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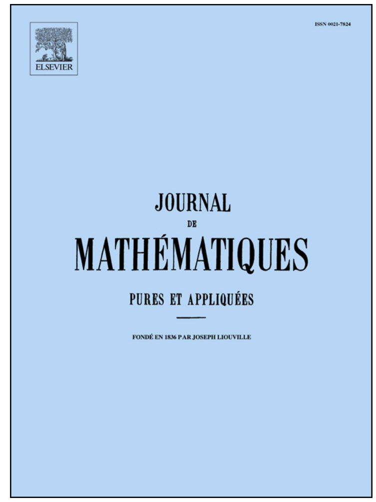

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The Choquet and Kellogg properties for the fine topology when $p = 1$ in metric spaces *

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Abstract

In the setting of a complete metric space that is equipped with a doubling measure and supports a Poincaré inequality, we prove the fine Kellogg property, the quasi-Lindelöf principle, and the Choquet property for the fine topology in the case $p = 1$.

Dans un contexte d'espace métrique complet muni d'une mesure doublante et supportant une inégalité de Poincaré, nous démontrons la propriété fine de Kellogg, le quasi-principe de Lindelöf, et la propriété de Choquet pour la topologie fine dans le cas $p = 1$.

1 Introduction

Much of nonlinear potential theory, for $1 < p < \infty$, deals with p-harmonic functions, which are local minimizers of the L^p-norm of $|\nabla u|$. Such minimizers can be defined also in metric measure spaces by using *upper gradients*, and the notion can be extended to the case $p = 1$ by considering *functions of least gradient*, which are functions of bounded variation (BV functions) that minimize the total variation locally — see Section 2 for definitions. Nonlinear potential theory for $1 < p < \infty$ has by now reached a mature state even

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in the general setting of metric spaces that are equipped with a doubling measure and support a Poincaré inequality, see especially the monograph [3] and e.g. [4, 9, 10, 23, 38]. Functions of least gradient have been studied less, see however [11, 33, 34, 40] for some previous works in the Euclidean setting, and [17, 24] in the metric setting.

In the case $1 < p < \infty$, it is known that the *p-fine topology* is the coarsest topology that makes p-superharmonic functions continuous. For nonlinear fine potential theory and its history in the Euclidean setting, for $1 < p < \infty$, see especially the monographs [1, 20, 31]. In the metric setting, fine potential theory for $1 < p < \infty$ has been studied recently in [6, 7, 8]. In these papers, the so-called Cartan and Choquet properties for the p-fine topology were proved, and the latter was then used to deduce two further facts: first that every p-finely open set is *p-quasiopen*, and second that an arbitrary set is *p-thick* at p-quasi-every of its points. The latter fact is called the fine Kellogg property.

Few results of fine potential theory seem to have been considered in the case $p = 1$, see however [41] for some results in weighted Euclidean spaces. The case $p = 1$ is quite different since cornerstones of the theory for $p > 1$, such as comparison principles and continuity of p-harmonic functions, are not available. However, the author has previously studied some aspects of fine potential theory for $p = 1$ in metric spaces in [25, 26, 27]. The setting in these papers as well as the current one is a complete metric space that is equipped with a doubling measure and supports a Poincaré inequality. In [27], the author proved a weak Cartan property in the case $p = 1$, see Theorem 6.6 below. In this paper we prove, in the case $p = 1$, the fine Kellogg property and the Choquet property in forms that are exactly analogous with the case $p > 1$, see Corollary 4.7 and Theorem 6.14. Moreover, in Theorem 5.2 we prove the so-called quasi-Lindelöf principle for 1-finely open sets, also in a form that is exactly analogous with the case $p > 1$.

In the case $1 < p < \infty$, the different properties are deduced as follows, see [7, 19, 31].

$$\text{Cartan property} \implies \text{Choquet property}$$
$$\implies \begin{cases} \text{Fine Kellogg property} \implies \text{Quasi-Lindelöf principle} \\ \text{Finely open sets are quasiopen} \end{cases}$$

For $p > 1$, the fine Kellogg property is closely related to the (usual) Kellogg property, which states that p-quasi every boundary point of an open set is *regular*, meaning that p-harmonic solutions of the Dirichlet problem are continuous up to these boundary points when the boundary data is continuous. According to the Wiener criterion, every point where the complement of an open set is p-thick is a regular boundary point, which combined with the fine Kellogg property implies the (usual) Kellogg property.

In the case $p = 1$, since we lack various tools such as comparison principles but on the other hand have access to other more geometric tools, we deduce the various properties in a rather different order, as follows.

$$\begin{array}{ccl} & & \text{Weak Cartan property} \\ & & \quad\Downarrow \\ \text{Quasi-Lindelöf principle} & \Longrightarrow & \text{Finely open sets are quasiopen} \\ \Uparrow & & \quad\Downarrow \\ \text{Fine Kellogg property} & \Longrightarrow & \text{Choquet property} \end{array}$$

Easy examples such as that of a square show that in the case $p = 1$ there can be many irregular boundary points and a Kellogg property does not hold. We retain the term "fine Kellogg property" for $p = 1$, but this property is a starting point for proving the other properties rather than an end in itself, in contrast to the case $p > 1$. In terms of applications, one of our main motivations is that the quasi-Lindelöf principle will be useful in further research when considering *1-strict subsets* and partition of unity arguments in 1-finely open sets.

2 Preliminaries

In this section we introduce the notation, definitions, and assumptions employed in the paper.

Throughout this paper, (X, d, μ) is a complete metric space that is equipped with a metric d and a Borel regular outer measure μ satisfying a doubling property, meaning that there exists a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B(x, r) := \{y \in X : d(y, x) < r\}$. We also assume that X consists of at least 2 points. Sometimes we abbreviate $B = B(x, r)$ and $aB := B(x, ar)$ for $a > 0$. Note that in metric spaces, a ball (as a set) does not necessarily have a unique center and radius, but we will always understand these to be prescribed for the balls that we consider.

A complete metric space equipped with a doubling measure is proper, that is, closed and bounded sets are compact. Since X is proper, for any open set $\Omega \subset X$ we define $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ to be the space of functions that are Lipschitz in every open $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of Ω. Other local spaces of functions are defined analogously.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension one is defined to be

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : \, A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \, r_i \le R \right\}.$$

(We interpret $B(x, 0) = \emptyset$ and $\mu(B(x, 0))/0 = 0$, so that finite coverings are also allowed.) The codimension one Hausdorff measure of $A \subset X$ is then defined to be

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).$$

All functions defined on X or its subsets will take values in $[-\infty, \infty]$. By a curve we mean a nonconstant rectifiable continuous mapping from a compact interval of the real line into X. A nonnegative Borel function g on X is an upper gradient of a function u on X if for all curves γ, we have

$$|u(x) - u(y)| \le \int_\gamma g \, ds,$$

where x and y are the end points of γ and the curve integral is defined by using an arc-length parametrization, see [21, Section 2] where upper gradients were originally introduced. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite. By only considering curves γ in a set $A \subset X$, we can talk about a function g being an upper gradient of u in A.

Given an open set $\Omega \subset X$, we let

$$\|u\|_{N^{1,1}(\Omega)} := \|u\|_{L^1(\Omega)} + \inf \|g\|_{L^1(\Omega)},$$

where the infimum is taken over all 1-weak upper gradients g of u in Ω. The substitute for the Sobolev space $W^{1,1}$ in the metric setting is the Newton-Sobolev space

$$N^{1,1}(\Omega) := \{u : \|u\|_{N^{1,1}(\Omega)} < \infty\},$$

which was first considered in [39]. We understand every Newton-Sobolev function to be defined at every $x \in \Omega$ (even though $\|\cdot\|_{N^{1,1}(\Omega)}$ is then only a seminorm).

We will assume throughout the paper that X supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for every ball $B(x,r)$, every $u \in L^1_{\mathrm{loc}}(X)$, and every upper gradient g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C_P r \fint_{B(x,\lambda r)} g\, d\mu,$$

where

$$u_{B(x,r)} := \fint_{B(x,r)} u\, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u\, d\mu.$$

The 1-capacity of a set $A \subset X$ is given by

$$\mathrm{Cap}_1(A) := \inf \|u\|_{N^{1,1}(X)},$$

where the infimum is taken over all functions $u \in N^{1,1}(X)$ such that $u \geq 1$ on A. We know that Cap_1 is an outer capacity, meaning that

$$\mathrm{Cap}_1(A) = \inf\{\mathrm{Cap}_1(W) : W \supset A \text{ is open}\}$$

for any $A \subset X$, see e.g. [3, Theorem 5.31]. If a property holds outside a set $A \subset X$ with $\mathrm{Cap}_1(A) = 0$, we say that it holds at 1-quasi-every point.

