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A NEW CARTAN-TYPE PROPERTY AND STRICT QUASICOVERINGS WHEN $p = 1$ IN METRIC SPACES

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Abstract. In a complete metric space that is equipped with a doubling measure and supports a Poincaré inequality, we prove a new Cartan-type property for the fine topology in the case $p = 1$. Then we use this property to prove the existence of 1-finely open *strict subsets* and *strict quasicoverings* of 1-finely open sets. As an application, we study fine Newton–Sobolev spaces in the case $p = 1$, that is, Newton–Sobolev spaces defined on 1-finely open sets.

1. Introduction

Nonlinear fine potential theory in metric spaces has been studied in several papers in recent years, see [7, 8, 6]. Much of nonlinear potential theory, for $1 < p < \infty$, deals with p-harmonic functions, which are local minimizers of the L^p-norm of $|\nabla u|$. Such minimizers can be defined also in metric measure spaces by using *upper gradients*, and the notion can be extended to the case $p = 1$ by considering functions of least gradient, which are BV functions that minimize the total variation locally; see Section 2 for definitions.

Nonlinear fine potential theory is concerned with studying p-harmonic functions and related superminimizers by means of the *p-fine topology*. For nonlinear fine potential theory and its history in the Euclidean setting, for $1 < p < \infty$, see especially the monographs [1, 15, 23], as well as the monograph [3] in the metric setting. The typical assumptions of a metric space, which we make also in this paper, are that the space is complete, equipped with a doubling measure, and supports a Poincaré inequality.

A central result in fine potential theory is the *(weak) Cartan property* for superminimizer functions. In [21] we proved the following formulation of this property in the case $p = 1$.

Theorem 1.1. [21, Theorem 1.1] *Let $A \subset X$ and let $x \in X \setminus A$ such that A is 1-thin at x. Then there exist $R > 0$ and $u_1, u_2 \in \mathrm{BV}(X)$ that are 1-superminimizers in $B(x, R)$ such that $\max\{u_1^\wedge, u_2^\wedge\} = 1$ in $A \cap B(x, R)$ and $u_1^\vee(x) = 0 = u_2^\vee(x)$.*

In [22] we used this property to prove the so-called Choquet property concerning finely open and *quasiopen* sets in the case $p = 1$, similarly as can be done when $1 < p < \infty$ (see [8]). On the other hand, it is natural to consider an alternative version of the weak Cartan property. In the case $p > 1$, superminimizers are Newton–Sobolev functions, but in the case $p = 1$ they are only BV functions and so the question arises whether the functions u_1, u_2 above can be replaced by a Newton–Sobolev function

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(even though it would no longer be a superminimizer). In Theorem 3.11 we show that such a new Cartan-type property indeed holds.

It is said that a set A is a *p-strict* subset of a set D if there exists a Newton–Sobolev function $u \in N^{1,p}(X)$ such that $u = 1$ on A and $u = 0$ on $X \setminus D$. In [7] it was shown that if U is a p-finely open set $(1 < p < \infty)$ and $x \in U$, then there exists a p-finely open strict subset $V \Subset U$ such that $x \in V$. The proof was based on the weak Cartan property. In Theorem 4.3 we show that the analogous result is true in the case $p = 1$. Here we need the Cartan-type property involving a Newton–Sobolev function (instead of the BV superminimizer functions).

This result on the existence of 1-strict subsets can be combined with the *quasi-Lindelöf principle* to prove the existence of *strict quasicoverings* of 1-finely open sets, that is, countable coverings by 1-finely open strict subsets. We do this in Proposition 5.4, and it is again analogous to the case $1 < p < \infty$, see [7]. Such coverings will be useful in future research when considering partition of unity arguments in finely open sets. In this paper, we apply strict quasicoverings in defining and studying *fine Newton–Sobolev spaces*, that is, Newton–Sobolev spaces defined on finely open or quasiopen sets. In the case $1 < p < \infty$, these were studied in [7]. In Section 5 we show that the theory we have developed allows us to prove directly analogous results in the case $p = 1$.

2. Preliminaries

In this section we introduce the notation, definitions, and assumptions used in the paper. Throughout this paper, (X, d, μ) is a complete metric space that is equipped with a metric d and a Borel regular outer measure μ that satisfies a doubling property, meaning that there exists a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B(x, r) := \{y \in X : d(y, x) < r\}$. We also assume that X supports a $(1, 1)$-Poincaré inequality defined below, and that X contains at least 2 points. For a ball $B = B(x, r)$ and $a > 0$, we sometimes abbreviate $aB := B(x, ar)$; note that in metric spaces, a ball (as a set) does not necessarily have a unique center and radius, but we will always understand these to be predetermined for the balls that we consider. By iterating the doubling condition, we obtain for any $x \in X$ and any $y \in B(x, R)$ with $0 < r \leq R < \infty$ that

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq \frac{1}{C_d^2} \left(\frac{r}{R} \right)^Q, \tag{2.1}$$

where $Q > 1$ only depends on the doubling constant C_d. When we want to state that a constant C depends on the parameters $a, b, \ldots$, we write $C = C(a, b, \ldots)$. When a property holds outside a set of μ-measure zero, we say that it holds almost everywhere, abbreviated a.e.

As a complete metric space equipped with a doubling measure, X is proper, that is, closed and bounded sets are compact. For any μ-measurable set $D \subset X$, we define $\mathrm{Lip}_{\mathrm{loc}}(D)$ to be the space of functions u on D such that for every $x \in D$ there exists $r > 0$ such that $u \in \mathrm{Lip}(D \cap B(x, r))$. For an open set $\Omega \subset X$, a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is then in $\mathrm{Lip}(\Omega')$ for every open $\Omega' \Subset \Omega$; this notation means that $\overline{\Omega'}$ is a compact subset of Ω. Other local spaces of functions are defined analogously.

For any $A \subset X$ and $0 < R < \infty$, the restricted Hausdorff content of codimension one is defined to be

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \colon A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \le R \right\}.$$

The codimension one Hausdorff measure of $A \subset X$ is then defined to be

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).$$

All functions defined on X or its subsets will take values in $[-\infty, \infty]$. By a curve we mean a nonconstant rectifiable continuous mapping from a compact interval of the real line into X. A nonnegative Borel function g on X is an upper gradient of a function u on X if for all curves γ, we have

$$|u(x) - u(y)| \le \int_\gamma g\, ds, \tag{2.2}$$

where x and y are the end points of γ and the curve integral is defined by using an arc-length parametrization, see [16, Section 2] where upper gradients were originally introduced. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite.

Let $1 \le p < \infty$; we are going to work solely with $p = 1$, but we give definitions that cover all values of p where it takes no extra work. We say that a family of curves Γ is of zero p-modulus if there is a nonnegative Borel function $\rho \in L^p(X)$ such that for all curves $\gamma \in \Gamma$, the curve integral $\int_\gamma \rho\, ds$ is infinite. A property is said to hold for p-almost every curve if it fails only for a curve family with zero p-modulus. If g is a nonnegative μ-measurable function on X and (2.2) holds for p-almost every curve, we say that g is a p-weak upper gradient of u. By only considering curves γ in a set $D \subset X$, we can talk about a function g being a (p-weak) upper gradient of u in D.

