JYVÄSKYLÄ STUDIES IN COMPUTING
240

Timur Mokaev

Localization and Dimension Estimation of Attractors in the Glukhovsky-Dolzhansky System

JYVÄSKYLÄN YLIOPISTO

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Esitetään Jyväskylän yliopiston informaatioteknologian tiedekunnan suostumuksella julkisesti tarkastettavaksi yliopiston Agora-rakennuksen Beeta-salissa kesäkuun 17. päivänä 2016 kello 14.

Academic dissertation to be publicly discussed, by permission of the Faculty of Information Technology of the University of Jyväskylä, in building Agora, Beeta hall, on June 17, 2016 at 14 o'clock.

UNIVERSITY OF JYVÄSKYLÄ

JYVÄSKYLÄ 2016

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Publishing Unit, University Library of Jyväskylä

URN:ISBN:978-951-39-6690-4
ISBN 978-951-39-6690-4 (PDF)

ISBN 978-951-39-6689-8 (nid.)
ISSN 1456-5390

Jyväskylä University Printing House, Jyväskylä 2016

ABSTRACT

Mokaev, Timur
Localization and dimension estimation of attractors in the Glukhovsky-Dolzhansky system
Jyväskylä: University of Jyväskylä, 2016, 50 p.(+included articles)
(Jyväskylä Studies in Computing
ISSN 1456-5390; 1456-5390; 240)
ISBN 978-951-39-6689-8 (nid.)
ISBN 978-951-39-6690-4 (PDF)
Finnish summary
Diss.

This thesis studies chaotic attractors in a Glukhovsky-Dolzhansky (GD) system, which describes fluid convection inside an ellipsoidal cavity, under the influence of external heating. In contrast to the Lorenz system, which describes convective fluid flows in 2D, the suggested GD system describes convective fluid flows in 3D. The GD system can be viewed as an approximate model to the Earth's ocean or atmosphere.

Computationally, these attractors can be classified according to their basin of attraction in the phase space. An attractor is called a *self-excited attractor* if its basin-of-attraction intersects with small equilibria neighborhoods of a dynamical system; otherwise, the attractor is called a *hidden attractor*. Self-excited attractors can be localized with little computational effort: after determining all equilibria states of the system, a trajectory is computed via integration starting from a point in an unstable equilibrium neighborhood, using numerical methods. After a transient process, the computed trajectory will reach the attractor and visualize the attractor. The aforementioned procedure will not work for finding hidden attractors, hidden attractors being difficult to localize.

So far, only self-excited attractors have been found in Lorenz systems. This thesis demonstrates the existence and localization of hidden chaotic attractors in a GD system. The demonstration is done via numerical methods which were developed in this thesis.

In addition to the numerical methods developed, this thesis proves the Eden conjecture for GD systems that is defining the maximum Lyapunov dimension on an attractor in an equilibrium point.

Keywords: Glukhovsky-Dolzhansky system, Lorenz-like system, chaotic attractor, hidden attractor, Lyapunov dimension, Lyapunov exponents

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ACKNOWLEDGEMENTS

This thesis was completed in the Doctoral School of the Department of Mathematical Information Technology, University of Jyväskylä.

I would like to express my sincere gratitude to my supervisors Prof. Pekka Neittaanmäki, Prof. Nikolay V. Kuznetsov and Prof. Gennady A. Leonov for their guidance and continuous support. I am also very grateful to the reviewers of the thesis, Prof. Sergei Abramovich and Prof. Marius-F. Danca, for their valuable comments. I sincerely thank Prof. Tiihonen for the fruitful discussions and additional comments.

I am very much obliged to Steve Legrand and Billy Braithwaite for their improvements of English and Finnish languages and grammars, as well as for some helpful remarks.

This work would not have been possible without support from the Faculty of Information Technology and Academy of Finland, COMAS doctoral program and support from the Russian Science Foundation (project 14-21-00041).

I would like to extend my deepest gratitude to my parents Ludmila Mokaeva and Nazir Mokaev and to my brother Ruslan Mokaev for their love and endless support for everything I do.

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PII G. A. Leonov, N. V. Kuznetsov, T. N. Mokaev. Homoclinic orbit and hidden attractor in the Lorenz-like system describing the fluid convection motion in the rotating cavity. *Communications in Nonlinear Science and Numerical Simulation, Vol. 28, No. 1, pp. 166–174, doi:10.1016/j.cnsns.2015.04.007*, 2015.

PIII G. A. Leonov, N. V. Kuznetsov, T. N. Mokaev. Homoclinic orbits, and self-excited and hidden attractors in a Lorenz-like system describing convective fluid motion. *The European Physical Journal Special Topics, Multistability: Uncovering Hidden Attractors, Vol. 224, pp. 1421–1458, doi:10.1140/epjst/e2015-02470-3*, 2015.

PIV N. V. Kuznetsov, G. A. Leonov, T. N. Mokaev. The Lyapunov dimension formula of self-excited and hidden attractors in the Glukhovsky-Dolzhansky system. *arXiv preprint arXiv:1509.09161, http://arxiv.org/pdf/1509.09161v1.pdf*, 2015.

PV G. A. Leonov, T. N. Mokaev. Formula of the Lyapunov dimension of attractor of the Glukhovsky-Dolzhansky system. *Doklady Mathematics, Vol. 93, No. 1, pp. 42–45. doi:10.7868/S0869565216030063*, 2016.

1 INTRODUCTION AND STRUCTURE OF THE WORK

1.1 Introduction

Theoretical investigations of various hydrodynamic phenomena are typically performed with the help of hydrodynamic models defined by the Euler or Navier-Stokes equations with an infinite number of degrees of freedom. An important problem has proved to be the problem of *turbulence*, i.e., random motion of the medium accompanied by chaotic property changes. The problem arose in the middle of the 20^{th} century when there were many gaps between theoretical hydrodynamics and applied problems of fluid dynamics (Reynolds, 1883). In order to fill these gaps, various theories of turbulence (e.g., (Richardson, 1922; Kolmogorov, 1941)) and onset of turbulence (e.g., (Landau, 1944; Hopf, 1948)) were developed, but they turned out to be insufficient. A breath of fresh air in the study of the onset of turbulence was the discovery by Ruelle, Takens (Ruelle and Takens, 1971), and Smale (Smale, 1967) of *strange attractor*, i.e., a chaotic attractive set in the phase space of a dynamical system, which consists of unstable trajectories with a complex behavior. According to this notion, a strange attractor is a mathematical prototype of stochastic oscillations and turbulence in the system.

Nevertheless, no general theory of turbulence has yet been developed. This is caused by the inability to obtain general solutions for the Navier-Stokes equations[1] and, as a consequence, to the necessity of reducing them to simpler equations based on the observations and physical concepts of the studied phenomena. For this reason, individual sections of hydrodynamics, e.g., aerodynamics,

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1 A great advance in the study of hydrodynamic systems, defined by the Navier-Stokes equations, was made by Olga Ladyzhenskaya. In the common case for initial boundary-value problems, she proved the unique solvability for small enough time, as well as the global unique solvability for small enough data, and, in the two-dimensional case, she proved the global unique solvability (Ladyzhenskaya, 1969). Later she investigated the case when the two-dimensional Navier-Stokes equation generates a dynamical system (Ladyzhenskaya, 1972) and proved the finite-dimensionality of its attractor (Ladyzhenskaya, 1982).

magnetohydrodynamics and sound theory, now form separate sciences. For particular interest in all these sciences is geophysical hydrodynamics that studies fluid (or other medium) rotation and covers a wide range of phenomena, e.g., ocean and atmosphere of the Earth and atmospheres of other planets. Within this branch of hydrodynamics, Edward Lorenz suggested a crude three-dimensional mathematical model for atmospheric convection (Lorenz, 1963). The obtained model describes a two-dimensional Rayleigh-Bénard convective flow (Getling, 1998). Later, in 1980, Glukhovsky and Dolzhansky suggested a three-mode mathematical model that allowed investigating the process of fluid convection contained within a rotating ellipsoidal cavity under a horizontal exterior heating (Glukhovskii and Dolzhanskii, 1980). Unlike the Lorenz model, it describes a three-dimensional convective flow and can be interpreted as an approximate model of the World Ocean or Earth atmosphere. Both Lorenz and Glukhovsky-Dolzhansky systems were obtained using the Galerkin method for reduction of initial hydrodynamic systems with infinite degrees of freedom to corresponding finite-dimensional autonomous dissipative nonlinear systems. The main idea of this method is to expand the fluid dynamical fields into infinite series of time-independent basis functions. The series are then truncated and substituted into initial PDEs, yielding a system of ODEs describing time evolution of the truncated expansion coefficients. In some cases, the obtained ODEs properly describe the behavior of original models and possess similar fundamental properties. Note that there also exist non-autonomous ocean models (see, e.g., (Pierini et al., 2016)).

The key feature of the Lorenz discovery is that for certain parameter values, the suggested simple convection model possesses a chaotic attractor (Lorenz, 1963; Stewart, 2000). This attractor can be obtained numerically if, for a trajectory that describes system's behavior, one chooses the initial point on an unstable manifold in a neighborhood of an unstable equilibrium. After a transient process, this trajectory reaches the attractor, visualizing it. In general, for numerical localization of attractor, it is necessary to explore its basin of attraction and choose an initial point in it. If for a particular attractor its basin of attraction is connected with the unstable manifold of unstable equilibrium, then the localization procedure is quite simple. From this perspective, the following classification of attractors is suggested (Leonov and Kuznetsov, 2009; Kuznetsov et al., 2010; Leonov et al., 2011, 2012; Leonov and Kuznetsov, 2013; PIII; Kuznetsov, 2016,a): an attractor is called a *self-excited attractor* if its basin of attraction intersects with any open neighborhood of equilibrium; otherwise, it is called a *hidden attractor*. Numerical localization of hidden attractors is much more challenging and requires the development of special methods (Zelinka, 2015, 2016).

The problem of hidden attractors is connected with the second part of *Hilbert's 16th problem* that arose in 1900 and is related to the question of the number and possible mutual disposition of limit cycles in two-dimensional polynomial systems (Hilbert, 1901-1902). Later, this problem came up in engineering tasks: in the investigation of widely known *Markus-Yamabe's* (Markus and Yamabe, 1960), *Aizerman's* (Aizerman, 1949) and *Kalman's conjectures* (Kalman, 1957) on absolute stability of automatic control systems, where a unique stable equilibrium can co-exist with a stable periodic solution (see (Pliss, 1958; Fitts, 1966; Barabanov, 1988; Bernat and Llibre, 1996; Bragin et al., 2011; Leonov and Kuznetsov, 2011a,b; Kuznetsov et al., 2011); the corresponding discrete examples were considered in (Alli-Oke et al., 2012)), in the problem of simulation of *phase-locked loops* (Kuznetsov et al., 2014, 2015). At the end of the 20^{th} century, the problem of the numerical analysis of hidden oscillations arose in simulations of aircraft control systems (Leonov et al., 2012; Andrievsky et al., 2013b,a, 2015), and in simulations of *drilling systems* (Kiseleva et al., 2012; Leonov et al., 2014). Existence of such vibrations in systems with external perturbation determines, in addition to the expected stable solution corresponding to the desired system behavior, other stable and unstable solutions, which correspond to undesirable and dangerous behaviors, often leading to a system crash. Another stimulus to study hidden oscillations was the discovery, in 2010, of the *chaotic hidden attractor* in the *Chua circuit* — a simple electric circuit with nonlinear feedback (Leonov and Kuznetsov, 2009; Kuznetsov et al., 2010, 2011; Leonov et al., 2011; Kuznetsov et al., 2011; Leonov et al., 2012; Kuznetsov et al., 2013). Until that moment, only self-excited attractors had been detected in the Chua circuit (Chua, 1992).

Today, the concept of hidden attractors has become widespread. Let us remember that in 2016 *"Localization of hidden Chuas attractors"* (Leonov et al., 2011) became the most cited *Physics Letters A* article published since 2011[2]; *"Hidden attractor in smooth Chua systems"* made the list of the most cited *Physica D: Nonlinear Phenomena* articles published since 2011[3]; and *"Hidden Attractors in Dynamical Systems. From Hidden Oscillations in Hilbert-Kolmogorov, Aizerman and Kalman Problems to Hidden Chaotic Attractors in Chua Circuits"* (Leonov and Kuznetsov, 2013) was the most read and one of the most cited *International Journal of Bifurcation and Chaos* articles[4]. A special edition of *The European Physical Journal Special Topics: Multistability: Uncovering Hidden Attractors* was published in 2015. It includes the recent results in this area obtained by scientists from 14 countries all over the world (see (Shahzad et al., 2015; Brezetskyi et al., 2015; Jafari et al., 2015; Zhusubaliyev et al., 2015; Saha et al., 2015; Semenov et al., 2015; Feng and Wei, 2015; Li et al., 2015; Feng et al., 2015; Sprott, 2015; Pham et al., 2015; Vaidyanathan et al., 2015)). Also, in 2015, plenary lectures devoted to the topic of hidden attractors were presented at 4th IFAC Conference on Analysis and Control of Chaotic Systems (Japan, 2015) by Guanrong Chen (Chen, 2015) and at International Conference on Advanced Engineering Theory and Applications 2015 (Vietnam, 2015) by Nikolay Kuznetsov (Kuznetsov, 2016). In 2016, a review article, "Hidden Attractors in Dynamical Systems", was accepted for publication in the leading international scientific journal "Physics Reports" (JCR 2015 Impact Factor 20.033) (Dudkowski et al., 2016).

All known chaotic attractors in the Lorenz system are self-excited. In the case when all three equilibria are unstable, the attractor is self-excited with respect to three equilibria. In the case when only zero equilibrium is unstable, it is

2 http://www.journals.elsevier.com/physics-letters-a/most-cited-articles/
3 http://www.journals.elsevier.com/physica-d-nonlinear-phenomena/most-cited-articles/
4 http://www.worldscientific.com/worldscinet/ijbc?null=&&journalTabs=cited

self-excited with respect to that one equilibrium. The existence of a hidden attractor in the Lorenz system is still an open question. The Glukhovsky-Dolzhansky system, which in comparison with the Lorenz system has one additional nonlinear term, also possesses a self-excited attractor (Glukhovskii and Dolzhanskii, 1980). This work also studies the possible existence of hidden attractors in the Glukhovsky-Dolzhansky system. A numerical procedure of hidden attractor localization for the Glukhovsky-Dolzhansky system is presented in Section 2.2 (see also PII; PIII).

One of the main characteristics of the system's chaotic behavior is the Lyapunov dimension of its attractor (see, e.g., (Farmer et al., 1983; Frederickson et al., 1983; Kuznetsov, 2016b)). The concept of Lyapunov dimension was introduced by Kaplan and Yorke (Kaplan and Yorke, 1979) and was further developed in (Constantin and Foias, 1985; Eden, 1989, 1990; Eden et al., 1991). Along with commonly used numerical methods for estimating and computing the Lyapunov dimension (see, e.g., MATLAB realizations of methods based on QR and SVD decompositions in (PI; PIII)), there is an effective analytical approach proposed by Gennady Leonov in 1991 (Leonov, 1991) (see also (Leonov and Boichenko, 1992; Boichenko et al., 2005; Leonov, 2008; Leonov et al., 2015; Kuznetsov, 2016b)). It is based on the invariance of dimensions with respect to diffeomorphisms and the direct Lyapunov method. The advantage of this method is that it allows one to estimate the Lyapunov dimension of an invariant set without localization of the set in the phase space. In the past two decades, the application of this method has helped obtain the upper estimate of dimension of attractors for the Rössler, Hénon, Chirikov, Lorenz and Shimizu-Morioka systems (Leonov, 2012; Leonov and Kuznetsov, 2015; Leonov et al., 2015, 2016). For some of these systems, following the Eden ideas (Eden, 1990), this method allows one to prove the conjecture of achieving the maximum Lyapunov dimension at the equilibrium and, thereby, to obtain the exact formula of the Lyapunov dimension of the B-attractor.

In this work, using the Leonov method, the upper estimate of the Lyapunov dimension of its attractor for the Lorenz-like system was obtained and, in the special case when this system coincides with the Glukhovsky-Dolzhansky system, the Eden conjecture was proved, and the exact formula of the Lyapunov dimension of corresponding attractor was obtained (PIII; PIV; PV). This analytical result was justified with the help of the numerical procedure described in PI.

1.2 Structure of the work

This work consists of an introduction, a chapter that presents the problem statement and the main results of the work, a conclusion, a list of references, three appendices, and included articles (Figure 1). The first section of the main chapter is based on (Dolzhansky et al., 1974; Glukhovskii and Dolzhanskii, 1980; Gledzer et al., 1981), and is devoted to the statement of the physical problem and to obtaining a corresponding mathematical model. The second and third sections present the main results of the thesis. The second section presents a localization of the hidden attractor in the considered Glukhovsky-Dolzhansky system (PII; PIII). In the third section for the Lorenz-like system, the upper estimate of the Lyapunov dimension of its attractor is obtained and, in the special case when the Lorenz-like system can be transformed to the Glukhovsky-Dolzhansky system, the exact formula of Lyapunov dimension is obtained (PIII; PIV; PV). The first appendix is based on (Leonov, 1991; Kuznetsov, 2016b) and describes the Leonov method for estimation of Lyapunov dimension. The second and third appendices give the MATLAB implementations of algorithms for numerical approximation of Lyapunov exponents, for numerical justification of the Eden conjecture and for hidden attractor localization in the Lorenz-like system.

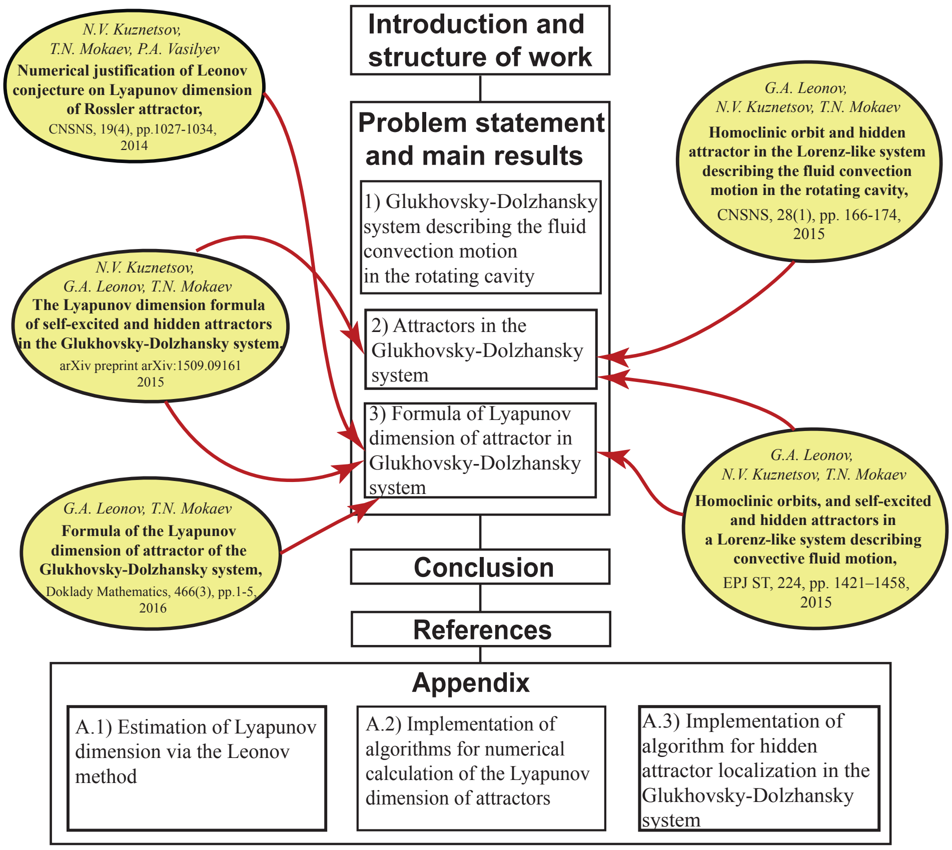


FIGURE 1 Structure of the chapters and their connection with included articles.

1.3 Included articles and author's contribution

The main results were published in the five included articles. In article PI, the author developed the algorithm of numerical justification of the Eden conjecture and implemented this algorithm for two of the three Rössler systems. In articles (PII; PIII), the algorithm for hidden attractor localization in the Lorenz-like system is implemented by the author. In articles (PIII; PIV; PV), the theorems about the Lyapunov dimension of attractors are due to the author.

The results of this study were also reported at the Seventh International Conference on Differential and Functional Differential Equations" (Russia, 2014) and the 13th International Conference of Numerical Analysis and Applied Mathematics (Greece, 2015).

2 PROBLEM STATEMENT AND MAIN RESULTS

2.1 Glukhovsky-Dolzhansky system describing fluid convection motion in the rotating cavity

In this section, following (Dolzhansky et al., 1974; Glukhovskii and Dolzhanskii, 1980; Gledzer et al., 1981), let us consider the physical problem of fluid convection inside an ellipsoidal cavity under external heating and present a rigorous derivation of the Lorenz-like system for this problem. In general form, the statement of the physical problem is as follows. Viscous incompressible fluid bounded by an ellipsoid surface

$$S(x_1,\, x_2,\, x_3) \equiv \left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 - 1 = 0, \quad a_1 > a_2 > a_3 > 0,$$

is under the condition of stationary inhomogeneous external heating. We assume that the ellipsoid as well as the heat sources rotate with constant velocity $\mathbf{\Omega}_0$ around the axis that crosses its center of mass and have a constant angle, α, with gravity vector $\mathbf{g}$. Vector $\mathbf{g}$ is stationary with respect to the ellipsoid motion. The value of $\mathbf{\Omega}_0$ is assumed to be such that the centrifugal forces can be neglected in comparison with the influence of the gravitational field.

Behavior of this hydrodynamic system can be described by Navier-Stokes equations under the Boussinesq approximation (Dolzhansky et al. (1974)):

$$\frac{\partial \mathbf{v}}{\partial \tau} + (\mathbf{v}\cdot\nabla)\mathbf{v} + 2\mathbf{\Omega}_0 \times \mathbf{v} = -\frac{1}{\rho}\nabla p - \beta \mathbf{g} T + \mathbf{f},$$
$$\frac{\partial T}{\partial \tau} + (\mathbf{v}\cdot\nabla)T = \frac{\kappa}{c_p}, \quad \nabla\cdot\mathbf{v} = 0 \tag{1}$$

with the following border condition

$$(\mathbf{v}\cdot\nabla)S = 0, \quad \text{if} \quad S = 0, \tag{2}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (v_1(\mathbf{x}, t),\, v_2(\mathbf{x}, t),\, v_3(\mathbf{x}, t))$ — fluid velocity vector field, $\mathbf{x} = (x_1, x_2, x_3)$, ρ — average density, β — coefficient of volume expansion of

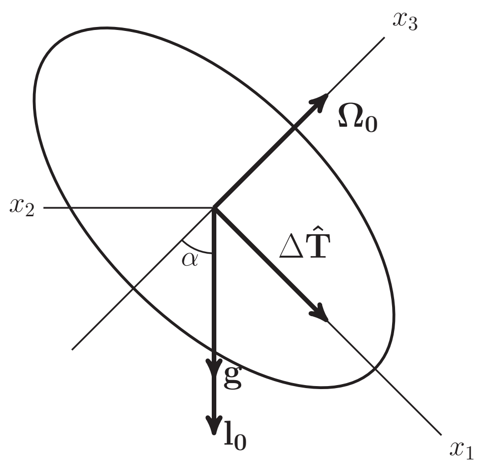


FIGURE 2 Illustration of the problem.

the liquid, $T = T(\mathbf{x}, t)$ — temperature deviation from a certain constant value T_0 that is defined by the specific conditions of the problem; hence, term $\beta \mathbf{g} T$ defines the resultant of the Archimedes and gravity forces, $\mathbf{f}$ — internal viscous forces for which $\nabla \times \mathbf{f} \neq 0$, κ — heat transfer per unit mass of liquid caused by the external heating and thermal conductivity, κ supposed to be a linear function of the spatial coordinates (x_1, x_2, x_3), c_p — specific heat at a constant pressure, ∇ — nabla operator, τ — time. The energy flux from external sources to fluid is defined by the Newton-Richman's law (Kays and Crawford (1993))

$$\frac{\kappa}{c_p} = \mu(\hat{T} - T),$$

where μ is the heat-transfer coefficient and $1/\mu$ defines a characteristic damping time in a stationary medium for deviations from steady-state temperature $\hat{T}$. The latter is assumed to be a linear function of the spatial coordinates.

Using the Galerkin method (Thompson, 1961; Monin, 1972), the solution of system (1) can be reduced to the solution of a finite system of ordinary differential equations. Under certain conditions, the motion of a real fluid can be described by a spatially linear velocity and temperature fields: therefore, the solution of initial system (1) is sought in the class of functions satisfying the following conditions

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = 0, \quad \frac{\partial^2 T}{\partial x_j \partial x_k} = 0. \tag{3}$$

As reference fields in the Galerkin method the following three solenoidal linear vector fields were used:

$$\mathbf{w}_1 = -\frac{a_2}{a_3} x_3 \,\mathbf{j} + \frac{a_3}{a_2} x_2 \,\mathbf{k}, \quad \mathbf{w}_2 = -\frac{a_3}{a_1} x_1 \,\mathbf{k} + \frac{a_1}{a_3} x_3 \,\mathbf{i}, \quad \mathbf{w}_3 = -\frac{a_1}{a_2} x_2 \,\mathbf{i} + \frac{a_2}{a_1} x_1 \,\mathbf{j},$$

satisfying the boundary condition, $(\mathbf{w}_i \cdot \nabla) S = 0$, on surfaces $S = \text{const}$. These fields are orthogonal in the following sense:

$$\int_V \mathbf{w}_i \cdot \mathbf{w}_k \, d^3\mathbf{x} = 0 \quad \text{for} \quad i \neq k.$$

Here V denotes the volume bounded by the ellipsoid. Let us write velocity field $\mathbf{v}$ and temperature field T in the following form

$$\mathbf{v}(\mathbf{x},t) = \omega_1(t)\mathbf{w}_1(\mathbf{x}) + \omega_2(t)\mathbf{w}_2(\mathbf{x}) + \omega_3(t)\mathbf{w}_3(\mathbf{x}), \tag{4}$$

$$T(\mathbf{x},t) = \mathbf{x}\cdot\nabla T = x_1\frac{\partial T}{\partial x_1} + x_2\frac{\partial T}{\partial x_2} + x_3\frac{\partial T}{\partial x_3}, \quad \nabla T = \nabla T(t), \quad T(0,t) = 0, \tag{5}$$

where Poincare parameters $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$ have the following relation with the components of vorticity vector $\mathbf{\Omega} \equiv \nabla\times\mathbf{v}$

$$\omega_1 = \frac{a_2a_3}{a_2^2+a_3^2}\Omega_1, \quad \omega_2 = \frac{a_1a_3}{a_1^2+a_3^2}\Omega_2, \quad \omega_3 = \frac{a_1a_2}{a_1^2+a_2^2}\Omega_3.$$

Multiplying both sides of system (1) by $\mathbf{w}_i$ and using conditions (3), expansions (4), (5), and orthogonality of $\mathbf{w}_i$, system (1) can be transformed (Dolzhansky et al., 1974) to the following simplified fluid convection system:

$$\frac{d\mathbf{M}}{d\tau} = \boldsymbol{\omega}\times(\mathbf{M}+2\mathbf{M}_0) + g\beta(\mathbf{l}_0\times\mathbf{q}) - \lambda\mathbf{M}, \qquad \frac{d\mathbf{q}}{d\tau} = \boldsymbol{\omega}\times\mathbf{q} + \mu(\hat{\mathbf{q}}-\mathbf{q}), \tag{6}$$

where λ — effective coefficient of viscosity, $\boldsymbol{\omega} = (\omega_1,\, \omega_2,\, \omega_3)$, vector $\mathbf{M}$ has the following components $M_i = \sum_{k=1}^{3} I_{ik}\omega_k$, $\{I_{ik}\}_{i,k=1,2,3}$ — diagonal matrix with the following elements

$$I_{11} = a_2^2 + a_3^2, \quad I_{22} = a_1^2 + a_3^2, \quad I_{33} = a_1^2 + a_2^2.$$

The components $M_{0i} = \sum_{k=1}^{3} I_{ik}\omega_{0s}$ of vector $\mathbf{M_0}$ can be expressed via components of $\mathbf{\Omega}_0$

$$\omega_{01} = \frac{a_2a_3}{a_2^2+a_3^2}\Omega_{01}, \quad \omega_{02} = \frac{a_1a_3}{a^2+a_3^2}\Omega_{02}, \quad \omega_{03} = \frac{a_1a_2}{a^2+a_2^2}\Omega_{03}.$$

Components of vectors $\mathbf{q}$ and $\hat{\mathbf{q}}$ denote temperature differences and steady-state temperature differences along the principal axis of the ellipsoid, respectively

$$\mathbf{q} = \left(a_1\frac{\partial T}{\partial x_1},\, a_2\frac{\partial T}{\partial x_2},\, a_3\frac{\partial T}{\partial x_3}\right), \quad \hat{\mathbf{q}} = \left(a_1\frac{\partial \hat{T}}{\partial x_1},\, a_2\frac{\partial \hat{T}}{\partial x_2},\, a_3\frac{\partial \hat{T}}{\partial x_3}\right).$$

Vector $\mathbf{l}_0 = (a_1\cos\gamma_1,\, a_2\cos\gamma_2,\, a_3\cos\gamma_3)$ determines the orientation of the ellipsoid and has the same direction as $\mathbf{g}$, the gravity vector. Here $\cos\gamma_i$ — direction cosines of the gravity vector.

Term $2\boldsymbol{\omega}\times\mathbf{M}_0$ in the first equation of (6) represents the Coriolis force in the considered velocity fields class. For initial system (1), this force is represented by term $2\mathbf{\Omega}_0\times\mathbf{v}$ in its left hand side (see, e.g., (Greenspan, 1968)).

Consider the case when the rotation of the ellipsoid occurs about the x_3 axis and vector $\mathbf{g}$ is placed in plane x_1x_3 and steady-state temperature difference $\Delta\hat{T} = (q_0,\, 0,\, 0)$ is generated along the x_1 axis (Figure 2). Then

$$\mathbf{l}_0 = (a_1\sin\alpha,\, 0,\, -a_3\cos\alpha), \quad \mathbf{\Omega}_0 = (0,\, 0,\, \Omega_0),$$

and equations (6) written in the coordinate form are as follows:

$$\begin{cases} \frac{d\omega_1}{d\tau} & = I_{11}^{-1}[(I_{33}-I_{22})\omega_2\omega_3 + 2a_3a_2\Omega_0\omega_2 + g\beta a_3\cos\alpha q_2] - \lambda\omega_1, \\ \frac{d\omega_2}{d\tau} & = I_{22}^{-1}[(I_{11}-I_{33})\omega_1\omega_3 - 2a_1a_2\Omega_0\omega_1 - g\beta a_3\cos\alpha q_1 - g\beta a_1\sin\alpha q_3] - \lambda\omega_2, \\ \frac{d\omega_3}{d\tau} & = I_{33}^{-1}[(I_{22}-I_{11})\omega_1\omega_2 + g\beta a_1\sin\alpha q_2] - \lambda\omega_3, \\ \frac{dq_1}{d\tau} & = \omega_2q_3 - \omega_3q_2 + \mu(q_0 - q_1), \\ \frac{dq_2}{d\tau} & = \omega_3q_1 - \omega_1q_3 - \mu q_2, \\ \frac{dq_3}{d\tau} & = \omega_1q_2 - \omega_2q_1 - \mu q_3. \end{cases} \tag{7}$$

Let

$$E = \frac{1}{2}\,\boldsymbol{\omega}\cdot\mathbf{M} + g\beta\,\mathbf{l}_0\cdot\mathbf{q}$$

denotes the full energy of the system.

In order to simplify system (7) in the study of geophysical flows, the co-called geostrophic wind relation is often used (Obukhov, 1949; Lorenz, 1967; Greenspan, 1968). It sets the approximate balance (geostrophic balance) between the Coriolis forces and the pressure gradient. Here, after expansion in the small parameter $\varepsilon = \bar{\omega}/\Omega_0$, where $\bar{\omega} = \sqrt{E/(a_1^2 + a_2^2 + a_3^3)}$, for equations (6), we get the following relations up to terms of order ε (Glukhovskii and Dolzhanskii, 1980)

$$2(\boldsymbol{\omega}\times\mathbf{M}_0) + g\beta(\mathbf{l}_0\times\mathbf{q}) = 0,$$

or

$$\omega_1 = -\frac{g\beta a_3}{2a_1a_2\Omega_0}\cos\alpha q_1 - \frac{g\beta a_1}{2a_1a_2\Omega_0}\sin\alpha q_3, \qquad \omega_2 = -\frac{g\beta a_3}{2a_1a_2\Omega_0}\cos\alpha q_2, \tag{8}$$

expressing geostrophic balance for the considered model. After substitution of expressions (8) into equations (7), we obtain the geostrophic model, which has the following equations of motion

$$\begin{cases} \frac{dX}{dt} = AYZ - BZU + CZ + DU - \sigma X, \\ \frac{dY}{dt} = -XZ + R_a - Y, \\ \frac{dZ}{dt} = XY + PU^2 - Z, \\ \frac{dU}{dt} = -PZU - U, \end{cases} \tag{9}$$

in dimensionless variables

$$X = \mu^{-1}\left(\omega_3 + \frac{g\beta a_3\cos\alpha}{2a_1a_2\Omega_0}q_3\right),\ Y = \frac{g\beta a_3}{2a_1a_2\lambda\mu}q_1,\ Z = \frac{g\beta a_3}{2a_1a_2\lambda\mu}q_2,\ U = \frac{g\beta a_3}{2a_1a_2\lambda\mu}q_3.$$

Here derivatives are taken with respect to dimensionless time $t = \mu\tau$, the parameters being:

$$A = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}\cos^2\alpha T_a^{-1}, \quad B = \frac{a_1a_2}{a_3(a_1^2 + a_2^2)}\sin 2\alpha T_a^{-1}, \quad C = \frac{2a_1^2a_2}{a_3(a_1^2 + a_2^2)}\sigma\sin\alpha,$$

$$D = (\sigma - 1)\cos\alpha T_a^{-1/2}, \quad P = \frac{a_1}{a_3}T_a^{-1/2}\sin\alpha,$$

and the following numbers

$$\sigma = \frac{\lambda}{\mu}, \quad T_a = \frac{\Omega_0^2}{\lambda^2}, \quad R_a = \frac{g\beta a_3 q_0}{2a_1 a_2 \lambda\mu}$$

are external parameters, which are naturally interpreted as Prandtl, Rayleigh and Taylor numbers, respectively.

