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The Light Ray Transform in Stationary and Static Lorentzian Geometries

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Abstract

Given a Lorentzian manifold, the light ray transform of a function is its integrals along null geodesics. This paper is concerned with the injectivity of the light ray transform on functions and tensors, up to the natural gauge for the problem. First, we study the injectivity of the light ray transform of a scalar function on a globally hyperbolic stationary Lorentzian manifold and prove injectivity holds if either a convex foliation condition is satisfied on a Cauchy surface on the manifold or the manifold is real analytic and null geodesics do not have cut points. Next, we consider the light ray transform on tensor fields of arbitrary rank in the more restrictive class of static Lorentzian manifolds and show that if the geodesic ray transform on tensors defined on the spatial part of the manifold is injective up to the natural gauge, then the light ray transform on tensors is also injective up to its natural gauge. Finally, we provide applications of our results to some inverse problems about recovery of coefficients for hyperbolic partial differential equations from boundary data.

Keywords Inverse problems · Light ray transform · Wave equation

Mathematics Subject Classification 53C65

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1 Introduction

Let $(\mathcal{N}, \bar{g})$ be a smooth Lorentzian manifold of dimension $1+n$, $n \geq 2$, with signature $(-,+,\ldots,+)$. We assume that $(\mathcal{N}, \bar{g})$ is oriented, connected and time oriented, and satisfies the strong causality condition. Here, by strong causality we mean that given any $p \in \mathcal{N}$ and any neighborhood $\mathcal{U}$ of p, there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p such that any causal curve segment with endpoints in $\mathcal{V}$ lies entirely in $\mathcal{U}$. We are interested in studying the injectivity of the so-called light ray transform on functions and tensors over such Lorentzian manifolds.

To formulate the problem precisely we introduce some notations. For each $m = 0, 1, \ldots$, let $\mathcal{S}^m = \mathcal{S}^m(\mathcal{N})$ denote the vector bundle of symmetric tensors of rank m on $\mathcal{N}$. In local coordinates each $\alpha \in C_c^\infty(\mathcal{N}; \mathcal{S}^m)$ can be written as

$$\alpha(y, w) = \alpha_{j_1\ldots j_m}(y) w^{j_1} \ldots w^{j_m}, \quad \forall\, (y, w) \in T\mathcal{N},$$

where we are using the Einstein summation convention. Next, let β be a maximal null geodesic in $(\mathcal{N}, \bar{g})$, namely an inextendible geodesic whose tangent vector at each point is lightlike:

$$\nabla^{\bar{g}}_{\dot{\beta}(s)} \dot{\beta}(s) = 0, \quad \text{and} \quad \bar{g}(\dot{\beta}(s), \dot{\beta}(s)) = 0. \tag{1}$$

Observe that Eq. (1) defines the parametrization of β uniquely up to a group of affine reparametrizations. We also note that such parametrizations can depend on the null geodesic itself. Given any choice of such parametrization along β, we define the light ray transform of $\alpha \in C_c^\infty(\mathcal{N}; \mathcal{S}^m)$ along β as follows:

$$L_\beta \alpha = \int_{\mathbb{R}} \alpha(\beta(s), \dot{\beta}(s))\, \mathrm{d}s. \tag{2}$$

Note that, by the strong causality condition the null geodesic $\beta(s)$ will lie outside of any compact set $K \subset \mathcal{N}$, for $|s|$ large enough (see [21, Lemma 13, p. 408]), and therefore the integral in (2) is well defined for compactly supported α. Note also that the domain of integration in (2) is also justified even when β is not complete since α is compactly supported.

Let us observe that an affine reparametrization of β results in the integral (2) to be scaled. Together with the linearity of the map L, this implies that the choice of the parametrization in (1) is of no significance provided that we are concerned with injectivity of the light ray transform on $\mathcal{N}$.

1.1 The Case of Stationary Geometries

The first result in our paper is concerned with the injectivity of the light ray transform on scalar functions under the additional assumption that $(\mathcal{N}, \bar{g})$ is both *globally hyperbolic* and *stationary*. For the purposes of this paper, it suffices to recall that global hyperbolicity is equivalent with $(\mathcal{N}, \bar{g})$ having a smooth spacelike Cauchy hypersurface, that is, a hypersurface which is intersected by every maximal timelike curve exactly once, see, e.g., [26, Corollary 11.19] and [21, Corollary 39, p. 422]. Also, by stationary, we mean that there exists a smooth complete timelike Killing vector field.

Let $N \subset \mathcal{N}$ denote a fixed, smooth, spacelike Cauchy hypersurface in $\mathcal{N}$, write $g = \bar{g}|_N$ and observe that (N, g) is a Riemannian manifold. It is well known (for example [14, Lemma 3.3]) that the manifold $(\mathcal{N}, \bar{g})$ admits an isometric embedding

$$\Phi : \mathbb{R} \times N \to \mathcal{N}$$

such that

$$\Phi^* \bar{g} = -c\, \mathrm{d}t^2 + \mathrm{d}t \otimes \eta + \eta \otimes \mathrm{d}t + g, \tag{3}$$

where c is a smooth positive function on N and η is a smooth covector field on N. For the convenience of the reader we show this in Sect. 2.1. In the more restrictive case where the one-form η in (3) vanishes identically in N, the manifold $\mathcal{N}$ is said to be *static*.

Let us remark that given a spacelike Cauchy surface N and a timelike Killing vector field $\mathcal{E}$, it is possible to fix the parametrization for the null geodesics in a natural way as follows. Given any maximal null geodesic β there is a unique point of intersection between β and N (see [21, Lemmas 29 and 42, on pp. 415 and 425]). We can then fix the affine parametrization by requiring that

$$\beta(0) \in N \quad \text{and} \quad \bar{g}(\dot{\beta}(0), \mathcal{E}) = -1.$$

Note that the quantity $\bar{g}(\dot{\beta}(s), \mathcal{E})$ is in fact a constant of motion along null geodesics and as such it can be fixed at any arbitrary point along β.

Before formulating our injectivity results in the setting of stationary globally hyperbolic Lorentzian manifolds $(\mathcal{N}, \bar{g})$, we give injectivity results in the model setting $(\mathcal{M}, \bar{g})$, where $\mathcal{M} = \mathbb{R} \times M$, M is an n-dimensional manifold with a smooth boundary, and $\bar{g}$ has the form (3) on $\mathbb{R} \times M$. In this setting, we prove injectivity of the light ray transform on scalar functions under one of the two hypotheses that we will formulate next.

To state the first hypothesis, we recall some concepts from Lorentzian geometry, namely, the notion of *time-separation* and *null cut locus*. The time-separation function, $\tau(p, q)$, between two points p and q is defined as the supremum of the semi-Riemannian length of all future-pointing causal curves connecting p to q, and zero if there is no such path. Next, let $p \in \mathcal{M}$, let $\beta : I \to \mathcal{M}$ be a future-pointing null geodesic with $\beta(0) = p$, and set

$$s_0 = \sup\{s \in I \mid \tau(p, \beta(s)) = 0\}.$$

If $s_0 \in I^{\mathrm{int}}$, we call $\beta(s_0)$ the future null cut point of p along β (see [3, Sect. 9.2]). Finally, the null cut locus $C_N^+(p)$ is then defined as the set of all future null cut points of p.

The geodesic β is the only causal path from p to $\beta(s)$ for $s \in (0, s_0)$, see, e.g., [3, Lemma 9.13]. On the other hand, for any $s > s_0$ there is a timelike curve from p to $\beta(s)$. As an example, let us consider the ultrastatic case $\mathcal{M} = \mathbb{R} \times M$ and $\bar{g} = -\mathrm{d}t^2 + g$ where (M, g) is a Riemannian manifold with boundary. If $\gamma(s)$ is a geodesic on M and $\gamma(s_0)$ is a cut point along γ in the Riemannian sense, then $\beta(s_0)$ is the future null cut point of $\beta(0)$ along $\beta(s) = (s, \gamma(s))$.

Hypothesis 1 *Suppose that M, g, c and η are real analytic and that the metric $\bar{g}$ given by (3) on $\mathbb{R} \times M$ has empty null cut locus.*

Before stating the second hypothesis, we need to make more definitions. We introduce the conformally scaled metric $\bar{g}_c$ on $\mathcal{M}$ by

$$\bar{g}_c = -\mathrm{d}t^2 + \mathrm{d}t \otimes \eta_c + \eta_c \otimes \mathrm{d}t + g_c, \tag{4}$$

where $\eta_c = c^{-1}\,\eta$, $g_c = c^{-1}\,g$. Next, we define $\mathscr{G}$ to be the set of smooth curves b on M that satisfy the following ordinary differential equation:

$$\nabla^{g_c}_{\dot{b}}\dot{b} = G(b, \dot{b}), \tag{5}$$

subject to the initial data $(b(0), \dot{b}(0)) \in TM$. The function $G(z, v)$ is defined for each $(z, v) \in TM$ as follows:

$$G(z, v) = -\left(\frac{c}{c + |\eta|_g^2}\right)\left((\nabla^{g_c}_v \eta_c)\, v\right)\, \eta_c^\sharp - (\eta_c v + \sqrt{(\eta_c v)^2 + |v|_{g_c}^2})\, F(z, v). \tag{6}$$

Here $\eta_c^\sharp$ is the vector dual to η_c with respect to g_c and the terms $(\nabla^{g_c}_v \eta_c)\, v$ and $\eta_c v$ denote the natural pairing between the one-forms $\nabla^{g_c}_v \eta_c$ and η_c with the vector v, respectively. All the terms in (6) are evaluated at the point $z \in M$. Finally, the term $F(z, v)$ is the vector field defined through

$$F(z, v) = d\eta_c(\cdot, v)^\sharp - \left(\frac{c}{c + |\eta|_g^2}\right) \mathrm{d}\eta_c(\eta_c^\sharp, v)\, \eta_c^\sharp.$$

As we will later see in Sect. 3, projections of null geodesics on M satisfy Eq. (5). This is due to the fact the null geodesics are translation invariant along the flow of the Killing field given by ∂_t.

The second hypothesis relies on a notion of foliation by a family of *strictly convex* hypersurfaces with respect to curves in $\mathscr{G}$, and can be stated as follows.

