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QUASISPHERES AND METRIC DOUBLING MEASURES

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Abstract. Applying the Bonk-Kleiner characterization of Ahlfors 2-regular quasispheres, we show that a metric two-sphere X is a quasisphere if and only if X is linearly locally connected and carries a *weak metric doubling measure*, i.e., a measure that deforms the metric on X without much shrinking.

1. Introduction

A homeomorphism $f\colon (X,d) \to (Y,d')$ between metric spaces is *quasisymmetric*, if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{d(x_1,x_2)}{d(x_1,x_3)} \leqslant t \ \text{ implies } \ \frac{d'(f(x_1),f(x_2))}{d'(f(x_1),f(x_3))} \leqslant \eta(t)$$

for all distinct $x_1, x_2, x_3 \in X$. Applying the definition with $t = 1$ shows that quasisymmetric homeomorphisms map all balls to sets that are uniformly round. Therefore, the condition of quasisymmetry can be seen as a global version of conformality or quasiconformality.

Starting from the work of Tukia and Väisälä [26], a rich theory of quasisymmetric maps between metric spaces has been developed. An overarching problem is to characterize the metric spaces that can be mapped to a given space S by a quasisymmetric map.

This problem is particularly appealing when S is the two-sphere $\mathbb{S}^2$. There are connections to geometric group theory, (cf. [3], [5], [6]), complex dynamics ([7], [8], [13]), as well as minimal surfaces ([17]).

Bonk and Kleiner [4] solved the problem in the setting of two-spheres with "controlled geometry", see also [17], [18], [22], [23], [29]. We say that (X,d) is a *quasisphere*, if there is a quasisymmetric map from (X,d) to $\mathbb{S}^2$. See Section 2 for further definitions.

THEOREM 1.1 ([4], Theorem 1.1)**.** *Suppose (X,d) is homeomorphic to $\mathbb{S}^2$ and Ahlfors 2-regular. Then (X,d) is a quasisphere if and only if it is linearly locally connected.*

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Finding generalizations of the Bonk-Kleiner theorem beyond the Ahlfors 2-regular case and to fractal surfaces is important; applications include Cannon's conjecture on hyperbolic groups, cf. [2], [16] (by [9] the boundary of a hyperbolic group is Ahlfors Q-regular with Q greater than or equal to the topological dimension of the boundary). A characterization of general quasispheres in terms of combinatorial modulus is given in [4, Theorem 11.1]. However, this result is difficult to apply in practice and in fact an easily applicable characterization is not likely to exist. Several types of fractal quasispheres have been found (cf. [1], [12], [19], [27], [28], [30]), showing the difficulty of the problem.

In this paper we characterize quasispheres in terms of a condition related to *metric doubling measures* of David and Semmes [10], [11]. These are measures that deform a given metric in a controlled manner. More precisely, a (doubling) Borel measure μ is a metric doubling measure of dimension 2 on (X, d) if there is a metric q on X and $C \geqslant 1$ such that for all $x, y \in X$,

$$C^{-1}\mu(B(x, d(x, y)))^{1/2} \leqslant q(x, y) \leqslant C\mu(B(x, d(x, y)))^{1/2}. \tag{1}$$

It is well-known that metric doubling measures induce quasisymmetric maps $(X, d) \to (X, q)$. Our main result shows that quasispheres can be characterized using a weaker condition where we basically only assume the first inequality of (1). We call measures satisfying such a condition *weak metric doubling measures*, see Section 2.

THEOREM 1.2. *Let (X, d) be a metric space homeomorphic to $\mathbb{S}^2$. Then (X, d) is a quasisphere if and only if it is linearly locally connected and carries a weak metric doubling measure of dimension* 2.

To prove Theorem 1.2 we show, roughly speaking, that the first inequality in (1) actually implies the second inequality. It follows that μ induces a quasisymmetric map $(X, d) \to (X, q)$, and (X, q) is 2-regular and linearly locally connected. Applying Theorem 1.1 to (X, q) and composing then gives the desired quasisymmetric map. It would be interesting to find higher-dimensional as well as quasiconformal versions of Theorem 1.2. See Section 6 for further discussion.

2. Preliminaries

We first give precise definitions. Let $X = (X, d)$ be a metric space. As usual, $B(x, r)$ is the open ball in X with center x and radius r, and $S(x, r)$ is the set of points whose distance to x equals r.

We say that X is λ-*linearly locally connected* (LLC), if for any $x \in X$ and $r > 0$ it is possible to join any two points in $B(x, r)$ with a continuum in $B(x, \lambda r)$, and any two points in $X \setminus B(x, r)$ with a continuum in $X \setminus B(x, r/\lambda)$.

A Radon measure μ on X is *doubling*, if there exists a constant $C_D \geqslant 1$ such that for all $x \in X$ and $R > 0$,

$$\mu(B(x,2R)) \leqslant C_D \mu(B(x,R)), \tag{2}$$

and *Ahlfors s-regular*, $s > 0$, if there exists a constant $A \geqslant 1$ such that for all $x \in X$ and $0 < R < \operatorname{diam} X$,

$$A^{-1} R^s \leqslant \mu(B(x,R)) \leqslant A R^s.$$

Moreover, X is Ahlfors s-regular if it carries an s-regular measure μ.

