indicates the length of the planning horizon, x_t represents the amount of phosphorus pollution to be released in year t, which is limited to 0.1. As before, f_{ip} refers to objective function i ($i = 1, 2, 3, 4$) in scenario p and Ω is the scenario space.

4. Results

In this section, we illustrate the proposed multi-scenario MORO step by step, following the steps shown in Fig. 1 describes in Section 2.3. To assess the efficacy of the novel approach, we compare the results with those of the previous studies.

4.1. Steps 1 and 2: Problem setting, scenario selection and generating candidate solutions

We consider the formulation of the lake problem (10) with four objective functions in five scenarios for $T = 100$ years and $N = 100$ random realizations of the natural inflows. We follow the selection approach of Eker and Kwakkel (2018) and thus use the same scenarios. These five scenarios are presented in Table 1. Therefore, to generate solutions, we need to solve a multi-scenario multi-objective optimization problem with 100 decision variables and 20 objective functions (5 scenarios × 4 objectives per scenario).

Based on the above-mentioned settings and the estimated worst possible values of the objective functions (6)–(9), the nadir points for the second objective (pollution) were set as 15 in all five scenarios, and for the other three objectives in all selected scenarios 0. Ideal points, presenting in Table 2, were calculated by solving the relevant single-scenario single-objective optimization problems for each objective function in each considered scenario. The utopian values were calculated by adding (for objectives to be maximized) or subtracting (for objective to be minimized) a small scalar of 0.0001 to (from) the ideal points. An ideal point represents the optimal performance that can be reached for each objective in a given scenario. For instance, as shown in Table 2, the best possible performance for the first objective (utility) in the fourth scenario was 0.581, which is about two-third of the maximum potential performance in the first three scenarios, and around one-third

Table 1
Deeply uncertain parameters and five reference scenarios.

Deeply uncertain variables			Scenarios				
Notation	Description	Range	1	2	3	4	5
b	Pollution rate of removal through natural outflows	[0.1, 0.45]	0.193	0.141	0.111	0.272	0.420
q	Pollution recycling rate through natural processes	[2.0, 4.5]	3.049	2.585	2.969	2.971	2
μ	Mean of natural pollution inflows	[0.01, 0.05]	0.017	0.012	0.010	0.033	0.020
σ	Standard deviation of natural inflows	[0.001, 0.005]	0.0021	0.0024	0.0037	0.0046	0.0017
δ	Utility discount factor	[0.93, 0.99]	0.953	0.950	0.957	0.931	0.980

Table 2
Ideal values for each objective function in each selected scenario.

Scenarios	Objective functions			
	Utility	Pollution	Inertia	Reliability
1	0.846	0.097	1.0	1.0
2	0.802	0.111	1.0	1.0
3	0.908	0.102	1.0	1.0
4	0.581	0.132	1.0	1.0
5	1.735	0.057	1.0	1.0

of the performance in the best-case scenario (s_5). These ideal values are represented the effect of deep uncertainty and highlight some of the problem's limitations, even before identifying the solution candidate.

Given this setup, we use the achievement scalarizing function and solve the resulting optimization problem using the Sequential Least Squares Programming (SLSQP) algorithm (Kraft 1994) available from the SciPy module (Oliphant 2007). To generate different Pareto optimal solutions we followed the Mueller-Gritschneder et al. (2009) method to pre-specify With the 50 reference points and weights $w_{ip} = 1/(z_{ip}^{nadir} - z_{ip}^{uto})$ $(i = 1, \ldots, 4; p = 1, \ldots, 5)$ we get 50 different Pareto optimal solutions for the original multi-objective problem. In practice, the number of solutions is to be set by the decision maker. Here, we chose 50 for comparison purposes, as it is the number of solutions considered by Eker and Kwakkel (2018) in their robustness analysis. Note that the 50 solutions plotting in Eker and Kwakkel (2018) were the brushed solutions after all the filtering. However, a set of 50 (or even fewer) solutions generated by the proposed multi-scenario MORO can reasonably (out) perform these. Because of the simultaneous consideration of the five selected scenarios in multi-scenario MORO, there is no need for further filtering. In this way, we can simultaneously save computational resources and improve the quality of the solutions. We will return to this later in this section.

Each poly-line of the parallel coordinate plot in Fig. 2 represents the performance of a single solution on all four objectives in each of the five scenarios. Different colors, in Fig. 2, distinguish the solutions in terms of their performance on the utility objective in the fifth scenario (s_5). The higher the utility, the lighter the color (yellow in the colored version). A higher utility leads to a higher pollution, which results in lower reliability values. The opposite is visible for solutions with higher reliability values, shown with darker colors (dark purple in the colored version), which highlights the trade-off between pollution/reliability and utility.

Another observation is that there is no significant trade-off between scenarios in each objective, particularly between the performances in utility and inertia in different scenarios. For example, the highest values for utility in different scenarios are from the same solution, or inertia values in all five scenarios are similar in each solution (visible by horizontal lines for inertia over the five scenarios in Fig. 2). This particular characteristic of the lake problem may prevent the explicit study of the trade-offs between scenarios and cause some difficulty in robustness and trade-off comparisons. We return to this point in Section 5.

4.2. Step 3: Robustness analysis

To re-evaluate the 50 solutions and assess their robustness, we generate an ensemble of 1000 randomly generated scenarios (using Latin Hypercube Sampling). The 50 solutions are re-evaluated over these scenarios to analyze and compare their robustness across a broader range of scenarios. Based on this ensemble of 1000 scenarios, the robustness of the candidate solutions is determined using the domain

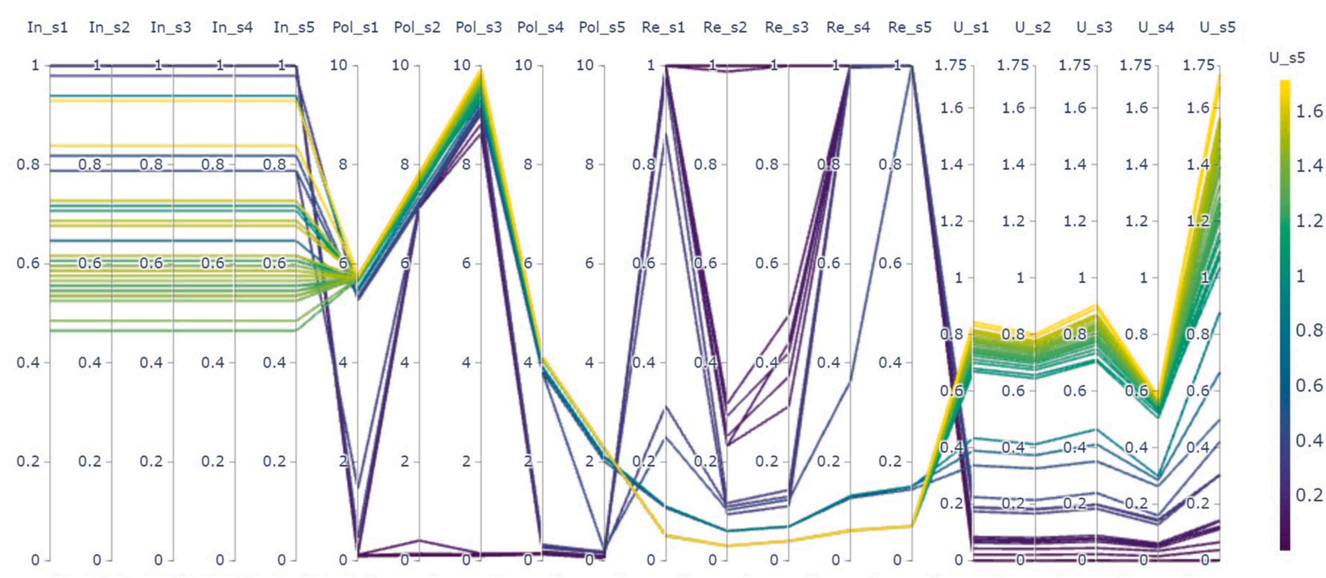

Fig. 2. Comparing candidate solutions based on all four objectives in five selected scenarios. U_sp: Utility in scenario p, Pol_sp: Pollution in scenario p, In_sp: Inertia in scenario p, Re_sp: Reliability in scenario p.

criterion and the mean/standard deviation.