Hypothesis 2 *The dimension n of M satisfies $n \geq 3$, and there is a function $\rho : M \to [0, l]$, so that the following conditions hold:*

(i) $d\rho \neq 0$ *when* $\rho > 0$, $\rho^{-1}(l) = \partial M$ *and* $\rho^{-1}(0)$ *has empty interior.*
(ii) *For any* $b \in \mathscr{G}$, *if* $\frac{\mathrm{d}}{\mathrm{d}t}\rho(b(t)) = 0$, *then* $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\rho(b(t)) > 0$.

Our main theorem in the model case can now be stated as follows.

Theorem 1 *Let M be a smooth compact manifold with boundary and consider a Lorentzian metric $\bar{g}$ of form (3) on $\mathcal{M} = \mathbb{R} \times M$. Suppose one of the following:* (i) *there is a metric $\bar{g}_\star$ of form (3) on $\mathcal{M}$ such that $\bar{g}_\star$ satisfies Hypothesis 1 and $\bar{g}$ is in a small enough C^3-neighborhood of $\bar{g}_\star$,* (ii) *$(\mathcal{M}, \bar{g})$ satisfies Hypothesis 2. Then the light ray transform in $(\mathcal{M}, \bar{g})$ is injective on scalar functions. In other words, given any $f \in C_c^\infty(\mathcal{M})$, there holds*

$$L_\beta f = 0 \ \ \textit{for all maximal } \beta \textit{ in } \mathcal{M} \implies f \equiv 0.$$

As applications of Theorem 1, we state the following two perturbative examples:

(1) Suppose that $(M, g_\star)$ is a compact real-analytic Riemannian manifold with boundary without cut points. If c is close to 1, η is close to 0 and g is close to $g_\star$, then the light ray transform is injective on $(\mathbb{R} \times M, \bar{g})$ with $\bar{g}$ as in (3).
(2) Suppose that $(M, g_\star)$ is a compact Riemannian manifold with strictly convex boundary and suppose that there is a strictly convex function on $(M, g_\star)$. If c is close to 1, η is close to 0 and g is close to $g_\star$, then the light ray transform is injective on $(\mathbb{R} \times M, \bar{g})$ with $\bar{g}$ as in (3).

We will now give a corollary of Theorem 1 in the setting of globally hyperbolic stationary Lorentzian manifolds $(\mathcal{N}, \bar{g})$.

Corollary 1 *Let $(\mathcal{N}, \bar{g})$ be a globally hyperbolic stationary Lorentzian manifold, and let $\mathcal{N}$ have a non-compact Cauchy hypersurface N. Suppose one of the following:*

(i) *The Lorentzian manifold $\mathcal{N}$ and the Cauchy hypersurface N are real analytic, and there is a real-analytic Killing vector field. Moreover, N has empty null cut locus.*
(ii) *There exists a function $\rho : N \to [0, \infty)$, such that Hypothesis 2 holds on each $M_l = \{\rho \leq l\}$ with respect to the function $\rho|_{M_l}$.*

Then the light ray transform in $(\mathcal{N}, \bar{g})$ is injective on $C_c^\infty(\mathcal{N})$.

We note that under the assumption (i), the corollary follows immediately from Theorem 1, since the embedding Φ is also analytic in this case, see Sect. 2.1. In the case that (ii) holds, we note that due to the non-compactness assumption on N, given any scalar function f on $\mathcal{N}$ with compact support, there exists a large enough l such that $\operatorname{supp} f \subset \Phi(\mathbb{R} \times M)$ with $M = \{\rho \leq l\}$. The corollary then follows from Theorem 1 since M satisfies Hypothesis 2 with $\rho|_M$.

1.2 The Case of Static Geometries

Given a static globally hyperbolic Lorentzian manifold there exists an embedding $\Phi : \mathbb{R} \times N \to \mathcal{N}$ such that (3) holds with $\eta \equiv 0$. To simplify the statement of our results, we define $\mathcal{M} = \Phi(\mathbb{R} \times M)$ where $M \subset N$ is a compact manifold of dimension n with smooth boundary and study the injectivity of the light ray transform on tensors of arbitrary rank m over $\mathcal{M}$.

Before presenting the main result, we need to recall the definition of the geodesic ray transform on tensors in (M, g_c). To this end, suppose that γ is a unit-speed geodesic in (M, g_c). We define the bundle

$$\partial_{\text{in}} SM = \{(x, v) \in TM \mid x \in \partial M,\ v \in T_x M,\ |v|_{g_c} = 1,\ g_c(v, \nu) < 0\},$$

where ν denotes the unit outward pointing normal vector to ∂M at the point x. For each $(x, v) \in \partial_{\text{in}} SM$, we consider the unique geodesic γ with initial data (x, v) and define

$$\tau_+(x, v) = \inf\{r > 0 \mid \gamma(r; x, v) \in \partial M,\ \dot{\gamma}(r; x, v) \notin T_{\gamma(r;x,v)} \partial M\}.$$

We assume that the manifold (M, g_c) is non-trapping, that is, for all unit-speed geodesics $\gamma(\cdot; x, v)$ with $(x, v) \in \partial_{\text{in}} SM$, there holds $\tau_+(x, v) < \infty$. Finally, let $S^m = S^m(M)$ denote the bundle of symmetric tensors of rank m on M (not to be confused with $\mathcal{S}^m$, the corresponding bundle on $\mathcal{N}$) and define the geodesic ray transform of $\omega \in C_c^\infty(M; S^m)$ along γ in M as follows:

$$\mathcal{I}\omega(x, v) := \int_0^{\tau_+(x,v)} \omega(\gamma(\tau; x, v), \dot{\gamma}(\tau; x, v))\, \mathrm{d}\tau. \tag{7}$$

Here, analogously to the Lorentzian case, we have in local coordinates

$$\omega(y, w) = \omega_{j_1 \dots j_m}(y) w^{j_1} \dots w^{j_m}, \quad \forall\, (y, w) \in TM.$$

We require the following hypothesis to hold:

Hypothesis 3 *The geodesic ray transform on (M, g_c) is solenoidally injective. In other words, for any $\omega \in C_c^\infty(M; S^m)$, there holds*

$$\mathcal{I}\omega(x, v) = 0 \quad \forall\, (x, v) \in \partial_{in} SM \implies \exists\, \theta \quad \text{such that} \quad \omega = d^s \theta, \quad \theta|_{\partial M} = 0,$$

where d^s denotes the symmetrized covariant derivative on (M, g_c).

The study of the solenoidal injectivity of the geodesic ray transform on tensors of arbitrary rank has a rich literature. For example, Hypothesis 3 with a fixed $m = 0, 1$ is known to be true when (M, g_c) is a simple manifold [1,19,20] or has strictly convex boundary and admits a foliation by strictly convex hypersurfaces [36]. Under the latter condition it was later proved that Hypothesis 3 holds for all $m = 0, 1, 2$ [31], and subsequently that it holds for all $m = 0, 1, 2, \dots$ [8]. For more related results we refer the reader to [5,22–24,30] and the review article [13]. We can now state our main theorem for the injectivity of the light ray transform on tensors.

Theorem 2 *Let $(\mathcal{N}, \bar{g})$ be a static globally hyperbolic Lorentzian manifold of dimension $1+n$. Let Φ be an embedding satisfying (3) with $\eta = 0$ and let $\mathcal{M} = \Phi(\mathbb{R} \times M)$ where M is a compact n dimensional submanifold of N with smooth boundary such* *that Hypothesis 3 holds. Let $\alpha \in C_c^\infty(\mathcal{M}; \mathcal{S}^m)$. The following injectivity result holds for the light ray transform on $(\mathcal{M}, \bar{g})$:*

$$L_\beta\, \alpha = 0 \quad \text{for all maximal} \beta \textit{ in } \mathcal{M} \implies \exists\, T, U \quad s.t \quad \alpha \equiv \bar{d}^s T + U\, \bar{g},$$

where $\bar{d}^s$ denotes the symmetrized[1] *covariant derivative, $T \in C_c^\infty(\mathcal{M}; \mathcal{S}^{m-1})$, $U \in C_c^\infty(\mathcal{M}; \mathcal{S}^{m-2})$ and $U\,\bar{g}$ denotes the symmetrized tensor product of the tensors U and $\bar{g}$.*

Let us emphasize that the gauge appearing in the statement of Theorem 2 is the natural one since the light ray transform of any tensor of the form $\bar{d}^s T + U\,\bar{g}$ with T, U compactly supported in $\mathcal{M}$, vanishes. We refer the reader to Lemma 1 for the details. Observe also that, akin to Corollary 1, the result of Theorem 2 can be formulated for compactly supported tensor fields on a suitable non-compact Lorentzian manifold. Finally we mention that Theorem 2 extends analogous results obtained in [10, Proposition 1.4], where only the cases $m = 0, 1$ were considered.

To our knowledge, there are no results on tensor tomography along general flows of the form given by (5). For this reason, we leave the case of higher rank tensors on stationary spacetimes as a topic of future work.

1.3 Applications and Examples

We discuss some applications of our main results in general relativity that builds on the perturbative examples (1) and (2) in Sect. 1.1.

Indeed, Theorem 1 can be applied in the context of the Kerr black hole spacetime as discussed next, following the notations in [2, Sect. 5]. Recall that the Kerr geometry $(\mathbb{R}^4, \bar{g}_{\mathrm{Kerr}})$ is an exact solution to the Einstein field equations in general relativity and describes the geometry of vacuum spacetime around an axially symmetric black hole with a so-called quasi-spherical event horizon. The metric has the following form:

$$\bar{g}_{\mathrm{Kerr}} = -\mathrm{d}t^2 + \lambda\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) + (r^2 + a^2)\sin^2\theta\, d\phi^2 + \frac{2mr}{\lambda}(a\sin^2\theta\,\mathrm{d}\phi - \mathrm{d}t)^2. \tag{8}$$

Here, (r, θ, ϕ) are the usual spherical coordinates in $\mathbb{R}^3$, m denotes the mass and ma is the angular momentum as measured from infinity. The parameters Δ and λ are defined through

$$\lambda = \lambda(\theta, r) = r^2 + a^2\cos^2\theta \quad \text{and} \quad \Delta = \Delta(r) = r^2 - 2mr + a^2.$$

It is easy to see that both vector fields ∂_t and ∂_θ are Killing fields. Moreover, ∂_t is timelike outside the *ergosphere*, i.e., the region

$$\{(r, \theta, \phi) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2 \;:\; r > m + \sqrt{m^2 - a^2\cos^2\theta}\}.$$

[1] We refer the reader to expressions (12)–(13) in Sect. 2.2 for the definition of the symmetrized covariant derivative and symmetrized tensor product in local coordinates.

