Option Pricing and Uncertainties in the Black-Scholes Model

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The pricing of European call options traded on the Chicago Board of Options Exchange (CBOE) is studied in the Black-Scholes model. A method for treating uncertainties in option prices based on the propagation of uncertainties is presented. Practically implementable formulas for the uncertainties are derived. The Black-Scholes prices with uncertainties are compared to SPX options, where the underlying asset is the Standard&Poor 500 index consisting of 500 large US based companies, sold on the CBOE. Possible arbitrage opportunities for certain strike prices and maturities are found, showing that real markets may have pricing issues.

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Introduction

The present thesis deals with modeling the prices of financial instruments known as European call options, which give the holder the option, though not the obligation, to buy shares at a predetermined time at a predetermined price. Fischer Black and Myron Scholes showed in 1973 in their Nobel prize wining work that such instruments have a unique price in an arbitrage-free market [**1**]. This is quite surprising, since the underlying risky asset, often a stock, usually does not have an objective price, but individual investors' subjective attitude towards risk determines the price on the market place. The formula of Black and Scholes derived for option pricing is now widely used. The formula is very simple but the application of the formula to real world options can be problematic due to the limiting assumptions of the theory.

Understanding the Black-Scholes theory requires an understanding of stochastic analysis. In this work the basic tools of stochastics, such as Itô's formula, stochastic differential equations, etc. are presumed to be known. All the necessary concepts related to finance are presented.

This work provides a novel way for treating the short-comings of the Black-Scholes theory. Problems related to the determination of a risk-free interest rate as well as the future volatility of the underlying asset are treated similarly to independent measurement errors in physics. The option pricing with uncertainties is then implemented using C++ and compared with the prices of recently sold options.

The thesis is organized as follows: in chapter 1 I first present the financial concepts necessary for the modeling of a market with options, next a derivation of the Black-Scholes model for pricing European call options, and end this chapter by discussing the treatment of uncertainties related to the volatility and risk-free interest rates. In chapter 2 incomplete markets are briefly discussed with examples of situations where there is no unique price for an option. In chapter 3 the Black-Scholes model with uncertainties is put to the test when it is applied to S&P500 market index call options in the Chicago Board of Options Exchange.

CHAPTER 1

Black-Scholes Model

1.1. Financial markets

In this section the basic concepts and mathematical constructs needed for modeling financial markets are presented. The definitions of a European call option, portfolio strategy, self-financing portfolio, arbitrage opportunities as well as complete markets are given. The link between arbitrage-free markets and the existence of an equivalent martingale measure is given in the form of the fundamental theorem of asset pricing.

We wish to model the prices of financial instruments known as European call options. These are important tools for managing risks such as the market risk, i.e. the risk brought on by the possibility of a bear market.

For now, we assume there are no transaction costs, that there are no limits on the number of stocks we can buy, and that we can borrow and lend money with the same interest rate (known as the risk-free interest rate). A European call option gives us the option but not the obligation to by a given number of stocks (or other assets) exactly at time T, known as the maturity, for a predetermined price K, known as the strike price.

Definition 1.1. A *European call option* is a financial instrument with value at maturity T

$$\begin{cases} (S_T - K), & \text{if } S_T > 0 \\ 0, & \text{else,} \end{cases} \tag{1.1}$$

where $K > 0$ is the strike price and S_T is the price of the underlying option at time T.

The price of a European call option can be modeled as a stochastic process defined on the time interval $t \in [0, T]$. The word *European* refers to the fact that the option can only be exercised at T and does not have any geographical significance. An American call option can be exercised at any time before T and an Asian call option is a bit more complicated, with the value depending on the average price of the underlying asset over a certain period.

1.1.1. Asset classes. Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is a filtration of $\mathcal{F}$ and W_t a Brownian motion with respect to the filtration $\mathbb{F}$. Consider a financial market which consists of two types of assets which are a risk-free asset $S_0(t)$ and a risky asset $S(t)$. The risk-free asset has the dynamics

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt \\ S_0(0) = 1, \end{cases} \tag{1.2}$$

where $r(t) > 0$ is the instantaneous interest rate at time t. The risk-free asset can be for example a bank account or a zero-coupon bond with maturity T. A zero-coupon bond with maturity T is a financial instrument which delivers the initial capital $S_0(0) = 1$ with interest in a single payment at time T. Since $r(t)$ is strictly positive, we can formally solve this ordinary differential equation:

$$\begin{aligned} dS_0(t) &= r(t)S_0(t)dt \\ \frac{dS_0}{S_0} &= r(t)dt \\ \int_1^{S_0(t)} \frac{dy}{y} &= \int_0^t r(s)ds \\ \log S_0(t) - \log 1 &= \int_0^t r(s)ds \\ S_0(t) &= \exp\left(\int_0^t r(s)ds\right). \end{aligned}$$

If we assume that r is constant, we get

$$S_0(t) = e^{rt}. \tag{1.3}$$

Even though the risk-free interest rate is not exactly constant, there are several reasons why this assumption is justified. Firstly, the run times of traded options are usually some months or at most one to three years and the interest rates tend to move slowly. Secondly, for an investor with basically any amount of capital (from thousands to hundreds of millions) there are certificates of deposit (CDs) available, which have a fixed interest rate for given period $[0, T]$.

A risky asset which is usually referred to as a *stock* has the dynamics

$$\begin{cases} dS(t) = \sigma S(t)dW_t + \mu S(t)dt \\ S(0) = s_0, \end{cases} \tag{1.4}$$

where σ and μ are constants and W_t denotes the Brownian motion. We assume the process $S(t)$ to be continuous and adapted. Since σ and μ are constants, it is easy to see that the SDE has a unique strong solution. The parameter σ gives the standard deviation of the returns and is referred to as volatility or risk. The parameter μ is the mean return. Looking at the equation (1.4) a solution of the form $S(t) = s_0 e^{aW_t + bt}$ would make sense. In order to determine the constants a and b, we need the definition of an *Itô process* and apply the so-called *Itô's formula* (see [**2**] def. 3.4.8 and thm. 3.4.10).

Definition 1.2. (*Itô process*) Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis and $(W_t)_{t\geq 0}$ an $\mathcal{F}_t$-Brownian motion. $(X_t)_{t\in[0,T]}$ is an $\mathbb{R}$-valued Itô process if it can be written as

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s \qquad \mathbb{P}\text{-a.s.} \quad \text{for all } t \in [0, T], \tag{1.5}$$

where

- X_0 is $\mathcal{F}_0$-measurable
- $(K_t)_{t\in[0,T]}$ and $(H_t)_{t\in[0,T]}$ are $\mathcal{F}_t$-adapted

- $\int_0^T |K_s| ds < +\infty$ $\mathbb{P}$-a.s.
- $\int_0^T |H_s|^2 ds < +\infty$ $\mathbb{P}$-a.s.