According to the numerical experiments (Glukhovskii and Dolzhanskii, 1980), the invariant surface $U = 0$ (the case, when $q_3 \equiv 0$) is an attractor of system (9). On this surface, the motion of system (9) is described by the following equations:

$$\begin{cases} \dot{X} = -\sigma X + CZ + AYZ, \\ \dot{Y} = R_a - Y - XZ, \\ \dot{Z} = -Z + XY. \end{cases} \tag{10}$$

From the physical point of view, this corresponds to an indifferent vertical stratification.

The following affine change of coordinates

$$X \to x, \quad Y \to C^{-1}y, \quad Z \to C^{-1}z$$

transforms system (10) to the Glukhovsky-Dolzhansky system

$$\begin{cases} \dot{x} = -\sigma x + z + A_c yz, \\ \dot{y} = R_c - y - xz, \\ \dot{z} = -z + xy. \end{cases} \tag{11}$$

where

$$R_c = R_a C > 0, \quad A_c = \frac{A}{C^2} > 0.$$

Consider the following three-dimensional Lorenz-like system

$$\begin{cases} \dot{x} = -\sigma x + \sigma y - ayz \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy. \end{cases} \tag{12}$$

This system with $r, \sigma, b > 0$ was first introduced in (Rabinovich, 1978), and for $a = 0$ it coincides with the classical Lorenz system (Lorenz, 1963).

Now let us consider the connection between system (11) and system (12). Suppose that

$$\sigma > ar. \tag{13}$$

Then after the following affine change of coordinates

$$x \to x, \quad y \to \frac{1}{\sigma - ar}z, \quad z \to r - \frac{1}{\sigma - ar}y \tag{14}$$

system (12) can be transformed to system (11) with the following parameters:

$$R_c = r(\sigma - ar) > 0, \quad A_c = \frac{a}{(\sigma - ar)^2} > 0, \quad b = 1. \tag{15}$$

Let us note that if the following relations

$$a = \frac{A_c \sigma^2}{(A_c R_c + 1)^2}, \quad r = \frac{R_c}{\sigma}(A_c R_c + 1). \tag{16}$$

hold, then the inverse transformation has the following form:

$$x \to x, \quad y \to R_c - \frac{\sigma}{A_c R_c + 1} z, \quad z \to \frac{\sigma}{A_c R_c + 1} y \tag{17}$$

Our particular interest to systems (12) and (11) is connected with the existence of chaotic attractors in their phase spaces.

2.2 Attractors in the Glukhovsky-Dolzhansky system

2.2.1 Self-excited and hidden attractors

Following (Milnor, 1985; Ott, 2002; Leonov and Kuznetsov, 2013; PIII), from the computational perspective one can consider the following non-rigorous definition of attractor (rigorous definition are discussed, e.g., in PIII). An oscillation can generally be easily numerically localized if the initial data from its open neighborhood in the phase space (with the exception of a minor set of points) lead to a long-term behavior that approaches the oscillation. Such an oscillation (or set of oscillations) is called an *attractor*, and its attracting set is called the *basin of attraction* (i.e., a set of initial data from which the trajectories tend to the attractor).

The study of an autonomous system typically begins with an analysis of the equilibria, which are easily found numerically or analytically. Therefore, from a computational perspective, it is natural to suggest the following classification of attractors (Kuznetsov et al., 2010; Leonov et al., 2011, 2012; Leonov and Kuznetsov, 2013), which is based on the simplicity of finding their basins of attraction in the phase space:

Definition 1. *(Kuznetsov et al., 2010; Leonov et al., 2011, 2012; Leonov and Kuznetsov, 2013) An attractor is called a* self-excited attractor *if its basin of attraction intersects with any open neighborhood of a stationary state (an equilibrium), otherwise it is called a* hidden attractor*.*

The basin of attraction for a hidden attractor is not connected with any equilibrium. For example, hidden attractors are attractors in systems with no equilibria or with only one stable equilibrium (a special case of multistability: coexistence of attractors in multistable systems). Note that multistability can be inconvenient in various practical applications (see, for example, discussions on problems related to the synchronization of coupled multistable systems in (Kapitaniak, 1992, 1996; Pisarchik and Feudel, 2014; Kuznetsov and Leonov, 2014)). Coexisting self-excited attractors can be found with the help of a standard computational procedure, whereas there is no standard way of predicting the existence or coexistence of hidden attractors in a system.

2.2.2 Self-excited attractor localization in the Glukhovsky-Dolzhansky system

It can be shown (Leonov and Boichenko, 1992) that for positive parameters, if $r < 1$, system (12) has a unique equilibrium $S_0 = (0,0,0)$, which is globally asymptotically Lyapunov stable. If $r > 1$, then (12) possesses three equilibria: a saddle $S_0 = (0,0,0)$ and symmetric (with respect to $z = 0$) equilibria

$$S_{1,2} = (\pm x_1, \pm y_1, z_1), \tag{18}$$

where

$$x_1 = \frac{\sigma\sqrt{\xi}}{\sigma + a\xi}, \quad y_1 = \sqrt{\xi}, \quad z_1 = \frac{\sigma\xi}{\sigma 1 + a\xi},$$

and

$$\xi = \frac{\sigma}{2a^2}\left[a(r-2) - \sigma + \sqrt{(\sigma - ar)^2 + 4a\sigma}\right].$$

Following (Glukhovskii and Dolzhanskii, 1980), let us take $\sigma = 4$ and define the stability domain of equilibria $S_{1,2}$. Using the Routh-Hurwitz criterion, we can obtain the following (see PIII):

Lemma 1. *The equilibria $S_{1,2}$ are stable if*

$$8a^2r^3 + a(7a - 64)r^2 + (288a + 128)r + 256a - 2048 < 0. \tag{19}$$

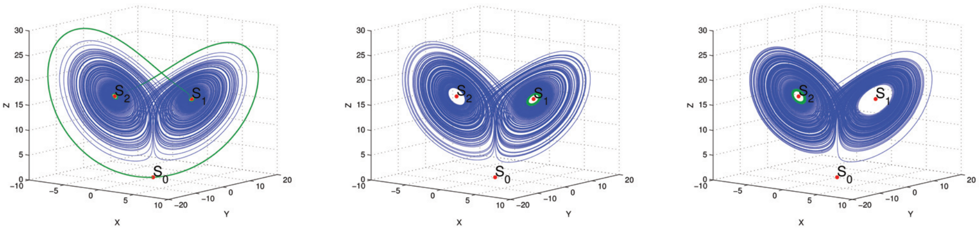
Consider the following parameters for system (12)

$$\sigma = 4, \quad a = 0.0052.$$

According to Lemma 1, if $r_1 \approx 16.4961242... < r < r_2 \approx 690.6735024$, the equilibria $S_{1,2}$ of system (12) become (unstable) saddle-focuses. For example, if $r = 17$, the eigenvalues of the equilibria of system (12) are the following

$$\begin{aligned} S_0: &\quad 5.8815, \quad -1, \quad -10.8815 \\ S_{1,2}: &\quad 0.0084 \pm 4.5643i, \quad -6.0168 \end{aligned}$$

and there is a self-excited chaotic attractor in the phase space of system (12). We can easily visualize this attractor (Figure 3) using the standard computational procedure with initial data in the vicinity of one of the equilibria, $S_{0,1,2}$, on the corresponding unstable manifolds. To improve the approximation of the attractor, one can consider its neighborhood and compute trajectories from a grid of points in this neighborhood.



(a) Initial data near equilibrium S_0. (b) Initial data near equilibrium S_1. (c) Initial data near equilibrium S_2.

FIGURE 3 Numerical visualization of the self-excited attractor in the Glukhovsky-Dolzhansky system with $r = 17$, $\sigma = 4$, $b = 1$, $a = 0.0052$ by using the trajectories that start in small neighborhoods of the unstable equilibria, $S_{0,1,2}$.

2.2.3 Hidden attractor localization in the Glukhovsky-Dolzhansky system

Following (Leonov et al., 2011; Leonov and Kuznetsov, 2013; PIII), we apply a special analytical-numerical procedure for localization of the hidden attractor of system (12). The main idea of this procedure is to construct a sequence of similar systems such that the initial data for numerical localization of the starting attractor for the first (starting) system can be obtained rather simply. For example, in some cases it is possible to consider a starting system with a self-excited starting attractor. Then we numerically track the transformation of the starting attractor while passing between systems.

Let us construct a line segment on the plane (a, r) that intersects a boundary of the stability domain of equilibria $S_{1,2}$ (see Figure 4). We choose point $P_1(r = 700, a = 0.0052)$ as the end point of the line segment. The eigenvalues for the equilibria of system (12) that correspond to parameters P_1 are the following:

$$\begin{aligned} S_0 : &\quad 50.4741, \quad -1, \quad -55.4741, \\ S_{1,2} : &\quad -0.1087 \pm 10.4543\mathrm{i}, \quad -5.7826. \end{aligned}$$

This means that equilibria $S_{1,2}$ become stable focus-nodes. Now we choose point $P_0(r = 687.5, a = 0.0052)$ as the initial point of the line segment. This point corresponds to the parameters for which system (12) has a self-excited attractor, which can be computed using the standard computational procedure. Then we choose a sufficiently small partition step for the line segment and compute a chaotic attractor in the phase space of system (12) at each iteration of the procedure[1]. The last computed point at each step is used as the initial point for the computation at the next step (the computation time must be sufficiently large).

In our experiment, the length of the path was 3.25 and there were 3 iterations. Here, for the selected path and partition, we can visualize a hidden attractor of system (12) (see Figure 5). The results of a continuation procedure are given in (PII; PIII).

[1] Here, for system (12) we use the so-called *stability of the basin of attraction* (see, e.g., (Menck et al., 2013)) with respect to variable parameter r.

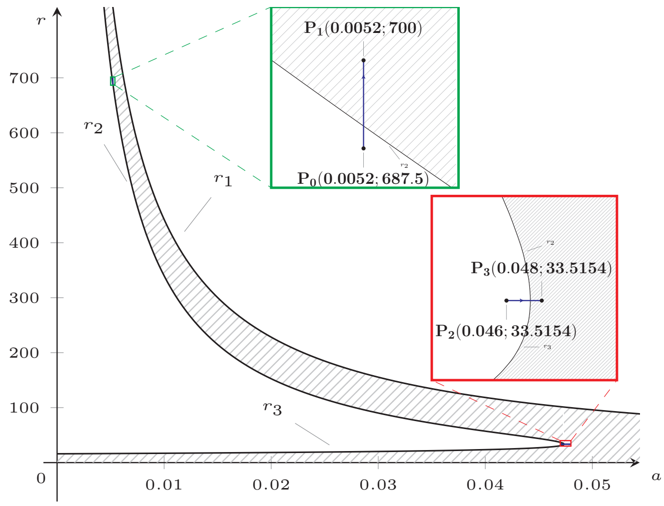


FIGURE 4 Paths $[P_0, P_1]$ and $[P_2, P_3]$ in the plane of parameters $\{a, r\}$ used in the continuation procedure.

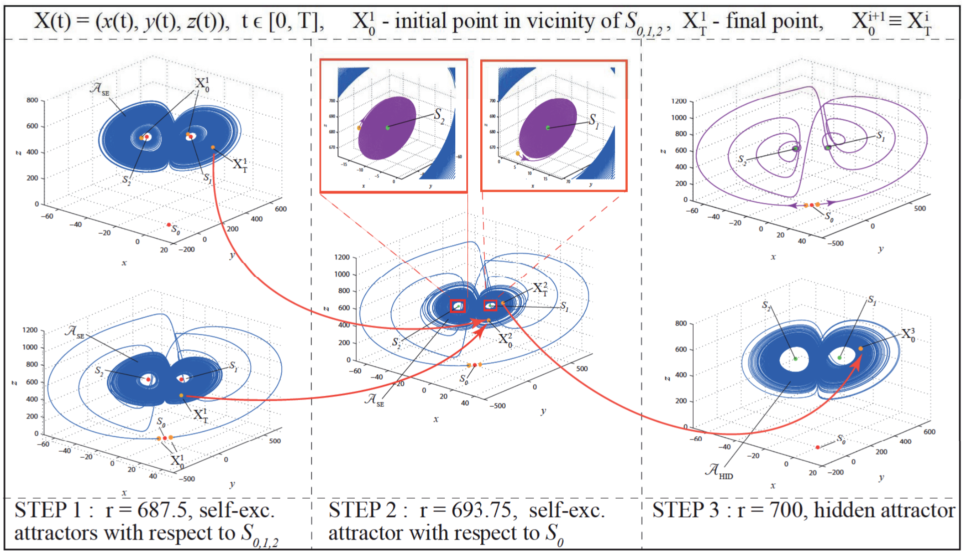


FIGURE 5 Chain of transformation in the continuation procedure.

Note that the choice of path and its partitions in the continuation procedure are not trivial. For example, a similar procedure does not lead to a hidden attractor for the following path on the plane (a, r). Consider $r = 33.51541181$, $a = 0.04735056...$ (the rightmost point on the stability domain) and take a starting point P_2: $r = 33.51541181$, $a = 0.046$ near it (Fig. 4). If we use partition step 0.001, then there are no hidden attractors after crossing the boundary of the stability domain. For example, if the end point is P_3: $r = 33.51541181$, $a = 0.048$, there is no chaotic attractor but only trivial attractors (equilibria $S_{1,2}$).

Also, let us note that a hidden attractor was localized in system (12) in the case when $a < 0$, $\sigma = -ar$ (Kuznetsov et al., 2015). For these parameters, system (12) coincides with the famous Rabinovich system that describes the interaction of waves in plasma (Rabinovich, 1978; Pikovski et al., 1978).

2.3 Formula of the Lyapunov dimension of attractor for the Glukhovsky-Dolzhansky system

One of the main characteristics of chaos in system is the Lyapunov dimension of its attractor. The concept of Lyapunov dimension was introduced by Kaplan and Yorke (Kaplan and Yorke, 1979). Their rigorous definitions are discussed in Appendix 1.

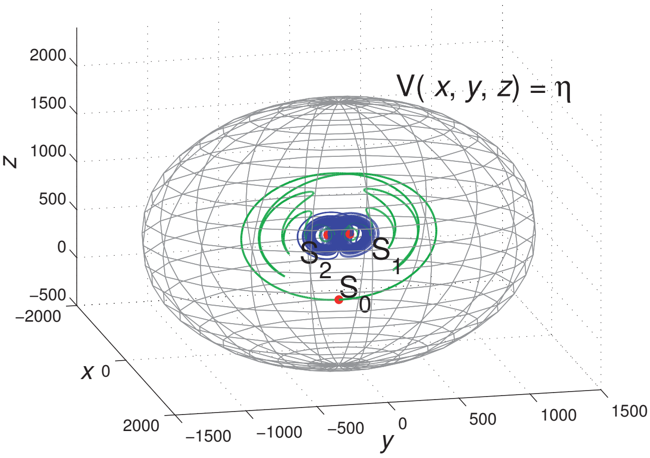


FIGURE 6 Attractors localization for a Lorenz-like system.

Using the Leonov method[2] (Leonov, 1991; Kuznetsov, 2016b), the assertion concerning the Lyapunov dimension $\dim_L \mathcal{K}$ of attractor $\mathcal{K}$ of system (12) can be formulated:

Theorem 1. *Suppose that either inequality $b < 1$ or inequalities $b \geqslant 1$, $\sigma > b$ are valid. If*

$$\left(r + \frac{\sigma}{a}\right)^2 < \frac{(b+1)(b+\sigma)}{a}, \tag{20}$$

then any solution of system (12)*, bounded on* $[0, +\infty)$*, tends to an equilibrium as* $t \to +\infty$.
If

$$\left(r + \frac{\sigma}{a}\right)^2 > \frac{(b+1)(b+\sigma)}{a}, \tag{21}$$

2 see a detailed discussion in Appendix 1.

then

$$\dim_L \mathcal{K} \leqslant 3 - \frac{2(\sigma+b+1)}{\sigma+1+\sqrt{(\sigma-1)^2+a\left(\frac{\sigma}{a}+r\right)^2}}. \tag{22}$$

Proof is presented in PIV. □

For the numerical computation of the Lyapunov dimension of attractor $\mathcal{K}$, it is possible to use the following procedure[3]. For numerical integration of the system that possesses $\mathcal{K}$, we consider a sufficiently large time T and choose a sufficiently dense grid of points $\mathcal{K}_{grid}$ on the attractor as the initial datum. For each grid point and obtained trajectory we compute the local Lyapunov dimension by the corresponding Kaplan-Yorke formula and then find the maximum over the computed values. It is also interesting to numerically check the Eden conjecture[4] (Eden, 1990) claiming that the maximum of the local Lyapunov dimension on the attractor is achieved in the equilibrium. Using this procedure with parameters $h = 0.5$, $k = 400$, $abs\_tol = rel\_tol = 10^{-8}$, we obtain the following results (see Table 1) for the Lorenz-like system (12) that numerically justify the Eden conjecture:

TABLE 1 Numerical justification of the Eden conjecture for a Lorenz-like system.

Syst.	Grid	Step	$\max_{x\in grid} \dim_L x$ (Benettin alg.)	$\max_{x\in grid} \dim_L x$ (Stewart alg.)	$\dim_L S_0$	(22)
(12)	$\frac{1}{2}\left(x^2+y^2+(a+1)\times \times\left(z-\frac{\sigma+r}{a+1}\right)^2\right) \leq \eta$	0.5	2.1466	2.1501	2.8917	2.8955

In the special case, when $b = 1$ and the Lorenz-like system (12) coincides with the Glukhovsky-Dolzhansky system (11) we can obtain another upper estimation which coincides with the local Lyapunov dimension of the zero equilibrium and analytically proves the Eden conjecture (see PV). This result is expressed by the following:

Theorem 2. *Let $b = 1$, $r > 2$ and the following relations hold*

$$\begin{cases} \sigma > \frac{-3+2\sqrt{3}}{3}ar, & \text{if} \quad 2 < r \leq 4 \\ \sigma \in \left(\frac{-3+2\sqrt{3}}{3}ar,\ \frac{3r+2\sqrt{r(2r+1)}}{r-4}ar\right), & \text{if} \quad r > 4. \end{cases} \tag{23}$$

Then

$$\dim_L \mathcal{K} = 3 - \frac{2(\sigma+2)}{\sigma+1+\sqrt{(\sigma-1)^2+4\sigma r}}. \tag{24}$$

Proof is presented in (PIII; PV). □

Note that this exact formula coincides with the formula for the classical Lorenz system (Leonov and Lyashko, 1993; Leonov, 2012; Leonov et al., 2016).

[3] see a description of the procedure in (PI) and MATLAB implementation in Appendix 2.
[4] see a detailed discussion in (PIII; Kuznetsov, 2016b).

3 CONCLUSION

This work studies the Glukhovsky-Dolzhansky system, which provides a mathematical model of the fluid convection in a rotating ellipsoid cavity under a horizontal external heating. An analytical-numerical procedure for localization of the hidden attractors is developed and implemented for this system. Using the Leonov method, the Eden conjecture on achieveness of maximum Lyapunov dimension on the attractor at the equilibrium is proved for this system and, thereby, an exact formula of the Lyapunov dimension of this attractor is obtained. By means of special procedure developed in this work, this analytical result is checked and justified numerically. Also we obtain the upper estimate of Lyapunov dimension of attractor for a Lorenz-like system, providing a generalization of the Glukhovsky-Dolzhansky system.

Further investigations are planned on the Lorenz-like system and on possible existence of hidden attractors in its phase space. An attempt to obtain the exact formula of the Lyapunov dimension of Lorenz-like system's attractors will be undertaken as well.

Also a deeper look at the correspondence of chaotic behavior of the Glukhovsky-Dolzhansky system to a hydrodynamic and atmospheric phenomena will be taken. Of a particular interest will be the studying of the connection between the chaos in this system and the El Niño Southern Oscillation phenomenon. It is known (Garay and Indig, 2015) that in some cases the Vallis' model for El Niño (Vallis, 1986, 1988) can be transformed to the Lorenz system.

YHTEENVETO (FINNISH SUMMARY)

Attraktoreiden lokalisointi ja dimension estimointi Glykhovsky-Dolzhansky-järjestelmässä

Tässä väitöstyössä tutkitaan kaaottisia attraktoreita Glukhovsky-Dolzhansky (GD) -järjestelmässä. GD-järjestelmä kuvaa nesteen lämmön virtausta soikion muotoisessa onkalossa, missä lämpö on ulkopuolinen vaikuttaja. Verrattuna Lorenz-järjestelmään, GD-järjestelmä kuvaa nesteen lämmön virtausta kolmiulotteisessa avaruudessa, ja järjestelmää voidaan käyttää esimerkiksi valtamerien tai ilmakehän mallintamiseen.

Attraktorit voidaan laskennallisesti luokitella kahteen luokkaan, riippuen attraktoreiden vetovoima-alan faasiavaruudesta. Jos attraktorin vetovoima-ala leikkaa dynaamisen järjestelmän tasapainopisteiden pienympäristöjä, attraktori luokitellaan *itsekiihtyväksi attraktoriksi* dynaamisessa järjestelmässä. Muissa tapauksissa attraktoria kutsutaan *piileväksi attraktoriksi*. Laskennallisesti itsekiihtyvien attraktoreiden lokalisointi on helppoa. Riittää kun määritellään kaikki tasapainopisteet järjestelmässä ja sitten numeerisia menetelmiä käyttäen integroidaan trajektori aloittaen jostain epävakaan tasapainoympäristön pisteestä. Piilevien attraktoreiden kohdalla edellä mainittu tapa ei onnistu, koska piilevien attraktoreiden lokalisointi on vaikeaa.

Lorenz-järjestelmissä on havaittu ainoastaan itsekiihtyviä attraktoreita. Tässä työssä osoitamme piilevien kaaottisten attraktoreiden olemassaolon GD-järjestelmissä. Näiden piilevien kaaottisten attraktoreiden lokaalisoinniksi kehiteltiin numeerisia menetelmiä, ja nämä numeeriset menetelmät ovat tämän työn kontribuutio.

Tässä työssä myös todistetaan Eden-konjektuuri GD-järjestelmissä, eli todistetaan ja määritellään attraktorin maksimaalinen Lyapunov-dimensio järjestelmän tasapainopisteessä.

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APPENDIX 1 ESTIMATION OF LYAPUNOV DIMENSION VIA THE LEONOV METHOD

In this section, following (Leonov, 1991; Kuznetsov, 2016b), the notions of the Hausdorff and Lyapunov dimensions are given, and the Leonov method for their upper estimation is described.

APPENDIX 1.1 Hausdorff and Lyapunov dimensions

Consider a compact metric space (X, ρ), subset $E \subset X$ and the following numbers $d \geq 0$, $\varepsilon > 0$. Consider covering of set E by open balls with a radius $r_j < \varepsilon$ and consider the following notion:

$$\mu_H(E, d, \varepsilon) = \inf \sum_j r_j^d,$$

where the infimum is taken over all ε-coverings of E. It is clear, that $\mu_H(E, d, \varepsilon)$ does not decrease while ε decreasing. Therefore, there is a limit (possibly infinite)

$$\mu_H(E, d) = \lim_{\varepsilon \to 0} \mu_H(E, d, \varepsilon).$$

Definition 2. *Function $\mu_H(\cdot, d)$ is called the Hausdorff d-measure.*

For a fixed set E, function $\mu_H(E, \cdot)$ has the following properties: there exist number $d_{kp} \in [0, \infty]$, such that $\mu_H(E, d) = \infty$, $\forall d < d_{kp}$ and $\mu_H(E, d) = 0$, $\forall d > d_{kp}$; if $X \subset \mathbb{R}^n$, then $d_{kp} \leq n$.

Definition 3. *The number*

$$\dim_H E = d_{kp} = \inf \{ d \mid \mu_H(E, d) = 0 \}$$

is called the Hausdorff dimension of set E.

Consider an autonomous differential equation

$$\dot{\mathrm{x}} = f(\mathrm{x}), \quad f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n, \tag{25}$$

where f is a continuously differentiable vector-function and U is an open set. Define by $\mathrm{x}(t, \mathrm{x}_0)$ a solution of (25) such that $\mathrm{x}(0, \mathrm{x}_0) = \mathrm{x}_0 \in U$, and consider evolutionary operator $\varphi^t(\mathrm{x}_0) = \mathrm{x}(t, \mathrm{x}_0)$. Assume the uniqueness and existence of solutions of (25) for $t \in [0, +\infty)$. Then system (25) generates dynamical system $\{\varphi^t\}_{t \geq 0}$. Let nonempty set $K \subset U \subseteq \mathbb{R}^n$ be invariant with respect to $\{\varphi^t\}_{t \geq 0}$, i.e., $\varphi^t(K) = K$ for all $t > 0$. Consider the linearization of system (25) along the solution $\varphi^t(\mathrm{x})$:

$$\dot{\mathrm{y}} = J(\varphi^t(\mathrm{x}))\mathrm{y}, \quad J(\mathrm{x}) = Df(\mathrm{x}), \tag{26}$$

where $J(\mathrm{x})$ is the $n \times n$ Jacobian matrix, which elements are continuous functions of x. Suppose that $\det J(\mathrm{x}) \neq 0 \quad \forall \mathrm{x} \in U$. Consider a fundamental matrix of linearized system (26) $D\varphi^t(\mathrm{x})$ such that $D\varphi^0(\mathrm{x}) = I$, where I is a unit $n \times n$ matrix. Let $\sigma_i(t, \mathrm{x}) = \sigma_i(D\varphi^t(\mathrm{x}))$, $i = 1, 2, ..., n$, be the singular values of $D\varphi^t(\mathrm{x})$ (i.e. $\sigma_i(t, \mathrm{x}) > 0$ and $\sigma_i(t, \mathrm{x})^2$ are the eigenvalues of the symmetric matrix $D\varphi^t(\mathrm{x})^* D\varphi^t(\mathrm{x})$ with respect to their algebraic multiplicity)[1] ordered so that $\sigma_1(t, \mathrm{x}) \geq ... \geq \sigma_n(t, \mathrm{x}) > 0$ for any x and $t \geq 0$. The singular value function of order $d \in [0, n]$ at $\mathrm{x} \in \mathrm{x}$ is defined as

$$\begin{aligned} &\omega_d(D\varphi^t(\mathrm{x})) = \sigma_1(t, \mathrm{x}) \cdots \sigma_{\lfloor d \rfloor}(t, \mathrm{x}) \sigma_{\lfloor d \rfloor + 1}(t, \mathrm{x})^{d - \lfloor d \rfloor}, \quad d \in [0, n), \\ &\omega_n(D\varphi^t(\mathrm{x})) = \sigma_1(t, \mathrm{x}) \cdots \sigma_n(t, \mathrm{x}), \end{aligned} \tag{27}$$

where $\lfloor d \rfloor$ is the largest integer less than or equal to d.

The concept of the Lyapunov dimension was suggested in the seminal paper by Kaplan and Yorke (Kaplan and Yorke, 1979), and later it was developed in a number of papers. The following definitions are considered in (Kuznetsov, 2016b) and inspirited by Douady-Oesterlé (Douady and Oesterle, 1980). The local Lyapunov dimension of map φ^t at point $\mathrm{x} \in \mathbb{R}^n$ is defined as

$$\dim_{\mathrm{L}}(\varphi^t, \mathrm{x}) = \max\{d \in [0, n] : \omega_d(D\varphi^t(\mathrm{x})) \geq 1\}$$

and the Lyapunov dimension of map φ^t with respect to invariant set K is defined as

$$\dim_{\mathrm{L}}(\varphi^t, K) = \sup_{\mathrm{x} \in K} \dim_{\mathrm{L}}(\varphi^t, \mathrm{x}) = \sup_{\mathrm{x} \in K} \max\{d \in [0, n] : \omega_d(D\varphi^t(\mathrm{x})) \geq 1\}.$$

In the paper of (Douady and Oesterle, 1980), it is rigorously proved that the Lyapunov dimension of map φ^t with respect to compact invariant set K is an upper estimate of the Hausdorff dimension of set K. Thus we have

$$\dim_{\mathrm{H}} K \leq \inf_{t \geq 0} \dim_{\mathrm{L}}(\varphi^t, K).$$

Here $\inf_{t \geq 0} \dim_{\mathrm{L}}(\varphi^t, K)$ is called *the Lyapunov dimension of dynamical system* $\{\varphi^t\}_{t \geq 0}$ with respect to invariant set K. For computations, it is important that (see, e.g., (Kuznetsov, 2016b))

$$\inf_{t \geq 0} \dim_{\mathrm{L}}(\varphi^t, K) = \liminf_{t \to +\infty} \dim_{\mathrm{L}}(\varphi^t, K). \tag{28}$$

Consider the finite-time Lyapunov exponents at point x:

$$\mathrm{LE}_i(t, \mathrm{x}) = \frac{1}{t} \ln \sigma_i(t, \mathrm{x}), \ i = 1, 2, ..., n, \quad t > 0.$$

If $n > \dim_{\mathrm{L}}(\varphi^t, \mathrm{x}) > 1$, $j(t, \mathrm{x}) = \lfloor \dim_{\mathrm{L}}(\varphi^t, \mathrm{x}) \rfloor$, $s(t, \mathrm{x}) = \dim_{\mathrm{L}}(\varphi^t, \mathrm{x}) - \lfloor \dim_{\mathrm{L}}(\varphi^t, \mathrm{x}) \rfloor$ then $0 = \frac{1}{t} \ln(\omega_{j(t,\mathrm{x}) + s(t,\mathrm{x})}(D\varphi^t(\mathrm{x}))) = \sum_{i=1}^{j(t,\mathrm{x})} \mathrm{LE}_i(t, \mathrm{x}) + s(t, \mathrm{x}) \, \mathrm{LE}_{j(t,\mathrm{x})+1}(t, \mathrm{x})$. The following representation

$$\dim_{\mathrm{L}}(\varphi^t, \mathrm{x}) = j(t, \mathrm{x}) + \frac{\mathrm{LE}_1(t, \mathrm{x}) + \cdots + \mathrm{LE}_{j(t,\mathrm{x})}(t, \mathrm{x})}{|\, \mathrm{LE}_{j(t,\mathrm{x})+1}(t, \mathrm{x})|} \tag{29}$$

[1] $*$ denotes matrix transposition.

corresponds to the Kaplan-Yorke formula (Kaplan and Yorke, 1979) with respect to the finite-time Lyapunov exponents. Remark that here $j(t, \mathrm{x}) = \max\{m : \sum_{i=1}^{m} \mathrm{LE}_i(t, \mathrm{x}) \geq 0\}$ and $\mathrm{LE}_{j(t,\mathrm{x})+1}(t, \mathrm{x}) < 0$ for $j(t, \mathrm{x}) < n$.

APPENDIX 1.2 Leonov method

In 1991 Leonov proposed an effective analytical method (Leonov, 1991) which allows one to estimate the Lyapunov dimension of invariant sets without localization of the set in the phase space. This method is based on the following theorems.

Consider a nonsingular $n \times n$ matrix, S. Let $\lambda_i(\mathrm{x}_0, S)$, $i = 1, 2, ..., n$, be the eigenvalues of the symmetrized Jacobian matrix

$$\frac{1}{2}\left(SJ(\mathrm{x}(t,\mathrm{x}_0))S^{-1} + (SJ(\mathrm{x}(t,\mathrm{x}_0))S^{-1})^*\right), \tag{30}$$

ordered so that $\lambda_1(\mathrm{x}_0, S) \geq \cdots \geq \lambda_n(\mathrm{x}_0, S)$ for any x_0.

Theorem 3. *Let $d = (j+s) \in [1, n]$, where integer $j = \lfloor d \rfloor \in \{1, \ldots, n\}$ and real $s = (d - \lfloor d \rfloor) \in [0, 1)$. If there are a differentiable scalar function $V : U \subseteq \mathbb{R}^n \to \mathbb{R}^1$ and a nonsingular $n \times n$ matrix S such that*

$$\sup_{\mathrm{x} \in K} \left(\lambda_1(\mathrm{x}, S) + \cdots + \lambda_j(\mathrm{x}, S) + s\lambda_{j+1}(\mathrm{x}, S) + \dot{V}(\mathrm{x})\right) < 0, \tag{31}$$

where $\dot{V}(\mathrm{x}) = (\mathrm{grad}(V))^* f(\mathrm{x})$, then for sufficiently large $T > 0$ we have

$$\dim_{\mathrm{L}}(\varphi^T, K) \leq j + s.$$

A proof of this theorem, based on the invariance of Lyapunov dimension with respect to diffeomorphisms, is given in (Kuznetsov, 2016b) (see also (Leonov et al., 2015; Kuznetsov et al., 2016)).

