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Title: Teacher Guidance in Mathematical Problem-Solving Lessons : Insights from Two Professional Development Programs

Year: 2019

Version: Accepted version (Final draft)

Please cite the original version:

Hähkiöniemi, M., & Francisco, J. (2019). Teacher Guidance in Mathematical Problem-Solving Lessons : Insights from Two Professional Development Programs. In P. Felmer, P. Liljedahl, & B. Koichu (Eds.), Problem Solving in Mathematics Instruction and Teacher Professional Development (pp. 279-296). Springer. Research in Mathematics Education. https://doi.org/10.1007/978-3-030-29215-7_15

Chapter x

Teacher Guidance in Mathematical Problem Solving Lessons – Insights from Two Professional Development Programs

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Abstract. When implementing a problem-solving lesson, the teacher needs to provide students appropriate guidance during problem solving. This demanding task requires understanding students' work in progress and giving them necessary help without constraining their thinking. In this article, we share insights from two professional development programs on how teachers guided students' problem solving and how they reflected on these instances. One of the programs included Finnish pre-service teachers while the other program included U.S. in-service teachers. We analyzed video-recorded problem-solving lessons from 16 Finnish and 2 U.S. teachers in grades 6–9. We found two themes about teacher guidance of student problem solving activity: focusing students' thinking and emphasizing justification. In the first theme, the teachers' ways to guide students differed depending on how much space they allowed for students' thinking and how much their guidance actually helped the students to focus on the targeted issue. In the second theme, we identified two patterns: asking repeatedly for justification and helping to build a justification. The elaboration on the themes is supported with the analysis of the teachers' reflection related to the themes.

Keywords: Guidance, Inquiry, Mathematics Teaching, Problem solving, Professional Development, Teacher Education

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X.1 Introduction

The purpose of this study is to share insights from two professional development programs on how teachers can guide students' problem solving in mathematics classrooms. Problem solving is a central aspect of reform in mathematics education (NCTM, 2000). Mathematics teachers are being encouraged to teach through problem solving. This is an instructional approach where students are expected to develop mathematical knowledge and skills as they explore and solve mathematics problems rather than teachers simply telling or showing them what to do and asking them to practice it. There is a general consensus in the mathematics education research community that teaching via problem solving or inquiry helps promote students' mathematical understanding. In an effort to support mathematics teachers in implementing such instruction, professional organizations such as the National Science Foundation have funded, over the years, the development of so-called standards-based curricula that teachers can use in mathematics classrooms (see e.g., TERC, 1998). These curricula usually provide teachers with thought-provoking and cognitively demanding mathematical tasks that teachers can use in mathematics classrooms along with detailed descriptions of mathematical ideas involved in the tasks, how students learn them, and some guidance on how to implement the lesson (Cengiz et al., 2011). Professional development programs are often offered to teachers to help them effectively implement these curricula (see Goldsmith et al., 2014).

Despite the efforts mentioned above, there is evidence that mathematics teachers at all grade levels struggle to teach through problem solving and, consequently, to engage students in thoughtful mathematical activity in mathematical classrooms (Cengiz et al., 2011). This suggests a need for more studies of how mathematics teachers implement problem-solving lessons to better understand the challenges they face when doing so. The present study addresses this issue by reporting on our observations of mathematics teachers attempting to implement such lessons in the context of two professional development programs: The Informal Mathematical Learning Project (IML) in USA, and the Inquiry-Based Mathematics Teaching project (IBMT) in Finland. We examined the video database of the two professional development programs and tried to describe how the teachers facilitated or guided the students' problem solving activity, the challenges they faced, and how they explained their interventions. We were aware of the different contexts of the two professional development programs. Therefore, the goal was not so much to compare or contrast the teachers' actions, but rather to have a larger database from where to gain a richer understanding of how mathematics teachers implement problem-solving activities in mathematics classrooms.

x.2 Problem Solving Lessons

A problem solving or inquiry lesson has a typical structure that consists of three main parts or phases (Stein, Engle, Smith, & Hughes, 2008; Van de Walle, 2004). In part one, the teacher *presents a mathematical problem* for students to work on. The problem has to be challenging to the students to stimulate their mathematical thinking. Several criteria have been developed for what counts as a good problem, and there is a wider agreement among mathematics educators on four of the criteria (Cai & Lester, 2010). The first is that the problem must contain important, useful mathematics. This is usually the mathematical concepts and skills that students are expected to learn from working on the problem. The other three criteria are that the problem must require high-level of thinking, contribute to students' conceptual development, and provide opportunities for the teacher to assess what students are learning and where they may be experiencing difficulties. Standard-based curricula usually provide such tasks to the teachers. However, teachers can also come up with their own tasks or modify existing tasks to make them meet the criteria (Cai & Lester, 2010). In the second phase or part of a problem-solving lesson, *students work on the mathematical problem,* individually or collaboratively. The teacher provides guidance to the students as they work on the problem without undermining the student's intellectual autonomy. In the third and final stage, the teacher engages the students in a *collective discussion* and reflection on their mathematical activity as a way to help them become aware of and make explicit the mathematical ideas that they are supposed to learn from the activity.