Let $D \subset X$ be a μ-measurable set. We define the norm

$$\|u\|_{N^{1,p}(D)} := \|u\|_{L^p(D)} + \inf \|g\|_{L^p(D)},$$

where the infimum is taken over all p-weak upper gradients g of u in D. The usual Sobolev space $W^{1,p}$ is replaced in the metric setting by the Newton–Sobolev space

$$N^{1,p}(D) := \{u \colon \|u\|_{N^{1,p}(D)} < \infty\},$$

which was first introduced in [25]. We understand every Newton–Sobolev function to be defined at every $x \in D$ (even though $\|\cdot\|_{N^{1,p}(D)}$ is then only a seminorm). It is known that for any $u \in N^{1,p}_{\mathrm{loc}}(D)$, there exists a minimal p-weak upper gradient of u in D, always denoted by g_u, satisfying $g_u \le g$ a.e. on D for any p-weak upper gradient $g \in L^p_{\mathrm{loc}}(D)$ of u in D, see [3, Theorem 2.25].

For any $D \subset X$, the space of Newton–Sobolev functions with zero boundary values is defined to be

$$N^{1,p}_0(D) := \{u|_D \colon u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus D\}.$$

This is a subspace of $N^{1,p}(D)$ when D is μ-measurable, and it can always be understood to be a subspace of $N^{1,p}(X)$.

The p-capacity of a set $A \subset X$ is

$$\operatorname{Cap}_p(A) := \inf \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u \ge 1$ on A. If a property holds outside a set $A \subset X$ with $\mathrm{Cap}_p(A) = 0$, we say that it holds p-quasieverywhere, or p-q.e. If $D \subset X$ is μ-measurable, then

$$\|u\|_{N^{1,p}(D)} = 0 \quad \text{if} \quad u = 0 \ p\text{-q.e. on } D, \tag{2.3}$$

see [3, Proposition 1.61].

We know that Cap_p is an outer capacity, meaning that

$$\mathrm{Cap}_p(A) = \inf_{\substack{W \text{ open} \\ A \subset W}} \mathrm{Cap}_p(W)$$

for any $A \subset X$, see e.g. [3, Theorem 5.31]. By [14, Theorem 4.3, Theorem 5.1], for any $A \subset X$ it holds that

$$\mathrm{Cap}_1(A) = 0 \quad \text{if and only if} \quad \mathcal{H}(A) = 0. \tag{2.4}$$

We say that a set $U \subset X$ is p-quasiopen if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\mathrm{Cap}_p(G) < \varepsilon$ and $U \cup G$ is open. We say that a function u defined on a set $D \subset X$ is p-quasicontinuous on D if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\mathrm{Cap}_p(G) < \varepsilon$ and $u|_{D\setminus G}$ is continuous (as a real-valued function). It is a well-known fact that Newton–Sobolev functions are quasicontinuous; for a proof of the following theorem, see [10, Theorem 1.1] or [3, Theorem 5.29].

Theorem 2.5. *Let $\Omega \subset X$ be open and let $u \in N^{1,p}(\Omega)$ (with $1 \le p < \infty$). Then u is p-quasicontinuous on Ω.*

The variational p-capacity of a set $A \subset D$ with respect to $D \subset X$ is given by

$$\mathrm{cap}_p(A, D) := \inf \int_X g_u^p \, d\mu,$$

where the infimum is taken over functions $u \in N_0^{1,p}(D)$ such that $u \ge 1$ on A, and g_u is the minimal p-weak upper gradient of u (in X). By truncation, we see that we can also assume that $0 \le u \le 1$ on X (and the same applies to the p-capacity). For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see [3, 4].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, following [24]. See also the monographs [2, 11, 12, 13, 26] for the classical theory in the Euclidean setting. Let $\Omega \subset X$ be an open set. Given $u \in L^1_{\mathrm{loc}}(\Omega)$, the total variation of u in Ω is defined to be

$$\|Du\|(\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_\Omega g_{u_i} \, d\mu \colon u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \right\},$$

where each g_{u_i} is the minimal 1-weak upper gradient of u_i in Ω. (In [24], local Lipschitz constants were used instead of upper gradients, but the properties of the total variation can be proved similarly with either definition.) We say that a function $u \in L^1(\Omega)$ is of bounded variation, and denote $u \in \mathrm{BV}(\Omega)$, if $\|Du\|(\Omega) < \infty$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf_{\substack{W \text{ open} \\ A \subset W}} \|Du\|(W).$$

If $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\|Du\|(\Omega) < \infty$, then $\|Du\|(\cdot)$ is a Radon measure on Ω by [24, Theorem 3.4]. A μ-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where χ_E is the characteristic function of E. The perimeter of E in Ω is also denoted by $P(E, \Omega) := \|D\chi_E\|(\Omega)$.

The lower and upper approximate limits of a function u on X are defined respectively by

$$u^\wedge(x) := \sup\left\{t \in \mathbf{R}\colon \lim_{r\to 0} \frac{\mu(B(x,r)\cap \{u<t\})}{\mu(B(x,r))} = 0\right\}$$

and

$$u^\vee(x) := \inf\left\{t \in \mathbf{R}\colon \lim_{r\to 0} \frac{\mu(B(x,r)\cap \{u>t\})}{\mu(B(x,r))} = 0\right\}.$$

Unlike Newton–Sobolev functions, we understand BV functions to be μ-equivalence classes. To consider fine properties, we need to consider the pointwise representatives $u^\wedge$ and $u^\vee$.

We will assume throughout the paper that X supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \ge 1$ such that for every ball $B(x,r)$, every $u \in L^1_{\mathrm{loc}}(X)$, and every upper gradient g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le C_P r \fint_{B(x,\lambda r)} g\, d\mu, \tag{2.6}$$

where

$$u_{B(x,r)} := \fint_{B(x,r)} u\, d\mu := \frac{1}{\mu(B(x,r))}\int_{B(x,r)} u\, d\mu.$$

The $(1,1)$-Poincaré inequality implies the following Sobolev inequality: if $x \in X$, $0 < r < \frac{1}{4}\operatorname{diam} X$, and $u \in N_0^{1,1}(B(x,r))$, then

$$\int_{B(x,r)} |u|\, d\mu \le C_S r \int_{B(x,r)} g_u\, d\mu \tag{2.7}$$

for some constant $C_S = C_S(C_d, C_P) \ge 1$, see [3, Theorem 5.51]. By applying this to approximating functions in the definition of the total variation, we obtain for any $x \in X$, $0 < r < \frac{1}{4}\operatorname{diam} X$, and any μ-measurable set $E \subset B(x,r)$

$$\mu(E) \le C_S r P(E,X). \tag{2.8}$$

Next we define the fine topology in the case $p = 1$.

Definition 2.9. We say that $A \subset X$ is 1-thin at the point $x \in X$ if

$$\lim_{r\to 0} r\frac{\operatorname{cap}_1(A\cap B(x,r), B(x,2r))}{\mu(B(x,r))} = 0.$$

We say that a set $U \subset X$ is 1-finely open if $X \setminus U$ is 1-thin at every $x \in U$. Then we define the 1-fine topology as the collection of 1-finely open sets on X (see [20, Lemma 4.2] for a proof of the fact that this is indeed a topology).

