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Abstract

A problem for finding optimal shape for systems governed by the mixed unilateral boundary value problem of Dirichlet-Signorini-type is considered. Conditions for the solvability of the problem are stated, when variational inequality formulation and penalty method is used for solving the state problem in question. The asymptotic relation of design problems based on these two formulations is presented. The optimal shape design problem is discretized by means of finite element method. The convergence results for the approximation are proved. The discretized versions are then formulated as a non-linear programming problem. Results of practical computations of the problem in question are reported.

Subject classification: AMS: 49, 65 K, 35 J

1. INTRODUCTION

During the past 15 years, optimization theory has been developed for the optimal design of many structural and mechanical systems. Generally speaking, an optimal shape design problem is an optimization problem that involves a function satisfying certain relations (ordinary or partial differential equations, e.g.) and optimization variables with geometrical structure of the problem in question. Optimal shape design problems, where the state problem is modelled by a partial differential equation, have been discussed widely in engineering literature [13, 14, 21, 22, 23], see especially the article of Haug [13], where a representative survey of appeared contributions can be found. Mathematical theory of such kind of problems, including the theory of their approximation by finite differences or finite elements, has been developed during the last ten years mainly by the French school of applied mathematics [3, 4, 5, 6, 17, 25, 29]. See also [1] and [28].

In practice, we often meet problems the behaviour of which is described by variational inequalities (see [7]). A natural question arises: what happens if the state problem is now given by these inequalities? Mathematical theory of problems with controlled right hand side has been given e.g. in [24], [17]. In the meantime, shape sensitivity analysis of design problems (control of coefficients) has been given in [26, 27]. A relatively small number of papers is devoted to optimal shape design problems with state inequalities. We mention here papers [15, 16] where the existence of a solution as well as the approximation are studied. It is well known that the relation between admissible designs and the corresponding solution of state inequality is not differentiable everywhere. From this point of view, some specific optimization approach has to be used for numerical realization. To overcome this difficulty in [10, 11] an alternative way has been proposed: instead of variational inequality (associated to a unilateral boundary value problem) a family of penalized variational equations is assumed. It can be shown (see Theorem 2.3, in [10]) that the corresponding optimal designs (associated with penalized problems) are close (in an appropriate sense) to an optimal design associated with the variational inequality formulation.

The aim of the present paper is to study the finite element approximation of our penalized design problems. The paper is organized as follows:

The problem in question can be regarded as a model example of the contact problem between an elastic body and a rigid foundation, which will be studied in a forthcoming paper [12].

2. STATEMENT OF THE OPTIMAL SHAPE DESIGN PROBLEM

Let us consider domains $\Omega = \Omega(v) \subset \mathbb{R}^2$ with the following geometrical structure

$$\Omega(v) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < v(x_2), 0 < x_2 < 1\},$$

where $v \in C^{0,1}([0,1])$ (i.e. v Lipschitz-function); $\partial\Omega(v) = \Gamma_1 \cup \Gamma(v)$ the boundary of $\Omega(v)$ with (see Figure 2.1)

$$\Gamma_1 = \partial\Omega(v) \setminus \Gamma(v),$$

$$\Gamma(v) = \{x \in \mathbb{R}^2 \mid x_1 = v(x_2), 0 < x_2 < 1\}.$$

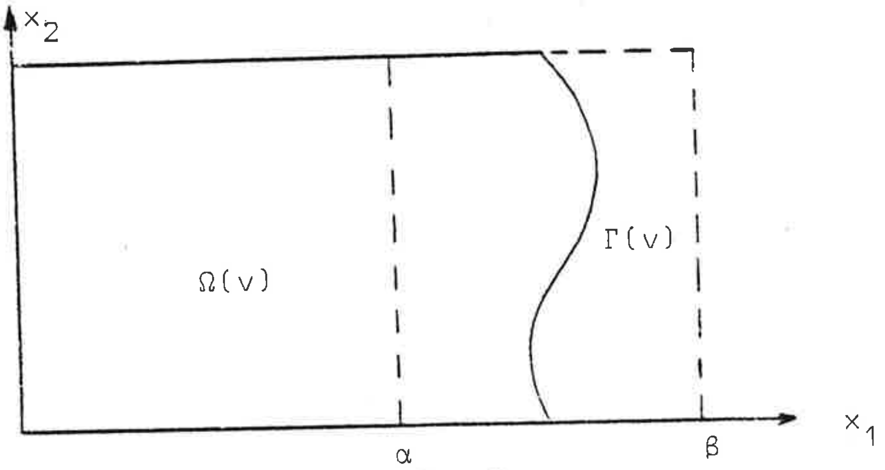

Figure 2.1. Domain $\Omega(v)$ with moving boundary $\Gamma(v)$

In this case the optimal shape design problem reduces to finding boundary $\Gamma(v)$ under certain criteria. Indeed, we can formulate our problem as follows :

Problem (P). (Optimal shape design) Find $w \in \mathcal{U}_{ad}$ such that

$$J(w) = \min_{v \in \mathcal{U}_{ad}} J(v), \tag{2.1}$$

where

$$J(v) = \int_{\Omega(v)} |y(v) - z_d|^2 \, dx \tag{2.2}$$

$z_d \in L^2(\tilde{\Omega})$ a given function $(\tilde{\Omega} := (0,\tilde{\beta}) \times (0,1),\ \tilde{\beta} > \beta)$ and $\mathcal{U}_{ad}$ is a set of admissible controls,

$$(2.3) \qquad \mathcal{U}_{ad} = \{v \in C^{0,1}([0,1]) \mid 0 < \alpha \leq v(x_2) \leq \beta\ ,$$
$$\left|\frac{dv}{dx_2}\right| \leq c_1\ ,\ \int_0^1 v(x_2)\,dx_2 = c_2\}\ ,$$

α, β, c_1 and c_2 are given positive constants. Function $y = y(v)$ is the solution of the unilateral boundary value problem: Find $y = y(v) \in K(\Omega(v))$ such that

$$(2.4) \qquad (\text{grad } y,\ \text{grad } (\xi-y))_{0,\Omega(v)} \geq (f,\ (\xi-y))_{0,\Omega(v)}$$

for all $\xi \in K(\Omega(v))$. Here $f \in L^2(\tilde{\Omega})$ is a given function and

$$V(\Omega(v)) = \{\xi \in H^1(\Omega(v)) \mid \xi = 0 \text{ on } \Gamma_1\}\ ,$$
$$K(\Omega(v)) = \{\xi \in V(\Omega(v)) \mid \xi \geq 0 \text{ on } \Gamma(v)\}\ ,$$

$(.,.)_{0,\Omega(v)}$ denotes the scalar product in $L^2(\Omega(v))$.

In [15] the following fundamental result is proved for the solvability of Problem $(\mathcal{P})$:

Theorem 2.1. *Problem $(\mathcal{P})$ has at least one solution.*

In practice Problem $(\mathcal{P})$ has to be solved approximately. The unknown part of the boundary is sought among piecewise linear arcs and the state problem is solved by finite element method in polygonal domain $\Omega(v_h)$. In $\Omega(v_h)$ a moving finite element grid is constructed. Another possibility would be the transformation technique: the problem is transformed into an equivalent one with the state problem defined on a unit square domain and then finite elements on a uniform mesh are employed.

Anyhow, in both cases the main trouble is that the mapping $v \mapsto y(v)$ is not differentiable. As, moreover, the cost functional $v \mapsto J(v)$ is not convex we are led to difficult optimization problem. To avoid the first difficulty in [10] another approach for (2.4) is utilized. Instead of (2.4) a family of penalized problems is considered: Find $y_\varepsilon = y_\varepsilon(v) \in V(\Omega(v))$ such that

$$(2.5) \qquad (\text{grad } y_\varepsilon, \text{grad } \xi)_{0,\Omega(v)} + \frac{1}{\varepsilon}(P(y_\varepsilon),\xi)_V$$

$$= (f,\xi)_{0,\Omega(v)} \quad \text{for all } \xi \in V(\Omega(v)) ,$$

where P denotes the penalty mapping

$$(P(y_\varepsilon),\xi)_V := - \int_0^1 y_\varepsilon^-(v(x_2),x_2)\xi(v(x_2),x_2)dx_2 ,$$

$y_\varepsilon^- = (|y_\varepsilon|-y_\varepsilon)/2$ is the negative part of y_ε and $\varepsilon > 0$ is a penalty factor.

The regularized optimal shape design problem reads:

<u>Problem $(P)_\varepsilon$</u>. Find $w_\varepsilon \in \mathcal{U}_{ad}$ such that

$$(2.6) \qquad J(w_\varepsilon) = \min_{v \in \mathcal{U}_{ad}} J(v) ,$$

where

$$J(v) = \int_{\Omega(v)} |y_\varepsilon(v) - z_d|^2 dx$$

and y_ε is the solution of (2.5).