Clearly the metric $\bar{g}_{\mathrm{Kerr}}$ is real analytic in this region, since Δ is non-vanishing there. It follows that given any $M \subset \mathbb{R}^3$ that is a sufficiently small compact submanifold with boundary outside the ergosphere, then the submanifold $(\mathbb{R} \times M, \bar{g}_{\mathrm{Kerr}})$ satisfies the conditions of Hypothesis 1. In fact, the further away M is from the origin of $\mathbb{R}^3$, the larger it can be since the Kerr geometry is close to the Minkowski geometry far away from the center of the black hole. Let us emphasize that this application is using the full strength of Theorem 1 in the sense that η is non-vanishing in (3). Only the case $a = 0$, corresponding to the Schwarzschild black hole, gives $\eta \equiv 0$.

Theorem 1 and Corollary 1 have applications to the recovery of zeroth order time-dependent coefficients for the wave equation from boundary data. It is well known that the canonical wave equation is well-posed on a globally hyperbolic Lorentzian manifold, see, e.g., [26]. If the manifold is also stationary, there is a rich theory on the solution space of the wave equation. This space is used for instance in the context of quantum field theory, see Sect. 4.3 of [37]. For another example we refer to the recent relativistic generalization of the Gutzwiller–Duistermaat–Guillemin trace formula for the wave group [34,35].

To keep the notation simple, we suppose that $(\mathcal{M}, \bar{g})$ is as in Theorem 1, and consider the following initial boundary value problem:

$$\begin{cases} \Box_{\bar{g}} u + q\, u = 0, & \text{on } \mathbb{R} \times M, \\ u = h, & \text{on } \mathbb{R} \times \partial M, \\ u = 0, & \text{on } (-\infty, 0) \times M, \end{cases} \tag{9}$$

where $\Box_{\bar{g}}$ denotes the wave operator on $(\mathcal{M}, \bar{g})$ given in local coordinates by the expression

$$\Box_{\bar{g}} u = -\sum_{i,j=0}^{n} |\det \bar{g}\,|^{-\frac{1}{2}} \frac{\partial}{\partial x^i} \left(|\det \bar{g}\,|^{\frac{1}{2}} \bar{g}^{ij} \frac{\partial}{\partial x^j} u \right)$$

and q is a smooth a priori unknown function with compact support in the set $(T, \infty) \times M$ with large enough $T > 0$. We consider the problem of recovering q from the Dirichlet-to-Neumann operator Λ_q that is defined for all h compactly supported in $\partial\mathcal{M}$ through

$$\Lambda_q : h \mapsto \partial_{\bar{\nu}} u|_{\partial\mathcal{M}}.$$

It can be shown that the question of unique recovery of q from Λ_q reduces to the question of injectivity of the light ray transform on $(\mathcal{M}, \bar{g})$ (see for example [33]). As an immediate consequence of Theorem 1, we deduce that Λ_q determines q uniquely, if Hypothesis 1 or 2 holds.

1.4 Previous Literature

The study of injectivity of the light ray transform on tensors of arbitrary rank is motivated in part due to its connection with coefficient determination problems for the wave equation on Lorentzian manifolds from boundary data, as shown for example in

[1,4,10,15,28,33,38] for the cases $m = 0, 1$. In the setting of Minkowski spacetime, invertibility of the light ray transform on scalar functions was proved by Stefanov in [28]. A local version of this injectivity result follows from the more general result [29] for scalar functions on real-analytic Lorentzian manifolds. An analogous local result for one-forms in the Minkowski space was later proved in [25]. In [18], motivated by a study of the Cosmic Microwave Background radiation, the light ray transform on two tensors was considered in Minkowski spacetime, and it was showed that the spacelike singularities of a two-tensor can be recovered from its light ray transform. In [39] this result was extended to recovery of some lightlike singularities.

Beyond the Minkowski space time the literature is sparse even in the scalar case $m = 0$. Stefanov proved the injectivity of the light ray transform for this case under the geometrical assumptions that the Lorentzian manifold is real analytic and that a convexity-type assumption holds [29]. In [10], injectivity of the light ray transform was proved for the cases $m = 0, 1$ when $(\mathcal{M}, \bar{g})$ is static and the transversal manifold has an injective geodesic ray transform. This result has been generalized to the case of non-smooth scalar functions and continuous one-forms [9]. Finally, we refer the reader to [17] for a microlocal study of the light ray transform on general Lorentzian manifolds. There it is proven that the spacelike singularities of a scalar function can be recovered from its light ray transform.

A fundamental open question in the field of inverse problems is the recovery of the potential q in (9) given the Dirichlet-to-Neumann map Λ_q when $\bar{g}$ is an arbitrary globally hyperbolic Lorentzian metric. In fact, this problem is open even when $\mathcal{M} = \mathbb{R} \times M$, with $M \subset \mathbb{R}^3$ a compact strictly convex set with smooth boundary, and $\bar{g}$ is assumed to be close to the Minkowski metric, say in any C^k-space. The present paper solves the problem in the near Minkowski case under the additional assumption that $\bar{g}$ is stationary. Indeed, as a corollary of Theorem 1, we can conclude that the light ray transform is injective for this class of Lorentzian manifolds (with $k = 3$), and the recovery of q follows then from the discussion in the end of Sect. 1.3.

Theorem 2 provides the generalization of [10] to the more general case of tensors of arbitrary rank $m \geq 2$ in static geometries. We mention that in the case $m = 2$, this theorem has applications in transmission ultrasound imaging of moving tissues and organs [17, Sect. 5]. As mentioned above, it is also related to analysis of the Cosmic Microwave Background radiation [18].

The analysis in this paper is based on reducing the question of injectivity of the light ray transform in the Lorentzian manifold $(\mathcal{M}, \bar{g})$ to the question of injectivity of a ray transform on the spatial part of the manifold (M, g_c). In the stationary case, the corresponding ray transform is a generalization of the geodesic ray transform, consisting of integrals over a family of curves that solve equations (5). Injectivity of such a ray transform has in fact been studied for a broader family of vector fields G than the specific one given by expression (6). We refer the reader to [11], and also the appendix section of [36] written by Hanming Zhou (see also [40]). In the static case, the corresponding ray transform is the geodesic ray transform on M. As discussed in Sect. 1.2, solenoidal injectivity on tensors of arbitrary rank is known to hold in several cases.

Before closing the section we mention that we have become aware of the independent preprint [16] that was uploaded within a few days of our work. There, the authors study the injectivity of two-tensors along null geodesics in Minkowski spacetime from partial data and show that uniqueness holds modulo the natural gauge.

1.5 Outline of the Paper

In Sect. 2, we begin with the derivation of (3). We then discuss the natural gauge for the injectivity of the light ray transform and the conformal invariance of this gauge, see Corollary 2. Section 3 is concerned with the proof of Theorem 1. Finally, Sect. 4 contains the proof of Theorem 2. The latter two sections are independent of each other.

2 Preliminaries

2.1 Geometry of Stationary Lorentzian Manifolds

The aim of this subsection is to construct the canonical embedding $\Phi : \mathbb{R} \times N \to \mathcal{N}$ corresponding to a Cauchy surface N in $\mathcal{N}$, such that the metric $\Phi^*\bar{g}$ takes the form (3). As discussed in the introduction, this is well known in the literature and we only provide it for the convenience of the reader.

We start by defining $\mathcal{E}$ as the complete Killing vector field on $\mathcal{N}$, and for each $x \in N$, define $\Phi(\cdot, x)$ as the integral curve

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x) = \mathcal{E}(\Phi(t,x)), \quad \forall t \in \mathbb{R} \quad \text{and} \quad \Phi(0,x) = x. \tag{10}$$

Existence of a solution $\Phi(t, x)$ for all $t \in \mathbb{R}$ is guaranteed by the completeness of the vector field $\mathcal{E}$. We will show that Φ is a diffeomorphism. By global hyperbolicity, any integral curve $\Phi(\cdot, x)$ cannot self-intersect. As two distinct integral curves cannot intersect either, we deduce that Φ is injective. To see surjectivity, let $y \in \mathcal{N}$ and consider the integral curve

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = \mathcal{E}(\Psi(t)) \quad \forall t \in \mathbb{R} \quad \text{and} \quad \Psi(0) = y.$$

Using the definition of a Cauchy hypersurface and the fact that $\mathcal{E}$ is timelike, it follows that $\Psi(s) \in N$ for some $s \in \mathbb{R}$. Hence $y = \Phi(-s, \Psi(s))$ and Φ is surjective. Finally, since $\mathcal{E}$ is smooth, it follows that Φ is a diffeomorphism.

Next, we study $\Phi^*\bar{g}$. Let (t, x) denote a local coordinate system near a point $p \in \mathbb{R} \times N$ and let $\bar{g}_{ij}$ represent the components of the metric in this coordinate system, with $i, j = 0, 1, \ldots, n$. Since $\mathcal{E}$ is a Killing vector field, it follows that the components $(\Phi^*\bar{g})_{ij}(t, x)$ are all independent of t. Therefore, we can write

$$\Phi^*\bar{g}(t,x) = (\Phi^*\bar{g})_{00}(x)\,\mathrm{d}t^2 + 2\,\underbrace{(\Phi^*\bar{g})_{0\alpha}(x)\,dx^\alpha}_{\eta}\,\mathrm{d}t + g(x),$$

where $g = \bar{g}|_N$ and the index α runs from 1 to n. Note that (N, g) is a Riemannian manifold since N is a spacelike hypersurface, in the sense that all of its tangent vectors are spacelike. Since $\partial_t = \Phi^*\mathcal{E}$, it is easy to see that $\eta = \partial_t^\flat = \Phi^*\mathcal{E}^\flat|_N$, where $\mathcal{E}^\flat$ is the dual covector associated with $\mathcal{E}$.

2.2 Conformal Invariance of the Gauge

We begin with a lemma that shows the gauge in Theorem 2 is the natural one for the injectivity of the light ray transform on tensors.

