Introducing Golden Section in the Mathematics Class to Develop Critical Thinking from the STEAM Perspective

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Abstract

The Golden Section is a mathematical concept that is one of the most famous examples of connections between mathematics and the arts. Despite its widespread references in various areas of nature, art, architecture, literature, music, or aesthetics, discussions of the golden ratio often turn out to be false or misleading. Most of the incorrect statements are based on approximations or stem from the lack of checking the facts, making scientific mistakes in verifying the original scientific, historical, cultural context, or performing arbitrary operations in the measurements. This article offers geometric data and measurements, which allow the students to explore the golden ratio in various contexts through problem-solving activities. At the same time, we encourage students and their teachers to initiate critical discussions based on multidisciplinary research in the areas of STEAM about their findings. Such research-based critical discussion can help to discover the context of their results from several other perspectives in addition to mathematics. It can also reflect both the cultural and scientific validity of the - otherwise mathematically correct - computations, as an essential expectation towards mathematics applied in a cultural or social context. For some of the topics described in this paper, we provide GeoGebra applets, which can let the reader explore the phenomena, and some pedagogical usage in classroom may yield examples for various populations of students. The topic is valuable in STEAM Education, with activities relying on European, Southeast Asian and Middle Eastern perspectives.

Keywords: Golden Ratio, STEAM education, Dynamic Geometry, cultural background

This article offers an introduction to the various appearances of the Golden Section in various fields of sciences and the arts. We selected examples from our own research and pedagogical experiences and offer an outline of how these Golden Section examples can be utilized in classroom teaching for motivating students and offer opportunities for research extended to all STEAM areas and initiating critical discussions on cultural topics in the mathematics class.

In general, we would like to introduce Golden Section as a topic, which holds several learning opportunities in upper secondary mathematics education in mathematical problem-solving. At the same time, learning about the Golden Section can be a great opportunity to debunk erroneous or mystical interpretations and start to think critically about the cultural role of geometric data and measurements. The choice of the points for measurement has an influence on the result and the measurements and computations often provide approximations of the Golden Ratio, not always an exact value. This paper is devoted to STEAM education, with a special emphasis on Technology, Arts and Mathematics connections, i.e., technology-assisted mathematical activities around artistic artefacts. Its main goal is to propose mathematical activities, which can be based on topics from the cultural background of the students, hence the variety of examples in the paper.

In addition, we would like to highlight the important connection between mathematics and arts as well as the interconnectedness of various fields in science to be able to offer a transdisciplinary way of learning. Furthermore, we developed all examples with technology files and enable readers to explore Golden Section with interactive worksheets created by the open-source mathematics software GeoGebra[1]. This offers various possibilities of activities in STEAM education. In the following sections, we will offer an introduction to Golden Section and then offer a variety of examples from a wide range of fields: within mathematics (with an example from trigonometric functions), music, the structure of musical instruments, geography (with an example inciting students to learn some spherical trigonometry), etc.

The Golden Section

An ancient question is “How can we divide a line segment into two segments in a harmonious way?” One answer is to divide it into two parts of equal length. Another answer has it that harmonious proportions can be obtained as follows (Figure 1).

Figure 1

Harmonious Division of a Segment

Denoted by *a* and *b* are the respective lengths of the segments AB and BC. These numbers should verify the following equation:

(1) $$\frac{a+b}{a}=\frac{a}{b},$$

i.e., the ratio of the total segment length over the largest sub segment is equal to the ratio of the two sub segments. This equation can be written as $1+\frac{b}{a}=\frac{a}{b}$. Now denoting $\varphi=\frac{a}{b}$, then Equation (1) is equivalent to

[1] Interactive resources for all examples can be found here: https://www.geogebra.org/m/qppfbnah

(2) $$\varphi^2 - \varphi - 1 = 0 .$$

Equation (2) has two solutions; only one of them is positive, namely $\varphi = \frac{1+\sqrt{5}}{2}$.

As $\sqrt{5}$ is an irrational number, ϕ is an irrational number too. A decimal development of ϕ is $\varphi \approx 1.618033988$.

Among the numerous mathematical properties of the number ϕ, we will mention a few:

- It has a surprising connection to the integer 5: $\varphi = \sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}}$.
- It is equal to the simplest continuous square root: $\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$
- It is equal to the simplest continuous fraction: $\varphi = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}$
- Its powers generate copies of the Fibonacci sequence[2]:

$$\begin{aligned}
\varphi^2 &= 1+\varphi \\
\varphi^3 &= 1+2\varphi \\
\varphi^4 &= 2+3\varphi \\
\varphi^5 &= 3+5\varphi \\
\varphi^6 &= 5+8\varphi \\
\varphi^7 &= 8+13\varphi \\
&\ldots
\end{aligned}$$

The number φ has been called the *Golden Section*, sometimes the *Divine Proportion*. A rectangle whose length a, and width b verify the condition $\frac{a}{b} = \varphi$ is called a *Golden Rectangle*. This *Divine Proportion* name was first used by Fra Luca Bartolomeo Pacioli, a Franciscan monk, in his book *De divina proportione* (illustrated by Leonardo da Vinci)[3].