The solvability of Problem $(P)_\varepsilon$ is proved in [10]:

<u>Theorem 2.2.</u> <u>For any</u> $\varepsilon > 0$ <u>Problem</u> $(P)_\varepsilon$ <u>has at least one solution.</u>

According to Theorem 2.2 there exists for any $\varepsilon_k \to 0+$ at least one optimal solution of Problem $(P)_{\varepsilon_k}$, which will be denoted by w_k and the corresponding state by $y_k(w_k)$. In [10] it is also proved that some solutions of Problem $(P)_{\varepsilon_k}$ are close to a solution of Problem (P). Indeed, it holds (see [10], Theorem 4.1):

Theorem 2.3. <u>There exist a subsequence</u> $\{w_{k_j}, y_{k_j}(w_{k_j})\}$ <u>of</u> $\{w_k, y_k(w_k)\}$ <u>and elements</u> $w \in \mathcal{U}_{ad}$, $y(w) \in K(\Omega(w))$ <u>such that</u>

$$w_{k_j} \rightrightarrows w \text{ (uniformly) in } [0,1] \text{ for } j \to \infty$$

$$y_{k_j}(w_{k_j}) \rightharpoonup y(w) \text{ (weakly) in } H^1(G_m(w)) \text{ for } j \to \infty ,$$

<u>and for any</u> m, <u>where</u>

$$(2.7) \quad G_m(w) = \{(x_1,x_2) \in \mathbb{R}^2 \mid 0 < x_1 < w(x_2) - 1/m , 0 < x_2 < 1\} ,$$

<u>and where</u> w <u>is the solution of Problem</u> $(\mathcal{P})$ <u>and</u> $y(w)$ <u>is the corresponding state, solving the unilateral boundary value problem</u> (2.4) <u>in</u> $\Omega(w)$.

3. FINITE ELEMENT APPROXIMATION OF THE PENALIZED PROBLEM

According to Theorem 2.3 solutions of Problem $(\mathcal{P})$ and Problem $(\mathcal{P})_\varepsilon$ are in certain sense close together. It can be shown that similar result holds, if the penalized problem $(\mathcal{P})_\varepsilon$ is replaced by a family of finite element approximations. This fact will be utilized in discretization.

Let $0 \equiv a_0 < a_1 < \dots < a_N \equiv 1$ be a partition of $[0, 1]$, $a_i - a_{i-1} \leq h$, and let

$$\mathcal{U}^h_{ad} = \{v_h \in C([0,1]) \mid v_h|_{[a_{i-1},a_i]} \in \mathcal{P}_1([a_{i-1},a_i])\} \cap \mathcal{U}_{ad} ,$$

where $\mathcal{P}_1$ denotes the set of all linear functions.

For any $v_h \in \mathcal{U}_{ad}$ we define

$$\Omega(v_h) = \{x \in \mathbb{R}^2 \mid x_1 \in (0, v_h(x_2)), x_2 \in (0,1)\} ,$$

i.e. the variable part of the boundary $\Gamma_2(v)$ is now approximated by piecewise linear arc $\Gamma_2(v_h)$. By $\mathcal{T}_h(v_h)$, $v_h \in \mathcal{U}_{ad}^h$, we denote the triangulation of $\Omega(v_h)$ such that the whole segment $I_i = \{(x_1,x_2) \mid x_1 = v_h(x_2),\ x_2 \in [a_{i-1},a_i]\}$ is the whole side of a triangle $T_i' \in \mathcal{T}_h(v_h)$ and satisfying usual requirements, concerning the mutual position of two triangles, belonging to $\mathcal{T}_h(v_h)$. Moreover, we shall assume such families of $\{\mathcal{T}_h(v_h)\}$, $h \to 0+$ only, which are regular uniformly with respect to $v_h \in \mathcal{U}_{ad}^h$, i.e. there exists $\alpha_0 > 0$ independently on $h > 0$ and $v_h \in \mathcal{U}_{ad}^h$ such that all interior angles of all triangles belonging to $\mathcal{T}_h(v_h)$ are greater or equal to α_0 (for practical applications some other technical restrictions will be added (see chapter 4.1)).
Finally, the symbol $\Omega_h(v_h)$ will denote the set $\Omega(v_h)$ with a given triangulation $\mathcal{T}_h(v_h)$; we also use abbreviation Ω_h for $\Omega_h(v_h)$.

To any $\mathcal{T}_h(v_h)$ a finite dimensional space $V_h(\Omega_h(v_h))$ will be associated:

$$V_h(\Omega_h(v_h)) = \{v_h \in C(\overline{\Omega_h(v_h)}) \mid v_h|_T \in P_1(T) ,$$

$$\forall\, T \in \mathcal{T}_h(v_h),\ v_h = 0 \quad \text{on} \quad \Gamma_1 = \partial\Omega_h(v_h) - \Gamma_2(v_h)\} .$$

The approximation of the penalized optimal shape design problem is now defined as follows:

<u>Problem $(P)_h$</u> Find $w_h \in \mathcal{U}_{ad}^h$ such that

$$(3.1) \qquad J_h(w_h) = \min_{v_h \in \mathcal{U}_{ad}^h} J_h(v_h) ,$$

where

$$J_h(v_h) = \int_{\Omega_h(v_h)} |y_h(v_h) - z_d|^2 dx$$

and $y_h = y_h(v_h) \in V_h(\Omega_h(v_h))$ is a solution of the nonlinear elliptic boundary value problem

$$(3.2) \qquad (\text{grad } y_h, \text{ grad } \xi_h)_{0,\Omega_h(v_h)} + \frac{1}{\varepsilon(h)} (P(y_h), \xi_h)_{V_h}$$

$$= (f, \xi_h)_{0,\Omega_h(v_h)} \quad \text{for all } \xi_h \in V_h(\Omega_h(v_h)) .$$

The penalty operator is defined like in the continuous case and the penalty parameter $\varepsilon = \varepsilon(h)$ is such that $\varepsilon(h) \to 0+$ for $h \to 0+$.

Using classical compactness arguments one can prove

Theorem 3.1. *For any* $h \in (0,1)$ *there exists a solution* $w_h \in \mathcal{U}_{ad}^h$ *of problem* $(\mathcal{P})_h$.

The main result of this section is

Theorem 3.2. *Let* $w_h \in \mathcal{U}_{ad}^h$ *be a solution of Problem* $(\mathcal{P})_h$ *and* $y_h(w_h)$ *the corresponding solution of the state equation* (3.2). *Then there exist a subsequence* $\{w_{h_j}\} \subset \{w_h\}$, *an element* $w \in \mathcal{U}_{ad}$ *and* $y(w) \in K(\Omega(w))$ *such that*

$$w_{h_j} \rightrightarrows w \text{ (uniformly) in } [0,1], \; h_j \to 0+$$

$$y_{h_j}(w_{h_j}) \rightharpoonup y(w) \text{ in } H^1(G_m(w)), \; h_j \to 0+$$

for any m, *where* w *is a solution of* Problem $(\mathcal{P})$ *and* $y(w)$ *is a solution of the corresponding state inequality* (2.4) *in* $\Omega(w)$.

For the proof of this theorem we need the following auxiliary result:

Lemma 3.3. *Let* $v_h \in \mathcal{U}_{ad}^h$ *be such that* $v_h \rightrightarrows v$ *(uniformly) in* $[0,1]$. *Let* $y_h = y_h(v_h)$ *be a solution of* (3.2) *on domain* $\Omega_h(v_h)$. *Then there exists a subsequence* $\{y_{h_j}(v_{h_j})\} \subset \{y_h(v_h)\}$ *such that*

(3.3) $$y_{h_j}(v_{h_j}) \rightharpoonup y(v) \quad \underline{\text{in}} \quad H^1(G_m(v))$$

<u>for</u> $j \to \infty$ <u>and for any natural m, where</u> $y(v) \in K(\Omega(v))$ <u>is the solution of variational inequality</u> (2.4) and $G_m(v)$ is defined analogously to (2.7).

<u>Proof</u>. To simplify notations we shall write Ω_h and y_h instead of $\Omega_h(v_h)$ and $y_h(v_h)$, when no confusion arises. The proof is similar to the one given for the continuous case [10] and it will be given in several steps.

<u>1°</u> First we prove that there exists $y = y(v) \in H^1(\Omega(v))$ and $\{y_{h_j}(v_{h_j})\} \subset \{y_h(v_h)\}$ such that

(3.4) $$y_{h_j}(v_{h_j}) \rightharpoonup y(v) \quad \text{in } H^1(G_m(v)) \text{ for any } m.$$

The sequence $\{\|y_h\|_{1,\Omega_h}\}$ is bounded. Indeed, from (3.1) it follows that

$$\|\text{grad } y_h\|^2_{0,\Omega_h} \leq (\text{grad } y_h, \text{ grad } y_h)_{0,\Omega_h} + \frac{1}{\varepsilon(h)} (P(y_h), y_h)_{V_h}$$

$$= (f, y_h)_{0,\Omega_h} .$$

From this, Poincare's generalized inequality and from the fact $0 < \alpha \leq v_h(x_2) \leq \beta$ it follows that there exists a constant $c > 0$ independent of h such that

$$c\|y_h\|^2_{1,\Omega_h} \leq \|f\|_{0,\Omega_h} \|y_h\|_{0,\Omega_h} \quad .^{1)}$$

---

1) In the sequel c denotes a generic constant, which doesn't depend on h.

Hence

(3.5) $\quad ||y_h||_{1,\Omega_h} \leq c \quad$ for any h .

Let m be fixed.

Then there exists $h_0 = h_0(m)$ such that $\overline{G_m(v)} \subset \Omega_h$ for all $h \leq h_0$. Consequently by (3.5)

(3.6) $\quad ||y_h||_{1,G_m(v)} \leq ||y_h||_{1,\Omega_h} \leq c \quad$ for all $h \leq h_0$.