Lemma 1 *Let $(\mathcal{N}, \bar{g})$ denote a globally hyperbolic Lorentzian manifold. Suppose that $T \in C_c^\infty(\mathcal{N}; \mathcal{S}^{m-1})$ and $U \in C_c^\infty(\mathcal{N}; \mathcal{S}^{m-2})$. Then,*

$$L_\beta(\bar{d}^s T + U\,\bar{g}) = 0 \quad \textit{for all maximal null geodesics } \beta \subset \mathcal{N}.$$

In other words, given T, U as above, L_β is invariant under the transformation

$$\alpha \to \alpha + \bar{d}^s T + U\,\bar{g}. \tag{11}$$

Proof First, let us recall that the symmetrized covariant derivative is defined through the expression

$$[\bar{d}^s T]_{i_1,\ldots,i_m} = \frac{1}{m!}\sum_{\pi\in S(m)} (\partial_{i_{\pi(1)}} T_{i_{\pi(2)},\ldots,i_{\pi(m)}} - \bar{\Gamma}^l_{i_{\pi(2)},i_{\pi(1)}} T_{l,i_{\pi(3)},\ldots,i_{\pi(m)}} \\ - \ldots - \bar{\Gamma}^l_{i_{\pi(m)},i_{\pi(1)}} T_{i_{\pi(2)},\ldots,l}), \tag{12}$$

where $\bar{\Gamma}^i_{jk}$ denotes the Christoffel symbols, $S(m)$ denotes the group of permutations of the set $\{1, \ldots, m\}$ and we are using the Einstein's summation convention with respect to the index l. We also recall that given any $U \in C_c^\infty(\mathcal{N}; \mathcal{S}^{m-2})$ the symmetrized tensor product $U\bar{g}$ is defined by

$$[U\,\bar{g}]_{i_1,\ldots,i_m} = \frac{1}{m!}\sum_{\pi\in S(m)} U_{i_{\pi(1)},i_{\pi(2)},\ldots,i_{\pi(m-2)}}\bar{g}_{i_{\pi(m-1)},i_{\pi(m)}}. \tag{13}$$

Since $\bar{g}(\dot{\beta}(s), \dot{\beta}(s)) = 0$ along any null geodesic, it follows trivially from (13) that $L_\beta(U\,\bar{g}) = 0$. Now applying the definition (12) together with the geodesic equation

$$\ddot{\beta}^i(s) + \bar{\Gamma}^i_{jk}(\beta(s))\dot{\beta}^j(s)\dot{\beta}^k(s) = 0,$$

it follows that

$$\partial_s T(\beta(s), \dot{\beta}(s)) = [\bar{d}^s T]_{i_1\ldots i_m}\dot{\beta}^{i_1}(s)\ldots\dot{\beta}^{i_m}(s)$$

and subsequently we have $L_\beta(\bar{d}^s T) = 0$ since T is compactly supported. □

Next, we aim to study the light ray transform on tensors under conformal rescalings of the metric and show that the natural gauge for the problem is conformally invariant. We consider a globally hyperbolic Lorentzian manifold $(\mathcal{N}, \bar{g})$ and use the notation $L^{\bar{g}}_\beta\alpha$

to emphasize the dependence of the light ray transform on the metric. Let $c \in C^\infty(\mathcal{N})$ be strictly positive valued and define $\tilde{g} = c\bar{g}$. Using [18, Sect. 6, Lemma 6.1], we observe that given a maximal null geodesic $\beta : \mathbb{R} \to \mathcal{N}$ satisfying (1) with respect to $\bar{g}$ and any non-zero $s_0 \in \mathbb{R}$, the same curve β parametrized as $\tilde{\beta}(s) = \beta(\sigma(s))$ satisfies (1) with respect to $\tilde{g}$, where

$$\sigma(s) = \int_{s_0}^{s} c(\beta(\tau))^{-1}\,\mathrm{d}\tau,$$

with $s \in \mathbb{R}$. This shows that given a $\alpha \in C_c^\infty(\mathcal{N}; \mathcal{S}^m)$, there holds

$$L_{\tilde{\beta}}^{\tilde{g}}\tilde{\alpha} = \int_{\mathbb{R}} \tilde{\alpha}(\tilde{\beta}(s), \dot{\tilde{\beta}}(s))\,\mathrm{d}s = \int_{\mathbb{R}} (c^{-m+1}\tilde{\alpha})(\beta(s), \dot{\beta}(s))\,\mathrm{d}s. \tag{14}$$

Using the above identity, it is clear that the injectivity of the light ray transform on scalar functions is conformally invariant. For tensors of rank $m \geq 1$ we have the following lemma that shows the natural gauge for the problem as seen in Theorem 2 is conformally invariant as well.

Lemma 2 *Let $(\mathcal{N}, \bar{g})$ be a Lorentzian manifold and consider $\tilde{g} = c\bar{g}$ for some smooth positive function c. Suppose $T \in C^\infty(\mathcal{N}; \mathcal{S}^{m-1})$ for some $m = 1, \ldots$. There exists $U \in C^\infty(\mathcal{N}; \mathcal{S}^{m-2})$, such that*

$$c^{-m+1}\tilde{d}^s T = \bar{d}^s(c^{-m+1}T) + U\,\bar{g}.$$

In the case $m = 1$, the tensor U is identically zero.

Proof We use the notations $\tilde{\Gamma}_{ij}^k$ (resp., $\tilde{d}^s$) and $\bar{\Gamma}_{ij}^k$ (resp., $\bar{d}^s$) to denote the Christoffel symbols (resp., symmetrized covariant derivative) on $\mathcal{N}$ with respect to the metrics $\tilde{g}$ and $\bar{g}$, respectively. By definition,

$$\begin{aligned}[\tilde{d}^s\tilde{T}]_{i_1,\ldots,i_m} = \frac{1}{m!}\sum_{\pi\in S(m)} &(\partial_{i_{\pi(1)}}\tilde{T}_{i_{\pi(2)},\ldots,i_{\pi(m)}} - \tilde{\Gamma}^l_{i_{\pi(2)},i_{\pi(1)}}\tilde{T}_{l,i_{\pi(3)},\ldots,i_{\pi(m)}}\\ &- \ldots - \tilde{\Gamma}^l_{i_{\pi(m)},i_{\pi(1)}}\tilde{T}_{i_{\pi(2)},\ldots,l}).\end{aligned} \tag{15}$$

Next, we define $\phi = -\frac{1}{2}\log c$ and recall the following identity that relates the Christoffel symbols $\bar{\Gamma}_{jk}^i$ and $\tilde{\Gamma}_{jk}^i$ (see [18, Lemma 6.3]):

$$\bar{\Gamma}_{jk}^i = \tilde{\Gamma}_{jk}^i + \delta_j^i\partial_k\phi + \delta_k^i\partial_j\phi - b^i\tilde{g}_{jk},$$

where $b = \nabla^{\tilde{g}}\phi$. Using the above identity together with the expression (15) we observe that the symmetrized derivative on tensors $\tilde{T} \in C_c^\infty(\mathcal{N}; \mathcal{S}^{m-1})$ transforms as

$$[\bar{d}^s\tilde{T}]_{i_1,\ldots,i_m} = [\tilde{d}^s\tilde{T}]_{i_1,\ldots,i_m} - \frac{1}{m!}\underbrace{\left[\sum_{\pi\in S(m)}\mathcal{S}_\pi\right]}_{I} - U\tilde{g}, \tag{16}$$

where

$$\begin{aligned}\mathcal{S}_\pi &= \Big((\partial_{i_{\pi(1)}}\phi)\,\tilde{T}_{i_{\pi(2)},\dots,i_{\pi(m)}} + (\partial_{i_{\pi(2)}}\phi)\,\tilde{T}_{i_{\pi(1)},i_{\pi(3)},\dots,i_{\pi(m)}}\Big)\\ &\quad + \dots + \Big((\partial_{i_{\pi(1)}}\phi)\,\tilde{T}_{i_{\pi(2)},\dots,i_{\pi(m)}} + (\partial_{i_{\pi(m)}}\phi)\,\tilde{T}_{i_{\pi(2)},\dots,i_{\pi(m-1)},i_{\pi(1)}}\Big).\end{aligned}$$

We can simplify I further, by considering the number of times that a fixed term of the form $(\partial_{i_{\tilde{\pi}(1)}}\phi)\,\tilde{T}_{i_{\tilde{\pi}(2)},\dots,i_{\tilde{\pi}(m)}}$ appears in I with $\tilde{\pi}\in S(m)$. Indeed, observe that

$$\begin{aligned}\sum_{\pi\in S(m)}(\partial_{i_{\pi(1)}}\phi)\,\tilde{T}_{i_{\pi(2)},\dots,i_{\pi(m)}} &= \sum_{\pi\in S(m)}(\partial_{i_{\pi(2)}}\phi)\,\tilde{T}_{i_{\pi(1)},i_{\pi(3)},\dots,i_{\pi(m)}}\\ &= \dots = \sum_{\pi\in S(m)}(\partial_{i_{\pi(m)}}\phi)\,\tilde{T}_{i_{\pi(2)},\dots,i_{\pi(1)}}.\end{aligned}$$

Consequently, Eq. (16) reduces to

$$[\bar{d}^s\tilde{T}]_{i_1,\dots,i_m} = [\tilde{d}^s\tilde{T}]_{i_1,\dots,i_m} - \frac{2(m-1)}{m!}\left[\sum_{\pi\in S(m)}(\partial_{i_{\pi(1)}}\phi)\tilde{T}_{i_{\pi(2)},\dots,i_{\pi(m)}}\right] - U\tilde{g}. \tag{17}$$

Next, we consider the tensor T in the statement of the lemma and define $\tilde{T}=c^{1-m}T$. We use the defining expression for the symmetrized derivative (15) and the definition of ϕ to obtain

$$\begin{aligned}[c^{-m+1}\tilde{d}^sT]_{i_1,\dots,i_m} &= [c^{-m+1}\tilde{d}^s(c^{m-1}\tilde{T})]_{i_1,\dots,i_m}\\ &= [\tilde{d}^s\tilde{T}]_{i_1,\dots,i_m} - \frac{2(m-1)}{m!}\left[\sum_{\pi\in S(m)}(\partial_{i_{\pi(1)}}\phi)\tilde{T}_{i_{\pi(2)},\dots,i_{\pi(m)}}\right].\end{aligned}$$

The claim follows from this identity and Eq. (17). □

Combining Lemma 2 together with Eq. (14) and Theorems 1–2 we have the following immediate corollary.