Problem solving lessons are often associated with attempts to implement mathematical instruction that is consistent with a constructivist perspective on learning, which states that learners build or construct their knowledge by reflectively abstracting and reorganizing (mathematical) activity (von Glasersfeld, 1995). This perspective has led to a reformulation of the teacher role in favor of instructional actions "centered on listening to students and attending to their reasoning" (Nelson, 2001). Several researchers argue that this has resulted in a number of misconceptions about constructivism (Fraivillig et al., 1999; Lobato et al., 2005). For example, one issue that often emerges in conversations about inquiry-based lessons is that the teachers should not intervene on students' mathematical activity or if they do, they should avoid telling or showing students what to do or how to think. The researchers have pointed out that this is a misunderstanding of constructivism as they claim that in inquiry lessons, teachers can and should intervene to advance children's thinking. Therefore, the question is not whether but rather how teachers can most effectively intervene to advance children's mathematical thinking without undermining the children's intellectual autonomy. This is what makes inquiry lessons challenging for teachers because it tends to rule out traditional methods of guidance based on simply telling or showing students what to do. There has been research on instructional practices that provide students with opportunities to explore mathematical concepts and build their own mathematical meanings (Fraivillig et al., 1999). However, the research has focused on teachers who were also researchers or classroom teachers with substantial day-to-day support from researchers. While this research has provided important results, it has not contributed as much insights into how regular teachers without much support implement problem solving lessons and the challenges they face when doing so (Cengiz et al., 2011). Our purpose in this article is to contribute to a better understanding of the issue by examining how a group of mathematics teachers implemented such lessons where they tried to advance students' mathematical thinking without undermining students' intellectual autonomy.

x.2 Promoting Student Mathematical Thinking

Several theoretical frameworks have been developed to model teaching interventions or actions that support students' mathematical thinking (see e.g., Franke et al., 2009; Fraivillig et al. 1999; Lobato et al., 2005; Staples, 2007). Fraivillig et al. developed the Advancing Children Thinking (ACT) as part of a part of a 5-year longitudinal study. The ACT framework distinguishes among three type of instructional actions that can help promote students' thinking: *Eliciting* students' mathematical thinking or methods, *supporting* students' conceptual understanding, and *extending* students' thinking. Eliciting and supporting strategies focus on "accessing and facilitating children's thinking about solution methods with which they are already familiar" (p.160). Extending students' thinking, on the other hand, involves challenging and further developing student thinking. A major finding of their study was that teachers engaged far less in eliciting and extending than in supporting student thinking. Fraivillig et al. claim that a "partial explanation for these differences lies in the kinds of pedagogical skills necessary for carrying out eliciting, supporting, and extending." (p. 167). They claim that supporting students' thinking involves skills with which most teachers are usually comfortable because those skills are similar to more common traditional notions of teaching. Eliciting and extending students thinking, on the other hand, are more complex tasks. Eliciting student mathematical thinking is time-consuming and requires patience, skill, and high levels of knowledge about individual children and about typical solution methods in major mathematical areas. It also requires effective classroom management to make sure all students participate in discussions, and the willingness and ability to relax intellectual control sufficiently for children to respond with their own solution methods. Similarly, extending student thinking requires information about each child's learning history and a realistic yet hopeful view of each child's potential. Fraivillig et al. argue that teachers need guidance with these processes and makes some several suggestions to help teachers in their study. Cengiz et al. (2011) proposed the Extending Student Thinking Framework (EST), which relates eliciting, supporting, and eliciting actions with different ways in which they can help extend students' mathematical thinking. Overall, their findings were similar to those in Fraivillig et al.'s study. Conner et al. (2014) proposed the Teachers' Support of Collective argumentation, a framework that focuses specifically on promoting mathematical argumentation. They argue that teachers can support collective argumentation in three main ways: providing parts or components of an argument, called *direct contributions* because they are made by the teacher and not the students, *asking questions* that elicit parts of arguments from students or through *other supportive actions* than questions or direct contributions.

x.2 Research Context

The data for this study came from two professional development projects that provided teachers with opportunities to guide students' problem-solving activities. A brief description of the goals and activities of the projects together with data collection and analysis is presented below to provide a context for the present study.

x.2.1 IML Project

The Informal Mathematical Learning Project (IML) was a 3-year NSF-funded after-school research project that took place in an Urban, low income, minority school district in USA. The goal of the project was to study how middle school students build mathematical ideas and different forms of reasoning as they worked on challenging mathematical problems from several content strands, including fractions, early algebra, combinatorics, probability, and statistics. The project included a professional development component aimed at helping mathematics teachers from the local school district improve their ability to design and implement learning environments that promoted thoughtful mathematical activity. Six in-service middle school mathematics teachers participated in the project. In the first year of the project, the teachers observed researchers lead research sessions on students' development of mathematical ideas with a group of 20-26 six grade students. During the session, the students were encouraged to work collaboratively and to always justify their solutions. Their contributions were always received positively and researchers avoided making judgments about their validity. Instead, the students were encouraged to evaluate their claims based on whether they thought the claims made sense. The students were also given extended time to work on the tasks and opportunities to revisit them if necessary.