As $H^1(G_m(v))$ is a Hilbert space, there exists a $y^{(m)} \in H^1(G_m(v))$ and a subsequence $\{y_{h^{(m)}}\} \subset \{y_h\}$ such that

$$y_{h^{(m)}} \rightharpoonup y^{(m)} \quad \text{in} \quad H^1(G_m(v)) \ .$$

Proceeding in the same way on $G_{m+1}(v)$ with $\{y_{h^{(m)}}\}$ one can choose $\{y_h^{(m+1)}\} \subset \{y_h^{(m)}\}$ such that

$$y_{h^{(m+1)}} \rightharpoonup y^{(m+1)} \quad \text{in} \quad H^1(G_{m+1}(v))$$

and $y^{(m+1)} = 0$ on $\partial G_{m+1}(v) \setminus I_m(v)$, where

$$I_m(v) = \{(x_1,x_2) \in \mathbb{R}^2 | \ x_1 = v(x_2) - 1/m, \quad x_2 \in [0,1]\} \ .$$

Moreover $y^{(m+1)} = y^{(m)}$ on G_m. Setting

$$y = y^{(m)} \qquad \text{on } G_m(v) \quad \text{for all } m$$

we obtain $y \equiv y(v) \in H^1(\Omega(v))$, $y = 0$ on $\partial\Omega(v) \setminus \Gamma(v)$.

A diagonal sequence, constructed from $\{y_{h^{(m)}}\}$ has the required property (3.4).

<u>2°</u> Let us prove that $y \in K(\Omega(v))$

For any $\xi \in H_0^1(\Omega_{\tilde{\beta}}) \cap H^2(\Omega_{\tilde{\beta}})$ we define $\xi_{h_j} = \pi_{h_j}(\xi|_{\Omega_{h_j}})$, where π_{h_j} denotes a mapping, which associates to any $\xi \in H^2(\Omega_{\tilde{\beta}})$ its piecewise linear interpolate from $V_{h_j}(\Omega_{h_j})$ (h_j is filter of indices for which (3.3) holds). Substituting ξ_{h_j} into (3.2), we get

$$(\text{grad}\, y_{h_j}, \text{grad}\, \xi_{h_j})_{0,\Omega_{h_j}} + \frac{1}{\varepsilon(h_j)}(P(y_{h_j}), \xi_{h_j})_{V_{h_j}} = (f, \xi_{h_j})_{0,\Omega_{h_j}} .$$

From this follows

$$(3.7) \qquad 0 \leq |(P(y_{h_j}), \xi_{h_j})_{V_{h_j}}|$$
$$\leq \varepsilon(h_j)\{\|f\|_{0,\Omega_{\tilde{\beta}}} + \|\text{grad}\, y_{h_j}\|_{0,\Omega_{h_j}}\}\|\xi_{h_j}\|_{1,\Omega_{h_j}}$$

By utilizing the approximating property of linear interpolate we find that

$$\|\xi_{h_j} - \xi\|_{1,\Omega_{h_j}} \leq c\, h_j|\xi|_{2,\Omega_{h_j}} \leq c\, h_j|\xi|_{2,\Omega_{\tilde{\beta}}} .$$

Consequently

$$\|\xi_{h_j}\|_{1,\Omega_{h_j}} \leq \|\xi_{h_j} - \xi\|_{1,\Omega_{h_j}} + \|\xi\|_{1,\Omega_{h_j}} \leq c .$$

Taking into account this, (3.5) and (3.7) we have

$$(3.8) \qquad \lim_{h_j \to 0+} (P(y_{h_j}), \xi_{h_j})_{V_{h_j}} = 0 .$$

On the other hand,

$$(3.9)\qquad |(P(y),\xi)_V - (P(y_{h_j}),\ \xi_{h_j})_{V_{h_j}}|$$

$$\leq |(P(y),\xi)_V - (P(y_{h_j}),\xi)_{V_{h_j}}| + |(P(y_{h_j}),\ \xi-\xi_{h_j})_{V_{h_j}}| .$$

The first term on the right hand side of (3.9) tends to zero, as follows from Lemma 3.2. in [10]. Let us estimate the second term:

$$|(P(y_{h_j}),\xi-\xi_{h_j})_{V_{h_j}}| = |\int_0^1 y^-_{h_j}\ (\xi(v_{h_j}) - \xi_{h_j}(v_{h_j}))dx_2|$$

$$\leq c\|y_{h_j}\|_{0,\Gamma(v_h)}\|\xi(v_{h_j}) - \xi_{h_j}(v_{h_j})\|_{0,\Gamma(v_h)}$$

$$\leq c\ \|y_{h_j}\|_{1,\Omega_{h_j}}\|\xi-\xi_{h_j}\|_{1,\Omega_{h_j}} \to 0 ,$$

if $h_j \to 0+$. In the last inequality we have used the fact that the norm of the trace mapping $\gamma : V(\Omega(v_h)) \to L^2(\Gamma(v_h))$ can be estimated uniformly with respect to h for sufficiently small h (in our situation this fact follows immediately from the proof of trace theorem, see [19], Th 2.1). Comparing (3.8) with (3.9) we see that

$$\lim_{h_j\to 0+} (P(y_{h_j}),\ \xi_{h_j})_{V_{h_j}} = (P(y),\ \xi)_V = 0$$

for any $\xi \in H^1_0(\Omega_{\tilde\beta}) \cap H^2(\Omega_{\tilde\beta})$, i.e. $y \geq 0$ on $\Gamma(v)$.

3^o. To establish the assertion of Lemma 3.3., it remains to verify that y is a solution of variational inequality (2.4).

Let $\xi \in K(\Omega(v))$. Then there exists a function $\psi \in H^1(\Omega_{\tilde\beta})$ such that

$$\psi|\partial\Omega(v) = \xi|\partial\Omega(v)$$

and $\psi \geq 0$ in $\Omega_{\tilde\beta}$. Setting

$$\eta = \xi - \psi$$

it is readily seen that $\eta \in H_0^1(\Omega(v))$ and therefore

$$\eta = \lim_{i\to\infty} \eta_i \qquad \text{in } H^1(\Omega(v))$$

with $\eta_i \in \mathcal{D}(\Omega(v))$. Similarly, one can prove the existence of $\psi_i \in C^\infty(\overline{\Omega}_{\tilde\beta})$ $\psi_i \geq 0$ in $\Omega_{\tilde\beta}$, such that

$$\psi_i \to \psi \qquad \text{in } H^1(\Omega_{\tilde\beta}) .$$

Let us define

$$\xi_i = \begin{cases} \psi_i + \eta_i & \text{in } \Omega(v) \\ \psi_i & \text{in } \Omega_{\tilde\beta} \setminus \Omega(v) . \end{cases}$$

Clearly $\xi_i \in C^\infty(\overline{\Omega_{\tilde\beta}})$,

$$\xi_i \to \psi + \eta = \xi \qquad \text{in } H^1(\Omega(v))$$

and

$$\xi_i \in K_m(\Omega(v)) = \{z \in H^1(\Omega_{\tilde\beta}) \mid z \geq 0 \text{ in } \Omega_{\tilde\beta} \setminus G_m(v)\}$$

for all sufficiently large i. Let

$$K_h(\Omega_h(v_h)) = \{\xi_h \in V_h(\Omega_h) \mid \xi_h \geq 0 \quad \text{on } \Gamma(v_h)\}$$

Then

$$\xi_{ih} = \pi_h(\xi_i|_{\Omega_h}) \in K_h(\Omega_h)$$

if i is large and h is small. Let h_j, $j \to \infty$ be filter of indices, for which (3.3) holds.
Consider the problem (3.2) in which a function ξ_h is replaced by $y_{h_j} - \xi_{ih_j}$. As $\xi_{ih_j} \in K_{h_j}(\Omega_{h_j})$ and $P(\xi_{ih_j}) = 0$, we obtain by utilizing the monotonity of P that

$$(3.10) \qquad (\operatorname{grad} y_{h_j}, \operatorname{grad} y_{h_j} - \operatorname{grad} \xi_{ih_j})_{0,\Omega_{h_j}} \leq (f, y_{h_j} - \xi_{ih_j})_{0,\Omega_{h_j}} .$$

Furthermore, it holds

$$(3.11) \qquad (\operatorname{grad} y_{h_j}, \operatorname{grad}(\xi_{ih_j} - y_{h_j}))_{0,\Omega_{h_j}}$$

$$= (\operatorname{grad} y_{h_j}, \operatorname{grad}(\xi_{ih_j} - y_{h_j}))_{0,G_m(v)}$$

$$+ (\operatorname{grad} y_{h_j}, \operatorname{grad}(\xi_{ih_j} - y_{h_j}))_{0,\Omega_{h_j} \setminus \Omega(v)}$$

$$+ (\operatorname{grad} y_{h_j}, \operatorname{grad}(\xi_{ih_j} - y_{h_j}))_{0,(\Omega(v) \setminus G_m(v)) \cap \Omega_{h_j}}$$

$$\leq (\operatorname{grad} y_{h_j}, \operatorname{grad}(\xi_{ih_j} - y_{h_j}))_{0,G_m(v)}$$

$$+ (\operatorname{grad} y_{h_j}, \operatorname{grad} \xi_{ih_j})_{0,\Omega_{h_j} \setminus \Omega(v)}$$

$$+ (\operatorname{grad} y_{h_j}, \operatorname{grad} \xi_{ih_j})_{0,(\Omega(v) \setminus G_m(v)) \cap \Omega_{h_j}} .$$

As

$$y_{h_j} \rightharpoonup y \qquad \text{in } H^1(G_m(v))$$

and

$$\xi_{ih_j} \to \xi_i \qquad \text{in } H^1(G_m(v)) \quad ,$$

then

(3.12) $$\limsup_{h_j\to 0+} (\operatorname{grad} y_{h_j}, \operatorname{grad}(\xi_{ih_j} - y_{h_j}))_{0,G_m(v)}$$

$$\leq (\operatorname{grad} y, \operatorname{grad}(\xi_i - y))_{0,G_m(v)} .$$

Because of (3.5) and the fact that $v_{h_j} \rightrightarrows v$ uniformly in [0,1] it holds