We now define weak metric doubling measures. In what follows, we use notation $B_{xy} = B(x, d(x,y)) \cup B(y, d(x,y))$.

Let μ be a doubling measure on (X,d). For $x, y \in X$ and $\delta > 0$, a finite sequence of points $x_0, x_1, \ldots, x_m$ in X is a *δ-chain from x to y*, if $x_0 = x$, $x_m = y$ and $d(x_j, x_{j-1}) \leqslant \delta$ for every $j = 1, \ldots, m$.

Now fix $s > 0$ and define a "μ-length" $q_{\mu,s}$ as follows: set

$$q^{\delta}_{\mu,s}(x,y) \coloneqq \inf \left\{ \sum_{j=1}^{m} \mu(B_{x_j x_{j-1}})^{1/s} \colon (x_j)_{j=0}^{m} \text{ is a } \delta\text{-chain from } x \text{ to } y \right\}$$

and

$$q_{\mu,s}(x,y) \coloneqq \limsup_{\delta \to 0} q^{\delta}_{\mu,s}(x,y) \in [0, \infty].$$

Definition 2.1. A doubling measure μ on (X,d) is a *weak metric doubling measure of dimension s*, if there exists $C_W \geqslant 1$ such that for all $x, y \in X$,

$$\frac{1}{C_W} \mu(B_{xy})^{1/s} \leqslant q_{\mu,s}(x,y). \tag{3}$$

In what follows, if the dimension s is not specified then it is understood that $s = 2$, and $q_{\mu,2}$ is shortened to q_μ.

3. Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2, assuming Proposition 3.1 to be proved in the following sections. First, it is not difficult to see that if there exists a quasisymmetric map $\varphi : X \to \mathbb{S}^2$, then X is LLC, and

$$\mu(E) \coloneqq \mathcal{H}^2(\varphi(E))$$

defines a weak metric doubling measure on X. Therefore, the actual content in the proof of Theorem 1.2 is the existence of a quasisymmetric parametrization, assuming LLC and the existence of a weak metric doubling measure (of dimension 2). The proof is based on the following result.

Proposition 3.1. *Let (X, d) be LLC and homeomorphic to $\mathbb{S}^2$. Moreover, assume that (X, d) carries a weak metric doubling measure μ of dimension 2. Then q_μ is a metric on X and μ is a metric doubling measure in (X, q_μ), that is there exists a constant $C_S \geqslant 1$ such that also the bound*

$$q_\mu(x, y) \leqslant C_S \mu(B_{xy})^{1/2}$$

holds for all $x, y \in X$.

We will apply the well-known growth estimates for doubling measures. The proof is left as an exercise, see [14, ex. 13.1].

Lemma 3.2. *Let X be as in Proposition 3.1 and let μ be a doubling measure on X. Then there exist constants $C, \alpha > 1$ depending only on the doubling constant C_D of μ such that*

$$\frac{\mu(B(x, r_2))}{\mu(B(x, r_1))} \leqslant C \max\left\{ \left(\frac{r_2}{r_1}\right)^{\alpha}, \left(\frac{r_2}{r_1}\right)^{1/\alpha} \right\}$$

for all $0 < r_1, r_2 < \operatorname{diam}(X)$.

Combining Proposition 3.1 and Lemma 3.2 shows that q_μ induces a quasisymmetric map. This is essentially Proposition 14.14 of [14]. We include a proof for completeness.

Corollary 3.3. *Let X and μ be as in Proposition 3.1. Then the identity mapping $i\colon (X, d) \to (X, q_\mu)$ is quasisymmetric, and (X, q_μ) is Ahlfors 2-regular.*

Proof. We denote $q = q_\mu$. We first show that i is a homeomorphism. Since (X, d) is a compact metric space, it suffices to show that i is continuous, i.e., that any q-ball $B^q(x, r)$ contains a d-ball $B^d(x, \delta)$ for some $\delta = \delta(x, r)$. Suppose that this does not hold for some $x \in X$ and $r > 0$. Then there exists a sequence $(x_n)_{n=1}^\infty$ such that $d(x_n, x) \to 0$ but $q(x_n, x) \geqslant r$ for all $n \in \mathbb{N}$. Now Proposition 3.1 implies

$$r \leqslant q(x_n, x) \leqslant C\mu(B^d(x, 2d(x, x_n)))^{1/2} \overset{n\to\infty}{\longrightarrow} 0,$$

which is a contradiction. Thus i is a homeomorphism. Let $x, y, z \in X$ be distinct. By Proposition 3.1 and Lemma 3.2 we have

$$\frac{q(x, y)}{q(x, z)} \leqslant C \frac{\mu(B_{xy})^{1/2}}{\mu(B_{xz})^{1/2}} \leqslant C \frac{\mu(B(x, 2d(x, y)))^{1/2}}{\mu(B(x, 2d(x, z)))^{1/2}} \leqslant \eta\left(\frac{d(x, y)}{d(x, z)}\right),$$

where $\eta : [0, \infty) \to [0, \infty)$ is the homeomorphism

$$\eta(t) = C \max\{t^{\alpha/2}, t^{1/2\alpha}\}.$$

Thus i is η-quasisymmetric.