Corollary 2 *Given a globally hyperbolic Lorentzian manifold $(\mathcal{N},\bar{g})$, injectivity of the light ray transform modulo the gauge (11) is conformally invariant. In particular, the injectivity results stated in Corollary 1 and Theorem 2 hold in the more general setting that $\mathcal{N}$ is conformally stationary or conformally static, respectively.*

3 Injectivity of *L* in Stationary Geometries

Suppose that $(\mathcal{M},\bar{g})$ is as in Theorem 1. We are interested in the question of injectivity of the light ray transform. Owing to the conformal invariance of the light ray transform on scalar functions (see Sect. 2.2), we will work with the conformally rescaled metrics

$\bar{g}_c$ and g_c as discussed in Sect. 1.1. For the remainder of this section, we abuse the notation slightly and write L to denote the light ray transform on $(\mathcal{M}, \bar{g}_c)$, where we recall that

$$\bar{g}_c = -\mathrm{d}t^2 + \eta_c \otimes \mathrm{d}t + \mathrm{d}t \otimes \eta_c + g_c.$$

Lemma 3 *Let $\beta : I \to \mathcal{M}$ be a maximal null geodesic on $(\mathcal{M}, \bar{g}_c)$ and write $\beta(s) = (a(s), b(s))$ where a and b are paths on $\mathbb{R}$ and M, respectively. Let $T \in \mathbb{R}$. Then $\beta_T : I \to \mathbb{R}$ defined through $\beta_T(s) = (a(s) + T, b(s))$ is a maximal null geodesic on $\mathcal{M}$.*

Proof This follows immediately from the fact that the components of $\bar{g}_c(t, x)$ are independent of the time-coordinate t. □

Let $f \in C_c^\infty(\mathcal{M})$ and suppose that $\beta : I \to \mathcal{M}$ is a maximal null geodesic. Define $\beta_T : I \to \mathcal{M}$ as translations of $\beta(s)$ along the time coordinate t analogously as above. Then,

$$\begin{aligned}\int_{\mathbb{R}} e^{-\iota\tau T} L_{\beta_T} f \, \mathrm{d}T &= \int_{\mathbb{R}} \int_I e^{-\iota\tau T} f(a(s) + T, b(s)) \, \mathrm{d}T \, \mathrm{d}s \\ &= \int_I e^{\iota\tau a(s)} \int_{\mathbb{R}} e^{-\iota\tau r} f(r, b(s)) \, \mathrm{d}r \, ds = \int_I e^{\iota\tau a(s)} \hat{f}(\tau, b(s)) \, \mathrm{d}s,\end{aligned} \tag{18}$$

with $\hat{f}$ denoting the Fourier transform[2] in t. We define the integral transform

$$\mathscr{I} f(b) = \int_I f(b(s)) \, \mathrm{d}s, \tag{19}$$

where $b = \pi \circ \beta$, with $\pi : \mathcal{M} \to M$ the natural projection and β a null geodesic on $\mathcal{M}$.

Let us analyze $\mathscr{I}$ further. Referring to $\beta(s) = (a(s), b(s))$ in Lemma 3, we use the shorthand notations

$$\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}s}, \quad \dot{b} = \frac{\mathrm{d}b}{\mathrm{d}s}, \quad |\dot{b}|_{g_c} = |\dot{b}|.$$

As $\dot{\beta}$ is lightlike, there holds

$$-\dot{a}^2 + 2\dot{a}\eta_c\dot{b} + |\dot{b}|^2 = 0.$$

Therefore

$$\dot{a} = \eta_c\dot{b} \pm \sqrt{(\eta_c\dot{b})^2 + |\dot{b}|^2}. \tag{20}$$

[2] We use the notation ι for the imaginary unit to avoid confusion with the indices.

Let $\bar{\Gamma}^i_{jk}$ denote the Christoffel symbols on $(\mathcal{M}, \bar{g}_c)$ and observe that $\bar{\Gamma}^i_{00} = 0$ for $i = 1, \ldots, n$. Using this and the definition of a null geodesic, we see that b satisfies Eq.

$$\frac{d^2 b^i}{\mathrm{d}s^2} + \bar{\Gamma}^i_{jk}(b(s))\dot{b}^j \dot{b}^k + 2\bar{\Gamma}^i_{0k}\dot{a}\dot{b}^k = 0, \tag{21}$$

for $i = 1, \ldots, n$. We can choose, without loss of generality, the positive sign in Eq. (20). Indeed, suppose that $(a_+(s), b_+(s))$ solves (20)–(21) with the positive sign in (20) for $s \in I$. Then, $(a_-(s), b_-(s)) := (a_+(-s), b_+(-s))$ with $s \in -I$ solves the same two equations with the negative sign in (20). Hence, the choice of sign corresponds to affine reparametrizations of a fixed null geodesic. For this reason, we will just consider the positive sign in (20).

Now, Eq. (21) can be recast in the form

$$\nabla^{g_c}_{\dot{b}} \dot{b} = G(b, \dot{b}), \tag{22}$$

where

$$G^i(b, \dot{b}) := (\Gamma^i_{jk} - \bar{\Gamma}^i_{jk})\dot{b}^j \dot{b}^k - 2\bar{\Gamma}^i_{0k}(\eta_c \dot{b} + \sqrt{(\eta_c \dot{b})^2 + |\dot{b}|^2})\dot{b}^k, \tag{23}$$

where Γ^i_{jk} denotes the Christoffel symbols on (M, g_c). We will now simplify the latter expression and show that the curves $b \in \mathscr{G}$ are coordinate invariant in $\mathcal{M}$. To see this, we first observe that

$$\bar{g}_c^{-1} = \left(\frac{c}{c + |\eta|_g^2}\right) \begin{bmatrix} -1 & \eta_c^\sharp \\ (\eta_c^\sharp)^T & \frac{c+|\eta|_g^2}{c}\, g_c^{-1} - \eta_c^\sharp \otimes \eta_c^\sharp, \end{bmatrix}$$

where $\eta_c^\sharp$ denotes the canonical vector that is dual to the one-form η_c and ^{T} denotes the transposition operation. Now, using the definition of the Christoffel symbols together with the fact that the coefficients of the metric are time-independent, we write

$$\begin{aligned} \Gamma^i_{jk} - \bar{\Gamma}^i_{jk} = & \underbrace{-\frac{1}{2}(\bar{g}_c)^{i0}\left((\bar{g}_c)_{0,j;k} + (\bar{g}_c)_{0,k;j}\right)}_{I} \\ & \underbrace{+\frac{1}{2}((g_c)^{im} - (\bar{g}_c)^{im})\left((g_c)_{mj;k} + (g_c)_{jm;k} - (g_c)_{jk;m}\right)}_{II}, \end{aligned}$$

where the term II involves a summation over the index $m = 1, \ldots, n$. The term I reduces as follows:

$$I = -\frac{1}{2}\left(\frac{c}{c + |\eta|_g^2}\right)(\eta_c^\sharp)^i \left((\eta_c)_{j;k} + (\eta_c)_{k;j}\right).$$

Similarly, the term II reduces as follows:

$$\begin{aligned}
II &= \frac{1}{2}((g_c)^{im} - (\bar{g}_c)^{im})\left((g_c)_{mj;k} + (g_c)_{jm;k} - (g_c)_{jk;m}\right)\\
&= \frac{1}{2}\left(\frac{c}{c+|\eta|_g^2}\right)(\eta_c^\sharp)^i(\eta_c^\sharp)^m\left((g_c)_{mj;k} + (g_c)_{jm;k} - (g_c)_{jk;m}\right)\\
&= \frac{1}{2}\left(\frac{c}{c+|\eta|_g^2}\right)(\eta_c^\sharp)^i(\eta_c)_l(g_c)^{ml}\left((g_c)_{mj;k} + (g_c)_{jm;k} - (g_c)_{jk;m}\right)\\
&= \left(\frac{c}{c+|\eta|_g^2}\right)(\eta_c^\sharp)^i(\eta_c)_l\Gamma^l_{jk}.
\end{aligned}$$

Combining the expressions for I and II we deduce that

$$(\Gamma^i_{jk} - \bar{\Gamma}^i_{jk})\dot{b}^j\dot{b}^k = -\left(\frac{c}{c+|\eta|_g^2}\right)\left((\nabla^{g_c}_{\dot{b}}\eta_c)\,\dot{b}\right)\,(\eta_c^\sharp)^i.$$

We now consider the last term in the expression for G. Using the definition of the Christoffel symbols again and the expression of the inverse matrix $\bar{g}_c^{-1}$ above, this reduces as follows:

$$\begin{aligned}
2\bar{\Gamma}^i_{0k}\dot{b}^k &= \bar{g}_c^{im}((g_c)_{m0;k} - (g_c)_{0k;m})\dot{b}^k = \bar{g}_c^{im}((\eta_c)_{m;k} - (\eta_c)_{k;m})\dot{b}^k\\
&= \left(g_c^{im} - \frac{c}{c+|\eta|_g^2}(\eta_c^\sharp)^i(\eta_c^\sharp)^m\right)\left((\eta_c)_{m;k} - (\eta_c)_{k;m}\right)\dot{b}^k.
\end{aligned}$$

Recalling that $(d\eta_c)_{mk} = (\eta_c)_{m;k} - (\eta_c)_{k;m}$, we conclude that G can be rewritten as given by Eq. (6), thus establishing that it is an invariantly defined vector field on M. Let us emphasize that the parametrization of the curve $b(s)$ in M with $s \in I$ is not a unit-speed parametrization and is directly induced by the initial choice of an affine parametrization for the null geodesic β in $\mathcal{M}$.

