Poincaré duality for open sets in Euclidean spaces

Terhi Moisala

February 23, 2016

Master's Thesis

(Minor Mathematics)

University of Jyväskylä
Department of Mathematics and Statistics

Supervisor: Pekka Pankka

Abstract

In this thesis we prove the Poincaré duality for open sets in Euclidean spaces. We start with a brief introduction to differential geometry and introduce then the de Rham cohomology. The actual proof begins with some auxiliary results. We prove first the Poincaré duality for sets that are diffeomorphic to $\mathbb{R}^n$. We then introduce the Mayer–Vietoris sequence for de Rham cohomology and show that the Poincaré duality holds for unions of open sets with some additional assumptions. Finally we prove the Poincaré duality for an arbitrary open set using the Whitney decomposition. We give also an illustrative example of the Poincaré duality in the punctured plane.

Tiivistelmä

Todistamme tässä työssä Poincarén dualiteetin Euklidisten avaruuksien avoimille joukoille. Annamme lyhyen johdatuksen differentiaaligeometriaan ja määrittelemme de Rham -kohomologian käsitteen. Itse Poincarén dualiteetin todistuksen aloitamme muutamalla aputuloksella. Näytämme ensin, että Poincarén dualiteetti pätee joukoille, jotka ovat diffeomorfisia avaruuteen $\mathbb{R}^n$. Todistamme sitten Poincarén dualiteetin avointen joukkojen yhdisteille erinäisten lisäoletusten vallitessa. Tätä varten esittelemme Mayer–Vietoris jonon de Rham -kohomologialle. Lopulta näytämme Poincarén dualiteetin mielivaltaiselle avoimelle joukolle käytten Whitney-jakoa. Annamme myös havainnollistavan esimerkin Poincarén dualiteetista punkteeratussa tasossa.

Acknowledgements

I would like to thank my supervisor Pekka Pankka for an interesting topic. I thank also my friends for support and motivating conversations. Especially I would like to thank Ville Kivioja for useful comments and advice.

Contents

1 Introduction

Ever since the concept of topology was defined, there have been attempts to classify topological spaces that are equivalent up to a homeomorphism. It was soon realized that finding out the (non-)existence of the homeomorphism is in general a hopeless task. This gave rise to algebraic topology, which assigns to a topological space some algebraic invariants that remain unchanged under a homeomorphism. Often it is easier to find out the nature of these invariants, since there is the whole machinery of algebra in service. Then if some topological invariants assigned to two topological spaces differ, there does not exist a homeomorphism between the spaces. Nowadays the most important concepts in algebraic topology are the homology, homotopy and cohomology groups.

Even though some concepts that are now regarded as a part of algebraic topology were already introduced by Enrico Betti and Bernhard Riemann in 19th century, the father of algebraic topology is considered to be Henri Poincaré (1854-1912). He created the foundation for this field in his paper *Analysis situs* in 1895, where he introduced the concepts of homology and fundamental group. In this paper he also gave the first version of the theorem called *Poincaré duality*, which he formulated in terms of the Betti numbers. The proof given by Poincaré was considered imperfect already at his time, but the content of the theorem appeared to be groundbreaking.

The branch of algebraic topology started to grow fast on the base Henri Poincaré created in the beginning of 20th century. Poincaré duality achieved new formulations as the theoretical background developed. However, it achieved its modern form only after the concept of cohomology was introduced about forty years later. In this work we consider the de Rham Cohomology, which describes the topological properties of a space in terms of differential forms and exterior derivative. Therefore the de Rham cohomology has several refined features compared to homology [1].

Lately the benefits of algebraic topology have been discovered also outside mathematics. About ten years ago it was discovered that some features of condensed matter can be explained by their topological properties. This has opened a new and fast growing field in the research of topological materials. Similar methods have been applied also in the field of cosmology.

In this work we prove the Poincaré duality for open sets in Euclidean spaces. The general statement is given for a connected orientable manifold, but we restrict the discussion to Euclidean spaces in order to keep the conversation brief. The proof is, however, in the general case in principle the same. First we briefly introduce the theoretical framework needed to formulate the statement of Poincaré duality. To be precise, in Section 2 we discuss some basic results in differential geometry. In Section 3 we define the de Rham cohomology and list some of its basic properties. The Poincaré duality is then considered in Section 4.

The actual proof of Poincaré duality consists of four parts. In Sections 5, 6 and 7 we prove the Poincaré duality with some additional assumptions. Finally in Section 8 we collect the results and prove the Poincaré duality for an arbitrary open set in Euclidean space. We conclude this work with an illustrative example of the Poincaré duality in the punctured plane in Section 9. The outline for this work is given by the lecture notes *Introduction to de Rham cohomology* by P. Pankka [5].

2 Preliminaries on differential geometry

In this section we give a brief insight to the most important concepts in differential geometry. The discussion is rather dense, since the goal is to give all the necessary tools for understanding the statement of Poincaré duality, which is the main topic of this work. To keep things simple, all the definitions are given in the Euclidean spaces. This section follows mostly the book *An Introduction to Manifolds* by Loring W. Tu [6, part I].

2.1 Tangent space of $\mathbb{R}^n$

We first introduce the concepts of tangent space and vector fields although they do not appear in the definition nor proof of the Poincaré duality. They offer an intuitive approach to the subject and play a central role when defining differential k-forms, which are the fundamental objects in the de Rham theory.

We first define the space of germs. Two differentiable functions in $\mathbb{R}^n$ have the same directional derivatives at point $p \in \mathbb{R}^n$ if they agree on some neighbourhood U of p. Thus it is convenient to define that, in this case, the functions are equivalent.

Definition 2.1. Let U and V be neighbourhoods of a point $p \in \mathbb{R}^n$ and let $f\colon U \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$ be smooth functions. We say that f and g are *equivalent* if there exists an open set $W \subset U \cap V$ containing p so that $f = g$ on W. We call the equivalence class $[f]_p$ the *germ* of f at p. We denote the (vector) space of all germs at p by C_p^∞.

Below we do not write the brackets but treat the equivalence classes as if they were functions.

Definition 2.2. The *tangent space* $T_p(\mathbb{R}^n)$ of $\mathbb{R}^n$ at a point $p \in \mathbb{R}^n$ is the vector space of all linear mappings $X_p\colon C_p^\infty \to \mathbb{R}$ that satisfy the Leibniz rule

$$X_p(fg) = X_p(f)g + fX_p(g) \quad \text{for all } f, g \in C_p^\infty.$$

The elements $X_p \in T_p(\mathbb{R}^n)$ are called *tangent vectors* at point p. Further, if U is an open subset of $\mathbb{R}^n$ and X is a function that assigns to each point $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$, we call X a *vector field* on U.

We notice that at least the directional derivatives $\partial/\partial x^i|_p$ for $i = 1, \ldots, n$ are elements in $T_p(\mathbb{R}^n)$. The next lemma tells that the tangent vectors of $\mathbb{R}^n$ can be considered as vectors themselves.

Lemma 2.3. *Let $p \in \mathbb{R}^n$. Then the mapping*

$$\phi\colon \mathbb{R}^n \to T_p(\mathbb{R}^n), \quad v \mapsto \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x_i}\right|_p$$

is an isomorphism of vector spaces.

See e.g. [6, Thm 2.3] for a proof. Especially we see that an element e_i of the standard basis $\{e_1, \ldots, e_n\}$ maps to the partial derivative $\partial/\partial x^i$. Hence the set $\{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$ forms a basis for $T_p(\mathbb{R}^n)$ and any tangent vector X_p on U can be expressed as

$$X_p = \sum_{i=1}^{n} a^i(p) \left.\frac{\partial}{\partial x_i}\right|_p, \tag{2.1}$$

where $a^i\colon U \to \mathbb{R}$ are real functions.

2.2 Dual space

Let us denote the space of linear mappings $f\colon U \to V$ between vector spaces U and V by $\mathrm{Hom}(U, V)$. In this work we assume that the vector spaces are over the field $\mathbb{R}$. It is an important fact that if U and V are isomorphic as vector spaces, this property passes down also to the level of linear mappings on U and V. This is formulated in the following lemma that will be useful later on.

Lemma 2.4. *Let U, V and W be vector spaces and let $\varphi\colon U \to V$ be a linear map. Then*

$$\varphi^*\colon \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W), \quad f \mapsto f \circ \varphi,$$

is a linear map. Moreover, if φ is an isomorphism, then also φ^* is an isomorphism.

Proof. Let $f, g \in \mathrm{Hom}(V, W)$ and $\lambda, \mu \in \mathbb{R}$. Then

$$\varphi^*(\lambda f + \mu g) = (\lambda f + \mu g) \circ \varphi = \lambda(f \circ \varphi) + \mu(g \circ \varphi) = \lambda \varphi^*(f) + \mu \varphi^*(g).$$

Suppose then that φ is an isomorphism. We show first that φ^* is injective. Let $f \in \mathrm{Hom}(V, W)$ so that $\varphi^*(f) = 0$. Hence $(f \circ \varphi)(u) = 0$ for all $u \in U$. Since φ is surjective, this implies that $f(v) = 0$ for all $v \in V$. Thus f is the zero map and φ^* is injective.

To show the surjectivity, take $g \in \mathrm{Hom}(U, W)$. Since also φ^{-1} is linear, the map $g \circ \varphi^{-1} \in \mathrm{Hom}(V, W)$. Additionally $\varphi^*(g \circ \varphi^{-1}) = g$, so φ^* is surjective and hence an isomorphism. □

An important special case is the space $\mathrm{Hom}(U, \mathbb{R})$, that is also called the *dual space* of U.

Definition 2.5. Let V be a vector space. Define the dual space V^* of V by

$$V^* = \{f\colon V \to \mathbb{R} : f \text{ is linear}\}.$$

With this notion we can formulate a corollary of Lemma 2.4 as follows.

Corollary 2.6. *If $\varphi\colon U \to V$ is a linear map (an isomorphism) between vector spaces, then also*

$$\varphi^*\colon V^* \to U^*, \quad f \mapsto f \circ \varphi,$$

is a linear map (an isomorphism) between vector spaces.

If φ and φ^* are as in the Corollary 2.6, we say that φ^* is the *dual map* of φ. The next theorem gives the correspondence between the bases of V and V^*.

Theorem 2.7. *Let V be a finite dimensional vector space. Then $V \cong V^*$. Also, if $\{e_1, \ldots, e_n\}$ is a basis of V, then the functions $\{a^1, \ldots, a^n\}$ defined by*

$$a^i(e_j) = \delta^i_j$$

are a basis of V^*.

Proof. Let $n = \dim V$ and $v = \sum_{i=1}^n v^i e_i \in V$. Then for any $f \in V^*$ holds

$$f(v) = \sum_{i=1}^n v^i f(e_i) = \sum_{i=1}^n f(e_i) a^i(v),$$

so $V^* = \mathrm{span}\{a^1, \ldots, a^n\}$. To show the linear independence, assume the coefficients $\lambda_i \in \mathbb{R}$ are such that $\sum_i \lambda_i a^i = 0$. Applying both sides to the vector e_j gives

$$0 = \sum_{i=1}^n \lambda_i a^i(e_j) = \sum_{i=1}^n \lambda_i \delta^i_j = \lambda_j.$$

Hence for each $i = 1, \ldots, n$ we have that $\lambda_i = 0$. Thus the functions $a^i, \ldots, a^n$ are linearly independent.

The vector spaces V and V^* are isomorphic via the linear isomorphism $I\colon V \to V^*$, $I(e_i) = a^i$, $i \in \{1, \ldots, n\}$. $\square$

2.3 Alternating k-linear functions

We next define multilinear algebra to be used to define differential forms.

Definition 2.8. Let V be vector a space and $V^k = V \times \ldots \times V$, the Cartesian product of k copies of V. A function $f\colon V^k \to \mathbb{R}$ is *k-linear* if it is linear in each of its components, that is, for all $(v_1, \ldots, v_k) \in V^k$, holds

$$\begin{aligned} f(v_1, \ldots, v_{j-1}, v_j + aw, v_{j+1}, \ldots, v_k) = f(v_1, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_k) \\ +af(v_1, \ldots, v_{j-1}, w, v_{j+1}, \ldots, v_k) \end{aligned}$$

for all $j \in \{1, \ldots, k\}$, $a \in \mathbb{R}$ and $w \in V_j$.

Definition 2.9. Let $f\colon V^k \to \mathbb{R}$ be a k-linear function and S_k the set of permutations $\{1, \ldots, k\} \to \{1, \ldots, k\}$. Function f is *alternating*, if

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sign}(\sigma) f(v_1, \ldots, v_k)$$

for any permutation $\sigma \in S_k$ and $(v_1, \ldots, v_k) \in V^k$. The space of alternating k-linear functions is denoted by $\operatorname{Alt}^k(V)$.

In the definition above we identify $\operatorname{Alt}^0(V) = \mathbb{R}$. Note also that

$$\operatorname{Alt}^1(V) = \{f\colon V \to \mathbb{R} : f \text{ is linear}\} = V^*.$$

Next we would like to define a product between elements $f \in \operatorname{Alt}^k(V)$ and $g \in \operatorname{Alt}^l(V)$ that preserves the alternating structure. The first try would be the tensor product $\otimes$ defined by

$$(f \otimes g)(v_1, \ldots, v_{k+l}) = f(v_1, \ldots, v_k) g(v_{k+1}, \ldots, v_{k+l}),$$

but now $f \otimes g$ is not necessarily alternating. The correct form is slightly more complicated and is called the *exterior product*.

Definition 2.10. Let $l, k \geq 1$. We call a permutation $\sigma \in S_{k+l}$ a *(k,l)-shuffle* if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \ldots < \sigma(k+l).$$

The space of all (k,l)-shuffles is denoted by $S(k,l)$.

Definition 2.11. Let $f \in \operatorname{Alt}^k(V)$ and $g \in \operatorname{Alt}^l(V)$. The exterior product $f \wedge g \in \operatorname{Alt}^{k+l}(V)$ is defined by

$$\begin{aligned} f \wedge g(v_1, \ldots, v_{k+l}) &= \sum_{\sigma \in S(k,l)} \operatorname{sign}(\sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \\ &= \sum_{\sigma \in S(k,l)} \operatorname{sign}(\sigma)(f \otimes g)(v_{\sigma(1)}, \ldots, v_{\sigma(k+l)}) \end{aligned}$$

where $v_1, \ldots, v_{k+l} \in V$.

We omit the verification of $f \wedge g \in \mathrm{Alt}^{k+l}(V)$ here, see e.g. [6, Prop 3.13]. We state as facts that the wedge product is associative and anticommutative in the following manner.

Lemma 2.12. *Let $k, l, p \in \mathbb{N}$. Let also $f \in \mathrm{Alt}^k(V)$, $g \in \mathrm{Alt}^l(V)$ and $h \in \mathrm{Alt}^p(V)$. Then*

(i) $f \wedge g = (-1)^{kl} g \wedge f$ *and*

(ii) $(f \wedge g) \wedge h = f \wedge (g \wedge h)$.

From (i) it follows immediately that $f \wedge f = -f \wedge f = 0$ for $f \in \mathrm{Alt}^k(V)$ if k is odd. The next theorem gives us the basis for $\mathrm{Alt}^k(V)$ whenever V is finite dimensional. The idea is rather simple: above we noted that $\mathrm{Alt}^1(V) = V^*$ and in the previous subsection we studied carefully the dual basis $\{a^1, \ldots, a^n\}$ in V^*. Also, since $a^i \wedge a^j \in \mathrm{Alt}^2(V)$ by definition, it is a good guess to form a basis for $\mathrm{Alt}^k(V)$ as an exterior product of dual vectors a^i.

Theorem 2.13. *Let V be an n-dimensional vector space with a basis $\{e_1, \ldots, e_n\}$ and let $\{a^1, \ldots, a^n\}$ be the corresponding dual basis. Then*

$$\{a^{\sigma(1)} \wedge \ldots \wedge a^{\sigma(k)} : \sigma \in S(k, n-k)\}$$

is a basis of $\mathrm{Alt}^k(V)$.

We skip the proof since it is similar to the proof of Theorem 2.7 but somewhat technical, see e.g. [4, Thm 2.15] for a proof.

2.4 Cotangent space

The dual of the tangent space of $\mathbb{R}^n$ at point $p \in \mathbb{R}^n$ is called the *cotangent space* of $\mathbb{R}^n$ and we denote it by $T_p^*(\mathbb{R}^n)$. The elements of $T_p^*(\mathbb{R}^n)$ are called *covectors*. Further, if $U \subset \mathbb{R}^n$ is an open set, a 1-*form* is a function that assigns to each point p on U a covector ω_p. So 1-form at point p is a linear function that takes a tangent vector X_p to a real number.

The cotangent space is closely related to the differentials of functions. Indeed, if f is a smooth function on some neighbourhood of $p \in \mathbb{R}^n$, we may construct a *differential* 1-*form* $\mathrm{d}f$ at point p as follows.

Definition 2.14. Let $f \in C_p^\infty$. The *differential* of f at point p is defined as

$$(\mathrm{d}f)_p(X_p) = X_p f$$

for any $X_p \in T_p(\mathbb{R}^n)$.

We observe that the differentials of coordinate functions give a basis of the cotangent space $T_p^*(\mathbb{R}^n)$.

Theorem 2.15. *Let $x^1, \dots, x^n$ be the standard coordinates of $\mathbb{R}^n$. Then at each point $p \in \mathbb{R}^n$ the set $\{\mathrm{d}x_p^1, \dots, \mathrm{d}x_p^n\}$ is the basis of the cotangent space $T_p^*(\mathbb{R}^n)$ dual to the basis $\{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p\}$ for the tangent space $T_p(\mathbb{R}^n)$.*

Proof. By the definition of differential

$$(\mathrm{d}x^i)_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) = \left. \frac{\partial}{\partial x^j} \right|_p x^i = \delta^i_j$$

for each $i, j \in \{1, \dots, n\}$ and $p \in \mathbb{R}^n$. The claim follows by Theorem 2.7. $\square$

The next lemma gives us a representation of the differential $\mathrm{d}f$ in terms of coordinates.