We next claim that μ is Ahlfors 2-regular on (X,q). Fix $x \in X$ and $0 < r < \operatorname{diam}(X,q)/10$. Since (X,q) is connected, there exists $y \in S^q(x,r)$. Now by Proposition 3.1,

$$C_S^{-2} r^2 \leqslant \mu(B_{xy}) \leqslant C_W^2 r^2.$$

On the other hand, the quasisymmetry of the identity map i and the doubling property of μ give

$$C^{-1}\mu(B^q(x,r)) \leqslant \mu(B_{xy}) \leqslant C\mu(B^q(x,r)),$$

where C depends only on C_D and η. Combining the estimates gives the 2-regularity. □

We are now ready to finish the proof of Theorem 1.2, modulo Proposition 3.1. Indeed, Corollary 3.3 shows that there is a quasisymmetric map from (X,d) onto the 2-regular (X,q_μ). It is not difficult to see that the quasisymmetric image of a LLC space is also LLC. Hence, by Theorem 1.1, there exists a quasisymmetric map from (X,q_μ) onto $\mathbb{S}^2$. Since the composition of two quasisymmetric maps is quasisymmetric, Theorem 1.2 follows.

4. Separating chains in annuli

We prove Proposition 3.1 in two parts. In this section we find short chains in annuli (Lemma 4.3). In the next section we take suitable unions of these chains to connect given points.

We first show that it suffices to consider δ-chains with sufficiently small δ. In what follows, we use notation

$$cB_{xy} = B(x, cd(x,y)) \cup B(y, cd(x,y)).$$

Lemma 4.1. *Let (X,d) be a compact, connected metric space admitting a weak metric doubling measure μ of some dimension $s > 0$. Then for any $r > 0$ there exists $\delta_r > 0$ such that if $x, y \in X$ with $d(x,y) \geqslant r$ then we have*

$$2C_W C_D^{2/s} q_{\mu,s}^{\delta_r}(x,y) \geqslant \mu(B_{xy})^{1/s}, \tag{4}$$

where C_W and C_D are the constants in (3) *and* (2)*, respectively.*

Proof. Suppose to the contrary that (4) does not hold for some $r > 0$. Then there exists a sequence of pairs of points $(x_j, y_j)_j$ for which $d(x_j,y_j) \geqslant r$ and

$$q_{\mu,s}^{1/j}(x_j,y_j) < \frac{1}{2C_W C_D^{2/s}} \mu(B_{x_j y_j})^{1/s}$$

for all $j = 1, 2, 3, \ldots$. Then by compactness we can, after passing to a subsequence, assume that $x_j \to x$ and $y_j \to y$ where also $d(x, y) \geqslant r$. Let then $k \in \mathbb{N}$ be arbitrary and $j \geqslant k$ so large that $B_{x_j y_j} \subset 4B_{xy}$,

$$d(x, x_j), d(y, y_j) \leqslant \frac{1}{k}$$

and

$$\mu(B_{xx_j})^{1/s} + \mu(B_{yy_j})^{1/s} < \frac{1}{3C_W}\mu(B_{xy})^{1/s}. \tag{5}$$

The last estimate is made possible by the fact that $\mu(\{z\}) = 0$ for every point z in the case of a doubling measure and a connected space, or more generally when the space is *uniformly perfect* (see [11, 5.3 and 16.2]). Now choose a $\frac{1}{j}$-chain $z_0, \ldots, z_m$ from x_j to y_j satisfying

$$\sum_{i=1}^{m} \mu(B_{z_i z_{i-1}})^{1/s} < \frac{1}{2C_W C_D^{2/s}}\mu(B_{x_j y_j})^{1/s} \leqslant \frac{1}{2C_W}\mu(B_{xy})^{1/s} \tag{6}$$

so that $x, z_0, \ldots, z_m, y$ is in particular a $\frac{1}{k}$-chain from x to y. Combining (5) and (6), we have

$$q_{\mu,s}^{1/k}(x, y) < \frac{5}{6C_W}\mu(B_{xy})^{1/s}.$$

This contradicts (3) when $k \to \infty$. $\square$

In what follows, we will abuse terminology by using a non-standard definition for separating sets.

Definition 4.2. Given $A, B, K \subset X$, we say that K *separates* A and B if there are distinct connected components U and V of $X \setminus K$ such that $A \subset U$ and $B \subset V$.