Corollary 1. *If for $d = j + s$ defined in Theorem 3 at equilibrium point $\mathrm{x}_{eq}^{cr} \equiv \varphi^t(\mathrm{x}_{eq}^{cr})$ the equality*

$$\dim_{\mathrm{L}}(\{\varphi^t\}_{t\geq 0}, \mathrm{x}_{eq}^{cr}) = j + s$$

holds, then for any invariant set $K \ni \mathrm{x}_{eq}^{cr}$ we get the formula of exact Lyapunov dimension

$$\dim_{\mathrm{H}} K \leq \inf_{t \geq 0} \dim_{\mathrm{L}}(\varphi^t, K) = \dim_{\mathrm{L}}(\{\varphi^t\}_{t\geq 0}, \mathrm{x}_{eq}^{cr}) = j + s.$$

For the study of continuous time dynamical system in $\mathbb{R}^3$, the following result is useful. Consider a certain open set, $K_\varepsilon \subset \mathbb{R}^n$, which is diffeomorphic to a ball, whose boundary $\partial\overline{K_\varepsilon}$ is transversal to vectors $f(\mathrm{x})$, $\mathrm{x} \in \partial\overline{K_\varepsilon}$. Let set K_ε be a positively invariant for the solutions of system (25).

Theorem 4 (see (Leonov, 1991, 2008))**.** *Suppose, a continuously differentiable function $V : U \subseteq \mathbb{R}^n \to \mathbb{R}^1$ and a non-degenerate matrix S exist such that*

$$\lambda_1(\mathrm{x}, S) + \lambda_2(\mathrm{x}, S) + \dot{V}(\mathrm{x}) < 0, \ \forall \mathrm{x} \in K_\varepsilon. \tag{32}$$

Then any solution of system (25) *with the initial data $\mathrm{x}_0 \in K_\varepsilon$ tends to the stationary set as $t \to +\infty$.*

APPENDIX 2 IMPLEMENTATIONS OF ALGORITHMS FOR NUMERICAL CALCULATION OF THE LYAPUNOV DIMENSION OF ATTRACTORS

APPENDIX 2.1 Benettin algorithm implementation

Let us briefly describe the algorithm for Lyapunov dimension calculation based on the classical Benettin algorithm (Benettin et al., 1980a,b) for numerical approximation of the Lyapunov exponents. For a certain initial point, x_0, let us integrate initial system (25) along with its variational equation. Let orthogonal $(n \times n)$-matrix Q_0 be the initial matrix for variational equation (usually, $Q_0 = I$). On the k-th iteration of the numerical procedure system (25) and its variational equation are numerically integrated on the small time interval $[0, h]$ with initial data $\{\mathrm{x}_{k-1}, Q_{k-1}\}$. This yields a new trajectory point, $\mathrm{x}_k \equiv \mathrm{x}(hk, \mathrm{x}_0)$, and $X_k \equiv D\varphi^{hk}(\mathrm{x}_0)$ — fundamental matrix of the linearized system (26) at point x_k.

Here the unique QR-decomposion of the obtained matrix, X_k, is considered: $X_k = Q_k R_k$, where Q_k — orthogonal matrix that will be taken as an initial datum for the variational equation on the next iteration, R_k — upper-triangular matrix with positive diagonal entries.

The described numerical procedure has k iterations (k — sufficiently large). Thus, the approximation of the finite-time Lyapunov exponents can be obtained [1] from

$$\mathrm{LE}_i(t, \mathrm{x}_0) \approx \frac{1}{kh} \sum_{j=1}^{k} \ln(R_j(i,i)), \quad i = 1, 2, ..., n.$$

LISTING 2.1 **computeLCEs.m** – computation of the Lyapunov characteristic exponents

```
function [t, trajectory, LCEs] = computeLCEs(extOde, initPoint, tBegin, ...
                                            tStep, iterNum, odeSolverOptions)

% For the given extended system represented by system ODEs and corresponding
% variational equation the function returns array of LCEs.
%
% Parameters:
%   extOde - extended system (ODE system + var. eq.);
%   initPoint -  initial point;
%   tBegin - initial time value;
%   t_step - time-step in the reorthogonalization procedure;
%   tStep - number of iterations in the reorthogonalization procedure;
%   odeSolverOptions - options for ode45 MATLAB solver;

% Dimension of the ODE :
dimOde = length(initPoint);

% Dimension of the extended ODE (ODE + Var. Eq.):
dimExtOde = dimOde * (dimOde + 1);

tBegin = 0; tEnd = tStep;
tSpan = [tBegin, tEnd];
initFundMatrix = eye(dimOde);
initCond = [initPoint(:); initFundMatrix(:)];

% Array of norms of vectors of fundamental matrix :
norms = zeros(1, dimOde);

% Array of sums of logarithms of norms :
```

[1] see also the corresponding discussion in (Kuznetsov et al., 2016; Kuznetsov, 2016b).

```
logSum = zeros(1, dimOde);

% Array of lyapunov exponents :
currLCEs = zeros(1, dimOde);

% Preallocations for output values :
LCEs = zeros(iterNum, dimOde);

% Main loop:
for iIteration = 1 : iterNum

    % Solving extended system :
    [currTime, extOdeSolution] = ode45(extOde, tSpan, initCond, odeSolverOptions);

    % Fundamental matrix X at the moment iIteration * tStep :
    X = reshape(extOdeSolution(end, (dimOde + 1) : dimExtOde), ...
                                                        dimOde, dimOde);

    % QR factorization of X :
    [Q, R] = qr(X);

    for  = iCoord : dimOde
        if R(iCoord, iCoord) < 0
            R(iCoord, iCoord) = (-1) * R(iCoord, iCoord);
            Q(:, iCoord) = (-1) * Q(:, iCoord);
        end
    end

    % Computing Lyapunov exponents at moment iIteration * tStep :
    for iCoord = 1 : dimOde
        norms(iCoord) = R(iCoord, iCoord);
        logSum(iCoord) = logSum(iCoord) + log( norms(iCoord) );
        currLCEs(iCoord) = logSum(iCoord) / (iIteration * tStep);
    end

    % Saving computations :
    t = [t; currTime(:)];
    trajectory = [trajectory; extOdeSolution(:, 1 : dimOde)];
    LCEs(iIteration, :) = currLCEs;

    % Updating :
    currInitPoint = extOdeSolution(end, 1 : dimOde);
    currInitFundMatrix = reshape(Q, 1, []);;

    tBegin = tBegin + tStep;
    tEnd = tEnd + tStep;
    tSpan = [tBegin, tEnd];
    initCond = [currInitPoint(:); currInitFundMatrix(:)];

end
end
```

APPENDIX 2.2 Stewart algorithm implementation

Singular value decomposition (*SVD*) of fundamental matrix $\mathrm{X}(t)$ has the following form

$$\mathrm{X}(t) = \mathrm{U}(t)\Sigma(t)\mathrm{V}^*(t): \quad \mathrm{U}(t)^*\mathrm{U}(t) \equiv I \equiv \mathrm{V}(t)^*\mathrm{V}(t),$$

where U, V — orthogonal matrices, consist of left and right singular vectors, respectively, $\Sigma(t) = \mathrm{diag}\{\sigma_1(t), ..., \sigma_n(t)\}$ — diagonal matrix with the positive diagonal elements equal to *singular values*. Singular values are the square roots of the eigenvalues of matrix $\mathrm{X}(t)^*\mathrm{X}(t)$ (Horn and Johnson, 1994).

Let us describe the Stewart algorithm (Stewart, 1997; Dieci and Elia, 2008) for the computation of the Lyapunov exponents based on the computation of the SVD decomposition for the large product of matrices.

Our setup is similar to that in the previous method. We want to calculate the singular value decomposition of fundamental matrix $X(t_k)$, where $t_k = hk$, $X(0) \equiv I$. For sufficiently small h, it is possible to write

$$X(t_k) = X(t_k, t_{k-1}) \cdots X(t_1, t_0),$$

where $X(t_k, t_{k-1})$ is the solution at t_k of variational equation

$$\dot{X}(t, t_{k-1}) = J(t)X(t, t_{k-1}), \quad X(t_{k-1}, t_{k-1}) = I.$$

The task is to compute the SVD of $X(t_k)$ without forming the product explicitly in order to avoid overflows and numerical dependency of the columns. The solution suggested in (Stewart, 1997) is based on the Rutishauser's LRCH algorithm for computing the eigenvalues of a symmetric positive semi-definite matrix, $B \equiv B_1$ (Rutishauser and Schwarz, 1963):

---

for $i := 1, 2, \ldots$ **do**
  $B_i = L_i L_i^*$ [Cholesky factorization]
  $B_{i+1} \leftarrow L_i^* L_i$
end for

Here L_i is a lower triangular matrix with positive diagonal entries. It is known that the limit matrix exists and has the eigenvalues of B on the diagonal.

Since computing the singular values of matrix A is equivalent to computing the square roots of the eigenvalues of matrix $B = A^*A$, one can use the LRCH algorithm for this purpose:

---

for $i := 1, 2, \ldots$ **do**
  $A_i = Q_i R_i^*$ [QR factorization]
  $A_{i+1} \leftarrow R_i^*$
end for

If $A = A^1 \cdot A^2 \cdots A^k$, then its QR factorization can be computed without forming the product of matrices:

$Q^{k+1} \leftarrow I$
for $j := k, k-1, \ldots, 1$ **do**
  $C \leftarrow A^j Q^{j+1}$
  $C = Q^j R^j$ [QR factorization]
end for

It is easy to see, that $A = Q^1 \cdot R^1 \cdots R^k$.

Thus, for the computation of the SVD factorization of $X(t_k)$ one should calculate all $X_1^{k-j+1} = X(t_j, t_{j-1})$, $j = 1, \ldots, k$ and then use the following algorithm:

$U_1 \leftarrow I; V_1 \leftarrow I$
for $i := 1, 2, \ldots$ **do**
  $Q_i^{k+1} \leftarrow I$

for $j := k, k-1, \ldots, 1$ **do**
  $C \leftarrow X_i^j Q^{j+1}$
  $C = Q_i^j R_i^j$ [QR factorization]
end for
if i odd **then**
  $U_{i+1} = U_i Q_i^1$
else
  $V_{i+1} = V_i Q_i^1$
end if
for $j := 1, 2, \ldots, k$ **do**
  $X_{i+1}^j \leftarrow (R_i^{k-j+1})^*$
end for
end for

As the initial LRCH procedure, this procedure also converges (Stewart, 1997; Dieci and Elia, 2008). Denote by U^k and V^k the obtained approximations to the limits of U_i and V_i, and by R^j the corresponding limits of R_i^j, $j = 1, \ldots, k$. The product SVD factorization of $X(t_k)$ is given by

$$X(t_k) \approx U^k \Sigma^k (V^k)^*, \quad \Sigma^k = R^1 \cdot R^2 \cdots R^k$$

and the approximation of the finite-time Lyapunov exponents can be obtained from

$$\text{LE}_i(t, \text{x}_0) \approx \frac{1}{t_k} \sum_{j=1}^{k} \ln(R^j(i,i)), \quad i = 1, 2, ..., n.$$

LISTING 2.2 **productSVD.m** – computation of the SVD decomposition for the product of matrices

```
function [U, R, V] = productSVD(initFactorization, nIterations)
% Parameters:
%   initFactorization - array containing factor matrices of the
%                       fundamental matrix X, such that:
%          X = initFactorization(:,:,1) * ... * initFactorization(:,:,end);
%   nIterations - number of iterations in the product SVD algorithm.

% dimOde - dimension of the ODEs, nFactors - number of factor matrices
[~, dimOde, nFactors] = size(initFactorization);

% A - 2D array of matrices storing the factor matrices at each iteration
A = zeros(dimOde, dimOde, nFactors, nIterations);
A(:, :, :, 1) = initFactorization;

% Q - array of matrices storing orhogonal matrices of the QR decomposition
Q = zeros(dimOde, dimOde, nFactors+1);

% U, V - orthogonal matrices in the SVD decomposition
U = eye(dimOde); V = eye(dimOde);

% R - array of upper triangular factor matrices, such that after
% the last iteration \Sigma = R(:,:,1) * ... * R(:,:,end)
R = zeros(dimOde, dimOde, nFactors);

% Main loop
for iIteration = 1 : nIterations
    Q(:, :, nFactors + 1) = eye(dimOde, dimOde);
    for jFactor = nFactors : -1 : 1
        C = A(:, :, jFactor, iIteration) * Q(:, :, jFactor+1);
        [Q(:, :, jFactor), R(:, :, jFactor)] = qr(C);
        for kCoord = 1 : dimOde
```

```
            if R(kCoord, kCoord, jFactor) < 0
                R(kCoord, :, jFactor) = -1 * R(kCoord, :, jFactor);
                Q(:, kCoord, jFactor) = -1 * Q(:, kCoord, jFactor);
            end;
        end;
    end;

    if mod(iIteration, 2) == 1
        U = U * Q(:, :, 1);
    else
        V = V * Q(:, :, 1);
    end

    for jFactor = 1 : nFactors
        A(:, :, jFactor, iIteration + 1) = R(:, :, nFactors-jFactor+1)';
    end
end

end
```

LISTING 2.3 **computeLEs.m** – Lyapunov exponents computation

```
function LEs = computeLEs(extOde, initPoint, tStep, ...
                                nFactors, nSvdIterations, odeSolverOptions)
% Parameters:
%   extOde - extended ODE system (system of ODEs + var. eq.);
%   initPoint -  initial point;
%   tStep - time-step in the factorization procedure;
%   nFactors - number of factor matrices in the factorization procedure;
%   nSvdIterations - number of iterations in the product SVD algoritm;
%   odeSolverOptions - solver options (sover = ode45);

% Dimension of the ODE :
dimOde = length(initPoint);

% Dimension of the extended ODE (ODE + Var. Eq.):
dimExtOde = dimOde * (dimOde + 1);

tBegin = 0; tEnd = tStep;
tSpan = [tBegin, tEnd];
initFundMatrix = eye(dimOde);
initCond = [initPoint(:); initFundMatrix(:)];

X = zeros(dimOde, dimOde, nFactors);

% Main loop : factorization of the fundamental matrix
for iFactor = 1 : nFactors
    [~, extOdeSolution] = ode45(extOde, tSpan, initCond, odeSolverOptions);

    X(:, :, iFactor) = reshape(...
                        extOdeSolution(end, (dimOde + 1) : dimExtOde), ...
                                                            dimOde, dimOde);
    currInitPoint = extOdeSolution(end, 1 : dimOde);
    currInitFundMatrix = eye(dimOde);

    tBegin = tBegin + tStep;
    tEnd = tEnd + tStep;
    tSpan = [tBegin, tEnd];
    initCond = [currInitPoint(:); currInitFundMatrix(:)];
end

% Product SVD of factorization X of the fundamental matrix
[~, R, ~] = productSVD(X, nSvdIterations);

% Computation of the Lyapunov exponents
LEs = zeros(1, dimOde);
for jFactor = 1 : nFactors
    LEs = LEs + log(diag(R(:, :, jFactor))');
end;
finalTime = tStep * nFactors;
LEs = LEs / finalTime;

end
```

LISTING 2.4 **lyapunovDim.m** – Local Lyapunov dimension computation

```
function LD = lyapunovDim( LEs )
% For the given array of Lyapunov exponents of a point the function
```

```
% compute local Lyapunov dimention of this point.

% Parameters:
%   LEs - array of Lyapunov exponents.

% Initialization of the local Lyupunov dimention :
LD = 0;

% Number of LCEs :
nLEs = length(LEs);

% Sorted LCEs :
sortedLEs = sort(LEs, 'descend');

% Main loop :
leSum = sortedLEs(1);
if ( sortedLEs(1) > 0 )
     for i = 1 : nLEs-1
        if sortedLEs(i+1) ~= 0
           LD = i + leSum / abs( sortedLEs(i+1) );
           leSum = leSum + sortedLEs(i+1);
           if leSum < 0
              break;
           end
        end
     end
end
end
```

LISTING 2.5 **rosslerSyst1.m** – Rössler system

```
function OUT = rosslerSyst1(t, x, a, b)

% Rossler system with parameters: a b

% Output vector that represents combined system :
OUT = zeros(12,1);

% Rossler equation:
OUT(1) = - x(2) - x(3);
OUT(2) = x(1);
OUT(3) = -b*x(3) + a*(x(2) - x(2)*x(2));

% Jacobian at the point [x(1), x(2), x(3)]
J = [-sigma, sigma-a*x(3),  -a*x(2);
    r-x(3),     -1,     -x(1);
    x(2),      x(1), -b];

X = [x(4), x(7), x(10);
    x(5), x(8), x(11);
    x(6), x(9), x(12)];

% Variational equation:
OUT(4:12) = J * X;
```

LISTING 2.6 **main.m** – numerical justification of the Lyapunov dimension hypothesis for the Rössler attractor

```
function main

% The procedure computes local lyapunov dimention of the fixed point and local Lyapunov dimentions
% of the points on the grid for the 1st Rossler attractor.

% Values of parameters :
a = 0.386; b = 0.2;

% T - time-step in iterative procedure :
T = 0.5;

% K - number of iterations of iterative procedure :
K = 400;

acc = 1e-8; RelTol = acc; AbsTol = acc; InitialStep = acc/10;
odeSolverOptions = odeset('RelTol', RelTol, 'AbsTol', AbsTol, ...
                        'InitialStep', InitialStep, 'NormControl', 'on');

% eps - step on the grid :
```

```
eps = 1e-1;

% Equilibrium :
equilibrium = [0 0 0];

% Attractor is located in cube :
xBegin = -1;    xEnd = 1.3;  % x \in [-1; 1.3];
yBegin = -0.7;  yEnd = 1.8;  % y \in [-0.7; 1.8];
zBegin = -1.1;  zEnd = 0;    % z \in [-1.1; 0];

xIterations = (xEnd - xBegin) / eps + 1;
yIterations = (yEnd - yBegin) / eps + 1;
zIterations = (zEnd - zBegin) / eps + 1;

% Infinity factor: if trajectory leaves cube with side 'infinity_factor',
% then we conclude, that trajectory will leave basin of attraction :
infinityFactor = 10;

% Result array :
gridResults = zeros(xIterations*yIterations*zIterations, 7);
iRes = 1;

% Computing local lyapunov dimentions of the grid points :
for i = 1 : xIterations
    for j = 1 : yIterations
        for k = 1 : zIterations

            currPoint = [xBegin+(i-1)*eps yBegin+(j-1)*eps zBegin+(k-1)*eps];
            [~, trajectory, LCEs] = computeLCEs(@(t, x) rosslerSyst1(t, x, a, b), ...
                                      currPoint, 0, T, K, odeSolverOptions);

            if (abs(trajectory(end, 1)) < infinityFactor ...
                        && abs(trajectory(end, 2)) < infinityFactor ...
                                   && abs(trajectory(end, 3)) < infinityFactor)

            % Saving results for current point :
            gridResults(iRes, :) = [currPoint lyapunovDim(LCEs(end, : )) ...
                                                                  LCEs(end, : )];
            iRes = iRes + 1;
            end
        end
    end
end

% Computing local Lyapunov dimention of the equilibrium :
[~, ~, LCEs] = computeLCEs(@(t, x) rosslerSyst1(t, x, a, b), equilibrium, 0, T, K, ...
                                                                    odeSolverOptions);
equilibLCEs = LCEs(end, : );

% Saving results in file :
fid = fopen('hypothesis_roessler_1.txt');
fprintf(fid, '%4s %4s %4s %10s %10s %10s %10s\r\n',...
                                     'x', 'y', 'z', 'dim_L', 'LCE1', 'LCE2', 'LCE3');
fprintf(fid, '%.2f, %.2f, %.2f, %.8f, %.8f, %.8f, %.8f\r\n', gridResults);
fprintf(fid, '\r\nLyapunov dimension in equilibrium:\r\n');
fprintf(fid, '%.2f, %.2f, %.2f, %.8f, %.8f, %.8f, %.8f\r\n', ...
                                       [x0 lyapunovDim(equilibLCEs) equilibLCEs]);
fclose(fid);
end
```

APPENDIX 3 IMPLEMENTATION OF ALGORITHM FOR HIDDEN ATTRACTOR LOCALIZATION IN THE GLUKHOVSKY-DOLZHANSKY SYSTEM

As it was mentioned in Section 2.1, if conditions

$$\sigma > ar, \quad b = 1$$

hold, then using the following affine change of coordinates

$$x \to x, \quad y \to \frac{1}{\sigma - ar}z, \quad z \to r - \frac{1}{\sigma - ar}y.$$

the Lorenz-like system (12) can be transformed to the Glukhovsky-Dolzhansky system (11). Thus, in numerical experiments system (12) as well as system (11) can be simulated (if the corresponding conditions on parameters are satisfied).

LISTING 3.1 **LorenzLikeSyst.m** – Lorenz-like system (12) combined with variational equation

```
function OUT = LorenzLikeSyst(t, x, r, sigma, b, a)

% Lorenz-like system with parameters: r sigma b a

OUT(1) = sigma*(x(2) - x(1)) - a*x(2)*x(3);
OUT(2) = r*x(1) - x(2) - x(1)*x(3);
OUT(3) = -b*x(3) + x(1)*x(2);

% Jacobian at the point [x(1), x(2), x(3)]
J = [-sigma, sigma-a*x(3),  -a*x(2);
    r-x(3),    -1,     -x(1);
    x(2),      x(1), -b];

X = [x(4), x(7), x(10);
    x(5), x(8), x(11);
    x(6), x(9), x(12)];

% Variational equation
OUT(4:12) = J*X;
```

LISTING 3.2 **continuation.m** – continuation procedure for localization of the hidden attractor in the Glukhovsky-Dolzhansky system

```
function out = continuation(ode, initPoint, tEnd, parameters, outDir)

% ODE solver parameters
acc = 1e-8;
RelTol = acc; AbsTol = acc; InitialStep = acc/10;
odeSolverOptions = odeset('RelTol', RelTol, 'AbsTol', AbsTol, ...
                        'InitialStep', InitialStep, 'NormControl', 'on');

% Current initial point
currInitPoint = initPoint;

numIter = size(parameters, 1);
out = zeros(numIter, 5);

% 1st equilibrium
S0 = [0 0 0];

% Create working directories
if ~exist(outDir, 'dir')
    mkdir(outDir);
end

% Routing the paths for plot directories
currProjDir = [outDir '/proj'];
if ~exist(currProjDir, 'dir')
```

```
    mkdir(currProjDir);
end

curr3dDir = [outDir '/3D'];
if ~exist(curr3dDir, 'dir')
    mkdir(curr3dDir);
end

% Main cycle
for iIteration = 1 : numIter

    % currParameters = [a sigma b r]
    currParameters = parameters(iIteration, :);

    a =  currParameters(1);
    sigma = currParameters(2);
    b = currParameters(3);
    r = currParameters(4);

    XSI = (sigma*b)/(2*a^2)*(a*(r-2)-sigma + sqrt((a*r-sigma)^2+4*a*sigma));
    X1 = (sigma*b*sqrt(XSI))/(sigma*b + a*XSI);
    Y1 = sqrt(XSI);
    Z1 = (sigma*XSI)/(sigma*b + a*XSI);


    % 2nd and 3d equilibria
    S1 = [X1 Y1 Z1];
    S2 = [-X1 -Y1 Z1];

    % Trajectory defining attractor
    [~,odeSolution] = ode45(ode, [0 tEnd], currInitPoint, odeSolverOptions);

    % Save starting point
    out(iIteration,:) = [currParameters(1) currParameters(4) currInitPoint];

    % Make a plot
    h = figure('Visible', 'off');
    % Plot equilibria
    plot3(S0(1), S0(2), S0(3), '.', 'markersize', 15, 'Color', 'red');
    text(S0(1), S0(2), S0(3),'S_0','fontsize', 18);
    hold on;
    plot3(S1(1), S1(2), S1(3), '.', 'markersize', 15, 'Color', 'red');
    text(S1(1), S1(2), S1(3),'S_1','fontsize', 18);
    hold on;
    plot3(S2(1), S2(2), S2(3), '.', 'markersize', 15, 'Color', 'red');
    text(S2(1), S2(2), S2(3),'S_2','fontsize', 18);
    hold on;
    % Plot attractor
    plot3(odeSolution(:,1), odeSolution(:,2), odeSolution(:,3));
    axis auto;
    grid on;
    xlabel('X');
    ylabel('Y');
    zlabel('Z');

    % Save current plot in different projections
    %   set(h, 'Visible', 'on');
    view(0,90), title('X-Y')
    saveas(h, [currProjDir '/' int2str(iIteration-1) ' - [' ...
            num2str(a,'%.6g') ' ' num2str(r,'%.6g') ']_XY.eps'],'epsc');

    view(0,0), title('X-Z')
    saveas(h, [currProjDir '/' int2str(iIteration-1) ' - [' ...
            num2str(a,'%.6g') ' ' num2str(r,'%.6g') ']_XZ.eps'],'epsc');

    view(90,0), title('Y-Z')
    saveas(h, [currProjDir '/' int2str(iIteration-1) ' - [' ...
            num2str(a,'%.6g') ' ' num2str(r,'%.6g') ']_YZ.eps'],'epsc');

    view([30,24]);
    saveas(h, [curr3dDir '/' int2str(iIteration-1) ' - [' ...
                num2str(a,'%.6g') ' ' num2str(r,'%.6g') '].eps'],'epsc');
    %   set(h, 'Visible', 'off');

    % Update starting point
    currInitPoint = odeSolution(end, : );

end

end
```

LISTING 3.3 **main.m** – start of the continuation procedure for the Glukhovsky-Dolzhansky system with specific parameters

```
function main

% Constant parameters:
sigma = 4; b = 1; a = 0.0052;

% Varying parameters:
rBegin = 687.5; rEnd = 700;

% Step number in continuation procedure
numSteps = 5;
stepSize = 1 / numSteps;

% Construct matrix of parameters
for i = 1 : numSteps
    rCurr = rBegin + i * stepSize * (rEnd - rBegin);
    rList = [rList rCurr];
end

paramMatrix = [repmat(a, numSteps+1, 1), ...
               repmat(sigma, numSteps+1, 1), ...
               repmat(b, numSteps+1, 1), ...
               rList'];

% Set initial data corresponding to a self-excited attractor
function OUT = J(X)
    OUT = [-sigma, sigma, -a * X(3) - a * X(2); ...
            rBegin - X(3), -1, -X(1); ...
            X(2), X(1), -b];
end

S0 = [0 0 0];
[V0, D0] = eig(J(S0));
[~, IX0] = sort(diag(D0), 'descend');
rEps0 = 0.1;

x0 = S0 + rEps0 * V0(:, IX0(1))';

% Set solver integration time
tEnd = 500;

% Set output directory
outDir = ['./OUT/A=' num2str(A_end, '%.4g') ', r=' num2str(rEnd, '%.4g')];

% Start continaution procedure
results = continuetion(@(t, x) LorenzLikeSyst(t, x, rBegin, sigma, b, a), ...
                                         x0, tEnd, paramMatrix, outDir);

% Save results
resFileName = ['./OUT/a=' num2str(a, '%.4g') ...
                            ', r=' num2str(rEnd, '%.4g') '.txt'];

save(resFileName, 'results', '-ascii','-double');

% Print results on screen
type(resFileName)

end
```

ORIGINAL PAPERS

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NUMERICAL JUSTIFICATION OF LEONOV CONJECTURE ON LYAPUNOV DIMENSION OF ROSSLER ATTRACTOR

by

N. V. Kuznetsov, T. N. Mokaev, P. A. Vasilyev 2014

Communications in Nonlinear Science and Numerical Simulation, Vol. 19, No. 4, pp. 1027–1034, doi:10.1016/j.cnsns.2013.07.026

Commun Nonlinear Sci Numer Simulat 19 (2014) 1027–1034

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Numerical justification of Leonov conjecture on Lyapunov dimension of Rossler attractor

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ARTICLE INFO

Article history:
Received 10 February 2013
Received in revised form 25 June 2013
Accepted 28 July 2013
Available online 9 August 2013

Keywords:
Rössler system
Lyapunov dimension
Strange attractor
Lyapunov exponent
Chaos
Leonov's conjecture

ABSTRACT

Exact Lyapunov dimension of attractors of many classical chaotic systems (such as Lorenz, Henon, and Chirikov systems) is obtained. While exact Lyapunov dimension for Rössler system is not known, Leonov formulated the following conjecture: *Lyapunov dimension of Rössler attractor is equal to local Lyapunov dimension in one of its stationary points.* In the present work Leonov's conjecture on Lyapunov dimension of various Rössler systems with standard parameters is checked numerically.

1. Introduction

Lyapunov exponents (LEs) play an important role in the description of dynamical systems behavior. They were introduced by Lyapunov [1] for the analysis of stability by the first approximation for *regular* time-varying linearizations, where the negativeness of the largest Lyapunov exponent indicates stability. Much later, in 1940s, Chetaev tried to prove that for *regular* time-varying linearizations, a positive Lyapunov exponent indicates instability in the sense of Lyapunov, but a gap in his proof was revealed and filled recently for more weak definition of instability [2]). Since there are no general methods for checking regularity of linearization and there are known Perron effects [3–5,2] of sign inversion of the largest Lyapunov exponent for nonregular time-varying linearizations, the computation of Lyapunov exponents for linearization of nonlinear autonomous system along nonstationary trajectories is widely used for investigation of chaos. In this case the positiveness of the largest Lyapunov exponent is often regarded as the indication of chaotic behavior in the considered nonlinear system. The various methods, used for the numerical computation of Lyapunov exponents, are described, e.g., in [6–9].

Nowadays various characteristics of attractors of dynamical systems (information dimension, metric entropy etc.) are studied based on Lyapunov exponents computation. In particular, Kaplan and Yorke defined a quantity they called *Lyapunov dimension* and conjectured that it was equal to information dimension [10].

In the work [11] Leonov considered exact formulas of Lyapunov dimension of Lorenz, Henon, and Chirikov attractors. By analogy with the results for these attractors he conjectured that Lyapunov dimension of Rössler attractor[1] is determined by a stationary point embedded in this attractor.

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[1] Following [12,13], an attractor is a bounded, closed, invariant, attracting subset of phase space of dynamical system. Since for the considered Rössler systems there are no analytical estimations of localization of their attractors, it is not feasible to check their boundness and closedness. Usually by Rössler attractor one means an attracting set obtained as a result of numerical experiments [14,15].

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http://dx.doi.org/10.1016/j.cnsns.2013.07.026

In the present paper Leonov's conjecture is checked numerically and it is demonstrated that this conjecture is true for three different types of Rössler systems. These three-dimensional systems are simplest and, in a sense, minimal models for continuous-time chaos. They have only a single nonlinear quadratic term and they generate chaotic attractors with a single "leaf" (in contrast to Lorenz attractor). Rössler systems arose as simplified prototypes of some chemical reactions while Otto Rössler researched different types of chaos in chemical kinetics.

2. Problem statement

2.1. Rössler systems

Consider the following three-dimensional Rössler systems [14,15]

$$(\mathbf{1.1})\begin{cases}\dot{u}=-y-z\\ \dot{y}=u\\ \dot{z}=a(y-y^2)-bz\end{cases}\quad(\mathbf{1.2})\begin{cases}\dot{u}=-y-z\\ \dot{y}=u+ay\\ \dot{z}=b-cz+uz\end{cases}\quad(\mathbf{1.3})\begin{cases}\dot{u}=-y-z\\ \dot{y}=u+ay\\ \dot{z}=bu-cz+uz\end{cases}\tag{1}$$

with the corresponding standard parameters

$$\begin{aligned}&(1.1): a=0,386;\ b=0,2;\\ &(1.2): a=0,2;\ b=0,2;\ c=5,7;\\ &(1.3): a=0,36;\ b=0,4;\ c=4,5.\end{aligned}\tag{2}$$

In the phase spaces of these systems, for parameters (2) there exist chaotic attractors and the corresponding stationary points

$$\begin{aligned}&x_0=(0,0,0)\quad\text{for systems } (1.1)\quad\text{and } (1.3),\\ &x_0=\left(\frac{c-\sqrt{c^2-4ab}}{2},-\frac{c-\sqrt{c^2-4ab}}{2a},\frac{c-\sqrt{c^2-4ab}}{2a}\right)\text{for system}(1.2)\end{aligned}\tag{3}$$

are embedded in these attractors [14,15].

2.2. Lyapunov dimension

Consider a topological characteristic — a *local Lyapunov dimension* of the point x_0 in the phase space U of dynamical system, which is associated with the Lyapunov spectrum $\lambda_1(x_0)\geqslant\ldots\geqslant\lambda_n(x_0)$ and is defined by formula

$$\dim_L x_0=j+\frac{\lambda_1(x_0)+\cdots+\lambda_j(x_0)}{|\lambda_{j+1}(x_0)|}.\tag{4}$$

Here $j\in[1,n]$ is the smallest natural number m such that

$$\lambda_1(x_0)+\cdots+\lambda_{m+1}(x_0)<0,\quad \lambda_{m+1}(x_0)<0,\quad \frac{\lambda_1(x_0)+\cdots+\lambda_m(x_0)}{|\lambda_{m+1}(x_0)|}<1.$$

Lyapunov dimension of invariant set $B\subset U$ of dynamical system is defined by the relation

$$\dim_L B=\sup_{x\in B}\dim_L x.\tag{5}$$

The properties of Lyapunov dimension are considered in details in the works [16–18]. In particular, it is proved that Lyapunov dimension is an upper bound for Hausdorff and fractal dimensions.

2.3. Leonov's conjecture

For Lorenz, Henon, and Chirikov systems a problem of computation of Lyapunov dimension of their attractors is solved in [19–24]. In these works it is obtained analytically exact Lyapunov dimension of attractors of these systems and in [11] it is given estimates of Lyapunov dimension of attractor of Rössler system (1.1). Based on these results, Leonov formulated the following.

Conjecture. If a stationary point x_0 is embedded in attractor A of Rössler systems (1), then

$$\dim_L A=\dim_L x_0.$$

In order to verify this conjecture for attractors of systems (1) with parameters (2) and stationary points (3), in the present work it is developed a special numerical procedure described below. Note that this procedure can be applied similarly to various modifications of Rössler system of higher orders (see, e.g., [15,25,26]).