Lemma 2.16. *Let $f \in C^\infty(U)$, where U is an open subset of $\mathbb{R}^n$. Then*

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

Proof. Since $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$ forms a basis for $T_p^*(\mathbb{R}^n)$, we may write $\mathrm{d}f$ at point $p \in \mathbb{R}^n$ as

$$(\mathrm{d}f)_p = \sum_{i=1}^n a_i(p) \mathrm{d}x_p^i,$$

where $a_i \colon U \to \mathbb{R}$ are functions. If we now apply both sides to the vector $\partial/\partial x^j|_p$, we get from the left hand side by the definition of differential

$$(\mathrm{d}f)_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) = \frac{\partial f}{\partial x^j}(p)$$

and from the right hand side

$$\sum_{i=1}^n a_i(p) \mathrm{d}x_p^i \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) = \sum_{i=1}^n a_i(p) \delta^i_j = a_j(p).$$

Hence $a_j = \partial f / \partial x^j$ and the claim follows. $\square$

2.5 Differential k-forms

Definition 2.17. Let $U \subset \mathbb{R}^n$ be an open set. A *k-form* ω on U is a function $U \to \sqcup_{p \in U} \mathrm{Alt}^k(T_p(\mathbb{R}^n))$ so that $\omega(p) \in \mathrm{Alt}^k(T_p(\mathbb{R}^n))$.

So given a k-form on U, if $p \in U$, then $\omega(p) =: \omega_p$ is an alternating mapping that maps k tangent vectors $X_p^1, \ldots, X_p^k$ to a real number. By Theorems 2.15 and 2.13, the basis of $\mathrm{Alt}^k(T_p(\mathbb{R}^n))$ is

$$\{\mathrm{d}x_p^{i_1} \wedge \ldots \wedge \mathrm{d}x_p^{i_k} : 1 \leq i_1 \leq \ldots \leq i_k \leq n\}.$$

We denote an element of the basis with $\mathrm{d}x_p^I$ for short. Hence any k-form ω at point p can be written as

$$\omega_p = \sum_I a_I(p)\mathrm{d}x_p^I, \tag{2.2}$$

where the summation is over $I \in \{(i_1, \ldots, i_k) : 1 \leq i_1 \leq \ldots \leq i_k \leq n\}$ and each a_I is a function $U \to \mathbb{R}$.

Definition 2.18. Let $U \subset \mathbb{R}^n$ be an open set and $\omega = \sum_I a_I \mathrm{d}x^I$ be a k-form on U. The form ω is called a *differential k-form* if all the functions a_I are smooth on U. The vector space of differential k-forms on U is denoted by $\Omega^k(U)$.

We note that Definition 2.17 is consistent with the definition of 1-forms given in Section 2.4: since $\mathrm{Alt}^1(T_p(\mathbb{R}^n)) = T_p^*(\mathbb{R}^n)$, a 1-form assigns to each point p in U an element in $T_p^*(\mathbb{R}^n)$. From now on we consider only the differential forms, and we call them just forms for short.

Another observation is that the space $\Omega^0(U)$ consists of smooth functions $U \to \mathbb{R}$, since we defined $\mathrm{Alt}^0(T_p(\mathbb{R}^n)) = \mathbb{R}$. Hence $\Omega^0(U) = C^\infty(U)$. By definition, for a given point in U every k-form is an alternating mapping. This implies that the exterior product $\wedge$ can be defined for differential forms pointwise.

Definition 2.19. Let $U \subset \mathbb{R}^n$ be an open set. Let also $\omega \in \Omega^k(U)$ and $\tau \in \Omega^l(U)$. Define the map

$$\omega \wedge \tau \colon U \to \bigsqcup_{p \in U} \mathrm{Alt}^{k+l}(T_p(\mathbb{R}^n)), \quad (\omega \wedge \tau)_p = \omega_p \wedge \tau_p,$$

to be the *exterior product* $\omega \wedge \tau \in \Omega^{k+l}(U)$ of ω and τ.

We note that the form $\omega \wedge \tau$ is indeed a differential form. Let $\omega \in \Omega^k(U)$ and $\tau \in \Omega^l(U)$ be forms

$$\omega_p = \sum_I a_I(p)\mathrm{d}x_p^I \quad \text{and} \quad \tau_p = \sum_J b_J(p)\mathrm{d}x_p^J,$$

where a_I and b_J are smooth for each I and J. Then

$$\omega_p \wedge \tau_p = \sum_{I,J} a_I(p)b_J(p)\mathrm{d}x_p^I \wedge \mathrm{d}x_p^J = \sum_K c_K(p)\mathrm{d}x_p^K,$$

where $c_K(p)$ is the sum of all products $a_I(p)b_J(p)$ (upto correct sign) for which multi-index K is a permutation of multi-indices I and J. Thus $p \mapsto C_K(p)$ is a smooth function if all functions $a_I(p)$ and $b_J(p)$ are smooth.

We have now defined the natural product between differential forms. Next we extend the differential of a function to an operator on general k-forms called exterior derivative. As the differential takes smooth functions, *i.e.* 0-forms, to 1-forms, also the exterior derivative lifts the degree of a form.

Definition 2.20. A family of linear operators $\mathrm{d}\colon \Omega^k(U) \to \Omega^{k+1}(U)$ is called the *exterior derivative*, if

(i) $\mathrm{d}f$ is the differential of $f \in C^\infty(U)$.

(ii) for any $\omega = \sum_I a_I \mathrm{d}x^I \in \Omega^k(U)$,

$$\mathrm{d}\omega = \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I.$$

The exterior derivative has the following properties.

Lemma 2.21. *Let $k, l \in \mathbb{N}$ and let $\omega \in \Omega^k(U)$ and $\tau \in \Omega^l(U)$. Then*

(i) $\mathrm{d}(\mathrm{d}\omega) = 0$ *and*

(ii) $\mathrm{d}(\omega \wedge \tau) = \mathrm{d}\omega \wedge \tau + (-1)^k \omega \wedge \mathrm{d}\tau$.

These two identities with the condition (i) from Definition 2.20 actually fully define the exterior derivative: they could have been taken as a definition and the property (ii) in Definition 2.20 would have followed. The proofs are straightforward calculations so they are omitted, see e.g. [6, Prop 4.13].

2.6 Pull-back and integration of forms

If we are given a smooth function $f\colon U' \to U$, where $U' \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^n$ are open sets, we can push forward some geometric objects in U' to objects living in U via f. Similarly, some objects in U' can be pulled back to U. We define first the push-forward of tangent vectors and use it to formulate the pull-back of differential forms.

Definition 2.22. Let $U' \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^n$ be open sets. Let also $f\colon U' \to U$ be a smooth function. The *push-forward* $f_*\colon T_p(U') \to T_{f(p)}(U)$ defined by f is

$$(f_*(X_p))(u) = X_p(u \circ f)$$

for $X_p \in T_p(U')$ and $u \in C^\infty_{f(p)}$.

Definition 2.23. Let $U' \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^n$ be open sets, $f\colon U' \to U$ a smooth function. The *pull-back* $f^*\colon \Omega^k(U) \to \Omega^k(U')$ defined by f is

$$(f^*\omega)_p(X^1_p, \dots, X^k_p) = \omega_{f(p)}(f_*X^1_p, \dots, f_*X^k_p)$$

for $\omega \in \Omega^k(U)$, $p \in U$ and $X^i_p \in T_p(U')$, $i = 1, \dots, k$.

Especially, for a 0-form $u \in \Omega^0(U) = C^\infty(U)$, we have $f^*(u) = u \circ f$. We state as a lemma some useful relations, see [6, Prop 18.7 and Thm 19.8] for proofs.

Lemma 2.24. *Let $U' \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^n$ be open sets and $f\colon U' \to U$ a smooth function. Then, for $\omega \in \Omega^k(U)$ and $\tau \in \Omega^l(U)$, holds*

(i) $f^*(\omega \wedge \tau) = (f^*\omega) \wedge (f^*\tau)$ *and*

(ii) $\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega)$.

The last operation on differential forms that we are about to need is the integration. In $\mathbb{R}^n$ the definition is simple. Since the basis of $\mathrm{Alt}^n(T_p(\mathbb{R}^n))$ consists of one element $(\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n)_p$, the space $\Omega^n(U)$ is isomorphic to $C^\infty(U)$. Hence the integration of an n-form boils down to an ordinary Lebesgue integral.

Definition 2.25. Let $U \subset \mathbb{R}^n$ be an open set and let $\omega = f\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \in \Omega^n(U)$ for some $f \in C^\infty(U)$. Then the integral of ω over the set U is

$$\int_U \omega = \int_U f \,\mathrm{d}m_n,$$

if the integral on the right hand side exists. Above m_n is the n-dimensional Lebesgue measure.

Sometimes it is convenient to restrict the discussion to the n-forms for which the integral over an open $U \subset \mathbb{R}^n$ is finite. This is the case for compactly supported n-forms. The compactly supported 0-forms are just compactly supported smooth functions $C_0^\infty(U)$; the general case is analogous.

Definition 2.26. The space of compactly supported differential k-forms $\Omega_c^k(U)$ is defined as

$$\Omega_c^k(U) = \{\omega \in \Omega^k(U) : \mathrm{spt}(\omega) \text{ is compact}, \mathrm{spt}(\omega) \subset U\},$$

where $\mathrm{spt}(\omega) = \mathrm{cl}\{p \in U : \omega_p \neq 0\}$.

In $\mathbb{R}^n$ this can be formulated as $\omega \in \Omega_c^k(U)$ if it is zero outside a bounded set contained in U. Clearly, if we write the exterior derivative for compactly supported forms as $\mathrm{d}\colon \Omega_c^k(U) \to \Omega_c^{k+1}(U)$, it is well-defined. Similarly, if $f\colon U' \to U$ is a smooth function, we may write the pull-back as $f^*\colon \Omega_c^k(U) \to \Omega_c^k(U')$.

We next formulate two useful lemmas for compactly supported $(n-1)$-forms, see e.g. [4, Lemma 10.15] for proofs.

Lemma 2.27. *Let $\tau \in \Omega_c^{n-1}(\mathbb{R}^n)$. Then $\int_{\mathbb{R}^n} \mathrm{d}\tau = 0$.*

The assumption of the compact support is crucial in the above lemma. The proof takes advantage of Fubini's theorem and the fundamental theorem of calculus. Indeed, if $\tau = \sum_{i=1}^n g_i \mathrm{d}x^1 \wedge \ldots \wedge \widehat{\mathrm{d}x^i} \wedge \ldots \mathrm{d}x^n$, where $\hat{}$ denotes the missing index, the functions we are about to integrate are just partial derivatives of smooth functions g_i. By Fubini's theorem it suffices to consider one dimensional integrals and the fundamental theorem of calculus leaves us with some vanishing boundary terms. Also the opposite result holds:

Theorem 2.28. *Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \omega = 0$. Then there exists $\tau \in \Omega_c^{n-1}(\mathbb{R}^n)$ for which $\mathrm{d}\tau = \omega$.*

3 De Rham cohomology

In the de Rham theory we are interested in k-forms that behave in a special way under the exterior derivative d. The crucial objects are the forms that are mapped to zero by d.

Definition 3.1. Let $U \subset \mathbb{R}^n$ be an open set. A form $\omega \in \Omega^k(U)$ is *closed* if $\mathrm{d}\omega = 0$ and it is *exact* if there exists a form $\tau \in \Omega^{k-1}(U)$ for which $\omega = \mathrm{d}\tau$.

In the previous section we noted that for any open set $U \subset \mathbb{R}^n$ the mapping $\mathrm{d} \circ \mathrm{d} \colon \Omega^{k-1}(U) \to \Omega^{k+1}(U)$ is a zero map if $k \geq 1$. Thus

$$\operatorname{Im}\left(\mathrm{d} \colon \Omega^{k-1}(U) \to \Omega^k(U)\right) \subset \operatorname{Ker}\left(\mathrm{d} \colon \Omega^k(U) \to \Omega^{k+1}(U)\right)$$

as a vector subspace. If we denote $\Omega^k(U) = \{0\}$ and $\mathrm{d} \colon \Omega^k(U) \to \Omega^{k+1}(U)$, $\mathrm{d} = 0$ for all $k < 0$, the above relation extends to all $k \in \mathbb{Z}$. Clearly the space $\operatorname{Ker}(\mathrm{d})$ is exactly the space of closed forms and $\operatorname{Im}(\mathrm{d})$ corresponds to the exact forms. Hence, *every exact form is also closed.* In de Rham theory we identify all the closed forms that differ by an exact form.

Definition 3.2. Let $U \subset \mathbb{R}^n$ be an open set. The quotient vector space

$$H^k(U) = \frac{\operatorname{Ker}(\mathrm{d} \colon \Omega^k(U) \to \Omega^{k+1}(U))}{\operatorname{Im}(\mathrm{d} \colon \Omega^{k-1}(U) \to \Omega^k(U))} = \frac{\{\text{closed } k\text{-forms in } U\}}{\{\text{exact } k\text{-forms in } U\}}$$

is the *kth de Rham cohomology group* of U.

Thus the elements $[\omega] \in H^k(U)$ are equivalence classes

$$[\omega] = \{\omega + \mathrm{d}\tau \in \Omega^k(U) \colon \tau \in \Omega^{k-1}(U)\}.$$

We can further define the *kth compactly supported cohomology group* $H^k_c(U)$ of U by requiring that all the forms in the definition of $H^k(U)$ are compactly supported; formally

$$H^k_c(U) = \frac{\operatorname{Ker}(\mathrm{d} \colon \Omega^k_c(U) \to \Omega^{k+1}_c(U))}{\operatorname{Im}(\mathrm{d} \colon \Omega^{k-1}_c(U) \to \Omega^k_c(U))}.$$

For an example and future purposes, we compute the zeroth cohomology class of a connected set.

Lemma 3.3. *Let $U \subset \mathbb{R}^n$ be a connected set. Then $H^0(U) \cong \mathbb{R}$.*

Proof. By definition

$$H^0(U) = \frac{\mathrm{Ker}(\mathrm{d}\colon \Omega^0(U) \to \Omega^1(U))}{\mathrm{Im}(\mathrm{d}\colon \Omega^{-1}(U) \to \Omega^0(U))}.$$

Since $\mathrm{d}\colon \Omega^{-1}(U) \to \Omega^0(U))$ is the zero map, we get simply

$$H^0(U) \cong \mathrm{Ker}(\mathrm{d}\colon \Omega^0(U) \to \Omega^1(U)).$$

Recalling that $\Omega^0(U)$ is the space of smooth functions on U, we have that

$$H^0(U) \cong \{f \in C^\infty(U) : \mathrm{d}f = 0\}.$$

Now the differential of a function f vanishes exactly when all the partial derivatives $\partial f/\partial x^i$ are identically zero for each $i \in \{1, \dots, n\}$. Hence f is constant in each connected component of U. Since U is connected, $H^0(U)$ is just the space of constant functions. Thus

$$H^0(U) \cong \mathbb{R}.$$

$\square$

Some operations on k-forms can be generalized to operate on the equivalence classes in $H^k(U)$. The most important ones are the pull-back and exterior product of the equivalence classes.

Lemma 3.4. *Let $U \subset \mathbb{R}^n$ and $U' \subset \mathbb{R}^m$ be open sets and let $f\colon U' \to U$ be a smooth function. The function $f^*\colon H^k(U) \to H^k(U')$,*

$$[\omega] \mapsto [f^*\omega],$$

is well-defined and linear.

See e.g. [6, Lemma 23.7] for a proof. The mapping $f^*\colon H^k(U') \to H^k(U)$ is the *pull-back* of f.

Lemma 3.5. *Let $U \subset \mathbb{R}^n$ be an open set. The mapping $\wedge\colon H^k(U) \times H^l(U) \to H^{k+l}(U)$,*

$$([\omega], [\tau]) \mapsto [\omega \wedge \tau],$$

is well-defined and bilinear.

See e.g. [6, p. 241] for a proof. The next result is known, in its general form for connected orientable manifolds, as the *de Rham's theorem.*

Lemma 3.6. *The map*

$$\int_{\mathbb{R}^n} : H_c^n(\mathbb{R}^n) \to \mathbb{R}, \quad [\omega] \mapsto \int_{\mathbb{R}^n} \omega,$$

is well-defined linear isomorphism.

Proof. We show first that the map is well-defined. Let $\omega, \omega' \in [\omega]$. Then there exists $\tau \in \Omega_c^{n-1}(\mathbb{R}^n)$ so that $\omega = \omega' + \mathrm{d}\tau$. By Lemma 2.27 and the linearity of integral, we have $\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} \omega'$. Hence the map is well-defined and linear.

To show injectivity, suppose $\int_{\mathbb{R}^n} \omega = 0$ for some $\omega \in \Omega_c^n(\mathbb{R}^n)$. By Theorem 2.28 ω is exact, *i.e.*, $[\omega] = 0$. We show then that $\int_{\mathbb{R}^n}$ is surjective. Let $a \in \mathbb{R}$ and choose any $f \in C_0^\infty(\mathbb{R}^n)$ so that $c := \int_{\mathbb{R}^n} f \mathrm{d}m_n \neq 0$. Then for an n-form

$$\omega = (a/c) f \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n,$$

we have $\int_{\mathbb{R}^n} \omega = a$. Since every n-form is closed and ω has compact support, $[\omega] \in H_c^n(\mathbb{R}^n)$ and the map $\int_{\mathbb{R}^n}$ is an isomorphism. □

Finally we introduce the *push-forward* of a compactly supported k-form via inclusion.

Definition 3.7. Let U and V be open sets in $\mathbb{R}^n$ such that $U \subset V$ and let $\iota: U \to V$ be the inclusion. The push-forward $\iota_*: \Omega_c^k(U) \to \Omega_c^k(V)$ is the map $\omega \mapsto \iota_*\omega$, where

$$(\iota_*\omega)_p = \begin{cases} \omega_p, & p \in U, \\ 0, & p \notin U. \end{cases}$$

Also the push-forward can be generalized to the cohomological level.