We denote the 1-fine interior of a set $H \subset X$, i.e. the largest 1-finely open set contained in H, by $\operatorname{fine-int} H$. We denote the 1-fine closure of $H \subset X$, i.e. the smallest 1-finely closed set containing H, by $\overline{H}^1$. We define the *1-base* $b_1 H$ of $H \subset X$ to be the set of points in X where H is *not* 1-thin. We say that a function u defined on a set $U \subset X$ is 1-finely continuous at $x \in U$ if it is continuous at x when U is equipped with the induced 1-fine topology on U and $[-\infty,\infty]$ is equipped with the usual topology.

By [3, Proposition 6.16], for all $x \in X$ and $0 < r < \frac{1}{8}\operatorname{diam} X$ (in fact, the second inequality holds for all $r > 0$)

$$\frac{\mu(B(x,r))}{2C_S r} \le \operatorname{cap}_1(B(x,r), B(x,2r)) \le \frac{C_d\mu(B(x,r))}{r}, \tag{2.10}$$

and so obviously $W \subset b_1 W$ for any open set $W \subset X$.

The following fact is given in [19, Proposition 3.3]:

$$\mathrm{Cap}_1(\overline{A}^1) = \mathrm{Cap}_1(A) \quad \text{for any } A \subset X. \tag{2.11}$$

The following result describes the close relationship between finely open and quasiopen sets.

Theorem 2.12. [22, Corollary 6.12] *A set $U \subset X$ is 1-quasiopen if and only if it is the union of a 1-finely open set and a $\mathcal{H}$-negligible set.*

For an open set $\Omega \subset X$, we denote by $\mathrm{BV}_c(\Omega)$ the class of functions $\varphi \in \mathrm{BV}(\Omega)$ with compact support in Ω, that is, $\operatorname{spt}\varphi \Subset \Omega$.

Definition 2.13. We say that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ is a 1-minimizer in Ω if for all $\varphi \in \mathrm{BV}_c(\Omega)$,

$$\|Du\|(\operatorname{spt}\varphi) \le \|D(u+\varphi)\|(\operatorname{spt}\varphi). \tag{2.14}$$

We say that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ is a 1-superminimizer in Ω if (2.14) holds for all nonnegative $\varphi \in \mathrm{BV}_c(\Omega)$.

More precisely, we should talk about $\operatorname{spt}|\varphi|^\vee$, since φ is only a.e. defined. In the literature, 1-minimizers are usually called functions of least gradient.

3. A new Cartan-type property

In this section we prove the new Cartan-type property, given in Theorem 3.11. First we take note of a few results that we will need in the proofs; the following is given in [3, Lemma 11.22].

Lemma 3.1. *Let $x \in X$, $r > 0$, and $A \subset B(x,r)$. Then for every $1 < s < t$ with $tr < \frac{1}{4}\operatorname{diam} X$, we have*

$$\mathrm{cap}_1(A, B(x,tr)) \le \mathrm{cap}_1(A, B(x,sr)) \le C_S\left(1 + \frac{t}{s-1}\right)\mathrm{cap}_1(A, B(x,tr)),$$

where C_S is the constant from the Sobolev inequality (2.7).

Theorem 3.2. [21, Theorem 3.11] *Let u be a 1-superminimizer in an open set $\Omega \subset X$. Then $u^\wedge \colon \Omega \to (-\infty,\infty]$ is lower semicontinuous.*

As mentioned in the introduction, in [21] we proved a weak Cartan property for $p = 1$, more precisely in the following form.

Theorem 3.3. [21, Theorem 5.2] *Let $A \subset X$ and let $x \in X \setminus A$ be such that A is 1-thin at x. Then there exist $R > 0$ and $E_0, E_1 \subset X$ such that $\chi_{E_0}, \chi_{E_1} \in \mathrm{BV}(X)$, χ_{E_0} and χ_{E_1} are 1-superminimizers in $B(x,R)$, $\max\{\chi_{E_0}^\wedge, \chi_{E_1}^\wedge\} = 1$ in $A \cap B(x,R)$, $\chi_{E_0}^\vee(x) = 0 = \chi_{E_1}^\vee(x)$, $\{\max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} > 0\}$ is 1-thin at x, and*

$$\lim_{r\to 0} r\frac{P(E_0, B(x,r))}{\mu(B(x,r))} = 0, \quad \lim_{r\to 0} r\frac{P(E_1, B(x,r))}{\mu(B(x,r))} = 0.$$

Now we collect a few facts that are not included in the above statement, but are given in the proof in [21]. Defining $B_j := B(x, 2^{-j}R)$ and $H_j := B_j \setminus \frac{9}{10}\overline{B_{j+1}}$ for $j = 0, 1, \ldots$, there exists an open set $W \supset A$ that is 1-thin at x,

$$W \cap \bigcup_{j=0,2,\ldots} H_j \subset E_0 \quad \text{and} \quad W \cap \bigcup_{j=1,3,\ldots} H_j \subset E_1, \tag{3.4}$$

and (see [21, Eq. (5.4)])

$$
\begin{aligned}
E_0 &\subset \left(\tfrac{3}{2}B_0 \setminus \tfrac{4}{5}B_1\right) \cup \bigcup_{j=2,4,\ldots}^{\infty} \tfrac{5}{4}B_j \setminus \tfrac{4}{5}B_{j+1} \quad \text{and} \\
E_1 &\subset \left(\tfrac{3}{2}B_1 \setminus \tfrac{4}{5}B_2\right) \cup \bigcup_{j=3,5,\ldots}^{\infty} \tfrac{5}{4}B_j \setminus \tfrac{4}{5}B_{j+1}.
\end{aligned} \tag{3.5}
$$

Moreover, by [21, Eq. (5.5)], for all $i = 2, 4, 6, \ldots$ we have

$$P(E_0 \cap \tfrac{5}{4}B_i, X) \le 5C_S \operatorname{cap}_1(W \cap B_i, 2B_i), \tag{3.6}$$

and similarly for all $i = 3, 5, 7, \ldots$,

$$P(E_1 \cap \tfrac{5}{4}B_i, X) \le 5C_S \operatorname{cap}_1(W \cap B_i, 2B_i). \tag{3.7}$$

From the proof it can also be seen that if $R > 0$ is chosen to be smaller, all of the above results still hold. The same will then apply to the conclusion of the next lemma. Let B_j and H_j be defined as above.

Lemma 3.8. *Let $A \subset X$ and let $x \in X \setminus A$ be such that A is 1-thin at x. Then there exists a number $R > 0$, an open set $W \supset A$ that is 1-thin at x, and open sets $F_j \supset W \cap H_j$ such that $F_j \subset \frac{5}{4}B_j \setminus \frac{3}{4}B_{j+1}$ for all $j = 0, 1, \ldots$, and*

$$\sum_{j=i}^{\infty} P(F_j, X) \le 50C_S^2 \operatorname{cap}_1(W \cap B_i, 2B_i) \tag{3.9}$$

for all $i = 0, 1, \ldots$.