We say that a set $U \subset X$ is 1-quasiopen if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\mathrm{Cap}_1(G) < \varepsilon$ and $U \cup G$ is open.

The variational 1-capacity of a set $A \subset D$ with respect to a set $D \subset X$ is given by

$$\mathrm{cap}_1(A, D) := \inf \int_X g_u\, d\mu,$$

where the infimum is taken over functions $u \in N^{1,1}(X)$ such that $u \geq 1$ on A and $u = 0$ on $X \setminus D$, and their upper gradients g_u. For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see e.g. [3, 5].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, following [35]. See also e.g. [2, 13, 14, 15, 42] for the classical theory in the Euclidean setting. Let $\Omega \subset X$ be an open set. Given a function $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the total variation of u in Ω by

$$\|Du\|(\Omega) := \inf\left\{\liminf_{i\to\infty} \int_\Omega g_{u_i}\, d\mu : \, u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega),\, u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\},$$

where each g_{u_i} is an upper gradient of u_i in Ω. (Note that in [35], local Lipschitz constants were used instead of upper gradients, but the theory can be developed with either definition.) If $u \in L^1(\Omega)$ and $\|Du\|(\Omega) < \infty$, we say that u is a function of bounded variation and denote $u \in \mathrm{BV}(\Omega)$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf\{\|Du\|(W) : A \subset W, W \subset X \text{ is open}\}.$$

If $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\|Du\|(\Omega) < \infty$, $\|Du\|(\cdot)$ is a Radon measure on Ω by [35, Theorem 3.4]. A μ-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where χ_E is the characteristic function of E. The perimeter of E in Ω is also denoted by

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

The lower and upper approximate limits of a function u on X are defined respectively by

$$u^\wedge(x) := \sup\left\{t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x,r) \cap \{u < t\})}{\mu(B(x,r))} = 0\right\} \tag{2.1}$$

and

$$u^\vee(x) := \inf\left\{t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x,r) \cap \{u > t\})}{\mu(B(x,r))} = 0\right\}. \tag{2.2}$$

Unlike Newton-Sobolev functions, we understand BV functions to be μ-equivalence classes. To study fine properties, we need to consider the pointwise representatives $u^\wedge$ and $u^\vee$. By Lebesgue's differentiation theorem (see e.g. [18, Chapter 1]), for any $u \in L^1_{\mathrm{loc}}(X)$ we have $u = u^\wedge = u^\vee$ μ-almost everywhere.

The BV-capacity of a set $A \subset X$ is defined by

$$\mathrm{Cap}_{\mathrm{BV}}(A) := \inf \|u\|_{\mathrm{BV}(X)}, \tag{2.3}$$

where the infimum is taken over all $u \in \mathrm{BV}(X)$ with $u \geq 1$ in a neighborhood of A. The BV-capacity has the following useful continuity property (not satisfied by the 1-capacity): by [16, Theorem 3.4] we know that if $A_1 \subset A_2 \subset \ldots \subset X$, then

$$\mathrm{Cap}_{\mathrm{BV}}\left(\bigcup_{i=1}^\infty A_i\right) = \lim_{i\to\infty} \mathrm{Cap}_{\mathrm{BV}}(A_i). \tag{2.4}$$

On the other hand, by [16, Theorem 4.3] we know that for some constant $C(C_d, C_P, \lambda) \ge 1$ and any $A \subset X$, we have

$$\mathrm{Cap}_{\mathrm{BV}}(A) \le \mathrm{Cap}_1(A) \le C\,\mathrm{Cap}_{\mathrm{BV}}(A); \tag{2.5}$$

whenever we want to state that a constant C depends on the parameters $a, b, \ldots$, we write $C = C(a, b, \ldots)$.

By combining this with [16, Theorem 5.1], we get for any $A \subset X$ that

$$\mathrm{Cap}_{\mathrm{BV}}(A) = 0 \quad \text{iff} \quad \mathrm{Cap}_1(A) = 0 \quad \text{iff} \quad \mathcal{H}(A) = 0. \tag{2.6}$$

Next we define the fine topology in the case $p = 1$.

Definition 2.7. We say that $A \subset X$ is 1-thin at the point $x \in X$ if

$$\lim_{r\to 0} r \frac{\mathrm{cap}_1(A \cap B(x,r), B(x,2r))}{\mu(B(x,r))} = 0.$$

We say that a set $U \subset X$ is 1-finely open if $X \setminus U$ is 1-thin at every $x \in U$. Then we define the 1-fine topology as the collection of 1-finely open sets on X.

We denote the 1-fine interior of a set $D \subset X$, i.e. the largest 1-finely open set contained in D, by $\operatorname{fine-int} D$. We denote the 1-fine closure of D, i.e. the smallest 1-finely closed set containing D, by $\overline{D}^1$.

Finally, we define the *1-base* $b_1 D$ of a set $D \subset X$ as the set of points where D is 1-*thick*, i.e. not 1-thin.

See [26, Section 4] for a proof of the fact that the 1-fine topology is indeed a topology. A rather differently formulated notion of 1-thinness was previously given in the weighted Euclidean setting in [41].

The support of a (μ-almost everywhere defined) function u on X is the closed set

$$\operatorname{spt} u := \{x \in X : \mu(\{u \neq 0\} \cap B(x,r)) > 0 \text{ for all } r > 0\}.$$

For an open set $\Omega \subset X$, we denote by $\mathrm{BV}_c(\Omega)$ the class of functions $\varphi \in \mathrm{BV}(\Omega)$ with compact support in Ω, that is, $\operatorname{spt}\varphi \Subset \Omega$.

Definition 2.8. Let $\Omega \subset X$ be an open set. We say that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ is a 1-minimizer in Ω if for all $\varphi \in \mathrm{BV}_c(\Omega)$,

$$\|Du\|(\operatorname{spt}\varphi) \le \|D(u+\varphi)\|(\operatorname{spt}\varphi). \tag{2.9}$$

We say that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ is a 1-superminimizer in Ω if (2.9) holds for all nonnegative $\varphi \in \mathrm{BV}_c(\Omega)$.

In the literature, 1-minimizers are usually called functions of least gradient.

3 Subsets of finite Hausdorff measure

It is a well known problem to find a subset of strictly positive but finite Hausdorff measure from a set of infinite Hausdorff measure, see [22, 29, 36]. We will need such a property when proving the fine Kellogg property, but unfortunately the existing results do not seem to directly apply to the codimension 1 Hausdorff measure $\mathcal{H}$. The reason is that the quantity $\mu(B(x,r))/r$ is not necessarily increasing with respect to r, which is usually taken as a standard assumption. Thus in this section we study the existence of subsets of finite Hausdorff measure $\mathcal{H}$. In doing this, we are nonetheless able to follow almost directly the argument presented in [32, Chapter 8], which is based on [22].

For any Radon measure ν on X (we understand measures to be positive) and $R > 0$, define the maximal function

$$\mathcal{M}_R\nu(x) := \sup_{0<r\le R} r\frac{\nu(B(x,r))}{\mu(B(x,r))}, \quad x \in X.$$

First we need a version of Frostman's lemma, which fortunately has been proved also for codimension Hausdorff measures — the following is a special case of [30, Theorem 6.1]. Note that we can always understand Radon measures on open (or more generally Borel) sets to be defined on the whole space X.

Theorem 3.1. *Let $W \subset X$ be an open set and $R > 0$. Then there exists a Radon measure ν on W such that $\mathcal{M}_R\nu \le 1$ on X and $\mathcal{H}_{10R}(W) \le C_F\nu(W)$ for some constant $C_F = C_F(C_d)$.*

We easily get the following version for compact sets.