3. Numerical justification of Leonov's conjecture

3.1. Lyapunov spectrum computation algorithm

To verify the conjecture, it is used an approach to the computation of Lyapunov spectrum, suggested in the works [6,7]. In [9] this approach was adapted to computer realization. This method is an iterative process and is a variation of standard QR algorithm for computation of eigenvalues and eigenvectors [27]. It is based on the following definitions and statements.

Consider system (1) in general form

$$\dot{x} = F(x), \tag{6}$$

where $x(t) \in \mathbb{R}^n$ for any $t \in \mathbb{R}$, $F : U \to \mathbb{R}^n$ is C^r-smooth function ($r \geqslant 1$) on the open set $U \subset \mathbb{R}^n$.

Denote by $A(t) = T_x F(f(t, x_0))$ the Jacobian matrix of system (6), where $f(t, x_0)$ is a solution of system (6).

Consider two close points x_0 and $(x_0 + u_0)$ in the phase space U, where u_0 is a small disturbance of the point x_0. Then the evolution of vector $u(t) = f(t, x_0 + u_0) - f(t, x_0)$ can be studied [28] by the following linearized system

$$\dot{u} = A(t)u. \tag{7}$$

The solution of equation (7) can be represented as $u(t) = \Phi(t)u_0$, where $\Phi(t) = T_{x_0} f(t, x_0)$ is a fundamental matrix of system (7). The exponential rate of divergence (or convergence) of nearby trajectories is given by formula

$$\lambda(x_0, u_0) \stackrel{\text{def}}{=} \overline{\lim}_{t\to\infty} \frac{1}{t} \ln \frac{\|u(t)\|}{\|u_0\|} = \overline{\lim}_{t\to\infty} \frac{1}{t} \ln \|\Phi(t)u_0\|. \tag{8}$$

This value is called Lyapunov exponent of order 1 (or, simply, Lyapunov exponent).

It can be considered a generalization of Lyapunov exponent of order 1 to the case of order p, $1 \leqslant p \leqslant n$. Let E_0^p be the p-dimensional subspace of tangent space E_0 and U_0 be the open parallelepiped generated by p linearly independent vectors $e_1, \ldots, e_p$ of E_0^p. Then Lyapunov exponent of order p is defined [6] as

$$\lambda^p(x_0, E_0^p) \stackrel{\text{def}}{=} \overline{\lim}_{t\to\infty} \frac{1}{t} \ln \mathrm{Vol}^p(T_{x_0} f(t, U_0)) = \overline{\lim}_{t\to\infty} \frac{1}{t} \ln \mathrm{Vol}^p[\Phi(t)e_1, \ldots, \Phi(t)e_p], \tag{9}$$

where Vol^p means p-dimensional volume induced in tangent space by scalar product.

If in (8) and (9) $\overline{\lim}_{t\to\infty}$ can be replaced by $\lim_{t\to\infty}$, then it is said that exact Lyapunov exponent exists.

It is known [1,29,6] that for regular linear systems there exist exact Lyapunov exponents[2] of order p, $1 \leqslant p \leqslant n$ and in the tangent space E_0 at the point x_0 it can be chosen p linearly independent vectors $e_1, \ldots, e_p$ such that

$$\lambda^p(x_0, E_0^p) = \lambda_1(x_0) + \cdots + \lambda_p(x_0), \tag{10}$$

where $\lambda_i(x_0) \stackrel{\text{def}}{=} \lambda(x_0, e_i)$, $i = 1 \ldots p$, and $\lambda_1(x_0) \geqslant \cdots \geqslant \lambda_p(x_0)$. That is, each Lyapunov exponent of order p is equal to the sum of p largest Lyapunov exponents of order 1.

In order to calculate all tangent vectors one can solve system (6) together with the matrix-valued variational equation [28]

$$\dot{\Phi}_t(x_0) = A(t)\,\Phi_t(x_0), \quad \Phi_0(x_0) = I, \tag{11}$$

where $\Phi_t(x_0) = T_{x_0} f(t, x_0)$ and I is identity matrix.

In this case one can go directly to the description of computation procedure. Choose the initial point x_0 and $(n \times n)$ matrix of orthonormal vectors $Q_0 = [q_1^0, \ldots, q_n^0]$. During the kth iteration, original system (6) is integrated together with variational equation (11) with the initial data $\{x_{k-1}, Q_{k-1}\}$ over the chosen small time interval h for obtaining $x_k = f(hk, x_0)$ and

$$U_k = [u_1^k, \ldots, u_n^k] = \Phi_{hk}(x_0).$$

Then the matrix U_k is QR decomposed, i.e. $U_k = Q_k R_k$, where Q_k is orthogonal matrix and R_k is upper triangular matrix. The p-dimensional volume, defined in (9), increases by the multiplier $R_k(1,1) \ldots R_k(p,p)$ since $\mathrm{V}^p\{u_1^k, \ldots, u_p^k\} = R_k(1,1) \ldots R_k(p,p)$, where $R_k(i,i)$ is a norm of the vector u_i^k, $i = 1 \ldots p$. The matrix Q_k is taken as the initial datum for variational equation at the following iteration.

So, formula (9) can be expressed as

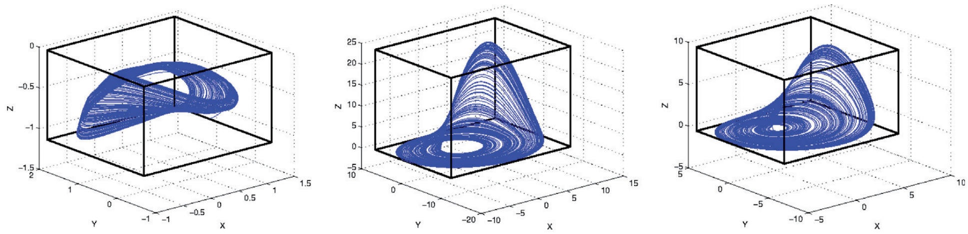
$$\lambda^p(x_0, U_0) = \lim_{k\to\infty} \frac{1}{kh} \sum_{i=1}^{k} \ln(R_i(1,1) \ldots R_i(p,p)), \quad 1 \leqslant p \leqslant n.$$

One repeats this iteration procedure K times. Subtracting λ^{p-1} from λ^p and using formula (10), one obtains approximate values of pth Lyapunov exponent of order 1 for the chosen trajectory. By formula (4) a local Lyapunov dimension can also be computed.

3.2. Discussion and results

The algorithm, described in the previous section, is used in the process of justification of Leonov's conjecture. The entire computational procedure is implemented in MATLAB. For the orthogonalization of fundamental matrix it is used MATLAB library function *qr*, which implements a factorization procedure by using the Householder transformation since a classical Gram–Schmidt algorithm is numerically unstable and its modified version requires more execution time.

[2] The opposite is not true: in the general case the existence of exact Lyapunov exponents does not imply regularity of the system [2].



(a) Attractor localization for (1.1) (b) Attractor localization for (1.2) (c) Attractor localization for (1.3)

Fig. 1. Localization of attractors of systems (1).

Table 1
The results of justification for the following parameters: $h = 1$, $K = 200$, $abs\_tol = rel\_tol = 10^{-8}$.

Rössler system	Cube	Grid step	$\max_{x \in grid} \dim_L x$	$\dim_L x_0$
(1.1)	$[-1; 1,3] \times [-0,7; 1,8] \times [-1,05; -0,03]$	0,1	2,4205	2,6042
(1.2)	$[-9; 12] \times [-11; 8] \times [-0,1; 23,9]$	0,5	2,0296	2,0341
(1.3)	$[-5; 7] \times [-7; 4] \times [-0,2; 9,8]$	0,5	2,0340	2,0620

For nonlinear systems (1) there are no exact formulas, describing the solutions of these systems in general form. In this case it is considered approximated solutions, obtained by numerical integration of this systems, which is based on various finite-difference and more complex methods [30,31]. For Rössler system (1.2) the problem of analysis of its analytical and numerical solutions is considered in [32].

In this paper for the integration of systems (1) it is used MATLAB realization (solver ode45) of Runge–Kutta finite-difference schemes of order 4–5 with an adaptive step. The absolute and relative tolerance are chosen equal to 10^{-8} since smaller values strongly influence a time of evaluation procedure. The parameter of procedure h, which determines integration time at each iteration, is chosen sufficiently small for the columns of fundamental matrix to be remained linearly independent. The parameter K – a number of iterations – must be sufficiently large in order that the trajectory, with the initial point in the neighborhood of attractor, covered this attractor. For the chosen parameters it was made the following: the number of iterations was increased by 2 times and a step was decreased by 2 times, in which case the result was qualitatively the same.

Since for Rössler systems (1) there are no analytical estimations of localization of their attractors, for estimation it is used computer experiments [33,34]. For the considered systems (1) their attractors are numerically localized in cubes (Fig. 1) by standard computational procedure[3]. On each cube it is chosen a grid with a certain step and at each grid point it is started the algorithm of computation of local Lyapunov dimension[4]. The obtained values are compared with a local Lyapunov dimension at stationary point. Then it is considered the grid points having the values of Lyapunov dimension, which are most close to a value at stationary point. Around each of these grid points it is considered a grid with a smaller step and at the points of this grid it is computed local Lyapunov dimensions. These values are also compared with a value at stationary point.

4. Conclusion

In this work Leonov's conjecture on Lyapunov dimension of various Rössler systems with standard parameters is verified numerically. While the data, given in Table (1), numerically confirm Leonov's conjecture, analytical proof of Leonov's conjecture is still an open problem.

Acknowledgments

This work was supported by the Academy of Finland, Russian Ministry of Education and Science (Federal target programm), Russian Foundation for Basic Research and Saint-Petersburg State University.

Appendix A. Computation of Lyapunov exponents and Lyapunov dimension in MATLAB

Here it is given the main parts of program, written in MATLAB, which implements the described above algorithm for the computation of Lyapunov dimension of three-dimensional dynamical system (f.e. it is considered Rössler system (1.1)).

[3] From a computational point of view, in nonlinear dynamical systems, attractors can be regarded as *self-excited* and *hidden attractors* [35–38]. Self-excited attractors can be localized numerically by *standard computational procedure, in which after a transient process a trajectory, started from a point of unstable manifold in a neighborhood of equilibrium, reaches a state of oscillation and therefore it can easily be identified*. In contrast, for a *hidden attractor, its basin of attraction does not intersect with small neighborhoods of equilibria*. While many classical attractors are self-exited attractors and therefore can be obtained numerically by standard computational procedure, for localization of hidden attractors it is necessary to develop special procedures since there are no similar transient processes leading to such attractors.

[4] Since numerical localization of attractors is considered and there is no effective way to prove ergodicity rigourously, one has to consider a mesh of initial conditions for investigation of Lyapunov exponents.

Listing 1: Computation of Lyapunov exponents

```
function [t, lces, trajectory] = lyapunov_exp(ode, x_start, t_start, ...
                                               t_step, k_iter, rel_tol, abs_tol)

% For the given combined system represented by system of differential
% equations and variational equation this function returns array of
% LCEs for the point x_start.
%
% Parameters:
%   ode - combined system (system of ODEs + var. eq.);
%   x_start -  initial point;
%   t_start - initial time value;
%   t_step - time-step in the reorthogonalization procedure;
%   k_iter - number of iterations in the reorthogonalization procedure;
%   rel_tol - relative error in Runge-Kutta 45 method;
%   abs_tol - absolute error in Runge-Kutta 45 method.

% n1 - size of the system of ODEs :
[~,n1] = size(x_start);

% n2 - size of the combined system :
n2 = n1*(n1+1);

% y - initial value for the combined system :
y = zeros(n2,1);

% Initializing y :
y(1:n1) = x_start(:);

for i = 1:n1
    y((n1+1)*i) = 1.0;
end

% norms - array of norms of vectors in Jacobi matrix :
norms = zeros(1,n1);

% log_sum - array of sums of logarithms of norms :
log_sum = zeros(1,n1);

% l_exp - array of lyapunov exponents :
lexp = zeros(1,n1);

% t_curr - current moment of time :
t_curr = t_start;

% tr_len - current index in the trajectory array :
tr_len = 1;

% Preallocations for output values :
t = zeros(k_iter,1);
lces = zeros(k_iter,3);

% Set options for MATLAB solver :
options = odeset('RelTol', rel_tol, 'AbsTol', abs_tol);

% Main loop:
for i = 1 : k_iter

    % Solving combined system :
    sol = ode45(ode, [t_curr t_curr+t_step], y, options);
    % i_last - the last moment :
    i_last = numel(sol.x);

    % Getting Jacobi matrix PhiT at the moment T
    % from vector Y :
    Y = transpose(sol.y);
    PhiT = reshape( Y(i_last, n1+1 : n2 ), n1, n1);

    % QR factorization of PhiT :
    [V, R] = qr(PhiT);

    for j = 1 : n1
        if R(j,j) < 0
            R(j,j) = (-1) * R(j,j);
            V(:,j) = (-1) * V(:,j);
        end
    end

    % Updating  y and t_curr :
    t_curr = t_curr + t_step;
    y( 1 : n1 ) = Y( i_last, 1:n1 );
    y( n1+1 : n2 ) = reshape(V, 1, []);

    % Computing lyapunov exponents (in moment t_curr) :
    for k = 1 : n1
        norms(k) = R(k,k);
        log_sum(k) = log_sum(k) + log( norms(k) );
        lexp(k) = log_sum(k) / (t_curr-t_start);
    end

    % Saving computations in corresponding vectors :
    t(i) = t_curr;
    lces(i, :) = lexp;

    for j = 1 : i_last
        trajectory(tr_len, :) = [sol.x(j) sol.y(1:n1, j)'];
        tr_len = tr_len + 1;
    end
end
end
```

Listing 2: Computation of Lyapunov dimension

```
function ld = lyapunov_dim(lces)
% For the given array of lyapunov exponents of some point this function
% compute local lyapunov dimention of this point.

% Parameters:
%   lces - array of lyapunov exponents.

% ld - local lyupunov dimention :
ld = 0;

% n - number of LCEs :
[~,n] = size(lces);

% lambda - sorted array of LCEs :
lambda = sort(lces, 'descend');

% Main loop :
le_sum = lambda(1);
if ( lambda(1) > 0 )
     for i = 1 : n-1
        if lambda(i+1) ~= 0
           ld = i + le_sum / abs( lambda(i+1) );
           le_sum = le_sum + lambda(i+1);
           if le_sum < 0
              break;
           end
        end
    end
end
end
```

Listing 3: Rössler system (1.1)

```
function OUT = rossler_syst_1(t, X)

% Parameters:
global a b

% Output vector that represents combined system :
OUT = zeros(12,1);

% Rossler equation:
OUT(1) = - X(2) - X(3);
OUT(2) = X(1);
OUT(3) = -b*X(3) + a*(X(2) - X(2)*X(2));

% Variational equation:
OUT(4:12) = [0 -1 -1; 1 0 0; 0 a*(1-2*X(2)) -b] ...
     * [X(4) X(7) X(10); X(5) X(8) X(11); X(6) X(9) X(12)];
```

Listing 4: Numerical procedure for Rössler system (1.1)

```
function run_rossler1

% This procedure computes local lyapunov dimention of the fixed point
% and local lyapunov dimentions of the points on the grid for
% the 1st Rossler attractor.

% Parameters :
global a b

% Values of parameters :
a = 0.386; b = 0.2;

% T - time-step in iterative procedure :
T = 1.0;

% K - number of iterations of iterative procedure :
K = 200;

% Relative and absolute errors for Runge-Kutta 45 method :
rel_tol = 1e-8;
abs_tol = 1e-8;

% eps - step on the grid :
eps = 1e-1;

% Fixed point :
x0 = [0 0 0];

% Attractor is located in cube :
x_begin = -1;    x_end = 1.3;  % x \in [-1; 1.3];
y_begin = -0.7;  y_end = 1.8;  % y \in [-0.7; 1.8];
z_begin = -1.1;  z_end = 0;    % z \in [-1.1; 0];

x_iterations = (x_end - x_begin) / eps + 1;
y_iterations = (y_end - y_begin) / eps + 1;
z_iterations = (z_end - z_begin) / eps + 1;

% Infinity factor: if trajectory leaves cube with side 'infinity_factor',
% then we conclude, that trajectory will leave basin of attraction :
infinity_factor = 10;

% Result array :
grid_results = zeros(x_iterations*y_iterations*z_iterations, 7);
i_res = 1;

% Computing local lyapunov dimentions of the grid points :
for i = 1 : x_iterations
    for j = 1 : y_iterations
        for k = 1 : z_iterations

            % Main logic :
            curr_point = [x_begin+(i-1)*eps y_begin+(j-1)*eps z_begin+(k-1)*eps];
            [~, lces, trajectory] = lyapunov_exp(@rossler_syst_1, curr_point, 0, ...
                                                        T, K, rel_tol, abs_tol);
            len = size(trajectory, 1);

            if (abs(trajectory(len, 2)) < infinity_factor ...
                        && abs(trajectory(len, 3)) < infinity_factor ...
                                    && abs(trajectory(len, 4)) < infinity_factor)

               % Saving results for current point :
               grid_results(i_res, :) = [curr_point lyapunov_dim(lces(end, : )) ...
                                                                            lces(end, : )];
               i_res = i_res + 1;
            end
        end
    end
end

% Computing local lyapunov dimention of6 the fixed point :
[~, lces, ~] = lyapunov_exp(@rossler_syst_1, x0, 0, T, K, rel_tol, abs_tol);
LCEs = lces(end, : );


% Saving results in file :
fid = fopen('hypothesis_roessler_1.txt');
fprintf(fid, '%4s %4s %4s %10s %10s %10s %10s\r\n',...
                                                                  'x', 'y', 'z', 'dim_L', 'lce1', 'lce3', 'lce3');
fprintf(fid, '%.2f, %.2f, %.2f, %.8f, %.8f, %.8f, %.8f\r\n', grid_results);
fprintf(fid, '\r\nLyapunov dimension in fixed point:\r\n');
fprintf(fid, '%.2f, %.2f, %.2f, %.8f, %.8f, %.8f, %.8f\r\n', ...
                                                             [x0 lyapunov_dim(LCEs) LCEs]);
fclose(fid);

end
```

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PII

HOMOCLINIC ORBIT AND HIDDEN ATTRACTOR IN THE LORENZ-LIKE SYSTEM DESCRIBING THE FLUID CONVECTION MOTION IN THE ROTATING CAVITY

by

G. A. Leonov, N. V. Kuznetsov, T. N. Mokaev 2015

Communications in Nonlinear Science and Numerical Simulation, Vol. 28, No. 1, pp. 166–174, doi:10.1016/j.cnsns.2015.04.007

Commun Nonlinear Sci Numer Simulat 28 (2015) 166–174

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Hidden attractor and homoclinic orbit in Lorenz-like system describing convective fluid motion in rotating cavity

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ARTICLE INFO

Article history:
Received 20 January 2015
Revised 6 March 2015
Accepted 7 April 2015
Available online 15 April 2015

Keywords:
Hidden attractor
Self-excited attractor
Multistability
Coexistence of attractors
Lorenz-like system
Homoclinic orbit
Lyapunov exponent
Lyapunov dimension

ABSTRACT

In this paper a Lorenz-like system, describing convective fluid motion in rotating cavity, is considered. It is shown numerically that this system, like the classical Lorenz system, possesses a homoclinic trajectory and a chaotic self-excited attractor. However, for the considered system, unlike the classical Lorenz system, along with self-excited attractor a hidden attractor can be localized. Analytical-numerical localization of hidden attractor is demonstrated.

1. Introduction

Consider the following physical problem: the convection of viscous incompressible fluid motion inside the ellipsoid

$$\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 = 1, \quad a_1 > a_2 > a_3 > 0,$$

under the condition of stationary inhomogeneous external heating. It is assumed that the ellipsoid together with heat sources rotates with the constant velocity $\mathbf{\Omega_0}$ around its axis. Vector $\mathbf{l_0}$ determines the orientation of the ellipsoid in the space and has the same direction as the gravity vector $\mathbf{g}$. Vector $\mathbf{g}$ is stationary with respect to the ellipsoid motion. The value $\mathbf{\Omega_0}$ is assumed to be such that the centrifugal forces can be neglected in comparison with the influence of the gravitational field. Consider the case when the ellipsoid rotates around the axis x_3 that has a constant angle α with the gravity vector $\mathbf{g}$ ($|\mathbf{g}| = g$) and the vector $\mathbf{g}$ is placed in the plane x_1x_3. Then $\mathbf{\Omega_0} = (0, 0, \Omega_0)$ and $\mathbf{l_0} = (a_1 \sin\alpha, 0, -a_3 \cos\alpha)$. Let the steady-state temperature difference $\Delta\hat{\mathbf{T}} = (q_0,\ 0,\ 0)$ be generated along the axis x_1 (Fig. 1). Corresponding mathematical model (three-mode model of convection)

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http://dx.doi.org/10.1016/j.cnsns.2015.04.007
1007-5704/

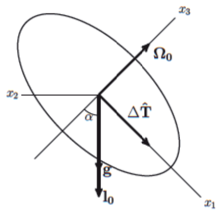


Fig. 1. Illustration of the problem setting.

was obtained by Glukhovsky and Dolzhansky [1] in the following form

$$\begin{cases} \dot{x} = Ayz + Cz - \sigma x, \\ \dot{y} = -xz + R_a - y, \\ \dot{z} = xy - z. \end{cases} \tag{1}$$

Here

$$\sigma = \frac{\lambda}{\mu}, \quad T_a = \frac{\Omega_0^2}{\lambda^2}, \quad R_a = \frac{g\beta a_3 q_0}{2a_1 a_2 \lambda\mu},$$
$$A = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}\cos^2\alpha\, T_a^{-1}, \quad C = \frac{2a_1^2 a_2}{a_3(a_1^2 + a_2^2)}\sigma\sin\alpha,$$
$$x(t) = \mu^{-1}\left(\omega_3(t) + \frac{g\beta a_3\cos\alpha}{2a_1a_2\Omega_0}q_3(t)\right), \quad y(t) = \frac{g\beta a_3}{2a_1a_2\lambda\mu}q_1(t),$$
$$z(t) = \frac{g\beta a_3}{2a_1a_2\lambda\mu}q_2(t),$$

and λ, μ, β are the coefficients of viscosity, heat conduction, and volume expansion, respectively; $q_1(t)$, $q_2(t)$, and $q_3(t)$ ($q_3(t) \equiv 0$) are temperature differences on the principal axes of ellipsoid a_1, a_2, and a_3, respectively; $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ are the projections of the vectors of fluid angular velocities on the axes x_1, x_2, and x_3, respectively. Here

$$\omega_1(t) = -\frac{g\beta a_3}{2a_1a_2\Omega_0}\cos\alpha\, q_1(t), \quad \omega_2(t) = -\frac{g\beta a_3}{2a_1a_2\Omega_0}\cos\alpha\, q_2(t).$$

The parameters σ, T_a, and R_a are the Prandtl, Taylor, and Rayleigh numbers, respectively.

After the linear transformation (see, e.g., [1,2]):

$$x \to x, \quad y \to R - \frac{\sigma}{a_0R+1}C^{-1}z, \quad z \to \frac{\sigma}{a_0R+1}C^{-1}y,$$

one obtains the following system

$$\begin{cases} \dot{x} = -\sigma(x-y) - ayz \\ \dot{y} = rx - y - xz \\ \dot{z} = -z + xy, \end{cases} \tag{2}$$

where $a_0 = A/C^2$, $R = R_a C$,

$$a = \frac{a_0\sigma^2}{(a_0R+1)^2}, \quad r = \frac{R}{\sigma}(a_0R+1). \tag{3}$$

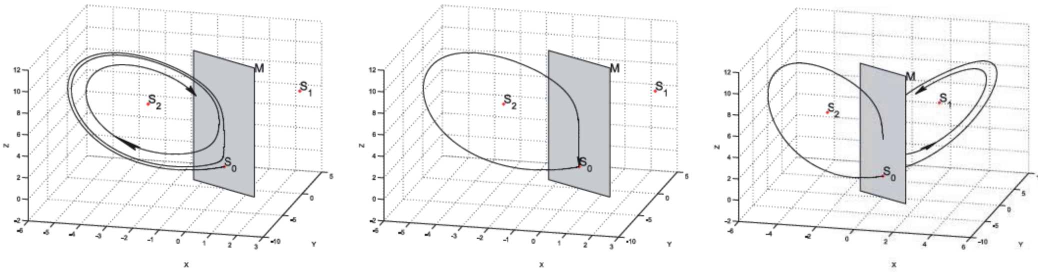
System (2) with $a = 0$ coincides with the classical Lorenz system [3] with $b = 1$. As it is discussed in [2], system (2) can also be used to describe the following physical processes: waves interaction in plasma [4–7], the convective fluid motion inside rotating ellipsoid [1], the rotation of rigid body in viscous fluid [8], the gyrostat dynamics [9,10], the convection of horizontal layer of fluid making harmonic oscillations [11], and the model of Kolmogorov's flow [12].

Note that the Glukhovsky–Dolzhansky system is sufficiently different from the classical Lorenz system. In the Lorenz system, the convective fluid motion in two dimensions is considered only. In the Glukhovsky–Dolzhansky system, the convective fluid motion in three dimensions is considered which can be interpreted as one of the models of ocean flow [1].

In [13] for system (2) in the case $\sigma = \pm ar$ a detailed analysis of the equilibria stability and asymptotic behavior of trajectories is given and the values of parameters are obtained for which system (2) is integrable.

In what follows system (2) will be considered under the condition that the parameter a is positive. In this case if $r < 1$, then (2) has a unique equilibrium $S_0 = (0, 0, 0)$, which is globally asymptotically Lyapunov stable [2,14]. If $r > 1$, then system (2) has three equilibria: $S_0 = (0, 0, 0)$ and

$$S_{1,2} = (\pm x_1, \pm y_1, z_1). \tag{4}$$



(a) $r = 7.44$ (b) $r = 7.4430045820796753...$ (c) $r = 7.445$

Fig. 2. The birth of homoclinic orbit in system (2) with $\sigma = 4$, $a = 0.0052$, $r \in [7.44, 7.445]$, $\delta = 0.01$.

Here

$$x_1 = \frac{\sigma\sqrt{\xi}}{\sigma + a\xi}, \quad y_1 = \sqrt{\xi}, \quad z_1 = \frac{\sigma\xi}{\sigma + a\xi},$$

and the number ξ is as follows

$$\xi = \frac{\sigma}{2a^2}[a(r-2) - \sigma + \sqrt{(\sigma - ar)^2 + 4a\sigma}].$$

The stability of equilibria $S_{1,2}$ depends on the parameters σ, r, a (see. Section 3.1).

For system (2) with the fixed σ, a it is possible to observe the classical scenario of transition to chaos similar to the scenario in the Lorenz system [15]. Next, increasing parameter $r > 1$, a homoclinic trajectory and a self-excited chaotic attractor are obtained numerically. Unlike the Lorenz system, for system (2) it is also possible to localize a hidden chaotic attractor.

2. Homoclinic orbit

Consider linearization of the system along the saddle point S_0 and denote corresponding Jacobian matrix by J. Consider a trajectory $x(t)$ of system (2) starting at a certain initial point x_0 on the eigenvector that corresponds to the positive eigenvalue of J, from a vicinity of the saddle point S_0. In order to visualize numerically a homoclinic trajectory $x_h(t)$ ($\lim_{t\to+\infty} x_h(t) = \lim_{t\to-\infty} x_h(t) = S_0$), one may integrate system (2) with sufficiently small initial data x_0.

For some parameters of system (2) this trajectory after a certain time intersects two-dimensional plane M spanned on the eigenvectors that correspond to negative eigenvalues of J. The parameters have to be chosen in such a way that the point of intersection belongs to δ-vicinity of S_0.

Let us fix the parameters: $\sigma = 4$ and $a = 0.0052$ (such values were considered in [1]). For $r = 7.44$ there is no intersection of the trajectory $x(t)$ with the plane M (see Fig. 2a) and for $r = 7.445$ the intersection occurs (see Fig. 2c). So, there exists an intermediate value $r^* \in [7.44, 7.445]$ for which one gets the approximation of a homoclinic orbit: $r^* = 7.4430045820796753\ldots$ (see Fig. 2b). The approximation of symmetric homoclinic orbit can be obtained by choosing the symmetric (with respect to S_0) initial data in the computational procedure (see Fig. 3).

From a theoretical point of view, the existence of homoclinic trajectory can be justified by the *Fishing principle* [16–19]. The Fishing principle is based on the construction of a special two-dimensional manifold such that the separatrix of the saddle point intersects or does not intersect the manifold for two different values of a parameter. The continuity implies the existence of some intermediate value of the parameter for which the separatrix touches the manifold. According to the construction the only possibility for separatrix is to touch the saddle and thus, one can numerically localize the birth of homoclinic orbit.

3. Chaotic attractor

3.1. Local stability analysis and computation of attractors

Let us study the stability of equilibria S_1, S_2 of system (2). By the Routh–Hurwitz criterion, one gets the following

Proposition 1. *If $\sigma > 2$ and the parameters r and a satisfy the inequality*

$$p_3(\sigma, a)r^3 + p_2(\sigma, a)r^2 + p_1(\sigma, a)r + p_0(\sigma, a) < 0, \tag{5}$$

where

$$p_3(\sigma, a) = a^2\sigma^2(\sigma - 2),$$
$$p_2(\sigma, a) = -a(2\sigma^4 - 4\sigma^3 - 3a\sigma^2 + 4a\sigma + 4a),$$

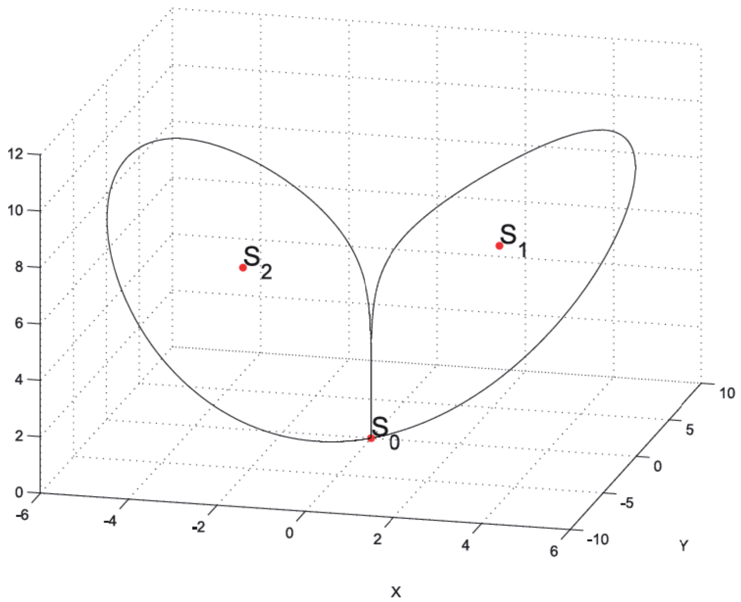


Fig. 3. Visualization of homoclinic butterfly for system (2).

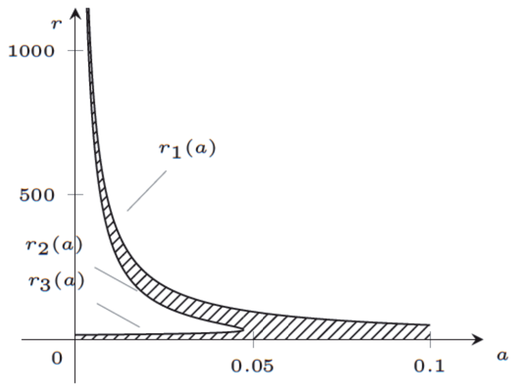


Fig. 4. Domain of stability of the equilibria $S_{1,2}$ of system (2) for $\sigma = 4$.

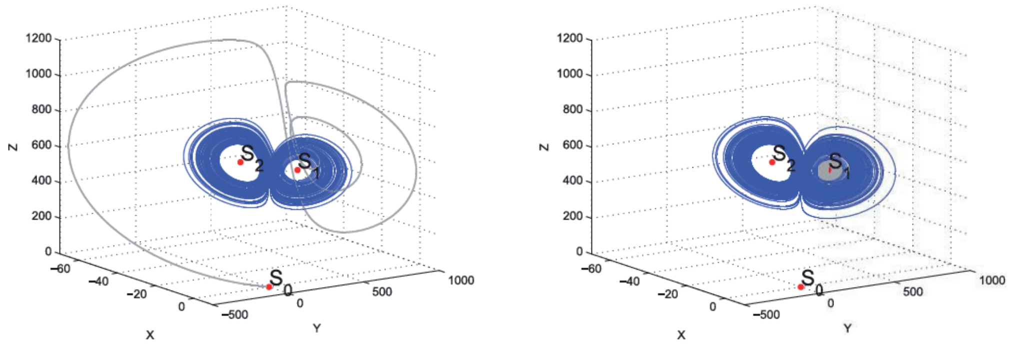
$$p_1(\sigma, a) = \sigma^2(\sigma^3 + 2(3a - 1)\sigma^2 - 8a\sigma + 8a),$$
$$p_0(\sigma, a) = -\sigma^3(\sigma^3 + 4\sigma^2 - 16a),$$

then the equilibria $S_{1,2}$ are stable.

Let us choose the parameter $\sigma = 4$ and, as in [1], construct the domains of stability of the equilibria $S_{1,2}$ in dependence on the values of parameters a and r. For $0 < a < a^*$, where $a^* = 0.04735\ldots$, there are three real positive roots: $r_1(a) > r_2(a) > r_3(a)$; for $a = a^*$—two real roots: $r_1(a)$ and $r_2(a) = r_3(a)$; and for $a > a^*$—one real root: $r_1(a)$.