The Golden Section became one of the most popular examples of connections between mathematics and the arts. Despite its widespread references in various areas of nature, art, architecture, literature, music, or aesthetics, discussions of the golden

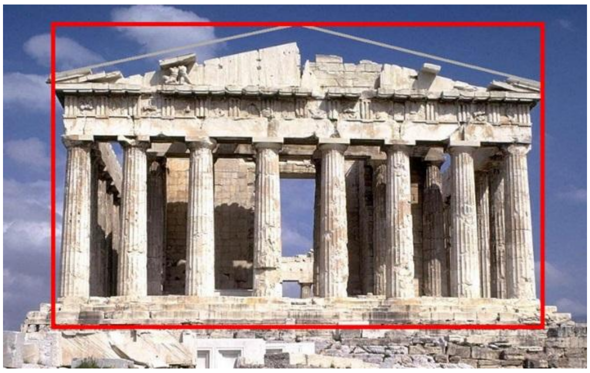
[2] A mathematical explanation of this fact is that φ generates a quadratic extension of the ring of integers Z, therefore, every power of ϕ is a linear combination of 1 and φ. It is easy to prove (by induction) that the coefficients of the successive powers are given by the Fibonacci sequence.

[3] https://en.wikipedia.org/wiki/De_divina_proportione

ratio often turn out to be false or misleading. Articles from Markowsky (1992), Gardner (1994), or Falbo (2005) serve as good guidance to study this famous geometrical proportion, but assess critically such photographs, which introduce, e.g., the main façade of temples in ancient Greece as golden rectangles, i.e., a rectangle whose length L and width l verify the condition $L/l=\phi$. The most famous example is the Parthenon on the Acropolis in Athens (see Figure 2). Actually, the rectangular part of the façade with the columns is not a golden rectangle, but the rectangle enclosing the whole of the façade is often depicted as such.

Figure 2

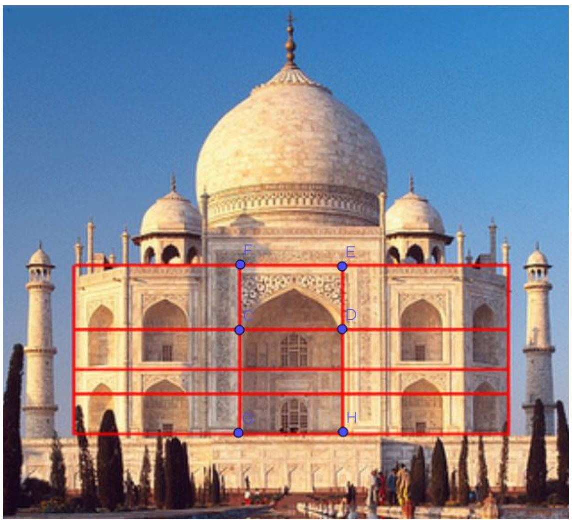
The Parthenon



Golden rectangles are often introduced as construction principles in other famous monuments as well, such as the Taj Mahal in India. Figure 3 shows a snapshot of a GeoGebra session to inscribe some golden rectangles in this monument and discuss the role of the photographic perspective and other variables in locating such proportions. We emphasize that this is a way for students to use GeoGebra as an exploration tool, sometimes using its features for *augmented reality* (AR).

Figure 3

Using GeoGebra to Project Golden Rectangles on a Photograph of the Taj Mahal



Similar kinds of technology-based activities can be developed to assert occurrences of Golden Rectangles in other monuments. As we will see in the next section, the number φ can be studied in relation to other ancient objects and monuments. For more examples, see Dana-Picard (2017), Frantilli et al. (2011), and numerous websites.

The Golden Section in the Bible: Objects

Noah's Ark

The dimensions of Noah's Ark are given in the verse Genesis 6, 15: *" And this is how thou shalt make it: the length of the ark three hundred cubits, the breadth of it fifty cubits, and the height of it thirty cubits".*

Figure 4

Noah's Ark (1846), by Edward Hicks[4]

The ratio of the breadth to the height is $\frac{50}{30} = \frac{5}{3} \approx 1.667$. This is obviously not the Golden Section, but it is an approximation of φ to one digit after the decimal period.

The Ark of the Covenant

The order to build the Ark of the Covenant (the cabinet containing scrolls of the Torah, written by Moses) is given in the following Biblical verse (Exodus 25, 10)[5]: *"And they shall make an ark of acacia-wood: two cubits and a half shall be the length thereof, and a cubit and a half the breadth thereof, and a cubit and a half the height thereof*." Figure 5 shows a reconstitution of the Ark with the cherubim (angels) on top. The dimensions indicated in the verse are those of the rectangular box, excepting the staves. The side faces are rectangles with respective length and width of 2.5 cubits and 1.5

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[4] Public domain. Retrieved from Wikipedia: https://en.wikipedia.org/wiki/Noah%27s_Ark#/media/File:Edward_Hicks,_American_-_Noah%27s_Ark_-_Google_Art_Project.jpg

[5] English translation from Hebrew by Machon Mamre: https://www.mechon-mamre.org

cubits. Their ratio is $\frac{2.5}{1.5} \simeq 1.667$, the same approximation as above of the Golden Section[6].