Theorem 3.2. *Let $K \subset X$ be compact and let $R > 0$. Then there exists a Radon measure ν on K such that $\mathcal{M}_R\nu \le 1$ on X and $\mathcal{H}_{10R}(K) \le C_F\nu(K)$.*

Proof. For any $a > 0$, let $K^a := \{x \in X : \operatorname{dist}(x,K) < a\}$. For each open set $W_i := K^{1/i}$, $i \in \mathbb{N}$, apply Theorem 3.1 to obtain a Radon measure ν_i on W_i such that $\mathcal{M}_R\nu_i \le 1$ on X and $\mathcal{H}_{10R}(W_i) \le C_F\nu_i(W_i)$. From the condition $\mathcal{M}_R\nu_i \le 1$ and the compactness of K it easily follows that $\nu_i(X)$ is a bounded sequence, and so there exists a subsequence (not relabeled) and a Radon measure ν on X such that $\nu_i \overset{*}{\rightharpoonup} \nu$ on X (see e.g. [2, Theorem 1.59]). By lower semicontinuity in open sets under weak* convergence (see e.g. [2, Proposition 1.62]), for every $j \in \mathbb{N}$ we have

$$\nu(X \setminus \overline{W_j}) \le \liminf_{i\to\infty} \nu_i(X \setminus \overline{W_j}) = 0,$$

and it follows that $\nu(X \setminus K) = 0$, that is, ν is a Radon measure on K. Moreover, for any $x \in X$ and $0 < r \le R$,

$$r\frac{\nu(B(x,r))}{\mu(B(x,r))} \le \liminf_{i\to\infty} r\frac{\nu_i(B(x,r))}{\mu(B(x,r))} \le 1,$$

and so $\mathcal{M}_R\nu \le 1$ on X. Finally, by upper semicontinuity in compact sets,

$$\begin{aligned}\nu(K) = \nu(\overline{W_1}) \ge \limsup_{i\to\infty} \nu_i(\overline{W_1}) &= \limsup_{i\to\infty} \nu_i(W_i)\\ &\ge \limsup_{i\to\infty} \frac{\mathcal{H}_{10R}(W_i)}{C_F} \ge \frac{\mathcal{H}_{10R}(K)}{C_F}.\end{aligned}$$

□

Now we prove the existence of subsets of positive finite $\mathcal{H}$-measure. We make no attempt to give the most general possible result (one that might cover e.g. Hausdorff measures of a different codimension), but rather just prove a version that will suffice for our purposes.

Theorem 3.3. *Let $K \subset X$ be a compact set with $\mathcal{H}(\{x\}) = 0$ for all $x \in K$. Then*

$$\mathcal{H}(K) = \sup\{\mathcal{H}(K_0) :\, K_0 \subset K \textit{ is compact with } \mathcal{H}(K_0) < \infty\}.$$

Proof. If $\mathcal{H}(K) < \infty$, the result is obvious. Thus we can assume that $\mathcal{H}(K) = \infty$. By the compactness of K, for each $k \in \mathbb{N}$ we find a finite family of balls $\{B_{k,j} := B(x_{k,j}, 2^{-k})\}_{j=1}^{m_k}$, $m_k \in \mathbb{N}$, such that the balls $\frac{1}{2}B_{k,j} = B(x_{k,j}, 2^{-k-1})$ cover K. Let

$$\mathcal{B} := \{B_{k,j} :\, k \in \mathbb{N},\, j \in \{1, \ldots, m_k\}\}.$$

Fix $0 < M < \infty$. Since $\mathcal{H}(K) = \infty$, we find and fix $0 < \delta < 1/2$ such that $\mathcal{H}_{10\delta}(K) > M$. By Theorem 3.2 we find a Radon measure ν_0 on K such that

$\mathcal{M}_\delta \nu_0 \le 1$ on X and $\mathcal{H}_{10\delta}(K) \le C_F \nu_0(K)$. For any ball $B = B(x,r)$, denote $\operatorname{rad}(B) := r$. Let $\mathcal{F}_\delta$ be the set of all Radon measures ν on K for which

$$\operatorname{rad}(B)\frac{\nu(B)}{\mu(B)} \le 1 \quad \text{for all } B \in \mathcal{B} \text{ with } \operatorname{rad}(B) \le \delta.$$

Let

$$h := \sup\{\nu(K) : \nu \in \mathcal{F}_\delta\}$$

and

$$\mathcal{G}_\delta := \{\nu \in \mathcal{F}_\delta : \nu(K) = h\}.$$

Then $h > M/C_F$, since $\nu_0 \in \mathcal{F}_\delta$. Pick a sequence $(\nu_i) \subset \mathcal{F}_\delta$ such that $\nu_i(K) \to h$. As $\nu_i(X)$ is clearly bounded (by the definition of $\mathcal{F}_\delta$ and the compactness of K), we find a subsequence (not relabeled) such that $\nu_i \overset{*}{\rightharpoonup} \widetilde{\nu}$ for some Radon measure $\widetilde{\nu}$ on X. By lower semicontinuity in open sets under weak* convergence,

$$\widetilde{\nu}(X \setminus K) \le \liminf_{i\to\infty} \nu_i(X \setminus K) = 0,$$

and for any $B \in \mathcal{B}$ with $\operatorname{rad}(B) \le \delta$,

$$\operatorname{rad}(B)\frac{\widetilde{\nu}(B)}{\mu(B)} \le \liminf_{i\to\infty} \operatorname{rad}(B)\frac{\nu_i(B)}{\mu(B)} \le 1,$$

and so $\widetilde{\nu} \in \mathcal{F}_\delta$. By upper semicontinuity in compact sets,

$$\widetilde{\nu}(K) \ge \limsup_{i\to\infty} \nu_i(K) = h,$$

and so necessarily $\widetilde{\nu}(K) = h$. Thus $\mathcal{G}_\delta$ is nonempty, it is easily seen to be convex, and by using the properties of weak* convergence as above it can be verified that $\mathcal{G}_\delta$ is compact with respect to the weak* topology. Then by the Krein-Milman theorem, see e.g. [37, Theorem 3.23], there exists an extreme point $\nu \in \mathcal{G}_\delta$; this means that if $\nu = t\nu_1 + (1-t)\nu_2$ for some $0 < t < 1$ and $\nu_1, \nu_2 \in \mathcal{G}_\delta$, then $\nu = \nu_1 = \nu_2$.

Fix $0 < \varepsilon < \delta$. Define

$$D_\varepsilon := \bigcup \left\{B \in \mathcal{B} : \operatorname{rad}(B) \le \varepsilon \text{ and } 2\nu(B) > \mu(B)/\operatorname{rad}(B)\right\}.$$

We claim that $\nu(K \setminus D_\varepsilon) = 0$.

Suppose that $\nu(K \setminus D_\varepsilon) > 0$ instead. Let $B_1, \ldots, B_m$ be all the balls in $\mathcal{B}$ with radius is strictly greater than ε. Define inductively for $i = 1, \ldots, m$,

$$\begin{aligned} A_1 &:= K \setminus D_\varepsilon, \\ A_{i+1} &:= A_i \setminus B_i \quad \text{if } \nu(A_i \setminus B_i) \geq \nu(A_i \cap B_i), \\ A_{i+1} &:= A_i \cap B_i \quad \text{if } \nu(A_i \setminus B_i) < \nu(A_i \cap B_i). \end{aligned}$$

Let $A := A_{m+1}$. Then $A \subset K$ is a Borel set and $\nu(A) > 0$. Moreover, either $A \subset B$ or $A \cap B = \emptyset$ for every $B \in \mathcal{B}$ with $\mathrm{rad}(B) > \varepsilon$. Recall that $\mathcal{H}(\{x\}) = 0$ for all $x \in K$; then by the doubling property of μ, necessarily

$$\liminf_{r \to 0} \frac{\mu(B(x,r))}{r} = 0. \tag{3.4}$$

Now if $x \in K$, for all balls $B \in \mathcal{B}$, $B \ni x$ with $\mathrm{rad}(B) \leq \delta$ we have

$$\nu(\{x\}) \leq \nu(B) \leq \frac{\mu(B)}{\mathrm{rad}(B)},$$

where the last quantity can be made arbitrarily small by (3.4). We conclude that $\nu(\{x\}) = 0$ for all $x \in K$. Thus we can find disjoint Borel sets H_1 and H_2 such that

$$A = H_1 \cup H_2 \quad \text{and} \quad \nu(H_1) = \nu(H_2) = \tfrac{1}{2}\nu(A),$$