In the second year of the project, the teachers were asked to implement similar sessions with a new cohort of six-grade students from the same school. Each session last one hour and half and was videotaped with the help of several cameras. The teachers chose to work in pairs and to implement the same sequence of activities as the researchers in the first year of the project. As a pair of teachers implemented a mathematical task, the other teachers and the researchers observed the lesson. At the end of all research sessions, there were debriefing meetings between the teachers and researchers. In these meetings, the teachers and researchers reflected jointly on the sessions that they had just led and planned subsequent sessions. The meetings lasted approximately one hour and were videotaped.

This paper focused on IML sessions led by two teachers involving the 4-tall Tower Problem. The problem asks students to find all possible towers 4-cubes tall when choosing from two colors, in this case yellow and blue, and to justify their solution. This was the first task implemented in the project. There were two teacher-lead sessions on the problem, and each was followed by a debriefing meeting. In this paper, we examined episodes from the two sessions and the debriefing meetings for insights on how the teachers provided guidance on the students' problem solving. The episodes reported in the paper were all from the first session.

x.2.2 IBMT Project

The Inquiry-Based Mathematics Teaching (IBMT) project involved Finnish secondary and upper secondary pre-service mathematics teachers. The goal of the project was to study how pre-service teachers implement inquiry-based mathematics teaching after participating in a unit about inquiry-based mathematics teaching.

Altogether, 29 pre-service teacher participated in the project at the final phase of the teacher-training program. They had taught several school lessons during the program. The inquiry-based mathematics teaching unit included nine 90-minute group work sessions. The unit contained discussing the basic ideas of inquiry-based mathematics teaching, solving as well as designing problems, discussing experiences and emerging ideas, considering possible teacher actions in different lesson phases and practicing to guide students in hypothetical teaching situations. Concerning teacher guidance, the purpose was to avoid showing or telling the students how to solve a problem but to interpret student ideas and activate them to develop or justify their ideas, for example, through questioning. Each pre-service teacher planned and implemented one inquiry-based mathematics lesson in grades 7–12. It should be noted that IBMT lessons differ from IML session with respect to available time: in IBMT, only one lesson was devoted to the activity and the next lesson was already planned to progress to another topic.

Each pre-service teacher participated in a video reflection meeting usually two months after the teacher had implemented the inquiry-based mathematics lesson. In this audio recorded meeting, researcher showed video clips of two episodes which were selected so that in one episode the teacher is asking questions to help students in building their solution and in the second episode the teacher is probing students' thinking. After each clip, the researcher asked about students' thinking, purposes of the questions, pros and cons of the questions as well as how satisfied the teacher is with the episode. Thus, the context for teacher reflection was slightly different from

IML, as video clips were used and the reflection did not happen right after the lesson.

For this paper, we selected secondary school lessons (grades 7–9) because they are close to U.S. middle school. There were 16 such lesson. The 45-minute lessons were video recorded with a video camera that was connected to a wireless microphone attached to the pre-service teacher. The camera followed the pre-service teacher as he or she moved around the classroom. In this paper, were examine those episodes which were selected to video reflection meetings in these 16 lessons.

x.2.3 Data Analysis

We used data-driven analysis to create categories or themes characterizing teacher guidance of students' problem solving (Miles & Huberman, 1994). At first, we described all of the episodes with regards to how students were thinking and what kind of difficulties they possibly had in the episodes as well as what the teachers did to try to guide the students. Then, we interpreted what pros and cons a particular teacher guidance had. Then, we examined whether the teacher showed awareness of the particularities of students' thinking and the possible effect of teacher guidance when reflecting on the episode.

After this, we grouped the episodes according to what kind of difficulty the students had and how the teacher guided them to go over the difficulty. By reviewing the groups, we noticed that some student difficulties (e.g., difficulty in noticing a pattern and having an incorrect idea or answer) were similar regarding how the teachers reacted on them (e.g., focus student attention to a pattern or mistake). Thus, we arranged the groups again and through this, we composed two main themes about teacher guidance of student problem solving: focusing students' thinking and emphasizing justification.