THEOREM 1.3. ***(Itô's formula)*** *Let $(X_t)_{t\in[0,T]}$ be an Itô process for which equation (1.5) holds and $(t,x) \to f(t,x)$ be a twice continuously differentiable function with respect to x and once with respect to t, and if the partial derivatives are continuous with respect to (x,t), it holds that*

$$\begin{aligned} f(t,X_t) =& f(0,X_0) + \int_0^t \frac{\partial f(u,X_u)}{\partial u} du + \int_0^t \frac{\partial f(u,X_u)}{\partial X_u} K_u du \\ &+ \int_0^t \frac{\partial f(u,X_u)}{\partial X_u} H_u dW_u + \frac{1}{2}\int_0^t \frac{\partial^2 f(u,X_u)}{\partial X_u^2} H_u^2 du. \end{aligned} \tag{1.6}$$

From the definition of Itô process it is clear that the Brownian motion $(W_t)_{t\geq 0}$ is an Itô process with $X_0 = 0$, $K_t = 0$, and $H_t = 1$. Let us consider $f(t,x) = s_0 \mathrm{e}^{ax+bt}$, with the Itô-process $x = W_t$. First we apply Itô's formula to $f(t, W_t)$ we get

$$\begin{aligned} f(t,W_t) = f(0,W_0) &+ \int_0^t \frac{\partial f(u,W_u)}{\partial u} du + \int_0^t \frac{\partial f(u,W_u)}{\partial W_u} K_u du \\ &+ \int_0^t \frac{\partial f(u,W_u)}{\partial W_u} H_u dW_u + \frac{1}{2}\int_0^t \frac{\partial^2 f(u,W_u)}{\partial W_u^2} H_u^2 du. \end{aligned}$$

For the stock price $S(t) = f(t, W_t)$ we have that that

$$\begin{aligned} S(t) = s_0 &+ \int_0^t bS(u)du + \int_0^t aS(u)\cdot 0du \\ &+ \int_0^t aS(u)\cdot 1dW_u + \frac{1}{2}\int_0^t a^2 S(u)\cdot 1^2 du \\ =& s_0 + \int_0^t bS(u)du + \int_0^t aS(u)dW_u + \frac{1}{2}\int_0^t a^2 S(u)du. \end{aligned}$$

On the other hand, the stochastic differential equation (1.4) is just a short hand notation for the integral equation

$$S(t) = s_0 + \int_0^t \sigma S(u)dW_u + \int_0^t \mu S(u)du.$$

Since integral terms must be the same, we have that $\sigma = a$ and $\mu = b + a^2/2$, or

$$\begin{aligned} a &= \sigma \\ b &= \mu - \frac{\sigma^2}{2}. \end{aligned}$$

The stock price is therefore given by

$$S(t) = s_0 \mathrm{e}^{\sigma W_t + \mu t - \frac{\sigma^2}{2} t}. \tag{1.7}$$

1.1.2. Mathematical definitions. Before we can move on with modeling financial markets, there are some definitions from stochastics that are needed (definitions here are from [**3**]). In order to define *a portfolio* we need the notion of a predictable process.

Definition 1.4. A stochastic process X is said to be $(\mathcal{F}_t)_{t\geq 0}$-*predictible* if the map $(\omega, t) \to X_t(\omega)$ is $\mathbb{P}$-measurable.

To be able to define the equivalent martingale measure needed for the derivation of the Black-Scholes equation, we also need the notion of a *local* martingale (for details see e.g. [**3**] section 1.2.4).

Definition 1.5. An adapted, right-continuous process M is an $(\mathcal{F}_t)_{t\in[0,T]}$-local martingale if there exists a sequence of stopping times (τ_n) such that

(1) The sequence τ_n is increasing and $\lim_{n\to\infty} \tau_n = \infty$, almost surely.
(2) For every n, the stopped process $M^{\tau_n}\chi_{\{\tau_n>0\}}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale.

In order to state the fundamental theorem of asset pricing, semi-martingales are needed.

Definition 1.6. An $(\mathcal{F}_t)_{t\geq 0}$-*semi-martingale* is a càdlàg process X which can be written as $X = M + A$, where M is an $(\mathcal{F}_t)_{t\geq 0}$-locat martingale and A is an $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process with finite variation and value 0 at $t = 0$.

More precisely the semi-martingales S needed are locally bounded, meaning that there exists a localizing sequence of stopping times such that S^{τ_n} is bounded. Finally let us make the notion of a *contingent claim* (or financial derivative), which the European call option is a form of, precise:

Definition 1.7. A *contingent claim* H is a square integrable $\mathcal{F}_T$ random variable, where T is a fixed time horizon known as the maturity.

1.1.3. Portfolios and arbitrage. For our model to make sense, there cannot be two risk-free assets with different interest rates r_1 and r_2 such that $r_1 \neq r_2$, since otherwise it would be possible to borrow money at the lower interest rate and lend it at the higher rate and thus make risk-free profit, or *arbitrage*. However, the number of different stocks in a market can be any natural number $d \in \mathbb{N}$. We can now define a *portfolio* as in [**3**] in a market consisting of a risk-free asset and d stocks.

Definition 1.8. A *portfolio* or a strategy is a $(d+1)$-dimensional $(\mathcal{F})_{t\geq 0}$-predictable process

$$(\hat{\pi}_t = (\pi_t^0, \pi_t^1, \dots, \pi_t^d), \quad t \geq 0),$$

where π_i denotes the number of asset i, with $i = 0$ being the risk-free asset. The value of the portfolio at time t is given by the value function

$$V_t(\hat{\pi}) := \sum_{i=0}^{d} \pi_t^i S_i(t).$$

We shall assume that short selling is possible, i.e. that π_t^i can be negative for all i. If $\pi_t^i > 0$, we say that we have a long position on the asset i, or we are long in i. The set of portfolios we want to consider are ones where all changes in the value of the portfolio are due to the changes in asset prices. This means that after the initial investment at $t = 0$ we do not invest more or withdraw funds before $t = T$. Such portfolios are called *self-financing*.

Definition 1.9. A portfolio $\hat{\pi}$ is called *self-financing* if it satisfies the condition,

$$dV_t(\hat{\pi}) = \sum_{i=0}^{d} \pi_t^i dS_i(t)$$

It turns out that the set of self-financing portfolios includes strategies which we cannot except if we want to model a viable market. The set we want to be looking at is the set of *admissible strategies*, which are self-financing portfolios where the value function $V_t(\hat{\pi}) \geq 0$ for all t. If we would not limit the amount of money that can be borrowed, we would admit an arbitrage opportunity.

Definition 1.10. An *arbitrage opportunity* is a self-financing strategy $\hat{\pi}$ with zero initial investment, terminal value $V_T^{\hat{\pi}} \geq 0$, such that $\mathbb{E}(V_T^{\hat{\pi}}) > 0$.

In continuous time with unlimited credit you could use a doubling strategy to make an unlimited amount of money on a finite interval $[0, T]$ (see theorem 10.1 in [**4**]). The idea here is to invest at times $(1 - 1/n)T$, starting by borrowing one dollar and investing it in the stock at $t = 0$, if you have not earned \$ 1 by $t = 0.5T$ you borrow double the money, i.e. \$ 2. If you have earned \$ 1 by $t = 0.75T$ you stop, otherwise you borrow again double the money, that is, \$ 4. Following this strategy you will make \$1 with probability 1 by some step $N \in \mathbb{N}$. Once you have made your dollar, repeat the process starting at time $(1 - 1/(N + 1))T$. It is easy to see that you would end up with an infinite amount of money at time $t = T$ if such a strategy would be permitted.

In order to state the fundamental theorem of asset pricing we need one more definition, that is, the definition of an *equivalent martingale measure* [**3**]. This turns out to be the link between the applied science of finance and stochastics.