(3.13) $$(\operatorname{grad} y_{h_j}, \operatorname{grad} \xi_{ih_j})_{0,\Omega_{h_j} \setminus \Omega(v)} \to 0 , \text{ for } j \to \infty .$$

As

$$|(\operatorname{grad} y_{h_j}, \operatorname{grad}(\xi_{ih_j} - \xi_i))_{0,(\Omega(v) \setminus G_m(v)) \cap \Omega_{h_j}}|$$

$$\leq c\|\xi_{ih_j} - \xi_i\|_{1,\Omega_{h_j}} \leq c\, h_j|\xi_i|_{2,\Omega(v)} \to 0 \quad \text{for } j \to \infty$$

we can conclude that

$$\limsup_{j\to\infty} (\operatorname{grad} y_{h_j}, \operatorname{grad} \xi_{ih_j})_{0,(\Omega(v) \setminus G_m(v)) \cap \Omega_{h_j}}$$

$$\leq \limsup_{j\to\infty} (\operatorname{grad} y_{h_j}, \operatorname{grad} \xi_i)_{0,(\Omega(v) \setminus G_m(v)) \cap \Omega_{h_j}}$$

$$+ \limsup_{j\to\infty} (\operatorname{grad} y_{h_j}, \operatorname{grad}(\xi_{ih_j} - \xi_i))_{0,(\Omega(v) \setminus G_m(v)) \cap \Omega_{h_j}} .$$

From this, (3.11), (3.12) and (3.13) we obtain that

(3.14) $$\limsup_{j\to\infty} (\operatorname{grad} y_{h_j}, \operatorname{grad}(\xi_{ih_j} - y_{h_j}))_{0,\Omega_{h_j}}$$

$$\leq (\operatorname{grad} y(v), \operatorname{grad}(\xi_i - y(v)))_{0,G_m(v)} + c\|\xi_i\|_{1,\Omega(v)\setminus G_m(v)} .$$

Finally, we estimate the linear term:

$$(f, \xi_{ih_j} - y_{h_j})_{0,\Omega_{h_j}} = (f, \xi_{ih_j} - y_{h_j})_{0,G_m(v)}$$

$$+ (f, \xi_{ih_j} - y_{h_j})_{0,\Omega_{h_j} \setminus \Omega(v)} + (f, \xi_{ih_j} - y_{h_j})_{0,(\Omega(v) \setminus G_m(v)) \cap \Omega_{h_j}} .$$

Hence,

$$\text{(3.15)} \qquad \liminf_{j\to\infty} (f, \xi_{ih_j} - y_{h_j})_{0,\Omega_{h_j}}$$

$$\geq (f, \xi_i - y)_{0,G_m(v)} - c \{\|f\|_{0,\Omega(v)\setminus G_m(v)} + \|\xi_i\|_{1,\Omega(v)\setminus G_m(v)}\} .$$

By summing (3.10)-(3.15) we have

$$\text{(3.16)} \qquad (\text{grad } y(v), \text{grad}(\xi_i - y(v)))_{0,G_m(v)} + c\|\xi_i\|_{1,\Omega(v)\setminus G_m(v)}$$

$$\geq (f, \xi_i - y(v))_{0,G_m(v)} - c \{\|f\|_{0,\Omega(v)\setminus G_m(v)} + \|\xi_i\|_{1,\Omega(v)\setminus G_m(v)}\} .$$

Letting $m \to \infty$ we have

$$(\text{grad } y(v), \text{grad}(\xi_i - y(v)))_{0,\Omega(v)} \geq (f, \xi_i - y(v))_{0,\Omega(v)} .$$

Thence, if $i \to \infty$ we have

$$(\text{grad } y(v), \text{grad}(\xi - y(v)))_{0,\Omega(v)} \geq (f, \xi - y(v))_{0,\Omega(v)}$$

for any $\xi \in K(\Omega(v))$. This completes the proof of Lemma 3.3.

Now we are able to prove Theorem 3.2.

Proof of Theorem 3.2. Let $\{w_h\}$, $h \to 0+$, $w_h \in \mathcal{U}^h_{ad}$ are solutions of Problem $(\mathcal{P})_h$. As $\mathcal{U}^h_{ad} \subset \mathcal{U}_{ad}$ for all h and $\mathcal{U}_{ad}$ is compact in C^0-topology, there exists a subsequence $\{w_{h_j}\} \subset \{w_h\}$ and $w \in \mathcal{U}_{ad}$ such that

$$w_{h_j} \rightrightarrows w \in \mathcal{U}_{ad} \quad \text{uniformly in } [0,\ 1] .$$

Let $\{y_{h_j}(w_{h_j})\}$ be corresponding solutions of state problem (3.2). In accordance with Lemma 3.3, there exists a subsequence $\{y_{h_j}(w_{h_j})\}$ (we use the same notation) and an element $y(w) \in K(w)$ such that

$$y_{h_j}(w_{h_j}) \rightharpoonup y(w) \qquad \text{in } H^1(G_m(w))$$

for any m, and moreover, y(w) solves the state inequality (2.4).

We now show that w solves Problem $(\mathcal{P})$. Let $v \in \mathcal{U}_{ad}$ be an arbitrary function. Then there exists $\overline{v}_h \in \mathcal{U}^h_{ad}$ such that

$$\overline{v}_h \rightrightarrows v \qquad \text{uniformly in } [0,1]$$

(see [3]). By definition of Problem $(\mathcal{P})_h$ one has:

$$\mathcal{J}_h(w_h) \leq \mathcal{J}_h(\overline{v}_h) \qquad \text{for all } v_h \in \mathcal{U}^h_{ad} .$$

Let us write

$$\mathcal{J}_{h_j}(w_{h_j}) = \|y_{h_j}(w_{h_j}) - z_d\|^2_{0,G_m(w)} + \|y_{h_j}(w_{h_j}) - z_d\|^2_{0,\Omega_{h_j}(w_{h_j}) \setminus G_m(w)}$$

$$\geq \|y_{h_j}(w_{h_j}) - z_d\|^2_{0,G_m(w)}$$

so that

$$(3.17)\qquad \|y_{h_j}(w_{h_j}) - z_d\|^2_{0,G_m(w)} \le J_{h_j}(w_{h_j}) \le J_{h_j}(\overline{v}_{h_j})$$

$$= \|y_{h_j}(v_{h_j})-z_d\|^2_{0,G_m(v)} + \|y_{h_j}(v_{h_j})-z_d\|^2_{0,\Omega_{h_j}(v_{h_j})\setminus G_m(v)} .$$

Here $y_{h_j}(\overline{v}_{h_j})$ denotes the solution of (3.2) on $\Omega_{h_j}(\overline{v}_{h_j})$.

Now the indices are chosen in such a way that both $\{y_{h_j}(w_{h_j})\}$ as well as $\{y_{h_j}(\overline{v}_{h_j})\}$ tend weakly to $y(w)$ and to $y(v)$ in $H^1(G_m(w))$ and $H^1(G_m(v))$, respectively. Applying Rellich's theorem to (3.17) we get

$$(3.18)\qquad \|y(w) - z_d\|^2_{0,G_m(w)}$$

$$\le \|y(v)-z_d\|^2_{0,G_m(v)} + \limsup_{j\to\infty} \|y_{h_j}(\overline{v}_{h_j})-z_d\|^2_{0,\Omega_{h_j}(\overline{v}_{h_j})\setminus G_m(v)} .$$

Let us analyse the last term on the right hand side of (3.18). One can write

$$(3.19)\qquad \|y_{h_j}(\overline{v}_{h_j}) - z_d\|^2_{0,\Omega_{h_j}(\overline{v}_{h_j})\setminus G_m(v)}$$

$$\le (\mathrm{meas}(\Omega_{h_j}(\overline{v}_{h_j})\setminus G_m(v)))^{1/2}\|y_{h_j}(\overline{v}_{h_j})-z_d\|^2_{L^4(\Omega_{h_j}(\overline{v}_{h_j}))} .$$

We define a mapping $F_h : \mathbb{R}^2 \to \mathbb{R}^2$ by means of the relation

$$F_h(\hat{x}_1, \hat{x}_2) = (\hat{x}_1 v_h(\hat{x}_2), \hat{x}_2) ,$$

$(\hat{x}_1, \hat{x}_2) \in \hat{\Omega} := (0,1) \times (0,1)$. Then $\Omega_{h_j}(\overline{v}_{h_j}) = F_{h_j}(\hat{\Omega})$.