At every stage of the analysis, interpretations were discussed between the two authors. We compared our interpretation of each episode to our interpretation of the other episodes (Glaser & Strauss, 1967). In addition, the context of each PD-program as well as the context of each episode were discussed between the authors to make sure that the interpretations are not made based on too narrow view of the data.

x.3 Findings

We found two common themes about teacher guidance of student problem solving activity: focusing students' thinking and emphasizing justification. We elaborate on the themes in this section along with supporting episodes from the two databases.

We use the labels teacher F1, F2 and F3 for the Finnish pre-service teachers and the label teacher U1 for the U.S. in-service teacher.

x.3.1 Focusing Students' Thinking

In this theme, the teachers tried to help students pay attention to or focus on some relevant issue in their mathematical activity such as noticing a mistake they had made or attending to an emerging pattern. The teachers did this in variety of ways depending on how much space they allowed for students' thinking and how much their guidance actually helped the students to focus on the targeted issue.

Helping Without Leaving Much Room for Thinking

This episode took place in an 8th grade class. The students were working on a problem that asked to find out how much money there would be after one, two, three and ten years when 3500 € were placed in a bank account at a 3.0% interest rate. A student determined how much money there would after one, two and three years and then asked help to determine the amount of money in ten years:

1 Student:	How is [the amount of money] after 10 years calculated?
2 Teacher F1:	So, now you have multiplied by 1.03 [amount of money after 1 year], haven't you? Then, what have you done here [money after 2 years]?
3 Student:	Multiplied this [amount of money after 1 year] by 1.03.
4 Teacher F1:	Mm. What about here [amount of money after 3 years]?
5 Student:	Multiplied this [amount of money after 2 years] by 1.03.
6 Teacher F1:	Yeah. So, uhm, what have you done like in total?
7 Student:	Well, three times this times this. Does it go like that?
8 Teacher F1:	Yeah. So you have multiplied by 1.03 three times here, so how like mathematically?
9 Student:	[In audible]
10 Teacher F1:	What?
11 Student:	This is multiplied 10 times by this.
12 Teacher F1:	Uhm. Uhm. Uhm. Almost, but, that not times but?
13 Student:	Raised to the 10th power.

In this episode, the teacher focused the student's attention on the fact that she had calculated the amount of money after three years by multiplying 3500 € three times by 1.03. As a result of the teacher's guidance, the student noticed the pattern and was able to determine the amount of money after 10 years. Thus, the teacher guidance helped the student to notice the pattern. The teacher was strongly leading even though the guidance was given mostly in the form of a series of questions. Even after the student got the idea of multiplying 10 times, the teacher continued to push the student to use the notion of exponent (turn 12). Thus, the teacher did not leave much room for the students' own thinking.

When reflecting on the lesson afterwards, teacher F1 did not mention anything about leading the student strongly and not leaving much room for students' own thinking. On the contrary, the teacher thought that the questions helped the student to think and to explain their thinking:

> Maybe it [questioning] helps the student to think about this more deeply here, like what is really done there. So that she is not just copying. […] [The student] learns to explain what she has done. (Teacher F1)

In addition, teacher F1 emphasized many times that through the kinds of questions asked in the episode a teacher gets information of students thinking:

> You can confirm what the student has been thinking, if it is not written on the paper. I don't recall if it was in this stage when I was there in helping her, but, or I did saw that she has multiplied by 1.03 because I knew the answers. But, a teacher can become sure that the student is thinking correctly.

The claim by teacher F1 that a teacher is able to "confirm what the student has been thinking" is in line with our interpretation that the teacher's guidance did not leave much space for students' thinking. It suggests that the teacher was observing whether the student was using a particular, expected way of thinking. The teacher said himself that in this situation "I think that it is good to clarify for the student, what has been thought here".

Partially Helping and Giving Room for Thinking

In this episode, teacher U1 talks to a pair of students as they worked on the 4-tall Tower Problem when choosing from two colors. The students had found eight towers and had organized them as shown in Figure 1. The students then told the teacher that they had found all towers and the following conversation took place between the teacher and the students:

1 Teacher U1:	How do you prove to me that you have all of them?
2 Student 1:	I came up with these [holds up a pair of towers and explains how they are opposites].
3 Teacher U1:	All right. How would you describe those? They are kind of like what?
4 Student 1:	They are kind of the same. They have the same shape and same size. It's just that the colors are in different order.
5 Teacher U1:	Ok. They are all the same.
6 Student 1:	You can see that that's like a diagonal [pointing to a zigzag pattern in two of the towers that helped show they were opposites]
7 Teacher U1:	Ok. Like a zigzag?
8 Student 1:	Yeah.
9 Teacher U1:	Ok. All right. So you would pair these two together [picks the two towers and puts them aside]. All Right. What else do you have?
10 Student 1:	[Pointing at student 2 in the table]. We had the same thing. So, she can show you the next.
11 Teacher U1:	[Looking at towers of student 2] So, you have yours grouped in pairs?