4.2.1. Robustness trade-offs with mean/standard deviation

In this section, we use the mean/standard deviation as the robustness measure to compare the results identified with multi-scenario MORO. solution (*j*) in each objective function (*i*), using (4). Fig. 3 shows the mean/standard robustness trade-offs of the generated candidate solutions over the 1000 random scenarios. The color code is similar to the previous figure; i.e., the higher the utility robustness, the lighter the color. Also, conflicts between the robustness values in reliability and utility are vivid when the poly-lines cross between the last two columns representing the robustness trade-offs between these two objectives. Note that since we are minimizing pollution, lower values of robustness are better for this objective. Thus, the relevant column, representing the robustness of pollution, are inverted in the plot to unify the robustness improvement direction, which is upwards (↑). Lines higher up in the plot describe solutions with higher robustness on all objectives.

Fig. 4 compares the robustness trade-offs of the solutions generated by the proposed multi-scenario MORO, the solutions of Eker and Kwakkel (2018), and the solutions produced by Quinn et al. (2017). As seen in this figure, the solutions generated by multi-scenario MORO result in a wider variety of robustness trade-offs compared to the solutions produced by the other methods. For example, the maximum robustness value for utility in multi-scenario MORDM (Eker and Kwakkel 2018) and MORDM (Quinn et al., 2017) were, respectively, 1.21 and 1.16, while almost a half of the solutions generated by multi-scenario MORO provided better values (up to 1.51). Similar patterns are valid for the robustness of all three other objectives. Moreover, as mentioned, the trade-offs between reliability and utility are evident among the solutions produced by multi-scenario MORO. These trade-offs are hardly visible with the solutions of the other methods. There are two reasons why the other methods could not find solutions with wider robustness trade-offs: 1) Scenario dependency of their solutions since their search area has a lower dimension (limiting the search to a hyperplane constructed by one scenario at a time). For instance, solutions with exceptionally low performance in one scenario (i.e., dominated solutions in that scenario) may have high performance in many other scenarios (i.e., non-dominated in many different scenarios). These solutions are not identified as non-dominated solutions when the search is confined to only a single-scenario space. 2) Some of the solutions (which represent wider trade-offs) may be eliminated from the final list after applying the reliability constraint (Quinn et al., 2017; Eker and Kwakkel 2018) (with a similar reason to the previous item).

4.2.2. Robustness trade-offs with domain criterion

The second robustness measure we use is the domain criterion. The following criteria are considered, based on previous studies using the lake problem (Quinn et al., 2017; Bartholomew and Kwakkel 2020):

1. Utility >0.2
2. Reliability >0.95
3. Pollution < Critical point (P_{crit})
4. Inertia >0.99.

In practice, these criteria would be set by a decision maker. For each criterion from the above list, we calculated the number of solutions meeting that criteria after re-evaluation over the ensemble of 1000 randomly generated scenarios in Section 4.2. Then ranked and sorted them based on their robustness in that criterion. Over the rank-sorted solutions, the robustness scores, on the following criteria, are described in Fig. 5.

Fig. 6 compares the distributions of the robustness for the 50 candidate solutions generated by multi-scenario MORO, the 50 brushed solutions of Eker and Kwakkel (2018), and the 86 brushed solutions produced by Quinn et al. (2017). Note that all the solutions are re-evaluated over the same ensemble of 1000 randomly generated scenarios. As seen in Fig. 6(b), in 60% of the generated solutions by multi-scenario MORO and in 26% of the solutions of the multi-scenario MORDM (Eker and Kwakkel 2018), the robustness values for utility were 1, meaning that the utility values met the domain criterion of 0.2 in all 1000 random scenarios for these solutions. In contrast, none of the solutions of the MORDM (Quinn et al., 2017) reached this value (the utility robustness value of 1). The maximum value of the utility robustness gained by a solution of Quinn et al. (2017) was 62.9%. The robustness value of 1 for inertia is observed in about 34% of the solutions generated by multi-scenario MORO, while, only one solution (amongst the ones produced by the MORDM (Quinn et al., 2017)) could obtain a similar value of robustness for inertia (see Fig. 6(d)). All the solutions generated by the multi-scenario MORDM (Eker and Kwakkel 2018) have the robustness value of 0 for inertia. Nevertheless, no solution (among the generated solutions by either approach) provides the robustness value 1 for reliability and pollution (Fig. 6(a) and (c)). The maximum robustness percentage of reliability and pollution, among the solutions of multi-scenario MORO, were 76.9% and 76.7%, respectively. Corresponding values amongst the solutions of Quinn et al. (2017) were, respectively, 63.4% and 63.1%. Also, the maximum robustness percentage of reliability and pollution obtained by the solutions of Eker and

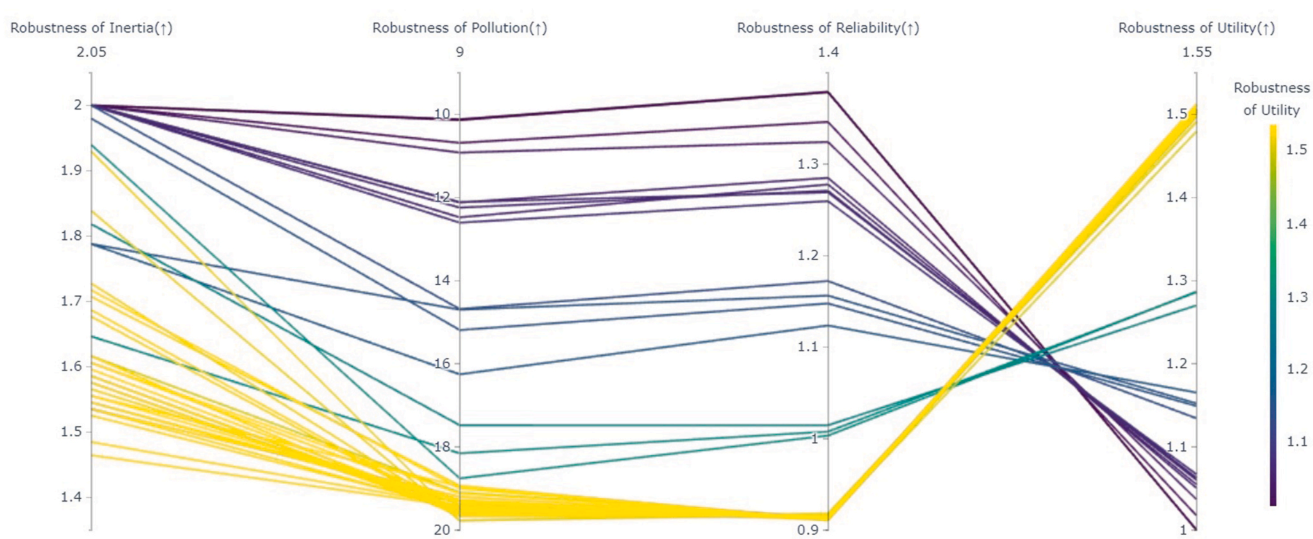

Fig. 3. The robustness trade-offs of the candidate solutions with the mean/std. deviation measure.