Definition 1.11. An *equivalent martingale measure (e.m.m.)* is a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ on $\mathcal{F}_T$, that is, $\{A \in \mathcal{F}_T \mid \mathbb{Q}(A) = 0\} = \{B \in \mathcal{F}_T \mid \mathbb{P}(B) = 0\}$, such that the discounted prices $\tilde{S}(t) = S_i(t)/S_0(t)$, $t < T$ are local $\mathbb{Q}$-martingales.

For a semi-martingale the *no-arbitrage condition* and the stricter *no-free lunch with vanishing risk* condition are intimately tied to the existence of an equivalent martingale measure. There are several different definitions of the *no-free lunch with vanishing risk* condition and they tend to be quite complicated. The one used here is definition 2.1.4.3 in [**3**]. The fundamental theorem can be stated as:

Theorem 1.12. ***(Fundamental Theorem.)*** *Let S be a locally bounded semi-martingale. There exists an equivalent martingale measure $\mathbb{Q}$ for S if and only if S satisfies the no-free lunch with vanishing risk condition.*

The proof is available in several sources but is quite lengthy, so it is not presented here. It can be found for example from the book of Delbaen and Schachermayer [**5**] (see Theorem 9.1.1.).

Before moving on to determining the proper prices of options, one more important concept is needed. This is the notion of complete markets. In a complete market any derivative or *contingent claim*, such as the options considered here, can be replicated with a self-financing portfolio. When a derivative can be replicated using a self-financing portfolio, the price of a derivative can be, given that the volatility off the underlying stock is understood, exactly determined. This is quite surprising, since the value of the underlying asset only has a subjective price. The exact definition of a complete market is quite straight forward [**3**].

Definition 1.13. Assume that the risk-free interest rate r is deterministic and $(\mathcal{F}_t)_{t\in[0,T]}$ be the natural filtration. The market is *complete* if any contingent claim (financial derivative) $H \in \mathcal{L}^2(\mathcal{F}_T)$ is the value of a self-financing strategy $\hat{\pi}$ at time $t = T$, i.e.

$$H = V_T(\hat{\pi}) = \sum_{i=0}^{d} \pi_T^i S_i(t).$$

While the arbitrage-free nature of a market was linked to the existence of an equivalent martingale measure, the completeness is linked to the uniqueness of this measure.

Theorem 1.14. *An* arbitrage-free *market is complete if and only if there exists a unique probability measure* $\mathbb{Q}$ *equivalent to the objective probability measure* $\mathbb{P}$ *such that the discounted stock prices* $\tilde{S}_i$*,* $i = 1, \ldots, d$ *are* $\mathbb{Q}$*-martingales.*

1.2. Derivation of the Black-Scholes equation

In the Black-Scholes model we have a market consisting of two assets: a risk-free asset with the dynamics of eq. (1.2) with constant r and a risky asset with the dynamics of eq. (1.4). There are several well-established ways to derive the Black-Scholes equation. I present here for completeness of presentation the important parts of the derivation based on the e.m.m. method. Further details on the derivation can be found for example in [**3, 4**]. The original derivation of Black and Scholes does not involve any e.m.m. and is not quite as elegant [**1**]. The necessary results, such as Girsanov's theorem and the martingale representation theorem can be found for example in the book of Karatzas and Shreve [**6**].

First we want to find the equivalent martingale measure, that is, a probability measure $\mathbb{Q}$ for which the discounted process $\tilde{S}(t) = S(t)/S_0(t)$, $t \in [0, T]$ is a $\mathbb{Q}$ martingale. An easy way to find this is to re-write the stochastic differential equation of $\tilde{S}(t)$

$$\begin{cases} d\tilde{S}(t) = \sigma\tilde{S}(t)dW_t + (\mu - r)\tilde{S}(t)dt \\ \tilde{S}(0) = s_0, \end{cases} \tag{1.8}$$

re-grouping the terms giving

$$d\tilde{S}(t) = \sigma\tilde{S}(t)(dW_t + \frac{(\mu - r)}{\sigma}dt) \tag{1.9}$$

Now by defining $\theta : [0, T] \times \Omega \to \mathbb{R}$ by

$$\theta = \frac{(\mu - r)}{\sigma} \tag{1.10}$$

and applying the Novikov's condition as well as Girsanov's theorem we see that $B(t) = W_t + \frac{(\mu-r)}{\sigma}t$ is a Brownian motion (and thus a martingale) under the measure $\mathbb{Q}$ which is given by

$$\mathbb{Q}(A) := \int_A \exp\big(-\theta W_t - \frac{1}{2}\theta^2 t\big) d\mathbb{P}, \tag{1.11}$$

and is the e.m.m. we were looking for.

The fact that this indeed is an equivalent martingale measure can be checked quite easily by hand. To see that $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$, it is enough to check that $Z_T := \exp(-\theta W_T - \frac{1}{2}\theta^2 T)$ can be used as a density:

$$\begin{aligned}\mathbb{E}Z_T =& e^{-\frac{\sigma^2\theta^2}{2}T}\mathbb{E}e^{-\sigma\theta W_T}\\ =& e^{-\frac{\sigma^2\theta^2}{2}T}\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty} e^{-\sigma\theta x}e^{-\frac{x^2}{2T}}dx\\ =& \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty} e^{-\frac{(x+\sigma\theta T)^2}{2T}}dx\\ =& 1.\end{aligned}$$

Since the exponential function is strictly positive, it holds that $Z_T > 0$ is a valid density, and thus $\{A \in \mathcal{F}_T \mid \mathbb{Q}(A) = 0\} = \{B \in \mathcal{F}_T \mid \mathbb{P}(B) = 0\}$.

It remains to show that the discounted price process $\tilde{S}$ is a martingale with respect to the measure $\mathbb{Q}$ given by equation (1.11). If $s < t$, the martingale property is

$$\mathbb{E}_{\mathbb{Q}}[\tilde{S}(t)|\mathcal{F}_s] = \tilde{S}(s) \quad \text{almost surely,} \tag{1.12}$$

or equivalently

$$\mathbb{E}Z_T\tilde{S}(t)\chi_G = \mathbb{E}Z_T\tilde{S}(s)\chi_G \tag{1.13}$$

for all $G \in \mathcal{F}_t$. To show this we start from the left hand side of equation (1.13) and apply the projection property of conditional expectations to get

$$\mathbb{E}Z_T\tilde{S}(t)\chi_G = \mathbb{E}\mathbb{E}[Z_T\tilde{S}(t)\chi_G|\mathcal{F}_t].$$

We can write this as

$$\mathbb{E}\mathbb{E}[Z_T\tilde{S}(t)\chi_G|\mathcal{F}_t] = \mathbb{E}\mathbb{E}[Z_{T-t}Z_t\tilde{S}(t)\chi_G|\mathcal{F}_t].$$

Taking out the $\mathcal{F}_t$-measurable part we get

$$\mathbb{E}Z_t\tilde{S}(t)\mathbb{E}[Z_{T-t}|\mathcal{F}_t]\chi_G = \mathbb{E}Z_t\tilde{S}(t)\chi_G \cdot 1.$$

Again, applying the projection property we get

$$\mathbb{E}Z_t\tilde{S}(t)\chi_G = \mathbb{E}\mathbb{E}[Z_t\tilde{S}(t)\chi_G|\mathcal{F}_s].$$

We can now pull χ_G out of the expected value ending up with

$$\begin{aligned}\mathbb{E}Z_s\tilde{S}(s)\chi_G &= \mathbb{E}Z_s\tilde{S}(s)\chi_G\mathbb{E}[Z_{T-s}|\mathcal{F}_s]\\ &= \mathbb{E}\exp(-\theta W_s - \frac{1}{2}\theta^2 s)\tilde{S}(s)\chi_G\mathbb{E}[\exp(-\theta W_{(T-s)} - \frac{1}{2}\theta^2(T-s))|\mathcal{F}_s]\\ &= \mathbb{E}\exp(-\theta W_T - \frac{1}{2}\theta^2 T)\tilde{S}(s)\chi_G = \mathbb{E}Z_T\tilde{S}(s)\chi_G,\end{aligned}$$

which proves the martingale property.