Theorem 3 *If $\mathscr{I}$ is injective, then L is also injective.*

Proof Suppose that $L_\beta f = 0$ where $\beta : I \to M$ denotes any maximal null geodesic in $\mathcal{M}$ with maximal interval I. Differentiating Eq. (18) k times with respect to τ and evaluating at $\tau = 0$, we obtain

$$0 = \sum_{j=0}^{k}\int_I (\iota a(s))^{k-j}\,\partial_\tau^j\hat{f}(0, b(s))\,\mathrm{d}s \quad \forall\, b \in \mathscr{G}.$$

Setting $k = 0$, we have

$$(\mathscr{I}\hat{f}(0,\cdot))(b) = 0 \quad \forall\, b \in \mathscr{G}.$$

By injectivity of $\mathscr{I}$, it holds that $\hat{f}(0, \cdot) = 0$. In a similar manner, by using induction on k together with the injectivity of $\mathscr{I}$, we deduce that

$$\partial_\tau^k \hat{f}(0, \cdot) = 0, \quad \forall k \in \mathbb{N}.$$

As $f(t, \cdot)$ is compactly supported in t, $\hat{f}(\tau, \cdot)$ is analytic in τ, and thus f vanishes everywhere. □

3.1 Proof of Theorem 1

It is clear that Theorem 1 follows, once we prove injectivity of the ray transform $\mathscr{I}$ along all maximal curves $b \in \mathscr{G}$. We prove this under the assumption that Hypothesis 1 or Hypothesis 2 holds. In fact, the transform $\mathscr{I}$ has been studied for more general vector fields $G(z, v)$ than the one given by expression (6) and invertibility is known to hold under some assumptions. When Hypothesis 2 holds on $(\mathcal{M}, \bar{g})$, injectivity of $\mathscr{I}$ follows from [36, Theorem 4.2] in the appendix by Hanming Zhou and the remarks immediately following that theorem.

To prove Theorem 1 under Hypothesis 1, we will use [11, Theorems 1–2]. There, injectivity of the map $\mathscr{I}$ is proved under the assumption that the manifold M is real analytic and G is in a sufficiently small C^2 neighborhood of a real-analytic vector field $G_\star$ and that the curves in $\mathscr{G}_\star$ do not contain any conjugate points. Hence, to conclude the proof, we need to show that if Hypothesis 1 holds in $\mathcal{M}$, the aforementioned assumptions are satisfied.

To this end, let us first recall the definition of *conjugate points* along curves $b \in \mathscr{G}$ (following [11]) and conjugate points along null geodesics β in $\mathcal{M}$. Given any $(s, \xi) \in TM$, we define the exponential map $\widetilde{\exp}_x(s, \xi) = b(s)$ where $b \in \mathscr{G}$ with $b(0) = x$ and $\dot{b}(0) = \xi$. Subsequently, we say that the point $b(s_0)$ is conjugate to x if $(D_{s,\xi}\widetilde{\exp}_x)(s_0, \xi_0)$ has rank less than n, where $\xi_0 = \dot{b}(0)$. The conjugate points on $\mathcal{M}$ are defined analogously, in terms of the exponential map, $\exp : T\mathcal{M} \to T\mathcal{M}$ of the Lorentzian manifold $(\mathcal{M}, \bar{g}_c)$ along null geodesics (see for example [21, Definition 10.9]).

We now return to verifying the assumptions of [11, Theorem 2] Under Hypothesis 1. It follows from (6) that G and $G_\star$ are in a small C^2 neighborhood of each other. To see that the curves in $\mathscr{G}_\star$ are real analytic we note that $(\mathcal{M}, \bar{g}_\star)$, $c_\star$ and $\eta_\star$ are assumed to be real analytic. Thus the curves $b \in \mathscr{G}_\star$ are also real analytic as they solve a second-order linear ordinary differential equation (5) with real-analytic coefficients.

In order to apply [11, Theorem 2] and deduce the injectivity of the ray transform $\mathscr{I}$, it remains to verify that the curves in $\mathscr{G}_\star$ do not have conjugate points. This will be proved in the following lemma.

Lemma 4 *If $(\mathcal{M}, \bar{g}_\star)$ has an empty null cut locus, then there are no conjugate points along any curve $b \in \mathscr{G}_\star$.*

Proof Suppose for contrary that there exists a curve $b \in \mathscr{G}_\star$ with a pair of conjugate points $b(0)$ and $b(s_0)$. The above definition of conjugate points implies in particular that there exists a one-parameter family of curves b_r in $\mathscr{G}_\star$ with r in a small neighborhood of origin, such that

$$b_r(0) = b(0), \quad \dot{b}_r(0) = \dot{b}(0) + rv$$

for some fixed $v \in T_{b(0)}M$ and

$$\mathrm{dist}(b_r(s_0), b(s_0)) \leq Cr^2 \tag{24}$$

for some uniform constant $C > 0$, where $\mathrm{dist}(\cdot, \cdot)$ is the Riemannian distance function on $(M, g_{\star c})$ (note that $b_0 \equiv b$). We define the functions $a_r(s)$ as the solutions to the following differential equation:

$$\frac{da_r}{\mathrm{d}s} = \eta_{\star c}\dot{b}_r \pm \sqrt{(\eta_{\star c}\dot{b}_r)^2 + |\dot{b}_r|^2} \quad \text{and} \quad a_r(0) = 0,$$

where the sign $\pm$ is chosen in order to make the curve $(a_r(s), b_r(s))$ future-pointing. Observe that the curves $\beta_r(s) = (a_r(s), b_r(s))$ define a family of maximal null geodesics in $(\mathcal{M}, \bar{g}_\star)$ (in what follows, we will drop the subscript r when $r = 0$).

Next, we observe that there exists a constant $\delta > 0$ depending only on $\eta_{\star c}$ and $g_{\star c}$, such that if

$$|\,\mathrm{dist}(b_{r_k}(s_0), b(s_0))\,| < \delta\,|a_{r_k}(s_0) - a(s_0)| \tag{25}$$

for a sequence $r_k \to 0$, then there exists a causal path between $\beta(s_0)$ and $\beta_{r_k}(s_0)$ for all k sufficiently large.

If (25) does not hold for any sequence $r_k \to 0$, then in particular it implies that $|a_r(s_0) - a(s_0)| < \frac{C}{\delta}r^2$ and all r sufficiently close to zero. But then the first variation of β among the family of null geodesics β_r must vanish at the point $\beta(s_0)$ and consequently the point $\beta(s_0)$ is a conjugate point to $\beta(0)$ along β. By [21, Proposition 10.48], there exists a future-pointing timelike curve connecting $\beta(0)$ to $\beta(s_0)$ and therefore there exists a null cut point on β corresponding to $\beta(0)$ which is a contradiction.

Thus, we assume that (25) holds, for a sequence $r_k \to 0$ and consequently that there exists a future-pointing causal path connecting $\beta(s_0)$ to $\beta_{r_k}(s_0)$, or vice versa, for some k. First, we consider the case where this future-pointing causal curve is from $\beta(s_0)$ to $\beta_{r_k}(s_0)$. Then the points $\beta(0)$ and $\beta_{r_k}(s_0)$ can be connected through the concatenation of the curve β that connects $\beta(0)$ to $\beta(s_0)$ and the causal curve that connects $\beta(s_0)$ to $\beta_{r_k}(s_0)$. By [21, Proposition 10.46], we conclude that $\tau(\beta_{r_k}(0), \beta_{r_k}(s_0)) \neq 0$, which implies that $C_N^+(\beta(0)) \neq \emptyset$. In the other case that the future-pointing causal curve connects $\beta_{r_k}(s_0)$ to $\beta(s_0)$, we can use a similar argument to conclude that $\tau(\beta(0), \beta(s_0)) \neq 0$ and subsequently that $C_N^+(\beta(0)) \neq \emptyset$. □

4 Proof of Theorem 2

We start by considering an embedding of the form (3) with $\eta \equiv 0$ and satisfying Hypothesis 3. Throughout this section and for the sake of brevity of notation we will assume without loss of generality that $c \equiv 1$ so as to discard the notations $\bar{g}_c$ and g_c (see Sect. 2.2). Observe that due to the more restrictive form of the metric (compared to the stationary case), null geodesics in $(\mathcal{M}, \bar{g})$ can conveniently be parameterized as

$$\beta(\cdot; r_0, x, v) = (r + r_0; \gamma(r; x, v)),$$

with $r_0 \in \mathbb{R}$, $(x, v) \in \partial_{\text{in}} SM$ and $\gamma(\cdot; x, v)$ denoting a unit-speed geodesic with initial data $(x, v) \in \partial_{\text{in}} SM$.

Owing to this identification of null geodesics, we can recast the light ray transform on $\mathbb{R} \times M$ for $\alpha \in C_c^\infty(\mathbb{R} \times M; \mathcal{S}^m)$ as

$$(L\alpha)(r_0, x, v) = \int_0^{\tau_+(x,v)} \alpha((r + r_0, \gamma(r; x, v)), (1, \dot{\gamma}(r; x, v)))\mathrm{d}r,$$

for all $(r_0, x, v) \in \mathbb{R} \times \partial_{\text{in}} SM$.

4.1 Notations

For symmetric tensors f and h, we denote the symmetrized tensor product simply by fh. In particular, if f and h are 1-forms, then

$$fh(v, w) = \frac{1}{2}(f(v)h(w) + f(w)h(w)), \quad v, w \in TM.$$

Following [7], we next define three operators. The operator

$$\boldsymbol{i} : C^\infty(M; S^m) \to C^\infty(M; S^{m+2})$$

is defined through $\boldsymbol{i} f = fg$, where we recall that S^m denotes the bundle of symmetric tensors of rank m on M. Next, the operator $\boldsymbol{j}$ is the trace with respect to g, that is,

$$\boldsymbol{j} : C^\infty(M; S^{m+2}) \to C^\infty(M; S^m)$$

is the adjoint of $\boldsymbol{i}$, and in local coordinates we can write, $(\boldsymbol{j} f)_{j_1 \dots j_n} = g^{jk} f_{jkj_1 \dots j_n}$. The composition $\boldsymbol{ji}$ is self-adjoint and positive definite [7, Lem. 2.3]. In particular, the inverse $(\boldsymbol{ji})^{-1}$ exists. Moreover, by the same lemma, the bundle S^m has the orthogonal decomposition into sub-bundles $S^m = \text{Ker}(\boldsymbol{j}) \oplus \text{Ran}(\boldsymbol{i})$. Finally, the operator

$$\boldsymbol{p} : C^\infty(M; S^m) \to C^\infty(M; S^m)$$

is defined to be the orthogonal projection from S^m to $\text{Ker}(\boldsymbol{j})$, and it can be written as

$$\boldsymbol{p} = 1 - \boldsymbol{i}(\boldsymbol{ji})^{-1}\boldsymbol{j},$$

see [7, Eq. (2.15)].