see e.g. [32, Lemma 8.20]. Now we define Radon measures ν_1 and ν_2 by

$$\begin{aligned} \nu_1(E) &:= 2\nu(E \cap H_1) + \nu(E \setminus A), \\ \nu_2(E) &:= 2\nu(E \cap H_2) + \nu(E \setminus A) \end{aligned}$$

for $E \subset K$. Then $\nu = (\nu_1 + \nu_2)/2$, $\nu_1(H_1) = 2\nu(H_1) > \nu(H_1)$, and $\nu_2(H_2) = 2\nu(H_2) > \nu(H_2)$. Thus by the extremality of ν, ν_1 and ν_2 cannot both belong to $\mathcal{G}_\delta$. Suppose $\nu_1 \notin \mathcal{G}_\delta$, the other case being treated analogously. Clearly $\nu_1(K) = \nu_2(K) = \nu(K) = h$. Thus $\nu_1 \notin \mathcal{F}_\delta$, that is, there exists $B \in \mathcal{B}$ with $\mathrm{rad}(B) \leq \delta$ such that $\nu_1(B) > \mu(B)/\mathrm{rad}(B)$. Then $\mathrm{rad}(B) \leq \varepsilon$ since otherwise either $A \subset B$ or $A \cap B = \emptyset$, and in both cases $\nu_1(B) = \nu(B) \leq \mu(B)/\mathrm{rad}(B)$. We have

$$2\nu(B) \geq 2\nu(B \cap H_1) + \nu(B \setminus A) = \nu_1(B) > \mu(B)/\mathrm{rad}(B),$$

whence $B \subset D_\varepsilon \subset X \setminus A$. Thus in total,

$$\frac{\mu(B)}{\operatorname{rad}(B)} < \nu_1(B) = \nu(B) \le \frac{\mu(B)}{\operatorname{rad}(B)},$$

a contradiction. Thus $\nu(K \setminus D_\varepsilon) = 0$.

Let $p \in \mathbb{N}$ such that $1/p < \delta$ and

$$D := \bigcap_{i=p}^{\infty} D_{1/i},$$

so that $\nu(K \setminus D) = 0$ by the claim. Fix a new $\varepsilon > 0$. For every $x \in D$ there exists $B \in \mathcal{B}$ such that $x \in B$, $\operatorname{rad}(B) \le \varepsilon/5$, and $2\nu(B) > \mu(B)/\operatorname{rad}(B)$. By the 5-covering theorem, we can extract a countable collection $\{B_i\}_{i=1}^{\infty}$ of pairwise disjoint balls such that the balls $5B_i$ cover D. Then

$$\mathcal{H}_\varepsilon(D) \le \sum_{i=1}^{\infty} \frac{\mu(5B_i)}{5\operatorname{rad}(B_i)} \le C_d^3 \sum_{i=1}^{\infty} \frac{\mu(B_i)}{\operatorname{rad}(B_i)} \le 2C_d^3 \sum_{i=1}^{\infty} \nu(B_i) \le 2C_d^3 \nu(K),$$

so that $\mathcal{H}(D) \le 2C_d^3\nu(K) < \infty$. Conversely, for any balls $\{\widetilde{B}_i\}_{i=1}^{\infty}$ with $\operatorname{rad}(\widetilde{B}_i) \le \delta/8$ covering $K \cap D$, such that $\widetilde{B}_i \cap K \ne \emptyset$ for all $i \in \mathbb{N}$, we find balls $\{B_i \in \mathcal{B}\}_{i=1}^{\infty}$ for which $B_i \supset \widetilde{B}_i$ and $4\operatorname{rad}(\widetilde{B}_i) \le \operatorname{rad}(B_i) \le 8\operatorname{rad}(\widetilde{B}_i)$ for all $i \in \mathbb{N}$, which gives

$$\begin{aligned} C_d^4 \sum_{i=1}^{\infty} \frac{\mu(\widetilde{B}_i)}{\operatorname{rad}(\widetilde{B}_i)} &\ge \sum_{i=1}^{\infty} \frac{\mu(B_i)}{\operatorname{rad}(B_i)} \\ &\ge \sum_{i=1}^{\infty} \nu(B_i) \ge \nu(K \cap D) = \nu(K) = h > M/C_F. \end{aligned}$$

Thus $M/(C_d^4 C_F) < \mathcal{H}(K \cap D) < \infty$. Then since $\mathcal{H}|_{K\cap D}$ is a Radon measure, we find a compact $K_0 \subset K \cap D$ such that $\mathcal{H}(K_0) > M/(C_d^4 C_F)$ (see e.g. [2, Proposition 1.43(i)]). This completes the proof since $0 < M < \infty$ was arbitrary. $\square$

4 The fine Kellogg property

In this section we prove the fine Kellogg property for $p = 1$; it states that an arbitrary set is 1-thick at 1-quasi-every point in the set. The proof will mostly consist of considerations of measurability, as well as exploiting the existence of subsets of finite Hausdorff measure.

Recall the definition of the BV-capacity from (2.3).

Lemma 4.1. *Let $D \subset X$. Then the functions*

$$x \mapsto r\frac{\mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,r))}{\mu(B(x,r))} \quad \text{for a fixed } r > 0,$$

as well as

$$x \mapsto \limsup_{r\to 0} r\frac{\mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,r))}{\mu(B(x,r))} \quad \text{and} \quad x \mapsto \liminf_{r\to 0} \frac{\mu(B(x,r))}{r}$$

are Borel measurable.

Proof. Fix $r > 0$. First we show that the function

$$x \mapsto \mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,r))$$

is lower semicontinuous. By using e.g. Lipschitz cutoff functions, we see that the function is finite for all $x \in X$. Fix $\varepsilon > 0$ and $x \in X$. By the continuity of $\mathrm{Cap}_{\mathrm{BV}}$ under increasing sequences of sets, see (2.4), we have for some $s < r$

$$\mathrm{Cap}_{\mathrm{BV}}(D\cap B(x,r)) \le \mathrm{Cap}_{\mathrm{BV}}(D\cap B(x,s))+\varepsilon \le \liminf_{y\to x} \mathrm{Cap}_{\mathrm{BV}}(D\cap B(y,r))+\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have shown lower semicontinuity. Similarly, it can be shown that

$$x \mapsto \mu(B(x,r))$$

is lower semicontinuous, and so we have shown that for a fixed $r > 0$, the function

$$x \mapsto r\frac{\mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,r))}{\mu(B(x,r))}$$

is Borel measurable. Moreover, again using the continuity of $\mathrm{Cap}_{\mathrm{BV}}$ under increasing sequences of sets we see that for any $x \in X$ and $r > 0$,

$$\sup_{0<s<r} s\frac{\mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,s))}{\mu(B(x,s))} = \sup_{\substack{0<s<r\\ s\in\mathbb{Q}}} s\frac{\mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,s))}{\mu(B(x,s))},$$

and so

$$x \mapsto \limsup_{r\to 0} r\frac{\mathrm{Cap}_{\mathrm{BV}}(D\cap B(x,r))}{\mu(B(x,r))} = \lim_{i\to\infty} \sup_{\substack{0<s<1/i\\ s\in\mathbb{Q}}} s\frac{\mathrm{Cap}_{\mathrm{BV}}(D\cap B(x,s))}{\mu(B(x,s))}$$

is Borel measurable. The Borel measurability of the third function is shown similarly. □

Lemma 4.2. *The set $A := \{x \in X : \mathcal{H}(\{x\}) > 0\}$ is Borel.*

Proof. It is easy to check that $\mathcal{H}(\{x\}) > 0$ if and only if

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{r} > 0.$$

Thus the result follows from Lemma 4.1. □

The BV-capacity has the following "Borel regularity".

Lemma 4.3. *Let $A \subset X$. Then there exists a Borel set $D \supset A$ such that $\mathrm{Cap}_{\mathrm{BV}}(D\cap B) = \mathrm{Cap}_{\mathrm{BV}}(A\cap B)$ for every ball $B = B(x,r)$.*

Proof. Take a countable dense set $\{x_j\}_{j=1}^{\infty}$ in X and let $\{B_k\}_{k=1}^{\infty}$ be the collection of all balls with center x_j for some j and a rational radius. Define the set function

$$\widetilde{\mathrm{Cap}}_{\mathrm{BV}}(H) := \sum_{k=1}^{\infty} 2^{-k}\frac{\mathrm{Cap}_{\mathrm{BV}}(H\cap B_k) - \mathrm{Cap}_{\mathrm{BV}}(A\cap B_k)}{\mathrm{Cap}_{\mathrm{BV}}(B_k)}, \quad H \supset A.$$

Let $\beta := \inf\{\widetilde{\mathrm{Cap}}_{\mathrm{BV}}(H) : H \supset A,\ H \text{ is Borel}\}$. If we take a sequence of Borel sets $D_i \supset A$ such that $\widetilde{\mathrm{Cap}}_{\mathrm{BV}}(D_i) < \beta + 1/i$, then clearly $D := \bigcap_{i=1}^{\infty} D_i$ is a Borel set satisfying $\widetilde{\mathrm{Cap}}_{\mathrm{BV}}(D) = \beta$.