Lemma 3.8. *The map* $\iota_*: H_c^k(U) \to H_c^k(V)$, $[\omega] \mapsto [\iota_*\omega]$, *is well-defined and linear.*

Proof. Since $\mathrm{spt}(\omega) = \mathrm{spt}(\iota_*\omega)$, we have that $[\iota_*\omega] \in H_c^k(V)$. Let $\omega, \omega' \in [\omega]$. So there exists $\tau \in \Omega_c^k(U)$ so that $\omega' = \omega + \mathrm{d}\tau$. Then

$$[\iota_*\omega'] = [\iota_*\omega + \mathrm{d}(\iota_*\tau)] = [\iota_*\omega],$$

since $\iota_*(\mathrm{d}\tau) = \mathrm{d}(\iota_*\tau)$. Hence the map is well-defined. For linearity, let $[\omega_1], [\omega_2] \in H_c^k(U)$ and $a \in \mathbb{R}$. Then

$$\begin{aligned} \iota_*([\omega_1] + a[\omega_2]) &= \iota_*[\omega_1 + a\omega_2] = [\iota_*(\omega_1 + a\omega_2)] = [\iota_*\omega_1 + a\iota_*\omega_2] \\ &= [\iota_*\omega_1] + a[\iota_*\omega_2] = \iota_*[\omega_1] + a\iota_*[\omega_2]. \end{aligned}$$

□

Together with Lemma 3.6, we get that we may define an integral of a cohomology of any open set in $\mathbb{R}^n$.

Lemma 3.9. *Let $U \subset \mathbb{R}^n$ be an open set. The map*

$$\int_U : H_c^n(U) \to \mathbb{R}, \quad [\omega] \mapsto \int_U \omega,$$

is well-defined and linear.

Proof. We argued above that the map $\iota_* : H_c^k(U) \to H_c^k(\mathbb{R}^n)$, $[\omega] \mapsto [\iota_*\omega]$, is well-defined. By Lemma 3.6 also $\int_{\mathbb{R}^n} : H_c^n(\mathbb{R}^n) \to \mathbb{R}$ is well-defined. Since

$$\int_U \omega = \int_{\mathbb{R}^n} \iota_*\omega$$

for each $\omega \in \Omega_c^n(U)$, the map $\int_U$ is a composition of two well-defined mappings $\int_{\mathbb{R}^n}$ and ι_*. Linearity follows immediately from linearity of the Lebesgue integral and ι_*. $\square$

4 Poincaré duality

We finally have all the necessary tools to formulate the statement of Poincaré duality in Euclidean spaces.

Theorem 4.1 (Poincaré duality)**.** *Let $U \subset \mathbb{R}^n$ be an open set. Then the map*

$$D_U \colon H^k(U) \to H_c^{n-k}(U)^*, \quad D_U([\xi])[\zeta] = \int_U \xi \wedge \zeta,$$

is an isomorphism.

We first ensure that the mapping D_U is well-defined. Let $\xi \in \Omega^k(U)$ and $\zeta \in \Omega_c^l(U)$. Then $\xi \wedge \zeta \in \Omega_c^{k+l}(U)$, since $\operatorname{spt}(\xi \wedge \zeta) \subset \operatorname{spt}(\xi) \cap \operatorname{spt}(\zeta)$. Hence, by Lemma 3.5, the map

$$\wedge \colon H^k(U) \times H_c^{n-k}(U) \to H_c^n(U), \quad ([\xi], [\zeta]) \mapsto [\xi \wedge \zeta],$$

is well-defined. Lemma 3.9 gives us that also the map

$$\int_U \colon H_c^n(U) \to \mathbb{R}, \quad [\omega] \mapsto \int_U \omega,$$

is well-defined. Then also the composed mapping

$$F \colon H^k(U) \times H_c^{n-k}(U) \to \mathbb{R}, \quad ([\xi], [\zeta]) \mapsto \int_U \xi \wedge \zeta,$$

is well-defined. Since $D_U([\xi])[\zeta] = F([\xi], [\zeta])$ for all $[\xi] \in H^k(U)$ and $[\zeta] \in H_c^{n-k}(U)$, also the map D_U is well-defined.

By Theorem 4.1, the map D_U is an injection, which implies that for every non-zero $[\xi] \in H^k(U)$ there is $[\zeta] \in H_c^{n-k}(U)$ so that $D_U([\xi])[\zeta] \neq 0$. Also the contrary holds: if $[\zeta] \in H_c^{n-k}(U)$ is non-zero, then there exists a linear map $\phi \colon H_c^{n-k}(U) \to \mathbb{R}$ for which $\phi([\zeta]) \neq 0$. Then, by surjectivity of D_U, there exists a form $[\xi] \in H^k(U)$ so that $D_U([\xi]) = \phi$ and $D_U([\xi])[\zeta] \neq 0$.

The proof of Poincaré duality consists of four parts. First, in Section 5, we prove the Poincaré duality for open subsets of $\mathbb{R}^n$ that are diffeomorphic to $\mathbb{R}^n$. In Section 6 we show that if the Poincaré duality is true for open sets U and V and it holds also for $U \cap V$, then it holds for the union $U \cup V$. After that, in Section 7, we prove some linear algebraic results for de Rham cohomology groups. We show that if pairwise disjoint open sets satisfy Poincaré duality, it holds also for their union. Using these three results we can prove the Poincaré duality for an arbitrary open subset U of $\mathbb{R}^n$. This proof covers Section 8.

5 Poincaré duality for $\mathbb{R}^n$

In this section we show the Poincaré duality for those sets in $\mathbb{R}^n$ that are diffeomorphic to $\mathbb{R}^n$. Let us first consider the space $H^k(U)$ when the open set $U \subset \mathbb{R}^n$ is *star-like*. The set U is star-like if there exists a point $x_0 \in U$ so that for every $x \in U$ the line segment between x_0 and x is contained in U. The point x_0 is called a *center* of U.

Theorem 5.1 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^n$ be an open star-like set. Then*

$$H^k(U) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0 \\ \{0\}, & \text{if } k > 0. \end{cases}$$

Proof. Since every star-like set is connected, the case $k = 0$ follows from Lemma 3.3. Suppose that $k > 0$. We need to show that

$$\mathrm{Ker}(\mathrm{d}\colon \Omega^k(U) \to \Omega^{k+1}(U)) = \mathrm{Im}(\mathrm{d}\colon \Omega^{k-1}(U) \to \Omega^k(U)).$$

In other words, if $\omega \in \Omega^k(U)$ is closed, we need to find $\eta \in \Omega^{k-1}(U)$ such that $\mathrm{d}\eta = \omega$. This follows immediately when we show that there exists a linear operator $S_k\colon \Omega^k(U) \to \Omega^{k-1}(U)$ for which

$$\omega = \mathrm{d}(S_k\omega) + S_{k+1}(\mathrm{d}\omega)$$

for each $\omega \in \Omega^k(U)$. Indeed, if also $\mathrm{d}\omega = 0$, we get $\omega = \mathrm{d}(S_k\omega)$.

We use next the fact that U is star-like. Denote by x_0 the center of U. Let $F\colon U \times \mathbb{R} \to U$ be a smooth map defined by

$$F(x,t) = x_0 + \lambda(t)(x - x_0),$$

where $\lambda(t)$ is a smooth function such that $\lambda(t) = 0$ for $t \le 0$, $0 \le \lambda(t) \le 1$ for $0 \le t \le 1$ and $\lambda(t) = 1$ for $t \ge 1$. This kind of function λ exists: choose for example the integral function of the extension by zero of $\exp(-1/(1-x^2))$ defined on $(-1,1)$. Since U is star-like, we have that $F(x,t) \in U$ for each $x \in U$ and $t \in \mathbb{R}$. Additionally $F(x,0) = x_0$ and $F(x,1) = x$ for any $x \in U$.

Now the pull-back $F^*\colon \Omega^k(U) \to \Omega^k(U \times \mathbb{R})$ takes a k-form on U to a k-form on $U \times \mathbb{R}$. Below we define an operator $\hat{S}_k\colon \Omega^k(U \times \mathbb{R}) \to \Omega^{k-1}(U)$ and verify that $S_k \coloneqq \hat{S}_k \circ F^*$ satisfies the required properties. We observe that with this definition

$$\begin{aligned} \mathrm{d}(S_k\omega) + S_{k+1}(\mathrm{d}\omega) &= \mathrm{d} \circ (\hat{S}_k \circ F^*) + (\hat{S}_{k+1} \circ F^*) \circ \mathrm{d} \\ &= (\mathrm{d} \circ \hat{S}_k) \circ F^* + \hat{S}_{k+1} \circ \mathrm{d} \circ F^* \\ &= (\mathrm{d}\hat{S}_k + \hat{S}_{k+1}\mathrm{d}) \circ F^* \end{aligned}$$

by Lemma 2.24.

Now each $\eta \in \Omega^k(U \times \mathbb{R})$ has a unique representation in the (standard) basis of $\Omega^k(U \times \mathbb{R})$ as

$$\eta = \sum_I a_I \mathrm{d}x^I + \sum_J b_J \mathrm{d}t \wedge \mathrm{d}x^J, \tag{5.1}$$

where $a_I, b_J \in C^\infty(U \times \mathbb{R})$ and the sums are over $I \in \{(i_1, \dots, i_k) : 1 \leq i_1 \leq \dots \leq i_k \leq n\}$ and $J = \{(j_1, \dots j_{k-1}) : 1 \leq j_1 \leq \dots \leq j_{k-1} \leq n\}$. We define now the map $\hat{S}_k \colon \Omega^k(U \times \mathbb{R}) \to \Omega^{k-1}(U)$ by

$$(\hat{S}_k \eta)_y = \sum_J \left(\int_0^1 b_J(y, s) \mathrm{d}s \right) \mathrm{d}x^J$$

for $\eta \in \Omega^k(U \times \mathbb{R})$ and $y \in U$.

Let $\eta \in \Omega^k(U \times \mathbb{R})$. Then the definition of the exterior derivative and $\mathrm{d}t \wedge \mathrm{d}t = 0$ gives us

$$\begin{aligned}
\mathrm{d}\eta &= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I + \sum_J \mathrm{d}b_J \wedge \mathrm{d}t \wedge \mathrm{d}x^J \\
&= \sum_I \left(\frac{\partial a_I}{\partial t} \mathrm{d}t + \sum_{l=1}^n \frac{\partial a_I}{\partial x^l} \mathrm{d}x^l \right) \wedge \mathrm{d}x^I \\
&+ \sum_J \left(\frac{\partial b_J}{\partial t} \mathrm{d}t + \sum_{l=1}^n \frac{\partial b_J}{\partial x^l} \mathrm{d}x^l \right) \wedge \mathrm{d}t \wedge \mathrm{d}x^I \\
&= \sum_I \frac{\partial a_I}{\partial t} \mathrm{d}t \wedge \mathrm{d}x^I + \sum_{I,l} \frac{\partial a_I}{\partial x^l} \mathrm{d}x^l \wedge \mathrm{d}x^I - \sum_{J,l} \frac{\partial b_J}{\partial x^l} \mathrm{d}t \wedge \mathrm{d}x^l \wedge \mathrm{d}x^I.
\end{aligned}$$

So, for $y \in U$,

$$\hat{S}_{k+1}(\mathrm{d}\eta)_y = \sum_I \left(\int_0^1 \frac{\partial a_I}{\partial t}(y, s) \mathrm{d}s \right) \mathrm{d}x^I - \sum_{J,l} \left(\int_0^1 \frac{\partial b_J}{\partial x^l}(y, s) \mathrm{d}s \right) \mathrm{d}x^l \wedge \mathrm{d}x^J.$$

On the other hand, for $y \in U$,

$$\mathrm{d}(\hat{S}_k \eta)_y = \mathrm{d} \sum_J \left(\int_0^1 b_J(y, s) \mathrm{d}s \right) \mathrm{d}x^J = \sum_{J,l} \left(\int_0^1 \frac{\partial b_J}{\partial x^l}(y, s) \mathrm{d}s \right) \mathrm{d}x^l \wedge \mathrm{d}x^J.$$

Thus

$$(\mathrm{d}\hat{S}_k + \hat{S}_{k+1}\mathrm{d})\eta(y) = \sum_I \left(\int_0^1 \frac{\partial a_I}{\partial t}(y, s) \mathrm{d}s \right) \mathrm{d}x^I = \sum_I (a_I(y, 1) - a_I(y, 0)) \mathrm{d}x^I$$

for every $y \in U$.

Our next goal is to show that $(F^*\omega)_{(x,0)} = 0$ and $(F^*\omega)_{(x,1)} = \omega_x$ for each $\omega \in \Omega^k(U)$ and $x \in U$. Let $X_1, \dots, X_k \in T_{F(x,t)}(U \times \mathbb{R})$. Then

$$X_l = \sum_{i=1}^{n} a_l^i \frac{\partial}{\partial x^i} + a_l^t \frac{\partial}{\partial t}$$

for each $l \in \{1 \dots k\}$, where $a_l^i, a_l^t \in C^\infty(U \times \mathbb{R})$. Hence we have that

$$\begin{aligned}
&(F^*\omega)_{(x,t)}(X_1, \dots, X_k) \\
&\quad = \omega_{F(x,t)} \left(\sum_{i=1}^{n} (X_1)_{(x,t)} F^i \left. \frac{\partial}{\partial x^i} \right|_{F(x,t)}, \dots, \sum_{i=1}^{n} (X_k)_{(x,t)} F^i \left. \frac{\partial}{\partial x^i} \right|_{F(x,t)} \right),
\end{aligned}$$

where each function $(X_l)F^i$ is given by formula

$$\begin{aligned}
(X_l)_{(x,t)} F^i &= (\sum_{j=1}^{n} a_l^j \frac{\partial}{\partial x^j} + a_l^t \frac{\partial}{\partial t})(x_0^i + \lambda(t)(x^i - x_0^i)) \\
&= a_l^i \lambda(t) + a_l^t \lambda'(t)(x^i - x_0^i).
\end{aligned}$$

Since λ is smooth and hence differentiable both at $t = 0$ and $t = 1$, the limit of the difference quotient exists and we may calculate the derivative of λ at $t = 1$ by approaching this value from above. So $\lambda'(1) = 0$, since $\lambda(t)$ is constant at $t \geq 1$. Similarly $\lambda'(0) = 0$. Thus for $t = 0$ and for $t = 1$ we have that

$$(F^*\omega)_{(x,t)}(X_1, \dots, X_k) = \omega_{F(x,t)} \left(\sum_{i=1}^{n} \lambda(t) a_1^i \left. \frac{\partial}{\partial x^i} \right|_{F(x,t)}, \dots, \sum_{i=1}^{n} \lambda(t) a_k^i \left. \frac{\partial}{\partial x^i} \right|_{F(x,t)} \right)$$

for $x \in U$. Since $\lambda(0) = 0$, we have

$$(F^*\omega)_{(x,0)} = 0$$

for each $x \in U$. Also, since $\lambda(1) = 1$ and $F(x,1) = x$, we obtain

$$\begin{aligned}
&(F^*\omega)_{(x,1)} \left(\sum_{i=1}^{n} a_1^i \frac{\partial}{\partial x^i} + a_1^t \frac{\partial}{\partial t}, \dots, \sum_{i=1}^{n} a_k^i \frac{\partial}{\partial x^i} + a_k^t \frac{\partial}{\partial t} \right) \\
&\quad = \omega_x \left(\sum_{i=1}^{n} a_1^i \frac{\partial}{\partial x^i}, \dots, \sum_{i=1}^{n} a_k^i \frac{\partial}{\partial x^i} \right)
\end{aligned}$$

for each $x \in U$. So at the point $(x,1)$ form $F^*\omega \in \Omega^k(U \times \mathbb{R})$ can be considered as ω_x, an element of $\Omega^k(U)$, with the identification given above. Hence if $(F^*\omega)_{(x,1)}$ is written as in (5.1), all functions b_J would equal zero. Thus

$$\left((\mathrm{d}\hat{S}_k + \hat{S}_{k+1}\mathrm{d}) \circ F^*\right)\omega(x) = (F^*\omega)_{(x,1)} - (F^*\omega)_{(x,0)} = \omega_x,$$

which completes the proof. □

In what follows we need the following corollary of the Poincaré lemma.

Corollary 5.2.

$$H^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0 \\ \{0\}, & \text{if } k > 0. \end{cases}$$

Above we were able to define precisely the nature of $H^k(U)$ for a star-like open subset U of $\mathbb{R}^n$. A corresponding result can be obtained also for compactly supported cohomology groups of $\mathbb{R}^n$.

Theorem 5.3.

$$H^k_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{R}, & \text{if } k = n \\ \{0\}, & \text{if } k < n. \end{cases}$$

The proof of this theorem is based on the following fact on cohomology groups, see e.g. [4, Example 9.29]. Note that in the following lemma and forthcoming proof of Theorem 5.3 we rely on some general theory of differential forms and de Rham cohomology on smooth manifolds. As this is the only such case, we refer the interested reader to [6, Parts II and V] and [4, Ch. 9] for details.

Lemma 5.4. *Let $n \geq 1$ and $S^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$ the unit n-sphere. Then*

$$H^k(S^n) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0 \,\text{or}\, n \\ \{0\}, & \text{if } 0 < k < n. \end{cases}$$

Proof of Theorem 5.3. We notice first that the case $k = n$ follows immediately from Lemma 3.6. Let then $k = 0$. Since $\mathbb{R}^n$ is not compact, any closed $f \in C_0^\infty(\mathbb{R}^n)$ must be zero at some point in $\mathbb{R}^n$. As we noticed in the proof of Lemma 3.3, each closed 0-form is a constant function whenever U is connected. Hence every closed 0-form is identically zero and $H^0_c(\mathbb{R}^n) \cong \{0\}$.

Consider now the case $0 < k < n$. We follow here the proof given in [4, Lemma 13.2]. Like in the proof of Poincaré lemma, it suffices to show that for every compactly supported closed k-form ω on $\mathbb{R}^n$ there exists a compactly supported $(k-1)$-form η for which $\mathrm{d}\eta = \omega$. Instead of $\mathbb{R}^n$ we consider the space $S^n \setminus \{p_0\}$, which is diffeomorphic to $\mathbb{R}^n$ via stereographic projection. Hence all the quantities defined via differential geometric objects are identical. In particular $H^k(S^n \setminus \{p_0\}) \cong H^k(\mathbb{R}^n)$ and further $H^k_c(\mathbb{R}^n) \cong H^k_c(S^n \setminus \{p_0\})$. Below we use the notation that forms without a prime are in $\Omega^k_c(S^n \setminus \{p_0\})$ and forms with a prime are in $\Omega^k(S^n)$.