Proof. By using the weak Cartan property (Theorem 3.3), choose $R > 0$ and $E_0, E_1 \subset X$ such that $\chi_{E_0}, \chi_{E_1} \in \mathrm{BV}(X)$ and χ_{E_0} and χ_{E_1} are 1-superminimizers in $B(x, R)$. We can assume that $R < \frac{1}{2} \operatorname{diam} X$. Also let $W \supset A$ be an open set that is 1-thin at x, as described above. Define

$$F_j := \{\chi_{E_0}^{\wedge} > 0\} \cap \tfrac{5}{4}B_j \setminus \tfrac{3}{4}\overline{B_{j+1}} \quad \text{for } j = 2, 4, \ldots,$$

and

$$F_j := \{\chi_{E_1}^{\wedge} > 0\} \cap \tfrac{5}{4}B_j \setminus \tfrac{3}{4}\overline{B_{j+1}} \quad \text{for } j = 3, 5, \ldots.$$

By (3.4), we have $F_j \supset W \cap H_j$ for all $j = 2, 3, \ldots$ as desired. The sets F_j are open by Theorem 3.2. By Lebesgue's differentiation theorem, the sets $\{\chi_{E_0}^{\wedge} > 0\}$ and $\{\chi_{E_1}^{\wedge} > 0\}$ differ from E_0 and E_1, respectively, only by a set of μ-measure zero. Thus by (3.5) and the fact that the sets F_j are at a positive distance from each other, we find that for all $i = 2, 4, \ldots$,

$$P(E_0 \cap \tfrac{5}{4}B_i, X) = P\left(\bigcup_{j=i,i+2,\ldots} F_j, X\right) = \sum_{j=i,i+2,\ldots} P(F_j, X),$$

and similarly for all $i = 3, 5, \ldots$,

$$P(E_1 \cap \tfrac{5}{4}B_i, X) = \sum_{j=i,i+2,\ldots} P(F_j, X).$$

Combining these with (3.6) and (3.7), and using Lemma 3.1, we have for all $i = 2, 3, \dots$

$$\begin{aligned}\sum_{j=i}^{\infty} P(F_j, X) &\le 5C_S(\operatorname{cap}_1(W \cap B_i, 2B_i) + \operatorname{cap}_1(W \cap B_{i+1}, 2B_{i+1}))\\ &\le 5C_S(\operatorname{cap}_1(W \cap B_i, 2B_i) + 5C_S \operatorname{cap}_1(W \cap B_{i+1}, 4B_{i+1}))\\ &\le 25C_S^2(\operatorname{cap}_1(W \cap B_i, 2B_i) + \operatorname{cap}_1(W \cap B_i, 4B_{i+1}))\\ &= 50C_S^2 \operatorname{cap}_1(W \cap B_i, 2B_i).\end{aligned}$$

Then by replacing R with $R/4$, we have the result. □

Recall the constant $\lambda \ge 1$ from the Poincaré inequality (2.6). We have the following boxing inequality from [18, Theorem 3.1]. Note that in [18] it is assumed that $\mu(X) = \infty$, but the proof reveals that we can alternatively assume $\mu(F) < \mu(X)/2$.

Theorem 3.10. *Let $F \subset X$ be an open set of finite perimeter with $\mu(F) < \mu(X)/2$ (in particular, $\mu(F)$ is finite). Then there exists a collection of balls $\{B_k = B(x_k, r_k)\}_{k\in\mathbf{N}}$ such that the balls λB_k are disjoint, $F \subset \bigcup_{k=1}^{\infty} 5\lambda B_k$,*

$$\frac{1}{2C_d} \le \frac{\mu(B_k \cap F)}{\mu(B_k)} \le \frac{1}{2}$$

for all $k \in \mathbf{N}$, and

$$\sum_{k=1}^{\infty} \frac{\mu(5\lambda B_k)}{5\lambda r_k} \le C_B P(F, X)$$

for some constant $C_B = C_B(C_d, C_P, \lambda)$.

Now we can show the following Cartan-type property.

Theorem 3.11. *Let $A \subset X$ and let $x \in X \setminus A$ be such that A is 1-thin at x. Then there exists a number $R > 0$, open sets $G \subset V \subset X$, and a function $\eta \in N_0^{1,1}(V)$ such that $A \cap B(x, R) \subset G$, V is 1-thin at x, $0 \le \eta \le 1$ on X, $\eta = 1$ on G, and*

$$\lim_{r\to 0} \frac{r}{\mu(B(x,r))} \|\eta\|_{N^{1,1}(B(x,r))} = 0. \tag{3.12}$$

Proof. Take $R > 0$, an open set $W \supset A$, and open sets $F_j \subset \frac{5}{4}B_j \setminus \frac{3}{4}B_{j+1}$ as given by Lemma 3.8. Let

$$\delta := \frac{1}{2^8(680\lambda)^Q C_d^3 C_S^3},$$

where $Q > 1$ is the exponent in (2.1). We can assume that $R \le \min\left\{1, \frac{1}{8}\operatorname{diam} X\right\}$. Since $\mu(\{x\}) = 0$ (see [3, Corollary 4.3]), we can also assume R to be so small that $\mu(\frac{5}{4}B_0) < \frac{1}{2}\mu(X)$, and so also $\mu(F_j) < \frac{1}{2}\mu(X)$ for all $j = 0, 1, \dots$. Since W is 1-thin at x, we can further assume that R is so small that

$$2^{-j} R \frac{\operatorname{cap}_1(W \cap B_j, 2B_j)}{\mu(B_j)} < \delta \tag{3.13}$$

for all $j = 0, 1, \dots$. Fix j. By the boxing inequality (Theorem 3.10) we find a collection of balls $\{B_k^j = B(x_k^j, r_k^j)\}_{k=1}^{\infty}$ such that the balls λB_k^j are disjoint, $F_j \subset \bigcup_{k=1}^{\infty} 5\lambda B_k^j$,

$$\frac{1}{2C_d} \le \frac{\mu(B_k^j \cap F_j)}{\mu(B_k^j)} \le \frac{1}{2} \tag{3.14}$$

for all $k \in \mathbf{N}$, and

$$\sum_{k=1}^{\infty} \frac{\mu(5\lambda B_k^j)}{5\lambda r_k^j} \le C_B P(F_j, X). \tag{3.15}$$

Thus we have

$$\begin{aligned}\mu(B_k^j) &\le 2C_d \mu(B_k^j \cap F_j) \le 2C_d \mu(F_j)\\ &\le 2^{2-j} R C_d C_S P(F_j, X) \quad \text{by (2.8)}\\ &\le 2^{8-j} R C_d C_S^3 \operatorname{cap}_1(W \cap B_j, 2B_j) \quad \text{by (3.9)}.\end{aligned}$$

Thus for all $k \in \mathbf{N}$,

$$\frac{\mu(B_k^j)}{\mu(B_j)} \le 2^8 C_d C_S^3 2^{-j} R \frac{\operatorname{cap}_1(W \cap B_j, 2B_j)}{\mu(B_j)} \le 2^8 C_d C_S^3 \delta \tag{3.16}$$

by (3.13). By (3.14) we necessarily have $F_j \cap B_k^j \neq \emptyset$ for all $k \in \mathbf{N}$, and so $\frac{5}{4}B_j \cap B_k^j \neq \emptyset$. Now if $r_k^j \ge 2^{-j}R$ for some $k \in \mathbf{N}$, then $B_j \subset 4B_k^j$ and so

$$\frac{\mu(B_k^j)}{\mu(B_j)} \ge \frac{1}{C_d^2},$$

contradicting (3.16) by our choice of δ. Thus $r_k^j \le 2^{-j}R$ for all $k \in \mathbf{N}$, so that $x_k^j \in 3B_j$, and thus by (2.1),

$$\left(\frac{r_k^j}{2^{-j+2}R}\right)^Q \le C_d^2 \frac{\mu(B_k^j)}{\mu(4B_j)} \le C_d^2 \frac{\mu(B_k^j)}{\mu(B_j)} \le 2^8 C_d^3 C_S^3 \delta$$

by (3.16), so that by our choice of δ,

$$r_k^j \le (2^8 C_d^3 C_S^3 \delta)^{1/Q} 2^{-j+2} R = \frac{2^{-j}R}{170\lambda}. \tag{3.17}$$