We show that $\beta = 0$; suppose instead $\beta > 0$. Then for some $k \in \mathbb{N}$ the k:th term in the definition of $\widetilde{\mathrm{Cap}}_{\mathrm{BV}}(D)$ is nonzero. By the definition of the BV-capacity, we find an open set $W \supset A\cap B_k$ with $\mathrm{Cap}_{\mathrm{BV}}(W) < \mathrm{Cap}_{\mathrm{BV}}(D\cap B_k)$. Then defining $D_0 := D \cap ((X\setminus B_k)\cup W)$, we have $\widetilde{\mathrm{Cap}}_{\mathrm{BV}}(D_0) < \widetilde{\mathrm{Cap}}_{\mathrm{BV}}(D)$, a contradiction. Thus $\beta = 0$.

Now take $x \in X$ and $r > 0$. Let $\varepsilon > 0$. By (2.4), for some $s < r$ we have

$$\mathrm{Cap}_{\mathrm{BV}}(D\cap B(x,r)) \le \mathrm{Cap}_{\mathrm{BV}}(D\cap B(x,s)) + \varepsilon,$$

and then for some $k \in \mathbb{N}$ we have $B(x,s) \subset B_k \subset B(x,r)$. Then

$$\begin{aligned}
\mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,r)) &\le \mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,s)) + \varepsilon \\
&\le \mathrm{Cap}_{\mathrm{BV}}(D \cap B_k) + \varepsilon \\
&= \mathrm{Cap}_{\mathrm{BV}}(A \cap B_k) + \varepsilon \quad \text{since } \beta = 0 \\
&\le \mathrm{Cap}_{\mathrm{BV}}(A \cap B(x,r)) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have the result. □

Proposition 4.4 ([27, Proposition 4.5])**.** *Let $x \in X$ with $\mathrm{Cap}_1(\{x\}) > 0$. Then $\{x\}$ is 1-thick at x, that is, $x \in b_1\{x\}$.*

According to [3, Proposition 6.16], if $x \in X$, $0 < r < \frac{1}{8}\operatorname{diam} X$, and $A \subset B(x,r)$, then for some constant $C = C(C_d, C_P, \lambda)$,

$$\frac{\mathrm{Cap}_1(A)}{C(1+r)} \le \mathrm{cap}_1(A, B(x,2r)) \le 2\left(1 + \frac{1}{r}\right)\mathrm{Cap}_1(A). \tag{4.5}$$

Now we prove the existence of a thickness point in any set of nonzero 1-capacity, which will immediately imply the fine Kellogg property. In the proof below, the reason for using the BV-capacity is that it is continuous with respect to increasing sequences of sets, which in particular allowed us to prove the measurability results of Lemma 4.1 and Lemma 4.3.

Theorem 4.6. *Let $A \subset X$ with $\mathrm{Cap}_1(A) > 0$. Then there exists a point $x \in A \cap b_1 A$.*

Proof. Suppose $A \cap b_1 A = \emptyset$, so that

$$\lim_{r\to 0} r \frac{\mathrm{cap}_1(A \cap B(x,r), B(x,2r))}{\mu(B(x,r))} = 0$$

for all $x \in A$. By Proposition 4.4 and (2.6) we know that $\mathcal{H}(\{x\}) = 0$ for all $x \in A$. By (2.5), $\mathrm{Cap}_{\mathrm{BV}}(A) > 0$. By (4.5) and (2.5),

$$\lim_{r\to 0} r \frac{\mathrm{Cap}_{\mathrm{BV}}(A \cap B(x,r))}{\mu(B(x,r))} = 0$$

for all $x \in A$. By Lemma 4.3 we find a Borel set $D \supset A$ such that $\mathrm{Cap}_{\mathrm{BV}}(D \cap B) = \mathrm{Cap}_{\mathrm{BV}}(A \cap B)$ for every ball $B = B(x,r)$. Thus for all $x \in A$,

$$\lim_{r\to 0} r \frac{\mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,r))}{\mu(B(x,r))} = 0.$$

Then define

$$F := \left\{ x \in X : \lim_{r\to 0} r \frac{\mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,r))}{\mu(B(x,r))} = 0 \right\} \supset A,$$

which is a Borel set by Lemma 4.1. Since we had $\mathcal{H}(\{x\}) = 0$ for all $x \in A$, we can define

$$H := D \cap F \cap \{x \in X : \mathcal{H}(\{x\}) = 0\} \supset A,$$

which is a Borel set by Lemma 4.2. For every $x \in H$ we now have

$$\lim_{r\to 0} r \frac{\mathrm{Cap}_{\mathrm{BV}}(H \cap B(x,r))}{\mu(B(x,r))} \le \lim_{r\to 0} r \frac{\mathrm{Cap}_{\mathrm{BV}}(D \cap B(x,r))}{\mu(B(x,r))} = 0.$$

Moreover, $\mathrm{Cap}_{\mathrm{BV}}(H) > 0$ since $A \subset H$. (The point so far has simply been to pass from A to the Borel set H.)

By the fact that $\mathrm{Cap}_{\mathrm{BV}}$ is a Choquet capacity, see [16, Corollary 3.8], we find a compact set $K \subset H$ with $\mathrm{Cap}_{\mathrm{BV}}(K) > 0$. By (2.6), $\mathcal{H}(K) > 0$. Now by Theorem 3.3 we find a compact set $K_0 \subset K$ with $0 < \mathcal{H}(K_0) < \infty$ and then clearly for every $x \in K_0$,

$$\lim_{r\to 0} r \frac{\mathrm{Cap}_{\mathrm{BV}}(K_0 \cap B(x,r))}{\mu(B(x,r))} = 0.$$

By Egorov's theorem, which we can apply by Lemma 4.1, we find $K_1 \subset K_0$ with $0 < \mathcal{H}(K_1) < \infty$ such that

$$r \frac{\mathrm{Cap}_{\mathrm{BV}}(K_1 \cap B(x,r))}{\mu(B(x,r))} \to 0 \quad \text{as } r \to 0$$

uniformly for all $x \in K_1$. Fix $\varepsilon > 0$. For some $\delta > 0$, the above quantity is less than ε whenever $r < \delta$. We find a covering $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of K_1, with $r_i < \delta/2$, such that

$$\sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} < \mathcal{H}(K_1) + \varepsilon.$$

We can assume that there exists $y_i \in B(x_i, r_i) \cap K_1$ for all $i \in \mathbb{N}$, and then the balls $B(y_i, 2r_i)$ also cover K_1. Thus

$$\begin{aligned}\mathrm{Cap}_{\mathrm{BV}}(K_1) &\le \sum_{i=1}^{\infty} \mathrm{Cap}_{\mathrm{BV}}(K_1 \cap B(y_i, 2r_i)) \\ &\le \varepsilon \sum_{i=1}^{\infty} \frac{\mu(B(y_i, 2r_i))}{2r_i} \\ &\le \varepsilon \sum_{i=1}^{\infty} \frac{\mu(B(x_i, 4r_i))}{2r_i} < C_d^2 \varepsilon(\mathcal{H}(K_1) + \varepsilon).\end{aligned}$$

Letting $\varepsilon \to 0$, we get $\mathrm{Cap}_{\mathrm{BV}}(K_1) = 0$, so that $\mathcal{H}(K_1) = 0$ by (2.6), which is a contradiction. □

Now we get the following fine Kellogg property for $p = 1$. For the case $p > 1$, see [7, Corollary 1.3].