Let $\omega \in \Omega^k_c(S^n \setminus \{p_0\})$ be closed. Then there exists an open neighbourhood $U \subset S^n$ of p_0 so that $\omega|_{U\setminus\{p_0\}} = 0$. Indeed, since $p_0 \notin \operatorname{spt}(\omega)$ and $\operatorname{spt}(\omega)$ is closed, $U = S^n \setminus \operatorname{spt}(\omega)$ is an open neighbourhood of p_0. We define $\omega' \in \Omega^k(S^n)$ so that $\omega'_p = \omega_p$ whenever $p \in S^n \setminus \{p_0\}$ and $\omega'_{p_0} = 0$. Then ω' is well-defined and smooth. Now ω' is also exact by Lemma 5.4. Let $\tau' \in \Omega^{k-1}(S^n)$ be a form for which $\mathrm{d}\tau' = \omega'$. We find next a form $\kappa \in \Omega^{k-1}_c(S^n \setminus \{p_0\})$ so that $\mathrm{d}\kappa = \omega$.

Let first $k = 1$. Then $\tau' \in C^\infty(S^n)$ and τ' is a constant function in U, since $\mathrm{d}\tau'|_U = 0$. Let $a \in \mathbb{R}$ be that constant. Let also $\tau := \tau'|_{S^n\setminus\{p_0\}}$. We then have that

$$\kappa := \tau - a \in \Omega^0_c(S^n \setminus \{p_0\}).$$

Indeed, since $p_0 \notin \operatorname{spt}(\kappa)$, the set $\operatorname{spt}(\kappa)$ is closed as a subset of S^n. So $\operatorname{spt}(\kappa)$ is compact as a closed subset of a compact set. Also, $\mathrm{d}\kappa = \mathrm{d}\tau = \omega$ and hence ω is exact.

Let then $1 < k < n$. We may assume that the neighbourhood U of p_0, $U \subset S^n$, is diffeomorphic to $\mathbb{R}^n$. Then $H^{k-1}(U) \cong H^{k-1}(\mathbb{R}^n) \cong \{0\}$ by Lemma 5.1, since $\mathbb{R}^n$ is a star-like set. Hence $\tau'|_U$ is exact and there exists $\eta' \in \Omega^{k-2}(U)$ so that $\mathrm{d}\eta' = \tau'|_U$. We fix now a smooth function $\varphi \colon S^n \to [0,1]$ such that $\operatorname{spt}(\varphi) \subset U$ and $\varphi|_V = 1$ for some open neighbourhood V of p_0 contained in U. Since U is diffeomorphic to $\mathbb{R}^n$ and such a function clearly exists in $\mathbb{R}^n$, we conclude that such a function also exists in U.

Define $\eta \in \Omega^{k-2}(S^n \setminus \{p_0\})$ so that $\eta_p = \varphi(p)\eta'_p$ for $p \in U \setminus \{p_0\}$ and $\eta_p = 0$ otherwise. Let again $\tau := \tau'|_{S^n\setminus\{p_0\}}$. Then the form

$$\kappa := \tau - \mathrm{d}\eta \in \Omega^{k-1}(S^n \setminus \{p_0\})$$

has a compact support by a similar deduction as in the case $k = 1$, since now $\kappa|_{V\setminus\{p_0\}} = 0$. Also

$$\mathrm{d}\kappa = \mathrm{d}\tau - \mathrm{d}\mathrm{d}\eta = \omega.$$

Hence every compactly supported closed k-form on $S^n \setminus \{p_0\}$ is exact whenever $0 < k < n$. So $H^k_c(\mathbb{R}^n) \cong \{0\}$ for $0 < k < n$ and the claim holds. □

Proposition 5.5. *Let $U \subset \mathbb{R}^n$ be diffeomorphic to $\mathbb{R}^n$. Then $D_U \colon H^k(U) \to H^{n-k}_c(U)^*$ is an isomorphism.*

Proof. Since U is diffeomorphic to $\mathbb{R}^n$, we have that $H^k(U) \cong H^k(\mathbb{R}^n)$ and $H_c^{n-k}(U) \cong H_c^{n-k}(\mathbb{R}^n)$. Hence we may apply the Corollary 5.2 and Theorem 5.3 directly to the set U.

Let $k = 0$. Now $\dim H^0(U) = \dim H_c^n(U)^* = 1$ by Corollary 5.2 and Theorem 5.3. Hence $[\chi_U] \in H^0(U)$ spans the space $H^0(U)$. We observe also that $D_U([\chi_U])$ is the integral $\int_U$, by definition. Since $\int_U : H_c^n(\mathbb{R}^n) \to \mathbb{R}$ is nontrivial and $\dim H_c^n(U)^* = 1$, we conclude that $D_U([\chi_U])$ spans $H_c^n(U)^*$. Thus D_U is surjective. Similarly, since $[\chi_U]$ spans $H^0(U)$, we conclude that D_U is injective. Thus D_U an isomorphism.

For $k > 0$ we know by Corollary 5.2 and Lemma 5.3 that $H^k(U) \cong \{0\} \cong H_c^{n-k}(U) \cong H_c^{n-k}(U)^*$. So D_U is trivially an isomorphism. $\square$

6 Mayer–Vietoris sequence

6.1 Chain complexes

In this section we prove that if Poincaré duality holds for some open sets U and V in $\mathbb{R}^n$ and for their intersection $U \cap V$, it holds also for the union $U \cup V$. The proof takes heavily advantage of the *Mayer–Vietoris sequence*, which is an important tool for calculating the cohomology groups of a given space. We begin with briefly introducing some central definitions on general chain complexes which we need for formulating the Mayer–Vietoris sequence.

Definition 6.1. Let A_k, $k \in \mathbb{Z}$, be vector spaces and let $\mathrm{d}_k \colon A_k \to A_{k+1}$ be linear operators so that $\mathrm{d}_{k+1} \circ \mathrm{d}_k = 0$ for all $k \in \mathbb{Z}$. Then the sequence $A_* = (A_k, \mathrm{d}_k)_{k \in \mathbb{Z}}$ is called a *chain complex.*

For example, let $U \subset \mathbb{R}^n$ be an open set. Then the vector spaces of k-forms $\Omega^k(U)$ together with the exterior derivative $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ form a chain complex $\Omega^*(U) = (\Omega^k(U), \mathrm{d})_{k \in \mathbb{Z}}$. We next define some terminology for sequences of linear maps between general vector spaces.

Definition 6.2. Let A, B, C be vector spaces and let $f \colon A \to B$ and $g \colon B \to C$ be linear maps. The sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* if $\operatorname{Ker} g = \operatorname{Im} f$.

In general we say that a sequence

$$\cdots \xrightarrow{f_{k-2}} A_{k-1} \xrightarrow{f_{k-1}} A_k \xrightarrow{f_k} A_{k+1} \xrightarrow{f_{k+1}} \cdots$$

is exact if $\operatorname{Ker} f_k = \operatorname{Im} f_{k-1}$ for all $k \in \mathbb{Z}$. An important special case of the exact sequences is the following.

Definition 6.3. Let A, B, C be vector spaces and $f \colon A \to B$, $g \colon B \to C$ linear maps. Let also $i : 0 \to A$ be the inclusion and $j : C \to 0$ the zero map. We call an exact sequence

$$0 \xrightarrow{i} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{j} 0$$

a *short exact sequence.*

Usually one does not write the mappings i and j above the arrows, since they are the only possible linear maps between those spaces. Beside the fact that $\operatorname{Ker} g = \operatorname{Im} f$ we can deduce that f is injective and g is surjective. Indeed, since the sequence is exact, we have that $\operatorname{Ker} f = \operatorname{Im} i = 0$ and $\operatorname{Im} g = \operatorname{Ker} j = C$.

The next task is to lift the concept of sequence to the level of chain complexes. First we define a map between chain complexes.

Definition 6.4. Let $A_* = (A_k, \mathrm{d}_k^A)_{k\in\mathbb{Z}}$ and $B_* = (B_k, \mathrm{d}_k^B)_{k\in\mathbb{Z}}$ be chain complexes. A *chain map* $f\colon A_* \to B_*$ is a sequence $(f_k\colon A_k \to B_k)_{k\in\mathbb{Z}}$ of linear maps satisfying $\mathrm{d}_k^B \circ f_k = f_{k+1} \circ \mathrm{d}_k^A$.

Alternatively, if f is a chain map, we say that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A_k \arrow[r, "\mathrm{d}_k^A"] \arrow[d, "f_k"'] & A_{k+1} \arrow[r] \arrow[d, "f_{k+1}"] & \cdots \\
\cdots \arrow[r] & B_k \arrow[r, "\mathrm{d}_k^B"] & B_{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

Definition 6.5. Let A_*, B_*, C_* be chain complexes and let $f\colon A_* \to B_*$ and $g\colon B_* \to C_*$ be chain maps. The sequence

$$A_* \xrightarrow{f} B_* \xrightarrow{g} C_*$$

of chain complexes is *exact* if $A_k \xrightarrow{f_k} B_k \xrightarrow{g_k} C_k$ is exact for each $k \in \mathbb{Z}$.

Similarly as above, an exact sequence

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

of chain complexes is called a short exact sequence of chain complexes.

We next consider an important special case of a short exact sequence of chain complexes of differential forms. Let U_1 and U_2 be open sets in $\mathbb{R}^n$. For $\nu = 1, 2$, let $j_\nu\colon U_1 \cap U_2 \to U_\nu$ and $i_\nu\colon U_\nu \to U_1 \cup U_2$ be the inclusions. Now the diagram

$$\begin{tikzcd}
 & U_1 \cap U_2 \arrow[dl, "j_1"'] \arrow[dr, "j_2"] & \\
U_1 \arrow[dr, "i_1"'] & & U_2 \arrow[dl, "i_2"] \\
 & U_1 \cup U_2 &
\end{tikzcd}$$

commutes. Let then $I = (I_k)_{k\in\mathbb{Z}} : \Omega^*(U_1 \cup U_2) \to \Omega^*(U_1) \oplus \Omega^*(U_2)$ be a chain map, where each I_k is defined as

$$I_k\colon \Omega^k(U_1 \cup U_2) \to \Omega^k(U_1) \oplus \Omega^k(U_2), \quad \omega \mapsto (i_1^*\omega, i_2^*\omega).$$

Let also $J = (J_k)_{k\in\mathbb{Z}} : \Omega^*(U_1) \oplus \Omega^*(U_2) \to \Omega^*(U_1 \cap U_2)$ be a chain map with

$$J_k\colon \Omega^k(U_1) \oplus \Omega^k(U_2) \to \Omega^k(U_1 \cap U_2), \quad (\omega_1, \omega_2) \mapsto j_1^*\omega_1 - j_2^*\omega_2,$$

for each $k \in \mathbb{Z}$. The mappings I and J are chain maps when we define $\mathrm{d}^{U_1 \oplus U_2} = \mathrm{d}^{U_1} \oplus \mathrm{d}^{U_2}$. Indeed, since the pull-back and exterior derivative commute by Lemma 2.24, we have

$$(\mathrm{d} \oplus \mathrm{d})(I(\omega)) = (\mathrm{d}(i_1^*\omega), \mathrm{d}(i_2^*\omega)) = (i_1^*(\mathrm{d}\omega), i_2^*(\mathrm{d}\omega)) = I(\mathrm{d}\omega)$$

and similarly for J. The next theorem plays an important role in the rest of this section.

Theorem 6.6. *Let U_1 and U_2 be open sets in $\mathbb{R}^n$ and let I and J be defined as above. The sequence*

$$0 \longrightarrow \Omega^*(U_1 \cup U_2) \xrightarrow{I} \Omega^*(U_1) \oplus \Omega^*(U_2) \xrightarrow{J} \Omega^*(U_1 \cap U_2) \longrightarrow 0$$

is a short exact sequence of chain complexes.

We omit the proof here, see e.g. [4, Thm 5.1]. We define now *homology* of a chain complex, which is a fundamental concept in algebraic topology.

Definition 6.7. Let $A_* = (A_k, \mathrm{d}_k)$ be a chain complex. The *kth homology* of A_* is the quotient space

$$H^k(A_*) = \frac{\operatorname{Ker} \mathrm{d}_k}{\operatorname{Im} \mathrm{d}_{k-1}}.$$

The elements of $H^k(A_*)$ are called *homology classes.*

We notice that the kth homology of the chain complex $\Omega^*(U)$ for some open $U \subset \mathbb{R}^n$ is just $H^k(U)$, the kth de Rham cohomology group of U. The next lemma implies that a chain map lifts to the homological level; we refer to [4, Lemma 4.3] for a proof.

Lemma 6.8. *Let $f\colon A_* \to B_*$ be a chain map. The induced map $f_*\colon H_k(A_*) \to H_k(B_*)$, $[c] \mapsto [f(c)]$, is well-defined and linear.*

The following theorem (Theorem 6.9) is the starting point of the upcoming proof for the Poincaré duality. By above lemma we may build sequences of homologies similarly as with chain complexes. The theorem tells that a short exact sequence of chain complexes lifts to a long exact sequence of homology classes. In other words, there exists a linear map called *connecting homomorphism* $\partial_k\colon H^k(C_*) \to H^{k+1}(A_*)$ that connects the exact sequence of kth homology classes to the sequence of $(k+1)$th homology classes. Moreover, the operator ∂_k leaves the sequence exact. We refer to [4, Thm 4.9] for a proof.

Theorem 6.9. *Let* $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ *be a short exact sequence of chain complexes. Then there exist linear maps* $\partial_k \colon H^k(C_*) \to H^{k+1}(A_*)$, $k \in \mathbb{Z}$ *so that the sequence*

$$\cdots \xrightarrow{\partial_{k-1}} H^k(A_*) \xrightarrow{f_*} H^k(B_*) \xrightarrow{g_*} H^k(C_*) \xrightarrow{\partial_k} H^{k+1}(A_*) \xrightarrow{f_*} \cdots$$

is exact.

Now Theorems 6.6 and 6.9 together give a corollary, which is called the Mayer–Vietoris sequence for de Rham cohomoloy.

Corollary 6.10. *Let* U_1, U_2 *be open sets in* $\mathbb{R}^n$ *and let* $I \colon \Omega^*(U_1 \cup U_2) \to \Omega^*(U_1) \oplus \Omega^*(U_2)$ *and* $J \colon \Omega^*(U_1) \oplus \Omega^*(U_2) \to \Omega^*(U_1 \cap U_2)$ *be the chain maps defined in Theorem 6.6. Then the sequence*

$$\cdots \xrightarrow{\partial_{k-1}} H^k(U_1 \cup U_2) \xrightarrow{I_*} H^k(U_1) \oplus H^k(U_2) \xrightarrow{J_*} H^k(U_1 \cap U_2) \xrightarrow{\partial_k} H^{k+1}(U_1 \cup U_2) \xrightarrow{I_*} \cdots$$

where ∂_k *is the connecting homomorphism of the sequence in Theorem 6.6, is exact.*

The Mayer–Vietoris sequence gives a fairly simple solution for the cohomologies of the punctured plane.

Example 6.11. We show using the Mayer–Vietoris sequence that

$$H^k(\mathbb{R}^2 \setminus \{0\}) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0 \text{ or } 1 \\ \{0\}, & \text{if } k = 2. \end{cases}$$

Let $U_+ := \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \leq 0\}$ and $U_- := \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \geq 0\}$. Now $U_+ \cup U_- = \mathbb{R}^2 \setminus \{0\}$ and $U_+ \cap U_- = \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, the disjoint union of the upper and lower half planes. Since $H^k(\mathbb{R}^2 \setminus \{0\}) = 0$ whenever $k < 0$ or $k > 2$, by Corollary 6.10 we have that the sequence

$$0 \longrightarrow H^0(\mathbb{R}^2 \setminus \{0\}) \xrightarrow{(I_0)_*} H^0(U_1) \oplus H^0(U_2) \xrightarrow{(J_0)_*} H^0(U_1 \cap U_2) \xrightarrow{\partial_0}$$

$$\xrightarrow{\partial_0} H^1(\mathbb{R}^2 \setminus \{0\}) \xrightarrow{(I_1)_*} H^1(U_1) \oplus H^1(U_2) \xrightarrow{(J_1)_*} H^1(U_1 \cap U_2) \xrightarrow{\partial_1}$$

$$\xrightarrow{\partial_1} H^2(\mathbb{R}^2 \setminus \{0\}) \xrightarrow{(I_2)_*} H^2(U_1) \oplus H^2(U_2) \xrightarrow{(J_2)_*} H^2(U_1 \cap U_2) \xrightarrow{\partial_2} 0$$

is exact. Since $\mathbb{R}^2\setminus\{0\}$ is connected, we have immediately that $H^0(\mathbb{R}^2\setminus\{0\}) \cong \mathbb{R}$ by Lemma 3.3. Also, the sets U_+ and U_- are star-like, so by the Poincaré lemma 5.1 $H^0(U_1) \cong H^0(U_2) \cong \mathbb{R}$. Hence

$$H^k(U_1) \oplus H^k(U_2) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R}, & \text{if } k = 0 \\ \{0\} & \text{otherwise.} \end{cases}$$

As we noticed above, $U_+ \cap U_-$ is a disjoint union of two star-like sets $\mathbb{R} \times \mathbb{R}_+$ and $\mathbb{R} \times \mathbb{R}_-$. Below we prove that the cohomology of a set is the direct product of the cohomologies of each pairwise disjoint component (see Theorem 7.1). Then Poincaré lemma gives us

$$H^k(U_1 \cap U_2) \cong H^k(\mathbb{R} \times \mathbb{R}_+) \oplus H^k(\mathbb{R} \times \mathbb{R}_-) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R}, & \text{if } k = 0 \\ \{0\} & \text{otherwise.} \end{cases}$$

With these results we may write the above exact sequence as

$$0 \longrightarrow \mathbb{R} \xrightarrow{\quad f_0 \quad} \mathbb{R} \oplus \mathbb{R} \xrightarrow{g_0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\partial_0}$$

$$\xrightarrow{\partial_0} H^1(\mathbb{R}^2 \setminus \{0\}) \xrightarrow{f_1} 0 \longrightarrow 0 \xrightarrow{\partial_1}$$

$$\xrightarrow{\partial_1} H^2(\mathbb{R}^2 \setminus \{0\}) \xrightarrow{f_2} 0,$$

where f_k is the linear map that corresponds to $(I_k)_*$ for each $k = 0, 1, 2$ via the isomorphisms, whose existence we observed above. Similarly g_0 corresponds to $(J_0)_*$ with this identification. For example, f_0 is the map for which the diagram

$$\begin{tikzcd}
H^0(\mathbb{R}^2 \setminus \{0\}) \arrow[r, "(I_0)_*"] \arrow[d, "\cong"] & H^0(U_1) \oplus H^0(U_2) \arrow[d, "\cong"] \\
\mathbb{R} \arrow[r, "f_0"] & \mathbb{R} \oplus \mathbb{R}
\end{tikzcd}$$

commutes.