Figure 5

The Ark of the Covenant[7]

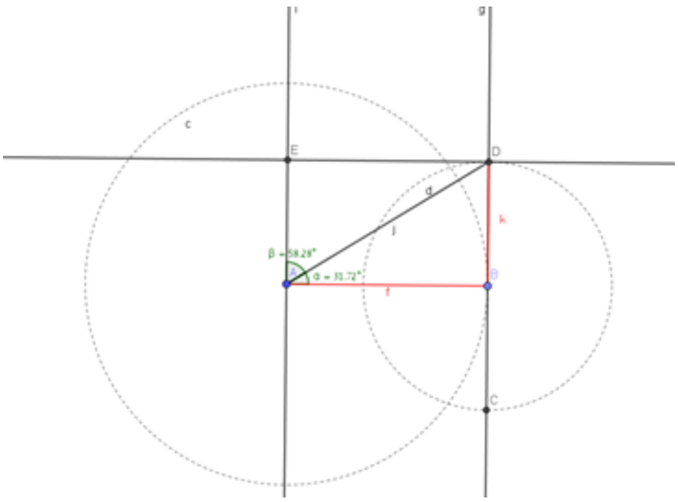
Students' discussion of these examples can assess the role of approximation in applications of mathematics.

Example: Jerusalem

Figure 6 shows a snapshot of a very easy GeoGebra session for the drawing of a golden rectangle.

Figure 6

Drawing of a Golden Rectangle with GeoGebra.



[6] Approximations to the 1st decimal digit are common in the Bible and in the Talmud (Garber, 2017).
[7] Credit: The Temple Institute, Jerusalem. Used by permission.

In an experimental way, this drawing provides a value of the two angles defined at each vertex of the rectangle by the diagonal exiting this vertex. Of course, a high school student should be able to compute these values using basic trigonometry. Here we have[8]**:** α=31.72° and β=58.28°. Note that:

- The number 31.72 is the latitude of Jerusalem and $\cot 31.72^o = 1.618$
- The number 58.28 is the culmination of the Sun (zenith) during the equinoxes in Jerusalem.

We can discuss with the students that it is hard to check the precision of these numbers, as the City of Jerusalem is a large area, so the question can be to which point in the town do these coordinates relate? It is worthwhile to discover and examine the assertion of various website authors who describe other connections between Jerusalem and the Golden Section. They are out of the scope of this paper, and sometimes they are based on inaccurate latitude-longitude coordinates.

The Golden Section: Measurements and the Pentagram

Consider a regular pentagon whose side length is equal to 1. An easy computation for the students shows that the (common) length of the diagonals is equal to ϕ. Figure 7 shows a snapshot of a GeoGebra[9] session to check this experimentally:

- Draw a segment AB of length 1;
- Draw a regular pentagon, AB being one of its sides;
- Draw the diagonals of this pentagon. The non-convex polygon formed by the diagonals is called a *Pentagram*.

The notations l, m, n, p, and q represent both the diagonal segments and their respective lengths; note that in the left column (called the algebraic window), they have a common value 1.618. We wish to make the following remarks:

- The accuracy of the approximation can be improved when requesting from the software to display more digits after the decimal period.
- When requesting from the software to display the object details, then the definition of q, for example, is Segment (C, A).

Now, compute the ratio of some of the segments. First, we obtain experimentally:

$$\frac{DB}{AB} = \varphi$$

Denote by F the point of intersection of the diagonals EB and AC. Then we have:

$$\frac{AB}{AF} = \varphi$$

Denote now by G the point of intersection of the parallel to AB through F. We have:

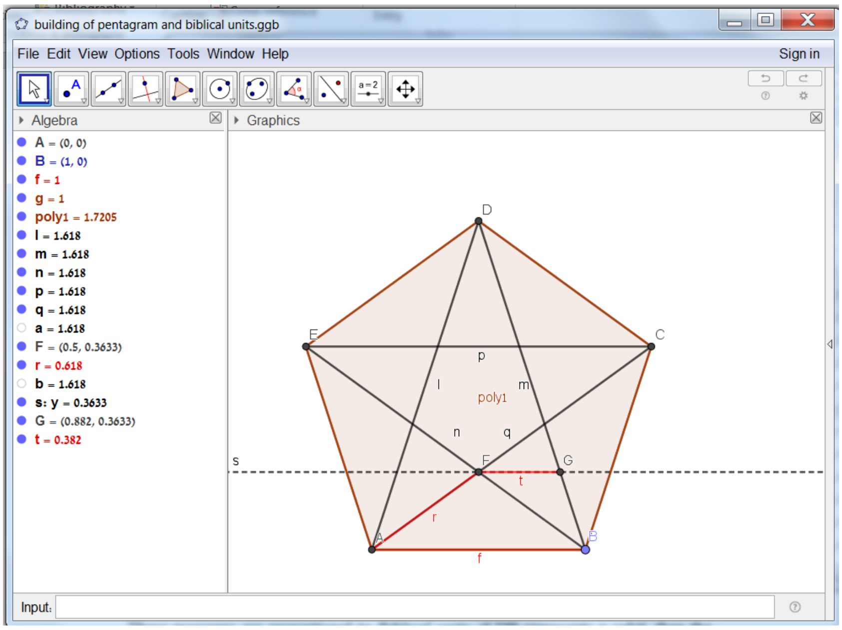
$$\frac{AF}{FG} = \varphi$$

[8] Of course, more digits can be obtained, using the **Options** menu of the software.
[9] GeoGebra is a freely downloadable software for Dynamic Geometry: www.geogebra.org .

Of course, this experimental discovery can be (and has to be) followed by exact computations, using trigonometry.

Figure 7

Building a Pentagram with GeoGebra

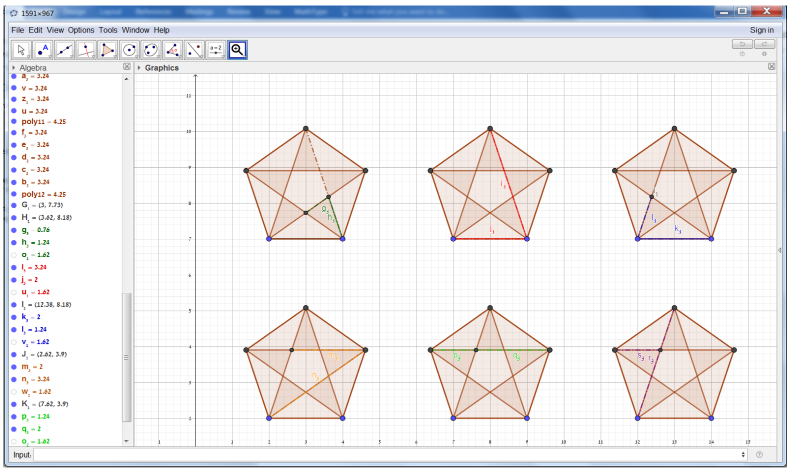


These measures are proportional to *Biblical length units*: if DB represents a cubit, then the following segments will represent a foot, a palm, etc.

Numerous other proportions of segments defined by the pentagon and the Pentagram and involving φ are illustrated in Figure 8. This is a snapshot of a GeoGebra session to check different ratios experimentally. For every pair of segments, the computer input and result appears in the left column (algebraic window).

Figure 8

Golden Ratios in Pentagrams shown with GeoGebra.



Octagonal buildings

The Castel del Monte, Italy

Castel del Monte was built by Emperor Frederick II on top of a hill in Apulia in Apulia (Southern Italy)[10]. It is shaped by several octagons. At first glance, Golden Rectangles may appear, as shown in Figure 10 (the main entrance). Once again, students can use GeoGebra as a tool for exploration. The students can check the details of Golden Rectangles with accuracy.

Figure 10

The Main Entrance to Castel del Monte

Now, let us look at the general structure of the castle. The main part is a ring enclosed by two concentric regular octagons (see Figure 11). The interior part of the smaller octagon is an open sky yard. At each vertex of the exterior octagon is an octagonal tower. We will focus on the main building.

Of course, the castle is divided into halls and rooms, as displayed in Figure 11b. Students can import the map as background of a GeoGebra session and re-build the map. The students can then check that the halls shaped as trapezoids (numbered with Roman numerals I-VIII in Figure 11b) have the following remarkable property[11]: the ratio of the long base measurement over the short one of each of these halls is equal to φ.

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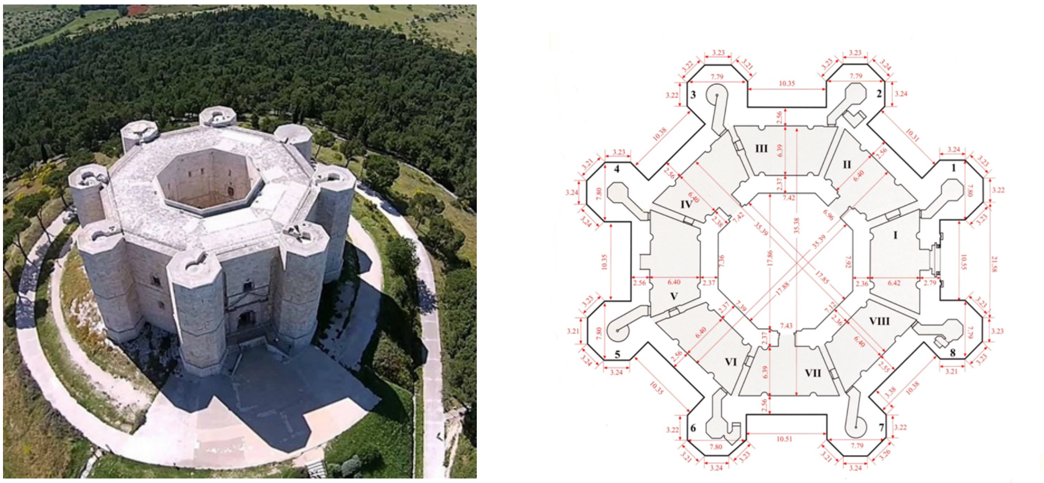
[10] For more info, see https://en.wikipedia.org/wiki/Castel_del_Monte,_Apulia