Corollary 4.7. *Let $A \subset X$. Then $\mathrm{Cap}_1(A \setminus b_1 A) = 0$.*

Proof. Let $F := A \setminus b_1 A$. Then clearly $b_1 F \subset b_1 A$, so that $F \cap b_1 F \subset b_1 F \setminus b_1 A = \emptyset$, and then by Theorem 4.6, $\mathrm{Cap}_1(F) = 0$. □

Remark 4.8. In this section we have not really studied or applied fine potential theory, but rather just basic properties of capacities and the measure-theoretic result of the previous section. By contrast, in the case $p > 1$, the fine Kellogg property is deduced from the Choquet property, which we only prove for $p = 1$ in Section 6. The kind of method we used in this section does not seem to be available in the case $p > 1$: we used the fact that $\mathcal{H}$ and Cap_1 have the same null sets but the analog of this is not true for $p > 1$.

5 The quasi-Lindelöf principle

In this section we prove the quasi-Lindelöf principle for the 1-fine topology, by using the fine Kellogg property.

We will need the following fact given in [3, Lemma 11.22].

Lemma 5.1. *Let $x \in X$, $r > 0$, and $A \subset B(x, r)$. Then for every $1 < s < t$ with $tr < \frac{1}{4} \operatorname{diam} X$, we have*

$$\mathrm{cap}_1(A, B(x, tr)) \le \mathrm{cap}_1(A, B(x, sr)) \le C_S \left(1 + \frac{t}{s-1}\right) \mathrm{cap}_1(A, B(x, tr)),$$

where $C_S = C_S(C_d, C_P, \lambda)$.

In proving the quasi-Lindelöf principle for $p = 1$, we follow the proof of [19, Theorem 2.3], where the property was shown for $p > 1$ (in the Euclidean setting and with slightly different definitions).

Theorem 5.2. *For every family $\mathcal{V}$ of 1-finely open sets there is a countable subfamily $\mathcal{V}'$ such that*

$$\mathrm{Cap}_1\left(\bigcup_{U\in\mathcal{V}} U \setminus \bigcup_{U'\in\mathcal{V}'} U'\right) = 0.$$

Proof. Take a countable dense set $\{x_j\}_{j=1}^\infty$ in X and let $\{B_k\}_{k=1}^\infty$ be the collection of all balls with center x_j for some j and a rational radius strictly less than $\frac{1}{16}\operatorname{diam} X$. Define the set function

$$\widetilde{\mathrm{Cap}}_1(A) := \sum_{k=1}^\infty 2^{-k}\frac{\mathrm{cap}_1(A\cap B_k, 2B_k)}{\mathrm{cap}_1(B_k, 2B_k)}, \quad A\subset X,$$

where the denominator is always strictly positive by (4.5). Take a collection of 1-finely open sets $\{U_i\}_{i\in\Lambda}$, and let $U := \bigcup_{i\in\Lambda} U_i$. Let

$$\beta := \inf\left\{\widetilde{\mathrm{Cap}}_1\left(U\setminus\bigcup_{i\in I} U_i\right) : I\subset\Lambda \text{ is countable}\right\}.$$

Choose countable sets $I_j\subset\Lambda$, $j\in\mathbb{N}$, such that

$$\widetilde{\mathrm{Cap}}_1\left(U\setminus\bigcup_{i\in I_j} U_i\right) < \beta + \frac{1}{j}.$$

Define the countable set

$$I_\infty := \bigcup_{j=1}^\infty I_j,$$

so that for

$$A := U\setminus\bigcup_{i\in I_\infty} U_i$$

we have $\widetilde{\mathrm{Cap}}_1(A) = \beta$. We show that $\beta = 0$; suppose instead $\beta > 0$. Then $\mathrm{Cap}_1(A) > 0$ by (4.5), and so by Theorem 4.6 there exists a point $x \in A\cap b_1 A$.

Choose $i \in \Lambda$ such that $x \in U_i$. Since $A \setminus U_i$ is 1-thin at x and A is 1-thick, we find $0 < r < \frac{1}{16} \operatorname{diam} X$ such that

$$r \frac{\operatorname{cap}_1((A \setminus U_i) \cap B(x,r), B(x,2r))}{\mu(B(x,r))} < \frac{1}{25 C_S^2 C_d} \frac{r}{2} \frac{\operatorname{cap}_1(A \cap B(x,r/2), B(x,r))}{\mu(B(x,r/2))},$$

and so

$$\operatorname{cap}_1((A \setminus U_i) \cap B(x,r), B(x,2r)) < \frac{1}{25 C_S^2} \operatorname{cap}_1(A \cap B(x,r/2), B(x,r)). \quad (5.3)$$

Then choose $k \in \mathbb{N}$ such that $B_k = B(y,s)$ with $d(y,x) < r/8$ and $3r/4 < s < 7r/8$. Now

$$\begin{aligned}
&\operatorname{cap}_1((A \setminus U_i) \cap B_k, 2B_k) \\
&\quad \le 5C_S \operatorname{cap}_1((A \setminus U_i) \cap B_k, 4B_k) \quad \text{by Lemma 5.1} \\
&\quad \le 5C_S \operatorname{cap}_1((A \setminus U_i) \cap B_k, B(x,2r)) \quad \text{since } B(x,2r) \subset 4B_k \\
&\quad \le 5C_S \operatorname{cap}_1((A \setminus U_i) \cap B(x,r), B(x,2r)) \quad \text{since } B_k \subset B(x,r) \\
&\quad < \frac{1}{5C_S} \operatorname{cap}_1(A \cap B(x,r/2), B(x,r)) \quad \text{by (5.3)} \\
&\quad \le \operatorname{cap}_1(A \cap B(x,r/2), B(x,2r)) \\
&\quad \le \operatorname{cap}_1(A \cap B(x,r/2), 2B_k) \\
&\quad \le \operatorname{cap}_1(A \cap B_k, 2B_k).
\end{aligned}$$

It follows that $\widetilde{\operatorname{Cap}}_1(A \setminus U_i) < \widetilde{\operatorname{Cap}}_1(A) = \beta$, a contradiction. Hence $\beta = 0$ and so $\widetilde{\operatorname{Cap}}_1\left(U \setminus \bigcup_{i \in I_\infty} U_i\right) = 0$. Then by (4.5) we see that $\operatorname{Cap}_1\left(U \setminus \bigcup_{i \in I_\infty} U_i\right) = 0$. □

6 The Choquet property

In this section we prove the fact that 1-finely open sets are 1-quasiopen, and then we prove the Choquet property for the 1-fine topology. To achieve these, we use the weak Cartan property proved in [27], as well as the fine Kellogg property and the quasi-Lindelöf principle of the previous sections.

Recall that a set $U \subset X$ is 1-quasiopen if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\operatorname{Cap}_1(G) < \varepsilon$ and $U \cup G$ is open. Quasiopen sets have the following stability.

Lemma 6.1. *Let $U \subset X$ be a 1-quasiopen set and let $A \subset X$ be $\mathcal{H}$-negligible. Then $U \setminus A$ and $U \cup A$ are 1-quasiopen sets.*

Proof. Let $\varepsilon > 0$. Take an open set $G \subset X$ such that $\mathrm{Cap}_1(G) < \varepsilon$ and $U \cup G$ is an open set. By (2.6) we know that $\mathrm{Cap}_1(A) = 0$, and since Cap_1 is an outer capacity, we find an open set $W \supset A$ such that $\mathrm{Cap}_1(G)+\mathrm{Cap}_1(W) < \varepsilon$. Now $(U \setminus A) \cup (G \cup W) = (U \cup G) \cup W$ is an open set with $\mathrm{Cap}_1(G \cup W) < \varepsilon$, so that $U \setminus A$ is a 1-quasiopen set. Similarly, $(U \cup A) \cup (G \cup W) = (U \cup G) \cup W$ is an open set, so that $U \cup A$ is also a 1-quasiopen set. □

The following fact about 1-finely open and 1-quasiopen sets is previously known.