We wish to derive a fair price for a European call option, which we can denote for short as $(S(T)-K)^+$. It is easy to see that the discounted price $(\widetilde{S(T)-K})^+$ as well as the martingale $M_t = \mathbb{E}_{\mathbb{Q}}[(S(T)-K)^+|\mathcal{F}_t^B]$, are square integrable, where B is the $\mathbb{Q}$-Brownian motion given by $B_t = W_t + \frac{\mu - r}{\sigma}t$. The martingale representation assures that there exists an $(\mathcal{F}_t^B)$-adapted process $(L_t)_{t\in[0,T]}$ for which $\mathbb{E}_{\mathbb{Q}}\int_0^T L_s^2 ds < \infty$, such that

$$(1.14) \qquad (\widetilde{S(T) - K})^+ = M_0 + \int_0^T L_s dB_s.$$

For a self-financing raiding strategy ϕ we could also write

$$(1.15) \qquad (\widetilde{S(T) - K})^+ = V_0(\phi) + \int_0^T \phi(u)\tilde{S}(u)\sigma dB_u.$$

Now the trick is to combine equations (1.14) and (1.15) and write

$$\begin{aligned} 0 =& \mathbb{E}_{\mathbb{Q}}\big|(\widetilde{S(T) - K})^+ - (\widetilde{S(T) - K})^+\big|^2 \\ =& \mathbb{E}_{\mathbb{Q}}\big(M_0 + \int_0^T L_s dB_s - V_0(\phi) - \int_0^T \phi(u)\tilde{S}(u)\sigma dB_u\big)^2 \end{aligned}$$

Applying Itô's isometry and the fact that the expected value of a stochastic integral vanishes, we end up with

$$(1.16) \qquad 0 = \mathbb{E}_{\mathbb{Q}}\big(M_0 - V_0(\phi)\big)^2 + \mathbb{E}_{\mathbb{Q}}\int_0^T \big|L_u - \phi(u)\tilde{S}(u)\sigma\big|^2 du.$$

From this we can deduce that $\mathbb{Q}\otimes dt$ almost everywhere that

$$\begin{aligned} V_0(\phi) &= M_0, \\ \phi(u) &= \frac{L_u}{\tilde{S}(u)\sigma}, \\ \tilde{V}_t(\phi) &= M_t. \end{aligned}$$

For the price $V_t(\phi)$ we can now write

$$\begin{aligned} V_t(\phi) =& e^{rt}\mathbb{E}_{\mathbb{Q}}\big[\frac{(S(T)-K)^+}{e^{rT}}\big|\mathcal{F}_t^B\big] \\ =& \mathbb{E}_{\mathbb{Q}}\big[e^{-r(T-t)}\big(S(t)\frac{S(T)}{S(t)} - K\big)^+\big|\mathcal{F}_t^B\big] \\ =& \mathbb{E}_{\mathbb{Q}}\big[e^{-r(T-t)}\big(x\frac{S(T)}{S(t)} - K\big)^+\big|\mathcal{F}_t^B\big]\Big|_{x=S(t)} \\ =& \mathbb{E}_{\mathbb{Q}}\big[e^{-r(T-t)}\big(x\frac{S(T)}{S(t)} - K\big)^+\big]\Big|_{x=S(t)}, \end{aligned}$$

where we used the fact that $S(T)/S(t)$ is independent from $\mathcal{F}_t^B$.

Next we can set

$$(1.17) \qquad F(t,x) := \mathbb{E}_{\mathbb{Q}}e^{-r(T-t)}\big(x\frac{S(T)}{S(t)} - K\big)^+,$$

which means that $V_t(\phi) = F(t, S(t))$. This gives us

$$\begin{aligned} F(t,x) =& \mathbb{E}_{\mathbb{Q}} e^{-r(T-t)} \big(x \frac{e^{rT}\tilde{S}(T)}{e^{rt}\tilde{S}(t)} - K \big)^+ \\ =& e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \big(x e^{r(T-t)} e^{\sigma B_{T-t} - \frac{\sigma^2(T-t)}{2}} - K \big)^+. \end{aligned}$$

Introducing $X = B_{T-t}/\sqrt{T-t}$, which is a standard Gaussian random variable, and the traditional short hand notation $a = T - t$, we get

$$\begin{aligned} F(t,x) =& e^{-ra} \mathbb{E}_{\mathbb{Q}} \big(x e^{ra} e^{\sigma\sqrt{a}X - \frac{\sigma^2 a}{2}} - K \big)^+ \\ =& e^{-ra} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \big(x e^{ra} e^{\sigma\sqrt{a}z - \frac{\sigma^2 a}{2}} - K \big)^+ e^{-\frac{z^2}{2}} dz \\ =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \big(x e^{\sigma\sqrt{a}z - \frac{\sigma^2 a}{2}} - K \big)^+ e^{-\frac{z^2}{2}} \chi_{\{z+d_2 \geq 0\}} dz, \end{aligned}$$

where we have

$$d_2 = d_1 - \sigma\sqrt{a} \quad \text{with} \quad d_1 = \frac{\log \frac{x}{K} + (r + \frac{\sigma^2}{2})a}{\sigma\sqrt{a}}. \tag{1.18}$$

Denoting the cumulative normal distribution by $\Phi(z)$, this can be written in the form known as the Black-Scholes formula:

$$\boxed{F(t,x) = x\Phi(d_1) - K\mathrm{e}^{-r(T-t)}\Phi(d_2)}. \tag{1.19}$$

1.3. Estimation of Uncertainties

Let us assume we do a calculation using the Black-Scholes formula for the price of a European call option for 100 Exxon stocks with maturity T and a strike price K. Suppose the calculation gives the proper price of the option at \$ 5. Now assume that you can buy these option from the market at \$ 4.5. If we keep the Black-Scholes model assumptions of liquidity and no friction in the market, we would have an arbitrage opportunity. However, the situation is in reality a bit more complicated. Since the volatility between the present and the maturity of the option is not known, it has to be estimated. The risk-free interest rate is also unknown for $t > 0$. The other quantities, namely the maturity and the strike price are known exactly.

While one option would be to do a simulation, this is not the right approach here. Computational cost is a priority, since real life arbitrage opportunities are very short lived. Instead of running a simulation with for example 10 000 random samples, we can derive an analytical expression for the uncertainty which only requires the execution of a few lines of code.