Thus, for $0 < a < a^*$ the equilibria $S_{1,2}$ are stable for $r < r_3(a)$ and $r_2(a) < r < r_1(a)$; for $a > a^*$ the equilibria $S_{1,2}$ are stable for $r < r_1(a)$ (see Fig. 4).

An oscillation in dynamical system can be easily localized numerically if the initial conditions from its open neighborhood lead to the long-time behavior that approaches the oscillation. Thus, from a computational point of view, it is natural to suggest the following classification of attractors, based on the simplicity of finding the basin of attraction in the phase space:



(a) Localization of self-excited attractor from an initial point in a vicinity of S_0

(b) Localization of self-excited attractor from an initial point in a vicinity of S_1

Fig. 5. A self-excited attractor of system (2) for $r = 687.5$, $\sigma = 4$, $a = 0.0052$.

Definition. ([20–23]) An attractor is called a ***hidden attractor*** if its basin of attraction does not intersect with small neighborhoods of the equilibria of the system, otherwise it is called a ***self-excited attractor***.

3.2. Self-excited attractor visualization

For a ***self-excited attractor*** its basin of attraction is connected with an unstable equilibrium and, therefore, self-excited attractors can be localized numerically by the ***standard computational procedure*** in which after a transient process a trajectory, started in a neighborhood of an unstable equilibrium (from a point of its unstable manifold), is attracted to the state of oscillation and traces it. Thus self-excited attractors can be easily visualized.

Using the obtained domain of stability (see Fig. 4), one can study qualitative behavior of trajectories of system (2) for the fixed $\sigma = 4$, $a = 0.0052$, and $r \in (16.4961242, 690.6735024)$. Consider $r = 687.5$.

For the above parameters the equilibrium S_0 is a saddle and $S_{1,2}$ are saddle-focuses. Having taken an initial point on the unstable manifold of one of the equilibria $S_{0,1,2}$ (Fig. 5), one can easily visualize a self-excited chaotic attractor by the standard computational procedure.

For $r \in (690.6735024, 830.4169122)$ the equilibria $S_{1,2}$ become stable and the trajectories, starting from the neighborhood of equilibrium S_0, are attracted to S_1 or S_2[1].

The question arises whether there exists a hidden chaotic attractor in system (2) for such values of parameters?

Next the computation of hidden attractor in system (2) is considered.

3.3. Hidden attractor visualization

For a hidden attractor its basin of attraction is not connected with equilibria. The hidden attractors, for example, are the attractors in the systems with no equilibria or with only one stable equilibrium (a special case of multistability–multistable systems and coexistence of attractors). One of the first well-known problems of analyzing hidden periodic oscillations arose in connection with the second part of Hilbert's 16th problem (1900) [27] on the number and mutual disposition of limit cycles in two-dimensional polynomial systems (see, e.g., [28,29]). Later the study of hidden attractors arose in connection with various fundamental problems and applied models (see, e.g., [30–34]). Recent examples of hidden attractors can be found, e.g., in [29,35–57]. Note that while coexisting self-excited attractors can be found by the standard computational procedure, there is no regular way to predict the existence or coexistence of hidden attractors.

One of the effective methods for numerical localization of hidden attractors in multidimensional dynamical systems is based on a ***homotopy*** and ***numerical continuation***: it is necessary to construct a sequence of similar systems such that for the first (starting) system the initial point for numerical computation of oscillating solution (starting oscillation) can be obtained analytically, e.g., it is often possible to consider the starting system with self-excited starting oscillation. Then the transformation of this starting oscillation is tracked numerically in passing from one system to another. The last system corresponds to the system in which hidden attractor is searched.

[1] In the Lorenz system for some values of the parameters (see, e.g., [15],[24]) one can numerically visualize a chaotic set which coexists with two stable equilibria. This chaotic set is self-excited and, as the classical Lorenz attractor [25,26], can be visualized by a trajectory started from an initial point in vicinity of unstable zero equilibrium.

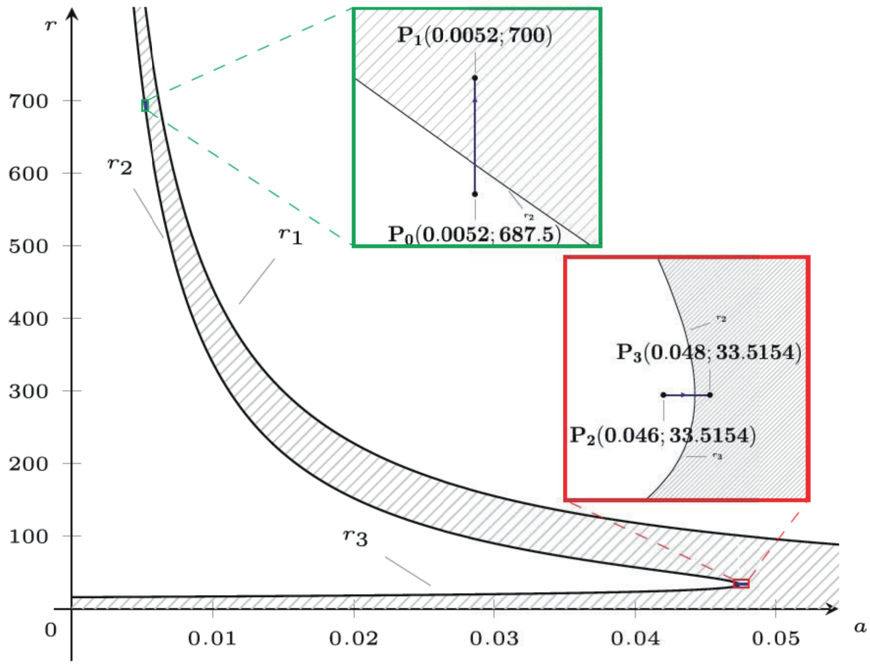


Fig. 6. (·) P_0, P_2 : self-excited attractors, (·) P_1 : hidden attractor, (·) P_3 : no chaotic attractors.

Let us construct on the plane (a, r) a line segment, intersecting a boundary of the domain of stability of the equilibria $S_{1,2}$ (see Fig. 6), with the end point $P_1(r = 700, a = 0.0052)$. The eigenvalues of equilibria $S_{0,1,2}$ of system (2) with parameters P_1 are the following:

$$S_0: \quad 50.4741, \quad -1, \quad -55.4741,$$
$$S_{1,2}: -0.1087 \pm 10.4543i, \quad -5.7826,$$

i.e. the equilibria $S_{1,2}$ become stable focus-nodes. Let us choose the point $P_0(r = 687.5, a = 0.0052)$ as the initial point of the line segment. This point corresponds to the parameters for which in system (2) there exists a self-excited attractor, such that it can be computed by the standard procedure. Then for the considered line segment a sufficiently small partition step is chosen and a chaotic attractor in the phase space of system (2) at each iteration step of the procedure is computed. The last computed point at each step is used as the initial point for the computation of the next step.

In our experiment the length of the line segment 2.5 and there are 6 iterations. At each iteration the largest Lyapunov exponent (LLE) and the Lyapunov dimension (LD) [14,58,59] are computed.[2]

The results of continuation procedure are shown in Fig. 8. Here for the selected path and selected partition it is possible to visualize a hidden attractor of system (2) (see Fig. 7).

Note also that in the work [66] the upper estimation of the Lyapunov dimension (LD) of attractor of system (2) is presented: for $r = 700$, $a = 0.0052$, $\sigma = 4$ it was obtained analytically the estimation LD < 2.8918 that is in a good agreement with the numerical result LD = 2.1322 (see Fig. 8 f).

Note that the choice of path and number of iterations in the continuation procedure is not trivial for the considered system. There is an example of the path on the plane (a, r) and the number of iterations such that they do not lead to a hidden attractor (Fig. 6). Consider $r = 33.51541181$, $a = 0.04735056\ldots$ – the rightmost point on the stability boundary and let us take the starting point $P_2 : r = 33.51541181$, $a = 0.046$ near it. If one takes the partition step 0.001 then there are no hidden attractors after

[2] There are two widely used definitions of Lyapunov exponents: the upper bounds of the exponential growth rate of the norms of linearized system solutions (LCEs) and the upper bounds of the exponential growth rate of the singular values of linearized system fundamental matrix (LEs). While often these two definitions give the same values, for a particular system, they may be different and there are examples in which Benettin algorithm [60] (see, e.g., its MatLab implementation in [61]) fails to compute the correct values (see corresponding discussion and examples in [59,62–64]). The existence of different definitions, computational methods, and related assumptions led to the appeal ***"Whatever you call your exponents, please state clearly how are they being computed"*** [65]. Here we use an algorithm based on SVD decomposition.

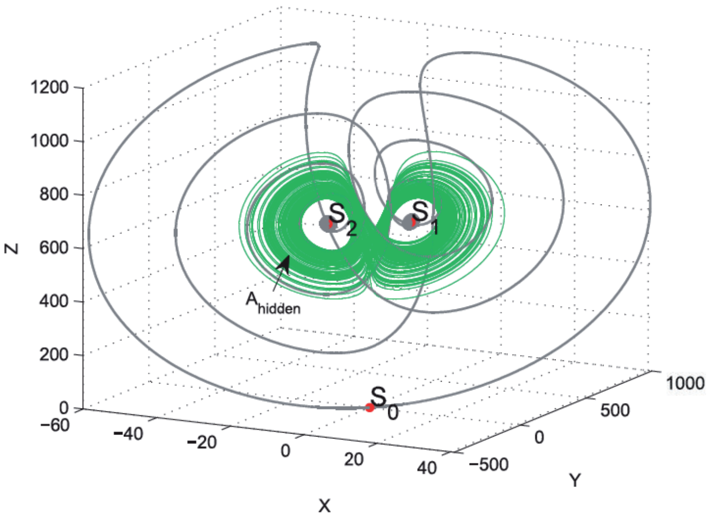
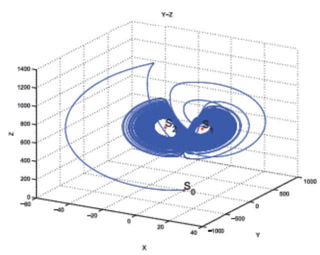
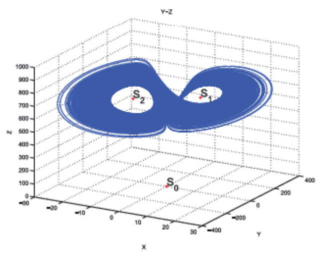


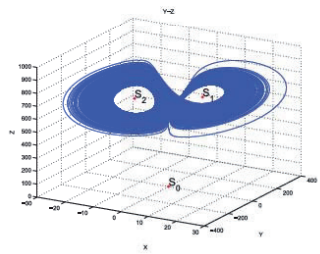
Fig. 7. Hidden attractor (green) in Lorenz-like system (2). Trajectories (gray) from the vicinity of unstable equilibrium S_0 tend to stable equilibria $S_{1,2}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



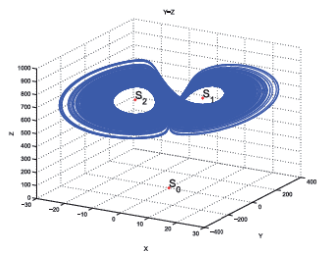
(a) $r = 687.5$, LLE $= 0.9774$, LD $= 2.1405$



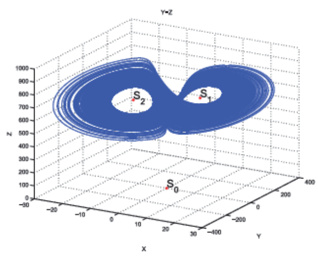
(b) $r = 690$, LLE $= 0.9649$, LD $= 2.1387$



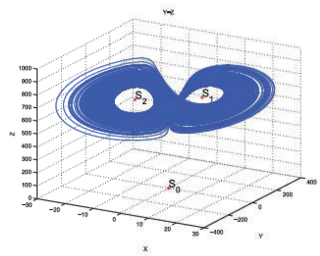
(c) $r = 692.5$, LLE $= 1.0315$, LD $= 2.1462$



(d) $r = 695$, LLE $= 1.0357$, LD $= 2.1471$

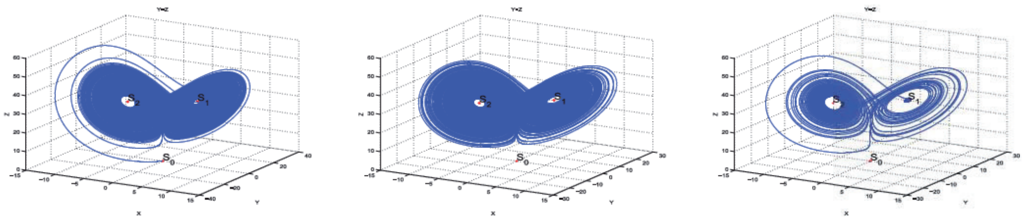


(e) $r = 697.5$, LLE $= 0.9824$, LD $= 2.1407$



(f) $r = 700$, LLE $= 0.9148$, LD $= 2.1322$

Fig. 8. The chain of transformations in continuation procedure for system (2) corresponding to the path $(0.0052, 687.5) \to (0.0052, 700)$.



(a) $a = 0.046$, LLE $= 0.4088$, LD $= 2.0642$. (b) $a = 0.047$, LLE $= 0.3824$, LD $= 2.0596$. (c) $a = 0.048$, LLE $= -0.0185$, LD $= 0$.

Fig. 9. The chain of transformations in continuation procedure for system (2) corresponding to the path (0.046, 33.51541181) → (0.048, 33.51541181).

crossing the border. For example, for parameters that correspond to the end point $P_3 : r = 33.51541181$, $a = 0.048$ there is no chaotic attractor and there are only trivial attractors—the equilibria $S_{1,2}$ (see Fig. 9).

4. Conclusions

In the present work by numerical methods the scenario of transition to chaos in physical model (2), describing a convective fluid motion, is demonstrated. Similarly to scenario in the classical Lorenz system, in system (2) a homoclinic trajectory and self-excited chaotic attractor are constructed. However, unlike the Lorenz system for system (2) it is possible to localize numerically a hidden attractor.

Acknowledgments

This work was supported by Russian Scientific Foundation (project 14-21-00041) and Saint-Petersburg State University.

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PIII

HOMOCLINIC ORBITS, AND SELF-EXCITED AND HIDDEN ATTRACTORS IN A LORENZ-LIKE SYSTEM DESCRIBING CONVECTIVE FLUID MOTION

by

G. A. Leonov, N. V. Kuznetsov, T. N. Mokaev 2015

The European Physical Journal Special Topics, Multistability: Uncovering Hidden Attractors, Vol. 224, pp. 1421–1458, doi:10.1140/epjst/e2015-02470-3

Eur. Phys. J. Special Topics **224**, 1421–1458 (2015)

DOI: 10.1140/epjst/e2015-02470-3

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Review

Homoclinic orbits, and self-excited and hidden attractors in a Lorenz-like system describing convective fluid motion

Homoclinic orbits, and self-excited and hidden attractors

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Received 17 March 2015 / Received in final form 20 May 2015
Published online 27 July 2015

Abstract. In this paper, we discuss self-excited and hidden attractors for systems of differential equations. We considered the example of a Lorenz-like system derived from the well-known Glukhovsky–Dolghansky and Rabinovich systems, to demonstrate the analysis of self-excited and hidden attractors and their characteristics. We applied the *fishing principle* to demonstrate the existence of a homoclinic orbit, proved the dissipativity and completeness of the system, and found absorbing and positively invariant sets. We have shown that this system has a self-excited attractor and a *hidden* attractor for certain parameters. The upper estimates of the Lyapunov dimension of self-excited and hidden attractors were obtained analytically.

1 Introduction: Self-excited and hidden attractors

When the theories of dynamical systems and oscillations were first developed (see, e.g., the fundamental works of Poincare and Lyapunov), researchers mainly focused on analyzing equilibria stability and the birth of periodic oscillations. The structures of many applied systems (see, e.g., the Rayleigh [148], Duffing [41], van der Pol [144], Tricomi [162], and Beluosov-Zhabotinsky [13] systems) are such that it is almost obvious that periodic oscillations exist, because the oscillations are excited by an unstable equilibrium. This meant that scientists of that time could compute such oscillations (called self-excited oscillations) by constructing a solution using initial data from a small neighborhood of the equilibrium, observing how it is attracted, and visualizing the oscillation (*standard computational procedure*). In this procedure, computational methods and the engineering notion of a *transient process* were combined to study oscillations.

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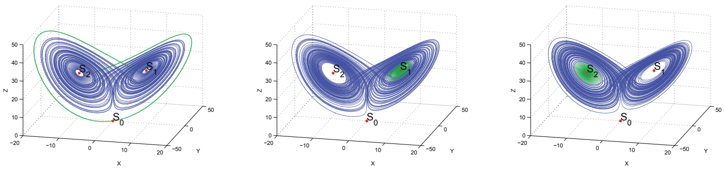


(a) Initial data near the equilibrium S_0 (b) Initial data near the equilibrium S_1 (c) Initial data near the equilibrium S_2

Fig. 1. Numerical visualization of the classical, self-excited, chaotic attractor in the Lorenz system $\dot{x} = 10(y - x), \dot{y} = 28x - y - xz, \dot{z} = -8/3z + xy$ by the trajectories that start in small neighborhoods of the unstable equilibria $S_{0,1,2}$. Here the separation of the trajectory into transition process (green) and approximation of attractor (blue) is rough.

At the end of the 19th century Poincare considered Newtonian dynamics of the three body problem, and revealed the possibility of more complicated behaviors of orbits "*so tangled that I cannot even begin to draw them*". He arrived at the conclusion that "*it may happen that small differences in the initial positions may lead to enormous differences in the final phenomena*". Further analyses and visualizations of such complicated "chaotic" systems became possible in the middle of the 20th century after the appearance of powerful computational tools.

An oscillation can generally be easily numerically localized if the initial data from its open neighborhood in the phase space (with the exception of a minor set of points) lead to a long-term behavior that approaches the oscillation. From a computational perspective, such an oscillation (or set of oscillations) is called an attractor, and its attracting set is called the basin of attraction (i.e., a set of initial data for which the trajectories tend to the attractor).

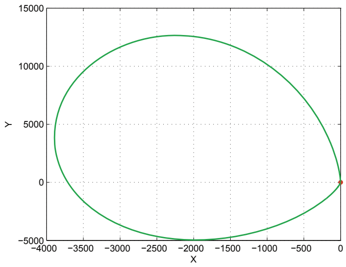
The first well-known example of a visualization of chaotic behavior in a dynamical system is from the work of Lorenz [122]. It corresponds to the excitation of chaotic oscillations from unstable equilibria, and could have been found using the standard computational procedure (see Fig. 1). Later, various self-excited chaotic attractors were discovered in many continuous and discrete systems (see, e.g., [23,26,31,50,123, 150,156]).

The study of an autonomous (unperturbed) system typically begins with an analysis of the equilibria, which are easily found numerically or analytically. Therefore, from a computational perspective, it is natural to suggest the following classification of attractors [80,108,111,112], which is based on the simplicity of finding their basins of attraction in the phase space:

Definition 1. *[80, 108, 111, 112] An attractor is called a* self-excited attractor *if its basin of attraction intersects with any open neighborhood of a stationary state (an equilibrium), otherwise it is called a* hidden attractor*.*

The basin of attraction for a hidden attractor is not connected with any equilibrium. For example, hidden attractors are attractors in systems with no equilibria or with only one stable equilibrium (a special case of the multistability: coexistence of attractors in multistable systems). Note that multistability can be inconvenient in various practical applications (see, for example, discussions on problems related to the synchronization of coupled multistable systems in [60,61,70,142]). Coexisting self-excited attractors can be found using the standard computational procedure[1],

[1] We have not discussed possible computational difficulties such as Wada and riddled basins.



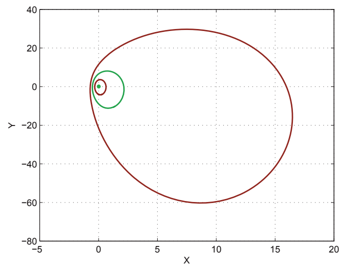


Fig. 2. Visualization of four limit cycles (green represents stable and red represents unstable) in a two-dimensional polynomial quadratic system $\dot{x} = -(a_1x^2+b_1xy+c_1y^2+\alpha_1x+\beta_1y), \dot{y} = -(a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y)$, for the coefficients $a_1 = b_1 = \beta_1 = -1$, $c_1 = \alpha_1 = 0$, $b_2 = -2.2$, and $c_2 = -0.7, a_2 = 10$, $\alpha_2 = 72.7778$, and $\beta_2 = -0.0015$. Localization of three nested limit cycles around the stable zero point (green dot) and one limit cycle to the left of the straight line $x = -1$.

whereas there is no standard way of predicting the existence or coexistence of hidden attractors in a system.

Hidden attractors arise in connection with various fundamental problems and applied models. The problem of analyzing hidden periodic oscillations first arose in the second part of Hilbert's 16th problem (1900), which considered the number and mutual disposition of limit cycles in two-dimensional polynomial systems [51]. The first nontrivial results were obtained by Bautin (see, e.g., [12]), which were devoted to the theoretical construction of three nested limit cycles around one equilibrium in quadratic systems. Bautin's method can only be used to construct nested, small-amplitude limit cycles, which can hardly be visualized. However, recently an analytical approach has been developed, which can be used to effectively visualize nested, normal amplitude limit cycles in quadratic systems [75,108,113].

Later, in the 1950s–1960s, studies of the well-known Markus-Yamabe's [125], Aizerman's [2], and Kalman's [59] conjectures on absolute stability led to the discovery of the possible coexistence of a hidden periodic oscillation and a unique stable stationary point in automatic control systems (see [10,15,20,45,79,104,105,143]; the corresponding discrete examples were considered in [4]).

The Rabinovich system [146] and the Glukhovsky-Dolghansky system [48] are among the first known chaotic systems that have hidden chaotic attractors [72,91]. The first one describes the interaction of plasma waves and was considered in 1978 by Rabinovich [140,146] Another is a model of convective fluid motion and was considered in 1980 by Glukhovsky and Dolghansky [48] (which we consider in the remainder of this paper).

Hidden oscillations appear naturally in systems without equilibria, describing various mechanical and electromechanical models with rotation, and electrical circuits with cylindrical phase space. One of the first examples is from a 1902 paper [154] in which Zommerfield analyzed the vibrations caused by a motor driving an unbalanced weight and discovered the so-called Zommerfield effect (see, e.g., [18,42]). Another well-known chaotic system without equilibria is the Nosè–Hoover oscillator [53,130] (see also the corresponding Sprott system, which was discovered independently [156,157]). In 2001, a hidden chaotic attractor was reported in a power system with no equilibria [165] (and references within).

After the idea of a "hidden attractor" was introduced and the first hidden Chua attractor was discovered [69,73,76,80,103,111,112], hidden attractors have received much attention. Results on the study of hidden attractors were presented in a number of invited survey and plenary lectures at various international conferences[2]. In 2012, an invited comprehensive survey on hidden attractors was prepared for the International Journal of Bifurcation and Chaos [108].

Many researchers are currently studying hidden attractors. Hidden periodic oscillations and hidden chaotic attractors have been studied in models such as phase-locked loops [68,71], Costas loops [16], drilling systems [65,110], DC-DC converters [176], aircraft control systems [5], launcher stabilization systems [6], plasma waves interaction [72], convective fluid motion [91], and many others models (see, e.g., [21,24,27,28,38,57,58,63,67,84,118–120,129,134–139,152,158,166–170,175]).

Similar to autonomous systems, when analyzing and visualizing chaotic behaviors of nonautonomous systems, we can consider the extended phase space and introduce various notions of attractors (see, e.g., [25,66]). Alternatively, we can regard time t as a phase space variable that obeys the equation $\dot{t} = 1$. For systems that are periodic in time, we can also introduce a cylindrical phase space and consider the behavior of trajectories on a Poincare section.

The consideration of system equilibria and the notions of self-excited and hidden attractors are natural for autonomous systems, because their equilibria can be easily found analytically or numerically. However, we may use other objects to construct transient processes that lead to the discovery of chaotic sets. These objects can be constructed for the considered system or its modifications (i.e., instead of analyzing the scenario of the system transiting into chaos, we can synthesis a new transition scenario). For example, we can use perpetual points [145] or the equilibria of the complexified system [134]. A periodic solution or homoclinic trajectory can be used in a similar way (some examples of theoretical studies can be found in [22,117,127,153]; however the presence of chaotic behavior in the considered examples may not imply the existence of a chaotic attractor, which can be numerically visualized using the standard computational procedure).

For nonautonomous systems, depending on the physical problem statement, the notion of self-excited and hidden attractors can be introduced with respect to either the stationary states ($x(t) \equiv x_0 \; \forall t$) of the considered nonautonomous system, the stationary points of the system at fixed initial time $t = t_0$, or the corresponding system without time-varying excitations. If the discrete dynamics of the system are considered on a Poincare section, then we can also use stationary or periodic points on the section that corresponds to a periodic orbit of the system (the consideration of periodic orbits is also natural for discrete systems).

In the following, we consider an example of a nonautonomous system (a forced Duffing oscillator), so that we can visualize the chaotic behavior. The classical example of a self-excited chaotic attractor (Fig. 3) in a Duffing system $\ddot{x}+0.05\dot{x}+x^3 = 7.5\cos(t)$ was numerically constructed by Ueda in 1961, but it become well-known much later [163]. To construct a self-excited chaotic attractor in this system, we use a transient process from the zero equilibrium of the unperturbed autonomous system (i.e., without $\cos(t)$) to the attractor (Fig. 3) in the forced system.

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[2] X Int. Workshop on Stability and Oscillations of Nonlinear Control Systems (Russia, 2008), Physics and Control [103] (Italy, 2009), 3rd International Conference on Dynamics, Vibration and Control (Hangzhou, China, 2010), IFAC 18th World Congress [105] (Italy, 2011), IEEE 5th Int. Workshop on Chaos-Fractals Theories and Applications [106] (Dalian, China, 2012), International Conference on Dynamical Systems and Applications (Ukraine, 2012), Nostradamus (Czech Republic, 2013) [107] and others.

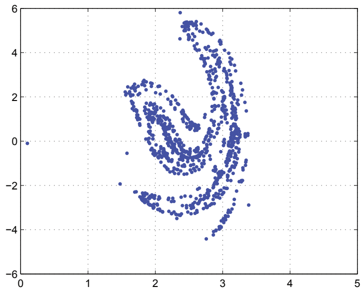


Fig. 3. Forced Duffing oscillator: $\ddot{x}+0.05\dot{x}+x^3 = 7.5\cos(t)$. The $(x, \dot{x})$ plane is mapped into itself by following the trajectory for time $0 \le t \le 2\pi$. After a transition process a trajectory from the vicinity of the zero stationary point of the unperturbed Duffing oscillator (without $7.5\cos(t)$) visualizes a self-excited chaotic attractor in the forced oscillator.

Note that if the attracting domain is the whole state space, then the attractor can be visualized by any trajectory and the only difference between computations is the timing of the transient process.

2 A Lorenz-like system

Consider a three-dimensional Lorenz-like system

$$\begin{cases} \dot{x} = -\sigma(x-y) - ayz \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases} \tag{1}$$

For $a = 0$, system (1) coincides with the classical Lorenz system [122]. For $\sigma > ar$ and $b = 1$ after a linear change of variables [99]

$$x \to x, \quad y \to \frac{C}{\sigma - ar} z, \quad z \to r - \frac{C}{\sigma - ar} y \tag{2}$$

system (1) takes the following form

$$\begin{aligned} \dot{x} &= -\sigma x + Cz + Ayz \\ \dot{y} &= R_a - y - xz \\ \dot{z} &= -z + xy \end{aligned} \tag{3}$$

with

$$C > 0, \quad R_a = \frac{r(\sigma - ar)}{C} > 0, \quad A = \frac{C^2 a}{(\sigma - ar)^2} > 0. \tag{4}$$

System (3) was suggested by Glukhovsky and Dolghansky [48], and describes convective fluid motion in an ellipsoidal rotating cavity, which can be interpreted as one of the models of ocean flows (see Appendix A for a description of this problem).

In [99], system (1) was obtained as a linear transformation of the Rabinovich system [146]. It describes interactions between waves in plasma [140,146]. Additionally, system (1) describes the following physical processes [99]: a rigid body rotation in a resisting medium, the forced motion of a gyrostat, a convective motion in harmonically oscillating horizontal fluid layer, and Kolmogorov flow. Systems (1) and (3) are interesting because of the discovery of chaotic attractors in their phase spaces. Moreover, system (1) was used to describe the specific mechanism of transition to chaos in low-dimensional dynamical systems (gluing bifurcations) [3].

For system (1) with $\sigma = \pm ar$, [44] contains a detailed analysis of equilibria stability and the asymptotic behavior of trajectories, and a derivation of the parameter values for which the system is integrable. Other researchers have also considered the analytical and numerical analysis of some extensions of system (1) [121,133].

Further, following [48], we consider system (1) with

$$b = 1, \quad a > 0, \quad r > 0, \quad \sigma > ar.$$

2.1 Classical scenario of the transition to chaos

For the Lorenz system [155], the following classical scenario of transition to chaos is known. Suppose that σ and b are fixed (let us consider the classical parameters $\sigma = 10$, $b = 8/3$), and that r varies. Then, as r increases, the phase space of the Lorenz system is subject to the following sequence of bifurcations. For $0 < r < 1$, there is globally asymptotically stable zero equilibrium S_0. For $r > 1$, equilibrium S_0 is a saddle, and a pair of symmetric equilibria $S_{1,2}$ appears. For $1 < r < r_h \approx 13.9$, the separatrices $\Gamma_{1,2}$ of equilibria S_0 are attracted to the equilibria $S_{1,2}$. For $r = r_h \approx 13.9$, the separatrices $\Gamma_{1,2}$ form two homoclinic trajectories of equilibria S_0 (homoclinic butterfly). For $r_h < r < r_c \approx 24.06$, the separatrices Γ_1 and Γ_2 tend to S_2 and S_1, respectively. For $r_c < r$, the separatrices $\Gamma_{1,2}$ are attracted to a self-excited attractor (see, e.g., [155,174]). For $r > r_a$, the equilibria $S_{1,2}$ become unstable. Finally, $r = 28$ corresponds to the classical self-excited Lorenz attractor (see Fig. (1)).

Furthermore, it has been shown that system (1) follows a similar scenario of transition to chaos. However, a substantial distinction of this scenario is the presence of hidden chaotic attractor in the phase space of system (1) for certain parameters values [91].

Let us determine the stationary points of system (1). We can show that for positive parameters, if $r < 1$, system (1) has a unique equilibrium $\mathrm{S}_0 = (0, 0, 0)$, which is globally asymptotically Lyapunov stable [19]. If $r > 1$, then (1) possesses three equilibria: a saddle $\mathrm{S}_0 = (0, 0, 0)$ and symmetric (with respect to $z = 0$) equilibria

$$\mathrm{S}_{1,2} = (\pm x_1, \pm y_1, z_1), \tag{5}$$

where

$$x_1 = \frac{\sigma\sqrt{\xi}}{\sigma + a\xi}, \quad y_1 = \sqrt{\xi}, \quad z_1 = \frac{\sigma\xi}{\sigma 1 + a\xi},$$

and

$$\xi = \frac{\sigma}{2a^2}\left[a(r-2) - \sigma + \sqrt{(\sigma - ar)^2 + 4a\sigma}\right].$$

The characteristic polynomial of the Jacobian matrix of system (1) at the point $(x,\, y,\, z)$ has the form

$$\chi(x,\, y,\, z) = \lambda^3 + p_1(x,\, y,\, z)\lambda^2 + p_2(x,\, y,\, z)\lambda + p_3(x,\, y,\, z),$$

where

$$p_1(x,\, y,\, z) = \sigma + 2, \quad p_2(x,\, y,\, z) = x^2 + ay^2 - az^2 + (\sigma + ar)z - r\sigma + 2\sigma + 1,$$
$$p_3(x,\, y,\, z) = \sigma x^2 + ay^2 - az^2 - 2axyz + (\sigma + ar)xy + (\sigma + ar)z - r\sigma + \sigma.$$

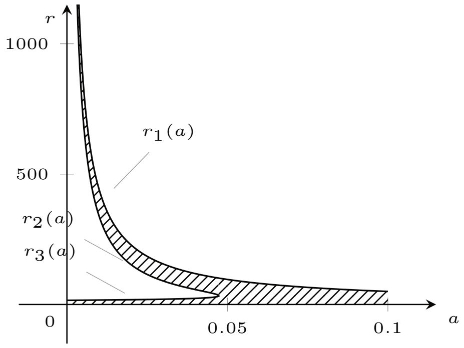


Fig. 4. The stability domain of equilibria $S_{1,2}$ for $\sigma = 4$.

Following [48], we let $\sigma = 4$ and define the stability domain of equilibria $S_{1,2}$. Using the Routh-Hurwitz criterion we can obtain the following (see Appendix B).

Proposition 1. *The equilibria* $S_{1,2}$ *are stable if*

$$8a^2r^3 + a(7a - 64)r^2 + (288a + 128)r + 256a - 2048 < 0. \tag{6}$$

The discriminant of the left-hand side of (6) has only one positive real root, $a^* \approx 0.04735$. So the roots of the polynomial in (6) are as follows. For $0 < a < a^*$, there are three real roots $r_1(a) > r_2(a) > r_3(a)$; for $a = a^*$, there are two real roots: $r_1(a)$ and $r_2(a) = r_3(a)$; for $a > a^*$, there is one real root $r_1(a)$. Thus, for $0 < a < a^*$, the equilibria $S_{1,2}$ are stable for $r < r_3(a)$ and for $r_2(a) < r < r_1(a)$; and for $a > a^*$ the equilibria $S_{1,2}$ are stable for $r < r_1(a)$ (see Fig. 4).