Fubini's theorem and the continuity of the imbedding $H^1(\hat{\Omega}) \hookrightarrow L^4(\hat{\Omega})$ imply

(3.20) $$\|y_{h_j}(\bar{v}_{h_j}) - z_d\|^2_{L^4(\Omega_{h_j}(\bar{v}_{h_j}))}$$

$$\leq c\,\|\hat{y}_{h_j} - \hat{z}_{d_j}\|^2_{L^4(\hat{\Omega})} \leq \hat{c}\,\|\hat{y}_{h_j} - \hat{z}_{d_j}\|^2_{1,\hat{\Omega}} \quad ,$$

when $\hat{y}_{h_j} = y_{h_j} \circ F_{h_j}$, $\hat{z}_{d_j} = z_d \circ F_{h_j}$ and $\hat{c} > 0$ is independent of h_j. Applying Fubini's theorem once again as well as (3.5) we get

$$\|\hat{y}_{h_j} - \hat{z}_{d_j}\|^2_{1,\hat{\Omega}} \leq c\, \|y_{h_j} - z_d\|^2_{1,\Omega_{h_j}} \leq c \; .$$

This and (3.19) imply the following estimate:

(3.21) $$\limsup_{j\to\infty} \|y_{h_j}(\bar{v}_{h_j}) - z_d\|^2_{0,\Omega_{h_j}(v_{h_j}) \setminus G_m(v)}$$

$$\leq c\,\{\text{meas}(\Omega(v) \setminus G_m(v))\}^{1/2} \; .$$

A combination of (3.18) and (3.21) yields

(3.22) $$\|y(w) - z_d\|^2_{0,G_m(v)}$$

$$\leq \|y(v) - z_d\|^2_{0,G_m(v)} + c\,\{\text{meas}\;\Omega(v) \setminus G_m(v)\}^{1/2} \; .$$

Passing to the limit with $m \to \infty$, we finally get

$$J(w) \leq J(v) \qquad \text{for all} \quad v \in \mathcal{U}_{ad} \; .$$

Thence Theorem 3.2 is proved.

4. NUMERICAL REALIZATION OF OPTIMAL SHAPE DESIGN PROBLEM

4.1 Construction of moving triangular grid

Let $\Omega_h(v_h) = \hat{\Omega}' \cup \Omega_h^r(v_h)$, where $\hat{\Omega}' = (0, \alpha') \times (0, 1)$, $\alpha' < \alpha$, is a part of $\Omega_h(v_h)$, where the moving boundary $\Gamma_2(v_h)$ cannot penetrate and $\Omega_h^r(v_h)$ the rest of $\Omega_h(v_h)$ (see Figure 4.1).

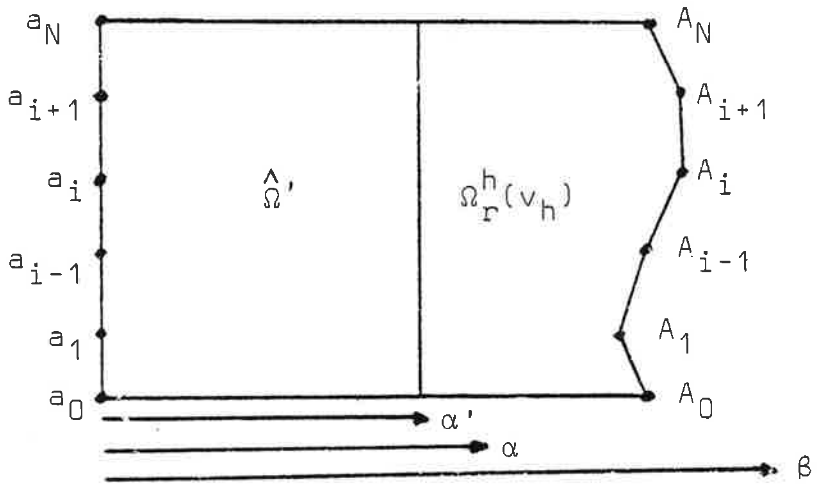

Figure 4.1. $\Omega_h(v_h) = \hat{\Omega}' \cup \Omega_h^r(v_h)$.

Let $\hat{T}_h$ and $T_h^r(v_h)$ resp. be a triangulation of $\hat{\Omega}'$ and of $\Omega_h^r(v_h)$ resp. $T_h^r(v_h)$ will be constructed my means of principle moving points

$$(4.1) \qquad A_i = (x_1^i, a_i), \; x_1^i = v_h(a_i), \; a_i = ih ,$$

further by means of associated moving points

$$A_i^j = (\varphi_i^j(x_1^i), a_i)$$

and of fixed points

$$\hat{A}_i = (\alpha', a_i)$$

$i = 0, \ldots, N, \quad j = 1, \ldots, M, \quad N = 1/h, \quad M = [\alpha/h]$.

It is readily seen that principal moving nodes and associate moving nodes may vary in x_1-direction only. The position of moving points is associated to principal moving points by means of function φ_i^j . If the dependence is supposed to be linear one may define

$$(4.2) \qquad \varphi_i^j(x_1^i) = \alpha' + \frac{x_1^i - \alpha'}{M} j \ , \quad j = 0, \ldots, M \ .$$

Finally, $T_h^r(v_h)$ will be constructed by points A_i, A_i^j and $\hat{A}_i$ such that triangulation of $\Omega_h^r(v_h)$ will be regular (see Figure 4.2)

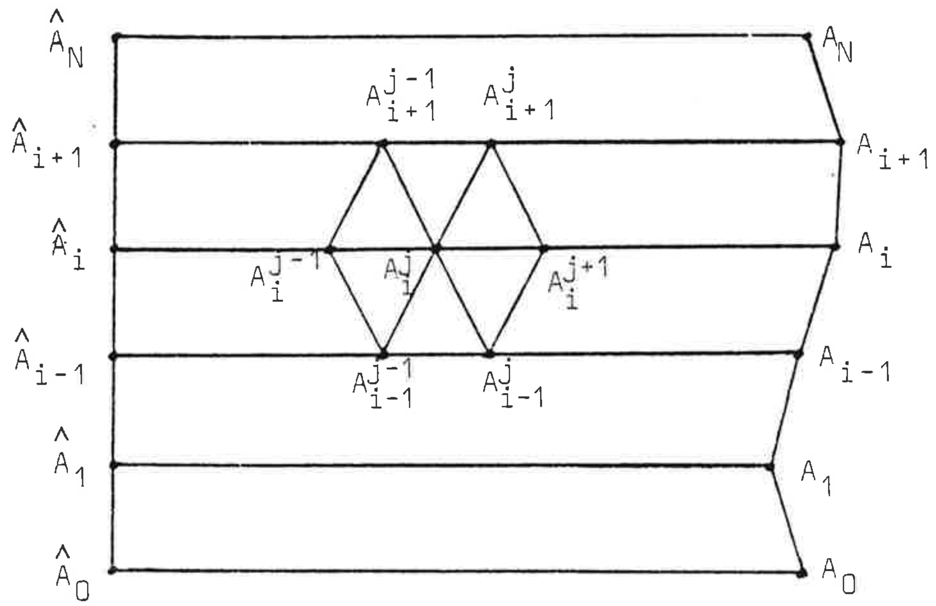

Figure 4.2. Triangulation of $\Omega_h^r(v_h)$.

Evidently $T_h(v_h) = \hat{T}_h \cup T_h^r(v_h)$ satisfy the assumptions desired in chapter 2 for regular triangulation of $\Omega(v_h)$.

4.2 Algebraic formulation of discretized shape design problem

Taking into account the geometry of $\Omega_h(v_h)$ and piecewise linearity of $\Gamma(v_h)$, for finding optimal $\Gamma(v_h)$ it is enough to find the x_1-coordinates of the principal moving nodes (determining $\Gamma(v_h)$) such that J_h is minimum. As for fixed h, the shape of $\Omega_h(v_h)$ and value of y_h resp. depends on design variables (x_1-coordinates of A_i)

$$X = (x_1^0, \ldots, x_1^N)$$

we shall write $\Omega_h(X)$ and $y_h(X)$ resp. and instead of $(x_1^0, \ldots, x_1^N)$ we shall write simply $(x_0, \ldots, x_N)$.

For fixed $\Omega_h(X)$, $y_h(X)$ can be obtained by solving the algebraic form of (3.2) for nodal displacements

$$Q(X) = (q_1, \ldots, q_p, q_{p+1}, \ldots, q_{p-N-1})^T$$

(p = number of interior nodes of $\mathcal{T}_h(v_h)$):

$$A(X)\; Q(X) - \frac{1}{\varepsilon} D(Q(X)) = F(X) . \tag{4.3}$$

Here $A(X)$ is the global stiffness matrix, which presents in $V_h(\Omega_h(X))$ the piecewise linear finite element discretization of $-\Delta$, $F(X)$ is the corresponding discrete right hand side; the dependence of these factors on design variables is emphasized here. Operator $D : \mathbb{R}^{K(h)} \to \mathbb{R}^{K(h)}$, $K(h) = \dim V_h(\Omega_h(X))$, $K(h) = p + N - 1$, is a nonlinear mapping corresponding to the second term of (3.2) with penalty operator $P(y_\varepsilon) = -(y_\varepsilon^-)^2$,

$$D(Q) = (0, \ldots, 0,\; h(q_{p+1}^-)^2, \ldots,\; h(q_{p+N-1}^-)^2) , \tag{4.4}$$

where $q_{p+1}, \dots, q_{p+N-1}$ are nodal displacements at principal moving nodes $(A_1, \dots, A_{N-1})$.

For the sake of smoothness, we have slightly modified the penalty operator from that of the previous chapters (this is allowed in the finite dimensional case, at least). The differentiability of the mapping $Q \to D(Q)$ will be utilized in sensitivity analysis (chapter 4.3).

In order to express the cost functional J_h by means of nodal displacements, we first find that it locally holds (for the sake of simplicity we assume $z_d \equiv 0$):

(4.5) $$J_h^T := \int_T (y_h(X))^2 dx$$

$$= \frac{\text{meas } T}{6} (q_i(X)^2 + q_j(X)^2 + q_k(X)^2$$

$$+ q_i(X)q_j(X) + q_j(X)q_k(X) + q_i(X)q_k(X)).$$

The dependence of the nodal displacements on design variables is emphasized here. We note that meas T depends on X, if vertices belong to principal moving nodes or to associated moving nodes.