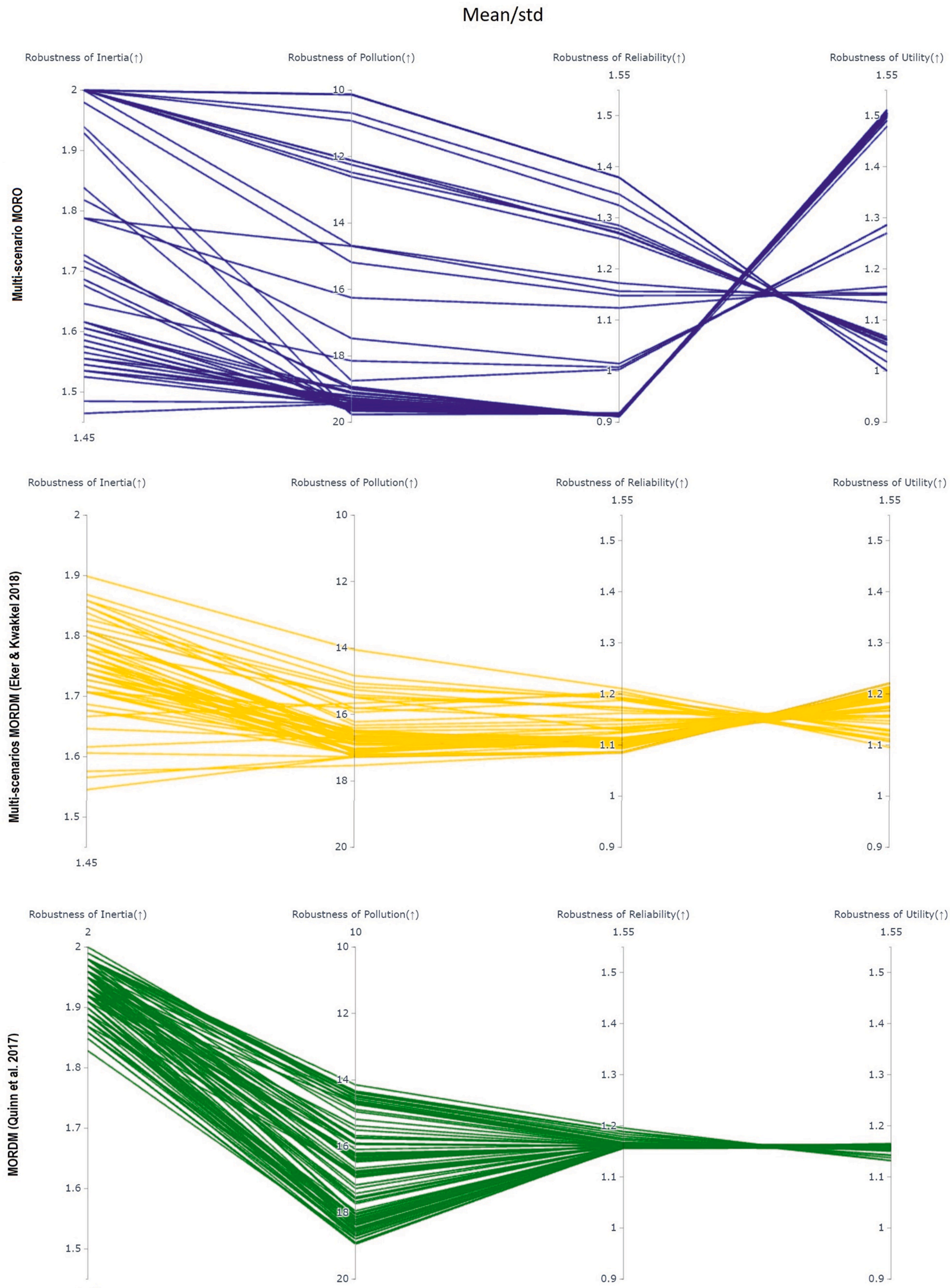

Fig. 4. Comparing the robustness trade-offs of the 50 candidate solutions generated by the proposed multi-scenario MORO, the 50 brushed solutions of Eker and Kwakkel (2018), and the 86 brushed solutions of Quinn et al. (2017) with the mean/std. deviation measure.

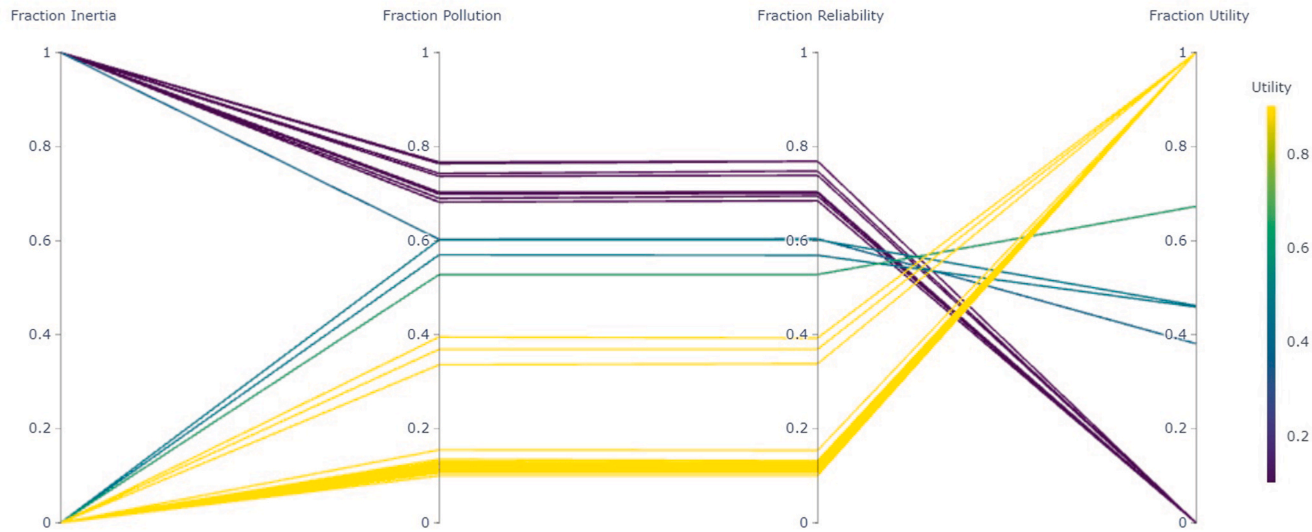

Fig. 5. The robustness trade-offs of the candidate solutions with the domain criterion measure for the ranked-sorted solutions (the higher the utility robustness, the lighter the colors).