Consider the problem of the existence of a homoclinic orbit, which is important in bifurcation theory and in scenarios of transition to chaos (see, e.g., [1]). For (1) and (3), we can prove the existence of homoclinic trajectories for the zero saddle equilibrium S_0 using the fishing principle [87,96,98,109]. The fishing principle is based on the construction of a special two-dimensional manifold such that a separatrix of a saddle point intersects or does not intersect the manifold for two different system parameter values. Continuity implies the existence of some intermediate parameter value for which the separatrix touches the manifold. According to the construction, the separatrix must touch a saddle point, so we can numerically localize the birth of a homoclinic orbit. A rigorous description is given in Appendix E.

For $\sigma = 4, a = 0.0052$, and $r \approx 7.443$ we numerically obtain a homoclinic trajectory (see Fig. 5).

We come now to the study of the limit behaviors of trajectories and attractors. We introduce some rigorous notions of a dynamical system and attractor and discuss the connection with the notions of self-excited and hidden attractors from a computational perspective.

3 Definitions of attractors

3.1 Dynamical systems and ordinary differential equations

Consider an autonomous system of the differential equations

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x}), \quad t \in \mathbb{R},\ \mathrm{x} \in \mathbb{R}^n, \tag{7}$$

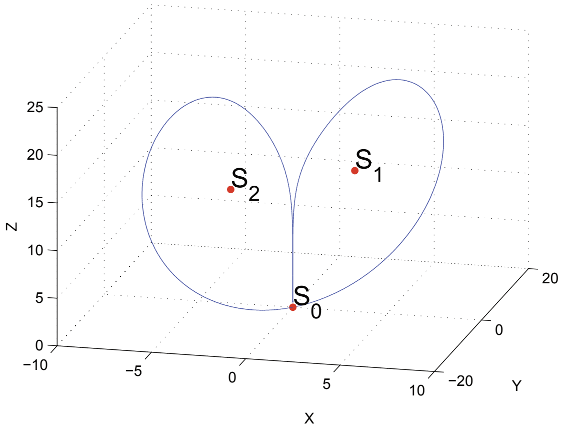


Fig. 5. Separatrices of the saddle $S_0 = (0, 0, 0)$ of system (1) for $\sigma = 4$, $a = 0.0052...$, and $r \approx 7.443$.

where $\mathrm{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector-function that satisfies *a local Lipschitz condition* in $\mathbb{R}^n$. The Picard theorem (see, e.g., [35,49]) for a local Lipschitz condition on the function f implies that, for any $\mathrm{x}_0 \in \mathbb{R}^n$, there exists a unique solution $\mathrm{x}(t, \mathrm{x}_0)$ to differential Eq. (7) with the initial data $\mathrm{x}(t_0, \mathrm{x}_0) = \mathrm{x}_0$, which is given on a certain finite time interval: $t \in I \subset \mathbb{R}$. The theorem regarding the continuous dependence on initial data [35,49][3] implies that the solution $\mathrm{x}(t, \mathrm{x}_0)$ continuously depends on x_0.

To study the limit behavior of trajectories and compute the limit values, characterizing trajectories, we consider the solutions of (7) for $t \to +\infty$ or $t \to \pm\infty$. For arbitrary quadratic systems, the existence of solutions for $t \in [t_0, +\infty)$ does not generally imply the existence of solutions for $t \in (-\infty, t_0]$ (see the classical one-dimensional example $\dot{x} = x^2$ or multidimensional examples from the work on the completeness of quadratic polynomial systems [47]). It is known that if f is continuously differentiable ($\mathrm{f} \in C^1$), then f is locally Lipschitz continuous in $\mathbb{R}^n$ (see, e.g., [52]). Additionally, if f is locally Lipschitz continuous, then for any $\mathrm{x}_0 \in \mathbb{R}^n$ the solution $\mathrm{x}(\cdot, \mathrm{x}_0) : I \to \mathbb{R}^n$ exists on maximal time interval $I = (t_-, t_+) \in \mathbb{R}$, where $-\infty \le t_- < t_+ \le +\infty$. If $t_+ < +\infty$, then $||\mathrm{x}(t, \mathrm{x}_0)|| \to \infty$ for $t \to t_+$, and if $t_- > -\infty$, then $||\mathrm{x}(t, \mathrm{x}_0)|| \to \infty$ for $t \to t_-$ (see, e.g., [161]). This implies that a solution of (7) is continuous if it remains bounded. For convenience, we introduce a set of time values $\mathbb{T} \in \{\mathbb{R}, \mathbb{R}_+\}$. The existence and uniqueness of solutions of (1) for all $t \in \mathbb{T}$ can be provided, for example, by a *global Lipschitz condition*.

Another effective method for studying the boundedness of solutions for all $t \in \mathbb{T}$ is to construct a Lyapunov function.

If the existence and uniqueness conditions for all $t \in \mathbb{T}$ are satisfied, then: 1) the solution of (7) satisfies the group property ([35,49])

$$\mathrm{x}(t + s, \mathrm{x}_0) = \mathrm{x}(t, \mathrm{x}(s, \mathrm{x}_0)), \quad \forall\, t, s \in \mathbb{T}, \tag{8}$$

[3] Similar theorems on the existence, unicity, and continuous dependence on the initial data for solutions of system with the discontinuous right-hand side are considered in [64,172].

and 2) $\mathrm{x}(\cdot,\cdot) : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping according to the theorem of the continuous dependence of the solution on the initial data. Thus, if the solutions of (7) exist and satisfy (8) for all $t \in \mathbb{T}$, the system generates a *dynamical system* [17] on the phase space $(\mathbb{R}^n, ||\cdot||)$. Here $||\mathrm{x}|| = \sqrt{x_1^2 + \cdots + x_n^2}$ is an Euclidean norm of the vector $\mathrm{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, which generates a metric on $\mathbb{R}^n$. We abbreviate *"dynamical system generated by a differential equation"* to *"dynamical system"*. Because the initial time is not important for dynamical systems, without loss of generality we consider

$$x(t, x_0): \; x(0, x_0) = x_0.$$

Consider system (1). Its right-hand side is continuously differentiable in $\mathbb{R}^n$, which means that it is locally Lipschitz continuous in $\mathbb{R}^n$ (but not globally Lipschitz continuous). Analogous with the results for the Lorenz system [36,126], we can prove that the solutions of (1) exist for all $t \in \mathbb{R}$, i.e. system (1) is invertible. For this purpose, we can use the Lyapunov function (Appendix C)

$$V(x, y, z) = \frac{1}{2}\left(x^2 + y^2 + (a+1)\left(z - \frac{\sigma + r}{a+1}\right)^2\right) \geq 0. \tag{9}$$

Then, system (1) generates a dynamical system and we can study its limit behavior and attractors.

3.2 Classical definitions of attractors

The notion of an attractor is connected with investigations of the limit behavior of the trajectories of dynamical systems. We define attractors as follows [9,19,32,34,82, 83,95,160].

Definition 2. *A set K is said to be* positively invariant *for a dynamical system if*

$$\mathrm{x}(t, K) \subset K, \;\; \forall t \geq 0,$$

and to be invariant *if*

$$\mathrm{x}(t, K) = K, \;\; \forall t \geq 0,$$

where $\mathrm{x}(t, K) = \{\mathrm{x}(t, \mathrm{x}_0) \mid \mathrm{x}_0 \in K, \; t \geq 0\}$.

Property 1. Invariant set K is said to be *locally attractive* for a dynamical system if, for a certain ε-neighborhood $K(\varepsilon)$ of set K,

$$\lim_{t \to +\infty} \rho(K, \mathrm{x}(t, \mathrm{x}_0)) = 0, \quad \forall \; \mathrm{x}_0 \in K(\varepsilon).$$

Here $\rho(K, \mathrm{x})$ is a distance from the point x to the set K, defined as

$$\rho(K, \mathrm{x}) = \inf_{\mathrm{z} \in K} ||\mathrm{z} - \mathrm{x}||,$$

and $K(\varepsilon)$ is a set of points x for which $\rho(K, \mathrm{x}) < \varepsilon$.

Property 2. Invariant set K is said to be *globally attractive* for dynamical system if

$$\lim_{t \to +\infty} \rho(K, \mathrm{x}(t, \mathrm{x}_0)) = 0, \quad \forall \; \mathrm{x}_0 \in \mathbb{R}^n.$$

Property 3. Invariant set K is said to be *uniformly locally attractive* for a dynamical system if for a certain ε-neighborhood $K(\varepsilon)$, any number $\delta > 0$, and any bounded set B, there exists a number $t(\delta, B) > 0$ such that

$$\mathrm{x}(t, B \cap K(\varepsilon)) \subset K(\delta), \quad \forall\, t \geq t(\delta, B).$$

Here

$$\mathrm{x}(t, B \cap K(\varepsilon)) = \{\mathrm{x}(t, \mathrm{x}_0) \mid \mathrm{x}_0 \in B \cap K(\varepsilon)\}.$$

Property 4. Invariant set K is said to be *uniformly globally attractive* for a dynamical system if, for any number $\delta > 0$ and any bounded set $B \subset \mathbb{R}^n$, there exists a number $t(\delta, B) > 0$ such that

$$\mathrm{x}(t, B) \subset K(\delta), \quad \forall\, t \geq t(\delta, B).$$

Definition 3. *For a dynamical system, a bounded closed invariant set K is:*

(1) an attractor *if it is a locally attractive set (i.e., it satisfies Property 1);*
(2) a global attractor *if it is a globally attractive set (i.e., it satisfies Property 2);*
(3) a B-attractor *if it is a uniformly locally attractive set (i.e., it satisfies Property 3); or*
(4) a global B-attractor *if it is a uniformly globally attractive set (i.e., it satisfies Property 4).*

Remark 1. In the definition of an attractor we assume closeness for the sake of uniqueness. This is because the closure of a locally attractive invariant set K is also a locally attractive invariant set (for example, consider an attractor with excluded one of the embedded unstable periodic orbits). The closeness property is sometimes omitted from the attractor definition (see, e.g., [8]). Additionally, the boundedness property is sometimes omitted (see, e.g., [29]). For example, a global attractor in a system describing a pendulum motion is not bounded in the phase space $\mathbb{R}^2$ (but it is bounded in the cylindrical phase space). Unbounded attractors are considered for nonautonomous systems in the extended phase space. Note that if a dynamical system is defined for $t \in \mathbb{R}$, then a locally attractive invariant set only contains the whole trajectories, i.e. if $x_0 \in K$, then $x(t, x_0) \in K$ for $\forall t \in \mathbb{R}$ (see [32]).

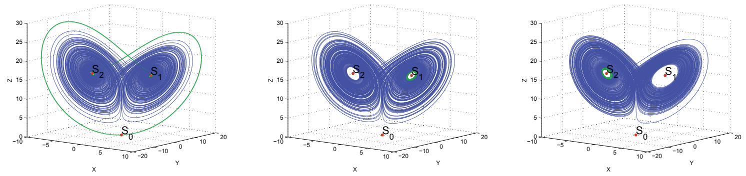
Remark 2. The definition considered here implies that a global B-attractor is also a global attractor. Consequently, it is rational to introduce the notion of a *minimal global attractor* (or *minimal attractor*) [32,34]. This is the smallest bounded closed invariant set that possesses Property 2 (or Property 1). Further, the attractors (global attractors) will be interpreted as minimal attractors (minimal global attractors).

Definition 4. *For an attractor K, the basin of attraction is a set $B(K) \subset \mathbb{R}^n$ such that*

$$\lim_{t \to +\infty} \rho(K, \mathrm{x}(t, \mathrm{x}_0)) = 0, \quad \forall\, \mathrm{x}_0 \in B(K).$$

Remark 3. From a computational perspective, it is not feasible to numerically check Property 1 for all initial states of the phase space of a dynamical system. A natural generalization of the notion of an attractor is consideration of the weaker attraction requirements: almost everywhere or on a set of the positive Lebesgue measure (see, e.g., [128]). See also *trajectory attractors* [30,33,151]. To distinguish an artificial computer generated chaos from a real behavior of the system one can consider the shadowing property of the system (see, e.g., the survey in [141]).

We can typically see an attractor (or global attractor) in numerical experiments. The notion of a B-attractor is mostly used in the theory of dimensions, where we consider invariant sets covered by balls. The uniform attraction requirement in Property 3



(a) Two trajectories with symmetric initial data near S_0 (b) A trajectory with initial data near S_1 (c) A trajectory with initial data near S_2

Fig. 6. Self-excited attractor of system (1) for $r = 17$, $\sigma = 4$, and $a = 0.0052$, computed from different initial points.

implies that a global B-attractor involves a set of stationary points ($\mathcal{S}$) and the corresponding unstable manifolds $W^u(\mathcal{S}) = \{\mathrm{x}_0 \in \mathbb{R}^n \mid \lim_{t\to-\infty} \rho(\mathcal{S}, \mathrm{x}(t, \mathrm{x}_0)) = 0\}$ (see, e.g., [32,34]). The same is true for B-attractor if the considered neighborhood $K(\varepsilon)$ in Property 3 contains some of the stationary points from $\mathcal{S}$. From a computational perspective, numerically checking Property 3 is also difficult. Therefore if the basin of attraction involves unstable manifolds of equilibria, then computing the minimal attractor and the unstable manifolds that are attracted to it may be regarded as an approximation of B-attractor. For example, consider the visualization of the classical Lorenz attractor from the neighborhood of the zero saddle equilibria. Note that a minimal global attractor involves the set $\mathcal{S}$ and its basin of attraction involves the set $W^u(\mathcal{S})$. Various analytical-numerical methods for computing attractors and their basins of attraction can be found in, for example, [7,39,46,132,164,177].

4 Self-excited attractor localization

In [48] system (3) with $\sigma = 4$ was studied. Consider the following parameters for system (1)

$$\sigma = 4, \quad a = 0.0052.$$

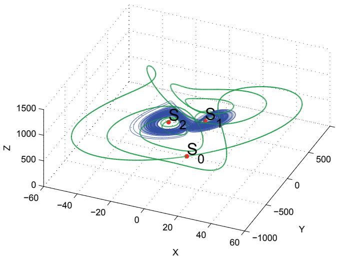
According to Proposition 1, if $r_1 \approx 16.4961242... < r < r_2 \approx 690.6735024$, the equilibria $S_{1,2}$ of system (1) become (unstable) saddle-focuses. For example, if $r = 17$, the eigenvalues of the equilibria of system (1) are the following

$$S_0\colon \; 5.8815, \quad -1, \quad -10.8815$$
$$S_{1,2}\colon 0.0084 \pm 4.5643\mathrm{i}, \quad -6.0168$$

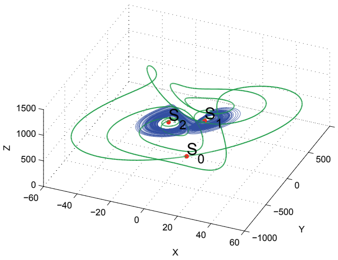
and there is a self-excited chaotic attractor in the phase space of system (1). We can easily visualize this attractor (Fig. 6) using the standard computational procedure with initial data in the vicinity of one of the equilibria $S_{0,1,2}$ on the corresponding unstable manifolds. To improve the approximation of the attractor one can consider its neighborhood and compute trajectories from a grid of points in this neighborhood.

5 Hidden attractor localization

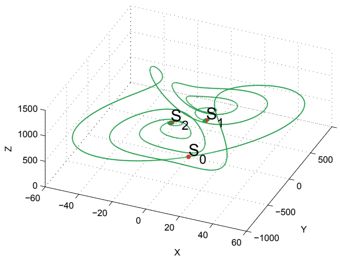
We need a special numerical method to localize the hidden attractor of system (1), because the basin of attraction does not intersect the small neighborhoods of the



(a) P_0: $r = 687.5$. Attractor (blue), B-attractor (green, blue and red)



(b) $r = 690$. Attractor (blue), B-attractor (green, blue and red)



(c) P_1: $r = 700$. B-attractor (green and red)

Fig. 7. B-attractor of system (1) for fixed $\sigma = 4$, $a = 0.0052$, and various r.

unstable manifolds of the equilibria. One effective method for the numerical localization of hidden attractors is based on a *homotopy* and *numerical continuation*. We construct a sequence of similar systems such that the initial data for numerically computing the oscillating solution (starting oscillation) can be obtained analytically for the first (starting) system. For example, it is often possible to consider a starting system with a self-excited starting oscillation. Then we numerically track the transformation of the starting oscillation while passing between systems.

In a scenario of transition to chaos in dynamical system there is typically a parameter $\lambda \in [a_1, a_2]$, the variation of which gives the scenario. We can also artificially introduce the parameter λ, let it vary in the interval $[a_1, a_2]$ (where $\lambda = a_2$ corresponds to the initial system), and choose a parameter a_1 such that we can analytically or computationally find a certain nontrivial attractor when $\lambda = a_1$ (often this attractor has a simple form, e.g., periodic). That is, instead of analyzing the scenario of a transition into chaos, we can synthesize it. Further, we consider the sequence $\lambda_j,\ \lambda_1 = a_1,\ \lambda_m = a_2,\ \lambda_j \in [a_1, a_2]$ such that the distance between λ_j and λ_{j+1} is sufficiently small. Then we numerically investigate changes to the shape of the attractor obtained for $\lambda_1 = a_1$. If the change in λ (from λ_j to λ_{j+1}) does not cause a loss of the stability bifurcation of the considered attractor, then the attractor for $\lambda_m = a_2$ (at the end of procedure) is localized.

Let us construct a line segment on the plane (a, r) that intersects a boundary of the stability domain of the equilibria $S_{1,2}$ (see Fig. 8). We choose the point $P_1(r = 700, a = 0.0052)$ as the end point of the line segment. The eigenvalues for

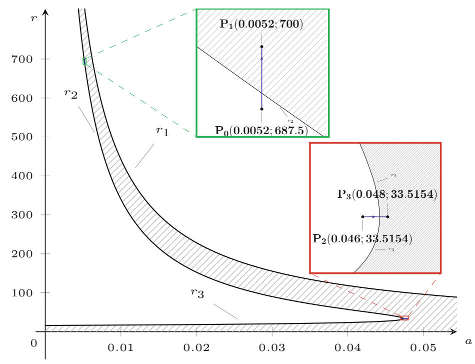


Fig. 8. Paths $[P_0, P_1]$ and $[P_2, P_3]$ in the plane of parameters $\{a, r\}$ used in the continuation procedure.

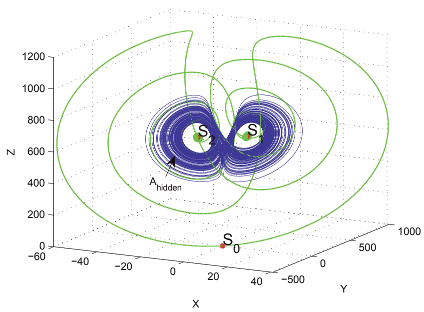


Fig. 9. Hidden attractor (blue) coexist with B-attractor (green outgoing separatrix of the saddle S_0 attracted to the red equilibria $S_{1,2}$).

the equilibria of system (1) that correspond to the parameters P_1 are the following:

$$S_0\colon \quad 50.4741, \quad -1, \quad -55.4741,$$
$$S_{1,2}\colon -0.1087 \pm 10.4543\mathrm{i}, \quad -5.7826.$$

This means that the equilibria $S_{1,2}$ become stable focus-nodes. Now we choose the point $P_0(r = 687.5, a = 0.0052)$ as the initial point of the line segment. This point corresponds to the parameters for which system (1) has a self-excited attractor, which can be computed using the standard computational procedure. Then we choose a sufficiently small partition step for the line segment and compute a chaotic attractor in the phase space of system (1) at each iteration of the procedure. The last computed point at each step is used as the initial point for the computation at the next step (the computation time must be sufficiently large).

In our experiment the length of the path was 2.5 and there were 6 iterations. Here for the selected path and partition, we can visualize a hidden attractor of system (1) (see Fig. 9). The results of continuation procedure are given in [91].

Note that the choice of path and its partitions in the continuation procedure is not trivial. For example, a similar procedure does not lead to a hidden attractor for the following path on the plane (a, r). Consider $r = 33.51541181$, $a = 0.04735056...$ (the rightmost point on the stability domain), and take a starting point P_2: $r = 33.51541181$, $a = 0.046$ near it (Fig. 8). If we use the partition step 0.001, then there are no hidden attractors after crossing the boundary of the stability domain. For example, if the end point is P_3: $r = 33.51541181$, $a = 0.048$, there is no chaotic attractor but only trivial attractors (the equilibria $S_{1,2}$).

6 Analytical localization of global attractor via Lyapunov functions

In the previous sections, we considered the numerical localization of various self-excited and hidden attractors of system (1). It is natural to question if these attractors (or the union of attractors) are global (in the sense of Definition 3) or if other coexisting attractors can be found.

The *dissipativity* property is important when proving the existence of a bounded global attractor for a dynamical system and gives an analytical localization of the global attractor in the phase space. The dissipativity of a system, on one hand, proves that there are no trajectories that tend to infinity as $t \to +\infty$ in the phase space and, on the other hand, can be used one to determine the boundaries of the domain that all trajectories enter within a finite time.

Definition 5. *A set $B_0 \subset \mathbb{R}^n$ is said to be* absorbing *for dynamical system (7) if for any $\mathrm{x}_0 \in \mathbb{R}^n$ there exists $T = T(\mathrm{x}_0)$ such that $\mathrm{x}(t, \mathrm{x}_0) \in B_0$ for any $t \geq T$.*

Note that the trajectory $\mathrm{x}(t, \mathrm{x}_0)$ with $\mathrm{x}_0 \in B_0$ may leave B_0 for only a finite time before it returns and stays inside for $t \geq T$.

Remark 4. In [116] the ball $B_R = \{\mathrm{x} \in \mathbb{R}^n : |\mathrm{x}| < R\}$ was regarded as an absorbing set. In this case, if there exists $R > 0$ such that

$$\limsup_{t\to\infty} |\mathrm{x}(t, \mathrm{x}_0)| < R, \quad \text{for any } \mathrm{x}_0 \in \mathbb{R}^n,$$

then it is said that a dynamical system is *dissipative in the sense of Levinson* . R is called *a radius of dissipativity*[4].

Definition 6 ([32,34]). *Dynamical system (7) is called* (pointwise) dissipative[5] *if it possesses a bounded absorbing set.*

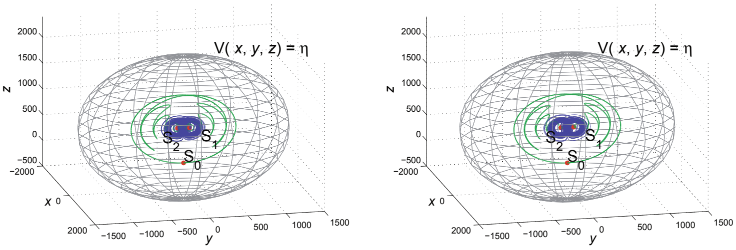
Theorem 1 ([32,34]). *If dynamical system (7) is dissipative, then it possesses a global B-attractor.*

We can effectively prove dissipativity by constructing the Lyapunov function [92,173]. Consider a sufficient condition of dissipativity for system (7).

---

[4] Because any greater radius also satisfies the definition, the minimal R is of interest for the problems of attractor localization and definition of ultimate bound.

[5] Together with the notions of an absorbing set and dissipative system, [19,83] also considered the definitions of a *B-absorbing set* and a *B-dissipative* system (uniform convergence of trajectories to the corresponding B-absorbing set). It is known [19] that if a dynamical system given on $(\mathbb{R}^n, \ ||\cdot||)$ is dissipative, then it is also B-dissipative.



(a) Absorbing set (grey), self-excited attractor (blue), and global B-attractor (blue, green and red) for system (1) with the parameters $r = 687.5$, $\sigma = 4$, and $a = 0.0052$. $\eta \approx 238541.3$.

(b) Absorbing set (grey), hidden attractor (blue), and global B-attractor (blue, green and red) for system (1) with the parameters $r = 700$, $\sigma = 4$, and $a = 0.0052$. $\eta \approx 247230.5$.

Fig. 10. Absorbing sets for system (1).

Theorem 2 ([149, 171]). *Suppose that there exists continuously differentiable function* $V(\mathrm{x}) : \mathbb{R}^n \to \mathbb{R}$*, possessing the following properties.*

(1) $\lim_{|\mathrm{x}| \to \infty} V(\mathrm{x}) = +\infty$, *and*
(2) there exist numbers R *and* $\varkappa$ *such that for any solution* $\mathrm{x}(t, \mathrm{x}_0)$ *of system* (7), *the condition* $|\mathrm{x}(t, \mathrm{x}_0)| > R$ *implies that* $\dot{V}(\mathrm{x}(t, \mathrm{x}_0)) \le -\varkappa$.

Then

(a) any solution $\mathrm{x}(t, \mathrm{x}_0)$ *to* (7) *exists at least on* $[0, +\infty)$, *so system* (7) *generates a dynamical system for any* $t \ge 0$ *and* $\mathrm{x}_0 \in \mathbb{R}^n$*; and*
(b) if $\eta > 0$ *is such that* $B_0 = \{\mathrm{x} \in \mathbb{R}^n \mid V(\mathrm{x}) \le \eta\} \supset \{\mathrm{x} \in \mathbb{R}^n \mid ||\mathrm{x}|| < R\}$, *then* B_0 *is a compact absorbing set of dynamical system* (7).

More general theorems, connected with the application of the Lyapunov functions to the proof of dissipativity for dynamical systems can be found in [101, 147].

It is known that the Lorenz system is dissipative (it is sufficient to choose the Lyapunov function $V(x, y, z) = \frac{1}{2}(x^2 + y^2 + (z - r - \sigma)^2)$). However, for example, one of the Rossler systems is not dissipative in the sense of Levinson [115] because the outgoing separatrix is unbounded. In the general case, there is an art in the construction of Lyapunov functions which prove dissipativity.

Lemma 1. *Dynamical system* (1) *is dissipative.*

The proof is based on Lyapunov function V from (9) (see Appendix C). If R, η are chosen as in the proof of Theorem 1, Appendix C, then dynamical system (1) has a compact absorbing set

$$B_0 = \left\{ (x, y, z) : V(x, y, z) = \frac{1}{2} \left(x^2 + y^2 + (a+1) \left(z - \frac{\sigma + r}{a+1} \right)^2 \right) \le \eta \right\}.$$

For example, for $\sigma > 1, r > 1, a < 1$ we can choose $R = \dfrac{\sigma + r}{a+1}$ and $\eta = 2(a + 1)R^2$ (see Fig. 10). Note that for system (1) the ellipsoidal absorbing set B_0 can be improved using special additional transformations and Yudovich's theorem (see, e.g., [14]), similarly to [93] for the Lorenz system.

There is also a cylindrical positively invariant set for system (1) [87],

$$C = \{|x| \le r + \frac{2a}{\sigma}r^2, \quad y^2 + (z-r)^2 \le r^2\}, \tag{10}$$

because

$$(y^2 + (z-r)^2)^\bullet \le -(y^2 + (z-r)^2) + r^2 < 0 \quad \forall x,\ |y| > r \text{ or } |z| > 2r$$

and

$$|x|^\bullet \le -\sigma|x| + \sigma|y| + a|y||z| < 0 \quad |y| \le r,\ |z| \le 2r,\ |x| > r + 2ar^2/\sigma.$$

Thus, as for the Lorenz system [100], we obtain three different estimates of the attractor: the ball B_R, ellipse B_0, and cylinder C.

7 Upper estimate of the Lyapunov dimension of attractor

7.1 Lyapunov exponents and Lyapunov dimension

Suppose that the right-hand side of system (7) is sufficiently smooth, and consider a linearized system along a solution $\mathrm{x}(t, \mathrm{x}_0)$. We have

$$\dot{\mathrm{u}} = J(\mathrm{x}(t, \mathrm{x}_0))\,\mathrm{u}, \quad \mathrm{u} \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \tag{11}$$

where

$$J(\mathrm{x}(t, \mathrm{x}_0)) = \left[\frac{\partial f_i(\mathrm{x})}{\partial \mathrm{x}_j}\big|_{\mathrm{x}=\mathrm{x}(t,\mathrm{x}_0)}\right]$$

is the $(n \times n)$ Jacobian matrix evaluated along the trajectory $\mathrm{x}(t, \mathrm{x}_0)$ of system (7). A fundamental matrix $X(t, \mathrm{x}_0)$ of linearized system (11) is defined by the variational equation

$$\dot{X}(t, \mathrm{x}_0) = J(\mathrm{x}(t, \mathrm{x}_0))\,X(t, \mathrm{x}_0). \tag{12}$$

We typically set $X(0, \mathrm{x}_0) = I_n$, where I_n is the identity matrix. Then $\mathrm{u}(t, \mathrm{u}_0) = X(t, \mathrm{x}_0)\mathrm{u}_0$. In the general case, $\mathrm{u}(t, \mathrm{u}_0) = X(t, \mathrm{x}_0)X^{-1}(0, \mathrm{x}_0)\mathrm{u}_0$. Note that if a solution of nonlinear system (7) is known, then we have

$$X(t, \mathrm{x}_0) = \frac{\partial \mathrm{x}(t, \mathrm{x}_0)}{\partial \mathrm{x}_0}.$$

Two well-known definitions of Lyapunov exponents are the upper bounds of the exponential growth rate of the norms of linearized system solutions (LCEs) [124] and the upper bounds of the exponential growth rate of the singular values of fundamental matrix of linearized system (LEs) [131].

Let $\sigma_1(X(t, \mathrm{x}_0)) \ge \cdots \ge \sigma_n(X(t, \mathrm{x}_0)) > 0$ denote the singular values of a fundamental matrix $X(t, \mathrm{x}_0)$ (the square roots of the eigenvalues of the matrix $X(t, \mathrm{x}_0)^* X(t, \mathrm{x}_0)$ are reordered for each t).

Definition 7. *The Lyapunov exponents (LEs) at the point* x_0 *are the numbers (or the symbols* $\pm\infty$*) defined by*

$$\mathrm{LE}_i(\mathrm{x}_0) = \limsup_{t \to \infty} \frac{1}{t} \ln \sigma_i(X(t, \mathrm{x}_0)). \tag{13}$$

LEs are commonly used[6] in the theories of dynamical systems and attractor dimensions [11,19,43,55,85,160].

Remark 5. The LEs are independent of the choice of fundamental matrix at the point x_0 [74] unlike the Lyapunov characteristic exponents (LCEs, see [124]). To determine all possible values of LCEs, we must consider a *normal fundamental matrix*.

We now define a Lyapunov dimension [62]

Definition 8. *A local Lyapunov dimension of a point* x_0 *in the phase space of a dynamical system is as follows:* $\dim_L x_0 = 0$ *if* $\mathrm{LE}_1(x_0) \leq 0$ *and* $\dim_L x_0 = n$ *if* $\sum_{i=1}^n \mathrm{LE}_i^o(x_0) \geq 0$*, otherwise*

$$\dim_L x_0 = j(x_0) + \frac{\mathrm{LE}_1(x_0) + \ldots + \mathrm{LE}_j(x_0)}{|\mathrm{LE}_{j+1}(x_0)|}, \tag{14}$$

where $\mathrm{LE}_1(x_0) \geq \ldots \geq \mathrm{LE}_n(x_0)$ *are ordered LEs and* $j(x_0) \in [1, n]$ *is the smallest natural number* m *such that*

$$\mathrm{LE}_1(x_0) + \ldots + \mathrm{LE}_m(x_0) > 0, \quad \mathrm{LE}_{m+1}(x_0) < 0, \quad \frac{\mathrm{LE}_1(x_0) + \ldots + \mathrm{LE}_m(x_0)}{|\mathrm{LE}_{m+1}(x_0)|} < 1.$$

The Lyapunov dimension of invariant set K of a dynamical system is defined as

$$\dim_L K = \sup_{x_0 \in K} \dim_L x_0. \tag{15}$$

Note that, from an applications perspective, an important property of the Lyapunov dimension is the chain of inequalities [19,55,56]

$$\dim_T K \leqslant \dim_H K \leqslant \dim_F K \leqslant \dim_L K. \tag{16}$$

Here $\dim_T K, \dim_H K$, and $\dim_F K$ are the topological, Hausdorff, and fractal dimensions of K, respectively.

Along with commonly used numerical methods for estimating and computing the Lyapunov dimension, there is an analytical approach that was proposed by Leonov [19,94,95,97,99,109]. It is based on the direct Lyapunov method and uses Lyapunov-like functions.

[6] The LCEs [124] and LEs [131] are "often" equal, e.g., for a "typical" system that satisfies the conditions of Oseledec theorem [131]. However, there are no effective methods for checking Oseledec conditions for a given system: "*Oseledec proof is important mathematics, but the method is not helpful in elucidating dynamics*" [37, p.118]). For a particular system, LCEs and LEs may be different. For example, for the fundamental matrix $X(t) = \begin{pmatrix} 1 & g(t) - g^{-1}(t) \\ 0 & 1 \end{pmatrix}$ we have the following ordered values: $\mathrm{LCE}_1 = \max\big(\limsup_{t\to+\infty} \mathcal{X}[g(t)], \limsup_{t\to+\infty} \mathcal{X}[g^{-1}(t)]\big), \mathrm{LCE}_2 = 0$; $\mathrm{LE}_{1,2} = \max, \min\big(\limsup_{t\to+\infty} \mathcal{X}[g(t)], \limsup_{t\to+\infty} \mathcal{X}[g^{-1}(t)]\big)$, where $\mathcal{X}(\cdot) = \frac{1}{t}\log|\cdot|$. Thus, in general, the Kaplan-Yorke (Lyapunov) dimensions based on LEs and LCEs may be different. Note also that positive largest LCE or LE, computed via the linearization of the system along a trajectory, do not necessary imply instability or chaos, because for non-regular linearization there are well-known Perron effects of Lyapunov exponent sign reversal [77,78,102]. Therefore for the computation of the Lyapunov dimension of an attractor one has to consider a grid of points on the attractor and corresponding local Lyapunov dimensions [81]. More detailed discussion and examples can be found in [74,102].