If we write $\mathcal{U}_{ad}^h$ in the form

(4.6) $$\mathcal{U}_{ad}^h = \{v_h \in C([0,1]) \mid v_h|_{[a_{i-1},a_i]} \in P_1,\ \alpha \leq v_h \leq \beta,$$

$$|v_h(t)-v_h(t')| \leq C_1|t-t'|,\ \int_0^1 v_h(t)dt = C_2\}$$

and take into account the local formula (4.5) we can formulate problem $(P)_h$ for fixed h and $\varepsilon(h)$ as the following nonlinear programming problem with linear constraints:

(4.7) $\quad$ Minimize $\psi(X,Q(X))$

subject to constraints

(4.8) $\quad \alpha \leq x_i \leq \beta \ , \quad i = 0,\ldots, N \ ,$

(4.9) $\quad C X \geq b \ .$

Here

$$\psi(X,Q(X)) = \sum_{i,j=1}^{K(h)} b_{ij}(X)\, q_i(X)\, q_j(X)$$

denotes the objective function obtained from $J_h(v_h) = \int_{\Omega_h(v)} (y_h(v_h))^2\, dx$ by the local formula (4.5); (4.8), (4.9) correspond to constraints, given in the definition of (4.6).

When the discretization of the continuous model has been done (by finite elements here), the backbone of a computer program for the numerical solution of the design problem is an optimization algorithm (quasi-Newton, conjugate gradient, projected gradient etc., as a rule a gradient algorithm). If the gradient with respect to design variables is not available, one must utilize nonsmooth optimization algorithms (methods without derivatives, derivatives with finite differences or subgradient methods). The benefit of the penalty method over variational inequality formulation of the state problem is that it enables us to use gradient methods.

It turns out that the gradient with respect to the design variables (variation of the boundary) is an involved step. The next section is devoted to this single step.

4.3 Computation of the gradient for the cost functional

By local formula (4.5), $\frac{\partial}{\partial x_l} \psi(X,Q(X))$, $l = 0,\dots, N$, consists of terms $\frac{\partial}{\partial x_l}$ meas T_i, $T_i \in \mathcal{T}_h^r(v_h)$, and $\frac{\partial}{\partial x_l} q_i(X)$:

$$(4.10) \qquad \frac{\partial}{\partial x_l} \psi(X,Q(X)) = \sum_{i,j=1}^{K(h)} \left(\frac{\partial}{\partial x_l} b_{ij}(X)\right) q_i(X) q_j(X)$$

$$+ \sum_{i,j=1}^{K(h)} \left[b_{ij}(X)\left(\frac{\partial}{\partial x_l} q_i(X) q_j(X) + q_i(X) \frac{\partial}{\partial x_l} q_j(X)\right)\right] .$$

As $\frac{\partial}{\partial x_l}$ meas T_i is evident to compute, the only problem in (4.10) is to find $\frac{\partial}{\partial x_l} q_i$, i.e. derivatives of nodal displacements with respect to design variables x_l, $l = 0,\dots, N$. We are led to question: how the nodal displacements are varying when the boundary $\Gamma(v_h)$ is varying? We shall make use of the implicit function theorem.

Let

$$(4.11) \qquad \Phi(Q,X) = A(X)Q - \frac{1}{\varepsilon} D(Q) - F(X)$$

and let X be fixed (i.e. $\Gamma(v_h)$ is fixed). Then the problem: Find $Q = Q(X)$ such that

$$(4.12) \qquad \Phi(Q,X) = 0$$

is equivalent to (4.3). Now

$$\frac{\partial}{\partial Q} \Phi \frac{\partial}{\partial x_l} Q + \frac{\partial}{\partial x_l} \Phi = 0, \qquad l = 0,\dots, N$$

is by (4.3) equivalent to equation

$$(4.13) \qquad (A(X) - \frac{1}{\varepsilon} D'(Q)) \frac{\partial}{\partial x_l} Q = \frac{\partial}{\partial x_l} F(X) - (\frac{\partial}{\partial x_l} A(X))Q$$

for $l = 0, \ldots, N$, where

$$D'(Q) = -2h \begin{bmatrix} 0 & & & & 0 \\ & & & & \\ & q^-_{p+1} & & & 0 \\ 0 & & q^-_{p+2} & & \\ & & & \ddots & \\ & 0 & & & q^-_{p+N-1} \end{bmatrix}$$

Thence, by (4.10) $\frac{\partial}{\partial x_l} Q(X)$ is obtained as a solution of linear system (4.13). As $A(X)$ is positive definite, the same is true for $(A(X) - \frac{1}{\varepsilon} D'(Q))$.

From the construction of $A(X)$ and $F(X)$ it is evident that the derivatives of $A(X)$ and $F(X)$ on the right hand side of (4.13) must be computed from corresponding derivatives of the <u>local stiffness matrix</u> and of the <u>local force vector</u>. For details see [20].

4.4 Algorithms for solving Problem $(\mathcal{P})_h$

The steps for solving the discretized shape design problem $(\mathcal{P})_h$ are: solve the discrete state equation (3.2) in order to obtain an objective function $\psi(X,Q(X))$, compute $\nabla_X \psi(X,Q(X))$ (if exists) and finally use an appropriate minimizing algorithm for finding a decreasing sequence: $\psi(X^i,Q(X^i)) \geq \psi(X^{i+1},Q(X^{i+1}))$. These steps are considered now in more detail.

1. <u>The state problem</u>. Because of the nonlinear part $D(Q)$ the state equation (4.3) must be solved iteratively. Both gradient methods and relaxations methods can be applied. Due to the high dimension of the problem we have used the relaxation method.

The nonlinear over relaxation algorithm for solving (4.3) with the penalty term ε reads:

<u>Algorithm (NSOR)</u> Let $Q^o \in \mathbb{R}^{K(h)}$ be arbitrarily chosen. Then Q^n being known, compute Q^{n+1}, component by component as follows. Let

(4.14) $$q_i^{n+1/2} = \frac{1}{a_{ii}} G_i^{n+1} \quad \text{for } i \notin I(\Gamma(v_h)) \text{ or if } G_i^{n+1} \geq 0$$

and

(4.15) $$q_i^{n+1/2} = \frac{a_{ii} - (a_{ii}^2 - 4h\varepsilon^{-1} G_i^{n+1})^{1/2}}{2\ h/\varepsilon}$$

for $i \in I\ (\Gamma(v_h))$ and $G_i^{n+1} < 0$, where

(4.16) $$G_i^{n+1} = F_i - \sum_{j=1}^{i-1} a_{ij}\ q_j^{n+1} - \sum_{j=i+1}^{K(h)} a_{ij}\ q_j^n$$

and where $I(\Gamma(v_h))$ refers to indices $i = p+1, \ldots, p+N-1$ corresponding to nodes of $\Gamma(v_h)$. Set

(4.17) $$q_i^{n+1} = q_i^n + \omega(q_i^{n+1/2} - q_i^n) .$$

The critical point of Algorithm (NSOR) is the optimal choice of relaxation parameter ω. In practice Q^0 will be chosen to be equal to the solution of the state problem of the previous iteration in the minimization procedure (step 3).

2. <u>The gradient for cost functional.</u> Form $\frac{\partial}{\partial x_l} F(X)$ and $\frac{\partial}{\partial x_l} A$ for $l = 0,\ldots, N$ by using local force vectors and local stiffness matrixes. Solve equation (4.13) by Cholesky method in order to obtain $\frac{\partial}{\partial x_l} Q(X)$ for $l = 0,\ldots, N$. Compute $\frac{\partial}{\partial x_l}$ meas (T), $T \in \mathcal{T}_h(v_h)$ in order to calculate the term $\frac{\partial}{\partial x_l} b_{ij}(X)$, $l = 0,\ldots, N$ (the majority of these terms will be identically zero). Finally $\frac{\partial}{\partial x_l} \psi(X,Q(X))$ is then obtained by equation (4.10).

3. <u>Minimizing of cost functional.</u> In a concrete choice of the optimization algorithms, specific features of the problem have to be taken into account:

i) The evaluation of cost functional and its gradient are time consuming (steps 1 and 2). The Hessian matrix cannot be given explicitly..

ii) Constraints are linear containing box constraints, inequality constraints (Lipschitz-condition) and an equality constraint ($\text{meas}(\Omega_h(v_h)) = C_2$). The Lipschitz condition reads

$$-\frac{C_1}{N} \leq x_l - x_{l-1} \leq \frac{C_1}{N}, \quad l = 1,\ldots, N \tag{4.18}$$

and the volume constraint

$$\sum_{l=1}^{N} (x_l + x_{l-1}) = 2\, C_2 N \,. \tag{4.19}$$

This implies that matrix (C_{ij}) in (4.9) is sparse.

iii) Function $v \to J(y(v))$ is not convex. Thence an initial guess plays an important role in minimizing procedure. Because of the complexity and the high dimensionality of the problem global optimization is not relevant.

For algorithms solving linearly constrained optimization problems we refer to [8, 9], to [18] (large scale linearly constrained problems) and to [30].

5. REMARKS ON ALTERNATIVE METHODS

5.1 Dual formulation for the state problem

By applying Green's formula to (2.4) one can easily prove that $y = y(v) \in K(\Omega(v))$ satisfies in $\Omega(v)$ the Poisson equation with mixed Dirichlet-Signorini boundary conditions:

$$(5.1) \qquad \begin{cases} -\Delta y = f & \text{in } \Omega(v) \\ y = 0 & \text{on } \Gamma_1 \\ y \geq 0 , \quad \frac{\partial}{\partial n} y \geq 0 , \; y \cdot \frac{\partial}{\partial n} y = 0 & \text{on } \Gamma(v) \end{cases}$$

For regularity of such type of a problem see [2] .