Lemma 4.3. *Suppose (X, d) is λ-LLC and homeomorphic to $\mathbb{S}^2$, and μ a weak metric doubling measure on X. Let k be the smallest integer such that $2^k > \lambda$. Then there exists $C > 1$ depending only on λ, C_D and C_W such that for any $x \in X, 0 < r < 2^{-8k}\operatorname{diam} X$ and $\delta > 0$ there exists a δ-chain $x_0, \ldots, x_p$ in the annulus $\overline{B(x, 2^{5k}r)} \setminus B(x, 2^{2k}r)$ such that*

$$\sum_{j=1}^{p} \mu(B_{x_j x_{j-1}})^{1/2} \leqslant C\mu(B(x, r))^{1/2}$$

and the union $\cup_j \overline{5B_{x_j x_{j-1}}}$ contains a continuum separating $B(x, r)$ and $X \setminus \overline{B(x, 2^{7k}r)}$.

Proof. Let $x \in X, 0 < r < 2^{-8k}\operatorname{diam} X$ and $\delta > 0$ be arbitrary. By Lemma 4.1 we may assume without loss of generality that

$$q_\mu^\delta(y,z) \geqslant \frac{1}{C'}\mu(B_{yz})^{1/2} \tag{7}$$

for any $y \in S(x, 2^{3k}r), z \in S(x, 2^{4k}r)$ and also $\delta < r$ by finding a finer chain than possibly asked.

Next we cover the annulus $A = \overline{B(x, 2^{5k}r)} \setminus B(x, 2^{2k}r)$ as follows: Let $\varepsilon > 0$ be small enough so that $\mu(B(w, \delta/10)) > \varepsilon^2$ for every $w \in X$ (see again [11, 16.2]). Then for every $w \in A$ we can choose a radius $0 < r_w < \delta/10$ with

$$\frac{\varepsilon^2}{2C_D} \leqslant \mu(B(w, r_w)) \leqslant \varepsilon^2.$$

Using the $5r$-covering theorem, we find a finite number m of pairwise disjoint balls $B_j = B(w_j, r_j)$, $r_j = r_{w_j}$ from the cover $\{B(w, r_w)\}_{w \in A}$, such that

$$A \subset \bigcup_{j=1}^m 5B_j \subset B(x, 2^{6k}r) \setminus \overline{B(x, 2^k r)}.$$

Observe that for any point z in the thinner annulus $A' = \overline{B(x, 2^{4k}r)} \setminus B(x, 2^{3k}r)$ there exists a continuum in A joining z to some point $y \in S(x, 2^{3k}r)$ by the LLC-property. Hence there exists a subcollection $B'_1, \ldots, B'_n$ of the cover $(5B_j)$ forming a ball chain from this y to z, meaning that $y \in B'_1, z \in B'_n$ and $B'_j \cap B'_{j+1} \neq \emptyset$. Thus we can define a "counting" function u for this cover on A' by setting $u(z)$ to be the smallest $n \in \{1, \ldots, m\}$ so that there exists a ball chain $(B'_i)_{i=1}^n$ from some $y \in S(x, 2^{3k}r)$ to z.

Using (7), we find a lower bound for u on $S(x, 2^{4k}r)$: Let $y \in S(x, 2^{3k}r), z \in S(x, 2^{4k}r)$ be arbitrary and $(B'_i)_{i=1}^n = (B(w'_i, 5r'_i))_{i=1}^n$ the corresponding chain. Then $y = w'_0, w'_1, \ldots, w'_n, z = w'_{n+1}$ is also a δ-chain. Hence

$$\mu(B_{yz})^{1/2} \leqslant C' \sum_{i=1}^{n+1} \mu(B_{w'_i w'_{i-1}})^{1/2} \leqslant C' C_D^3 n\varepsilon$$

as every $B_{w'_i w'_{i-1}}$ is contained in $B(w'_l, 20r_{w'_l})$, $l = i$ or $i-1$. On the other hand $B(x, 2^{7k}r) \subset B(y, 2^{7k+1}r)$, and since the balls B_j are disjoint,

$$m\varepsilon^2 \leqslant \mu(B(y, 2^{7k+1}r)) \leqslant C_D^{7k+1}\mu(B_{yz}),$$

implying $n^2 \geqslant m/C''$ or $u(z) \geqslant \sqrt{m}/C''$.

Let then n be the minimal value of u on $S(x, 2^{4k}r)$ and for $j = 1, 2, \ldots, n$ define

$$A_j = \bigcup_{5B_i \cap u^{-1}(j) \neq \emptyset} 5B_i.$$

By the definition of u each ball $5B_i$ can be contained in at most two "level sets" A_j and so we obtain a constant $C \geqslant 1$ such that

$$\begin{aligned}
\min_{1 \leqslant j \leqslant n} \sum_{5B_i \subset A_j} \mu(5B_i)^{1/2} &\leqslant \frac{1}{n} \sum_{j=1}^{n} \sum_{5B_i \subset A_j} \mu(5B_i)^{1/2} \\
&\leqslant \frac{1}{n} C_D^3 \varepsilon \cdot 2m \\
&\leqslant 2C_D^3 \frac{\sqrt{m}}{n} \sqrt{\varepsilon^2 m} \\
&\leqslant C\mu(B(x,r))^{1/2}.
\end{aligned}$$