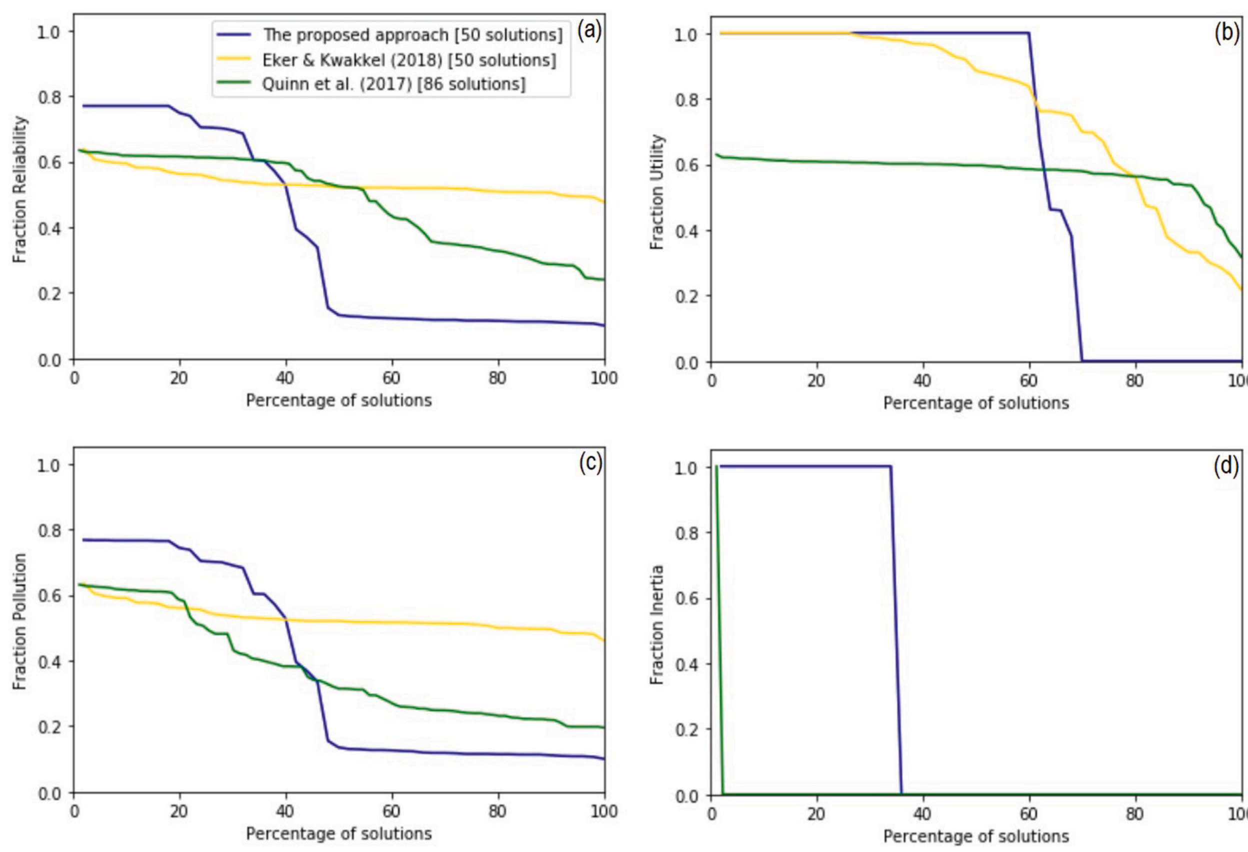

Fig. 6. Comparing the robustness distributions of the 50 candidate solutions generated by the proposed multi-scenario MORO, the 50 solutions of Eker and Kwakkel (2018), and the 86 solutions of Quinn et al. (2017), with the domain criterion measure.

Kwakkel (2018) were 63.7% and 63.4%, respectively. This result is also in line with the results of a similar analysis in Quinn et al. (2017) and Bartholomew and Kwakkel (2020). In general, two main reasons may cause these results. First, some parts of the Pareto front are left unexplored (i.e., the set of generated solutions is not diverse enough to cover the entire front). Second, there may be no feasible solution in those scenarios that can satisfy the given criterion. In the next section, through scenario analysis, we will show that there is no feasible solution meeting the domain criterion on reliability and pollution (as can also be seen in Fig. 6(a) and (c)) in about 23% of the scenarios. This means that in around 23% of the scenarios, the reliability criterion cannot be satisfied.

The values for the deeply uncertain parameters of these scenarios are presented in Fig. 8. Moreover, comparing the left-side plots (a and c) in Fig. 6, demonstrates a strong correlation between the robustness of reliability and pollution (particularly amidst the solutions of multi-scenario MORO) which is expected, as minimum pollution values give rise to high-reliability values. This strong correlation is even more visible by tracing the straight lines between pollution and reliability in Fig. 5. Apart from these insights, in Fig. 5, we also see that the existing trade-offs between reliability and utility are again visible. The color codes are the same as Fig. 3, i.e., the higher the utility robustness, the lighter the colors.

Comparing the robustness of the solutions generated by multi-scenario MORO and the ones produced by the two other methods (i.e., MORDM (Quinn et al., 2017) and multi-scenario MORDM (Eker and Kwakkel 2018)) with the domain criterion measure, as portrayed in Fig. 7, confirms the superiority of multi-scenario MORO. For instance, multi-scenario MORO identifies Pareto optimal solutions that provide a broader range of robustness trade-offs. They also help decision makers to gain more insights into the problem than the previous variants of MORDM.

In the next section, we investigate in more detail the feasibility of the domain criteria through scenario analysis. Furthermore, we analyze vulnerable scenarios to identify the combinations of deeply uncertain parameters causing poor performance in those scenarios.

4.3. Step 4: Scenario analysis/discovery

To check the feasibility of meeting the domain criterion for each objective function in any scenario, first, we calculate the ideal points for all four objective functions in all 1000 scenarios. Effectively, we are searching for the best possible values for each objective under the conditions of each scenario; i.e., *"what is the best that could happen in every scenario?"*. The ideal points can be calculated by solving a single-scenario single-objective problem using for each objective function in each scenario. This required solving $4 \times 1000 = 4000$ problems for our case study. However, the total computation time for solving all these problems is less than a couple of hours on a personal laptop and we only need to calculate the ideal points once. Indeed, the ideal point calculations are related to the best-/worst-case discovery (Halim et al., 2016).

Table 3 represents the minimum and maximum values for each objective function among the components of the ideal points across all 1000 scenarios (i.e., the best possible values for each objective in the best- and worst-case scenario), describing the best and the worst performances for each objective. As seen in this table, the corresponding ideal values for reliability in some (or at least in one) scenarios are very close to 0 ($min = 0.04$). This means that, in those scenarios, there is no feasible solution (even in the feasible region of the single-scenario single-objective problem) with a reliability higher than 0.04, which is far less than the domain criterion for reliability at 0.95. Similarly, the ideal values for pollution in some (or at least one) scenarios are more than 10, which is also far more than the maximum values of the critical points in any scenarios (i.e., 0.9165). Counting the scenarios with the reliability of less than 0.95 indicates 231/1000 scenarios, which confirms the claim that no feasible solutions meet the reliability criterion (reliability > 0.95) in about 23% of the scenarios.

Fig. 8 shows the combination of deeply uncertain parameter values (listed in Table 1) leading to poor performance in the ensemble of 1000 scenarios. Each dot represents a scenario. Orange dots (•) correspond to scenarios where the reliability criterion is not met, while blue dots (•) belong to scenarios that met the reliability criterion (reliability > 0.95). Similar to the results of the sensitivity analysis of Quinn et al. (2017), the first plot at the bottom left describes the area in which some nonlinear combination of small values of q and b results in poor performance on the reliability objective. Of course, other uncertain parameters also have some impacts. For instance, higher values of *mean* natural pollution can also contribute to a failure on reliability, even for higher values of q, if b is not large enough. Overall, it seems that for $b > 0.3$ we rarely have a failure on the reliability objective, giving the decision maker a new insight into the problem. This point is more visible in Fig. 9 showing the vulnerable combinations of deeply uncertain parameters. As seen in this figure, only few failures were observable for $b > 0.3$ and none for $b > 0.34$. Also, no failure was recorded for $\delta > 0.98$.

Investigating the feasible region and ideal values for each objective across an ensemble of scenarios fosters the understanding of the behavior of deeply uncertain parameters in combination with each other. The directed search (Kwakkel 2017; Moallemi et al., 2020) in some extreme areas of the uncertainty space, provides us with detailed insights into the system dynamics in these areas. Because calculating the ideal values and the proposed scenario analysis do not need any prior knowledge about the solutions and their robustness, this kind of analysis can be done even before solution generation. Thus, one can get more insight into the problem and modify the model or preferences, if needed, before determining solutions, potentially saving time and energy.

5. Discussion

As mentioned in the introduction, previous variants of MORDM only considered a single scenario at a time in the search for the candidate solutions–leaving MORO aside. Therefore, the feasibility of the generated solutions in a different scenario is questionable, and, in the best-case, the solutions are scenario-dependent, if not infeasible. For example, Quinn et al. (2017) added a hard constraint of reliability $>$ 0.85 to the intertemporal model of the lake problem in a reference scenario (which is the same as s_5 in this study). Then, they solved the four-objective optimization model with the BORG MOEA (Hadka and Reed 2013) to generate candidate solutions that were subsequently stress-tested (re-evaluated) across 1000 scenarios for robustness analysis. They used the domain criterion as their robustness measure and their second criterion was reliability $>$0.95. They showed that their generated solutions met this criterion only in around 60% of scenarios (see Figure 8 in (Quinn et al., 2017)), that is quite similar to the results described in Figs. 6(a) and 7. However, they could not find any failure mechanism on the reliability criterion. The reason is that there exists no feasible solution with a reliability $>$0.95 in around 23% of the scenarios, as described in Section 4.3. Identifying such a failure mechanism is almost impossible if only one scenario is considered in the search phase.