[11] As this is the main feature that the authors wished to have checked, a simplified version of the plan could have been used, without the inner part of the octagons centered at the vertices of the main one (corresponding to tower around the external wall).

Figure 11

The Castel del Monte

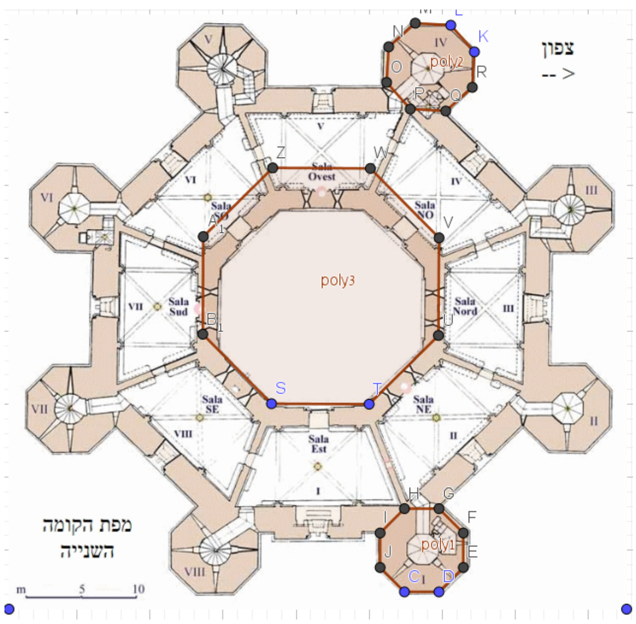


(a) Aerial view (b) Map with measurements

All of these halls have the same dimension because of the numerous symmetries of the building. Dana-Picard and Hershkovitz (2018) give details of a classroom activity for this. Remark: the actual measurements presented in Figure 12 have been obtained by Götze (1998). There exist other maps, with different focus of interest. Automated creation of regular polygons with GeoGebra enables us to check that the octagonal shapes of the castle are not actual regular octagons; such deviations are clear in Figure 12.

Figure 12

Checking the Regularity of Octagons

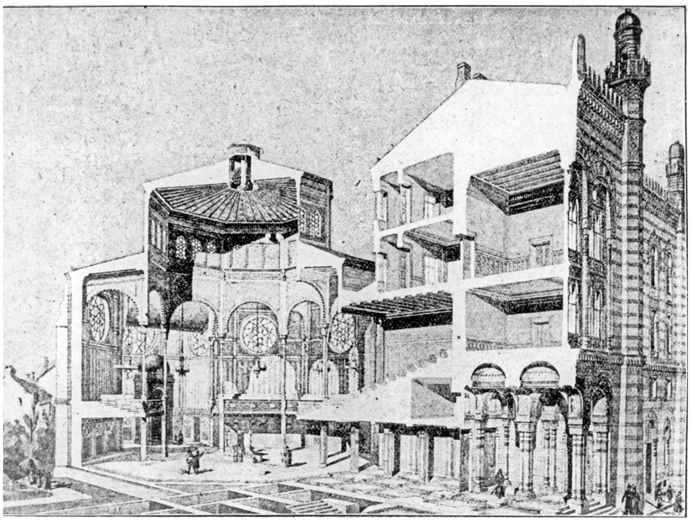


The Rumbach Synagogue, Hungary

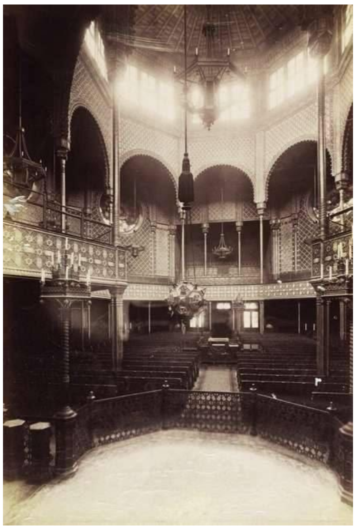
We look now at a 19th century synagogue on Rumbach Sebestyén street, Budapest. It was built on a piece of land enclosed in a non-regular polygon. The architects planned a building in which we can distinguish two different parts (see Figure 13a): one of them is divided into an apartment and classrooms; the other part, the westernmost, is an octagonal synagogue.