Now we may deduce the dimensions of the sets $H^1(\mathbb{R}^2\setminus\{0\})$ and $H^2(\mathbb{R}^2\setminus\{0\})$ using, by turns, the exactness of the sequence and the rank-nullity theorem: if $f\colon V \to W$ is a linear map between vector spaces V and W, then

$$\dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f) = \dim V.$$

Firstly, we notice that $\operatorname{Im} \partial_1 = \{0\}$ and $\operatorname{Ker} f_2 = H^2(\mathbb{R}^2 \setminus \{0\})$. Then, by the exactness of the sequence, we have that $H^2(\mathbb{R}^2 \setminus \{0\}) \cong \{0\}$.

Consider then the beginning of the sequence. By the exactness $\operatorname{Im} 0 = 0 = \operatorname{Ker} f_0$, so $\dim(\operatorname{Ker} f_0) = 0$. Hence $\dim(\operatorname{Im} f_0) = 1$ by the rank-nullity theorem. Using the exactness gives us further that also $\dim(\operatorname{Ker} g_0) = 1$. Again by the rank-nullity theorem $\dim(\operatorname{Im} g_0) = \dim(\mathbb{R} \oplus \mathbb{R}) - 1 = 1$. Since now $\operatorname{Ker} f_1 = H^1(\mathbb{R}^2 \setminus \{0\})$ and on the other hand $\operatorname{Ker} f_1 = \operatorname{Im} \partial_0$, we deduce that $\dim(H^1(\mathbb{R}^2 \setminus \{0\})) = 1$. Hence $H^1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$ and the claim holds.

A similar long exact sequence in homology exists for the dual spaces of compactly supported cohomologies. We discuss this in the next section.

6.2 Exact sequences for compactly supported de Rham cohomology and its dual

Theorem 6.12. *Let U_1 and U_2 be open sets in $\mathbb{R}^n$ and let $j_\nu \colon U_1 \cap U_2 \to U_\nu$ and $i_\nu \colon U_\nu \to U_1 \cup U_2$, where $\nu = 1, 2$, be inclusions. We define chain maps $J \colon \Omega_c^*(U_1 \cap U_2) \to \Omega_c^*(U_1) \oplus \Omega_c^*(U_2)$ and $I \colon \Omega_c^*(U_1) \oplus \Omega_c^*(U_2) \to \Omega_c^*(U_1 \cup U_2)$ by*

$$J_k \colon \omega \mapsto ((j_1)_*\omega, -(j_2)_*\omega) \quad \text{and} \quad I_k \colon (\omega_1, \omega_2) \mapsto (i_1)_*\omega_1 + (i_2)_*\omega_2,$$

where the push-forward $(i_\nu)_*$ induced by inclusion i_ν is as in Definition 3.7. Then the sequence

$$0 \longrightarrow \Omega_c^*(U_1 \cap U_2) \xrightarrow{J} \Omega_c^*(U_1) \oplus \Omega_c^*(U_2) \xrightarrow{I} \Omega_c^*(U_1 \cup U_2) \longrightarrow 0$$

is exact.

Proof. There are three parts in the proof. We need to show that, for each $k \in \mathbb{Z}$, J_k is injective, I_k is surjective and $\operatorname{Im} J_k = \operatorname{Ker} I_k$. To show the injectivity of J_k, let $\omega \in \Omega_c^k(U_1 \cap U_2)$ be so that $J_k(\omega) = (0, 0)$. Then ω is necessarily zero in $U_1 \cap U_2$, so ω is identically zero.

We show then that I_k is surjective. Let $\omega \in \Omega_c^k(U_1 \cup U_2)$ and let $\{\rho_1, \rho_2\}$ be a smooth partition of unity subordinate to U_1 and U_2. So $\rho_\nu \colon U_1 \cup U_2 \to \mathbb{R}$ are so that $\operatorname{spt}(\rho_\nu) \subset U_\nu$ and $\rho_1(x) + \rho_2(x) = 1$ for all $x \in U_1 \cup U_2$. Now

$$\begin{aligned} I_k\big((\rho_1\omega)|_{U_1}, (\rho_2\omega)|_{U_2} \big) &= (i_1)_* \, (\rho_1\omega)|_{U_1} + (i_2)_* \, (\rho_2\omega)|_{U_2} \\ &= \rho_1\omega + \rho_2\omega = \omega. \end{aligned}$$

It remains to show that $\operatorname{Im} J_k = \operatorname{Ker} I_k$. Let $(\omega_1, \omega_2) \in \operatorname{Im}(J_k)$. So there exists $\omega \in \Omega_c^k(U_1 \cap U_2)$ for which $((j_1)_*\omega, -(j_2)_*\omega) = (\omega_1, \omega_2)$. Then we have that

$$I_k(\omega_1, \omega_2) = (i_1)_*(j_1)_*\omega - (i_2)_*(j_2)_*\omega = 0,$$

since $((i_\nu)_*(j_\nu)_*\omega)_p = \omega_p$ for each $p \in U_1 \cap U_2$ and zero otherwise for both $\nu = 1, 2$. Hence $(\omega_1, \omega_2) \in \operatorname{Ker} I_k$ and $\operatorname{Im} J_k \subset \operatorname{Ker} I_k$.

Let then $(\omega_1, \omega_2) \in \operatorname{Ker}(I_k)$. We need to find $\eta \in \Omega_c^k(U_1 \cap U_2)$ so that $J_k(\eta) = (\omega_1, \omega_2)$. Now $(i_1)_*\omega_1 = -(i_2)_*\omega_2$. This implies that $(\omega_1)_p = -(\omega_2)_p$ when $p \in U_1 \cap U_2$ and $(\omega_1)_p = (\omega_2)_p = 0$ otherwise. Using these identities, we obtain that

$$\begin{aligned} J_k(\omega_1|_{U_1 \cap U_2}) &= ((j_1)_* \, \omega_1|_{U_1 \cap U_2}, -(j_2)_* \, \omega_1|_{U_1 \cap U_2}) \\ &= ((j_1)_* \, \omega_1|_{U_1 \cap U_2}, (j_2)_* \, \omega_2|_{U_1 \cap U_2}) \\ &= (\omega_1, \omega_2). \end{aligned}$$

So $\operatorname{Ker} I_k \subset \operatorname{Im} J_k$ and the sequence is exact. □

Now Theorems 6.9 and 6.12 give us the following corollary.

Corollary 6.13. *Let $J\colon \Omega_c^*(U_1 \cap U_2) \to \Omega_c^*(U_1) \oplus \Omega_c^*(U_2)$ and $I\colon \Omega_c^*(U_1) \oplus \Omega_c^*(U_2) \to \Omega_c^*(U_1 \cup U_2)$ be the chain maps defined in Theorem 6.12, $J_*\colon H_c^k(U_1 \cap U_2) \to H_c^k(U_1) \oplus H_c^k(U_2)$ and $I_*\colon H_c^k(U_1) \oplus H_c^k(U_2) \to H_c^k(U_1 \cup U_2)$ their induced maps and $\partial_*\colon H_c^k(U_1 \cup U_2) \to H_c^{k+1}(U_1 \cap U_2)$ the connecting homomorphism. Then the sequence*

$$\cdots \longrightarrow H_c^k(U_1 \cap U_2) \xrightarrow{J_*} H_c^k(U_1) \oplus H_c^k(U_2) \xrightarrow{I_*} H_c^k(U_1 \cup U_2) \xrightarrow{\partial_*} H_c^{k+1}(U_1 \cap U_2) \longrightarrow \cdots$$

is exact.

In order to obtain an analogous result for the dual spaces, we begin with a lemma on dual sequences.

Lemma 6.14. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of vector spaces. Then $C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^*$ is an exact sequence of dual spaces, where g^* and f^* are the dual maps of g and f as in Corollary 2.6.*

This lemma is purely algebraic and it is proven in [3, p. 72]. To simplify the notation in the forthcoming statements, we denote the dual map $(\iota_*)^*$ of the push-forward $\iota_*\colon \Omega_c^k(U) \to \Omega_c^k(V)$ by $\tilde{\iota}$, that is, we set

$$\tilde{\iota}\colon \Omega_c^k(V)^* \to \Omega_c^k(U)^*, \quad (\tilde{\iota}(L))(\omega) = L(\iota_*\omega),$$

for each $L \in \Omega_c^k(V)^*$ and $\omega \in \Omega_c^k(U)$. Now Lemma 6.14 and Corollary 6.13 give the following result as a consequence.

Theorem 6.15. *Let U_1 and U_2 be open sets in $\mathbb{R}^n$. Let $J_*\colon H_c^k(U_1\cap U_2)\to H_c^k(U_1)\oplus H_c^k(U_2)$, $I_*\colon H_c^k(U_1)\oplus H_c^k(U_2)\xrightarrow{I_*} H_c^k(U_1\cup U_2)$ and $\partial_*\colon H_c^k(U_1\cup U_2)\to H_c^{k+1}(U_1\cap U_2)$ be homomorphisms as in Corollary 6.13. Then the sequence*

$$\cdots\longrightarrow H_c^{k+1}(U_1\cap U_2)^*\xrightarrow{\tilde\partial} H_c^k(U_1\cup U_2)^*\xrightarrow{\tilde I} H_c^k(U_1)^*\oplus H_c^k(U_2)^*\xrightarrow{\tilde J} H_c^k(U_1\cap U_2)^*\longrightarrow\cdots$$

is exact, where

$$\begin{aligned}&\tilde I\colon H_c^k(U_1\cup U_2)^*\to H_c^k(U_1)^*\oplus H_c^k(U_2)^*, \quad \alpha\mapsto(\tilde\imath_1(\alpha),\tilde\imath_2(\alpha)),\\ &\tilde J\colon H_c^k(U_1)^*\oplus H_c^k(U_2)^*\to H_c^k(U_1\cap U_2)^*, \quad (\alpha_1,\alpha_2)\mapsto\tilde\jmath_1(\alpha_1)-\tilde\jmath_2(\alpha_2),\end{aligned}$$

and

$$\tilde\partial\colon H_c^{k+1}(U_1\cap U_2)^*\to H_c^k(U_1\cup U_2)^*, \quad \tilde\partial=(\partial_*)^*,$$

are homomorphisms.

6.3 Poincaré duality for unions

We have now obtained the Mayer–Vietoris sequence both for cohomology groups and the duals of the compactly supported cohomologies. These two sequences are connected with the maps $D_{U_1\cup U_2}$, $D_{U_1}\oplus D_{U_2}$ and $D_{U_1\cap U_2}$ given by the Poincaré duality. We next prove the main result of this section.

Proposition 6.16. *Let U_1 and U_2 be open sets in $\mathbb{R}^n$ so that $D_{U_\nu}\colon H^k(U_\nu)\to H_c^{n-k}(U_\nu)^*$ and $D_{U_1\cap U_2}\colon H^k(U_1\cap U_2)\to H_c^{n-k}(U_1\cap U_2)^*$ are isomorphisms for each $k\in\mathbb{Z}$ and $\nu=1,2$. Then $D_{U_1\cup U_2}\colon H^k(U_1\cup U_2)\to H_c^{n-k}(U_1\cup U_2)^*$ is an isomorphism for each k.*

We begin with some auxiliary results.

Lemma 6.17. *Let $V\subset U\subset\mathbb{R}^n$ be open sets and $i\colon V\to U$ the inclusion. Then the diagram*

$$\begin{tikzcd} H^k(U) \arrow[r, "i^*"] \arrow[d, "D_U"] & H^k(V) \arrow[d, "D_V"] \\ H_c^{n-k}(U)^* \arrow[r, "\tilde\imath"] & H_c^{n-k}(V)^* \end{tikzcd}$$

commutes.

Proof. Let $[\omega]\in H^k(U)$ and $[\tau]\in H_c^{n-k}(V)$. Then

$$(D_V\circ i^*)([\omega])[\tau]=D_V([i^*\omega])[\tau]=\int_V i^*\omega\wedge\tau.$$

On the other hand

$$(\tilde{i} \circ D_U)([\omega])[\tau] = (\tilde{i}(D_U[\omega]))[\tau] = D_U[\omega](i_*[\tau]) = \int_U \omega \wedge i_*\tau.$$

Now for each $p \in V$ we have that $(i^*\omega \wedge \tau)_p = (\omega \wedge i_*\tau)_p$. Indeed, let $X_p^1, \dots, X_p^k \in T_p(V)$. Then

$$(i^*\omega)_p(X_p^1, \dots, X_p^k) = \omega_{i(p)}(i_*X_p^1, \dots, i_*X_p^k) = \omega_p(X_p^1, \dots, X_p^k),$$

since $(i_*(X_p^j))(u) = X_p^j(u \circ i) = X_p^j(u)$ for each $u \in C_p^\infty$ and $j = 1, \dots, k$. Also, by definition $(i_*\tau)_p = \tau_p$ whenever $p \in V$. Since $(\omega \wedge i_*\tau)_p = 0$ for all $p \in U \setminus V$, we have that

$$\int_V i^*\omega \wedge \tau = \int_U \omega \wedge i_*\tau.$$

The claim follows. □

In the proof of Lemma 6.19 we use the duality of connecting homomorphisms. See e.g. [4, p.131] for proof.

Lemma 6.18. *Let U_1 and U_2 be open sets in $\mathbb{R}^n$. Let $\partial^*\colon H^k(U_1 \cap U_2) \to H^{k+1}(U_1 \cup U_2)$ and $\partial_*\colon H_c^{n-(k+1)}(U_1 \cup U_2) \to H_c^{n-k}(U_1 \cap U_2)$ be the connecting homomorphisms for the corresponding Mayer–Vietoris sequences. Then*

$$\int_{U_1 \cup U_2} \partial^*[\omega] \wedge [\tau] = (-1)^{k+1} \int_{U_1 \cap U_2} [\omega] \wedge \partial_*[\tau]$$

for each $[\omega] \in H^k(U_1 \cap U_2)$ and $[\tau] \in H_c^{n-(k+1)}(U_1 \cup U_2)$.

Lemma 6.19. *Let U_1, U_2 be open sets in $\mathbb{R}^n$. Then the diagram*

$$\begin{array}{ccc}
H^k(U_1 \cap U_2) & \xrightarrow{\partial^*} & H^{k+1}(U_1 \cup U_2) \\
\big\downarrow{\scriptstyle D_{U_1 \cap U_2}} & & \big\downarrow{\scriptstyle D_{U_1 \cup U_2}} \\
H_c^{n-k}(U_1 \cap U_2)^* & \xrightarrow{(-1)^{k+1}\tilde{\partial}} & H_c^{n-(k+1)}(U_1 \cup U_2)^*
\end{array}$$

commutes, where ∂_* and ∂^* are the connecting homomorphisms as in Lemma 6.18 and $\tilde{\partial} = (\partial_*)^*$ is the dual map of ∂_*.

Proof. Let $[\omega] \in H^k(U_1 \cap U_2)$ and $[\tau] \in H_c^{n-(k+1)}(U_1 \cup U_2)$. Following the upper route in the diagram gives us

$$D_{U_1 \cup U_2}(\partial^*[\omega])[\tau] = \int_{U_1 \cup U_2} \partial^*[\omega] \wedge [\tau].$$

Correspondingly the other route gives

$$\tilde{\partial}(D_{U_1\cap U_2}([\omega]))[\tau] = D_{U_1\cap U_2}([\omega])(\partial_*[\tau]) = \int_{U_1\cap U_2} [\omega] \wedge \partial_*[\tau].$$

By Lemma 6.18 we have that $D_{U_1\cup U_2} \circ \partial^* = (-1)^{k+1}\tilde{\partial} \circ D_{U_1\cap U_2}$. Thus the claim follows. $\square$

The proof of Proposition 6.16 takes advantage of the five lemma [2, Lemma 24.1].

Lemma 6.20 (Five lemma)**.** *Consider the commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "p"] & E \arrow[d, "q"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

If both rows are exact and homomorphisms f, g, p and q are isomorphisms, then h is an isomorphism.

Proof of Prop 6.16. Consider the diagram

$$\begin{tikzcd} H^k(U_1\cup U_2) \arrow[r, "I_*"] \arrow[d, "D_{U_1\cup U_2}"] & H^k(U_1)\oplus H^k(U_2) \arrow[r, "J_*"] \arrow[d, "D_{U_1}\oplus D_{U_2}"] & H^k(U_1\cap U_2) \arrow[r, "\partial_*"] \arrow[d, "D_{U_1\cap U_2}"] & H^{k+1}(U_1\cup U_2) \arrow[d, "D_{U_1\cup U_2}"] \\ H_c^{n-k}(U_1\cup U_2)^* \arrow[r, "\tilde{I}"] & H_c^{n-k}(U_1)^*\oplus H_c^{n-k}(U_2)^* \arrow[r, "\tilde{J}"] & H_c^{n-k}(U_1\cap U_2)^* \arrow[r, "(-1)^{k+1}\tilde{\partial}"] & H_c^{n-(k+1)}(U_1\cup U_2)^* \end{tikzcd}$$

By Corollaries 6.10 and 6.13 both rows in the diagram are exact. We state first that this exact diagram commutes. By Lemma 6.17 both of the diagrams

$$\begin{tikzcd} H^k(U_1\cup U_2) \arrow[r, "i_\nu^*"] \arrow[d, "D_{U_1\cup U_2}"] & H^k(U_\nu) \arrow[d, "D_{U_\nu}"] \\ H_c^{n-k}(U_1\cup U_2)^* \arrow[r, "\tilde{i}"] & H_c^{n-k}(U_\nu)^* \end{tikzcd}$$

and

$$\begin{tikzcd} H^k(U_\nu) \arrow[r, "j_\nu^*"] \arrow[d, "D_{U_\nu}"] & H^k(U_1\cup U_2) \arrow[d, "D_{U_1\cup U_2}"] \\ H_c^{n-k}(U_\nu)^* \arrow[r, "\tilde{j}_\nu"] & H_c^{n-k}(U_1\cup U_2)^* \end{tikzcd}$$

commute for $\nu = 1, 2$. So the left part of the exact diagram commutes. On the other hand, Lemma 6.19 gives directly the commutativity of the rightmost part. So the whole diagram commutes.