As another example, Bartholomew and Kwakkel (2020) set the domain criterion of utility $>$ 0.75. Let us set this criterion as a hard constraint in the model. Suppose we separately solve the lake problem (with this constraint) for each selected scenario. In that case, we cannot find any feasible solution for the optimization problem related to the fourth scenario in which the ideal values for the utility are less than 0.75 (see Table 2). Utility has undesired values in more than 33% of the scenarios. Therefore, if we generate solutions based on any other four scenarios, none of the solutions can meet this constraint 'utility $>$0.75' in the fourth scenario (i.e., they are infeasible in this scenario), even though they are feasible in all other four scenarios. Accordingly, some solutions identified by the previous variant of MORDM, which does not consider this scenario (or some similar scenarios), are infeasible in some scenarios in terms of satisfying the constraint of utility $>$ 0.75. In other words, if the set of scenarios considering as part of the search phase of MORDM does not contain scenarios causing a particular type of failure mechanism, we cannot find solutions that can cope with this failure.

In general, identifying solutions for some particular scenario cannot guarantee the feasibility of the solutions in any other scenario; i.e., scenario-dependent solutions may not be feasible in some other scenarios. This feasibility robustness (i.e., the solution is feasible in all (or in a wide variety of) scenarios) is an essential factor that must be somehow checked or guaranteed in the search phase. To the best of our knowledge, this concept of robustness has not received much attention from the authors developing methods for dealing with deep uncertainty. One reason for this may be the particular characteristics of the lake problem. As the most popular benchmark problem for methodological developments for decision making under deep uncertainty, it is weak in representing the trade-offs between scenarios. Consequently, the simultaneous consideration of multiple scenarios within the optimization problem, like multi-scenario MORO, helps in verifying the

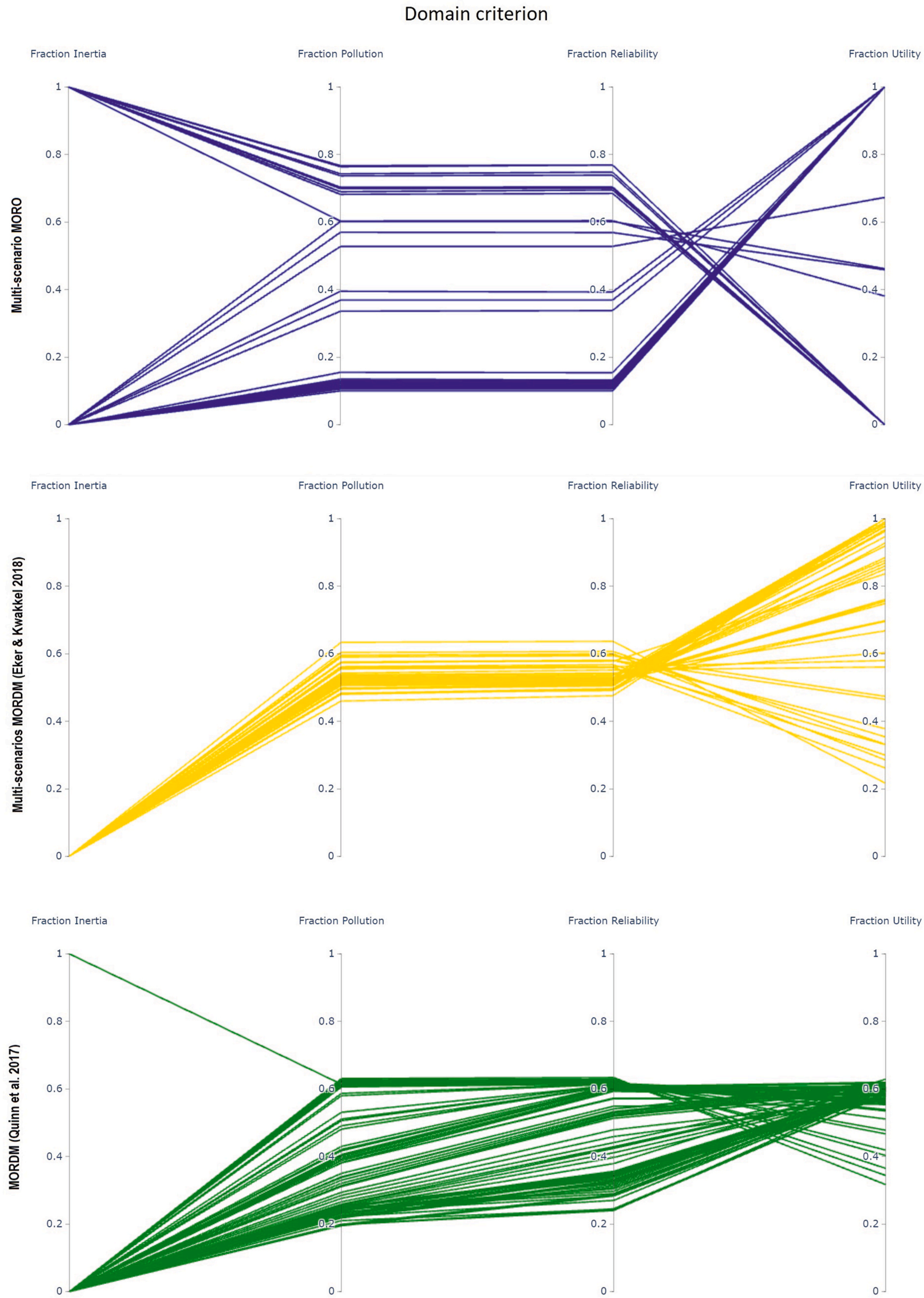

Fig. 7. Comparing the robustness of the 50 candidate solutions generated by the proposed multi-scenario MORO, the 50 brushed solutions of Eker and Kwakkel (2018), and the 86 brushed solutions of Quinn et al. (2017), with the domain criterion measure.All the solutions are re-evaluated over the same ensemble of 1000 random scenarios generated in this study.

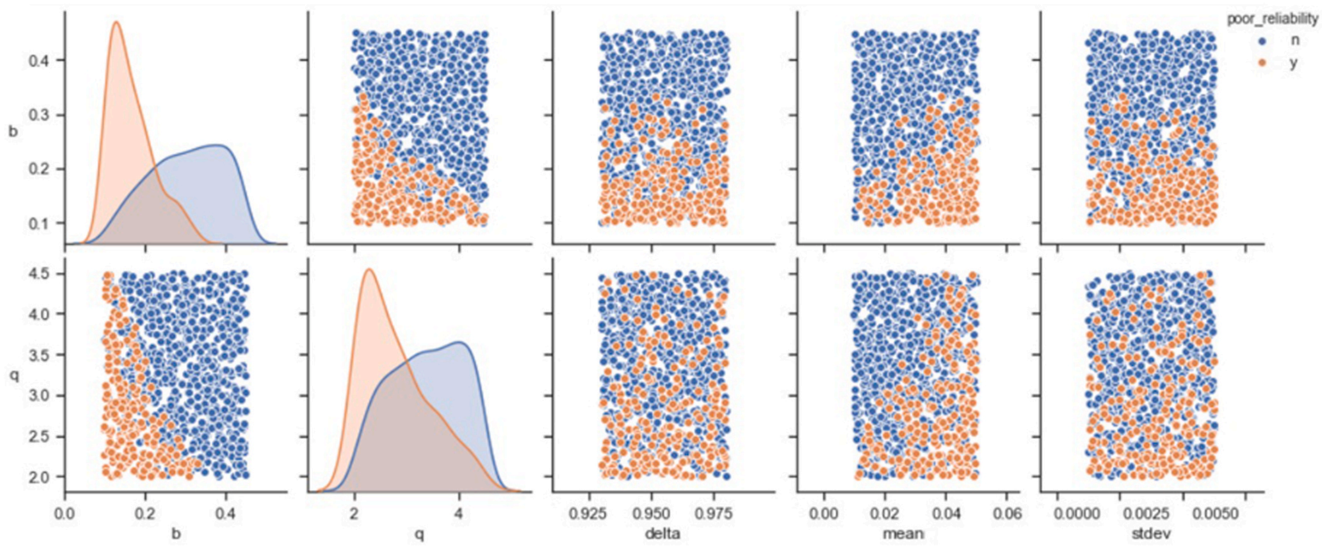

Fig. 8. Combinations of uncertain parameter values leading to failure in reliability.