4.2 Helmholtz Decomposition

Let us first recall the Helmholtz decomposition as proven in [27, Theorem 3.3.2], that is, given any $\omega \in C^\infty(M; S^m)$, there are unique $\omega^s \in C^\infty(M; S^m)$ and $h \in C^\infty(M; S^{m-1})$ satisfying

$$\omega = \omega^s + d^s h, \quad \delta^s \omega^s = 0, \quad h|_{\partial M} = 0,$$

where δ^s is the adjoint of d^s. We say that ω is *solenoidal* if $\omega = \omega^s$.

For a family $\omega \in C_c^\infty(\mathbb{R}; C^\infty(M; S^m))$ we define $\omega^s(t) = (\omega(t))^s$. As the corresponding potential $h(t)$ is obtained by solving the elliptic partial differential equation,

$$\delta^s d^s h(t) = \delta^s \omega(t), \quad h(t)|_{\partial M} = 0,$$

we see that $h \in C_c^\infty(\mathbb{R}; C^\infty(M; S^{m-1}))$ and $\omega^s \in C_c^\infty(\mathbb{R}; C^\infty(M; S^m))$.

We define also the Fourier transform in time by

$$\widehat{\omega}(\tau) = \int_{\mathbb{R}} e^{-\iota\tau t} \omega(t)\, \mathrm{d}t.$$

Then $d^s \widehat{\omega}(\tau) = \widehat{d^s \omega}(\tau)$ and $\delta^s \widehat{\omega}(\tau) = \widehat{\delta^s \omega}(\tau)$. In particular, $\widehat{\omega}(\tau) = \widehat{\omega^s}(\tau) + d^s \widehat{h}(\tau)$ and $\delta^s \widehat{\omega^s}(\tau) = 0$. As the Helmholtz decomposition of $\widehat{\omega}(\tau)$ is unique, we obtain

$$(\widehat{\omega}(\tau))^s = \widehat{\omega^s}(\tau). \tag{26}$$

4.3 Trace-Free Helmholtz Decomposition

We will next recall the trace-free Helmholtz decomposition as discussed for example in [7]. By [7, Theorem 1.5], for any $\omega \in C^\infty(M; S^m)$ there are unique $\omega^{\mathrm{tfs}} \in C^\infty(M; S^m)$, $h \in C^\infty(M; S^{m-1})$ and $\omega^{\mathrm{t}} \in C^\infty(M; S^{m-2})$ satisfying

$$\omega = \omega^{\mathrm{tfs}} + \boldsymbol{i}\omega^{\mathrm{t}} + d^s h, \quad \delta^s \omega^{\mathrm{tfs}} = 0, \quad h|_{\partial M} = 0, \quad \boldsymbol{j}\omega^{\mathrm{tfs}} = 0, \quad \boldsymbol{j}h = 0. \tag{27}$$

This decomposition is obtained by first solving the following elliptic partial differential equation for h,

$$\delta^s \boldsymbol{p} d^s h = \delta^s \boldsymbol{p}\omega, \quad h|_{\partial M} = 0.$$

Then $\omega^{\mathrm{t}} = (\boldsymbol{j}\boldsymbol{i})^{-1}\boldsymbol{j}(\omega - d^s h)$ and $\omega^{\mathrm{tfs}} = \omega - \boldsymbol{i}\omega^{\mathrm{t}} - d^s h$.

The last equation $\boldsymbol{j}h = 0$ in the decomposition (27) is in fact a consequence of the first four equations. That is, if

$$\omega = \omega_0 + \boldsymbol{i}\omega_1 + d^s \omega_2, \quad \delta^s \omega_0 = 0, \quad \omega_2|_{\partial M} = 0, \quad \boldsymbol{j}\omega_0 = 0, \tag{28}$$

then $\omega_0 = \omega^{\mathrm{tfs}}$, $\omega_1 = \omega^{\mathrm{t}}$ and $\omega_2 = h$. Indeed, writing $\omega_0' = \omega_0 - \omega^{\mathrm{tfs}}$, $\omega_1' = \omega_1 - \omega^t$ and $\omega_2' = \omega_2 - h$, we obtain $\omega_1' = -(\boldsymbol{ji})^{-1}\boldsymbol{j} d^s \omega_2'$. Then $\boldsymbol{p} d^s \omega_2' = -\omega_0'$ and ω_2' solves

$$\delta^s \boldsymbol{p} d^s \omega_2' = 0, \quad \omega_2'|_{\partial M} = 0.$$

Therefore $\omega_2' = 0$ and $\omega_2 = h$. Now also $\omega_0 = \omega^{\mathrm{tfs}}$ and $\omega_1 = \omega^{\mathrm{t}}$ by [7, Th. 1.5]. We record the following consequence that will be useful in what follows.

Remark 1 If $w = d^s h$ for some $h \in C^\infty(M; S^{m-1})$ satisfying $h|_{\partial M} = 0$, then $w^{\mathrm{tfs}} = 0$ and $w^{\mathrm{t}} = 0$.

Analogously to the previous section, for a family $\omega \in C_c^\infty(\mathbb{R}; C^\infty(M; S^m))$ we can define

$$\omega^{\mathrm{tfs}}(t) = (\omega(t))^{\mathrm{tfs}}, \quad \omega^{\mathrm{t}}(t) = (\omega(t))^{\mathrm{t}},$$

that gives smooth families of tensors that are compactly supported in time. Observe that $\boldsymbol{i}\widehat{\omega}(\tau) = \widehat{\boldsymbol{i}\omega}(\tau)$ and $\boldsymbol{j}\widehat{\omega}(\tau) = \widehat{\boldsymbol{j}\omega}(\tau)$, and analogously with (26), we have

$$(\widehat{\omega}(\tau))^{\mathrm{tfs}} = \widehat{\omega^{\mathrm{tfs}}}(\tau), \quad (\widehat{\omega}(\tau))^{\mathrm{t}} = \widehat{\omega^{\mathrm{t}}}(\tau).$$

4.4 Injectivity of the Light Ray Transform on Tensors

For the remainder of the paper and for the sake of brevity, we will abuse the notation slightly and identify tensors in $\mathcal{M}$ with their identical copies in $\Phi^{-1}(\mathcal{M})$ without explicitly writing the embedding. Let $\alpha \in C^\infty(\mathcal{M}; \mathcal{S}^m)$ and suppose that $L\alpha \equiv 0$. As $\bar{g} = -\mathrm{d}t^2 + g$ and α is symmetric, we write

$$\alpha = f\,\mathrm{d}t + \omega + b\,\bar{g}, \tag{29}$$

where

$$f \in C_c^\infty(\mathbb{R}; C^\infty(M; S^{m-1})) \quad \omega \in C_c^\infty(\mathbb{R}; C^\infty(M; S^m)) \quad b \in C_c^\infty(\mathcal{M}; \mathcal{S}^{m-2}).$$

We can simplify (29) further by considering the Helmholtz decomposition of f, that we denote by $f = f^s + d^s p$. To this end, we begin by writing

$$d^s p\,\mathrm{d}t = \bar{d}^s(p\,\mathrm{d}t) + \partial_t p\,\bar{g} - \partial_t p\,g.$$

Note that the term $\bar{d}^s(p\,\mathrm{d}t) + \partial_t p\,\bar{g}$ takes the form of the gauge (11) and by Lemma 1 lies in the kernel of L, while

$$-\partial_t p\,g \in C_c^\infty(\mathbb{R}; C^\infty(M; S^m)).$$

In particular, we can replace ω with $\omega - \partial_t p\, g$ in (29). As also $b\,\bar{g}$ is in the kernel of L, we can assume without loss of generality that

$$\alpha = f\,\mathrm{d}t + \omega, \quad f = f^s. \tag{30}$$

We have the following Fourier slicing lemma.

Lemma 5 *Suppose that* $\alpha \in C_c^\infty(\mathbb{R} \times M; S^m)$ *is of the form* (30)*. Then for* $k = 0, 1, \ldots,$ *and* $(x, v) \in \partial_{in} SM$ *it holds that*

$$\begin{aligned}\partial_\tau^k \widehat{L\alpha}(\tau, x, v)|_{\tau=0} &= \mathcal{I}(\partial_\tau^k \widehat{f}(\tau, \cdot)|_{\tau=0})(x, v) + \sum_{j=0}^{k-1} \binom{k}{j} \mathcal{R}_{k-j}(\partial_\tau^j \widehat{f}(\tau, \cdot)|_{\tau=0})(x, v)\\ &\quad + \mathcal{I}(\partial_\tau^k \widehat{\omega}(\tau, \cdot)|_{\tau=0})(x, v) + \sum_{j=0}^{k-1} \binom{k}{j} \mathcal{R}_{k-j}(\partial_\tau^j \widehat{\omega}(\tau, \cdot)|_{\tau=0})(x, v),\end{aligned} \tag{31}$$

where

$$\mathcal{R}_j \omega(x, v) = \int_0^{\tau_+(x,v)} (\imath r)^j\, \omega(\gamma(r; x, v), \dot{\gamma}(r; x, v))\, \mathrm{d}r, \quad \omega \in C_c^\infty(M; S^m).$$

We are now ready to prove the main theorem.

Proof of Theorem 2 As discussed above, we can write α in the form (30) with $f = f^s$. Now note that for any $(x, v) \in \partial_{\mathrm{in}} SM$ we have that $(y, w) \in \partial_{\mathrm{in}} SM$ as well, where

$$y = \gamma(\tau_+(x, v); x, v), \quad w = -\dot{\gamma}(\tau_+(x, v); x, v).$$

Moreover, we have that $\mathcal{I}\omega(x, v) = \mathcal{I}\omega(y, w)$ for any $\omega \in C_c^\infty(M; S^m)$ with even m but $\mathcal{I}\omega(x, v) = -\mathcal{I}\omega(y, w)$ for any $f \in C_c^\infty(M; S^m)$ with odd m.