By assumption each map $D_{U_1} \oplus D_{U_2}$ and $D_{U_1 \cap U_2}$ is an isomorphism. Thus we may apply the Five lemma 6.20 to each four steps part in the long exact sequence so that, using the notation in Lemma 6.20, $f = p = D_{U_1} \oplus D_{U_2}$, $g = q = D_{U_1 \cap U_2}$ and $h = D_{U_1 \cup U_2}$. Hence, by the Five lemma, $D_{U_1 \cup U_2}$ is an isomorphism. $\square$

7 Cohomology of disjoint unions

In this section we consider the cohomology groups of an open set $U \subset \mathbb{R}^n$ in the special case that U is a pairwise disjoint union of open sets U_α for $\alpha \in I$, where I is some countable index set. We start with some linear algebraic results. In the end of this section we show that U satisfies Poincaré duality if it holds for each set U_α.

7.1 Linear algebra for cohomology groups

Theorem 7.1. *Let I be a countable index set and let $U_\alpha \subset \mathbb{R}^n$ be pairwise disjoint open sets for each $\alpha \in I$. Let also $U = \sqcup_\alpha U_\alpha$. Then*

$$H^k(U) \cong \prod_{\alpha \in I} H^k(U_\alpha).$$

Proof. Let $\iota_\alpha \colon U_\alpha \to U$ be an inclusion. Recall that an element in $\prod_{\alpha\in I} H^k(U_\alpha)$ is a sequence $([\omega_\alpha])_{\alpha \in I}$, where $[\omega_\alpha] \in H^k(U_\alpha)$. We show that the map

$$\theta \colon H^k(U) \to \prod_{\alpha \in I} H^k(U_\alpha), \quad \theta([\omega]) = (\iota_\alpha^*[\omega])_\alpha = ([\iota_\alpha^*\omega])_\alpha, \tag{7.1}$$

is an isomorphism of vector spaces. Clearly θ is linear. Suppose $\theta([\omega]) = 0$. Then $\iota_\alpha^*\omega$ is exact for all $\alpha \in I$. In other words, using the notation $\omega = \sum_J a_J \mathrm{d}x^J$, then

$$\iota_\alpha^*\omega = \omega|_{U_\alpha} = \sum_J a_J|_{U_\alpha} \, \mathrm{d}x^J$$

for each $\alpha \in I$. Therefore, for all $\alpha \in J$, we find forms $\tau_\alpha = \sum_{J'} b_{\alpha J'} \mathrm{d}x^{J'} \in \Omega^{k-1}(U_\alpha)$ such that

$$\mathrm{d}\tau_\alpha = \sum_J a_J|_{U_\alpha} \, \mathrm{d}x^J.$$

Now for each $J' \in \{(i_1, \ldots, i_k)\} : 1 \le i_1 \le \ldots \le i_k \le n\}$, let $b_{J'} \colon U \to \mathbb{R}$ be the function satisfying $b_{J'}(p) = b_{\alpha J'}(p)$ for $p \in U_\alpha$ and each $\alpha \in I$. Since the sets U_α are pairwise disjoint and cover the set U, function $b_{J'}$ is well-defined. It is also smooth, since each $b_{\alpha J'}$ is smooth on the component U_α of U. Hence $\tau = \sum_{J'} b_{J'} \mathrm{d}x^{J'}$ is a $(k-1)$-form for which $\mathrm{d}\tau = \omega$. So ω is exact, *i.e.*, $[\omega] = 0$ and θ is injective.

To show surjectivity, let $([\omega_\alpha])_\alpha \in \prod_\alpha H^k(U_\alpha)$. Now each ω_α is closed and of the form

$$\omega_\alpha = \sum_J a_{\alpha J} \mathrm{d}x^J,$$

where functions $a_{\alpha J}: U_\alpha \to \mathbb{R}$ are smooth. Similarly as above we may define smooth functions $a_J: U \to \mathbb{R}$ as $a_J(p) = a_{\alpha J}(p)$ whenever $p \in U_\alpha$. Defining $\omega = \sum_J a_J \mathrm{d}x^J$ gives us then a well-defined k-form. Also for any $p \in U$ holds $\mathrm{d}\omega_p = (\mathrm{d}\omega_\alpha)_p = 0$ for some α, so ω is closed. So $[\omega] \in H^k(U)$ and clearly $\theta([\omega]) = ([\omega_\alpha])_\alpha$. Hence θ is also surjective and further an isomorphism. $\square$

Theorem 7.2. *Let I be a countable index set and let $U_\alpha \subset \mathbb{R}^n$ be pairwise disjoint open sets for each $\alpha \in I$. Denote $U = \sqcup_\alpha U_\alpha$. Then*

$$H_c^k(U)^* \cong \prod_{\alpha \in I} H_c^k(U_\alpha)^*.$$

The proof of this theorem consists of two steps. We introduce the direct sum of vector spaces and show that both $H_c^k(U)^*$ and $\prod_\alpha H_c^k(U_\alpha)^*$ are isomorphic to the dual of the direct sum of spaces $H_c^k(U_\alpha)$. The latter part follows from a general statement for vector spaces, which we write as a lemma below.

Definition 7.3. Let I be a set and let V_α be a vector space for each $\alpha \in I$. The *direct sum* of vector spaces V_α is

$$\bigoplus_{\alpha \in I} V_\alpha = \{(v_\alpha)_\alpha \in \prod_{\alpha \in I} V_\alpha : \#\mathrm{spt}((v_\alpha)_\alpha) < \infty\},$$

where $\mathrm{spt}((v_\alpha)_\alpha) = \{\alpha \in I : v_\alpha \neq 0\}$ is the *support* of $(v_\alpha)_\alpha$.

Since $\bigoplus_{\alpha \in I} V_\alpha$ is a vector subspace of $\prod_{\alpha \in I} V_\alpha$, we may define two natural operations between $\bigoplus_{\alpha \in I} V_\alpha$ and the individual vector spaces V_α. The first one is the inclusion

$$i_\beta: V_\beta \to \bigoplus_{\alpha \in I} V_\alpha, \quad v \mapsto (v_\alpha)_\alpha, \text{ where } v_\alpha = \begin{cases} v, & \text{if } \alpha = \beta \\ 0, & \text{otherwise} \end{cases}$$

and the second one is the projection

$$p_\beta: \bigoplus_{\alpha \in I} V_\alpha \to V_\beta, \quad (v_\alpha)_\alpha \mapsto v_\beta.$$

We get as an immediate result that

$$p_\beta \circ i_\alpha = \begin{cases} \mathrm{id}: V_\alpha \to V_\alpha, & \text{if } \alpha = \beta \\ 0, & \text{otherwise} \end{cases} \tag{7.2}$$

and

$$\sum_{\alpha \in I} i_\alpha \circ p_\alpha = \mathrm{id}: \bigoplus_{\alpha \in I} V_\alpha \to \bigoplus_{\alpha \in I} V_\alpha. \tag{7.3}$$

Lemma 7.4. *Let I be a set and let V_α be vector spaces for each $\alpha \in I$. Then the map*

$$\delta\colon \left(\bigoplus_{\alpha\in I} V_\alpha\right)^* \to \prod_{\alpha\in I} V_\alpha^*, \quad L \mapsto (L\circ i_\alpha)_\alpha, \tag{7.4}$$

is an isomorphism of vector spaces with the inverse

$$\delta'\colon \prod_{\alpha\in I} V_\alpha^* \to \left(\bigoplus_{\alpha\in I} V_\alpha\right)^*, \quad (L_\alpha)_\alpha \mapsto \sum_{\alpha\in I} L_\alpha \circ p_\alpha.$$

Proof. The map δ is clearly linear and since every $L\circ i_\alpha$ is a linear map $V_\alpha \to \mathbb{R}$, it is well-defined. Similarly δ' is linear. To show that δ' is well-defined, let $v \in \bigoplus_{\alpha\in I} V_\alpha$. Now v has finite support, so there are only finitely many indices α for which $(L_\alpha \circ p_\alpha)(v) \neq 0$. Hence $\sum_\alpha (L_\alpha\circ p_\alpha)(v) \in \mathbb{R}$. Thus $\sum_\alpha L_\alpha \circ p_\alpha \in (\bigoplus_{\alpha\in I} V_\alpha)^*$ and also δ' is well-defined.

Take then any $(L_\alpha)_\alpha \in \prod_{\alpha\in I} V_\alpha^*$. By (7.2) we have that

$$\begin{aligned}(\delta\circ\delta')((L_\alpha)_\alpha) &= \delta\left(\sum_{\beta\in I} L_\beta\circ p_\beta\right) = \left(\left(\sum_{\beta\in I} L_\beta\circ p_\beta\right)\circ i_\alpha\right)_\alpha \\ &= \left(\sum_{\beta\in I} L_\beta\circ p_\beta\circ i_\alpha\right)_\alpha = (L_\alpha)_\alpha.\end{aligned}$$

Also, by (7.3), for any $L \in (\bigoplus_\alpha V_\alpha)^*$ holds

$$(\delta'\circ\delta)(L) = \delta'((L\circ i_\alpha)_\alpha) = \sum_{\alpha\in I} (L\circ i_\alpha)\circ p_\alpha = L\circ\left(\sum_{\alpha\in I} i_\alpha\circ p_\alpha\right) = L,$$

since L is linear by the definition of dual space. We proved that $\delta' = \delta^{-1}$ and hence δ is an isomorphism of vector spaces. $\square$

Since $H_c^k(U_\alpha)$ are vector spaces, the dual space $\left(\bigoplus_\alpha H_c^k(U_\alpha)\right)^*$ is isomorphic to $\prod_\alpha H_c^k(U_\alpha)^*$ by Lemma 7.4. For the last part of the proof, we use the push-forward ι_* introduced in Definition 3.7.

Proof of Theorem 7.2. We show first that

$$\rho\colon \bigoplus_{\alpha\in I} H_c^k(U_\alpha) \to H_c^k(U), \quad ([\omega_\alpha])_\alpha \mapsto \sum_{\alpha\in I} (\iota_\alpha)_*[\omega_\alpha] \tag{7.5}$$

is an isomorphism. Suppose again $\rho(([\omega_\alpha])_\alpha) = 0$, *i.e.*, $\sum_\alpha (\iota_\alpha)_*\omega_\alpha$ is exact. We show that each ω_α is exact. Since any $p\in U$ belongs to exactly one U_β and hence at point p we have that $\sum_\alpha((\iota_\alpha)_*\omega_\alpha)_p = (\omega_\beta)_p$ in U_β. So $[\omega_\alpha] = 0$ for each α and the map ρ is injective.

Take then any $[\omega] \in H_c^k(U)$. Since ω has compact support, it is nonzero only on finitely many sets U_α. Then the sequence $([\omega_\alpha])_\alpha$, where $\omega_\alpha = \omega|_{U_\alpha}$, is an element in $\bigoplus_\alpha H_c^k(U_\alpha)$. Also

$$\rho(([\omega_\alpha])_\alpha) = \sum_{\alpha\in I}(\iota_\alpha)_*[\omega|_{U_\alpha}] = \sum_{\alpha\in I}[\chi_{U_\alpha}\omega] = [\omega],$$

so ρ is surjective. Hence by Corollary 2.6 the dual map

$$\rho^*\colon H_c^k(U)^* \to \left(\bigoplus_{\alpha\in I} H_c^k(U_\alpha)\right)^*, \quad L \mapsto L\circ\rho,$$

of ρ is an isomorphism. Thus Lemma 7.4 gives us that

$$\delta\circ\rho^*\colon H_c^k(U)^* \to \prod_{\alpha\in I} H_c^k(U_\alpha)^*$$

is an isomorphism, which proves the claim. □

7.2 Poincaré duality for disjoint unions

Proposition 7.5. *Let $\{U_\alpha\}_{\alpha\in I}$ be a pairwise disjoint collection of open sets in $\mathbb{R}^n$, where I is a countable index set. Suppose that $D_{U_\alpha}\colon H^k(U_\alpha) \to H_c^{n-k}(U_\alpha)^*$ is an isomorphism for each $k\in\mathbb{Z}$ and $\alpha \in I$. Then $D_U\colon H^k(U) \to H_c^{n-k}(U)^*$ is an isomorphism for each k, where $U = \sqcup_{\alpha\in I}U_\alpha$.*

Proof. Consider the diagram

$$\begin{tikzcd}
H^k(U) \arrow[rr, "\theta"] \arrow[d, "D_U"'] & & \prod_\alpha H^k(U_\alpha) \arrow[d, "\prod D_{U_\alpha}"] \\
H_c^{n-k}(U)^* \arrow[r, "\rho^*"] & (\bigoplus_\alpha H_c^{n-k}(U_\alpha))^* \arrow[r, "\delta"] & \prod_\alpha H_c^{n-k}(U_\alpha)^*
\end{tikzcd}$$

where θ is as in (7.1), ρ^* is the dual map of ρ introduced in (7.5) and δ is as in (7.4). So each map θ, ρ^* and δ is an isomorphism. Since we assume also the map $\prod D_{U_\alpha}$ to be an isomorphism, it suffices to show that the diagram commutes. We begin with some observations.

Let $\alpha\in I$ and $\iota_\alpha\colon U_\alpha \to U$ be the inclusion. By Lemma 6.17 the diagram

$$\begin{tikzcd}
H^k(U) \arrow[r, "\iota_\alpha^*"] \arrow[d, "D_U"] & H^k(U_\alpha) \arrow[d, "D_{U_\alpha}"] \\
H_c^{n-k}(U)^* \arrow[r, "\tilde{\iota}_\alpha"] & H_c^{n-k}(U_\alpha)^*
\end{tikzcd}$$

commutes. Hence we have that

$$D_{U_\alpha} \circ \iota_\alpha^* = \tilde{\iota}_\alpha \circ D_U. \tag{7.6}$$

Let then $p_\alpha \colon \bigoplus_\beta H_c^{n-k}(U_\beta) \to H_c^{n-k}(U_\alpha)$ be the canonical projection. We argue next that

$$\rho^*(L) = \sum_{\alpha \in I} \tilde{\iota}_\alpha(L) \circ p_\alpha \tag{7.7}$$

for each $L \in H_c^{n-k}(U)^*$. Let $([\omega_\alpha])_\alpha \in \bigoplus_\alpha H_c^{n-k}(U_\alpha)$. Then

$$\begin{aligned}
\rho^*(L)([\omega_\alpha])_\alpha = L(\rho(([\omega_\alpha])_\alpha)) = L\left(\sum_{\beta \in I} (\iota_\beta)_*[\omega_\beta]\right) = \sum_{\beta \in I} L((\iota_\beta)_*[\omega_\beta]) \\
= \sum_{\beta \in I} (\tilde{\iota}_\beta L)[\omega_\beta] = \left(\sum_{\beta \in I} (\tilde{\iota}_\beta L) \circ p_\beta\right)([\omega_\alpha])_\alpha,
\end{aligned}$$

where we used the linearity of L and the definitions of the maps ρ^*, $\tilde{\iota}_\alpha$ and p_α.

Now for each $[\omega] \in H^k(U)$ we have by (7.6), Lemma 7.4 and (7.7) that

$$\begin{aligned}
\left(\prod D_{U_\alpha} \circ \theta\right)([\omega]) &= \left(\prod D_{U_\alpha}\right)(\iota_\alpha^*[\omega])_\alpha = (D_{U_\alpha}(\iota_\alpha^*[\omega]))_\alpha \\
&= (\tilde{\iota}_\alpha D_U[\omega])_\alpha = \delta \circ \delta'(\tilde{\iota}_\alpha D_U[\omega])_\alpha \\
&= \delta\left(\sum_{\alpha \in I} (\tilde{\iota}_\alpha D_U[\omega]) \circ p_\alpha\right) = \delta(\rho^*(D_U[\omega])) \\
&= (\delta \circ \rho^* \circ D_U)([\omega]).
\end{aligned}$$

Hence the diagram commutes and D_U is an isomorphism. □

8 Proof of Poincaré duality

Definition 8.1. Let $I_1, \dots, I_n$ be intervals in $\mathbb{R}$. We call the set $I_1 \times \dots \times I_n \subset \mathbb{R}^n$ an *n-interval* or an *n-rectangle*. If all the intervals I_j, $j \in \{1, \dots, n\}$, are of the same length, we call it an *n-cube*.

We begin with the auxiliary result (Proposition 8.2) that an open set $U \subset \mathbb{R}^n$ is a finite union of pairwise disjoint unions of open n-intervals. Then the Poincaré duality follows for U. Namely, open n-intervals are diffeomorphic to $\mathbb{R}^n$, so they satisfy the Poincaré duality by Proposition 5.5 in section 5. Section 7 gives us that Poincaré duality also holds for the disjoint union of n-intervals. So the Poincaré duality holds for each pairwise disjoint union. Two pairwise disjoint unions may overlap, but since their intersection is only a pairwise disjoint union of n-rectangles, the Poincaré duality holds for their union by section 6. By Proposition 8.2 we need only finite number of the pairwise disjoint unions to cover U. Hence we may repeat the argument for larger unions of pairwise disjoint unions and gain after finite number of steps the whole set U.