Proposition 6.2 ([25, Proposition 4.3])**.** *Every 1-quasiopen set is the union of a 1-finely open set and a $\mathcal{H}$-negligible set.*

Lemma 6.3. *Let $V_0 \subset X$ be 1-quasiopen and let $x \in V_0 \setminus b_1(X \setminus V_0)$, that is, $X \setminus V_0$ is 1-thin at x. Then there exists a 1-finely open and 1-quasiopen set V such that $x \in V \subset V_0$.*

Proof. By Proposition 6.2, $V_0 = W \cup N$ where $\mathcal{H}(N) = 0$ and W is 1-finely open, and W is also 1-quasiopen by Lemma 6.1. Let $V := W \cup \{x\}$. If $\mathrm{Cap}_1(\{x\}) > 0$, necessarily $x \in W$ and so $V = W$ is 1-finely open and 1-quasiopen. Otherwise V is also 1-quasiopen by Lemma 6.1. It is 1-finely open since

$$B(x,r) \setminus V \subset B(x,r) \setminus W \subset (B(x,r) \setminus V_0) \cup (B(x,r) \cap N)$$

for all $r > 0$, and so

$$\begin{aligned}
\limsup_{r\to 0} r\frac{\mathrm{cap}_1(B(x,r) \setminus V, B(x,2r))}{\mu(B(x,r))}
&\le \limsup_{r\to 0} r\frac{\mathrm{cap}_1((B(x,r) \setminus V_0) \cup (B(x,r) \cap N), B(x,2r))}{\mu(B(x,r))} \\
&= \limsup_{r\to 0} r\frac{\mathrm{cap}_1(B(x,r) \setminus V_0, B(x,2r))}{\mu(B(x,r))} \quad \text{by (2.6) and (4.5)} \\
&= 0
\end{aligned}$$

since $x \notin b_1(X \setminus V_0)$. □

Recall the definitions of the lower and upper approximate limits $u^\wedge$ and $u^\vee$ from (2.1) and (2.2). From [28, Theorem 1.1] we get the following result, which was proved earlier in the Euclidean setting in [12, Theorem 2.5].

Proposition 6.4. *Let $u \in \mathrm{BV}(X)$ and $\varepsilon > 0$. Then there exists an open set $G \subset X$ such that $\mathrm{Cap}_1(G) < \varepsilon$ and $u^\wedge|_{X\setminus G}$ is real-valued lower semicontinuous.*

It is perhaps a curious fact that only now we need to talk about minimizers for the first time; recall the definitions of 1-minimizers and 1-superminimizers from Definition 2.8.

Theorem 6.5 ([27, Theorem 3.11])**.** *Let u be a 1-superminimizer in an open set $\Omega \subset X$. Then $u^\wedge \colon \Omega \to (-\infty, \infty]$ is lower semicontinuous.*

We have the following weak Cartan property.

Theorem 6.6 ([27, Theorem 5.2])**.** *Let $A \subset X$ and let $x \in X \setminus A$ be such that A is 1-thin at x. Then there exist $R > 0$ and $E_0, E_1 \subset X$ such that $\chi_{E_0}, \chi_{E_1} \in \mathrm{BV}(X)$, χ_{E_0} and χ_{E_1} are 1-superminimizers in $B(x,R)$, $\max\{\chi_{E_0}^\wedge, \chi_{E_1}^\wedge\} = 1$ in $A \cap B(x,R)$, $\{\max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} > 0\}$ is 1-thin at x, and*

$$\lim_{r\to 0} r\frac{P(E_0, B(x,r))}{\mu(B(x,r))} = 0, \qquad \lim_{r\to 0} r\frac{P(E_1, B(x,r))}{\mu(B(x,r))} = 0.$$

Now we can prove a result that is an analog of the existence of *p-strict subsets* for $p > 1$, see [6, Lemma 3.3]. In fact, later we will only need the existence of the sets V given in the proposition below, and not the functions v.

Proposition 6.7. *Let $U \subset X$ be 1-finely open and let $x \in U$. Then there exists a 1-finely open and 1-quasiopen set V such that $x \in V \subset U$, and a function $w \in \mathrm{BV}(X)$ such that $0 \le w \le 1$ on X, $w^\wedge = 1$ on V, and $\mathrm{spt}\, w \Subset U$.*

Proof. The set $X \setminus U$ is 1-thin at x. Take $R > 0$ and $E_0, E_1 \subset X$ as given by Theorem 6.6 with the choice $A = X \setminus U$. Let $u := \max\{\chi_{E_0}^\wedge, \chi_{E_1}^\wedge\} \in \mathrm{BV}(X)$ (here we understand u to be pointwise defined). Now $0 \le u \le 1$, u is lower semicontinuous in $B(x,R)$ by Theorem 6.5, $u = 1$ on $B(x,R) \setminus U$, and $u^\vee(x) = 0$ since the set $\{\max\{\chi_{E_0}^\wedge, \chi_{E_1}^\wedge\} > 0\} \subset \{\max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} > 0\}$ is

1-thin at x, and so it also has zero measure density at x, see [25, Lemma 3.1]. Moreover, $\{u^\vee > 0\} \subset \{\max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} > 0\}$ and so

$$\lim_{r\to 0} r \frac{\mathrm{cap}_1(\{u^\vee > 0\} \cap B(x,r), B(x,2r))}{\mu(B(x,r))} = 0. \tag{6.8}$$

Let $\eta \in \mathrm{Lip}_c(B(x,R))$ such that $0 \le \eta \le 1$ on X and $\eta = 1$ on $B(x,R/2)$. Let $v := \eta(1-u) \in \mathrm{BV}(X)$ (still understood to be pointwise defined). Then $0 \le v \le 1$, v is upper semicontinuous on X, $v = 0$ on $X \setminus U$, and $v^\wedge(x) = 1$. Moreover, by (6.8),

$$\lim_{r\to 0} r \frac{\mathrm{cap}_1(\{v^\wedge < 1\} \cap B(x,r), B(x,2r))}{\mu(B(x,r))} = 0. \tag{6.9}$$

The set $V_0 := \{v^\wedge > 1/2\}$ contains x and is 1-quasiopen by Proposition 6.4. Moreover, $x \notin b_1(X \setminus V_0)$ by (6.9). By Lemma 6.3 we find a 1-finely open and 1-quasiopen set V such that $x \in V \subset V_0$. Let

$$w(\cdot) := \min\left\{1, (4v(\cdot) - 1)_+\right\} \in \mathrm{BV}(X)$$

(now understood in the usual sense of a BV function, i.e. as a μ-equivalence class). Then $0 \le w \le 1$ and $w^\wedge = 1$ on V (since it is a subset of V_0). Moreover, $\mathrm{spt}\, w \subset \{v \ge 1/4\}$ by the upper semicontinuity of v. Thus $\mathrm{spt}\, w \Subset U$, and this also guarantees that $V \subset U$. □

Remark 6.10. The above proof would be somewhat more straightforward if we knew that the set V_0 itself was 1-finely open; then we would not need Lemma 6.3. This would be the case if the superminimizer functions $\chi_{E_0}^\vee$ and $\chi_{E_1}^\vee$ were upper semicontinuous with respect to the 1-fine topology, but in general they are not, see [27, Example 5.8]. This is in contrast with the case $p > 1$, where the p-fine topology makes all p-superharmonic functions continuous. However, the fact that $\{\max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} > 0\}$ is 1-thin at x means that $\chi_{E_0}^\vee$ and $\chi_{E_1}^\vee$ are 1-finely continuous at the point x, and this was enough for the proof to run through.

Theorem 6.11. *Every 1-finely open set is 1-quasiopen.*

Proof. Let $U \subset X$ be 1-finely open. For every $x \in U$, by Proposition 6.7 we find a 1-finely open and 1-quasiopen set V_x such that $x \in V_x \subset U$. By the quasi-Lindelöf principle (Theorem 5.2) we find a countable subcollection $\{V_i\}_{i=1}^\infty$ and a Cap_1-negligible set $N \subset U$ such that $U = \bigcup_{i=1}^\infty V_i \cup N$. The set N is 1-quasiopen since Cap_1 is an outer capacity, and since a countable union of 1-quasiopen sets is easily seen to be 1-quasiopen, U is 1-quasiopen. □

Combining this with Proposition 6.2 and Lemma 6.1, we have a characterization of 1-quasiopen and 1-finely open sets by means of each other.

Corollary 6.12. *A set $U \subset X$ is 1-quasiopen if and only if it is the union of a 1-finely open set and a $\mathcal{H}$-negligible set.*

Proposition 6.13. *If $A \Subset D \subset X$, then*

$$\operatorname{cap}_1(A, D) = \operatorname{cap}_1(\overline{A}^1, D).$$

Proof. By [25, Proposition 3.3], we have

$$\operatorname{cap}_1(A, D) = \operatorname{cap}_1(\overline{A}^1 \cap D, D).$$

Since clearly $\overline{A}^1 \subset \overline{A} \subset D$, we have the result. □

Now we prove the Choquet property for $p = 1$; for the case $1 < p < \infty$ see [7, Theorem 1.2].