Figure 13

The Rumbach Synagogue, Budapest[12]



(a) Drawing by the architect Otto Wagner, 1869



(b) The synagogue in 1895

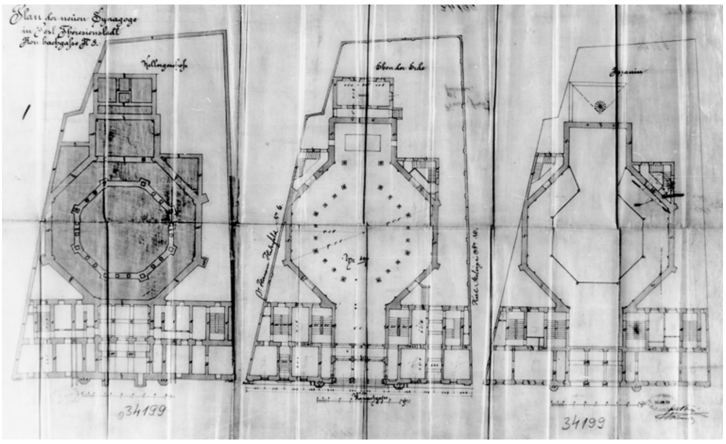
In Figure 13b, a photo from the year 1895 is displayed. This is a view from the center of the prayer hall in the direction of the entrance. This explains the fact that in Figure 13a, on the right, the octagon is not complete, one of the missing sides being a side of an external rectangle. Symmetrically, the same phenomenon occurs on the opposite side, where the "Holy Ark" containing the Torah (the Pentateuch, or the Five Books written by Moses on Mount Sinai) scrolls is situated.

Figure 14 shows the original plans. On the left is the plan of the women's gallery (Two floors above the men's prayer hall (ground floor, on the right). Two concentric octagons are apparent: the external one for the walls, the internal one is actually not full. The last one is made of separate columns, which bear the women's gallery, visible in the plan on the right.

[12] The authors wish to thank Kőnig Tamás, the architect in charge of the re-building of the synagogue, for a wonderful lecture on the history of the synagogue. They are also thankful to him for having shared documents.

Figure 14

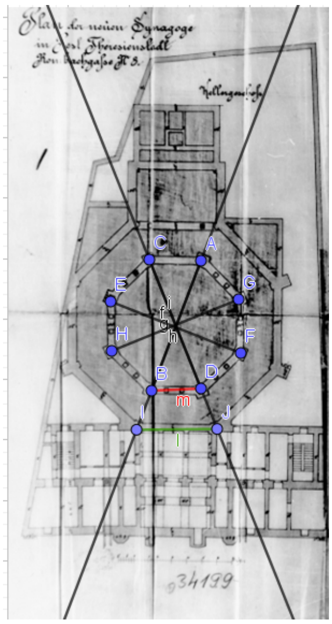
The Original Plans of the Rumbach Synagogue



Students can perform the same exercise as in the previous section, using the augmented reality (AR) features of the software in order to check the proportions of the building **(**see Figure 15). Augmented reality applications are already used in GeoGebra[13] and there are numerous efforts to experiment with AR in teaching mathematics and other related topics. We are currently running some experiments with AR and will report this in future papers.

Figure 15

Golden Proportions in Rumbach Synagogue



[13] https://www.geogebra.org/ar

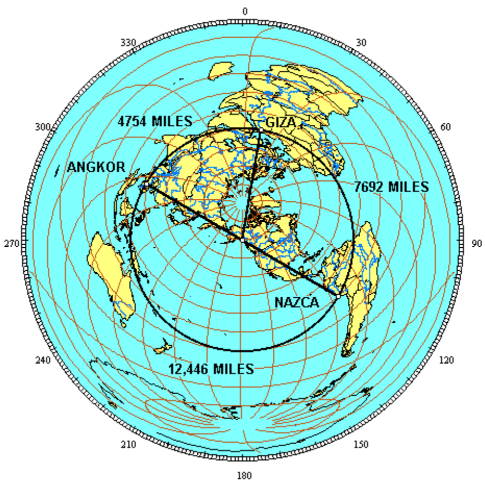
The columns supporting the women's gallery are not connected by walls and the whole of the volume on the three floors is open. Nevertheless, if students draw the diagonals through the center of the octagon, they define trapezoids whose bases are external walls (long bases) and the "virtual" segments between columns (short bases). An experimental checking using the software shows that the length ratio of the bases is equal to φ .

Angkor Wat and the Giza Pyramids

Angkor Wat was originally constructed in the early 12th century by the Khmer King Suryavarman II in what is now Cambodia. Several strange numbers arising from the monument, sometimes related to space measurements are given by Spivey (2016).