Thus recalling that $F_j \cap B_k^j \neq \emptyset$, so that $(\frac{5}{4}B_j \setminus \frac{3}{4}B_{j+1}) \cap B_k^j \neq \emptyset$, we conclude that $20\lambda B_k^j \subset B_{j-1} \setminus B_{j+2}$ (let $B_{-1} := B(x, 2R)$). Now, define Lipschitz functions

$$\xi_k^j := \max\left\{0, 1 - \frac{\operatorname{dist}(\cdot, 10\lambda B_k^j)}{10\lambda r_k^j}\right\}, \quad k \in \mathbf{N},$$

so that $\xi_k^j = 1$ on $10\lambda B_k^j$ and $\xi_k^j = 0$ on $X \setminus 20\lambda B_k^j$. Using the basic properties of 1-weak upper gradients, see [3, Corollary 2.21], we obtain

$$\int_X g_{\xi_k^j}\, d\mu \le \frac{\mu(20\lambda B_k^j)}{10\lambda r_k^j}.$$

Define $V := \bigcup_{j=0}^{\infty}\bigcup_{k=1}^{\infty} 10\lambda B_k^j$. Now for every $i = 1, 2, \ldots$,

$$\begin{aligned}\operatorname{cap}_1(V\cap B_i, 4B_i) &\le \operatorname{cap}_1\left(\bigcup_{j=i-1}^{\infty}\bigcup_{k=1}^{\infty} 10\lambda B_k^j, 4B_i\right)\\ &\le \sum_{j=i-1}^{\infty}\sum_{k=1}^{\infty}\operatorname{cap}_1\left(10\lambda B_k^j, 4B_i\right)\\ &\le \sum_{j=i-1}^{\infty}\sum_{k=1}^{\infty}\int_X g_{\xi_k^j}\,d\mu \le \sum_{j=i-1}^{\infty}\sum_{k=1}^{\infty}\frac{\mu(20\lambda B_k^j)}{10\lambda r_k^j}\\ &\le C_d^2 C_B \sum_{j=i-1}^{\infty} P(F_j, X) \quad \text{by (3.15)}\\ &\le 50 C_d^2 C_B C_S^2 \operatorname{cap}_1(W\cap B_{i-1}, 2B_{i-1}) \quad \text{by (3.9).}\end{aligned} \tag{3.18}$$

Thus

$$2^{-i}R\frac{\operatorname{cap}_1(V\cap B_i, 4B_i)}{\mu(B_i)} \le 50 C_d^3 C_B C_S^2 2^{-i+1} R\frac{\operatorname{cap}_1(W\cap B_{i-1}, 2B_{i-1})}{\mu(B_{i-1})} \to 0$$

as $i\to\infty$, since W is 1-thin at x. By Lemma 3.1 it is then straightforward to show that V is also 1-thin at x. Let us also define the Lipschitz functions

$$\eta_k^j := \max\left\{0, 1-\frac{\operatorname{dist}(\cdot, 5\lambda B_k^j)}{5\lambda r_k^j}\right\}, \quad j = 0, 1, \ldots,\ k = 1, 2, \ldots,$$

so that $\eta_k^j = 1$ on $5\lambda B_k^j$ and $\eta_k^j = 0$ on $X\setminus 10\lambda B_k^j$, and then let

$$\eta := \sup_{j=0,1,\ldots,\ k=1,2,\ldots} \eta_k^j.$$

Recall from Lemma 3.8 that $\bigcup_{j=0}^{\infty} F_j \supset W\cap B(x,R)$; thus $\eta\ge 1$ on

$$G := \bigcup_{j=0}^{\infty}\bigcup_{k=1}^{\infty} 5\lambda B_k^j \supset \bigcup_{j=0}^{\infty} F_j \supset W\cap B(x,R) \supset A\cap B(x,R).$$

By [3, Lemma 1.52] we know that $g_\eta \le \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} g_{\eta_k^j}$. Thus for any $i = 1, 2, \ldots$,

$$\begin{aligned}\int_{B_i} g_\eta\,d\mu &\le \sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\int_{B_i} g_{\eta_k^j}\,d\mu \le \sum_{j=i-1}^{\infty}\sum_{k=1}^{\infty}\int_X g_{\eta_k^j}\,d\mu\\ &\le 50 C_d C_B C_S^2 \operatorname{cap}_1(W\cap B_{i-1}, 2B_{i-1}),\end{aligned}$$

where the last inequality follows just as in the last four lines of (3.18). Since we assumed $R\le 1$ and so $5\lambda r_k^j \le 1$ by (3.17), we similarly get

$$\|\eta\|_{L^1(B_i)} \le 50 C_d C_B C_S^2 \operatorname{cap}_1(W\cap B_{i-1}, 2B_{i-1}).$$

Using the fact that W is 1-thin at x and the doubling property of μ, we get (3.12). Estimating just as in the last four lines of (3.18), now with $i = 1$, we get

$$\int_X g_\eta\,d\mu \le \sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\int_X g_{\eta_k^j}\,d\mu \le 50 C_d C_B C_S^2 \operatorname{cap}_1(W\cap B_0, 2B_0) < \infty.$$

Thus $\eta\in N^{1,1}(X)$. Clearly $\eta = 0$ on $X\setminus V$, and so $\eta\in N_0^{1,1}(V)$. □

4. 1-strict subsets

In this section we study 1-strict subsets which are defined as follows.

Definition 4.1. A set $A \subset D$ is a *1-strict subset* of $D \subset X$ if there is a function $u \in N_0^{1,1}(D)$ such that $u = 1$ on A.

Equivalently, A is a 1-strict subset of D if $\operatorname{cap}_1(A, D) < \infty$. In [22, Proposition 6.7] we proved the following result by using the weak Cartan property (Theorem 3.3).

Proposition 4.2. *Let $U \subset X$ be 1-finely open and let $x \in U$. Then there exists a 1-finely open set W such that $x \in W \subset U$, and a function $w \in \mathrm{BV}(X)$ such that $0 \le w \le 1$ on X, $w^\wedge = 1$ on W, and $\operatorname{spt} w \Subset U$.*

This kind of formulation is sufficient for some purposes, but now we are able to improve it by replacing $w \in \mathrm{BV}(X)$ with $w \in N^{1,1}(X)$. The following is our main result on the existence of 1-strict subsets.