What we want is a probability distribution of the proper price of the option $F(t, x)$. Firstly, we want to estimate such σ, $\delta\sigma$, r, and δr that the true volatility has the probability distribution $\mathcal{N}(\sigma, (\delta\sigma)^2)$ and the true risk-free interest rate has the distribution $\mathcal{N}(r, (\delta r)^2)$. Assuming that the uncertainties are independent and random, we get by the general propagation of uncertainty [**7**] that the proper value of the option is Gaussian with mean $F(t, x)$ and standard deviation

$$\delta F(t,x,\sigma,r) = \sqrt{\left(\frac{\partial F(t,x,\sigma,r)}{\partial \sigma}\delta\sigma\right)^2 + \left(\frac{\partial F(t,x,\sigma,r)}{\partial r}\delta r\right)^2}, \tag{1.20}$$

where the partial derivatives are evaluated with the best guess values σ and r. If the uncertainties were correlated, it would still hold that

$$\delta F(t,x,\sigma,r) \leq \left|\frac{\partial F(t,x,\sigma,r)}{\partial \sigma}\right|\delta\sigma + \left|\frac{\partial F(t,x,\sigma,r)}{\partial r}\right|\delta r. \tag{1.21}$$

The partial derivatives of the price function of eq. (1.19) are

$$\begin{aligned}
\frac{\partial F(t,x,\sigma,r)}{\partial \sigma} &= x\phi(d_1(\sigma))d_1'(\sigma) - K\mathrm{e}^{-r(T-t)}\phi(d_2(\sigma))d_2'(\sigma) \\
&= \frac{1}{\sqrt{2\pi}}\left(x\mathrm{e}^{-\frac{d_1(\sigma)^2}{2}}(\sqrt{T-t} - \frac{d_1(\sigma)}{\sigma}) + K\mathrm{e}^{-r(T-t)}\mathrm{e}^{-\frac{d_2(\sigma)^2}{2}}\frac{d_1(\sigma)}{\sigma}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial F(t,x,\sigma,r)}{\partial r} &= x\phi(d_1(r))d_1'(r) + K(T-t)\mathrm{e}^{-r(T-t)}\Phi(d_2(r)) \\
&\quad - K\mathrm{e}^{-r(T-t)}\phi(d_2(r))d_2'(r) \\
&= \frac{1}{\sqrt{2\pi}}\left(x\mathrm{e}^{-\frac{d_1(r)^2}{2}}\frac{\sqrt{T-t}}{\sigma} + K(T-t)\mathrm{e}^{-r(T-t)}\int_{-\infty}^{d_2(r)}\mathrm{e}^{-\frac{z^2}{2}}\,dz\right. \\
&\quad \left. - K\mathrm{e}^{-r(T-t)}\mathrm{e}^{-\frac{d_2(\sigma)^2}{2}}\frac{\sqrt{T-t}}{\sigma}\right).
\end{aligned}$$

CHAPTER 2

Incomplete Markets

In this section I expand the discussion to *incomplete markets*, that is, to situations where the prices of derivatives cannot be uniquely determined. By the definition of market completeness, in a complete market any contingent claim must be the final value of a self-financing portfolio. A simple example of an incomplete market would be a case where there is a stochastic process determining the pay-out of a contingent claim which is not the value of an underlying asset. A toy example illustrating this kind of a situation would be a weather insurance, where the pay-out depends on the temperature, say exactly in one year in Jyväskylä (see Björk Chap. 15 [**4**] for a similar example). Assume that the pay-out in euros of our insurance be

$$\begin{cases} 0, & \text{if } T \geq 25°\text{C}, \\ 25 - T, & \text{otherwise,} \end{cases}$$

which looks just like the pay-out of a European call option. However, this "option" does not have a unique price, unlike the options in the Black-Scholes model. The underlying reason is that this option is not replicable, since we can not buy degrees of warmth on a given day in order to hedge this claim.

For a deeper example which gives insights in to what is going on in general for incomplete markets, we can use theorem 1.14. To find an example of an incomplete market, we only need to find a situation where the equivalent martingale measure is not unique.

Example 2.1. (**Incomplete markets, non-unique e.m.m.**) Let us assume the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))_{t\in[0,T]}$ satisfying the usual conditions (see [**6**]), and a two-dimensional Brownian motion $(W_t, \tilde{W}_t)$ with respect to the filtration. Pick the price process to be $S(t) = e^{-\sigma W_t - \frac{\sigma^2}{2}t}$ for $t \in [0, T]$ be for the stock.

It is clear that $S(t)$ is a $\mathbb{P}$-martingale with respect to the filtration $(\mathcal{F}_t))_{t\in[0,T]}$. If we take the risk-free interest rate to be $r = 0$, the discounted prices are the same as the non-discounted prices and so $\mathbb{P}$ is itself an equivalent martingale measure (to see this, just replace r with 0 in eq. (1.11)).

Let $\theta > 0$ be arbitrary and set

$$H_t^\theta = w^{-\theta\tilde{W}_t - \frac{\theta^2}{2}t}.$$

Note that W_t and $\tilde{W}_t$ are independent. Now

$$\mathbb{Q}^\theta(A) = \mathbb{E}(\chi_A H_T^\theta)$$

is an equivalent martingale measure for any $\theta > 0$, i.e. there are an infinite number of equivalent martingale measures.

Indeed the equivalence of the measures $\mathbb{Q}^\theta(A)$ and $\mathbb{P}$ can be seen by noting that $0 < H_T^\theta < \infty$ $\mathbb{P}$-almost surely. Therefore, if $\mathbb{P}(A) = 0$, this means that $\chi_A = 0$ $\mathbb{P}$-a.s. and vice versa, and clearly $\mathbb{Q}^\theta(A)$ is zero $\mathbb{P}$-a.s. if and only if $\chi_A = 0$ $\mathbb{P}$-almost surely. The martingale property $\mathbb{E}_{\mathbb{Q}^\theta}[S(t)|\mathcal{F}_s] = S(s)$ can be seen by applying the projection property of expected values.

CHAPTER 3

Application of the Black-Scholes Model in the Chicago Board of Options Exchange

In this chapter the Black-Scholes model is applied to options traded on the Chicago Board of Options Exchange (CBOE). The derivative product considered here is the SPX, which is an European call option, where the underlying risky asset is the Standard&Poor 500 index. These options are traded in large volumes and options with a wide range of maturities and strike prices are available. The options are settled in dollars instead of actually delivering any stocks, making this the closest thing to the definition of a European call option used in mathematical modeling. We compare the calculated option prices to the final prices of sold SPX options traded on June 18, 2019.

3.1. Risk-free interest rate and volatility

For the risk-free interest rate I use is the United States Government Treasury bill rate, also known as T-bill rate. US government can be seen as having the lowest default risk, since it has the ability to tax the world's largest economy in order to pay off its debt obligations. The current and historical rates are provided by the US government in [**8**]. The rates fluctuate, so for the purposes here we pick the risk-free interest rate to be

$$r = 2.19 \pm 0.22, \tag{3.1}$$

a range which allows for slow drift in interest rates over the maturities considered in this work, with 2.19% representing the current interest rate on June 18.

The determination of the volatility is more problematic, since the volatility between $t = 0$ and $t = T$ is not known in advance. Since volatility is not constant over time a commonly used educated guess is to use the historical volatility (which is exactly known) for a period which is approximately as long as the time to maturity. Here, I used for maturities of one month or less the historical volatility of the S&P500 for the past month, for maturities between 1 and 3 months that of the past three months, for $T = 3$–6 months that of past six months, for T=6–12 months that of the past year, and finally for the longer maturities that of the past three years. The uncertainty related to the volatility is assumed to be 10 %.