The shortest distance between two points, A and B, on a sphere is measured along an arc of a *great circle*, namely a circle passing through A and B and whose center is the center of the sphere (Jennings, 1994, Chap. 2). There exists a single great circle through Angkor Wat and the Giza plateau, with its pyramids. The distance from Angkor Wat to Cheops pyramid is equal to 4,754 miles. If a point, A, were to denote the location of Angkor Wat, then the second endpoint of the diameter of this great circle is situated 35 miles northwest from the Peruvian city of Nazca, where an outstanding archeological site is located.

Figure 16
The Great Circle: From Angkor Wat to Cheops Pyramid[14]



The distance from Cheops Pyramid to one of the main sites in Nazca is equal to 7,692 miles. Note that $7,692 = 4754 \cdot 1.618005889$, a 4-digit accurate approximation of the Golden Section. This can be an excellent problem-solving analysis challenge for students.

There are other ancient sites either on this great circle or close to it. For example, Preah Vihear in Cambodia and Passover Island, are on the circle. Petra and

[14] Credit: http://home.hiwaay.net/~jalison/phi2.html[15]
https://www.ticketmachupicchu.com/alignment-ollantaytambo-other-ancient-sites-planet/

Persepolis are within 10 kilometers. They are listed on a website[15] dedicated to Machu Picchu. Their relation to the Golden Section has still to be studied. Students can explore these structures and their relationships on the great circle mentioned.

We propose a canvas for some classroom activities:

- Import a suitable picture such as the background of Figure 16 as a background for a GeoGebra session.
- On this background, plot the points corresponding to the monuments under consideration.
- Plot the circle through these points. Interaction using the mouse may be highlighting.
- A nice challenge for the students: find new monuments/traditional sites close to the circle.
- Checking with the DGS the distance from these sites to the great circle may provide an opportunity to check the importance of approximations in applied situations.
  More advanced students may have the benefit of this example to broaden previous knowledge about spherical geometry, as explained by Jennings (1994). We emphasize that this is especially important for applications.

Calendars

This section is also based mostly on approximations and has multiple goals:

- Computations are based mostly on approximations. The topic yields another opportunity to assess the role of approximations.
- The topic under study is always seen as purely geometric. We show here a more abstract occurrence of the Golden Section. Other abstract situations exist; they will be the topic of further work.
- The connection between concrete objects (planets and their motions) and more abstracts notions (time) can be highlighted. An extension towards history of the students' traditional calendar is natural.

Numerous calendars are used in the world. Generally, the best known are the Gregorian calendar, the Muslim calendar, the Buddhist calendar, and the Jewish calendar.

Several definitions of a *solar year exist*, dependent on the astronomical phenomenon taken as a reference. For example, a *sidereal year* is the time such that, when observed from the Earth, the Sun is at the same position with respect to the stars on the celestial sphere. The *Gregorian calendar* is solar. One such year is the time interval of two passages of the Earth at the same point on its orbit around the Sun. It is computed so that the spring equinox is as close as possible to the 21st day of March. The length of a Gregorian year is 365.256363051 days (i.e., 365 days 6 hours 9 minutes and 9.767 6 seconds). Because of the fraction of a day (the digits after the decimal period), a

[15] https://www.ticketmachupicchu.com/alignment-ollantaytambo-other-ancient-sites-planet/

correction is needed and an extra day is added every 4 years, but every 100 years this extra day is not added[16].

A lunar calendar is based on lunar months. One *lunation* is the interval between two *new Moons*. It is approximatively equal to 29.53058885 days. The Muslim calendar is lunar, i.e., fully based on the Moon.

Figure 17

The Phases of the Moon along one Lunation

The two kinds of calendars have very different features. In a solar calendar, the notion of year is meaningful but months are arbitrary. In the lunar case, the notion of a month is (almost) well defined, but the notion of a year is arbitrary.

The Jewish and Buddhist calendars are lunisolar, i.e., they take into account both astronomical phenomena. The notion of a year is important, but the basic unit is the lunar month. As one lunar month is not made of an integer number of days, the equilibrium is made in the Jewish calendar by using months of 29 days alternating with months of 30 days. Two of the months may have either 29 or 30 days (separately) according to the specific year (The Buddhist calendar, which is based on lunar cycles within a sidereal year, is still used for religious observances, while the official governmental calendars are Gregorian, thus religious observances fall on different days each [Gregorian] calendar year).

Twelve such months are not synchronized with a solar year: there is a gap of 12 days. This is compensated when adding a 13th month from time to time. How is this performed?

The Greek astronomer Meton[17] of Athens observed that 19 solar years (i.e., 19 x 365 days) correspond (almost) exactly to 235 lunations. This corresponds to 6,940 days, with an error of a few hours. With this approximation, Meton obtained that 125 long months (30 days) and 110 short months (29 days) were needed for his calendar.

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[16] There are other exceptions, but they are out of the scope of this paper.

[17] We must mention that the results of Meton's observations were already known in Babylon. The Jewish calendar as we know it has been established in Babylon, because the traditional way of fixing the calendar based on testimony on the New Moon was vanishing. Recall that at that time, the Jewish people were in exile, mostly in Babylon, and his national structure (among others, we mean the Sanhedrin who in particular in charge of the calendar) did not work anymore.