Theorem 6.14. *Let $A \subset X$ and let $\varepsilon > 0$. Then there exists an open set $W \subset X$ such that $W \cup b_1 A = X$ and $\operatorname{Cap}_1(W \cap A) < \varepsilon$.*

Proof. Let $x \in X \setminus b_1 A$. Clearly for all $r > 0$,

$$b_1 A \cap B(x, r) \subset b_1(A \cap B(x, r)) \subset \overline{A \cap B(x, r)}^1,$$

and so by Proposition 6.13,

$$\begin{aligned}
&\limsup_{r\to 0} r \frac{\operatorname{cap}_1(b_1 A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} \\
&\quad \le \limsup_{r\to 0} r \frac{\operatorname{cap}_1(\overline{A \cap B(x, r)}^1, B(x, 2r))}{\mu(B(x, r))} \\
&\quad = \limsup_{r\to 0} r \frac{\operatorname{cap}_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0.
\end{aligned}$$

Thus $X \setminus b_1 A$ is a 1-finely open set, and then it is also 1-quasiopen by Theorem 6.11. Take an open $G \subset X$ such that $\operatorname{Cap}_1(G) < \varepsilon$ and $(X \setminus b_1 A) \cup G =: W$ is open. Now $W \cup b_1 A = X$, and by the fine Kellogg property (Corollary 4.7),

$$\operatorname{Cap}_1(W \cap A) \le \operatorname{Cap}_1((A \setminus b_1 A) \cup G) = \operatorname{Cap}_1(G) < \varepsilon.$$

□

Remark 6.15. In the case $p > 1$, one seems to need a strong version of the Cartan property (involving only one superminimizer function, instead of two) to deduce the Choquet property; see [7, 8]. For $p = 1$ we do not know whether such a Cartan property holds, though a result in that vein was given in the proof of [25, Proposition 5.8]. However, the weak Cartan property is enough for proving the existence of 1-strict subsets as in Proposition 6.7, and this combined with the quasi-Lindelöf principle and the fine Kellogg property is then enough for proving the Choquet property. However, the same would not work in the case $p > 1$, since there the Choquet property is needed for proving the fine Kellogg property.

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References

[1] D. Adams and L. I. Hedberg, *Function spaces and potential theory*, Grundlehren der Mathematischen Wissenschaften, 314. Springer-Verlag, Berlin, 1996. xii+366 pp.

[2] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[3] A. Björn and J. Björn, *Nonlinear potential theory on metric spaces*, EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich, 2011. xii+403 pp.

[4] A. Björn and J. Björn, *Obstacle and Dirichlet problems on arbitrary nonopen sets in metric spaces, and fine topology*, Rev. Mat. Iberoam. 31 (2015), no. 1, 161–214.

[5] A. Björn and J. Björn, *The variational capacity with respect to nonopen sets in metric spaces*, Potential Anal. 40 (2014), no. 1, 57–80.

[6] A. Björn, J. Björn, and V. Latvala, *Sobolev spaces, fine gradients and quasicontinuity on quasiopen sets*, Ann. Acad. Sci. Fenn. Math. 41 (2016), no. 2, 551–560.

[7] A. Björn, J. Björn, and V. Latvala, *The Cartan, Choquet and Kellogg properties for the fine topology on metric spaces*, J. Anal. Math. 135 (2018), no. 1, 59–83.

[8] A. Björn, J. Björn, and V. Latvala, *The weak Cartan property for the p-fine topology on metric spaces*, Indiana Univ. Math. J. 64 (2015), no. 3, 915–941.

[9] A. Björn, J. Björn, and N. Shanmugalingam, *The Dirichlet problem for p-harmonic functions on metric spaces*, J. Reine Angew. Math. 556 (2003), 173–203.

[10] A. Björn, J. Björn, and N. Shanmugalingam, *The Dirichlet problem for p-harmonic functions with respect to the Mazurkiewicz boundary, and new capacities*, J. Differential Equations 259 (2015), no. 7, 3078–3114.

[11] E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. 7 1969 243–268.

[12] M. Carriero, G. Dal Maso, A. Leaci, and E. Pascali, *Relaxation of the nonparametric plateau problem with an obstacle*, J. Math. Pures Appl. (9) 67 (1988), no. 4, 359–396.

[13] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics series, CRC Press, Boca Raton, 1992.

[14] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969 xiv+676 pp.

[15] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984. xii+240 pp.

[16] H. Hakkarainen and J. Kinnunen, *The BV-capacity in metric spaces*, Manuscripta Math. 132 (2010), no. 1-2, 51–73.

[17] H. Hakkarainen, R. Korte, P. Lahti, and N. Shanmugalingam, *Stability and continuity of functions of least gradient*, Anal. Geom. Metr. Spaces 3 (2015), 123–139.

[18] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext. Springer-Verlag, New York, 2001. x+140 pp.

[19] J. Heinonen, T. Kilpeläinen, and J. Malý, *Connectedness in fine topologies*, Ann. Acad. Sci. Fenn. Ser. A I Math. 15 (1990), no. 1, 107–123.

[20] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, 2006. xii+404 pp.

[21] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. 181 (1998), no. 1, 1–61.

[22] J. D. Howroyd, *On dimension and on the existence of sets of finite positive Hausdorff measure*, Proc. London Math. Soc. (3) 70 (1995), no. 3, 581–604.

[23] J. Kinnunen and O. Martio, *Nonlinear potential theory on metric spaces*, Illinois J. Math. 46 (2002), no. 3, 857–883.

[24] R. Korte, P. Lahti, X. Li, and N. Shanmugalingam, *Notions of Dirichlet problem for functions of least gradient in metric measure spaces*, to appear in Revista Matemática Iberoamericana.

[25] P. Lahti, *A Federer-style characterization of sets of finite perimeter on metric spaces*, Calc. Var. Partial Differential Equations 56 (2017), no. 5, 56:150.

[26] P. Lahti, *A notion of fine continuity for BV functions on metric spaces*, Potential Anal. 46 (2017), no. 2, 279–294.

[27] P. Lahti, *Superminimizers and a weak Cartan property for $p = 1$ in metric spaces*, to appear in Journal d'Analyse Mathématique.

ACCEPTED MANUSCRIPT

[28] P. Lahti and N. Shanmugalingam, *Fine properties and a notion of quasicontinuity for* BV *functions on metric spaces*, Journal de Mathématiques Pures et Appliquées, Volume 107, Issue 2, February 2017, Pages 150–182.

[29] D. G. Larman, *On Hausdorff measure in finite-dimensional compact metric spaces*, Proc. London Math. Soc. (3) 17 1967 193–206.

[30] J. Malý, *Coarea integration in metric spaces*, NAFSA 7–Nonlinear analysis, function spaces and applications. Vol. 7, 148–192, Czech. Acad. Sci., Prague, 2003.

[31] J. Malý and W. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997. xiv+291 pp.

[32] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995. xii+343 pp.

[33] J. M. Mazón, J. D. Rossi, and S. Segura de León, *Functions of least gradient and* 1*-harmonic functions*, Indiana Univ. Math. J. 63 No. 4 (2014), 1067–1084.

[34] A. Mercaldo, S. Segura de León, and C. Trombetti, *On the solutions to* 1*-Laplacian equation with* L^1 *data*, J. Funct. Anal. 256 (2009), no. 8, 2387–2416.

[35] M. Miranda, Jr., *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) 82 (2003), no. 8, 975–1004.

[36] C. A. Rogers, *Hausdorff measures*, Cambridge University Press, London-New York 1970 viii+179 pp.

[37] W. Rudin, *Functional analysis. Second edition*, International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991. xviii+424 pp.

[38] N. Shanmugalingam, *Harmonic functions on metric spaces*, Illinois J. Math. 45 (2001), no. 3, 1021–1050.

[39] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana 16(2) (2000), 243–279.

[40] P. Sternberg, G. Williams, and W. Ziemer, *Existence, uniqueness, and regularity for functions of least gradient*, J. Reine Angew. Math. 430 (1992), 35–60.

[41] B. O. Turesson, *Nonlinear potential theory and weighted Sobolev spaces*, Lecture Notes in Mathematics, 1736. Springer-Verlag, Berlin, 2000. xiv+173 pp.

[42] W. P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.

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