Using a saddle-point formulation to a unilateral boundary value problem (5.1) on $K(\Omega(v))$ ($v \in \mathcal{U}_{ad}$ being fixed) we obtain a problem: Find $(y, \frac{\partial}{\partial n} y) \in V(\Omega(v)) \times \Lambda$ such that

$$(5.2) \qquad L(y, \frac{\partial}{\partial n} y) = \min_{\varphi \in V(\Omega(v))} \sup_{\mu \in \Lambda} L(\varphi,\mu) = \max_{\mu \in \Lambda} \inf_{\varphi \in V(\Omega(v))} L(\varphi,\mu) ,$$

where $L : V(\Omega(v)) \times \Lambda \to \mathbb{R}$ is a lagrangian defined by means of

$$L(y(v),\mu) = \frac{1}{2} \|\text{grad } y(v)\|^2_{0,\Omega(v)} - (f,y(v))_{0,\Omega(v)} - \langle \mu, y(v) \rangle$$

and

$$\Lambda = \{\mu \in H^{-1/2}((0,1)) \mid \mu \geq 0\} .$$

The symbol $\langle \, , \rangle$ denotes the duality pairing between $H^{-1/2}((0,1))$ and $H^{1/2}((0,1))$, which is an extension of a scalar product in $L^2((0,1))$, i.e.

$$\langle \mu, y(v) \rangle = \int_0^1 \mu(x_2)\, y(v(x_2))\, dx_2,$$

$$\mu \in L^2((0,1)).$$

By utilizing the classical Uzawa method to (5.2), we have the following iterative procedure for solving the original unilateral boundary value problem (5.1) in $\Omega(v)$:

Algorithm 5.1. Let $\lambda^0 \in \Lambda$ be given. If $\lambda^{(k)} \in \Lambda$ is known, find $y^{(k)} \in V(\Omega(v))$ such that

(5.3) $$L(y^{(k)}, \lambda^{(k)}) \leq L(\varphi, \lambda^{(k)}) \quad \text{for all} \quad \varphi \in V(\Omega(v)).$$

Replace $\lambda^{(k)}$ by a new value $\lambda^{(k+1)}$ as follows

(5.4) $$\lambda^{(k+1)} = P_\Lambda(\lambda^{(k)} - \rho\, y^{(k)}), \quad \rho > 0,$$

where P_Λ is a projection onto the convex set Λ.

The convergence of the above type of algorithm is studied in [4]. Step (5.3) consists of a mixed elliptic boundary value problem

(5.5) $$\begin{cases} -\Delta y(v)^{(k)} = f & \text{in } \Omega(v) \\ y(v)^{(k)} = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial}{\partial n} y(v(x_2))^{(k)} = \dfrac{\lambda^{(k)}(x_2)}{(1+(v'(x_2))^2)^{1/2}}, & x_2 \in (0,1). \end{cases}$$

The algorithm we propose here for solving the optimal shape design problem (2.1) with the state problem (5.1) proceeds as follows:

Algorithm 5.2. Start with arbitrarily given $\lambda^{(0)} \in \Lambda$. Let $\lambda^{(k)} \in \Lambda$ be already known. Then solve the optimal shape design problem (2.1) with (5.5) as a state equation.

Let $v^{(k)} \in \mathcal{U}_{ad}$ be its solution and $y^{(k)}(v^{(k)})$ be the corresponding state, respectively. Finally, replace $\lambda^{(k)}$ by $\lambda^{(k+1)}$ by means of

$$\lambda^{(k+1)} = P_{\Lambda}(\lambda^{(k)} - \rho y^{(k)}(v^{(k)})), \quad \rho > 0 .$$

Let us note in this place that the convergence of Algorithm 5.2 remains still an open problem.

5.2 Adjoint state for the penalized problem

In chapter 4.4 the gradient for the cost functional J_ε is computed in algebraic form. Due to the simple geometry it is quite straightforward. We note however on the use of more general methods in sensitivity analysis of shape design problems (material (or speed) derivatives, see [6] , [29]). An essential point is to find the adjoint state. In connection of the penalized problem (2.5) it reads: Find $p \in V(\Omega(v))$ such that

$$(5.6) \qquad \int_{\Omega(v)} \operatorname{grad} p \cdot \operatorname{grad} \xi \, dx + \frac{2}{\varepsilon} \int_0^1 p|_{\Gamma(v)} \, y_\varepsilon^-|_{\Gamma(v)} \, \xi|_{\Gamma(v)} \, dx_2$$

$$= 2 \int_{\Omega(v)} (y_\varepsilon - z_d) \, \xi \, dx \quad \text{for all } \xi \in V(\Omega(v)) .$$

By utilizing the adjoint state equation (5.6), the material derivative for cost functional J_ε can be computed by evident modifications of methods presented in [6, 29]. See also [26, 27].

6. NUMERICAL EXAMPLES

Problem (P) has been solved numerically by using three different methods for the state problem:

(i) variational inequality formulation
(ii) method of penalization
(iii) dual formulation.

We consider the minimization of the functional for total displacements:

(6.1) $$J_1(v) = \int_{\Omega(v)} (y(v))^2 \, dx$$

as well as for boundary displacements:

(6.2) $$J_2(v) = \int_0^1 (y(v))^2 \, dx_2 \ .$$ ← This cost functional

We have considered in the theoretical part only the case of the first cost functional, but the results are straightforward to extend to cover the second one as well.

The algorithms presented above were implemented to UNIVAC 1100/60 computer of Computer Centre of the University of Jyväskylä. In minimization module of design procedure the subroutine VE01A of Harwell Subroutine Library was utilized. The authors are indebted to B.Sc. T. Tiihonen for his assistance in numerical experiments.

Example 6.1. In this example we have applied the above mentioned three different types of variational methods for minimizing the cost functional (6.1) over U^h_{ad} with $h = 1/8$.

Let $f(x_1,x_2) = 4 \sin 2\pi x_2$ and assume in the definition of $\mathcal{U}^h_{ad}$ (see formula (4.6)) $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $C_1 = 1$, $C_2 = 1$.
The initial guess is $X^0 = (1,\ldots,1)$, i.e. $\Omega^0 = (0,1) \times (0,1)$.
In variational inequality formulation difference approximation is used to provide $\nabla_X \psi_1(X,Q(X))$ ($\psi_1(X,Q(X))$ is the discretized form of $J_1(v)$, see chapter 4.2).
In penalty method and dual method $\nabla_X \psi_1(X,Q(X))$ was computed by formulae presented in chapter 4.3. Penalty parameter ε was chosen to be $= 10^{-6}$ and parameter ρ in Uzawa algorithm $= 5$.
Single precision was utilized. The obtained results can be seen in Table 6.1.

	Variational ineq. formul.	Penalty method	Dual method
Initial value of ψ_1	.0035647	.0035645	.00356710
Final value of ψ_1	.0027075	.0026696	.0026683
x_1-coordinates of the unknown boundary	1.0736472	1.0631734	1.0624992
	.9486468	.9381738	.9374992
	.8236468	.8131729	.8124990
	.9486468	.9381729	.9374999
	1.0736468	1.0631734	1.0625005
	1.1796846	1.1866188	1.1875005
	1.0546840	1.0616407	1.0625006
	.9296840	.9366406	.9375006
	1.0090733	1.0616407	1.0625005
Number of gradient evaluations (iterations)	36	31	28
CPU-time is seconds	175	58	70

Table 6.1. Comparision of different methods.

It can be seen that the penalty method gives the best result. The higher CPU-time in variational inequality formulation is due to the fact that difference method is used for approximating $\nabla_X \psi_1(X,Q(X))$. Every iteration in this connection means ten function evaluations. The function evaluation in dual method is slightly more time consuming than in penalty method (Uzawa-NSOR). In both methods the gradient of the cost functional is evaluated by methods presented in chapter 4.4.

The solution of the state equation in penalty method with Ω^{31} (last iteration), the triangulation for Ω^{31} and the final position of design nodes $A_i = (x_i, ih)$ can be seen in Figure 6.2.

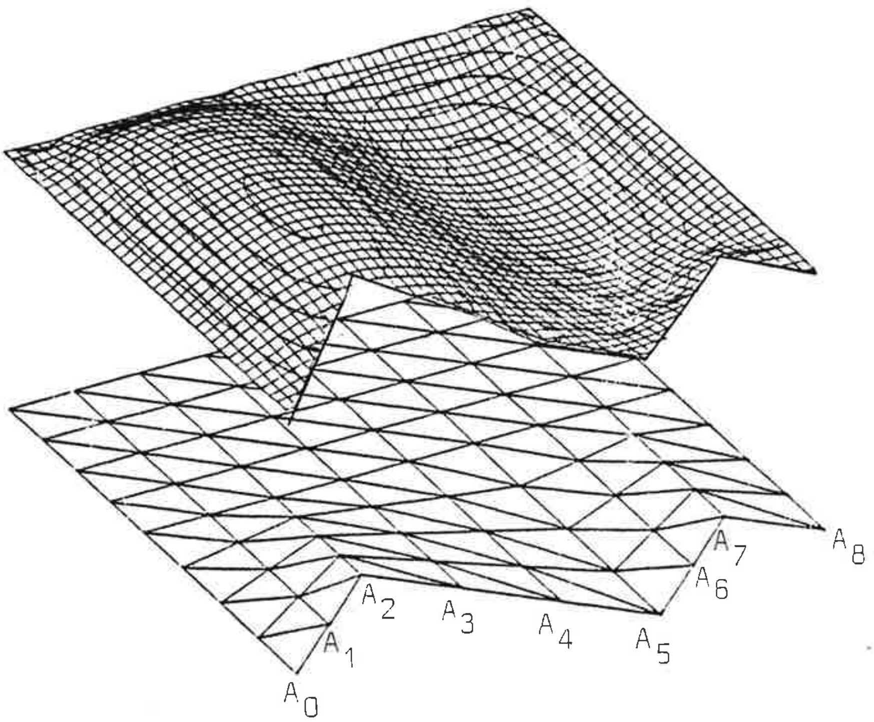

Figure 6.2. Spline-smoothed FE-solution of the state equation (3.2) with Ω^{31}, triangulation of Ω^{31} and final position of design nodes A_i $(i = 0,\ldots,8)$.