Table 3
Maximum and minimum values for each objective functions among the components of the ideal points across 1000 randomly generated scenarios.

	Objective functions			
	Utility	Pollution	Inertia	Reliability
min	0.5712	0.0159	1.00	0.04
max	1.7348	10.2174	1.00	1.00

feasibility of the generated solutions in various scenarios during the search process. However, one cannot consider an infinite number of scenarios. Therefore, it is vital to study the effects of the number of scenarios considered within the optimization problem and explore the trade-offs between the overall robustness and optimality in any given scenario. This investigation is performed in the next sections.

5.1. Effects of the number of scenarios

In this section, we study the effects of different numbers of scenarios within multi-scenario MORO. Again, we consider the lake problem (10) with four objectives, but with a different number of scenarios, namely 1, 5, 9, and 50 scenarios. These problems, which respectively include 4, 20, 36, and 200 meta-objectives, are solved utilizing achievement scalarizing functions with 50 reference points to generate different Pareto optimal solutions. Note that, when we compare all generated solutions in a particular dimension of the scenario space (e.g., 1D or single scenario comparisons in Table 4), some solutions that were non-dominated in the higher dimension may dominate in the lower dimensions and no longer lie in the Pareto set. These solutions are removed and, therefore, the number of solutions considered in comparisons may be less than fifty.

We compare the solutions found through the above-mentioned models for the five scenarios described in Table 1. In the model with

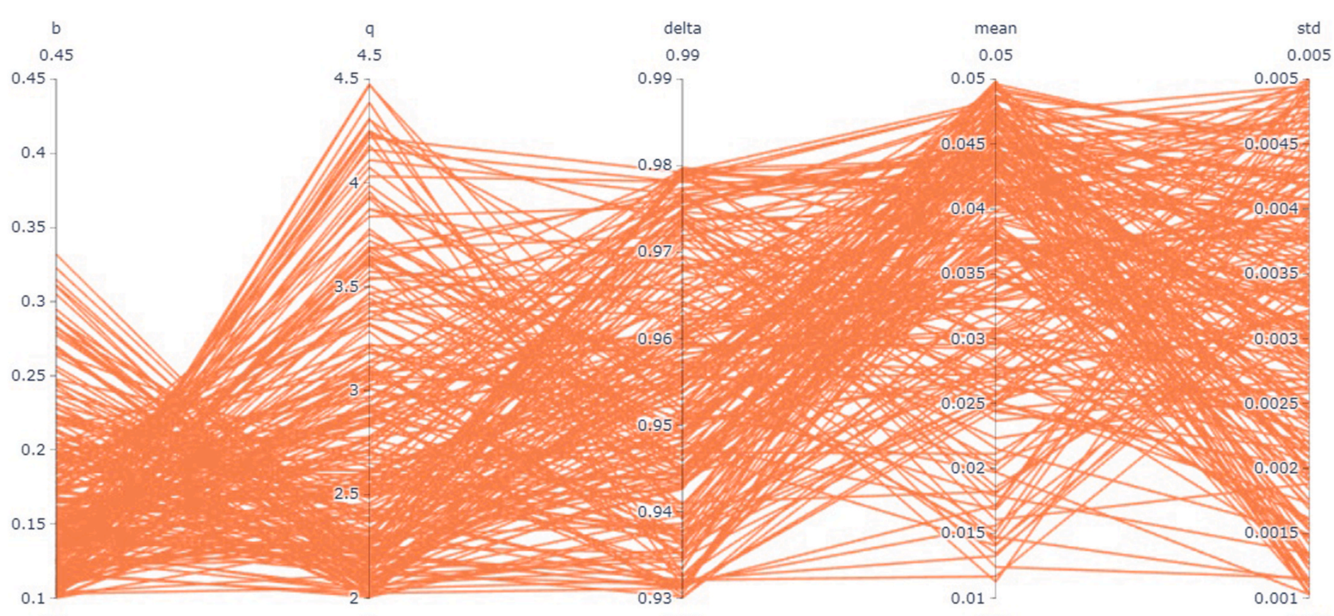

Fig. 9. Combinations of uncertain parameter values leading to failure in reliability.

Table 4
Percentage of the solutions that remain in the Pareto set for each scenario for each model. Red and green fonts describe the worst and the best, respectively.

Number of scenarios in multi-scenario MORO		Scenarios				
		s_1	s_2	s_3	s_4	s_5
50 $(s_1, .., s_{50})$	38	24%	18%	16%	16%	32%
9 $(s_1, .., s_9)$	46	35%	30%	35%	26%	26%
5 $(s_1, .., s_5)$	49	61%	63%	61%	55%	29%
1 (s_1)	39	48%	39%	39%	39%	33%
1 (s_2)	34	64%	74%	51%	62%	41%
1 (s_3)	33	38%	41%	47%	38%	35%
1 (s_4)	32	63%	55%	52%	79%	45%
1 (s_5)	33	47%	53%	40%	56%	63%

nine scenarios, in addition to these five scenarios, four extra scenarios (the same scenarios utilized by Bartholomew and Kwakkel (2020, Table 2, page 127)) are also considered. Moreover, we consider 41 additional scenarios from the set of randomly generated scenarios in the model with 50 scenarios. Table 4 portrays the results of this comparison. The number of scenarios utilizing in the optimization problem has been shown in column 1, while the second column shows the total number of solutions in the Pareto set for each model in the 5D scenario space constructing by the first five scenarios. The percentage of the solutions that remain in the Pareto set when evaluated in each scenario is described in columns 3 to 7. We merge all the solutions found by all models (multi-scenario multi-objective optimization problems with 1,5, 9, and 50 scenarios), identify the non-dominated solutions in each scenario, and classify them according to the models that generated them.

As seen in Table 4, in general, the percentage of the solutions that remain in the Pareto set for any single scenario decreases by increasing the number of scenarios considered within the optimization problem, displaying the price of robustness. This observation is also in-line with the results of Bartholomew and Kwakkel (2020). Moreover, the solutions generated by the five scenarios optimization problem perform relatively well, especially in the first four scenarios in comparison with the results from the other optimization problems. Therefore, there is not much loss in the optimality in the first four scenarios compared to the solutions generating by the other models. In contrast, the optimality loss (the price of robustness) is high in the model with 9 and 50 scenarios.

Note that not all solutions to each single-scenario optimization problem remain in the Pareto set of that particular scenario, mainly because of the stochastic nature of the natural flows in the lake problem. This random variation of the natural flows causes some dominance issues in the non-dominated sorting calculation, i.e., multiple evaluations of a decision may give rise to some close but not the same values for the objective functions. In fact, some solutions are dominated because of the random values set by the model for the natural flows in each evaluation, not because of the existence of any better solutions. This issue also questions the suitability of the lake problem for robustness and trade-off comparisons.