Let $j \in \{1, \ldots, n\}$ be the index for which the above left hand sum is smallest. Since by construction A_j necessarily intersects any curve joining $B(x, 2^k r)$ and $X \setminus \overline{B(x, 2^{6k} r)}$, it separates $B(x, r)$ and $X \setminus \overline{B(x, 2^{7k} r)}$ by the LLC-property as $2^k > \lambda$. Hence the closed set $\overline{A_j}$ contains a continuum K separating these sets by topology of $\mathbb{S}^2$, see for example [20] V 14.3.. Now K is covered by a ball chain $\overline{B(w'_0, 5r'_0)}, \ldots, \overline{B(w'_p, 5r'_p)}$ of closures of balls $5B_i$ contained in A_j. Hence these points $w'_0, \ldots, w'_p$ are the desired δ-chain, since clearly $d(w'_i, w'_{i+1}) \leqslant 5r'_i + 5r'_{i+1} < \delta$ and

$$\sum_{i=1}^{p} \mu(B_{w'_i w'_{i-1}})^{1/2} \leqslant C\mu(B(x,r))^{1/2}$$

by our choice of j. □

Remark 4.4. Note that in the claim of the above lemma the constant C is uniform with respect to the required step δ of the chain; we can in fact find arbitrarily fine chains and have the same estimate from above for $\sum \mu(B_j)^{1/2}$. This is essentially obtained by the doubling property and the $5r$-covering theorem. We also work with dimension $s = 2$, since passing from the lower estimate of 4.1 to the upper in the claim we actually switch the power $1/s$ of the measure to $(s-1)/s$, both $1/2$ in the proof. Thus this argument seems not to apply for higher dimension (see Question 6.3). Moreover the topology of $\mathbb{S}^2$ is used for finding a single separating component, which is not always possible for example on a torus.

5. Proof of Proposition 3.1

In this section (X, d, μ) satisfies the assumptions of Proposition 3.1. Lemma 4.3 and the $5r$-covering lemma then give the following: For any given $B = B(x, R) \subset X$ and $\delta > 0$ there is a cover of the x-component U of B by at most $M = M(\lambda, C_D, L)$ balls $\{B_i\}_{i=1}^m$ with centers in U such that for every i

(1) $L^{-2}\mu(B) \leqslant \mu(B_i) \leqslant L^{-1}\mu(B)$
(2) A continuum $K_i \subset \overline{2^{7k}B_i} \setminus B_i$ separates B_i and $X \setminus \overline{2^{7k}B_i}$
(3) $K_i \subset \bigcup_p \overline{5B_{x_p^i x_{p-1}^i}}$, where $(x_p^i)_p$ is a δ-chain
(4) $\sum_p \mu(B_{x_p^i x_{p-1}^i})^{1/2} \leqslant C\mu(B_i)^{1/2}$.

Here k is as in Lemma 4.3, $L > C_D^{8k}$ and $C = C(\lambda, C_D, C_W)$.

We would like to take unions of the continua K_i to join points. However, the union $\cup_i K_i$ need not be a connected set. The following lemma takes care of this problem. We denote by $\hat{K}_i$ the *interior* of K_i, i.e., the component of $X \setminus K_i$ that contains B_i.

Lemma 5.1. *Let $i \in \{1, 2\}$. Let $B_i = B(x_i, r_i) \subset X$ be a (small) ball and let $K_i \subset \overline{2^{7k}B_i} \setminus B_i$ be a continuum that separates B_i and $X \setminus \overline{2^{7k}B_i}$. Suppose $\hat{K}_1 \cap \hat{K}_2 \neq \emptyset$. If $K_1 \cap K_2 = \emptyset$, then either $K_1 \subset \hat{K}_2$ and $\hat{K}_1 \subset \hat{K}_2$ or $K_2 \subset \hat{K}_1$ and $\hat{K}_2 \subset \hat{K}_1$.*

Proof. Since X is homeomorphic to $\mathbb{S}^2$, path components of an open set in X are exactly its components. In addition such components are open. Since K_1 and K_2 are nonempty disjoint compact sets, there exist path connected open sets $U_1, U_2 \subset X$ such that $K_1 \subset U_1 \subset X \setminus K_2$ and $K_2 \subset U_2 \subset X \setminus K_1$. Let $w \in \hat{K}_1 \cap \hat{K}_2$. Let $\gamma : [0,1] \to X$ be a path from w to $z \in X \setminus (\overline{2^{7k}B_1} \cup \overline{2^{7k}B_2})$. By the separation properties $\gamma([0,1])$ intersects K_1 and K_2. Let

$$s = \inf\{t \in [0,1] \mid \gamma(t) \in K_1 \cup K_2\}.$$

Now $s > 0$ and $\tilde{\gamma} := \gamma|_{[0,s]}$ is a path that intersects $K_1 \cup K_2$ exactly once. Without loss of generality we may assume $\gamma(s) \in K_1$. By construction of U_1 the point w can be connected to any point in K_1 inside $X \setminus K_2$. Thus $K_1 \subset \hat{K}_2$. Now let $y \in \hat{K}_1$. It suffices to show that there exists a path in $\hat{K}_2$ from y to w. Suppose there is no such path. Now the argument of the first part of this proof implies that $K_2 \subset \hat{K}_1$. Let S be the number obtained by changing the infimum in the definition of s to the respective supremum. Necessarily $\gamma(S) \in K_2$, since otherwise we could construct a path in $\hat{K}_2$ from w to z. Since $K_2 \subset U_2 \subset \hat{K}_1$, there exists a path connecting w to $\gamma(S)$ in $\hat{K}_1$, i.e., there exists a path from w to z in $\hat{K}_1$, which is impossible. Thus $\hat{K}_1 \subset \hat{K}_2$. $\square$