Theorem 4.3. *Let $U \subset X$ be 1-finely open and let $x \in U$. Then there exists a 1-finely open set W such that $x \in W \subset U$, and a function $w \in N_0^{1,1}(U)$ such that $0 \le w \le 1$ on X, $w = 1$ on W, and $\operatorname{spt} w \Subset U$. Moreover, if $\operatorname{Cap}_1(\{x\}) = 0$, then $\|w\|_{N^{1,1}(X)}$ can be made arbitrarily small.*

Proof. Applying Theorem 3.11 with the choice $A = X \setminus U$, we find a number $R > 0$, open sets $G \subset V \subset X$, and a function $\eta \in N_0^{1,1}(V)$ such that $B(x,R) \subset G \cup U$, V is 1-thin at x, $0 \le \eta \le 1$ on X, $\eta = 1$ on G, and

$$\lim_{r \to 0} \frac{r}{\mu(B(x,r))} \|\eta\|_{N^{1,1}(B(x,r))} = 0.$$

Choose $0 < r \le R$ such that $r\|\eta\|_{N^{1,1}(B(x,r))}/\mu(B(x,r)) \le 1$ and let

$$\rho := \max\left\{0, 1 - \frac{4\operatorname{dist}(\cdot, B(x,r/2))}{r}\right\} \in \operatorname{Lip}(X),$$

so that $0 \le \rho \le 1$ on X, $\rho = 1$ on $B(x,r/2)$, and $\operatorname{spt} \rho \Subset B(x,r)$. Then let $w := (1-\eta)\rho$. By the Leibniz rule (see [3, Theorem 2.15]), we have $w \in N^{1,1}(X)$ and

$$\int_X g_w \, d\mu = \int_{B(x,r)} g_w \, d\mu \le \int_{B(x,r)} g_\eta \, d\mu + \int_{B(x,r)} g_\rho \, d\mu \le \frac{\mu(B(x,r))}{r} + \frac{4\mu(B(x,r))}{r}.$$

Thus $\|w\|_{N^{1,1}(X)} \le (5/r + 1)\mu(B(x,r))$. If $\operatorname{Cap}_1(\{x\}) = 0$, then also $\mathcal{H}(\{x\}) = 0$ by (2.4), and so we can make $\mu(B(x,r))/r$ as small as we like by choosing suitable r. Then we can also make $\|w\|_{N^{1,1}(X)}$ arbitrarily small.

Regardless of the value of $\operatorname{Cap}_1(\{x\})$, the set V is 1-thin at x, that is, $x \notin b_1 V$. Since V is open we have $V \subset b_1 V$; recall (2.10) and the comment after it. We know that $\overline{V}^1 = V \cup b_1 V$ by [19, Corollary 3.5], so in conclusion $x \notin \overline{V}^1$. Thus

$$W := B(x,r/2) \setminus \overline{V}^1 \subset \{w = 1\}$$

is a 1-finely open neighborhood of x. Finally, $\operatorname{spt} w$ is compact and

$$\operatorname{spt} w \subset \operatorname{spt} \rho \setminus G \subset (U \cup G) \setminus G \subset U,$$

so that $\operatorname{spt} w \Subset U$. Clearly now $w \in N_0^{1,1}(U)$. □

Let us make a few more observations concerning 1-strict subsets. In general it is not clear which subsets A of a set D are 1-strict subsets. If A is a compact subset of an open set D, we obviously have $\operatorname{cap}_1(A,D)<\infty$, and the test function can even be chosen to be Lipschitz. When A is a compact subset of a 1-*quasiopen* set D, we cannot necessarily choose a Lipschitz test function but one might nonetheless suspect that $\operatorname{cap}_1(A,D)<\infty$. However, this is not always the case.

Example 4.4. Let $X=\mathbf{R}^2$ (unweighted), denote the origin by 0, and let

$$A:=\bigcup_{j=1}^{\infty}A_j\cup\{0\}\quad\text{with}\quad A_j:=\{2^{-j}\}\times[-1/(2j),1/(2j)].$$

Denoting $A_j^{\varepsilon}:=\{x\in X\colon\ \operatorname{dist}(x,A_j)<\varepsilon\}$, with $\varepsilon>0$, let

$$D:=\bigcup_{j=1}^{\infty}D_j\cup\{0\}\quad\text{with}\quad D_j:=A_j^{2^{-3j}}.$$

Since all the sets D_j are disjoint, it is straightforward to check that

$$\operatorname{cap}_1(A,D)=\sum_{j=1}^{\infty}\operatorname{cap}_1(A_j,D_j)=\sum_{j=1}^{\infty}\frac{1}{j}=\infty.$$

Now A is clearly a compact set, and D is 1-quasiopen since $D\cup B(0,r)$ is an open set for every $r>0$.

One can also make the sets A,D connected by adding the line $(0,1/2]\times\{0\}$ to A, and by adding e.g. the sets $(2^{-j-1},2^{-j})\times(-2^{-j-1},2^{-j-1})$ to D; then we still have $\operatorname{cap}_1(A,D)=\infty$ but the calculation is somewhat more complicated.

The variational 1-capacity is an outer capacity in the following weak sense.

Proposition 4.5. *Let $A\subset D\subset X$. Then*

$$\operatorname{cap}_1(A,D)=\inf_{\substack{V\text{ 1-quasiopen}\\ A\subset V\subset D}}\operatorname{cap}_1(V,D).$$

Proof. We can assume that $\operatorname{cap}_1(A,D)<\infty$. Fix $0<\varepsilon<1$. Take $u\in N_0^{1,1}(D)$ such that $u=1$ on A and $\int_X g_u\,d\mu<\operatorname{cap}_1(A,D)+\varepsilon$. The set $V:=\{u>1-\varepsilon\}$ is 1-quasiopen by Theorem 2.5, and

$$\operatorname{cap}_1(V,D)\le\int_X g_{u/(1-\varepsilon)}\,d\mu=\frac{\int_X g_u\,d\mu}{1-\varepsilon}\le\frac{\operatorname{cap}_1(A,D)+\varepsilon}{1-\varepsilon}.$$

Since $0<\varepsilon<1$ was arbitrary, we have the result. □

Even though 1-quasiopen sets and 1-finely open sets are very closely related (recall Theorem 2.12), it is not clear whether the following holds.

Open Problem. If $D\subset X$ and $A\subset\operatorname{fine-int}D$, do we have

$$\operatorname{cap}_1(A,D)=\inf_{\substack{V\text{ 1-finely open}\\ A\subset V\subset D}}\operatorname{cap}_1(V,D)?$$

Note that according to Theorem 4.3, the above property does hold in the very special case when A is a point with 1-capacity zero.

Let us say that a set $K\subset X$ is *1-quasiclosed* if $X\setminus K$ is 1-quasiopen. Now we can show that 1-strict subsets have the following continuity.

Proposition 4.6. *Let $D\subset X$ and let $K_1\supset K_2\supset\ldots$ be bounded 1-quasiclosed subsets of D such that $\operatorname{cap}_1(K_1,D)<\infty$. Then for $K:=\bigcap_{i=1}^{\infty}K_i$ we have*

$$\operatorname{cap}_1(K,D)=\lim_{i\to\infty}\operatorname{cap}_1(K_i,D).$$

We will show in Example 4.7 below that the assumption $\operatorname{cap}_1(K_1, D) < \infty$ is needed.