5.2. Robustness over the randomly generated scenarios

The domain criterion robustness of the solutions, generated by 50-, 9-, 5- and single-scenario models, is presented in Fig. 10. The larger the number of scenarios considered within the optimization problem (especially for more than five scenarios), the higher the robustness values on reliability, pollution, and inertia objectives after re-evaluation. This means that the optimization problem involving 50 scenarios largely dominates the solutions found for the 9-scenario optimization problem. The inverse is observed for the utility objective. It seems that by increasing the number of scenarios that is simultaneously considered within the optimization problem, the solutions are increasingly biased towards higher reliability at the expense of utility. The individual performance of the solutions found for the 5-scenario optimization formulation, as shown in Fig. 10 with blue lines, show a

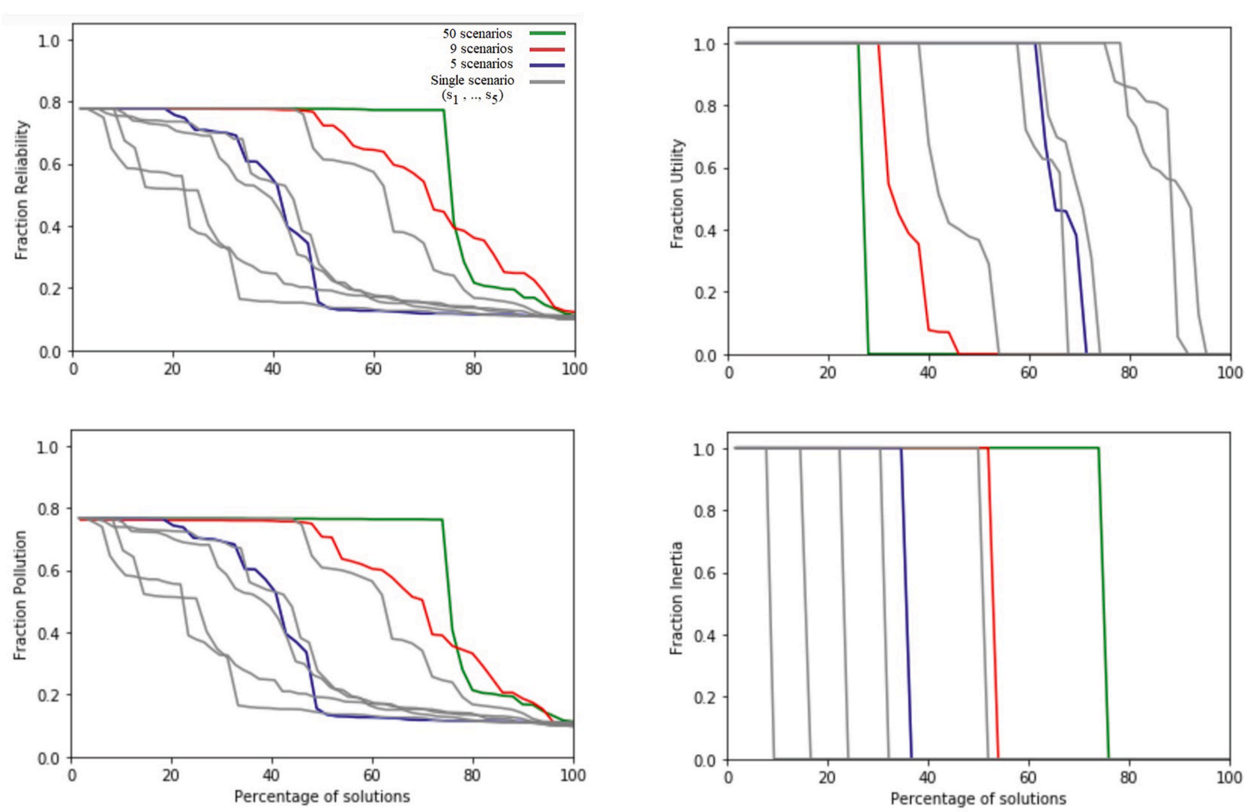

Fig. 10. Comparing the robustness of solutions generated by 50-, 9-, 5- and single-scenario models (domain criterion measure).

moderate behavior. In all objective functions, the robustness of the solutions produced by the 5-scenario formulation lies somewhere in the middle of the others, demonstrating more balanced solutions. However, utilizing different reference points for each optimization formulation can generate various solutions that can completely change the story. Indeed, the decision maker can steer the solution process towards the area of interest by providing relevant reference points. Therefore, determining the reference points is an important step in multi-scenario MORO. Furthermore, the trade-off diversity between the objectives and the robustness ranges is not visible in Fig. 10, highlighting the need for a different visualization. Accordingly, Fig. 11 is used to investigate these two issues.

As seen in Fig. 11, a similar pattern is observed across the different optimization formulations, for both the domain criterion and the mean/standard deviation robustness measure. The robustness ranges are almost the same across formulations, while the trade-off diversity between objectives varies. The main reason, again, is because of the set of chosen reference points. To pre-determine the reference points, we use the method proposed by Mueller-Gritschneder et al. (2009), in which the extreme points together with some evenly distributed points on the convex hull of all extreme points are considered as the reference points to ensure diversity. Nonetheless, in higher dimensions (i.e., when we consider more scenarios in the optimization problem), to cover the whole space, one needs to generate more solutions compared to lower dimensional formulations. For example, the number of extreme points in the model with four objectives and 1, 5, 9, and 50 scenarios is 4, 20, 36, and 200, respectively. Therefore, as we generate 50 solutions for each formulation, we cannot cover the whole objective space in the higher dimensions, particularly for the 50-scenario formulation. This explanation justifies the extreme distribution of the robustness of the solutions generated by the 50 scenarios formulation. The lack of diversity in the solutions produced by some single-scenario formulations can be justified by the limited search area comparing to the optimization formulations for five and nine scenarios. Nevertheless, once again, the particular characteristics of the lake problem prevent further investigations and comparison in the robustness of the solutions generating by different models.

To sum up, there is no significant difference in the robustness of the solutions generated by various models since there is no significant trade-off between scenarios in each objective function in the lake problem. Because the feasible region in all scenarios stays the same, any Pareto optimal solution can be generated by any model if an appropriate reference point is set. Therefore, determining the reference points and the number of solutions to be generated are more vital than the number of scenarios considered in the optimization model of the lake problem, as a special case.

5.3. Computation cost

Another vital matter is the computational cost of multi-scenario MORO (Bartholomew and Kwakkel 2020; Giudici et al., 2020). We examine the effects of considering more scenarios within the optimization phase of the proposed approach on computational cost. The number of function evaluations (NFE) and the processing time for generating 50 solutions by multi-scenario MORO with 1, 5, 9 and 50 scenarios are described in Table 5. All these models were solved 50 times (once for each reference point) in a laptop with *Intel CORE i7 CPU* and 16 GB RAM.

Overall, as expected, the computation costs increased when the number of scenarios grew. During the experiments, we noticed that the number of function evaluations and/or the time of evaluation varied from one reference point to another. For example, the most time-consuming calculations were related to the reference points that directed the search into the area in which the utility objective is maximizing. Also, the computational cost in most of the extremes was lower than the cost of identifying the more balanced solutions. This is the reason why the computation cost in the 50-scenario model was lower than in some other models.

As discussed earlier in Section 4, the solutions generated by multi-scenario MORO (without any additional filtering) have a similar robustness to the solutions produced by the previous variants of MORDM (after extra filtering). The proposed multi-scenario MORO, however, identified these solutions with fewer function evaluations. As seen in Table 5, the number of function evaluations was less than 145 000 in s_5 (the reference scenario), which is much less than 200 000 function evaluations that were used by Quinn et al. (2017) in MORDM. Moreover, the MORDM approach, used in Quinn et al. (2017), can hardly generate more robust solutions (if ever) even with more function evaluations mainly because no information about other scenarios can be considered within its optimization model. This is also true in the case of separate consideration of multiple scenarios as performed in multi-scenario MORDM (Eker and Kwakkel 2018). In contrast, by increasing the computational resources (like NFE) in the proposed multi-scenario MORO, one can include more scenarios within the optimization model that boost the robustness of the generated solutions. Furthermore, as mentioned above, we ran the proposed multi-scenario MORO on a personal laptop, which is significantly slower than the high-performance computer resources often used to solve different variants of MORDM (e.g., as utilized in Bartholomew and Kwakkel (2020)). Therefore, multi-scenario MORO is computationally more efficient than the previous approaches for the search phase of MORDM while considering more scenarios within the optimization formulation and has other advantages such as representing a wider variety of robustness trade-offs.