LEs and the Lyapunov dimension are invariant under linear changes of variables (see, e.g., [74]). Therefore we can apply the linear variable change $\mathrm{y} = S\mathrm{x}$ with a nonsingular $n \times n$-matrix S. Then system (7) is transformed into

$$\dot{\mathrm{y}} = S\,\dot{\mathrm{x}} = S\,\mathrm{f}(S^{-1}\mathrm{y}) = \tilde{\mathrm{f}}(\mathrm{y}).$$

Consider the linearization along corresponding solution $\mathrm{y}(t, \mathrm{y}_0) = S\mathrm{x}(t, S^{-1}\mathrm{x}_0)$, that is,

$$\dot{\mathrm{v}} = \tilde{J}(\mathrm{y}(t, \mathrm{y}_0))\,\mathrm{v}, \quad \mathrm{v} \in \mathbb{R}^n. \tag{17}$$

Here the Jacobian matrix is as follows

$$\tilde{J}(\mathrm{y}(t, \mathrm{y}_0)) = S\,J(\mathrm{x}(t, \mathrm{x}_0))\,S^{-1} \tag{18}$$

and the corresponding fundamental matrix satisfies $Y(t, \mathrm{y}_0) = SX(t, \mathrm{x}_0)$.

For simplicity, let $J(\mathrm{x}) = J(\mathrm{x}(t, \mathrm{x}_0))$. Suppose that $\lambda_1(\mathrm{x}, S) \geqslant \cdots \geqslant \lambda_n(\mathrm{x}, S)$ are eigenvalues of the symmetrized Jacobian matrix (18)

$$\frac{1}{2}\left(SJ(\mathrm{x})S^{-1} + (SJ(\mathrm{x})S^{-1})^*\right). \tag{19}$$

Theorem 3 ([86,97]). *Given an integer $j \in [1, n]$ and $s \in [0, 1]$, suppose that there are a continuously differentiable scalar function $\vartheta : \mathbb{R}^n \to \mathbb{R}$ and a nonsingular matrix S such that*

$$\lambda_1(\mathrm{x}, S) + \cdots + \lambda_j(\mathrm{x}, S) + s\lambda_{j+1}(\mathrm{x}, S) + \dot{\vartheta}(\mathrm{x}) < 0, \ \forall\, \mathrm{x} \in K. \tag{20}$$

Then $\dim_L K \leqslant j + s$.

Here $\dot{\vartheta}$ is the derivative of ϑ with respect to the vector field f:

$$\dot{\vartheta}(\mathrm{x}) = (\mathrm{grad}(\vartheta))^*\mathrm{f}(\mathrm{x}).$$

The introduction of the matrix S can be regarded as a change of the space metric.

Theorem 4 ([19,94,97,99]). *Assume that there are a continuously differentiable scalar function ϑ and a nonsingular matrix S such that*

$$\lambda_1(\mathrm{x}, S) + \lambda_2(\mathrm{x}, S) + \dot{\vartheta}(\mathrm{x}) < 0, \ \forall\, \mathrm{x} \in \mathbb{R}^n. \tag{21}$$

Then any solution of system (7) bounded on $[0, +\infty)$ tends to an equilibrium as $t \to +\infty$.

Thus, if (21) holds, then the global attractor of system (7) coincides with its stationary set.

Theorems 3 and 4 give the following results for system (1).

Theorem 5. *Suppose that $\sigma > 1$.*

If

$$\left(r + \frac{\sigma}{a}\right)^2 < \frac{2(\sigma + 1)}{a}, \tag{22}$$

then any solution of system (1) bounded on $[0, +\infty)$ tends to an equilibrium as $t \to +\infty$.

If

$$\left(r + \frac{\sigma}{a}\right)^2 > \frac{2(\sigma + 1)}{a}, \tag{23}$$

then

$$\dim_L K \leqslant 3 - \frac{2(\sigma + 2)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + a\left(\frac{\sigma}{a} + r\right)^2}}. \tag{24}$$

Proof. We use the matrix

$$S = \begin{pmatrix} -a^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the eigenvalues of the corresponding matrix (19) are the following

$$\lambda_2 = -1,$$
$$\lambda_{1,3} = -\frac{\sigma+1}{2} \pm \frac{1}{2}\left[(\sigma-1)^2 + a\left(2z - \frac{\sigma+ar}{a}\right)^2\right]^{\frac{1}{2}}.$$

To check property (20) of Theorem 3 and property (21) of Theorem 4, we can consider the Lyapunov-like function

$$\vartheta(x,y,z) = \frac{2(1-s)V(x,y,z)}{\left[(\sigma-1)^2 + a\left(\frac{\sigma}{a}+r\right)^2\right]^{\frac{1}{2}}},$$

where

$$V(x,y,z) = \frac{\gamma}{\sigma}x^2 + \gamma y^2 + \gamma\left(1+\frac{a}{\sigma}\right)z^2 - 2\gamma(r-1)z, \quad \gamma = \frac{\sigma+ar}{2(r-1)}.$$

Finally, for system (1) with given S and ϑ, if condition (23) is satisfied and

$$s > \frac{-(\sigma+3) + \sqrt{(\sigma-1)^2 + a\left(\frac{\sigma}{a}+r\right)^2}}{\sigma+1+\sqrt{(\sigma-1)^2 + a\left(\frac{\sigma}{a}+r\right)^2}},$$

then Theorem 3 gives (24). If condition (22) is valid and $s = 0$, then the conditions of Theorem 4 are satisfied and any solution bounded on $[0, +\infty)$ tends to an equilibrium as $t \to +\infty$. □

Note that for $\sigma = 4$, $r = 687.5$, and $a = 0.0052$ the analytical estimate of the Lyapunov dimension of the corresponding self-excited attractor is as follows

$$\dim_L K < 2.890997461...$$

and the values of the local Lyapunov dimension at equilibria are

$$\dim_L S_0 = 2.890833450..., \quad \dim_L S_{1,2} = 2.009763700....$$

Numerically, by an algorithm in Appendix D, the Lyapunov dimension of the self-excited attractor is LD = 2.1405.

The analytical estimate of the Lyapunov dimension of the hidden attractor for $\sigma = 4$, $r = 700$, and $a = 0.0052$ is as follows

$$\dim_L K < 2.891882349...,$$

and the local Lyapunov dimension at the stationary points are the following

$$\dim_L S_0 = 2.891767634..., \ \dim_L S_{1,2} = 1.966483617...$$

Numerically, the Lyapunov dimension of the hidden attractor is LD = 2.1322.

Thus, the Lyapunov dimensions of B-attractor (which involve equilibrium S_0) and the global attractor are very close to the analytical estimate.

In the general case the coincidence of the analytical upper estimate with the local Lyapunov dimension at a stationary point gives the exact value of the Lyapunov dimension of the global attractor (see, e.g., studies of various Lorenz-like systems [90,94,97,99,109,114]).

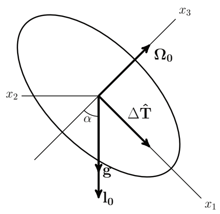


Fig. 11. Illustration of the problem.

Appendix

A Description of the physical problem

Consider the convection of viscous incompressible fluid motion inside the ellipsoid

$$\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 = 1, \quad a_1 > a_2 > a_3 > 0$$

under the condition of stationary inhomogeneous external heating. We assume that the ellipsoid and heat sources rotate with constant velocity $\Omega_{\mathbf{0}}$ around the axis. Vector $\mathbf{l_0}$ determines the orientation of the ellipsoid and has the same direction as the gravity vector $\mathbf{g}$. Vector $\mathbf{g}$ is stationary with respect to the ellipsoid motion. The value $\Omega_{\mathbf{0}}$ is assumed to be such that the centrifugal forces can be neglected when compared with the influence of the gravitational field. Consider the case when the ellipsoid rotates around the axis x_3 that has a constant angle α with gravity vector $\mathbf{g}$ ($|\mathbf{g}| = g$). The vector $\mathbf{g}$ is placed in the plane x_1x_3. Then, $\Omega_{\mathbf{0}} = (0,\, 0,\, \Omega_0)$ and $\mathbf{l_0} = (a_1 \sin\alpha,\, 0,\, -a_3\cos\alpha)$. Let the steady-state temperature difference $\mathbf{\Delta\hat{T}} = (q_0,\, 0,\, 0)$ be generated along the axis x_1 (Fig. 11). The corresponding mathematical model (three-mode model of convection) was obtained by Glukhovsky and Dolzhansky [48] in the form (see (3))

$$\begin{cases} \dot{x} = -\sigma x + Cz + Ayz, \\ \dot{y} = R_a - y - xz, \\ \dot{z} = -z + xy. \end{cases}$$

Here

$$\sigma = \frac{\lambda}{\mu}, \qquad T_a = \frac{\Omega_0^2}{\lambda^2}, \quad R_a = \frac{g\beta a_3 q_0}{2a_1a_2\lambda\mu},$$

$$A = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}\cos^2\alpha\, T_a^{-1}, \quad C = \frac{2a_1^2a_2}{a_3(a_1^2 + a_2^2)}\sigma\sin\alpha,$$

$$x(t) = \mu^{-1}\left(\omega_3(t) + \frac{g\beta a_3\cos\alpha}{2a_1a_2\Omega_0}q_3(t)\right), \quad y(t) = \frac{g\beta a_3}{2a_1a_2\lambda\mu}q_1(t),$$

$$z(t) = \frac{g\beta a_3}{2a_1a_2\lambda\mu}q_2(t),$$

and λ, μ, β are the coefficients of viscosity, heat conduction, and volume expansion, respectively; $q_1(t)$, $q_2(t)$, and $q_3(t)$ ($q_3(t) \equiv 0$) are temperature differences on the principal axes of the ellipsoid; $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ are the projections of the vectors of fluid angular velocities on the axes x_1, x_2, and x_3, respectively. Here

$$\omega_1(t) = -\frac{g\beta a_3}{2a_1 a_2 \Omega_0}\cos\alpha\, q_1(t), \quad \omega_2(t) = -\frac{g\beta a_3}{2a_1 a_2 \Omega_0}\cos\alpha\, q_2(t).$$

The parameters σ, T_a, and R_a are the Prandtl, Taylor, and Rayleigh numbers, respectively.

The linear change of variables [48]

$$x \to x, \quad y \to C^{-1}y, \quad z \to C^{-1}z,$$

transforms system (3) into the system

$$\begin{cases} \dot{x} = -\sigma x + z + A_c yz, \\ \dot{y} = R_c - y - xz, \\ \dot{z} = -z + xy. \end{cases} \tag{25}$$

with

$$R_c = R_a C, \quad A_c = \frac{A}{C^2}.$$

After the linear transformation (see, e.g., [99]):

$$x \to x, \quad y \to R_c - \frac{\sigma}{R_c A_c + 1}z, \quad z \to \frac{\sigma}{R_c A_c + 1}y, \tag{26}$$

system (25) takes the form of (1) with

$$a = \frac{A_c \sigma^2}{(R_c A_c + 1)^2}, \quad r = \frac{R_c}{\sigma}(R_c A_c + 1). \tag{27}$$

B Proof of Proposition 1

For system (1) the characteristic polynomial of the Jacobian matrix of the right-hand side at the point $\mathrm{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ has the form

$$\chi(\mathrm{x}_0) = \lambda^3 + p_1(\mathrm{x}_0)\lambda^2 + p_2(\mathrm{x}_0)\lambda + p_3(\mathrm{x}_0),$$

where

$$\begin{aligned} p_1(\mathrm{x}_0) &= \sigma + 2, \\ p_2(\mathrm{x}_0) &= x_0^2 + ay_0^2 - az_0^2 + (\sigma + ar)z_0 - r\sigma + 2\sigma + 1, \\ p_3(\mathrm{x}_0) &= \sigma x_0^2 + ay_0^2 - az_0^2 - 2ax_0y_0z_0 + (\sigma + ar)x_0y_0 + (\sigma + ar)z_0 - r\sigma + \sigma. \end{aligned}$$

Applying the Hurwitz criterion, the necessary and sufficient stability conditions for stationary point x_0 are the following

$$p_1(\mathrm{x}_0) > 0, \tag{28}$$

$$p_2(\mathrm{x}_0) > 0, \tag{29}$$

$$p_3(\mathrm{x}_0) > 0, \text{ and} \tag{30}$$

$$p_1(\mathrm{x}_0)p_2(\mathrm{x}_0) - p_3(\mathrm{x}_0) > 0. \tag{31}$$

Equilibria $S_{1,2}$ exist for $r > 1$, and we can check that $\chi(S_1) = \chi(S_2)$. For further analysis we can introduce

$$D = a\left(\sqrt{(\sigma - ar)^2 + 4\sigma a} - (\sigma - ar)\right) > 0. \tag{32}$$

Then for stationary points (5), condition (29) takes the form

$$p_2(S_{1,2}) = \frac{2}{D}\left(C_1\sqrt{(\sigma - ar)^2 + 4\sigma a} - C_1(\sigma - ar) - 2\sigma^2 a(\sigma - ar)\right) > 0, \tag{33}$$

where

$$C_1 = \sigma(\sigma - ar)^2 + a^2 r + a\sigma^2 > 0.$$

Because $\sigma > ar > 0$, we have

$$\begin{aligned} C_1\sqrt{(\sigma - ar)^2 + 4\sigma a} &> C_1(\sigma - ar) + 2\sigma^2 a(\sigma - ar) \quad \text{iff} \\ (\sigma - ar)^2 + 4\sigma a &> \left((\sigma - ar) + \frac{2\sigma^2 a(\sigma - ar)}{C_1}\right)^2 \quad \text{iff} \\ 4\sigma a &> \frac{4\sigma^2 a(\sigma - ar)^2}{C_1} + \frac{4\sigma^4 a^2(\sigma - ar)^2}{C_1^2} \quad \text{iff} \\ C_1^2 &> \sigma(\sigma - ar)^2 C_1 + \sigma^3 a(\sigma - ar)^2. \end{aligned}$$

The last inequality is satisfied because

$$\frac{1}{a}\left(C_1^2 - \sigma(\sigma - ar)^2 C_1 - \sigma^3 a(\sigma - ar)^2\right) = \sigma ar(\sigma - ar)^2 + a(\sigma^2 + ar)^2 > 0.$$

This implies (33).

Condition (30) for $S_{1,2}$ takes the form

$$\begin{aligned} p_3(S_{1,2}) = \frac{2\sigma}{D}&\left(\sqrt{(\sigma - ar)^2 + 4\sigma a} - (\sigma - ar + 2a)\right) \cdot \\ &\cdot \left((\sigma - ar)^2 + 4\sigma a - (\sigma - ar)\sqrt{(\sigma - ar)^2 + 4\sigma a}\right) > 0. \end{aligned} \tag{34}$$

Since

$$(\sigma - ar)^2 + 4\sigma a - (\sigma - ar + 2a)^2 = 4a^2(r - 1) > 0$$

and

$$(\sigma - ar)^2 + 4\sigma a > (\sigma - ar)\sqrt{(\sigma - ar)^2 + 4\sigma a} \quad \text{iff} \quad \sqrt{(\sigma - ar)^2 + 4\sigma a} > (\sigma - ar),$$

condition (34) is also satisfied.

Condition (31) for stationary points $S_{1,2}$ is as follows

$$\begin{aligned} p_1(S_{1,2})p_2(S_{1,2}) - p_3(S_{1,2}) = \frac{2}{D}&\left(C_2\sqrt{(\sigma - ar)^2 + 4\sigma a} - \right. \\ &\left. - C_2(\sigma - ar) - 2\sigma^2 a(\sigma(\sigma - ar) - 4a)\right) > 0, \end{aligned} \tag{35}$$

where

$$C_2 = \left((\sigma(\sigma - ar) - a)^2 + a^2(r\sigma + 2r - 1) + a\sigma^2(\sigma - 2)\right).$$

If $\sigma > 2$, then $C_2 > 0$ and we can derive a chain of inequalities for (35):

$$C_2\sqrt{(\sigma - ar)^2 + 4\sigma a} > C_2(\sigma - ar) + 2\sigma^2 a(\sigma(\sigma - ar) - 4a) \quad \text{iff}$$

$$(\sigma - ar)^2 + 4\sigma a > \left((\sigma - ar) + \frac{2\sigma^2 a(\sigma(\sigma - ar) - 4a)}{C_2}\right)^2 \quad \text{iff}$$

$$4\sigma a > \frac{4\sigma^2 a(\sigma - ar)(\sigma(\sigma - ar) - 4a)}{C_2} + \frac{4\sigma^4 a^2(\sigma(\sigma - ar) - 4a)^2}{C_2^2} \quad \text{iff}$$

$$C_2^2 > \sigma(\sigma - ar)(\sigma(\sigma - ar) - 4a)C_2 + \sigma^3 a(\sigma(\sigma - ar) - 4a)^2.$$

We can divide the last inequality by $\left(-a^2\right)$ and rewrite it in the form of polynomial

$$a^2\sigma^2(\sigma - 2)r^3 - a\left(2\sigma^4 - 4\sigma^3 - 3a\sigma^2 + 4a\sigma + 4a\right)r^2 + \sigma^2\left(\sigma^3 + 2(3a - 1)\sigma^2 - \right.$$
$$\left. -8a\sigma + 8a\right)r - \sigma^3\left(\sigma^3 + 4\sigma^2 - 16a\right) < 0.$$

This inequality corresponds to the stability condition for the equilibria $S_{1,2}$.
□

C Proofs of Lemma 1 and the completeness of system (1)

Suppose that the Lyapunov function has the form

$$V(x, y, z) = \frac{1}{2}\left[x^2 + y^2 + (a + 1)\left(z - \frac{\sigma + r}{a + 1}\right)^2\right]. \qquad (36)$$

Here $V(x, y, z) \to \infty$ as $|(x, y, z)| \to \infty$. For an arbitrary solution $\mathrm{x}(t) = (x(t), y(t), z(t))$ of system (1) we have

$$\begin{aligned}\dot{V}(x, y, z) &= x(-\sigma x + \sigma y - ayz) + y(rx - y - xz) + ((a + 1)z - (\sigma + r))(-z + xy)\\ &= -\sigma x^2 - y^2 - (a + 1)z^2 + (\sigma + r)z.\end{aligned}$$

Suppose that $\varepsilon \in (0, (a + 1))$ and $c = \min\{\sigma, 1, (a + 1) - \varepsilon\} > 0$. Then

$$\begin{aligned}\dot{V}(x, y, z) &= -\sigma x^2 - y^2 - ((a + 1) - \varepsilon)z^2 - \varepsilon z^2 + (\sigma + r)z\\ &= -\sigma x^2 - y^2 - ((a + 1) - \varepsilon)z^2 - \left(\sqrt{\varepsilon}z - \frac{(\sigma + r)}{2\sqrt{\varepsilon}}\right)^2 + \frac{(\sigma + r)^2}{4\varepsilon}\\ &\le -c(x^2 + y^2 + z^2) + \frac{(\sigma + r)^2}{4\varepsilon}.\end{aligned}$$

Suppose that $x^2 + y^2 + z^2 \ge R^2$. Then a positive $\varkappa$ exists such that

$$\dot{V}(x, y, z) \le -cR^2 + \frac{(\sigma + r)^2}{4\varepsilon} < -\varkappa \quad \text{for} \quad R^2 > \frac{1}{c}\frac{(\sigma + r)^2}{4\varepsilon}.$$

We choose a number $\eta > 0$ such that

$$\{(x, y, z) \mid V(x, y, z) \le \eta\} \supset \left\{(x, y, z) \mid x^2 + y^2 + z^2 \le R^2\right\},$$

i.e., the relation $x^2 + y^2 + z^2 \le R^2$ implies that

$$x^2 + y^2 + (a + 1)\left(z - \frac{\sigma + r}{a + 1}\right)^2 = x^2 + y^2 + z^2 + az^2 - 2(\sigma + r)z + \frac{(\sigma + r)^2}{a + 1} \le 2\eta.$$

Since

$$-2(\sigma+r)z \ \leq\ 2(\sigma+r)|z| \ \leq\ 2(\sigma+r)R,$$

it is sufficient to choose $\eta > 0$ such that

$$(a+1)R^2 + 2(\sigma+r)R + \frac{(\sigma+r)^2}{a+1} \leq 2\eta, \quad \text{i.e.} \quad \eta \geq \frac{1}{2}(a+1)\left(R + \frac{\sigma+r}{a+1}\right)^2.$$

Further, we can apply Theorem 2, which implies the Lemma.

Using Lyapunov function (36), we can prove the boundedness of solutions of system (1) for $t \leq 0$. Note that

$$\left(2\left(z - \frac{\sigma+r}{a+1}\right)^2 + \frac{(\sigma+r)^2}{2(a+1)^2}\right) - \left(z - \frac{\sigma+r}{2(a+1)}\right)^2 = \left(z - \frac{3(\sigma+r)}{2(a+1)}\right)^2 \geq 0,$$

so the inequality

$$\left(z - \frac{\sigma+r}{2(a+1)}\right)^2 \leq \left(2\left(z - \frac{\sigma+r}{a+1}\right)^2 + \frac{(\sigma+r)^2}{2(a+1)^2}\right)$$

is satisfied. This implies that

$$\dot{V} \geq 2\sigma\left(-\frac{1}{2}x^2\right) + 2\left(-\frac{1}{2}y^2\right) - 2(a+1)\left[\left(z - \frac{\sigma+r}{a+1}\right)^2 + \frac{(\sigma+r)^2}{4(a+1)^2}\right] + \frac{1}{4}\frac{(\sigma+r)^2}{a+1}$$

$$= 2\sigma\left(-\frac{1}{2}x^2\right) + 2\left(-\frac{1}{2}y^2\right) + 4\left(-\frac{1}{2}(a+1)\left(z - \frac{\sigma+r}{a+1}\right)^2\right) - \frac{1}{4}\frac{(\sigma+r)^2}{a+1}$$

$$\geq 2\sigma(-V) + 2(-V) + 4(-V) - \frac{1}{4}\frac{(\sigma+r)^2}{a+1}.$$

Suppose that $k = 2\sigma + 2 + 4$, and $m = \frac{1}{4}\frac{(\sigma+r)^2}{a+1}$. Then

$$\dot{V} + kV \geq -m.$$

This implies that

$$\frac{d}{dt}(e^{kt}V) = e^{kt}\dot{V} + ke^{kt}V \geq -e^{kt}m.$$

Thus for $t \leq 0$ we have

$$V(0) - e^{kt}V(t) \geq (me^{kt} - m)/k$$

or

$$V(t) \le e^{-kt}V(0) + (me^{-kt} - m)/k.$$

This implies that V does not tend to infinity in a finite negative time. Therefore, any solution $(x(t), y(t), z(t))$ of system (1) does not tend to infinity in a finite negative time. Thus, differential Eq. (1) generates a dynamical system for $t \in \mathbb{R}$.

D Computation of Lyapunov exponents and Lyapunov dimension using MATLAB

The singular value decomposition (SVD) of a fundamental matrix $\mathrm{X}(t)$ has the from

$$\mathrm{X}(t) = \mathrm{U}(t)\Sigma(t)\mathrm{V}^T(t): \quad \mathrm{U}(t)^T\mathrm{U}(t) \equiv I \equiv \mathrm{V}(t)^T\mathrm{V}(t),$$

where $\Sigma(t) = \mathrm{diag}\{\sigma_1(t), ..., \sigma_n(t)\}$ is a diagonal matrix with positive real diagonal entries known as *singular values*. The singular values are the square roots of the eigenvalues of the matrix $\mathrm{X}(t)^*\mathrm{X}(t)$ (see [54]). Lyapunov exponents are defined as the upper bounds of the exponential growth rate of the singular values of the fundamental matrix of linearized system (see Eq. (13)).

We now give a MATLAB implementation of the discrete SVD method for computing Lyapunov exponents based on the product SVD algorithm (see, e.g., [40,159]).

Listing 1: **productsSVD.m** – product SVD algorithm.

```
function [U, R, V] = productSVD(initFactorization , nIterations)
% Parameters:
%   initFactorization - array containing factor matrices of the
%                       fundamental matrix X, such that:
%           X = initFactorization(:,:,1) * ... * initFactorization(:,:,end);
%   nIterations - number of iterations in the product SVD algorithm.

% dimOde - dimension of the ODEs, nFactors - number of factor matrices
[~, dimOde, nFactors] = size(initFactorization);

% A - 2D array of matrices storing the factor matrices at each iteration
A = zeros(dimOde, dimOde, nFactors, nIterations);
A(:, :, :, 1) = initFactorization;

% Q - array of matrices storing orhogonal matrices of the QR decomposition
Q = zeros(dimOde, dimOde, nFactors+1);

% U, V - orthogonal matrices in the SVD decomposition
U = eye(dimOde); V = eye(dimOde);

% R - array of upper triangular factor matrices, such that after
% the last iteration \Sigma = R(:,:,1) * ... * R(:,:,end)
R = zeros(dimOde, dimOde, nFactors);

% Main loop
for iIteration = 1 : nIterations
    Q(:, :, nFactors + 1) = eye(dimOde, dimOde);
    for jFactor = nFactors : -1 : 1
        C = A(:, :, jFactor, iIteration) * Q(:, :, jFactor+1);
        [Q(:, :, jFactor), R(:, :, jFactor)] = qr(C);
        for kCoord = 1 : dimOde
            if R(kCoord, kCoord, jFactor) < 0
                R(kCoord, :, jFactor) = -1 * R(kCoord, :, jFactor);
                Q(:, kCoord, jFactor) = -1 * Q(:, kCoord, jFactor);
            end;
        end;
    end;

    if mod(iIteration, 2) == 1
        U = U * Q(:, :, 1);
    else
        V = V * Q(:, :, 1);
    end

    for jFactor = 1 : nFactors
        A(:, :, jFactor, iIteration + 1) = R(:, :, nFactors-jFactor+1)';
    end
end

end
```

Listing 2: **computeLEs.m** – computation of the Lyapunov exponents.

```
function LEs = computeLEs(extOde, initPoint, tStep, ...
                              nFactors, nSvdIterations, odeSolverOptions)
% Parameters:
%   extOde - extended ODE system (system of ODEs + var. eq.);
%   initPoint -  initial point;
%   tStep - time-step in the factorization procedure;
%   nFactors - number of factor matrices in the factorization procedure;
%   nSvdIterations - number of iterations in the product SVD algoritm;
%   odeSolverOptions - solver options (sover = ode45);

% Dimension of the ODE :
dimOde = length(initPoint);

% Dimension of the extended ODE (ODE + Var. Eq.):
dimExtOde = dimOde * (dimOde + 1);

tBegin = 0; tEnd = tStep;
tSpan = [tBegin, tEnd];
initFundMatrix = eye(dimOde);
initCond = [initPoint(:); initFundMatrix(:)];

X = zeros(dimOde, dimOde, nFactors);

% Main loop : factorization of the fundamental matrix
for iFactor = 1 : nFactors
    [~, extOdeSolution] = ode45(extOde, tSpan, initCond, odeSolverOptions);

    X(:, :, iFactor) = reshape(...
                        extOdeSolution(end, (dimOde + 1) : dimExtOde), ...
                                                           dimOde, dimOde);
    currInitPoint = extOdeSolution(end, 1 : dimOde);
    currInitFundMatrix = eye(dimOde);

    tBegin = tBegin + tStep;
    tEnd = tEnd + tStep;
    tSpan = [tBegin, tEnd];
    initCond = [currInitPoint(:); currInitFundMatrix(:)];
end

% Product SVD of factorization X of the fundamental matrix
[~, R, ~] = productSVD(X, nSvdIterations);

% Computation of the Lyapunov exponents
LEs = zeros(1, dimOde);
for jFactor = 1 : nFactors
    LEs = LEs + log(diag(R(:, :, jFactor))');
end;
finalTime = tStep * nFactors;
LEs = LEs / finalTime;

end
```

Listing 3: **lyapunovDim.m** – computation of the Lyapunov dimension.

```
function LD = lyapunovDim( LEs )
% For the given array of Lyapunov exponents of a point the function
% compute local Lyapunov dimention of this point.

% Parameters:
%   LEs - array of Lyapunov exponents.

% Initialization of the local Lyupunov dimention :
LD = 0;

% Number of LCEs :
nLEs = length(LEs);

% Sorted LCEs :
sortedLEs = sort(LEs, 'descend');

% Main loop :
leSum = sortedLEs(1);
if ( sortedLEs(1) > 0 )
    for i = 1 : nLEs-1
        if sortedLEs(i+1) ~= 0
            LD = i + leSum / abs( sortedLEs(i+1) );
            leSum = leSum + sortedLEs(i+1);
            if leSum < 0
                break;
            end
        end
    end
end
end
```

Listing 4: **genLorenzSyst.m** – generalized Lorenz system (1) along with the variational equation.

```
function OUT = genLorenzSyst(t, x, r, sigma, b, a)

% Generalized Lorenz system with
% parameters: r sigma b a

OUT(1) = sigma*(x(2) - x(1)) - a*x(2)*x(3);
OUT(2) = r*x(1) - x(2) - x(1)*x(3);
OUT(3) = -b*x(3) + x(1)*x(2);

% Jacobian at the point [x(1), x(2), x(3)]
J = [-sigma, sigma-a*x(3),  -a*x(2);
     r-x(3),     -1,      -x(1);
     x(2),      x(1), -b];

X = [x(4), x(7), x(10);
     x(5), x(8), x(11);
     x(6), x(9), x(12)];

% Variational equation
OUT(4:12) = J*X;
```

Listing 5: **main.m** – computation of the Lyapunov exponents and local Lyapunov dimension for the hidden attractor of generalized Lorenz system (1).

```
function main

% Parameters of generalized Lorenz system
% that correspond to the hidden attractor
r = 700; sigma = 4; b = 1; a = 0.0052;

% Initial point for trajectory that visualizes the hidden attractor
x0 = [-14.551336132013954 -173.86811769236883 718.92035664071227];

tStep = 0.1;
nFactors = 10000;
nSvdIterations = 3;

% ODE solver parameters
acc = 1e-8; RelTol = acc; AbsTol = acc; InitialStep = acc/10;
odeSolverOptions = odeset('RelTol', RelTol, 'AbsTol', AbsTol, ...
                          'InitialStep', InitialStep, 'NormControl', 'on');

LEs = computeLEs(@(t, x) genLorenzSyst(t, x, r, sigma, b, a), ...
                 x0, tStep, nFactors, nSvdIterations, odeSolverOptions);

fprintf('Lyapunov exponents: %6.4f, %6.4f, %6.4f\n', LEs);

LD = lyapunovDim(LEs);

fprintf('Lyapunov dimension: %6.4f\n', LD);

end
```

E Fishing principle and the existence of a homoclinic orbit in the Glukhovsky–Dolghansky system

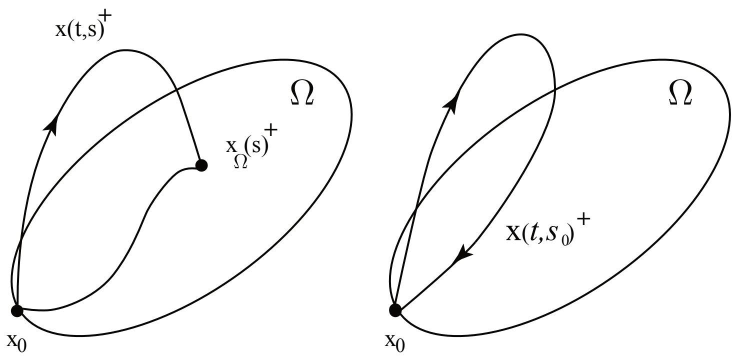
E.1 Fishing Principle

Consider autonomous system of differential equations (1) with the parameter

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x}, q), \quad t \in \mathbb{R},\ \mathrm{x} \in \mathbb{R}^n,\ q \in \mathbb{R}^m. \tag{37}$$

Let $\gamma(s), s \in [0,1]$ be a smooth path in the space of the parameter $\{q\} = \mathbb{R}^m$. Consider the following Tricomi problem [162]: *Is there a point $q_0 \in \gamma(s)$ for which system (37) with q_0 has a homoclinic trajectory?*

Consider system (37) with $q = \gamma(s)$, and introduce the following notions. Let $\mathrm{x}(t,s)^+$ be an outgoing separatrix of the saddle point x_0 (i.e. $\lim\limits_{t\to-\infty} \mathrm{x}(t,s)^+ = \mathrm{x}_0$) with



(a) Separatrix $x(t,s)^+$, where $s \in [0, s_0)$.
(b) Separatrix $x(t,s)^+$, where $s = s_0$.

Fig. 12. Separatrix $x(t,s)^+$, where $s \in [0, s_0]$.

a one-dimensional unstable manifold. Define by $x_\Omega(s)^+$ the point of the first crossing of separatrix $x(t,s)^+$ with the closed set Ω:

$$x(t,s)^+ \,\overline{\in}\, \Omega, \quad t \in (-\infty, T),$$

$$x(T,s)^+ = x_\Omega(s)^+ \in \Omega.$$

If there is no such crossing, we assume that $x_\Omega(s)^+ = \emptyset$ (the empty set).