Proof. Fix $\varepsilon > 0$. By Proposition 4.5 we find a 1-quasiopen set V such that $K \subset V \subset D$ and $\operatorname{cap}_1(V, D) < \operatorname{cap}_1(K, D) + \varepsilon$. For each $j \in \mathbf{N}$ we find an open set $\widetilde{G}_j \subset X$ such that $V \cup \widetilde{G}_j$ is open and $\operatorname{Cap}_1(\widetilde{G}_j) \to 0$ as $j \to \infty$. For each $i, j \in \mathbf{N}$, we find an open set $G_{i,j} \subset X$ such that $K_i \setminus G_{i,j}$ is compact and $\operatorname{Cap}_1(G_{i,j}) < 2^{-i-j}$. Letting $G_j := \widetilde{G}_j \cup \bigcup_{i=1}^{\infty} G_{i,j}$ for each $j \in \mathbf{N}$, we have that each $V \cup G_j$ is open, each $K_i \setminus G_j$ is compact, and $\operatorname{Cap}_1(G_j) \to 0$ as $j \to \infty$. Then for each $j \in \mathbf{N}$ we find a function $w_j \in N^{1,1}(X)$ such that $0 \le w_j \le 1$ on X, $w_j = 1$ on G_j, and $\|w_j\|_{N^{1,1}(X)} \to 0$ as $j \to \infty$. Passing to a subsequence (not relabeled), we can assume that $w_j \to 0$ a.e.

Since $\operatorname{cap}_1(K_1, D) < \infty$, we find $v \in N_0^{1,1}(D)$ such that $0 \le v \le 1$ on X and $v = 1$ on K_1. For each $j \in \mathbf{N}$, let $\rho_j := v w_j$. Then $\|\rho_j\|_{L^1(X)} \to 0$ as $j \to \infty$, and by the Leibniz rule (see [3, Theorem 2.15]),

$$\int_X g_{\rho_j}\, d\mu \le \int_X g_{w_j}\, d\mu + \int_X w_j g_v\, d\mu \to 0$$

as $j \to \infty$; for the second term this follows from Lebesgue's dominated convergence theorem. Thus $\operatorname{cap}_1(G_j \cap K_1, D) \to 0$. Fix $j \in \mathbf{N}$ such that $\operatorname{cap}_1(G_j \cap K_1, D) < \varepsilon$. Since every $K_i \setminus G_j$ is compact and $V \cup G_j$ is open, for some $i \in \mathbf{N}$ we have $K_i \setminus G_j \subset V \cup G_j$. Thus $K_i \subset V \cup G_j$. Then

$$\begin{aligned}\operatorname{cap}_1(K_i, D) &\le \operatorname{cap}_1(V \cup (G_j \cap K_1), D) \le \operatorname{cap}_1(V, D) + \operatorname{cap}_1(G_j \cap K_1, D)\\ &\le \operatorname{cap}_1(K, D) + \varepsilon + \operatorname{cap}_1(G_j \cap K_1, D) \le \operatorname{cap}_1(K, D) + 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof is concluded. ☐

Example 4.7. In the notation of Example 4.4, let $K_i := \bigcup_{j=i}^{\infty} A_j \cup \{0\}$ for each $i \in \mathbf{N}$. These are compact sets and similarly as in Example 4.4 we find that $\operatorname{cap}_1(K_i, D) = \infty$ for every $i \in \mathbf{N}$. However, $\operatorname{cap}_1(K, D) = 0$ for $K := \bigcap_{i=1}^{\infty} K_i = \{0\}$, by the fact that a point has 1-capacity zero and by using (2.3).

5. Application to fine Sobolev spaces

Björn–Björn–Latvala [7] have studied different definitions of Newton–Sobolev spaces on quasiopen sets in metric spaces in the case $1 < p < \infty$. As an application of the theory we have developed, we show that the analogous results hold for $p = 1$.

First we prove the following fact in a very similar way as it is proved in the case $1 < p < \infty$, see [8, Theorem 1.4(b)] and [6, Theorem 4.9(b)]. Recall that a function u defined on a set $U \subset X$ is 1-quasicontinuous on U if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\operatorname{Cap}_1(G) < \varepsilon$ and $u|_{U \setminus G}$ is continuous (as a real-valued function).

Theorem 5.1. *A function u on a 1-quasiopen set U is 1-quasicontinuous on U if and only if it is finite 1-q.e. and 1-finely continuous 1-q.e. on U.*

Proof. To prove one direction, suppose there is a set $N \subset U$ such that $\operatorname{Cap}_1(N) = 0$ and u is finite and 1-finely continuous at every point in $V := U \setminus N$. By Theorem 2.12, we can assume that V is 1-finely open. Let $\{(a_j, b_j)\}_{j=1}^{\infty}$ be an enumeration of all intervals in $\mathbf{R}$ with rational endpoints and let

$$V_j := \{x \in V : a_j < u(x) < b_j\}.$$

By the 1-fine continuity of u, the sets V_j are 1-finely open. Hence by Theorem 2.12, they are also 1-quasiopen. Fix $\varepsilon > 0$. There are open sets $G_j \subset X$ such that $\mathrm{Cap}_1(G_j) < 2^{-j-1}\varepsilon$ and each $V_j \cup G_j$ is open. Since Cap_1 is an outer capacity, there is also an open set $G_N \supset N$ such that $\mathrm{Cap}_1(G_N) < \varepsilon/2$. Now

$$G := G_N \cup \bigcup_{j=1}^{\infty} G_j$$

is an open set such that $\mathrm{Cap}_1(G) < \varepsilon$, and $u|_{U\setminus G}$ is continuous since $V_j \cup G$ are open sets.

To prove the converse direction, by Theorem 2.12 we know that $U = V \cup N$, where V is 1-finely open and $\mathcal{H}(N) = 0$, and then also $\mathrm{Cap}_1(N) = 0$ by (2.4). By the quasicontinuity of u, for each $j \in \mathbf{N}$ we find an open set $G_j \subset X$ such that $\mathrm{Cap}_1(G_j) < 1/j$ and $u|_{V\setminus G_j}$ is continuous. By (2.11), we have $\mathrm{Cap}_1(\overline{G_j}^1) = \mathrm{Cap}_1(G_j)$ for each $j \in \mathbf{N}$, and so the set

$$A := N \cup \bigcap_{j=1}^{\infty} \overline{G_j}^1$$

satisfies $\mathrm{Cap}_1(A) = 0$. If $x \in U \setminus A$, then $x \in V \setminus \overline{G_j}^1$ for some $j \in \mathbf{N}$. Since $V \setminus \overline{G_j}^1$ is a 1-finely open set and $u|_{V\setminus \overline{G_j}^1}$ is continuous, it follows that u is finite and 1-finely continuous at x. □

We will need the following quasi-Lindelöf principle from [22].

Theorem 5.2. [22, Theorem 5.2] *For every family $\mathcal{V}$ of 1-finely open sets there is a countable subfamily $\mathcal{V}'$ such that*

$$\mathrm{Cap}_1\left(\bigcup_{V\in\mathcal{V}} V \setminus \bigcup_{V'\in\mathcal{V}'} V'\right) = 0.$$

From now on, $U \subset X$ is always a 1-quasiopen set. Note that 1-quasiopen sets are μ-measurable by [5, Lemma 9.3].