3.2. Transaction costs and dividends

In order to make meaningful comparisons to actually traded options, we need to include two price-affecting factors. The first one is transaction costs. This can be simply included by subtracting the cost associated with the purchase of the option from the calculated value. For the fees I used the ones given by the brokerage firm

Interactive Brokers, which trades the SPX for the lowest commercially available price [**9**]. The fees used in this work were

$$\begin{cases} \$0.35, & \text{for options priced under } \$1 \\ \$0.44, & \text{for options priced above or at } \$1. \end{cases}$$

One problem with the Black-Scholes model as it is, is that it does not take into account dividends, which any viable company should pay. Since the underlying asset of the SPX is the S&P 500 index, which includes 500 companies which all pay their dividends at different times; some monthly, some quarterly, some annually, the dividends can be modeled by assuming that the asset pays continuous dividends. The dividend yield of the S&P 500 was taken to be the current 1.9%. In the Black-Scholes formula continuous dividends can be included by replacing the stock price $S(t) \to S(t)e^{-qa}$ and $r \to r - q$ in d_1, where q is the continuously paid dividend (see chapter 18 in [**4**]).

3.3. Results

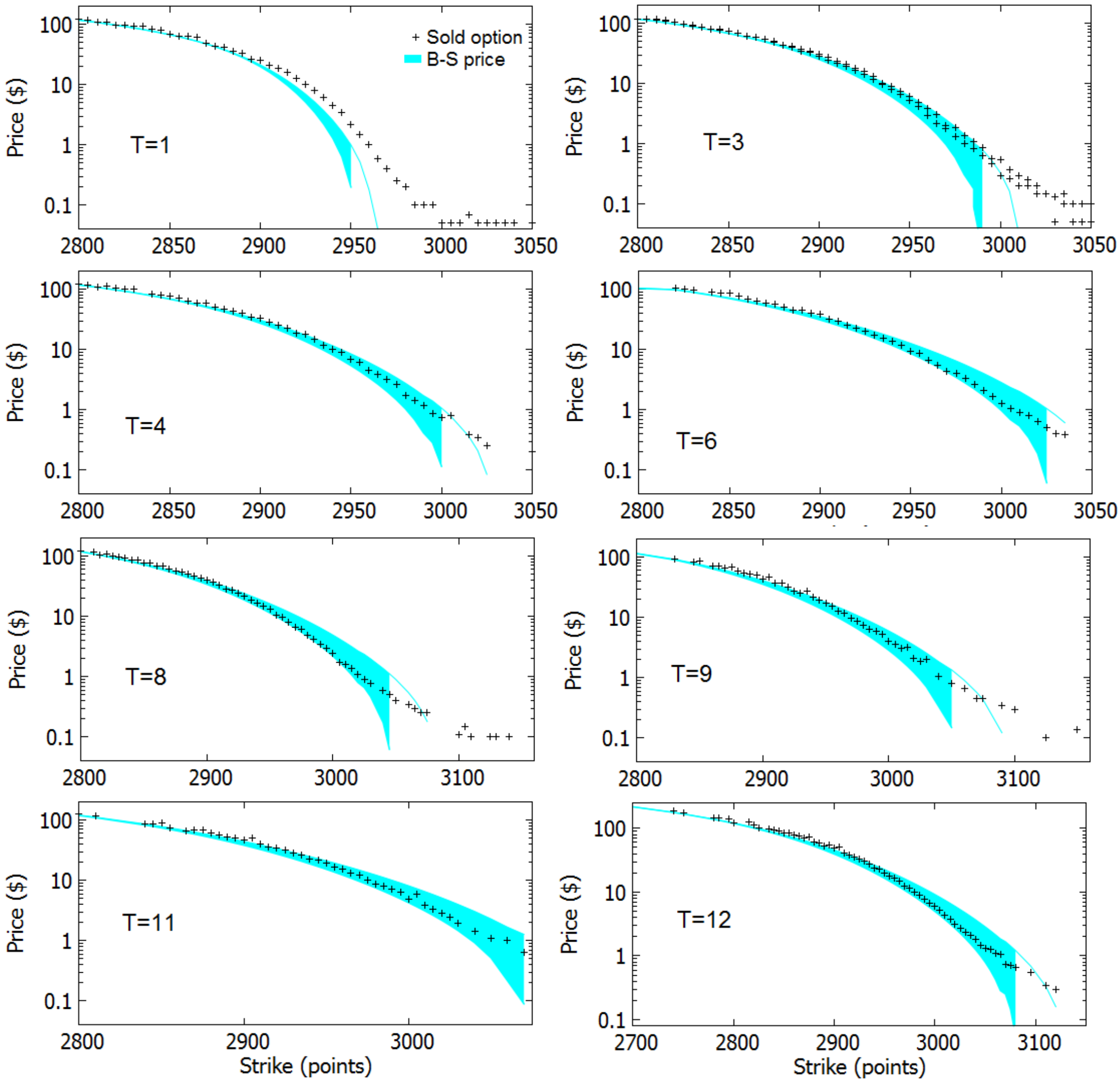
The calculations were done for derivatives with all combinations of maturities and strike prices exchanged in the CBOE on June 18, 2019. This includes 1586 different combinations of maturities and strike prices. The maturities were measured in banking days, that is, roughly 252 days a year. Banking holidays were taken into account. The maturities ranged from 1 day to 629 days (2.5 years). The results are shown as plots in Appendix A. The black plus-signs denote a sold option, while the blue area is the price according to the Black-Scholes model to 1σ accuracy according to the eq. (1.20).