Proposition 8.2. *Let $U \subset \mathbb{R}^n$ be an open set. Then there is a countable collection $\mathcal{G} \coloneqq \{Q_i \subset \mathbb{R}^n : i \in \mathbb{N}\}$ of open n-intervals so that $U = \bigcup_{i=1}^{\infty} Q_i$. Moreover, these n-intervals can be divided to a finite number of subcollections $\mathcal{F}_1, \dots, \mathcal{F}_{P(n)}$ of $\mathcal{G}$ so that in every $\mathcal{F}_j$ the n-intervals are pairwise disjoint.*

The idea of the proof is to first obtain a Whitney decomposition for the set U with closed n-cubes whose interiors are disjoint but the boundaries overlap. Then we divide these cubes into pairwise disjoint subcollections. We show that the closed n-cubes can be enlarged to open cubes so that they remain pairwise disjoint within each collection. First we prove the proposition for a bounded set U and after that in the general case.

Lemma 8.3 (Whitney decomposition)**.** *Let $U \subset \mathbb{R}^n$ be an open and bounded set. Then there is a collection $\mathcal{G}$ of closed n-cubes Q such that the side length of each Q is 2^{-k} for some $k \in \mathbb{N}$, $U = \bigcup_{Q \in \mathcal{G}} Q$, and the interiors of the n-intervals in $\mathcal{G}$ are pairwise disjoint.*

Proof. For each $k \in \mathbb{N}$ we define a grid of points with dyadic spacing

$$A_k \coloneqq \{(m_1 2^{-k}, \dots, m_n 2^{-k}) \in \mathbb{R}^n : m_1, \dots, m_n \in \mathbb{Z}\}$$

and the corresponding n-intervals (or n-cubes)

$$Q_x^k = [x^1, x^1 + 2^{-k}] \times \ldots \times [x^n, x^n + 2^{-k}],$$

where $x = (x^1, \ldots, x^n) \in A_k$. Now $\bigcup_{x \in A^k} Q^k_x = \mathbb{R}^n$ and $\mathrm{int}(Q^k_x) \cap \mathrm{int}(Q^k_y) = \emptyset$ whenever $x \neq y$. We denote $Q^k := \{Q^k_x \subset \mathbb{R}^n : x \in A_k\}$.

Let now $\mathcal{G}^1 := \{Q \in Q^1 : \mathrm{dist}(Q, \mathbb{R}^n \setminus U) > \sqrt{n}/2\}$. Then $\mathcal{G}^1$ is the collection of all cubes of side length $1/2$ contained in U, whose distance to $\mathbb{R}^n \setminus U$ is big enough. Choose then

$$\mathcal{G}^2 := \{Q \in Q^2 : \mathrm{dist}(Q, \mathbb{R}^n \setminus U) > \sqrt{n}/4, \mathrm{int}(Q) \cap \mathrm{int}(Q') = \emptyset \text{ for all } Q' \in \mathcal{G}^1\},$$

all the cubes fitted well inside U with side length $1/4$ that are not nested with any of the cubes already chosen to $\mathcal{G}^1$. We continue by induction and obtain, for an arbitrary $k \in \mathbb{N}$, a collection

$$\mathcal{G}^k := \{Q^k : \mathrm{dist}(Q^k, \mathbb{R}^n \setminus U) > \sqrt{n}\, 2^{-k},\ \mathrm{int}(Q^k) \cap \mathrm{int}(Q) = \emptyset \text{ for all } Q \in \bigcup_{l=1}^{k-1} \mathcal{G}^l\}.$$

Below we need the observation, that U is bounded and hence every $\mathcal{G}^k$ is finite. We define a countable collection of n-intervals

$$\mathcal{G} := \bigcup_{k=1}^{\infty} \mathcal{G}^k$$

and show that $\bigcup_{Q \in \mathcal{G}} Q = U$. By construction, $\bigcup_{Q \in \mathcal{G}} Q \subset U$. We still need to ensure that this collection indeed covers the set U. Let $x = (x_1, \ldots, x_n) \in U$. Since U is open, there is $r > 0$ so that $B(x, r) \subset U$. Let then $k_r \in \mathbb{N}$ be such that $2^{-k_r} < r/\sqrt{n}$. Now there is a point $y = (y_1, \ldots, y_n) \in A_{k_r}$ for which $0 < x^i - y^i < 2^{-k_r}$ for every coordinate $i = 1, \ldots, n$. Hence x is contained in a cube $Q^{k_r}_y$. Furthermore $Q^{k_r}_y \subset B(x, r)$, since

$$\max\{d(x, z) : z \in Q^{k_r}_y\} \leq \sqrt{n}\, 2^{-k_r} < r.$$

Since now $Q^{k_r}_y \subset U$, either $Q^{k_r}_y$ belongs to the collection $\mathcal{G}^{k_r}$ or there is $k < k_r$ so that $Q^{k_r}_y \subset Q^k$ for some $Q^k \in \mathcal{G}^k$. We conclude that U is covered by the collection of chosen n-cubes, which proves the claim. $\square$

Lemma 8.4. *Proposition 8.2 holds for a bounded open set $U \subset \mathbb{R}^n$.*

Proof. Consider the dyadic n-cubes given by Whitney decomposition and the collections $\mathcal{G}^k$ and $\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}^k$ in the proof of the lemma. We next construct collections $\mathcal{F}_1, \ldots, \mathcal{F}_m \subset \mathcal{G}$, where each $\mathcal{F}_j$, $j = 1, \ldots, m$, consists of pairwise disjoint n-intervals and $\mathcal{G} = \sqcup_{j=1}^{m} \mathcal{F}_j$. We start with the collection $\mathcal{F}_1$. The idea is to choose cubes starting from the largest ones and make the collection maximal. So we aim that for all $Q \in \mathcal{G} \setminus \mathcal{F}_1$ there exists $Q' \in \mathcal{F}_1$ so that $Q \cap Q' \neq \emptyset$.

Choose now any cube $Q_1 \in \mathcal{G}^1$ to the collection $\mathcal{F}_1$. Pick then any other cube $Q_2 \in \mathcal{G}^1$ so that $Q_1 \cap Q_2 = \emptyset$, if possible. Continue inductively by taking cubes in $\mathcal{G}^1$ so that the chosen cubes do not intersect any of the already picked ones. This process ends after finite number of steps, since $\mathcal{G}^1$ is finite. Then we have obtained a collection $\{Q_1, \ldots, Q_h\}$, $h \in \mathbb{N}$, that is maximal in $\mathcal{G}^1$. In other words, for every $Q \in \mathcal{G}^1$ there is Q_j, $j = 1, \ldots, h$, so that $Q \cap Q_j \neq \emptyset$.

Then pick a cube $Q \in \mathcal{G}^2$ that does not intersect any of the cubes $\{Q_1, \ldots, Q_h\}$ chosen into $\mathcal{F}_1$, if there exists one. Keep choosing cubes from collection $\mathcal{G}^2$ until no cubes can be chosen so that the collection $\mathcal{F}_1$ remains pairwise disjoint. Again this happens after finitely many steps. Continue similarly with the collections $\mathcal{G}^3, \mathcal{G}^4, \ldots$. We end up with a maximal collection $\mathcal{F}_1$ where all the cubes are pairwise disjoint. Indeed, the collection $\mathcal{F}_1$ clearly has the property that for each $Q \in \mathcal{G}$ there exists $Q' \in \mathcal{F}_1$ so that $Q \cap Q' \neq \emptyset$.

We next form collection $\mathcal{F}_2$ by repeating the above procedure for the sets $Q \in \mathcal{G} \setminus \mathcal{F}_1$. Choose any cube $Q \in \mathcal{G}^1 \setminus \mathcal{F}_1$ into collection $\mathcal{F}_2$. Then pick another $Q' \in \mathcal{G}^1 \setminus \mathcal{F}_1$ so that $Q \cap Q' = \emptyset$, if possible. Continue until no other cubes can be chosen from collection $\mathcal{G}^1 \setminus \mathcal{F}_1$. Then proceed to collection $\mathcal{G}^2 \setminus \mathcal{F}_1$ and continue inductively for each $\mathcal{G}^k \setminus \mathcal{F}_1$, $k \in \mathbb{N}$.

The collections $\mathcal{F}_3, \mathcal{F}_4, \ldots, \mathcal{F}_m$ are obtained like above, where each collection $\mathcal{F}_j$, $j = 1, \ldots, m$, consists of cubes $Q \in \mathcal{G} \setminus (\bigcup_{l=1}^{j-1} \mathcal{F}_l)$. In this way every $Q \in \mathcal{G}$ is chosen to some $\mathcal{F}_j$. Indeed, for each $Q' \in \mathcal{G}$, the set

$$\mathcal{G}' := \{Q \in \mathcal{G} : \operatorname{diam}(Q) \geq \operatorname{diam}(Q')\}$$

is finite. Denote $l := \#\mathcal{G}'$. Then we pick Q' to some collection $\mathcal{F}_j$, $j \leq l + 1$, since we choose cubes to collections starting from the biggest ones. If there exists cubes $Q_1 \in \mathcal{G}'$ and $Q_2 \in \mathcal{G}'$ so that $Q_1 \cap Q_2 \neq \emptyset$, we have $j < l + 1$. Namely, then some collection $\mathcal{F}_i$, $i \leq l$, contains both of the cubes Q_1 and Q_2 by construction.

We next show that with above procedure we may choose only finitely many collections $\mathcal{F}_m$. Pick a cube $Q^k \in \mathcal{F}_m$ for some large m, where k denotes the side length 2^{-k} of Q^k. Now there exists $Q^{k_1} \in \mathcal{F}_1$ for which $Q^{k_1} \cap Q^k \neq \emptyset$ and $k_1 \leq k$, so Q^{k_1} is larger than or of equal size to Q^k. Otherwise we would have picked Q^k to collection $\mathcal{F}_1$. Similarly for each $j = 1, \ldots, m-1$ there is a cube $Q^{k_j} \in \mathcal{F}_j$ so that $Q^{k_j} \cap Q^k \neq \emptyset$ and $k_j \leq k$. So there are $m - 1$ cubes, which intersect Q^k and whose side length is at least the side length of Q^k. But since the interiors of all the cubes are disjoint, there is a maximum number of cubes that can be fitted around Q^k. The number of surrounding cubes is maximal if the side length of these cubes is as small as possible, *i.e.*, $k_j = k$ for all $j = 1, \ldots, m-1$. Then there are $S(n)$ cubes that can intersect Q^k, where $S(n)$ is the number of all l-faces of an n-cube, $l = 0, \ldots, n-1$. Hence $m - 1 \leq S(n)$ and we get an upper bound $L(n) = S(n) + 1$.

We have now obtained a finite number of collections of closed n-intervals that are pairwise disjoint in each collection and decompose the bounded open set U. The next task is to find open n-intervals with the same properties. We are going to enlarge the closed n-intervals to open cubes so that there will not happen new overlaps. Take any cube $Q^k \in \mathcal{G}^k$ chosen in Whitney decomposition and define $d = \mathrm{dist}(Q^k, \mathbb{R}^n \setminus U)$. By construction $d > \sqrt{n}\, 2^{-k}$. Let then $Q^{k'} \in \mathcal{G}^{k'}$ be such that $Q^k \cap Q^{k'} \neq \emptyset$. We show next that $k' \leq k+1$, or in other words, $\mathrm{diam}(Q^{k'}) \geq \mathrm{diam}(Q^k)/2$.

In Whitney decomposition we start the construction with biggest appropriate cubes and continue to smaller ones. Then $Q^{k'}$ is the largest cube that has distance at least $\sqrt{n}\, 2^{-k'}$ to the boundary of U. Now if $k' = k+1$, then $Q^{k'}$ has this property, since

$$\mathrm{dist}(Q^{k+1}, \mathbb{R}^n \setminus U) \geq d - \mathrm{diam}(Q^{k+1}) > \sqrt{n}\, 2^{-k} - \sqrt{n}\, 2^{-(k+1)} = \sqrt{n}\, 2^{-(k+1)}.$$

The first inequality follows form the fact that $Q^{k+1} \cap Q^k \neq \emptyset$. So a cube with side length $2^{-(k+1)}$ can always be fitted beside Q^k and chosen to collection $\mathcal{G}^{k+1}$. Therefore any smaller cube does not intersect Q and hence $k' \leq k+1$.

By definition Q^k is of the form

$$Q^k = [x^1, x^1 + 2^{-k'}] \times \ldots \times [x^n, x^n + 2^{-k'}]$$

for some $(x^1, \ldots, x^n) \in U$. We define

$$\tilde{Q}^k = (x^1 - 2^{-(k+2)}, x^1 + 2^{-k} + 2^{-(k+2)}) \times \ldots \times (x^n - 2^{-(k+2)}, x^n + 2^{-k} + 2^{-(k+2)}).$$

Then $\tilde{Q}$ reaches less than half of the smallest cube intersecting it. If every n-interval given by Whitney decomposition is enlarged in this way, those cubes that in the beginning did not intersect each other, remain disjoint. Since also each $\tilde{Q} \subset U$, replacing the closed cubes Q with the corresponding open cubes $\tilde{Q}$ gives us the desired collections $\mathcal{F}_1, \ldots, \mathcal{F}_{L(n)}$ for any bounded open set U. $\square$

It still remains to generalize the result of Lemma 8.4 to hold for an unbounded open set U. We formulate as a lemma the following useful result.

Lemma 8.5. *The n-intervals $Q_j = [j_1, j_1 + 1] \times \ldots \times [j_n, j_n + 1]$, where $j = (j_1, \ldots, j_n) \in \mathbb{Z}^n$, cover $\mathbb{R}^n$. Also, they can be divided into collections $\mathcal{F}_1, \ldots, \mathcal{F}_{2^n}$ so that in every $\mathcal{F}_i$ the n-intervals are pairwise disjoint.*

Proof. We prove the lemma by induction. If $n = 1$, the claim clearly holds: choose the collections $I_1 = \{[2j, 2j+1] : j \in \mathbb{Z}\}$ and $I_2 = \{[2j+1, 2(j+1)] : j \in \mathbb{Z}\}$. Suppose then that there are collections $K_1, \ldots, K_{2^n}$ of n-intervals Q_j given by the induction assumption. Now every $(n+1)$-interval $Q'_j =$

$[j_1, j_1+1] \times \ldots \times [j_{n+1}, j_{n+1}+1]$ in $\mathbb{R}^{n+1}$ is of the form $Q_j \times [j_{n+1}, j_{n+1}+1]$, where $Q_j = [j_1, j_1+1] \times \ldots \times [j_n, j_n+1] \subset \mathbb{R}^n$ and $j_{n+1} \in \mathbb{Z}$. Define collections

$$K_1 \times I_1, \ldots, K_{2^n} \times I_1, K_1 \times I_2, \ldots, K_{2^n} \times I_2,$$

where

$$K_i \times I_l = \{Q \times J : Q \in K_i,\ J \in I_l\}.$$

Now each collection $K_i \times I_l$ consists of pairwise disjoint $(n+1)$-intervals, since we assumed that in each collection K_i the cubes $Q \in K_i$ are pairwise disjoint. Also each $(n+1)$-interval $[j_1, j_1+1] \times \ldots \times [j_{n+1}, j_{n+1}+1]$ belongs to one of these collections. Hence we found $2 \cdot 2^n = 2^{n+1}$ collections that partition the $(n+1)$-intervals covering $\mathbb{R}^{n+1}$. □

Proof of Prop. 8.2. Let U be any open set in $\mathbb{R}^n$ and let Q_j be a closed n-interval of side length 1 in Lemma 8.5 for $j \in \mathbb{Z}^n$. We replace Q_j for each $j \in \mathbb{Z}^n$ with an open n-interval $\tilde{Q}_j = (j_1 - 1/10, j_1 + 1 + 1/10) \times \ldots \times (j_n - 1/10, j_n + 1 + 1/10)$. Then collections $K_1, \ldots, K_{2^n}$ remain pairwise disjoint and the cubes $\tilde{Q}_j$, for $j \in \mathbb{Z}^n$, cover $\mathbb{R}^n$.

For $j \in \mathbb{Z}^n$, let $U_j = U \cap \tilde{Q}_j$. Then each U_j is an open and bounded set. By Lemma 8.4 we find for each $j \in \mathbb{Z}^n$ pairwise disjoint collections $\mathcal{F}_1^j, \ldots, \mathcal{F}_{L(n)}^j$ of open n-cubes so that $U_j = \bigcup_{k=1}^{L(n)} \bigcup_{Q \in \mathcal{F}_k^j} Q$. Note that since $L(n)$ is only an upper bound, some of the collections $\mathcal{F}_i^j$ may be empty. We define, for $l = 1, \ldots, 2^n$, the collection

$$\mathcal{F}_{li} = \bigcup_{\{j : Q_j \in K_l\}} \mathcal{F}_i^j$$

of n-cubes. Then each collection $\mathcal{F}_{li}$ for $l = 1, \ldots, 2^n$ and $i = 1, \ldots, L(n)$ is countable and pairwise disjoint. By renaming each $\mathcal{F}_{li}$ we obtain collections $\mathcal{F}_1, \ldots, \mathcal{F}_{P(n)}$, where $P(n) = 2^n \cdot L(n)$. Also

$$\bigcup_{j=1}^{P(n)} \bigcup_{Q \in \mathcal{F}_j} Q = U,$$

where each Q is an open n-cube. The proof is complete. □

It only remains to collect the results and apply Proposition 8.2.

Proof of Theorem 4.1 (Poincaré duality). Let $\mathcal{F}_1, \ldots, \mathcal{F}_{P(n)}$ be the collections of open n-cubes $Q \subset U$ given by Proposition 8.2. So we may present U as

$$U = \bigcup_{k=1}^{P(n)} U_k,$$

where each

$$U_k = \bigsqcup_{Q \in \mathcal{F}_k} Q.$$

The open n-cubes Q are diffeomorphic to $\mathbb{R}^n$, so by Proposition 5.5 each Q satisfies the Poincaré duality. Then Proposition 7.5 gives us that the Poincaré duality holds for each U_k.

We prove next by induction that the set $U_1 \cup \dots \cup U_m$ satisfies the Poincaré duality for $m \leq P(n)$. Consider first the set $U_1 \cap U_2$. It is a union of pairwise disjoint n-rectangles, which are also diffeomorphic to $\mathbb{R}^n$. By Propositions 5.5 and 7.5 the Poincaré duality holds for the set $U_1 \cap U_2$. Hence by Proposition 6.16 also the union $U_1 \cup U_2$ satisfies the Poincaré duality.