Actually, concerning years, the following formula, known as *Meton's formula*[18], holds:

$$12 \cdot 12 + 7 \cdot 13 = 235$$

In other words, in a cycle of 19 years, take 12 years with 12 months each and 7 years with an intercalary month (a 13th month), and this performs the needed synchronization. A 13-month year is called embolismic.

This is the way the Jewish calendar is built: 12 years of 12 months each and years 3, 6, 8,11,14,17, and 19 of each cycle are embolismic.

Meton noted that if we were to share the 6940 days equally between the 19 years, we would obtain 365 + 1/4 +1/76 days. This reinforces the need for embolismic years. He noted that the proportion of years in each lunation is given by: 19/235 ≈ 0.0809. By denoting this number as α, we have: α ≈ 0.0809, 2α ≈ 0.1618 ≈ φ/10, … 20 α≈ φ.

Based on this, we can make a thought experiment on the approximation to a mathematical concept, i.e., the Golden Proportion, which we understood to be a geometric characteristic, that it can be projected to time.

Further connections

The *Fibonacci sequence*[19] is defined by $\begin{cases} F_1 = F_2 = 1 \\ F_{n+2} = F_{n+1} + F_n, n \geq 1 \end{cases}$. It is generally presented to students as a first example of a sequence defined with an induction formula using two terms to determine the next one. This is feasible with junior high school (lower secondary/middle school) students.

It is connected to the Golden Section by the following property: $\lim_{n \to +\infty} \frac{F_{n+1}}{F_n} = \varphi$.

The notion of the limit of a sequence comes later in the syllabus, therefore activities connected to the ratio of two consecutive elements of the Fibonacci sequence as an approximation of the Golden Section can appear only later.

Practical applications of the mathematical notions are of the utmost importance for many students. The Fibonacci sequence appears in numerous cases, some of them are handcrafted such as Judaica objects; others exist in nature, such as various flowers. These have already been described on many occasions. For specific groups of students, it may be interesting to look after occurrences of elements of the Fibonacci sequence in the monuments described in the previous sections. For example, the authors found all of the first nine elements in the Rumbach synagogue in Budapest. (See also Dana-Picard and Hershkovitz [2018]). Finding Fibonacci numbers when counting architectural

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[18] Once again, we can work here with approximations. Actually, there are 12.368 lunations in a solar year and 234.992 lunations in a solar cycle of 19 years. These more precise computations may be the basis of educational activities, using GeoGebra animations.

[19] For a thorough treatment of the Fibonacci sequence and applications, we refer to Koshy's book (2001).

elements is purely descriptive, thus accessible to any age. Nevertheless, the computations in the "Octagonal Buildings" section and the Dynamic Geometry System-based activities can be proposed only after some geometric background has been acquired (See Dana-Picard and Hershkovitz [2019]).

The geometric and trigonometric aspects of our study address students of different ages. With basic knowledge of trigonometry, the study of the various ratios in regular pentagons and pentagrams is accessible. The next example, in "The Golden Section: Measurements and the Pentagrams" section, is aimed at more advanced students, as it requires acquaintance with the arcsine function.

The part of our study devoted to astronomy requires at least some interest of the students for this kind of topic. The numerical data are available over the web; it is the educator's duty either to provide the data or to guide the students for their web search. Computations for the Hebrew lunisolar calendar involve more details than what we presented in the "Calendars" section. Nevertheless, they are accessible already in primary school. A colleague of the authors teaches this topic to 5^{th} grade students. Figure 18 shows an activity proposed by the second author to such students.

Discussion

In this paper, we offered a variety of examples for critical discussion on the Golden Section together with interactive resources and suggested some uses in education.

We believe that using some of these examples for developing students' abilities for critical explorations and debunking myths and understanding the role of approximation in mathematical applications could offer motivation for learning and could enable teachers to engage in interesting learning experiences. We hope that this paper also offers motivations for others to develop further examples within this topic and/or explore other mathematical ideas with technology and in relation to the arts. We believe that transdisciplinary learning, often highlighted in STEAM education could become a powerful approach in education.

The examples offer new perspectives to explore cultural phenomena. We are planning to develop further examples specific to Asian cultures and would like to encourage teachers and researchers in the various Asian regions to examine possible uses of local arts and culture for STEAM education. We would like to use these examples to use with further teaching and learning of mathematics and other STEAM subjects in a transdisciplinary matter thus enriching classroom activities and offering new ways to engage and contribute to students' learning. Furthermore, Dana-Picard and Hershkovitz (2019) showed how to use examples from the cultural background of students. They developed activities for exploration with a Dynamic Geometry System of Jewish monuments and their geometric properties, aimed at a specific student population. With the examples described in the "Calendars" section, we show that this point of view can be suitable for other students, building activities on each student's cultural background.

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