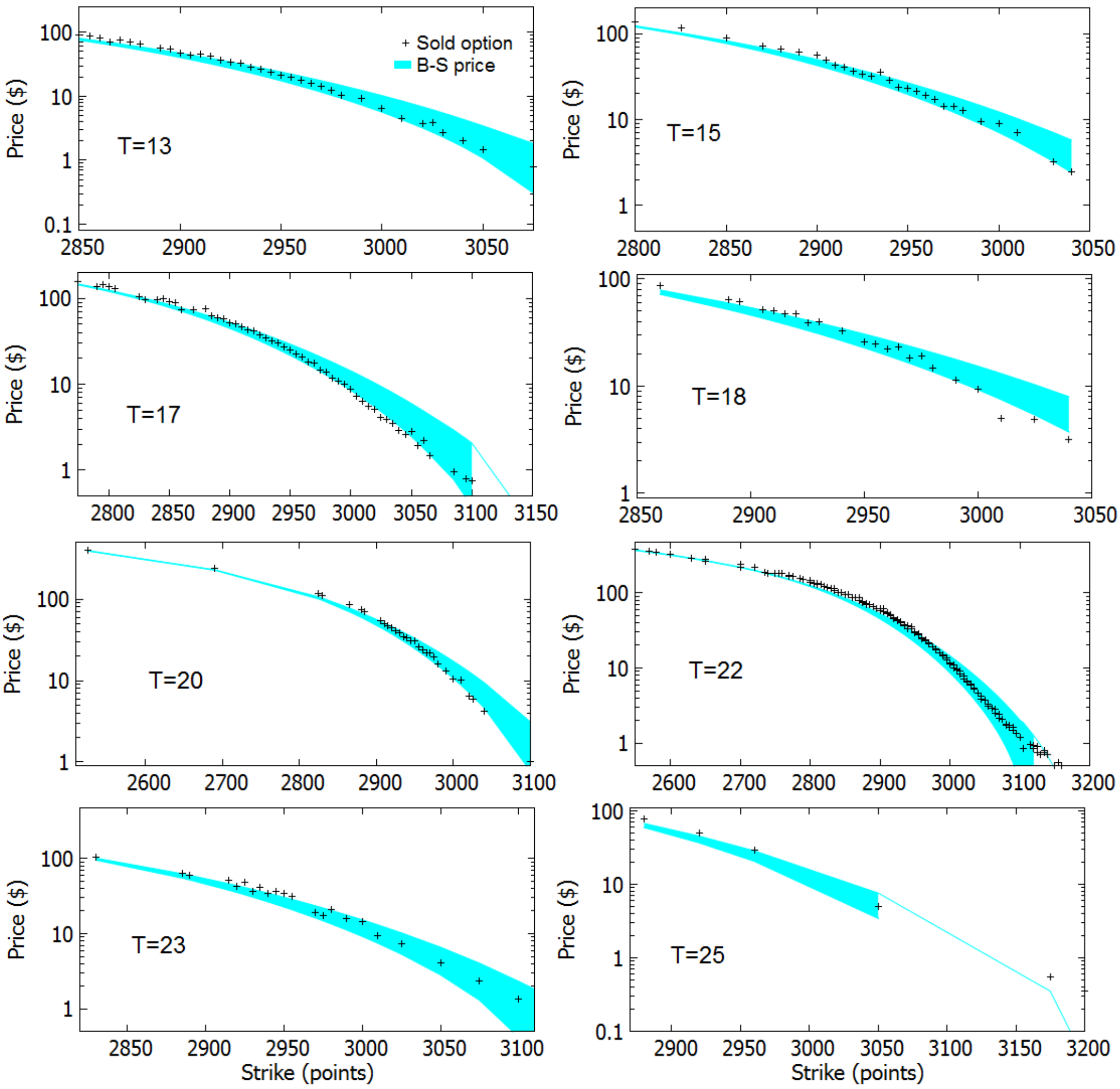
For the $T = 1$ maturities the higher strike price options seem to have been sold for too high of a price, while for the lower strike prices the model agrees with the real prices. For T=3–52 the model agrees with the data well, but slight over pricing can be seen for the lower strike prices. The high strike options with $T = 3$–9 look to be over priced. This is where the effect of the fees come into play. These options are traded at 5–10 cents but a fee of 35 cents makes buying these undesirable. Between T =66–252 i.e. 3–12 months there seems to have been significant arbitrage opportunities. Options with strike prices above 3100 points have been systematically under estimated. Another way to look at this is that the implied volatility of the market is not what we estimate it to be. For the longer maturities there is no significant difference between the model prices and the real world prices.

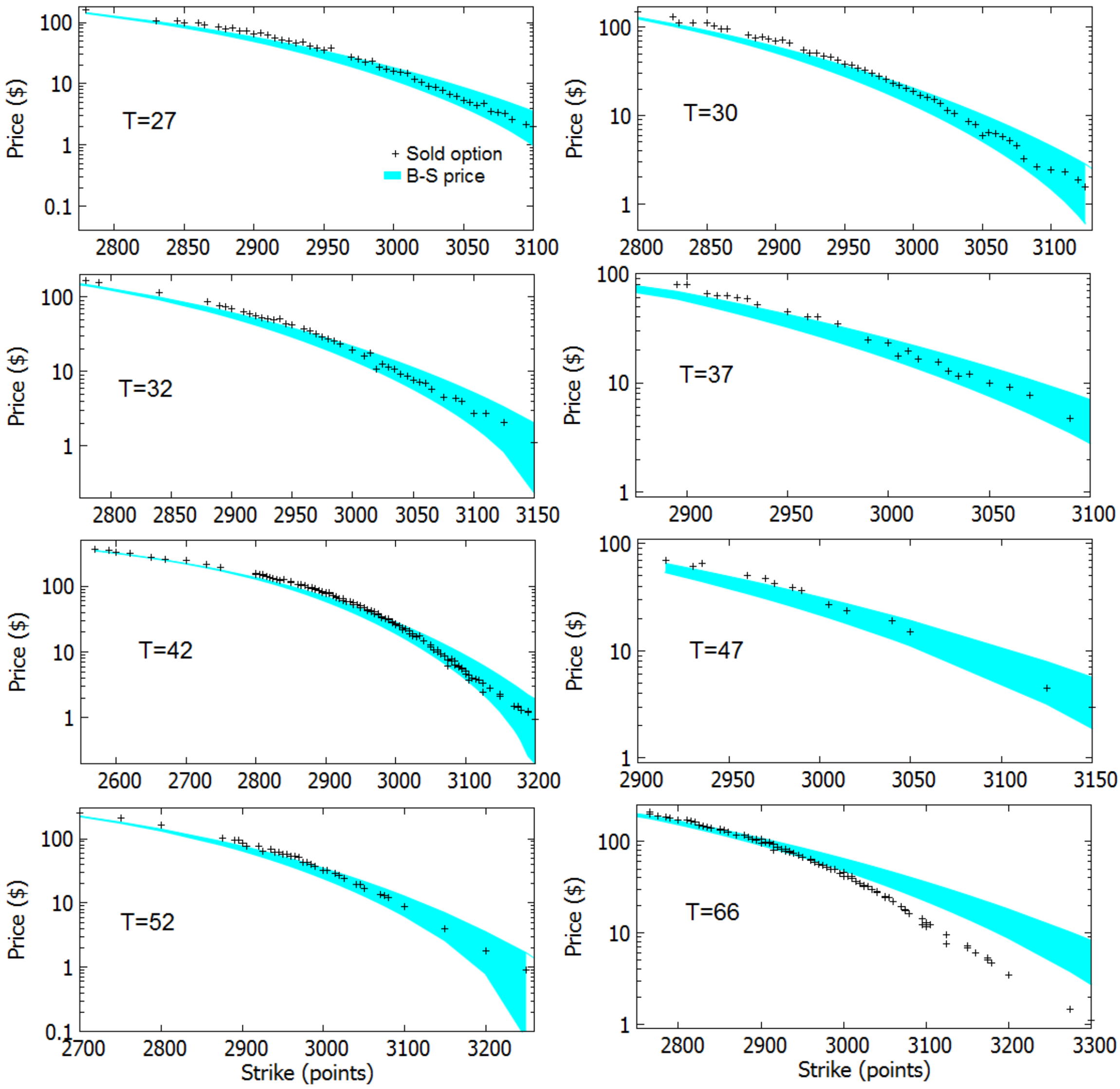
In conclusion, the SPX options traded in the CBOE seem to be largely arbitrage free. However, an investor with the possibility to short the S&P 500 could have possibly made arbitrage on June 18, 2019 by buying the SPX options with $K > 3100$ and T=3–12 months.