Suppose now that the set $U_1 \cup \dots \cup U_{m-1}$ satisfies the Poincaré duality. In order to use Proposition 6.16 we need to show that the set

$$(U_1 \cup U_2 \cup \dots \cup U_{m-1}) \cap U_m = (U_1 \cap U_m) \cup (U_2 \cap U_m) \cup \dots \cup (U_{m-1} \cap U_m)$$

satisfies Poincaré duality. Now

$$(U_1 \cap U_m) \cap (U_2 \cap U_m) = U_1 \cap U_2 \cap U_m,$$

which is a pairwise disjoint union of n-rectangles as a finite intersection of pairwise disjoint n-intervals. So it satisfies the Poincaré duality. Hence also the set $(U_1 \cap U_m) \cup (U_2 \cap U_m)$ satisfies the Poincaré duality by Proposition 6.16. Consider then the set

$$((U_1 \cap U_m) \cup (U_2 \cap U_m)) \cap (U_3 \cap U_m) = (U_1 \cap U_3 \cap U_m) \cup (U_2 \cap U_3 \cap U_m).$$

Again the set

$$(U_1 \cap U_3 \cap U_m) \cap (U_2 \cap U_3 \cap U_m) = U_1 \cap U_2 \cap U_3 \cap U_m$$

satisfies the Poincaré duality by Propositions 5.5 and 7.5. So the union $(U_1 \cap U_3 \cap U_m) \cup (U_2 \cap U_3 \cap U_m)$ and further the set

$$((U_1 \cap U_m) \cup (U_2 \cap U_m)) \cup (U_3 \cap U_m)$$

satisfy Poincaré duality by Proposition 6.16. We may repeat argument for each set $(U_k \cap U_m)$, $k < m$. Thus the Poincaré duality holds for the set $(U_1 \cup U_2 \cup \dots \cup U_{m-1}) \cap U_m$. Then by Proposition 6.16 the Poincaré duality is true for $U_1 \cup \dots \cup U_m$. Choosing $m = P(n)$ gives us that the Poincaré duality holds for

$$U_1 \cup \dots \cup U_{P(n)} = U,$$

which proves the claim. $\square$

9 Punctured plane

We give an illustrative example of the Poincaré duality in the punctured plane $\mathbb{R}^2 \setminus \{0\}$. We are about to find out explicit representatives for the cohomology classes in $H^k(\mathbb{R}^2 \setminus \{0\})$ and $H_c^k(\mathbb{R}^2 \setminus \{0\})$, and see how the isomorphism $D_{\mathbb{R}^2\setminus\{0\}}$ connects the basis elements.

In Example 6.11 we proved using the Mayer–Vietoris sequence that

$$H^k(\mathbb{R}^2 \setminus \{0\}) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0 \text{ or } 1 \\ \{0\}, & \text{if } k = 2. \end{cases}$$

Now the Poincaré duality (Theorem 4.1) states that

$$H^k(\mathbb{R}^2 \setminus \{0\}) \cong H_c^{n-k}(\mathbb{R}^2 \setminus \{0\})^*.$$

On the other hand, by Theorem 2.7, we have that

$$H_c^{n-k}(\mathbb{R}^2 \setminus \{0\})^* \cong H_c^{n-k}(\mathbb{R}^2 \setminus \{0\}).$$

Hence

$$H_c^k(\mathbb{R}^2 \setminus \{0\}) \cong \begin{cases} \{0\}, & \text{if } k = 0 \\ \mathbb{R}, & \text{if } k = 1 \text{ or } 2. \end{cases}$$

The duality of $H^2(\mathbb{R}^2 \setminus \{0\})$ and $H_c^0(\mathbb{R}^2 \setminus \{0\})$ is thus trivial. We find next the basis elements for the non-trivial cohomology groups.

As we have argued in the proof of Lemma 3.3,

$$H^0(\mathbb{R}^2 \setminus \{0\}) \cong \{f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} : f \text{ is constant}\}.$$

Thus we have

$$H^0(\mathbb{R}^2 \setminus \{0\}) = \operatorname{span}([\chi_{\mathbb{R}^2\setminus\{0\}}]).$$

Consider then the compactly supported 2-forms on $\mathbb{R}^2\setminus\{0\}$. Since $\dim(H_c^2(\mathbb{R}^2\setminus \{0\})) = 1$, any non-trivial element forms a basis for this space. Now

$$\Omega_c^2(\mathbb{R}^2 \setminus \{0\}) = \{f \, \mathrm{d}x \wedge \mathrm{d}y : f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})\}.$$

Hence the function f must be zero in some neighbourhood of the origin. To simplify the notation, let us take into use the polar coordinates. We treat the coordinates x and y from now on as smooth functions $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ of variables r and θ such that

$$x(r, \theta) = r \cos\theta$$

and

$$y(r, \theta) = r \sin\theta. \tag{9.1}$$

By the definition of differential, we have that

$$\mathrm{d}x = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta$$

and

$$\mathrm{d}y = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta. \tag{9.2}$$

A direct calculation shows that $\mathrm{d}x \wedge \mathrm{d}y = r\,\mathrm{d}r \wedge \mathrm{d}\theta$, so $\mathrm{d}r$ and $\mathrm{d}\theta$ are indeed linearly independent and the elements of $\Omega^2(\mathbb{R}^2 \setminus \{0\})$ can be written as $f(r,\theta)\,\mathrm{d}r \wedge \mathrm{d}\theta$.

Let now $f \in C_0^\infty(\mathbb{R}_+)$ such that $\int_0^\infty f(t)\mathrm{d}t =: C > 0$. We claim that $[f(r)\,\mathrm{d}r \wedge \mathrm{d}\theta]$ spans $H_c^2(\mathbb{R}^2 \setminus \{0\})$. Since any 2-form defined on $\mathbb{R}^2 \setminus \{0\}$ is closed, the class $[f(r)\,\mathrm{d}r \wedge \mathrm{d}\theta]$ is indeed an element in $H_c^2(\mathbb{R}^2 \setminus \{0\})$. By Lemma 3.6 a compactly supported 2-form ω is exact if and only if the integral $\int_{\mathbb{R}^2} \omega = 0$. Since the form $f(r)\,\mathrm{d}r \wedge \mathrm{d}\theta$ can now be pushed forward to $\mathbb{R}^2$ via inclusion, we deduce that $f(r)\,\mathrm{d}r \wedge \mathrm{d}\theta$ is exact in $\mathbb{R}^2 \setminus \{0\}$ only if the integral vanishes. However, by Fubini's theorem for polar coordinates

$$\int_{\mathbb{R}^2\setminus\{0\}} f(r)\,\mathrm{d}r \wedge \mathrm{d}\theta = \int_0^{2\pi} \left(\int_0^\infty f(r)\,\mathrm{d}r \right) \mathrm{d}\theta = 2\pi C > 0.$$

Hence

$$H_c^2(\mathbb{R}^2 \setminus \{0\}) = \mathrm{span}([f(r)\,\mathrm{d}r \wedge \mathrm{d}\theta]).$$

To study the duality of $H^0(\mathbb{R}^2 \setminus \{0\})$ and $H_c^2(\mathbb{R}^2 \setminus \{0\})$, let $[a\chi_{\mathbb{R}^2\setminus\{0\}}] =: [a] \in H^0(\mathbb{R}^2 \setminus \{0\})$ and $[bf(r)\,\mathrm{d}r \wedge \mathrm{d}\theta] \in H_c^2(\mathbb{R}^2 \setminus \{0\})$, where $a, b \in \mathbb{R}$. Now

$$D_{\mathbb{R}^2\setminus\{0\}}([a])[bf(r)\,\mathrm{d}r \wedge \mathrm{d}\theta] = \int_{\mathbb{R}^2\setminus\{0\}} a \cdot bf(r)\,\mathrm{d}r \wedge \mathrm{d}\theta = ab2\pi C.$$

We see that for any non-zero $[a]$, the map $D_{\mathbb{R}^2\setminus\{0\}}([a]) \in H_c^2(\mathbb{R}^2 \setminus \{0\})^*$ is non-trivial. On the other hand, Let L be a linear map in $H_c^2(\mathbb{R}^2 \setminus \{0\})^*$. Then

$$L([f(r)\,\mathrm{d}r \wedge \mathrm{d}\theta]) = s$$

for some $s \in \mathbb{R}$ and we have that $D_{\mathbb{R}^2\setminus\{0\}}([s/(2\pi C)]) = L$.

We are now left with the case $k = 1$, which is perhaps the most interesting. The spanning element of $H^0(\mathbb{R}^2 \setminus \{0\})$ is any 1-form defined on $\mathbb{R}^2 \setminus \{0\}$ that is closed but not exact. Since any $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$ is of the form

$$\omega = f_1\mathrm{d}x + f_2\mathrm{d}y,$$

where $f_1, f_2 \in C^\infty(\mathbb{R}^2 \setminus \{0\})$, we have that the differential of ω is

$$\mathrm{d}\omega = \mathrm{d}f_1 \wedge \mathrm{d}x + \mathrm{d}f_2 \wedge \mathrm{d}y = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y.$$

Therefore a 1-form ω is closed whenever

$$\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y}. \tag{9.3}$$

Let us consider the form

$$\eta := \frac{1}{x^2 + y^2}(-y\,\mathrm{d}x + x\,\mathrm{d}y).$$

Since

$$\frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{y^2-x^2}{(y^2+x^2)^2} = \frac{\partial}{\partial y}\frac{-y}{x^2+y^2},$$

we have that η is closed and thus $[\eta] \in H^1(\mathbb{R}^2 \setminus \{0\})$. If η is also exact, there exists a smooth function $g\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $\mathrm{d}g = \eta$. In other words, for this g we have that

$$\frac{\partial g}{\partial x} = \frac{-y}{x^2+y^2} \quad \text{and} \quad \frac{\partial g}{\partial y} = \frac{x}{x^2+y^2}.$$

However, there does not exist a smooth function that satisfies these conditions. Suppose that this kind of g exists. Then we have by the fundamental theorem of calculus that

$$g(x,y) = \int \frac{-y}{x^2+y^2}\,\mathrm{d}x + A(y) = -\arctan\left(\frac{x}{y}\right) + A(y),$$

where $A(y)$ is some function depending only on y. On the other hand

$$g(x,y) = \int \frac{x}{x^2+y^2}\,\mathrm{d}y + B(x) = \arctan\left(\frac{y}{x}\right) + B(x),$$

where $B(x)$ is a function of x. Clearly this is a contradiction. Thus η is not exact and

$$H^1(\mathbb{R}^2 \setminus \{0\}) = \operatorname{span}([\eta]).$$

To gain some intuition about the form η, we may again transfer to the polar coordinates. Substitution of equations (9.1) and (9.2) into the definition of η gives us

$$\eta = \mathrm{d}\theta. \tag{9.4}$$

From this identity one could think that η is an exact form, since it can be written as a differential of a function. This is, however, not correct reasoning, since the function $\theta(x,y)$ is not continuous, much less smooth. Therefore the notation $\mathrm{d}\theta$ is misleading. In $\mathbb{R}^2 \setminus \{0\}$ the 1-form η is well-defined and the equation (9.4) should be interpreted so that η agrees with the differential of θ, when the branch of θ is correctly chosen. In complex analysis, the integral of $\mathrm{d}\theta$ over a line is the *winding number* of the line around the origin.

Now that we have found a non-trivial element $[\eta] \in H^1(\mathbb{R}^2 \setminus \{0\})$, the Poincaré duality suggests us immediately a basis element for $H^1_c(\mathbb{R}^2 \setminus \{0\})$. Let $f \in C_0^\infty(\mathbb{R}_+)$ be the function in the basis element of $H^2_c(\mathbb{R}^2 \setminus \{0\})$ for which $\int_0^\infty f(t)\mathrm{d}t = C$. Then for a compactly supported 1-form $f(r)\,\mathrm{d}r$ we have that

$$\int_{\mathbb{R}^2\setminus\{0\}} \eta \wedge (f(r)\,\mathrm{d}r) = -\int_{\mathbb{R}^2\setminus\{0\}} f(r)\,\mathrm{d}r \wedge \mathrm{d}\theta = -2\pi C.$$

We claim that $f(r)\,\mathrm{d}r$ defines a non-trivial compactly supported cohomology class if it is closed. Indeed, the map $D_{\mathbb{R}^2\setminus\{0\}}([\eta]) \in H^1_c(\mathbb{R}^2 \setminus \{0\})^*$ is linear, so for any $[\tau] \in H^1_c(\mathbb{R}^2 \setminus \{0\})$ we have that $D_{\mathbb{R}^2\setminus\{0\}}([\eta])[\tau] \neq 0$ exactly when $[\tau]$ is non-trivial. If $f(r)\,\mathrm{d}r$ is closed, the equivalence class $[f(r)\,\mathrm{d}r]$ is well-defined. Then by the above calculation $D_{\mathbb{R}^2\setminus\{0\}}([\eta])[f(r)\,\mathrm{d}r] = -2\pi C \neq 0$.

We show next that $f(r)\,\mathrm{d}r$ is closed. Now $r(x,y) = \sqrt{x^2+y^2}$ is a smooth function on $\mathbb{R}^2 \setminus \{0\}$. Therefore, in the Cartesian coordinates we have that

$$f(r)\,\mathrm{d}r = f(r)\left(\frac{\partial r}{\partial x}\mathrm{d}x + \frac{\partial r}{\partial y}\mathrm{d}y\right) = \frac{f(r)}{\sqrt{x^2+y^2}}(x\,\mathrm{d}x + y\,\mathrm{d}y).$$

Also, the chain rule gives us

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial f}{\partial r}\frac{x}{\sqrt{x^2+y^2}}$$

and similarly

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{y}{\sqrt{x^2+y^2}}.$$

Using these identities we get after a straightforward calculation that

$$\frac{\partial}{\partial x}\frac{f(r)y}{\sqrt{x^2+y^2}} = \frac{xy}{r^2}\left(\frac{\partial f}{\partial r} + \frac{f(r)}{r}\right) = \frac{\partial}{\partial y}\frac{f(r)x}{\sqrt{x^2+y^2}}.$$

By condition (9.3) we have that $f(r)\,\mathrm{d}r$ is closed and hence $[f(r)\,\mathrm{d}r] \in H^1_c(\mathbb{R}^2 \setminus \{0\})$. As we argued above, by Poincaré duality $[f(r)\,\mathrm{d}r]$ is non-trivial and thus

$$H^1_c(\mathbb{R}^2 \setminus \{0\}) = \mathrm{span}([f(r)\,\mathrm{d}r]).$$

It is an interesting fact that the basis elements of $H^1(\mathbb{R}^2 \setminus \{0\})$ and $H^1_c(\mathbb{R}^2 \setminus \{0\})$ have representatives which are pointwise linearly independent. One could think that the non-trivial elements in $H^1_c(\mathbb{R}^2 \setminus \{0\})$ could be obtained from the elements in $H^1(\mathbb{R}^2 \setminus \{0\})$ by multiplying with a compactly supported smooth function. This is, however, not correct reasoning. For example, a direct calculation shows that the form $g(r)\eta$, where $g \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, satisfies the condition (9.3) if and only if the function g is constant. Hence the 1-form $g(r)\eta$ does not define a (non-zero) compactly supported cohomology class.

Let us finally study closer the exactness of $f(r)\,\mathrm{d}r$. By the fundamental theorem of calculus, we have that $f(r)\,\mathrm{d}r = \mathrm{d}F$ for

$$F(r) = \int_0^r f(t)\,\mathrm{d}t + A,$$

where $A \in \mathbb{R}$. Since F is a smooth function, $[f(r)\,\mathrm{d}r]$ is indeed trivial as an element of $H^1(\mathbb{R}^2 \setminus \{0\})$. Nevertheless, F is not compactly supported for any $A \in \mathbb{R}$. To show this, suppose it is. Since F must equal zero in some neighbourhood of the origin, necessarily $A = 0$. Then, since f is compactly supported, there exists $r' \in \mathbb{R}$ such that $F(r) = \int_0^\infty f(t)\,\mathrm{d}t = C$ for all $r > r'$, which is a contradiction. Hence the form $[f(r)\,\mathrm{d}r]$ is non-trivial as an element of $H^1_c(\mathbb{R}^2 \setminus \{0\})$.

10 Conclusion

The Poincaré duality is one of the main theorems in algebraic and differential topology. Its proof consists of several independent auxiliary results, which all require different approach. The proof of Poincaré Lemma and the compactly supported case in $\mathbb{R}^n$ (Section 5) are good exercises on differential geometry. Section 6 is an outlook to the algebraic methods of algebraic topology. The general definition of homology, exact sequences and working with commutative diagrams are all basic tools in algebraic topology.

The rest of the thesis does not rely on the properties of de Rham cohomology as much. The proof for disjoint unions in Section 7 was a lesson on linear algebra and the actual proof of the Poincaré duality was totally different to the other parts by nature. The proof of Poincaré duality was about basic analysis, mimicing the proof of the Besicovitch covering theorem, and it took advantage of the Whitney decomposition.

Most of all the proof to the Poincaré duality serves as an introduction to de Rham cohomology. Especially the worked out example in a low-dimensional, simple case clarified the difference between the cohomology groups and compactly supported cohomologies.

References

[1] Jean Dieudonné. *A History of Algebraic and Differential Topology 1900-1960*. Birkhäuser, Boston, 1989.

[2] William Fulton. *Algebraic Topology: A First Course*. Springer-Verlag, New York, 1995. Graduate Texts in Mathematics.

[3] Werner Greub. *Linear Algebra*. Springer-Verlag, New York, fourth edition, 1981.

[4] Ib Madsen and Jørgen Tornehave. *From Calculus to Cohomology*. Cambridge University Press, Cambridge, 1997. De Rham cohomology and characteristic classes.

[5] Pekka Pankka. Introduction to de Rham cohomology. Lecture notes, 2013.

[6] Loring W. Tu. *An Introduction to Manifolds*. Springer, New York, 2008.