Applying (31) with $k = 0$ and using the above observation implies that

$$\mathcal{I}(\widehat{f}(0)) = 0, \quad \mathcal{I}(\widehat{\omega}(0)) = 0. \tag{32}$$

Using Hypothesis 3 together with $f = f^s$ and Remark 1 we deduce that

$$\widehat{f}(0) = 0, \quad \widehat{\omega^{\mathrm{tfs}}}(0) = 0, \quad \widehat{\omega}^{\mathrm{t}}(0) = 0.$$

Let us define

$$a_0(t, x) = \int_{-\infty}^t \omega^{\mathrm{t}}(t', x)\, \mathrm{d}t'. \tag{33}$$

As ω is compactly supported in time, $a_0(t)$ vanishes for t sufficiently small. Moreover, for large t,

$$a_0(t) = \int_{-\infty}^{\infty} \omega^{\mathrm{t}}(t')\mathrm{d}t' = \widehat{\omega}^{\mathrm{t}}(0) = 0.$$

Thus $a_0 \in C_c^\infty(\mathbb{R}; C^\infty(M; S^{m-2}))$. Observe also that $\partial_t a_0 = \omega^{\mathrm{t}}$ and hence $\iota\tau\widehat{a}_0 = \widehat{\omega}^{\mathrm{t}}$. In particular,

$$\partial_\tau^k \widehat{\omega}^{\mathrm{t}}(0) = \iota k \partial_\tau^{k-1}\widehat{a}_0(0), \quad k = 0, 1, \ldots.$$

In what follows, we will write

$$\omega = \omega^{\mathrm{tfs}} + \boldsymbol{i}\omega^{\mathrm{t}} + d^s a_1, \quad a_1 = a_1^s + d^s h,$$

to denote the trace-free Helmholtz decomposition of ω and the Helmholtz decomposition of a_1, respectively. We will use the fact that for any $u \in C_c^\infty(M; S^m)$,

$$\begin{aligned}
\mathcal{R}_j(d^s u) &= \iota^j \int_0^{\tau_+} r^j\, d^s u(\gamma(r), \dot\gamma(r))\mathrm{d}r \\
&= \iota^j \int_0^{\tau_+} r^j \partial_r(u(\gamma(r), \dot\gamma(r)))\mathrm{d}r \\
&= -\iota j \mathcal{R}_{j-1}(u).
\end{aligned}$$

When $k = 1$, Eq. (31) reduces to the fact that

$$\begin{aligned}
&\mathcal{I}(\partial_\tau \widehat{f}) + \mathcal{I}(\partial_\tau \widehat{\omega^{\mathrm{tfs}}} + \iota\widehat{a}_0) + \mathcal{R}_1(d^s\widehat{a}_1) \\
&\quad = \mathcal{I}(\partial_\tau \widehat{f}) + \mathcal{I}(\partial_\tau \widehat{\omega^{\mathrm{tfs}}} + \iota\widehat{a}_0) - \iota\mathcal{I}(\widehat{a}_1) \\
&\quad = \mathcal{I}(\partial_\tau \widehat{f} - \iota\widehat{a_1^s}) + \mathcal{I}(\partial_\tau \widehat{\omega^{\mathrm{tfs}}} + \iota\widehat{a}_0 g)
\end{aligned}$$

vanishes at $\tau = 0$. Note that $\partial_\tau \widehat{f} - \iota\widehat{a_1^s}$ is solenoidal and of rank $m - 1$. Moreover, the tensor

$$w := \partial_\tau \widehat{\omega^{\mathrm{tfs}}} + \iota\widehat{a}_0 g$$

is of rank m and satisfies

$$w^{\mathrm{t}} = \iota\widehat{a}_0 g \quad \text{and} \quad w^{\mathrm{tfs}} = \partial_\tau \widehat{\omega^{\mathrm{tfs}}}.$$

Hence at $\tau = 0$,

$$\partial_\tau \widehat{f} = \iota\widehat{a_1^s}, \quad \partial_\tau \widehat{\omega^{\mathrm{tfs}}} = 0, \quad \widehat{a}_0 = 0.$$

We will now proceed with an induction argument to show that for all $j \in \mathbb{N}$, it holds at $\tau = 0$ that

$$\partial_\tau^j \widehat{f} = \iota j \partial_\tau^{j-1} \widehat{a_1^s}, \quad \partial_\tau^j \widehat{\omega^{\text{tfs}}} = 0, \quad \partial_\tau^{j-1} \widehat{a}_0 = -\iota (j-1) \partial_\tau^{j-2} \widehat{h}. \tag{34}$$

Indeed, let us suppose that this hypothesis holds for all $j = 1, \ldots, k-1$. Together with (31) this implies that

$$\begin{aligned}
&\mathcal{I}(\partial_\tau^k \widehat{f}) + \iota \sum_{j=0}^{k-1} \binom{k}{j} j \mathcal{R}_{k-j}(\partial_\tau^{j-1} \widehat{a_1^s}) + \mathcal{I}(\partial_\tau^k \widehat{\omega^{\text{tfs}}} + \iota k \partial_\tau^{k-1} \widehat{a}_0) \\
&\quad + \sum_{j=0}^{k-1} \binom{k}{j} \mathcal{R}_{k-j}(\iota j \partial_\tau^{j-1} \widehat{a}_0 + d^s \partial_\tau^j \widehat{a}_1)
\end{aligned}$$

vanishes at $\tau = 0$. As a_1 vanishes on $\mathbb{R} \times \partial M$, we have

$$\begin{aligned}
\mathcal{R}_{k-j}(d^s \partial_\tau^j \widehat{a}_1) &= -\iota (k-j) \mathcal{R}_{k-(j+1)}(\partial_\tau^j \widehat{a}_1) \\
&= -\iota (k-j) \mathcal{R}_{k-(j+1)}(\partial_\tau^j \widehat{a_1^s}) - \iota (k-j) \mathcal{R}_{k-(j+1)}(d^s \partial_\tau^j \widehat{h}).
\end{aligned} \tag{35}$$

Next, using the identity

$$-\iota \sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!} (k-j) \mathcal{R}_{k-(j+1)}(\partial_\tau^j \widehat{a_1^s}) = -\iota \sum_{j=1}^{k} \binom{k}{j} j \mathcal{R}_{k-j}(\partial_\tau^{j-1} \widehat{a_1^s}),$$

together with (35), we see that

$$\begin{aligned}
&\mathcal{I}(\partial_\tau^k \widehat{f}) + \mathcal{I}(\partial_\tau^k \widehat{\omega^{\text{tfs}}} + \iota k \partial_\tau^{k-1} \widehat{a}_0) - \iota k \mathcal{I}(\partial_\tau^{k-1} \widehat{a_1^s}) \\
&\quad + \sum_{j=0}^{k-1} \binom{k}{j} \mathcal{R}_{k-j}(\iota j \partial_\tau^{j-1} \widehat{a}_0) - \iota \sum_{j=0}^{k-1} \frac{k!}{j!(k-(j+1))!} \mathcal{R}_{k-(j+1)}(d^s \partial_\tau^j \widehat{h})
\end{aligned}$$

vanishes at $\tau = 0$. As h vanishes on $\mathbb{R} \times \partial M$, we have for $j = 0, \ldots, k-2$,

$$\mathcal{R}_{k-(j+1)}(d^s \partial_\tau^j \widehat{h}) = -\iota (k-(j+1)) \mathcal{R}_{k-(j+2)}(\partial_\tau^j \widehat{h}),$$

and for $j = k-1$,

$$\mathcal{R}_{k-(j+1)}(d^s \partial_\tau^j \widehat{h}) = \mathcal{I}(d^s \partial_\tau^j \widehat{h}) = 0.$$

We rewrite

$$-\iota\sum_{j=0}^{k-1}\frac{k!}{j!(k-(j+1))!}\mathcal{R}_{k-(j+1)}(d^s\partial_\tau^j\widehat{h}) = \sum_{j=0}^{k-2}\frac{k!}{j!(k-(j+2))!}\mathcal{R}_{k-(j+2)}(\partial_\tau^j\widehat{h})$$
$$= \sum_{j=2}^{k}\frac{k!}{(j-2)!(k-j)!}\mathcal{R}_{k-j}(\partial_\tau^{j-2}\widehat{h}),$$

and, using $\partial_\tau^{j-1}\widehat{a}_0 = -\iota(j-1)\partial_\tau^{j-2}\widehat{h}$,

$$\sum_{j=0}^{k-1}\binom{k}{j}\mathcal{R}_{k-j}(\iota j\partial_\tau^{j-1}\widehat{a}_0) = \sum_{j=2}^{k-1}\frac{k!}{(j-2)!(k-j)!}\mathcal{R}_{k-j}(\partial_\tau^{j-2}\widehat{h}).$$

Therefore

$$\mathcal{I}(\partial_\tau^k\widehat{f}) + \mathcal{I}(\partial_\tau^k\widehat{\omega^{\mathrm{tfs}}} + \iota k\partial_\tau^{k-1}\widehat{a}_0) - \iota k\mathcal{I}(\partial_\tau^{k-1}\widehat{a_1^s}) + k(k-1)\mathcal{I}(\partial_\tau^{k-2}\widehat{h})$$

vanishes at $\tau = 0$. We obtain at $\tau = 0$,

$$\partial_\tau^k\widehat{f} = \iota k\partial_\tau^{k-1}\widehat{a_1^s}, \quad \partial_\tau^k\widehat{\omega^{\mathrm{tfs}}} = 0, \quad \iota\partial_\tau^{k-1}\widehat{a}_0 = (k-1)\partial_\tau^{k-2}\widehat{h},$$

and this closes the induction argument.

We can now use (34) to deduce that

$$f = \partial_t a_1^s, \quad \omega^{\mathrm{tfs}} = 0, \quad a_0 = -\partial_t h.$$

To see this, recall that since the functions $f, a_1^s, \omega^{\mathrm{tfs}}, a_0$ and h are compactly supported in time, their Fourier transforms in t are real analytic. Hence,

$$\hat{f}(\tau,\cdot) = \sum_{k=0}^{\infty}\partial_\tau^k\hat{f}(0,\cdot)\frac{\tau^k}{k!} = \iota\tau\sum_{k=0}^{\infty}\partial_\tau^k\widehat{a_1^s}(0,\cdot)\frac{\tau^k}{k!}$$

implying that $f = \partial_t a_1^s$. The other two claims follow similarly. Recalling also that $\omega^{\mathrm{t}} = \partial_t a_0$, Eq. (30) can be rewritten as

$$\alpha = \partial_t a_1^s\,\mathrm{d}t + d^s a_1 + \partial_t a_0\,g.$$

This expression can be further simplified to obtain

$$\alpha = \bar{d}^s\underbrace{(a_0\,\mathrm{d}t + a_1)}_{T} + \underbrace{(\partial_t a_0)}_{U}\,\bar{g}, \tag{36}$$

thus concluding the proof of the theorem. □

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