6. Conclusions

In various disciplines, there has been a growing interest in robust multi-objective optimization. This topic has in parallel been explored by researchers in both mathematical multi-objective optimization, and decision making under deep uncertainty. The former focuses mostly on theory developments, while the latter concentrates on practical problems. We believe that integrating the developments of these two fields can address some current issues in robust multi-objective optimization. One of the widely used model-based decision support frameworks in DMDU is many-objective robust decision making (MORDM). A critical step within the MORDM framework is the search phase, where candidate solutions are identified using multi-objective optimization. In this step, typically, one solves one or more single-scenario multi-objective optimization problems to produce a large set of promising solutions to be stress-tested under uncertainty. However, this solution set might not be feasible or be dominated in other scenarios. As an alternative, others have proposed to optimize robustness directly, but this leaves the trade-off between optimality within individual scenarios and robustness over the scenario set unexplored. To address these gaps, in this paper, we have proposed a new multi-scenario multi-objective robust optimization approach (called multi-scenario MORO) drawing on the concept of scalarizing functions from mathematical multi-objective optimization.

In the novel approach, the performance of solutions in terms of all objectives in all selected scenarios is evaluated within a single optimization problem. Therefore, the generated solutions are feasible in all selected scenarios and robust efficient. Furthermore, the proposed multi-scenario MORO enhances the robustness of the generated solutions, reduces scenario dependency, and produces a wider variety of robustness trade-offs than the previous variants of MORDM. Multi-scenario MORO also provides the opportunity of exploring trade-offs between optimality/feasibility in any given scenario and robustness over a broader range of scenarios by considering different numbers of scenarios within the optimization problem, which helps the decision maker in discovering balanced solutions. The computation cost of multi-scenario MORO is low compared to previous approaches.

However, there is a need for more experience in different real-life environmental problems. As we observed in this study, the lake

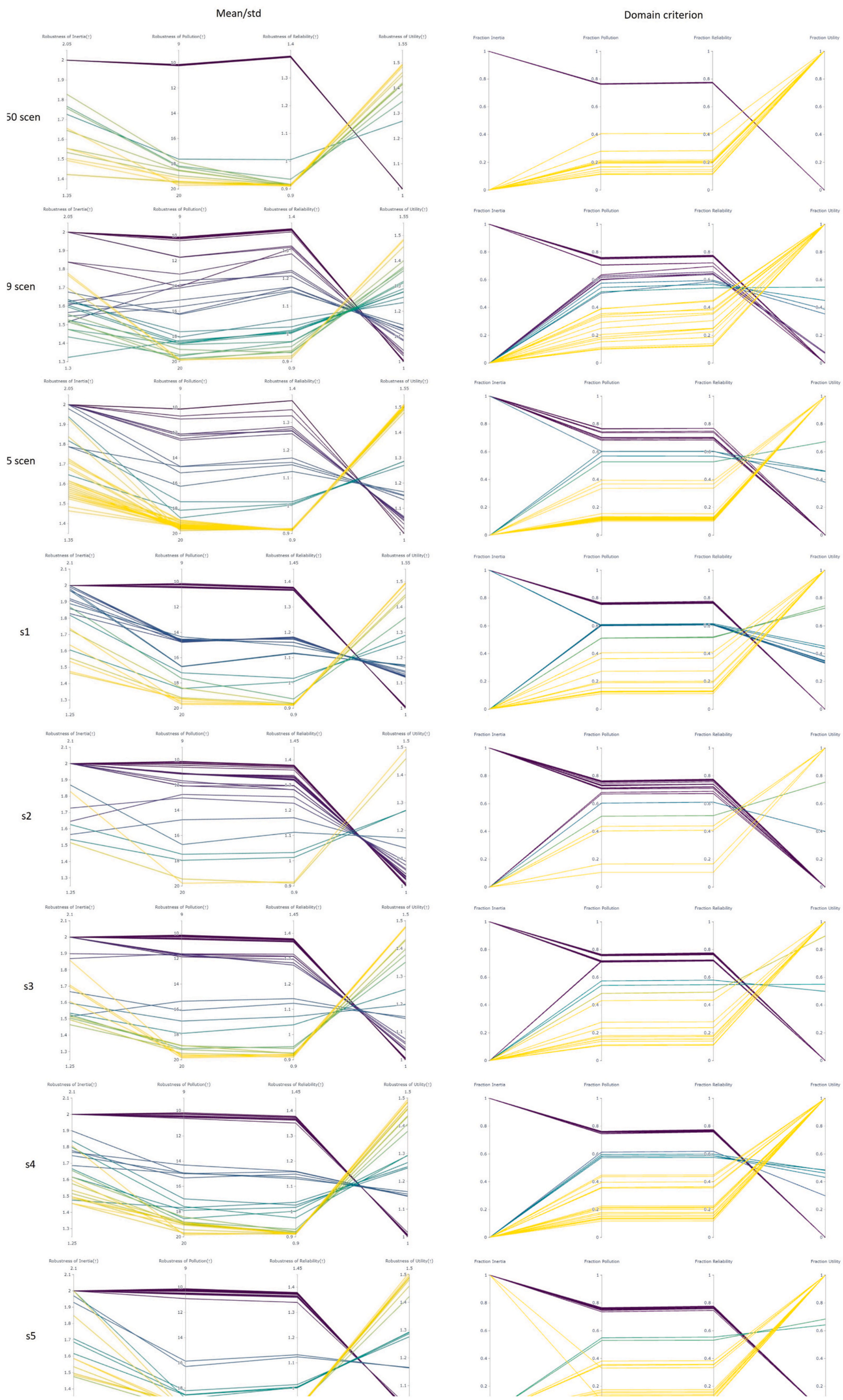

Fig. 11. Comparing the robustness of candidate solutions generated by 50, 9, and 5 scenarios.

Table 5
The total number of function evaluations (NFE) and the processing time for generating 50 solutions by multi-scenario MORO with a different number of scenarios within the optimization model.

	Multi-scenario MORO							
Number of scenarios	1(s_1)	1(s_2)	1(s_3)	1(s_4)	1(s_5)	5	9	50
Total NFE	184 724	101 486	133 815	162 049	144 457	147 912	156 360	25 394
Time (sec)	5 092	2 785	3 543	4 688	4 303	20 673	39 378	35 625
Non-dominated/ non-duplicated	39/ 50	34/ 50	33/ 50	32/ 50	33/ 50	49/ 50	50/ 50	50/ 50

problem, as a widely used benchmark problem in robustness comparisons, cannot reflect the trade-offs between scenarios. Therefore, there is a need for new benchmark problems that reflect trade-offs between scenarios. This will be one of our future research directions.

We also proposed a novel approach for scenario discovery based on the basic concepts of mathematical multi-objective optimization to determine vulnerable scenarios before generating any solution. The decision maker can learn about vulnerability and the sources of failures even before policy determination, paving the way for considering other solution methods like a priori and particularly interactive multi-objective optimization methods (Miettinen et al. 2008, 2016; Miettinen 1999). This interesting topic is also in our future research interests.

Last but not least, in this study, we used the inter-temporal open-loop formulation (including static periodical decision variables) of the lake problem for demonstrating our method and for comparisons because it is easy to understand and is supposed to present the relationship between scenarios and robustness of solutions. Nonetheless, the proposed multi-scenario MORO can also be applied to solve adaptive formulations, such as direct policy search (Quinn et al., 2017) and planned adaptive direct policy search (Bartholomew and Kwakkel 2020). We also believe that the best way to deal with deep uncertainty is dynamic robustness and adaptive approaches. As another interesting future direction, our proposed multi-scenario MORO can also be combined with dynamic adaptive policy pathways (Haasnoot et al., 2013). In this way, we can design a dynamic multi-stage multi-scenario MORO approach to identify the best combination of the initial decisions and scenario-relevant possible adaptation decisions (Shavazipour and Stewart 2019; Shavazipour et al., 2020).

Software availability

All code used for this research can be found at https://github.com/industrial-optimization-group/Multi-scenario-Multi-objective-Robust-Optimization-under-Deep-Uncertainty.git.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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