Motivated by Lemma 5.1 we say that a continuum K_i is *maximal* (in $\{K_i\}_{i=1}^m$) if it is not contained in the interior of some other K_j. Define K to be the union of all maximal continua in $\{K_i\}_{i=1}^m$. Clearly K is compact. Let us show that it is also connected. Suppose K_i and K_j are distinct maximal continua. Let $B_{(i)}$ and $B_{(j)}$ be the balls in $\{B_i\}$ that are contained in the interiors $\hat{K}_i$ and $\hat{K}_j$, respectively. Since $\{B_i\}$ is a cover of the x-component of B, we can find a chain of balls in $\{B_i\}$ connecting any pair of points in the component. On the other hand, every ball B_i intersects the x-component, so it suffices to consider the case where $B_{(i)} \cap B_{(j)} \neq \emptyset$. By Lemma 5.1 either $K_i \cap K_j \neq \emptyset$ or we may assume that $K_i \subset \hat{K}_j$, but the latter contradicts maximality. Thus K is a continuum. We have now proved the following proposition.

Proposition 5.2. *Fix $L > C_D^{8k}$, $\delta > 0$, and $B = B(x, R) \subset X$. Then there are at most $M = M(\lambda, C_D, L) < \infty$ balls B_i centered at the x-component U of B such that*

1. *$U \subset \cup_i B_i$*
2. *$\mu(B_i) \leqslant \frac{1}{L}\mu(B)$ for all i*
3. *For every i there is a continuum $K_i \subset \overline{2^{7k} B_i} \setminus B_i$ which separates B_i and $X \setminus \overline{2^{7k} B_i}$*
4. *$K_i \subset \bigcup_p \overline{5B_{x_p^i x_{p-1}^i}}$, where $(x_p^i)_p$ is a finite δ-chain*
5. *$\sum_p \mu(B_{x_p^i x_{p-1}^i})^{1/2} \leqslant C\mu(B)^{1/2}$, $C = C(\lambda, C_D, C_W)$*
6. *the union K of all maximal continua in $\{K_i\}$ is a continuum.*

Now we can finish the proof of Proposition 3.1 with the following:

Lemma 5.3. *There exists a constant $C = C(\lambda, C_D, C_W)$ such that for any $\delta > 0$ and $x, y \in X$,*

$$q_\mu^\delta(x, y) \leqslant C\mu(B_{xy})^{1/2}.$$

Proof. Fix $x, y \in X$ and apply Proposition 5.2 to $B^1 = B(x, 2^{2k} d(x, y))$ with $L = C_D^{15k}$. Note that x and y belong to the same component of B^1. Let $z = x$ or $z = y$. Let us define balls $B^{l,z}$ recursively for $l \geqslant 2$. Define $B^{1,z} = B^1$. Suppose we have defined the set $B^{n,z}$ for all $n \leqslant l$. Apply Proposition 5.2 with the same L to $B^{l,z}$ to find a ball $B_j^{l,z}$ which contains z. By Lemma 5.1 $B_j^{l,z}$ is contained in the interior of some maximal continuum $K_{j'}^{l,z}$. Define $B^{l+1,z} = 2^{7k} B_{j'}^{l,z}$. Note that Proposition 5.2 also yields the balls $B^{n,z}$ and $B_i^{n,z}$ and continua $K_i^{n,z}$

and $K^{n,z}$. Also, by the separation properties and Lemma 5.1

$$z \in B_j^{l,z} \subset \hat{K}_j^{l,z} \subset \hat{K}_{j'}^{l,z} \subset 2^{7k} B_{j'}^{l,z} = B^{l+1,z}.$$

Let $\varepsilon > 0$ and let $B_z = B(z, r_z)$ be a ball with $r_z \leqslant 6\delta$ and $\mu(B_z) \leqslant C_D^{-1}\varepsilon^2$. Define

$$K_z := \bigcup_{n=1}^{l_z^\varepsilon} K^{n,z}$$

where l_z^ε is the smallest integer l that satisfies $K^{l,z} \subset B(z, 100^{-1} r_z)$. Such a number exists, since $z \in B^{l,z}$ for all l. Moreover, our choice of L gives $C_D^{7k} L^{-1} = \tau < 1$ and

$$\mu(B^{l,z}) \leqslant C_D^{7k} L^{-1} \mu(B^{l-1,z}) \leqslant \tau \mu(B^{l-1,z}) \leqslant \ldots \leqslant \tau^{(l-1)} \mu(B^1). \tag{8}$$