Theorem 6 (Fishing Principle [87,96,98]). *Suppose that for the path $\gamma(s)$ there is an $(n-1)$-dimensional bounded manifold Ω with a piecewise-smooth edge $\partial\Omega$ that possesses the following properties.*

1. *For any $x \in \Omega \setminus \partial\Omega$ and $s \in [0,1]$, the vector $f(x, \gamma(s))$ is transversal to the manifold $\Omega \setminus \partial\Omega$,*
2. *for any $s \in [0,1]$, $f(x_0, \gamma(s)) = 0$, the point $x_0 \in \partial\Omega$ is a saddle;*
3. *for $s = 0$ the inclusion $x_\Omega(0)^+ \in \Omega \setminus \partial\Omega$ is valid (Fig. 12a),*
4. *for $s = 1$ the relation $x_\Omega(1)^+ = \emptyset$ is valid (i.e. $x_\Omega(1)^+$ is an empty set),*
5. *for any $s \in [0,1]$ and $y \in \partial\Omega \setminus x_0$ there exists a neighborhood $U(y, \delta) = \{x \,|\, |x - y| < \delta\}$ such that $x_\Omega(s)^+ \,\overline{\in}\, U(y, \delta)$.*

If conditions 1)–5) are satisfied, then there exists $s_0 \in [0,1]$ such that $x(t, s_0)^+$ is a homoclinic trajectory of the saddle point x_0 (Fig. 12b).

The fishing principle can be interpreted as follows. Figure 12a shows a fisherman at the point x_0 with a fishing rod $x(t,s)^+$. The manifold Ω is a lake surface and $\partial\Omega$ is a shore line. When $s = 0$, a fish has been caught by the fishing rod. Then, $x(t,s)^+, s \in [0, s_0)$ is a path taken by the fishing rod when it brings the fish to the shore. Assumption 5) implies that the fish cannot be taken to the shore $\partial\Omega\backslash x_0$, because $\partial\Omega\backslash x_0$ is a forbidden zone. Therefore, only the situation shown in Fig. 12b is possible (i.e., at $s = s_0$ the fisherman has caught a fish). This corresponds to a homoclinic orbit.

Now let us describe the numerical procedure for defining the point Γ on the path $\gamma(s)$, which corresponds to a homoclinic trajectory. Here we assume that conditions

1), 2), and 5) of the fishing Principle are satisfied. Consider a sequence of paths $\gamma_j(s) \subset \{\gamma_{j-1}(s),\ s \in [0,1]\} \subset \{\gamma(s),\ s \in [0,1]\}$, $\forall s \in [0,1]$ such that the length $\{\gamma_j(s)\}$ tends to zero as $j \to +\infty$. Condition 3) is satisfied for $\gamma_j(0)$ and condition 4) is satisfied for $\gamma_j(1)$. This sequence can be obtained if the paths γ and γ_j are sequentially divided into two paths of the same length and we choose the path such that for its end points condition 3) is satisfied and condition 4) is not satisfied (or vice versa). Obviously, the sequence $\gamma_j(s)$, $s \in [0,1]$ is contracted to the point $\Gamma \in \{\gamma_j(s),\ s \in [0,1]\}$, $\forall j$. This point corresponds to a homoclinic trajectory of system (37).

Now, consider the conditions of the non-existence of a homoclinic orbit. Consider the Jacobian matrix of system (37)

$$J(\mathrm{x}, s) = \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(\mathrm{x}, \gamma(s)).$$

Let $\lambda_1(\mathrm{x}, s, S) \geqslant \cdots \geqslant \lambda_n(\mathrm{x}, s, S)$ denote the eigenvalues of the symmetrized matrix

$$\frac{1}{2}\left(SJ(\mathrm{x}, s)S^{-1} + (SJ(\mathrm{x}, s)S^{-1})^*\right),$$

where S is a nonsingular matrix.

Suppose system (37) has a saddle point $\mathrm{x}_0 \equiv \mathrm{x}_0(s)$, $\forall s \in [0,1]$, the point x_0 belongs to a positively invariant bounded set K, and $J(\mathrm{x}_0, s)$ has only real eigenvalues.

Theorem 7 ([87]). *Assume that there are a continuously differentiable scalar function $\vartheta(\mathrm{x}, s)$ and a nonsingular matrix S such that for system (37) with $q = \gamma(s)$, the inequality*

$$\lambda_1(\mathrm{x}, S) + \lambda_2(\mathrm{x}, S) + \dot{\vartheta}(\mathrm{x}) < 0,\ \forall\, \mathrm{x} \in K, \quad \forall s \in [0,1] \tag{38}$$

is satisfied. Then system (37) has no homoclinic trajectories for all $s \in [0,1]$ such that

$$\lim_{t \to -\infty} \mathrm{x}(t) = \lim_{t \to +\infty} \mathrm{x}(t) = \mathrm{x}_0. \tag{39}$$

E.2 Existence of a homoclinic trajectories in the Glukhovsky-Dolzhansky system

Consider the separatrix $x^+(t), y^+(t), z^+(t)$ of the zero saddle point of system (1), where $x(t)^+ > 0$, $\forall t \in (-\infty, \tau)$, τ is a number, and $\lim\limits_{t \to -\infty} x(t)^+ = 0$ (i.e. positive outgoing separatrix is considered).

Define the manifold Ω as

$$\Omega = \{x = 0,\ y \le 0,\ y(\sigma - az) \le 0,\ y^2 + z^2 \le 2r^2\}.$$

Check condition 1).

Inside the set $\Omega \backslash \partial\Omega$ we have

$$\dot{x} = y(\sigma - az) < 0.$$

Check condition 5).

a) On $B_1 = \{x = 0,\ y = 0,\ -\sqrt{2}r \le z \le \sigma/a\}$ system (1) has the solution

$$x(t) \equiv y(t) \equiv 0, \quad z(t) = z(0)\exp(-t).$$

b) On $B_2 = \{x = 0,\ y < 0,\ z = \sigma/a,\ y^2 + z^2 \le 2r^2\}$ we have

$$\ddot{x} = \sigma y.$$

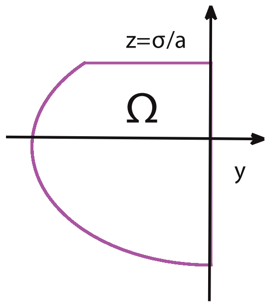


Fig. 13. Manifold Ω.

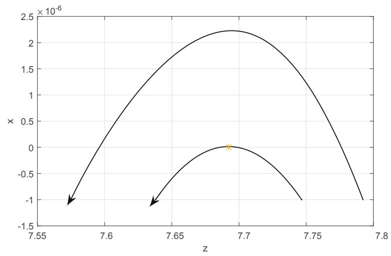


Fig. 14. Local behavior of the trajectories of system (1) in the neighborhood of set B_2 ($\sigma = 4$, $a = 0.52$, $r = 10^5$).

Therefore the local behavior of trajectories in the neighborhood of B_2 is shown in Fig. 14.

c) The set $B_3 = \{x = 0,\, y < 0,\, -\sqrt{2}r \leq z \leq \sigma/a,\, y^2 + z^2 = 2r^2\}$ is located outside of the positively invariant cylinder C (see Eq. (10)). Thus, the separatrices of the zero saddle point (which belongs to C) can not reach the set B_3.

Check condition 3).

Consider the development of the asymptotic integration of system (1) [88]. Assume that

$$ar = c - \lambda\varepsilon + O(\varepsilon^2), \tag{40}$$

where c and λ are some numbers and $\varepsilon = 1/\sqrt{r}$ is a small parameter.

Lemma 2. *For any $\sigma > c, \sigma > 1$ there exists a time $T > 0$ such that for sufficiently large r, $(x^+(T), y^+(T), z^+(T)) \in \Omega \setminus \partial\Omega$ (i.e. condition 3) of the fishing principle is valid).*

Proof (sketch). Using the transformation

$$t \to \frac{t}{\sqrt{r}}, \quad x \to \sqrt{r}x, \quad y \to ry, \quad z \to rz, \tag{41}$$

we can obtain

$$\begin{aligned}
\dot{x} &= \sigma y - \varepsilon\sigma x - (c - \lambda\varepsilon + O(\varepsilon^2))yz \\
\dot{y} &= x - \varepsilon y - xz \\
\dot{z} &= -\varepsilon z + xy.
\end{aligned} \tag{42}$$

a) Consider the zero approximation of system (42) (system (42) without $\lambda\varepsilon$ and $\varepsilon = 0$) and its solution $(x_0(t), y_0(t), z_0(t))$. There are two independent integrals

$$V(x_0(t), y_0(t), z_0(t)) = (\sigma - c)z_0(t)^2 + \sigma y_0(t)^2 - x_0(t)^2 = C_1,$$

$$W = y_0(t)^2 + z_0(t)^2 - 2z_0(t) = C_2.$$

Thus, the positive outgoing separatrix $x_0^+(t), y_0^+(t), z_0^+(t)$ of the saddle point $(x = y = z = 0)$ of zero approximation of system (42) belongs to the intersection of surfaces $V = 0$ and $W = 0$, i.e.

$$V(x_0^+(t), y_0^+(t), z_0^+(t)) = 0 = W(x_0^+(t), y_0^+(t), z_0^+(t)), \forall t \in (-\infty, +\infty). \tag{43}$$

From (43) it follows that $x_0^+(t) \neq 0$, $\forall t \in \mathbb{R}^n$ and

$$\lim_{t\to+\infty} x_0^+(t) = \lim_{t\to+\infty} y_0^+(t) = \lim_{t\to+\infty} z_0^+(t) = 0.$$

b) Consider the first approximation of system (42) (system (42) without $O(\varepsilon^2)$). For the small values of ε the outgoing separatrix $(x_1^+(t), y_1^+(t), z_1^+(t))$ of the zero saddle point of the first approximation of system (42) is close to $(x_0^+(t), y_0^+(t), z_0^+(t))$ on $(-\infty, \tau)$. Therefore for sufficiently small ε and some τ the separatrix $(x_1^+(t), y_1^+(t), z_1^+(t))$ reaches δ-vicinity of the zero saddle. Then there exists finite $\tau = \tau(\varepsilon, \delta)$ such that

$$|x_1^+(\tau(\varepsilon, \delta))| < \delta, \quad |y_1^+(\tau(\varepsilon, \delta))| < \delta, \quad |z_1^+(\tau(\varepsilon, \delta))| < \delta.$$

Consider two functions

$$V_\varepsilon(x, y, z) = (\sigma - c + \lambda\varepsilon)z^2 + \sigma y^2 - x^2, \quad W = y^2 + z^2 - 2z. \tag{44}$$

For the derivatives of (44) along the positive outgoing separatrix we have

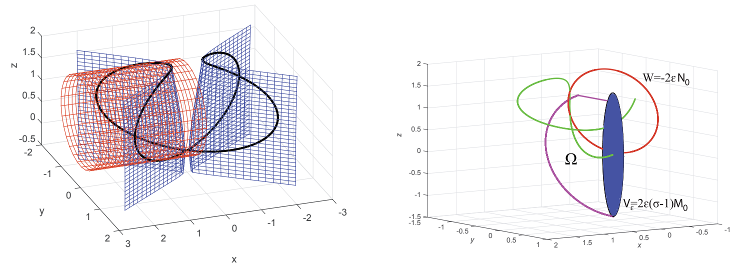
$$\begin{aligned}
\frac{d}{dt}V_\varepsilon(t) &\equiv \frac{d}{dt}V_\varepsilon(x_1^+(t), y_1^+(t), z_1^+(t)) = -2\varepsilon V_\varepsilon(x_1^+(t), y_1^+(t), z_1^+(t)) + 2\varepsilon(\sigma - 1)x_1^+(t)^2, \\
\frac{d}{dt}W(t) &\equiv \frac{d}{dt}W(x_1^+(t), y_1^+(t), z_1^+(t)) = -2\varepsilon W(x_1^+(t), y_1^+(t), z_1^+(t)) - 2\varepsilon z_1^+(t).
\end{aligned} \tag{45}$$

Integrating (45) from $-\infty$ to τ and taking into account

$$\lim_{\tau_0\to-\infty} V_\varepsilon(\tau_0) = \lim_{\tau_0\to-\infty} W(\tau_0) = 0,$$

we obtain

$$V_\varepsilon(\tau) = 2\varepsilon(\sigma - 1)\int_{-\infty}^{\tau} e^{-2\varepsilon(\tau - s)}(x_1^+(t))^2\, dt = 2\varepsilon(\sigma - 1)M_0 + o(\varepsilon), \tag{46}$$



(a) Zero approximation of system (42): integrals (V – blue, W – red) and separatrices (black)

(b) First approximation of system (42): for $x_1^+(T) = 0$ the positive outgoing separatrix is located inside magenta domain Ω, and near red circuit and the border of blue ellipsoid (48)

Fig. 15. $\sigma = 2.3445$, $a = 0.0065$, $r = 300$, $c = 2$, $\lambda = 1$.

$$W(\tau) = -2\varepsilon \int\limits_{-\infty}^{\tau} e^{-2\varepsilon(\tau - s)} (z_1^+(s))^2 \, ds = -2\varepsilon N_0 + o(\varepsilon), \tag{47}$$

where M_0 and N_0 are some positive numbers. If z_1 and z_2 satisfy

$$V_\varepsilon(0, 0, z_1) = 2\varepsilon(\sigma - 1)M_0, \quad W(0, 0, z_2) = -2\varepsilon N_0, \tag{48}$$

then

$$z_1 = \sqrt{\varepsilon}\sqrt{\frac{2M_0(\sigma - 1)}{\sigma - c + \lambda\varepsilon}}, \quad z_2 = \varepsilon \frac{2N_0}{1 + \sqrt{1 - 2N_0\varepsilon}}. \tag{49}$$

Hence the situation shown in Fig. 15b occurs in the neighborhood of the saddle point ($x = y = z = 0$) for the surfaces

$$V_\varepsilon(x, y, z) = 2\varepsilon(\sigma - 1)M_0, \quad W(x, y, z) = 2\varepsilon N_0. \tag{50}$$

The separatrix $(x_1^+(\tau), y_1^+(\tau), z_1^+(\tau)$ has to be near the surfaces $V_\varepsilon(x, y, z)$ and $W(x, y, z)$. From the mutual disposition of surfaces (50) and different order of smallness in (49) it follows that if $x_1^+(\tau) > 0$, then $y_1^+(\tau) < 0$ and $\dot{x}_1^+(\tau) < 0$ for sufficiently small ε. Moreover, $\dot{x}_1^+(t) < 0$ for $t > \tau$ and $x_1^+(t) \leq 0$. This implies the existence of $T > \tau$ such that $x_1^+(T) = 0$. We can obtain similar results for the case $x_1^-(\tau) < 0$ (then $y_1^-(\tau) > 0$). The behavior of separatrix $(x^+(\tau), y^+(\tau), z^+(\tau)$ is consistent with the behavior of $(x_1^+(\tau), y_1^+(\tau), z_1^+(\tau)$.

Check condition 4). We now check condition 4) for system (1) with parameters (27). For this system it was proved [19, pp. 276–277, 269–272] that if

$$R_c < \frac{4(\sigma + 1)}{1 + \sqrt{1 + 8A_c(\sigma + 1)}}, \tag{51}$$

then condition (38) of Theorem 7 is satisfied for $S = I, V(x, s) \equiv 0$.

Now we can show that if (51) holds, then condition 4) for system (1) is also satisfied. Consider the path

$$\begin{aligned}
&R_c(s), \quad A_c(s), \quad \sigma(s) \equiv \sigma, \\
&A_c(0) = 0, \quad A_c(1) = A_c, \\
&R_c(s) \in \left(\frac{2\sigma}{1+\sqrt{1+4A_c(s)\sigma}}, \frac{4(\sigma+1)}{1+\sqrt{1+8A_c(s)(\sigma+1)}} \right), \\
&R_c(0) = \sigma(1+\delta),
\end{aligned} \tag{52}$$

where δ is a small positive number.

For $s = 0$ condition 4) is satisfied (see, e.g., [89,96]). If for some $s_1 \in [0,1]$ condition 4) is not satisfied, then condition 3) is satisfied for s_1. In this case Theorem 6 implies that there exists $s_2 \in [0, s_1]$ for which a homoclinic trajectory exists. But $R_c(s)$ is chosen in such a way that conditions of Theorem 7 are valid and hence the homoclinic trajectories do not exist. This contradiction proves the fulfillment of condition 4) of Theorem 6 for all $s \in [0,1]$. Condition 6) is checked.

Check condition 2). From (52) it is obvious that condition 2 is satisfied.

Remark 6. For $c = \sigma$ Lemma 2 is not valid since a positive outgoing separatrix of the zero approximation of system (42) follows a heteroclitic orbit

$$\lim_{t\to+\infty} x_0^0(t) = \lim_{t\to+\infty} y_0^0(t) = \lim_{t\to+\infty} z_0^0(t) = 2.$$

In this case we may consider a sequence of systems close to (1). For example, instead of (27) we can consider $a = a(\beta_k) = \frac{A_c\sigma(\sigma-\beta_k)}{(R_cA_c+1)^2}$, where β_k are a small positive numbers and $\lim\limits_{k\to+\infty} \beta_k = 0$, such that path (52) satisfies condition 4) of the fishing principle. Then, using Lemma 2 and the fishing principle, we get the sequences of r_k^h and corresponding homoclinic orbits. Choosing a convergent subsequence from r_k^h and using Arzela–Ascoli theorem, we can justify the existence of a homoclinic orbit in the initial system with $a(0)$.

Note also that since a and r are varying in the asymptotic integration, $\partial\Omega$ is also varying.

Finally we get the following

Theorem 8. *For any fixed $A_c > 0, \sigma > 1$ there exists a number*

$$R_c \in \left(\frac{2\sigma}{1+\sqrt{1+4A_c\sigma}}, +\infty \right)$$

such that system (1) with parameters (27) has a homoclinic trajectory of the zero saddle point.

This work was supported by Russian Scientific Foundation (project 14-21-00041) and Saint-Petersburg State University.

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PIV

THE LYAPUNOV DIMENSION FORMULA OF SELF-EXCITED AND HIDDEN ATTRACTORS IN THE GLUKHOVSKY-DOLZHANSKY SYSTEM

by

N. V. Kuznetsov, G. A. Leonov, T. N. Mokaev 2015

arXiv preprint arXiv:1509.09161, http://arxiv.org/pdf/1509.09161v1.pdf

The Lyapunov dimension formula of self-excited and hidden attractors in the Glukhovsky-Dolzhansky system

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Abstract

In the past two decades Lyapunov functions are used for the estimation of attractor dimensions. By means of these functions the upper estimate of Lyapunov dimension for Rössler attractor and the exact formulas of Lyapunov dimension for Hénon, Chirikov, and Lorenz attractors are obtained.

In this report the simplest model, suggested by Glukhovsky and Dolzhansky, which describes a convection process in rotating fluid, is considered. A system of differential equations for this model is a generalization of Lorenz system. For the Lyapunov dimension of attractor of the model, the upper estimate is obtained.

Keywords: chaos, chaotic attractor, generalized Lorenz system, Lyapunov dimension, Lyapunov function, fluid convection

1. Introduction

In the present paper the simplest three-dimensional system, describing the convection of fluid within an ellipsoidal rotating cavity, is considered. This system, suggested by Glukhovsky and Dolghansky [4], is as follows

$$\begin{cases} \dot{x} = -\sigma x + z + a_0 yz \\ \dot{y} = R - y - xz \\ \dot{z} = -z + xy, \end{cases} \tag{1}$$

where σ, R, a_0 are positive numbers.

Preprint submitted to Elsevier *October 1, 2015*

After the change of variables

$$x \to x, \quad y \to R - \frac{\sigma}{a_0 R + 1} z, \quad z \to \frac{\sigma}{a_0 R + 1} y \tag{2}$$

system (1) takes the form

$$\begin{cases} \dot{x} = -\sigma x + \sigma y - Ayz \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy, \end{cases} \tag{3}$$

where

$$b = 1, \quad A = \frac{a_0 \sigma^2}{(a_0 R + 1)^2}, \quad r = \frac{R}{\sigma}(a_0 R + 1). \tag{4}$$

This system is a generalization of a classical Lorenz system [20].

System (3) with the parameters r, σ, $b > 0$ is mentioned first in [23] and after the corresponding change of variables [14] can be transformed to the Rabinovich system of waves interaction in plasma [22, 6].

Consider system (3) under the assumption that r, σ, b, A are positive. In this case by [14] one obtains the following: if $r < 1$, then (3) has a unique equilibrium $S_0 = (0, 0, 0)$ (the trivial case). If $r > 1$, then (3) has three equilibria: $S_0 = (0, 0, 0)$ and $S_{1,2} = (\pm x_1, \pm y_1, z_1)$, where

$$x_1 = \frac{\sigma b \sqrt{\xi}}{\sigma b + A\xi}, \quad y_1 = \sqrt{\xi}, \quad z_1 = \frac{\sigma \xi}{\sigma b + A\xi},$$

and the real number ξ is defined as

$$\xi = \frac{\sigma b}{2A^2}\left[A(r-2) - \sigma + \sqrt{(Ar - \sigma)^2 + 4A\sigma}\right].$$

2. Lyapunov functions in the dimension theory

Consider a differential equation

$$\frac{d\mathrm{x}}{dt} = f(\mathrm{x}), \quad \mathrm{x} \in \mathbb{R}^n \tag{5}$$

with a continuously differentiable vector function $f(\mathrm{x})$. Assume that for any initial value x_0 there exists a unique solution of (5) $\mathrm{x}(t, \mathrm{x}_0)$, defined for

$t \in [0, +\infty)$. Here $\mathrm{x}(0, \mathrm{x}_0) = \mathrm{x}_0$. Let K be an invariant set, i.e. $\mathrm{x}(t, K) = K$ for all $t \in [0, +\infty)$.

Suppose $J(\mathrm{x})$ is the Jacobian matrix of $f(\mathrm{x})$

$$J(\mathrm{x}) = \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}}.$$

Consider a nonsingular $(n \times n)$-matrix S. Suppose that $\lambda_1(\mathrm{x}, S) \geqslant \cdots \geqslant \lambda_n(\mathrm{x}, S)$ are eigenvalues of the matrix

$$\frac{1}{2}\left(SJ(\mathrm{x})S^{-1} + (SJ(\mathrm{x})S^{-1})^*\right). \qquad (6)$$

Here * denotes a matrix transposition.

Theorem 1. [8, 13] *Given an integer $j \in [1, n]$ and $s \in [0, 1)$, there is a continuously differentiable scalar function $\vartheta : \mathbb{R}^n \to \mathbb{R}$ and a nonsingular matrix S such that*

$$\lambda_1(\mathrm{x}, S) + \cdots + \lambda_j(\mathrm{x}, S) + s\lambda_{j+1}(\mathrm{x}, S) + \dot{\vartheta}(\mathrm{x}) < 0, \ \forall \mathrm{x} \in \mathbb{R}^n. \qquad (7)$$

In this case the Lyapunov dimension of the compact set K is estimated as follows

$$\dim_L K \leqslant j + s.$$

Here $\dot{\vartheta}$ is a derivative of the function ϑ with respect to the vector field f:

$$\dot{\vartheta}(\mathrm{x}) = (grad(\vartheta))^* f(\mathrm{x}).$$

Theorem 2. [12, 13] *Suppose that there is a continuously differentiable scalar function ϑ and a nonsingular matrix S such that*

$$\lambda_1(\mathrm{x}, S) + \lambda_2(\mathrm{x}, S) + \dot{\vartheta}(\mathrm{x}) < 0, \ \forall \mathrm{x} \in \mathbb{R}^n. \qquad (8)$$

Then any solution of system (5)*, bounded on* $[0, +\infty)$*, tends to an equilibrium as* $t \to +\infty$.

Thus, if condition (8) holds, then an attractor of system (5) coincides with its stationary set.

3. Estimation of Lyapunov dimension

In [14] it is proved that system (3) is dissipative, i.e. it possesses a bounded absorbing set and, thus, has an attractor.

By Theorems 1 and 2 it can be formulated the assertion, concerning the Lyapunov dimension of an attractor (denote it by K) of system (3). This assertion is a generalization of the result obtained in [11] for $b = 1$.

Theorem 3. *Suppose that either the inequality $b < 1$ or the inequalities $b \geqslant 1$, $\sigma > b$ are valid.*

If

$$\left(r + \frac{\sigma}{A}\right)^2 < \frac{(b+1)(b+\sigma)}{A}, \tag{9}$$

then any solution of system (3)*, bounded on* $[0, +\infty)$*, tends to an equilibrium as* $t \to +\infty$*.*

If

$$\left(r + \frac{\sigma}{A}\right)^2 > \frac{(b+1)(b+\sigma)}{A}, \tag{10}$$

then

$$\dim_L K \leqslant 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + A\left(\frac{\sigma}{A} + r\right)^2}}. \tag{11}$$

Sketch of the proof. We use the matrix

$$S = \begin{pmatrix} -A^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the eigenvalues of the corresponding matrix (6) are the following

$$\lambda_2 = -b,$$

$$\lambda_{1,3} = -\frac{\sigma + 1}{2} \pm \frac{1}{2}\left[(\sigma - 1)^2 + A\left(2z - \frac{\sigma + Ar}{A}\right)^2\right]^{\frac{1}{2}}.$$

The goal is to obtain a Lyapunov-like function that allows one to check property (7) of Theorem 1 and property (8) of Theorem 2.

For considered eigenvalues one can obtain the following relation

$$2(\lambda_1 + \lambda_2 + s\lambda_3) \leqslant -(\sigma + 1 + 2b) - s(\sigma + 1) + (1 - s)\left[(\sigma - 1)^2 + A\left(\frac{\sigma}{A} + r\right)^2\right]^{\frac{1}{2}} + $$
$$+ \frac{2(1 - s)}{\left[(\sigma - 1)^2 + A\left(\frac{\sigma}{A} + r\right)^2\right]^{\frac{1}{2}}}\,\omega(x, y, z),$$

where

$$\omega(x, y, z) = -(\sigma + Ar)z + Az^2.$$

Using the following Lyapunov-like function

$$\vartheta(x, y, z) = \frac{2(1 - s)V(x, y, z)}{\left[(\sigma - 1)^2 + A\left(\frac{\sigma}{A} + r\right)^2\right]^{\frac{1}{2}}},$$

where

$$V(x, y, z) = \frac{\gamma}{\sigma}x^2 + \gamma y^2 + \gamma\left(1 + \frac{A}{\sigma}\right)z^2 - 2\gamma(r - 1)z, \qquad \gamma = \frac{\sigma + Ar}{2b(r - 1)},$$

one obtains that

$$\dot{V}(x, y, z) + \omega(x, y, z) < 0, \qquad \forall\, x, y, z \in K.$$

Therefore, if condition (10) is satisfied and

$$s > \frac{-(\sigma + 1 + 2b) + \sqrt{(\sigma - 1)^2 + A\left(\frac{\sigma}{A} + r\right)^2}}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + A\left(\frac{\sigma}{A} + r\right)^2}},$$

then for system (3) with the given S and ϑ Theorem 1 gives estimation (11). If condition (9) is valid and $s = 0$, then the conditions of Theorem 2 are satisfied and any solution bounded on $[0, +\infty)$ tends to an equilibrium as $t \to +\infty$. □

For system (3) with $b = 1$ we can obtain another upper estimation [18].

Theorem 4. *Let $b = 1$, $r > 2$ and the following relations hold*

$$\begin{cases} \sigma > \frac{-3+2\sqrt{3}}{3}\,Ar, & \text{if} \quad 2 < r \leq 4 \\ \sigma \in \left(\frac{-3+2\sqrt{3}}{3}\,Ar,\ \frac{3r+2\sqrt{r(2r+1)}}{r-4}\,Ar\right), & \text{if} \quad r > 4. \end{cases} \tag{12}$$

Then

$$\dim_L K \le 3 - \frac{2(\sigma+2)}{\sigma+1+\sqrt{(\sigma-1)^2+4\sigma r}}. \qquad (13)$$

Sketch of the proof. We use the same idea but choose the following matrix

$$S = \begin{pmatrix} -\sqrt{\frac{r}{\sigma}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the following Lyapunov-like function

$$\vartheta(x,y,z) = \frac{2(1-s)V(x,y,z)}{\sqrt{(\sigma-1)^2+4\sigma r}},$$

where

$$V(x,y,z) = \gamma_1 x^2 + \gamma_2 y^2 + (A\gamma_1+\gamma_2)\,z^2 - (\sigma+Ar)z.$$

Further it is shown that under conditions of the theorem it is always possible to choose the parameters γ_1, γ_2 such that

$$\dot V(x,y,z) + \omega(x,y,z) < 0, \qquad \forall\, x,y,z \in K,$$

where

$$\omega(x,y,z) = -(\sigma+Ar)z + \frac{(\sigma+Ar)^2}{4\sigma r}z^2 + \frac{(\sigma-Ar)^2}{4\sigma r}y^2.$$

Hence, for the chosen S and ϑ the main inequality of Theorem 1

$$2(\lambda_1+\lambda_2+s\lambda_3+\dot\vartheta) < -(\sigma+3)-s(\sigma+1)+(1-s)\sqrt{(\sigma-1)^2+4\sigma r}+\frac{2(1-s)\left(\omega+\dot V\right)}{\sqrt{(\sigma-1)^2+4\sigma r}} < 0.$$

is satisfied and this completes the proof. □

Corollary 1. *If*

1. $\sigma = Ar$, $4\sigma r > (b+1)(b+\sigma)$
   or
2. $b = 1$, $r > 2$ *and*

$$\begin{cases} \sigma > \frac{-3+2\sqrt{3}}{3}Ar, & \textit{if} \quad 2 < r \le 4, \\ \sigma \in \left(\frac{-3+2\sqrt{3}}{3}Ar,\ \frac{3r+2\sqrt{r(2r+1)}}{r-4}Ar\right), & \textit{if} \quad r > 4, \end{cases}$$

then the Lyapunov dimension of the zero equilibrium of system (3) *with* $b = 1$ *or* $\sigma = Ar$ *coincides with* (13). *Thus for* $K \supset (0, 0, 0)$ *we have*

$$\dim_L K = 3 - \frac{2(\sigma+2)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4\sigma r}},$$

Note that this exact formula coincides with the formula for the classical Lorenz system [13].

4. Convection of the rotating fluid

In this section the above results, concerning generalization of Lorenz system (3), are applied to system (1), describing the convection of fluid within an ellipsoidal rotating cavity. Nonsingular linear transformation (2), obviously, does not change the Lyapunov dimension. In general case, it is known (see, e.g. [7, 9]) the invariance of the Lyapunov dimension under a diffeomorphism.

It is known [4] that for system (1) if $R < h$, then there exists one equilibrium $S_0 = (0,\, R,\, 0)$. It is stable if $0 < R < h$ or $a_0 R^2 + R < \sigma$. If $R > h$, then there exist two additional equilibria $S_{1,2} = \left(\pm\sqrt{\frac{R-h}{h}},\, h,\, \pm\sqrt{(R-h)h}\right)$ and they are stable if $a_0 \geqslant \gamma$ or $a_0 < \gamma$, $h < R < R_0$. Here

$$h = \frac{\sqrt{1 + 4a_0\sigma} - 1}{2a}, \quad R_0 = \frac{\sigma h(h+4)}{\sigma - 2 - \sigma^2 + \sigma h}, \quad \gamma = \frac{\sigma(\sigma-2)}{(\sigma^2 - \sigma + 2)^2}.$$

In [4] by means of numerical simulations for the case when parameter $\sigma = 4$ it was found certain values of the parameters a_0 and R for which in system (1) it is observed either a periodic regime (i.e. there exists a limit cycle) or a chaotic regime (and for system (1) there exists a chaotic attractor).

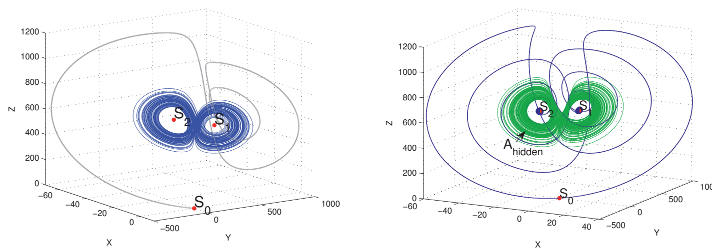
In [10, 11] for parameters

$$b = 1, \quad \sigma = 4, \quad A = 0.0052, \quad r = 687.5 \tag{14}$$

in the phase space of system (3) there was obtained a chaotic self-excited attractor[1] and for parameters

$$b = 1, \quad \sigma = 4, \quad A = 0.0052, \quad r = 700 \tag{15}$$

there was localized numerically a chaotic hidden attractor (Fig. 1). Using linear transformation (2) and relation (4), one can obtain corresponding parameters for system (1) such that there exists self-excited or hidden attractor.



(a) A – local self-excited attractor (b) A_{hidden} – local hidden attractor

Figure 1: Self-excited and hidden attractors of system (3).

By numerical methods [11] one can get the following values of Lyapunov dimension for the local self-excited attractor of system (1):

$$\dim_L A = 2.1405$$

and for the local hidden attractor of system (1):

$$\dim_L A_{\text{hidden}} = 2.1322.$$

[1] Recently a concept of self-excited and hidden attractors was suggested [16, 17, 15, 11]: an attractor is called a *self-excited attractor* if its basin of attraction overlaps with neighborhood of an equilibrium, otherwise it is called a *hidden attractor*. For example, hidden attractors are attractors in systems with no equilibria or with only one stable equilibrium (a special case of multistability and coexistence of attractors). While coexisting self-excited attractors can be found using the standard computational procedure, there is no standard way of predicting the existence or coexistence of hidden attractors in a system. Recent examples of hidden attractors can be found in *The European Physical Journal Special Topics: Multistability: Uncovering Hidden Attractors*, 2015 (see [26, 1, 5, 29, 24, 25, 3, 19, 2, 27, 21, 28]).

These values correspond to the values of Lyapunov dimension for the global attractor $K \supset (0,0,0)$ of system (1) obtained in Corollary 1. For (14) (self-excited local attractor) we have

$$\dim_L K = \dim_L S_0 = 2.8908.$$

and for (15) (hidden local attractor) we have

$$\dim_L K = \dim_L S_0 = 2.8917.$$

Acknowledgements

This work was supported by the Russian Science Foundation (14-21-00041)

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by

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Doklady Mathematics, Vol. 93, No. 1, pp. 42–45. doi:10.7868/S0869565216030063