For finer triangulation ($h = 1/16$, $h = 1/32$ etc.) the number of design variables is 17, 33 etc. In optimization procedure the number of iterations for finding as good design as above remains almost the same. The function evaluation is however highly more time consuming: for example about 8 times more expensive for h = 1/16 than for h = 1/8. In this connection no essentially new information can be obtained by finer triangulation.

One possibility is to begin the design procedure with a rough triangulation and to take the obtained optimal design nodes plus nodes between these nodes as a new initial guess.

Example 6.2. In this example the criteria function is (6.2). Let

$$f(x_1,x_2) = 8 \sin 2\pi x_1 \sin 2\pi x_2 .$$

We assume that $h = 1/8$, $\alpha = 3/4$, $\beta = 3/2$, $C_1 = 1$, $C_2 = 1$.

As already mentioned, the criteria function is not convex, so that the initial guess has great influence for the design obtained when standard minimization routines are utilized (local minimum). To illustrate this situation let us consider three initial guesses for x_1-coordinates of design variables (see Table 6.3).

x_1-coordinates of design variables	value of criteria function (6.2)
$X_1^0 = (1.0,.9,.8,.9,1.0,1.1,1.2,1.1,1.0)$	$J_2 = .31898E-3$
$X_2^0 = (1.0,1.0,1.0,1.0,1.0,1.0,1.0,1.0,1.0)$	$J_2 = .38345E-2$
$X_3^0 = (1.0,1.1,1.2,1.1,1.0,.9,.8,.9,1.0)$	$J_2 = .23762E-2$

Table 6.3. Value of criteria function for different initial guesses.

In Table 6.4 we see the final values of the criteria functions for the above three initial guesses. Penalty method with $\varepsilon = 10^{-6}$ is used for solving the state problem.

	Guess X_1^0	Guess X_2^0	Guess X_3^0
Value of cost function J_2 for initial guess	.31898 E-3	.38345 E-2	.23762 E-2
Final value of cost functional	.24012 E-4	.16546 E-4	.15238 E-2
Final x_1-coordinates of design variables	.7500000 .7499999 .8324477 .9574477 1.0811422 1.2061422 1.1228118 1.1583375 1.0333421	.7500000 .7500000 .8234432 .9484432 1.0621496 1.1871496 1.2409772 1.1168916 .9918914	1.0969671 1.1935951 1.1329431 1.1250003 1.0000002 .8750000 .7500000 .8750000 .9999559
Values of nodal displacements on final design points (principal moving points)	.0000000 .0000000 -.0007669 -.0009479 .0004957 .0126108 -.0002110 .0055680 -.0003415 .0000000	.0000000 .0000000 -.0007498 .0009452 -.0005171 .0113655 -.0001061 -.0007284 .0004256 .0000000	.0000000 .0000000 .0135833 .0088125 .0026262 -.0004898 .0440756 .0841414 .0538423 .0000000
Number of iterations	60	60	19
CPU-time (seconds)	77	76	23

Table 6.4. Influence of initial guess for design obtained

In Table 6.4 it can be seen that the first two initial guesses lead roughly speaking to the same minimum. For initial guess X_3^0 the algorithm can reduce the cost functional relatively less than for initial guess X_2^0 for example (47 % - 99.9 %). The final design is also different from that obtained for the other two initial guesses.

The solution of the state problem for Ω^{60} (with initial guess X_2^0), final triangulation and final position of design nodes $A_i = (x_i, ih)$ can be seen in Figure 6.6.

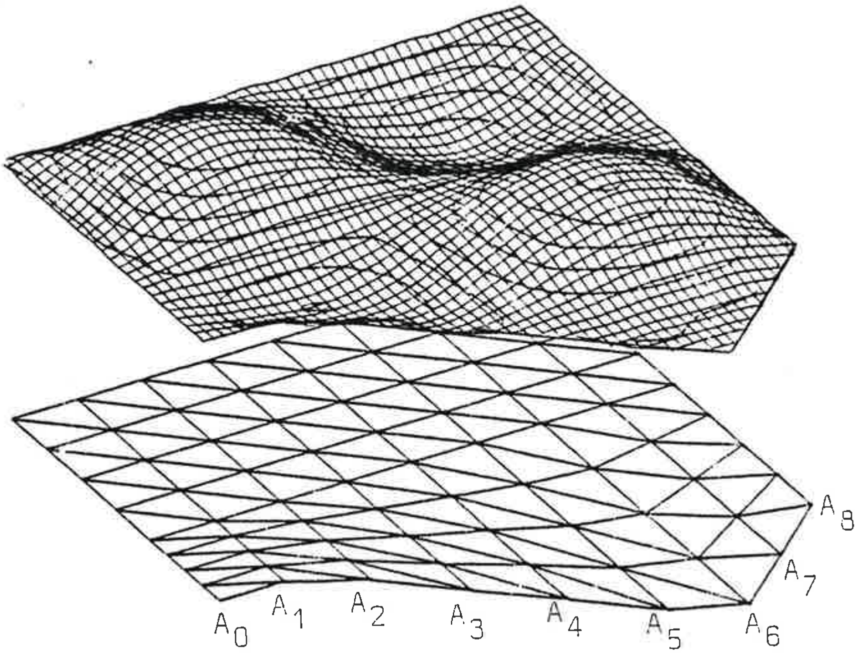

Figure 6.5. Spline-smoothed FE-solution of the state problem in Example 6.2. with Ω^{60} corresponding triangulation for Ω^{60} and final position of design nodes A_i = $i = 0,\dots, 8$.

It can be seen that the constraint $x_1 \geq \alpha = 3/4$ is active in the first two design nodes $A_0 = (.75,0)$ and $A_1 = (.75,.125)$.
At node $A_6 \approx (1.24,.75)$ the Lipschitz constraint is active.
By choosing the constraints for U_{ad}^h in an appropriate way we can find design points A_i such that $u_h(A_i) = 0$, $\frac{\partial}{\partial n} u_h \geq 0$.

Example 6.1 as well as other methods for solving the state problem (variational inequality formulation and dual method) gave essentially the same "optimal design". The only difference is in CPU-time, which for penalty method and dual method is roughly speaking the same, but for variational inequality formulation about three times higher.

Finally, we consider the diminution of the criteria function J_2 in the design process. In Figure 6.6 we can see the decreasing of J_2 versus iterations; as above, the penalty method with $\varepsilon = 10^{-6}$ for solving the state problem and the subroutine VE01A of Harwell Subroutine Library for the minimization process were utilized. X_2^0 was chosen as the initial guess (see Table 6.3).

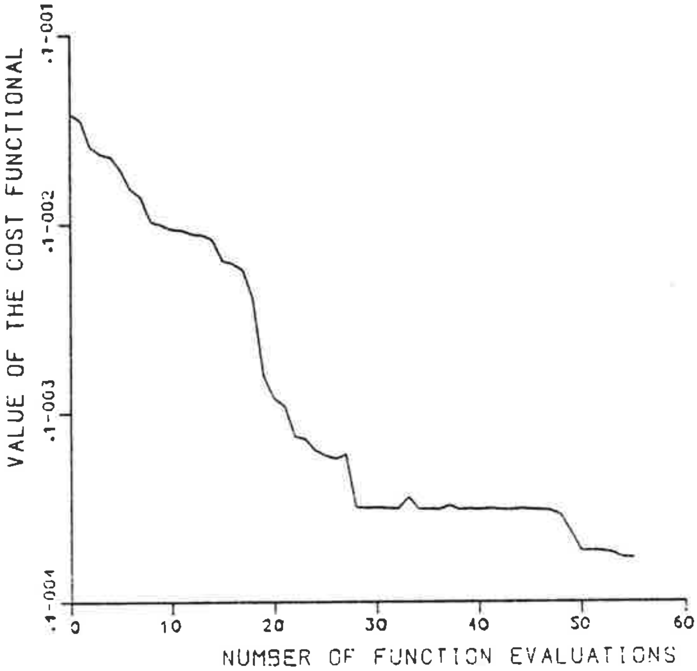

Figure 6.6. Decreasing of criteria function versus iteration

In Figure 6.6 the value of J_2 is given by a logarithmic scale. We can find that the criteria function has been reduced from Ω^0 (corresponding to initial guess X_2^0 in Table 6.3) to Ω^{28} about a factor of 100 . From step Ω^{28} to Ω^{56} (28 iterations) the value of the criteria function has been reduced a relatively small amount. As the total CPU-time is here 76 seconds, about 50 % of the CPU-time was used without obtaining any essential new information.

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