In particular, $\operatorname{diam}(B^{l,z}) \overset{l\to\infty}{\longrightarrow} 0$. We next show that K_z is a continuum. It is clearly compact, and connectedness follows if

$$K^{n,z} \cap K^{n+1,z} \neq \emptyset. \tag{9}$$

Let j be the index for which $2^{7k} B_j^{n,z} = B^{n+1,z}$. To show (9) it suffices to show that $K_j^{n,z} \cap K_i^{n+1,z} \neq \emptyset$ for some maximal $K_i^{n+1,z}$. By Lemma 5.1 there exists a maximal continuum $K_i^{n+1,z}$ such that the interiors of $K_i^{n+1,z}$ and $K_j^{n,z}$ intersect. Moreover either (9) holds or one of $K_i^{n+1,z} \subset \hat{K}_j^{n,z}$, $K_j^{n,z} \subset \hat{K}_i^{n+1,z}$ is true for any such i. Suppose $K_j^{n,z} \subset \hat{K}_i^{n+1,z}$. By separation properties $B_j^{n,z} \subset 2^{7k} B_i^{n+1,z}$, which together with our choice of L leads to a contradiction:

$$\begin{aligned} \mu(B_j^{n,z}) &\leqslant \mu(2^{7k} B_i^{n+1,z}) \leqslant C_D^{7k} \mu(B_i^{n+1,z}) \leqslant C_D^{7k} L^{-1} \mu(B^{n+1,z}) \\ &= C_D^{7k} L^{-1} \mu(2^{7k} B_j^{n,z}) \leqslant C_D^{14k} L^{-1} \mu(B_j^{n,z}) < \mu(B_j^{n,z}). \end{aligned}$$

Now if (9) were not true, $K_i^{n+1,z} \subset \hat{K}_j^{n,z}$ for every i for which the interiors of $K_i^{n+1,z}$ and $K_j^{n,z}$ intersect. This is impossible, since *every* ball $B_i^{n+1,z}$ lies in the interior of some maximal continuum and at least one of them intersects $K_j^{n,z}$. Hence (9) holds and K_z is a continuum.

Finally, define

$$K = K_x \cup K_y.$$

Note that K is a continuum, since by construction $K^{1,x} = K^{1,y}$. Recall that for all i, j, z there exists a finite δ-chain $(x_p^{i,j,z})_p$ in $2^{7k} B_j^{i,z} \setminus B_j^{i,z}$ such that

$$K_j^{i,z} \subset \bigcup_p \overline{5B_{x_p^{i,j,z} x_{p-1}^{i,j,z}}} \subset \bigcup_p 6B_{x_p^{i,j,z} x_{p-1}^{i,j,z}},$$

and

$$\sum_p \mu(B_{x_p^{i,j,z} x_{p-1}^{i,j,z}})^{1/2} \leqslant C\mu(B_j^{i,z})^{1/2}.$$

Since the set of balls

$$\mathcal{B} := \left\{B(x_p^{i,j,z}, 6d(x_p^{i,j,z}, x_{p-1}^{i,j,z})), B(x_{p-1}^{i,j,z}, 6d(x_p^{i,j,z}, x_{p-1}^{i,j,z}))\right\}_{i,j,p,z}$$

forms an open cover for the continuum K, we may extract a finite chain of balls $(A_i)_{i=1}^{N-1}$ of the set $\mathcal{B}$ so that, denoting $A_0 = B_x$, $A_N = B_y$ we have $A_i \cap A_{i-1} \neq \emptyset$ for $i = 1, \ldots N$. Let $x_0 = x$, $x_{2N} = y$ and for other indices choose $x_{2i} \in A_i$ so that $A_i = B(x_{2i}, r_i)$ for some $r_i \leqslant 6\delta$. Let $x_{2i-1} \in A_i \cap A_{i-1}$ for $i = 1, \ldots, N$. Now $(x_i)_{i=0}^{2N}$ is a 6δ-chain between the points x and y. Moreover, by (8)

$$\begin{aligned}
&\sum_{i=1}^{2N} \mu(B_{x_i x_{i-1}})^{1/2} \leqslant 2\sum_{i=0}^{N} \mu(2A_i)^{1/2} \leqslant C\sum_{i=1}^{N-1} \mu(A_i)^{1/2} + 4\varepsilon \\
&\leqslant C\sum_{B\in\mathcal{B}} \mu(B)^{1/2} + 4\varepsilon \leqslant C\sum_{z,i,j,p} \mu(B(x_p^{i,j,z}, d(x_p^{i,j,z}, x_{p-1}^{i,j,z})))^{1/2} + 4\varepsilon \\
&\leqslant C\sum_{z,i,j}\sum_{p} \mu(B_{x_p^{i,j,z} x_{p-1}^{i,j,z}})^{1/2} + 4\varepsilon \leqslant C\sum_{z,i}\sum_{j} \mu(B_j^{i,z})^{1/2} + 4\varepsilon \\
&\leqslant C\sum_{z}\sum_{i} M\mu(B^{i,z})^{1/2} + 4\varepsilon \leqslant CM\sum_{z}\sum_{i} \tau^{(i-1)/2}\mu(B^1)^{1/2} + 4\varepsilon \\
&\leqslant CM\mu(B^1)^{1/2} + 4\varepsilon = CM\mu(B(x, 2^{2k}d(x,y)))^{1/2} + 4\varepsilon \\
&\leqslant CM\mu(B_{xy})^{1/2} + 4\varepsilon.
\end{aligned}$$

Since ε is arbitrary, the claim follows.