12 Student 2: [Nods]. I got these two because they are no other possible ways we can put them in order. Then I said since you can use two colors you can put three blues and one yellow and then three yellow and one blue.

13 Teacher U1: Ok. [Looking at towers of student 1] Do you have that? You have that too right here, right? Ok. All right. [to student 1] Put those over here.

14 Student 2: I got these two since you can use two colors. I put two colors in the middle and two on the outside. That is the same with this. You just switch them around.

15 Teacher U1: Ok.

16 Student 1: Then I was thinking. I noticed that it couldn't just be six. I had to find at least one more and she found it. Then I said you have to multiply four by two because there are four cubes in a tower and there's two colors. I mean you have to multiply the building [with four cubes] and the [number of] color.

17 Student 2: And I said that how you can find out how many cubes we got [organizes the towers in two rows of four opposite colors] you can put say two [towers] times four [groups] equals eight.

18 Teacher U1: [Looking toward student 2] You said there are two colors and they are four tall so therefore there should be eight towers 'cause it 2 times four is eight. [Turning to student 1] So you said they are four tall and two colors and you said four times two equals eight. So you are convinced that there are 8 altogether? [Student 1 and 2 say yes.] Ok. Now. Let's go back to the task on the overhead. [Reads the problem.] Does that mean that every tower has to have two colors?

19 Students: [After a pause] Oh. [Teacher U1 smiles. Student 1 and 2 build two extra towers BBBB and YYYY.]

20 Teacher U1: Now there are ten [towers]. Humm… Now you agree that there are ten. But what happens to that two times four is eight and four times two is eight, that mathematical thing that you were talking about?

21 Student 2: [Reorganizing her towers in five groups of two] I get the same because [pauses] you still can do it my way but it will just be five on the side and two. You will do five times two.

22 Teacher U1: [Looking toward student 2] I agree but before you talked about the fact that it was four tall and there are two colors and your ideas was that four times two equals eight therefore eight must be the total. But now you are you are saying that I can build ten.

23 Student 1: Now I am saying that these two [YYYY and BBBB] they are the same colors. They really don't count.

24 Teacher U1: Or they don't?

25 Student 1: They count to the [total number of] four tall towers, but they don't count to the [towers with] different colors.

26 Teacher U1: Ok. They don't count to the four times two?

27 Student 1: No.

28 Teacher U1: Ok. All right. All Right. I will be back.

Fig. 1 Student 1's towers (left) and student 2's towers (right)

In the above episode, teacher U1 asked for the students' ideas, repeated them to students, and listened to students. By doing so, the teacher elicited students' thinking and showed interest in them. Because of the careful eliciting, it became visible that although the two students proposed the same 4 × 2 multiplication rule, their explanations of the rule were different. Student 1's rule was based on the number of cubes and number of colors whereas student 2's rule was based on having four groups of two towers that had opposite colors. The teacher did not notice the difference, and thought that both students were thinking about the number of cubes times the number of colors (see turns 18 and 22). This episode shows that careful eliciting offered the teacher the opportunity to recognize the difference in the students' thinking process leading to the same result. However, the teacher did not recognize the difference.

Teacher U1 tried to help the students to notice that their rule is not correct. The teacher did this in an indirect way by having students notice that there could be more than eight towers. The teacher's question "does that mean that every tower has to have exactly two colors?" (turn 18) led the students to notice that the towers did not have to have two colors. After this, the students built two more towers (one tower all yellow and the other tower all blue). However, the students did not abandon their rules. Instead, they adjusted their rules to include the two extra towers. Student 2 reorganized her in five groups of two opposite towers (turn 21). Student 1 applied the 4 × 2 rule only to the initial set of eight towers and did not include the two extra one-color towers in the group.

The episode of teacher U1 shows that a teacher may be able to elicit students' ideas, give room for students' thinking and still struggle to offer guidance that helps the students to notice their mistake.

In the debriefing meeting, teacher U1 reflected on this episode:

> And they [students] also came up with this mathematical idea…Student 1 and Student 2 said originally… Well there are four towers tall, there are two colors and 4 × 2 is 8. Therefore 8 is the number (of towers). There are only 8 that you can make. And then when we clarified [that the problem said that that there are] two colors that area available and the fact that they could have one completely yellow and one completely blue, they then had 10. We said well. How does that 4 × 2 equals 8 work if now you have 10 towers? And they were like…then all of sudden they kind of threw that it off it didn't work for them. (Teacher U1)

The teacher's reflection confirms our interpretation that the teacher's aim in getting the students to notice that there could also be towers with one color was to have the students evaluate their 4 × 2 rule. Thus, the teacher seemed to be aware of the importance of having students critically examine their results without just telling that they had an incorrect rule. The teacher also recognized the need to give the students further guidance because the students just modified their rules to include two more towers. However, the teacher did not figure out how to do this. Also when reflecting on the episode, the teacher seemed not to pay attention to the difference between student 1's and student 2's rules.