Definition 5.3. A family $\mathcal{B}$ of 1-quasiopen sets is a *1-quasicovering* of U if it is countable, $\bigcup_{V\in\mathcal{B}} V \subset U$, and $\mathrm{Cap}_1\left(U \setminus \bigcup_{V\in\mathcal{B}} V\right) = 0$. If every $V \in \mathcal{B}$ is a 1-finely open 1-strict subset of U with $V \Subset U$, then $\mathcal{B}$ is a *1-strict quasicovering* of U. Moreover, we say that

1. $u \in N^{1,1}_{\mathrm{fine-loc}}(U)$ if $u \in N^{1,1}(V)$ for every 1-finely open 1-strict subset $V \Subset U$;
2. $u \in N^{1,1}_{\mathrm{quasi-loc}}(U)$ (respectively $L^1_{\mathrm{quasi-loc}}(U)$) if there is a 1-quasicovering $\mathcal{B}$ of U such that $u \in N^{1,1}(V)$ (respectively $L^1(V)$) for every $V \in \mathcal{B}$.

Proposition 5.4. *There exists a 1-strict quasicovering $\mathcal{B}$ of U. Moreover, the associated Newton–Sobolev functions can be chosen compactly supported in U.*

Proof. By Theorem 2.12, we have $U = V \cup N$, where V is 1-finely open and $\mathcal{H}(N) = 0$, and then also $\mathrm{Cap}_1(N) = 0$ by (2.4). For every $x \in V$, by Theorem 4.3 we find a 1-finely open set $V_x \ni x$ such that $V_x \Subset V$ and an associated function $v_x \in N^{1,1}_0(V)$ such that $0 \le v_x \le 1$ on X, $v_x = 1$ on V_x, and $\operatorname{spt} v_x \Subset V$. The collection $\mathcal{B}' := \{V_x\}_{x\in V}$ covers V, and by the quasi-Lindelöf principle (Theorem 5.2) and the fact that $\mathrm{Cap}_1(U \setminus V) = 0$, there exists a countable subcollection $\mathcal{B} \subset \mathcal{B}'$ such that $\mathrm{Cap}_1\left(U \setminus \bigcup_{V_x\in\mathcal{B}} V_x\right) = 0$. □

It follows that $N^{1,1}_{\mathrm{fine-loc}}(U) \subset N^{1,1}_{\mathrm{quasi-loc}}(U)$. From now on, since the proofs given in [7] in the case $1 < p < \infty$ apply almost verbatim also in our setting, we will only point out the differences with [7].

Theorem 5.5. *Let $u \in N^{1,1}_{\mathrm{quasi-loc}}(U)$. Then u if finite* 1*-q.e. and* 1*-finely continuous* 1*-q.e. on U. Thus u is also* 1*-quasicontinuous on U.*

Proof. Follow verbatim the proof of [7, Theorem 4.4], except that replace the reference to [7, Proposition 4.2] by Proposition 5.4, and the references to [6, Theorem 4.9(b)] and [8, Theorem 1.4(b)] by Theorem 5.1. □

Definition 5.6. A nonnegative function $\widetilde{g}_u$ on U is a *1-fine upper gradient* of $u \in N^{1,1}_{\mathrm{quasi-loc}}(U)$ if there is a quasicovering $\mathcal{B}$ of U such that for every $V \in \mathcal{B}$, $u \in N^{1,1}(V)$ and $\widetilde{g}_u = g_{u,V}$ a.e. on V, where $g_{u,V}$ is the minimal 1-weak upper gradient of u in V.

Lemma 5.7. *If $u \in N^{1,1}_{\mathrm{quasi-loc}}(U)$, then it has a unique (in the a.e. sense)* 1*-fine upper gradient $\widetilde{g}_u$.*

Proof. Follow verbatim the proof of [7, Lemma 5.2]. □

Theorem 5.8. *If $u \in N^{1,1}_{\mathrm{quasi-loc}}(U)$ and $\widetilde{g}_u$ is a* 1*-fine upper gradient of u, then $\widetilde{g}_u$ is a* 1*-weak upper gradient of u in U.*

Proof. Follow verbatim the proof of [7, Theorem 5.3]. □

Proposition 5.9. *If $u \in N^{1,1}_{\mathrm{quasi-loc}}(U)$, then there is a* 1*-strict quasicovering $\mathcal{B}$ of U such that for every $V \in \mathcal{B}$, there exists $u_V \in N^{1,1}(X)$ with $u = u_V$ on V.*

Proof. Follow verbatim the proof of [7, Proposition 5.5], except that replace the reference to [7, Theorem 4.4] by Theorem 5.5, and [7, Proposition 4.2] by Proposition 5.4. □

The following definition is originally from Kilpeläinen–Malý [17].

Definition 5.10. Let $U \subset \mathbf{R}^n$. A function $u \in L^1(U)$ is in $W^{1,1}(U)$ if

1. there is a quasicovering $\mathcal{B}$ of U such that for every $V \in \mathcal{B}$ there is an open set $G_V \supset V$ and $u_V \in W^{1,1}(G_V)$ such that $u = u_V$ on V, and
2. the *fine gradient* ∇u, defined by $\nabla u = \nabla u_V$ a.e. on each $V \in \mathcal{B}$, also belongs to $L^1(U)$.

Moreover, let

$$\|u\|_{W^{1,1}(U)} := \int_U (|u| + |\nabla u|)\, dx.$$

Recall that we constantly assume U to be a 1-quasiopen set.

Theorem 5.11. *Let $U \subset \mathbf{R}^n$. Then $u \in W^{1,1}(U)$ if and only if there exists $v \in N^{1,1}(U)$ such that $v = u$ a.e. on U. Moreover, $g_v = |\nabla u|$ a.e. on U and $\|v\|_{N^{1,1}(U)} = \|u\|_{W^{1,1}(U)}$.*

Here g_v is the minimal 1-weak upper gradient of v in U.

Proof. Follow verbatim the proof of [7, Theorem 5.7], except that replace the reference to [7, Proposition 5.5] by Proposition 5.9, [3, Proposition A.12] by [3, Corollary A.4], and [7, Theorem 5.4] by Theorem 5.8. □

Returning momentarily to the metric space setting, define the space

$$\widehat{N}^{1,1}(U) := \{u\colon u = v \text{ a.e. for some } v \in N^{1,1}(U)\}.$$

Theorem 5.12. *Let $u \in \widehat{N}^{1,1}(U)$. Then $u \in N^{1,1}(U)$ if and only if u is 1-quasicontinuous on U.*

Proof. Assume that u is 1-quasicontinuous on U. There is $v \in N^{1,1}(U)$ such that $u = v$ a.e. on U. By Theorem 5.5, v is 1-quasicontinuous on U. By [3, Proposition 5.23] and [9, Proposition 4.2], $u = v$ 1-q.e. on U, and thus $u \in N^{1,1}(U)$ by (2.3).

The converse follows from Theorem 5.5. □

Theorem 5.13. *Let $U \subset \mathbf{R}^n$, and let u be an everywhere defined function on U. Then $u \in N^{1,1}(U)$ if and only if $u \in W^{1,1}(U)$ and u is 1-quasicontinuous. Moreover, then $\|u\|_{N^{1,1}(U)} = \|u\|_{W^{1,1}(U)}$.*

Proof. This follows from Theorems 5.11 and 5.12. □

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