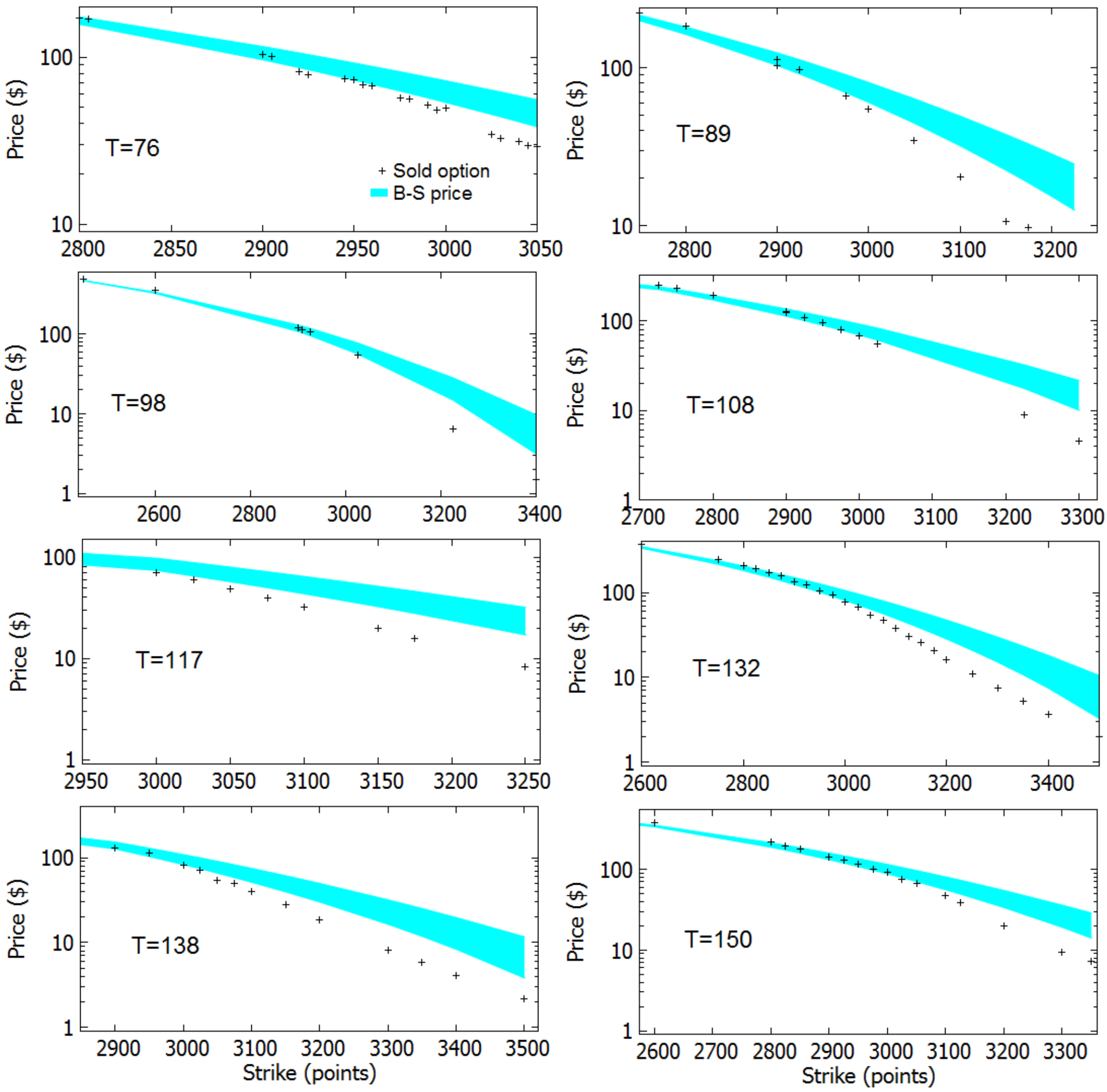
APPENDIX A

Data

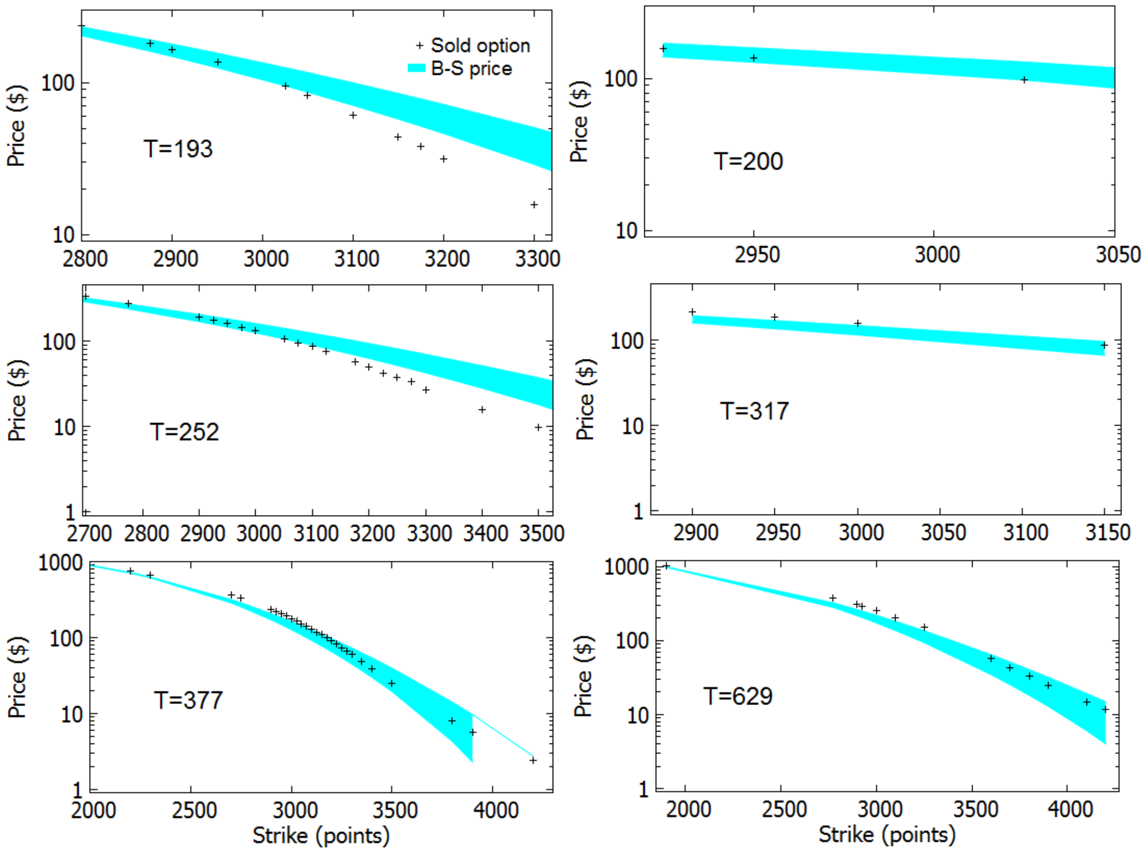



Sold option
B-S price
T=13
T=15
T=17
T=18
T=20
T=22
T=23
T=25
Price ($)
Strike (points)


T=27
T=30
T=32
T=37
T=42
T=47
T=52
T=66
Sold option
B-S price
Price ($)
Strike (points)



T=76
Sold option
B-S price
T=89
T=98
T=108
T=117
T=132
T=138
T=150
Price ($)
Strike (points)


Sold option
B-S price
T=193
T=200
T=252
T=317
T=377
T=629
Price ($)
Strike (points)

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