Helping and Giving Room for Thinking

Teacher F2's lesson introduced the notion of exponent to 7th grade students using a task that involved folding a piece of paper. When folding the paper once, it formed two pieces. Then, when folded again, the paper is divided into four pieces. When folded three times, there are eight pieces, and so on. The students were asked to construct a rule by which you can calculate the amount of parts when the paper is folded a) 30, b) *x* times. A student pair had noticed a recursive pattern, but had difficulties in finding a general rule when the teacher approached them. The students first said that they doubted that they had found a good pattern. The teacher read their paper and a discussion started:

1 Teacher F2:	If you think like that, then you can continue forward with that.
2 Student 1:	But it isn't because you say [inaudible]
3 Student 2:	Yeah.
4 Teacher F2:	Go to the next task when you feel like that.
5 Student 2:	[Inaudible]
6 Teacher F2:	Think here, could you think here, when you are multiplying the previous result by two, what is the previous result? How did it? How did you?
7 Students:	Well, the one that has been multiplied by two.
8 Teacher F2:	Yeah, how did you get the previous result?
9 Student 2:	Two times two times two times two times two times two
10 Teacher F2:	Write it down. Write it down. [Teacher leaves]

In this episode, the teacher guided the students to notice a pattern and the students did actually notice it. The teacher did this by asking questions that helped focus the students' attention on the fact that they were multiplying several times by two (turn 6). After the students noticed this preliminary idea, the teacher provided only some guidance by requesting them to write their idea (turn 10). Thus, the teacher left it to the students to find that they should think about how many times they are multiplying by two. By asking them to write, the teacher helped students begin to pay attention to this. In the students' written solution, they finally mentioned multiplying 30 times. Thus, teacher F2's guidance seemed to be helpful for the students and left room for students' own thinking.

Teacher F2 seemed also to reflect on the episode differently than teacher F1 whose guidance was leading:

> Not only saying to students how it goes but more like getting them to justify, reason about the thing by themselves. Like when questioning, I feel that one important thing is that students themselves process the information. They would explain the issue. […] My point of view is that, things should be justified. Not always perfectly, but trying to justify. And that students would learn that things should be justified. […] Another point is that I think you learn better if you process the issue. Like if you only do calculation without never thinking what you are doing, then I think that it is not good. (Teacher F2)

Teacher F2 emphasized the importance of students processing information by themselves without teacher doing this for them. In addition, the teacher mentioned that "maybe I could had listened more to the students and let them have more time to explain." Thus, listening students' ideas was important for the teacher because he would like to listen to students more even though he already did that in the episode. Thus, the teacher seemed to be aware of the importance of giving room for students' thinking.

x.3.2 Emphasizing Justification

Within this theme, we observed two patterns about how teacher guidance emphasized the importance of justification in mathematics. One pattern was to simply ask repeatedly students to justify their claim. Another pattern is that after asking for a justification, the teacher helped students build a justification.

Repeatedly Asking Students to Justify Their Idea

In case of the 4-tall Tower Problem, at some point all students had 16 different towers. However, even though 16 is the right answer, the students had difficulties in justifying the answer. One group claimed that the answer is 16 because it is 4 times 4. The teacher U1 tried to get them to justify this claim.

1 Teacher U1:	Why are you saying 4 times 4?
2 Student 3:	I am saying that you can divide the 16 towers into a group of four towers each [pause and organizes the towers into four groups of four towers]
3 Teacher U1:	[Asking student 4] So what does that have to do with four times four equals sixteen. Student 3 said he has 16 and he can put them in four groups. Ok. Sixteen divided by four is four but how does 4 times 4 represent what you did now that you know you have all of them?
4 Student 4:	Because I smart.
5 Teacher U1:	Well, you know what? Now you have to convince me and you have to prove to it to me [Student 3 signals that he wants to say something] Go ahead.
6 Student 3:	Because each group, has four towers and each tower contains four cubes. And. And…
7 Teacher U1:	I know that each tower contains four cubes.

8 Student 3:	That is how it comes to four times four because we got four groups [of towers], four [towers in] each one [group], times how many cubes are in the tower and that makes it 16.
9 Teacher U1:	But, Student 3 how do you know that you have all of the possible combinations? Is there a way you can arrange them so that you can be sure you have everything?

This is a case of a teacher repeatedly trying to get students to justify their answers by insisting that the student explain how 4×4 gives the number of all towers. Despite the teacher's efforts, the teacher was not successful and did not seem to know how to help the students come up with a valid justification.