□

6. Concluding remarks

It is natural to ask if Theorem 1.2 remains valid with weak metric doubling measures of dimension $s \neq 2$. The two lemmas below show that it does not.

Lemma 6.1. *Let (X, d) be a linearly locally connected metric space homeomorphic to $\mathbb{S}^2$, and $0 < s < 2$. Then X does not carry weak metric doubling measures of dimension s.*

Proof. Assume towards a contradiction that X carries such as measure μ. Then there exists $C > 0$ such that for every $x, y \in X$ the following holds: if $(x_i)_{i=0}^m$ is a δ-chain from x to y and if δ is small enough, then

$$\begin{aligned} \mu(B_{xy})^{1/2} &= \mu(B_{xy})^{1/2-1/s}\mu(B_{xy})^{1/s} \leqslant C\mu(B_{xy})^{1/2-1/s}\sum_{i=1}^m \mu(B_{x_ix_{i-1}})^{1/s} \\ &\leqslant C\mu(B_{xy})^{1/2-1/s}\max_i \mu(B_{x_ix_{i-1}})^{1/s-1/2}\sum_{i=1}^m \mu(B_{x_ix_{i-1}})^{1/2}. \end{aligned}$$

Notice that

$$\max_i \mu(B_{x_ix_{i-1}})^{1/s-1/2} \to 0 \quad \text{as } \delta \to 0.$$

Applying the estimates to all δ-chains and letting $\delta \to 0$, we conclude that μ is a weak metric doubling measure of dimension 2 and

$$\mu(B_{xy})^{1/2} \leqslant \epsilon q_{\mu,2}(x,y) \quad \text{for all } \epsilon > 0.$$

Since $\mu(B_{xy}) > 0$ for all distinct x and y, if follows that $q_{\mu,2}(x,y) = \infty$. This contradicts Theorem 1.2. $\square$

Lemma 6.2. *Fix $s > 2$. Then there exists a metric space (X,d), homeomorphic to $\mathbb{S}^2$ and LLC, such that X carries a weak metric doubling measure of dimension s but there is no quasisymmetric $f : X \to \mathbb{S}^2$.*

Proof. Let $(\mathbb{R}^2, d)$ be a Rickman rug; d is the product metric

$$d((x_1,y_1),(x_2,y_2)) = \Big(|x_1-x_2|^2 + |y_1-y_2|^{2/(s-1)}\Big)^{1/2}.$$

It is well-known that there are no quasisymmetric maps from $(\mathbb{R}^2, d)$ onto the standard plane. Moreover, it is not difficult to show that $\mu = \mathcal{H}^1 \times \mathcal{H}^{s-1}$ is a weak metric doubling measure of dimension s on $(\mathbb{R}^2, d)$. To construct a similar example homeomorphic to $\mathbb{S}^2$, one can apply a suitable stereographic projection. $\square$

It would be interesting to extend Theorem 1.2 to higher dimensions. Recall that the Bonk-Kleiner theorem (Theorem 1.1) does not extend to dimensions higher than 2, see [24], [15], [21].

Question 6.3. Let (X,d) be a metric space homeomorphic to $\mathbb{S}^n$, $n \geqslant 3$. Assume that X is linearly locally contractible and carries a weak metric doubling measure of dimension n. Is there a quasisymmetric $f : (X,d) \to (X,d')$, where (X,d') is Ahlfors n-regular?

Recall that (X,d) is linearly locally contractible if there exists $\lambda' \geqslant 1$ such that $B(x,R) \subset X$ is contractible in $B(x,\lambda' R)$ for every $x \in X$, $0 < R < \operatorname{diam} X/\lambda'$. Linear local contractibility is equivalent to the LLC condition when X is homeomorphic to $\mathbb{S}^2$, see [4].

The basic tool in the proof of Theorem 1.2 was a coarea-type estimate for real-valued functions. Extending our method to higher dimensions would require similar estimates for suitable maps with values in $\mathbb{R}^{n-1}$, which are difficult to construct when $n \geqslant 3$. This problem is related to the deep results of Semmes [25] on Poincaré inequalities in Ahlfors n-regular and linearly locally contractible n-manifolds.

Finally, it is also desirable to characterize the metric spheres that can be uniformized by *quasiconformal* homeomorphisms which are more flexible than quasisymmetric maps, see [22]. However, it is not clear which definition of quasiconformality should be used in the generality of possibly fractal surfaces. Our methods suggest a measure-dependent modification to the familiar geometric definition. More precisely, given a measure μ, conformal modulus should be defined applying not the usual path length but a *μ-length* as in Section 2.

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