In the debriefing meeting that followed, the teacher acknowledged not knowing what to do to help the student realize that the 4×4 rule was not a valid justification:

> A lot of them have pairs but Student 4 has groups of four and they are opposites but he has them in a diagonal but the question is how does he articulate what he knows and that's the suggestion. How do you get them to do that? His response is just because I know. I am smart. That's why I know. And 4 times 4. Even Student 3 said 16 divided by four is four. I am like well what does that have to do with what we are doing? So, the question I have as a facilitator is what do I do in order to elicit the convincing argument. Cause even with Student 4 he is getting at a point where he is getting annoyed with me because I keep saying how do you know. (Teacher U1)

It seems that teacher U1 was aware of the importance of justification, recognized that students were proposing incorrect justification based on spotting numbers and operations which give the number 16 and tried to emphasize the importance of justification by asking the students repeatedly to justify their claims. However, the teacher just kept repeating the same general question "how do you know you have them all" and found no way to help the students to build a better justification.

Helping Students to Formulate Justification

In the teacher F3's ninth-grade lesson about divisibility rules, a student pair claimed that if a number is divisible by two, the last digit is 0, 2, 4, 6 or 8. When visiting the students early in the lesson, the teacher asked them many times to justify their claim. When the teacher revisited the students later, they had not yet justified their claim. Thus, the teacher started to guide them more strongly in building a justification.

1 Teacher F3:	What are you wondering about?
2 Student 2:	These justifications. Impossible. We do not have creativity for this. All these tasks. We just have answers. […] We can write that proved by calculating.
3 Teacher F3:	Look at some number, 342 for instance.
4 Student 2:	[Inaudible] I don't understand.
5 Teacher F3:	Well, write her. [The student 2 writes.] […] Then, write it as I have on the slide down there [points to $532 = 5 \cdot 100 + 3 \cdot 10 + 2$ on a whiteboard].
6 Student 2:	Down there.
7 Teacher F3:	There are ones, tenths, hundreds. Would it be?

8 Student 2:	Well, no. I don't get it. I don't [inaudible] anything about that. I write them. So? 300, 40, 2 [writes the numbers 300, 40 and 2]. So.
9 Teacher F3:	How do you see from here that you don't have to? Why you don't have to look at this and this [points to the numbers 300 and 40]? Why it is enough to look at the last number?
10 Student 2:	I don't know. I have been taught.
11 Teacher F3:	What if you divide this by two [points to number 300]?
12 Student 2:	It will be 150.
13 Student1:	[Yawning aloud]
14 Teacher F3:	And here. You have like four times ten [points to number 40]. If you divide it by two, the ten here?
15 Student 2:	So like, uhm, uhm. I can't speak anymore. So that they are composed of tenths, hundreds, and they are always divisible, because they end with zero.
16 Teacher F3:	Well, maybe you should not use ending with zero as a justification because we are just thinking about that, but you do know that ten is divisible by two and it tells that you don't have to look at this number anymore although it is 40 because there is the ten as a factor.

In this episode, the teacher guided the students to build a justification by specializing the theorem that they are proving to a case of number 342. Finally, this resulted in a generic justification stated by the student A. Instead of asking students why 342 is divisible by two, the teacher asked more specific questions that are related to detailed properties of the number (turns 9, 11 and 14).

When comparing the two justification-episodes, it can be seen that while in the first episode, Teacher U1 emphasized students' own justification, the students were not able to build a justification. On the contrary, in the second episode, the student finally produced a justification but the justification was composed together with teacher F3 and was not totally student's own.

When reflecting on the lesson afterwards, teacher F3 showed awareness of the students' difficulty being in understanding what needs to be justified when the claim is already known to be true:

> I suppose that at this stage they did not yet really understand what needed to be justified. Because they knew it for sure that in case of number two [divisibility rule for number 2], it ends with even number. (Teacher F3)

In addition, the teacher raised the issue of how strong hints should be given to students:

1 Researcher:	What are the benefits of these kinds of questions for the teacher and for the student?
2 Teacher F3:	Well, these are quite leading so that it is pretty much going toward me telling how this should be thought and how this should be justified and do like this to get this justified. Like very leading. There is really not much, like very creative students thinking. But on the other hand, but if this had been understood well, then they would be able to do the other ones [other divisibility rules]. Like here is a model for you, do the other proof tasks like this, like the same method.
3 Researcher:	Why would this be bad?

4 Teacher F3: Because it is against the ideology of inquiry mathematics for sure, I think. It would be like here is an example, do like this. To certain point, you can do proof tasks like that as well as other calculation tasks, so that here is a model example, do the others like this. But it is not. […] If you think about whether you want to teach proof tasks or only calculation tasks. If you do want to have also proof tasks, then it is not quite. Is it better than nothing to teach through model examples? In some sense, it must be that too, so that you must see models of how you can write these.

The teacher seemed to be unsatisfied with not giving enough room for the student's own thinking. Thus, the teacher was aware of giving the students strong hints. Particularly, the teacher recognized that the last question (turn 14) should have been less leading and leave more space for students' own thinking:

> I don't know if I gave the students enough time to think about the answer or to build an answer or how much I always asked more questions, those kind of leading questions. At least the last question, "you have like four times ten, if you divide it by two, the ten here," is very leading. I could have asked that more openly. (Teacher F3)

On the other hand, the teacher said that it is not good, if students remain stuck the whole lesson:

> At some point, you cannot expect having energy to be stuck forever. So that you give, you make it easier for the student so that he does not become totally frustrated and stays on the mood and can still have the feeling that I could do this be myself although it wouldn't really be like that. Like he would do the thinking process although someone helps very much. Then he could have a good feeling about that, which would help to do these by himself in the future. (Teacher F3)

Thus, the teacher seemed to be aware of balancing between letting the students to build their own justification and the teacher introducing ideas for the justification.

x.4 Discussion

We found that that when the teachers were focusing students' thinking, it varied how much teacher guidance actually helped students in focusing their thinking. Along with offering help, the teachers gave more or less room for the students' own thinking. It is well known that teachers in their daily work have to find a balance between offering help and leaving it for students to figure out. Of course, when a teacher leads strongly (teacher F1), students might be able to notice something, but strong leading may reduce the cognitive challenge too much, which is not the purpose in problem solving (Cai & Lester, 2010; Stein, Grover, & Henningsen, 1996). On the other hand, our study shows that it is also possible that, when trying to give room for students' own thinking, a teacher may not be able to help students' to notice an issue (teacher U1). Our results support the importance of noticing and interpreting students' ideas as well as advancing their thinking based on their ideas (e.g., Jacobs, Lamb & Philipp, 2010). As shown in the case of teacher U1, although a teacher may elicit students' ideas thoroughly, the teacher may still struggle to make sense of the students' ideas, which affects how helpful the teacher's guidance is for the students. In best scenario, the teacher guidance should help the students, but at the same time leave room for student thinking (Teacher F2). In other words, teacher guidance or scaffolding should be faded so that when students can do something on their own, the teacher does not anymore provide help. According to Van de Pol, Volman and Beishuizen (2010), this is one of the main principles in scaffolding.

Previous studies have suggested that students do not often spontaneously start to build justification that is more mathematical if they already have an empirical justification (Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2004; Martino & Maher, 1999). Thus, it is important that teachers push students to build mathematical justifications. The teachers in our study did this by asking students questions that emphasized the need for justification. However, according to our results, this may not be enough in promoting justification. The case of teacher U1 shows that a teacher may be asking for justification but does not find ways to help students start to build justifications. Furthermore, the analysis of teacher's reflection shows that the teacher was aware of the importance of getting students to justify and of her inability to do this. We take this awareness as a first step in teacher development.

In another episode, we found that in addition to asking for justification several times, the teacher F3 helped the student to build a justification. Teacher F3 did this by asking more specific questions about the situation than general questions such as "how do you know you have them all" asked by U1. Thus, this study supports the results by Franke et al. (2009) that series of specific questions can help students to give better explanations than general question.

The issue of finding a balance between helping students and leaving room for their own thinking came up in the both themes. The case of teacher U1 shows that a teacher may show awareness of the importance of giving room for students' thinking but, at the same time, the teacher may be unable to offer students help. On the other hand, the case of teacher F1 shows that a teacher may be able to help students but, at the same time, the teacher may not be aware of not giving much room for students' own thinking. This suggest that both awareness of the importance of giving room for students' own thinking as well as actually helping students without constraining their thinking are both issues that are to be taken into account when designing teacher professional development programs. This is also important in preventing the misconception that a teacher should not intervene on students thinking. Also when teaching through problem solving, teachers should guide children's thinking (Fraivillig et al., 1999; Lobato et al., 2005).

Our purpose in this study was not to compare the two professional development programs. However, we would like to speculate on the issue that has to do with how much help should be provided for students. The IBMT lessons were ordinary school lessons where usually only limited time is available for this long problem-solving activity. The IML lessons, on the other hand, were part of an after school program where several lessons can be devoted to dealing with the same problem if needed.

This practical difference may explain why IBMT teachers aimed for closure within one lesson whereas IML teachers often ended the lessons without having reached the closure. This is evident, for example, in the fact that IML teachers only asked for justification whereas IBMT teachers offered students guidance in building justification. This timescale difference may also provide some affordances and constraints for professional development programs. When teachers are learning to give room for students' thinking, after-school setting may provide teachers more time. On the other hand, in real classrooms there are time limitations and teachers have to learn to support students within a certain time.

Acknowledgement

The analysis of the data from IBMT project was partially funded by the Academy of Finland (project number 318010 & 286576). IML project was funded by the National Science Foundation (Grant REC-0309062).

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