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CALDERÓN'S PROBLEM FOR p -LAPLACE TYPE EQUATIONS

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Abstract

We investigate a generalization of Calderón's problem of recovering the conductivity coefficient in a conductivity equation from boundary measurements. As a model equation we consider the p -conductivity equation

$$\operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0$$

with $1 < p < \infty$, which reduces to the standard conductivity equation when $p = 2$.

The thesis consists of results on the direct problem, boundary determination and detecting inclusions. We formulate the equation as a variational problem also when the conductivity σ may be zero or infinity in large sets. As a boundary determination result we recover the first order derivative of a smooth conductivity on the boundary. We use the enclosure method of Ikehata to recover the convex hull of an inclusion of finite conductivity and find an upper bound for the convex hull if the conductivity within an inclusion is zero or infinite.

Tiivistelmä

Calderónin ongelma kysyy: voidaanko johtavuuden arvo johtavuusyhtälössä määrittää reunamittauksia käyttäen? Tutkimme Calderónin ongelman yleistystä tilanteeseen, jossa suoraa ongelmaa kuvaa p -johtavuusyhtälö

$$\operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0,$$

missä $1 < p < \infty$. Yhtälö on tavallinen johtavuusyhtälö, jos $p = 2$.

Väitöskirjan tulokset koskevat suoraa ongelmaa, reunamäärittystä ja sisältymän (eli inklusion tai esteen) etsimistä. Yleistämme suoran ongelman tilanteeseen, jossa johtavuus σ voi saada arvon nolla tai ääretön suurissa joukoissa. Reunamäärittystuloksena määritämme siistin johtavuuden gradientin tutkitun alueen reunalla. Käytämme Ikehatan kotelointimenetelmää määrittääksemme äärellisjohtavuuksisen sisältymän kuperan verhon, ja ylärajan verholle, jos johtavuus sisältymässä on nolla tai ääretön.

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List of included articles

This thesis is based on the work contained within the following publications:

- [I] **Enclosure method for the p -Laplace equation**
T. Brander, M. Kar, and M. Salo,
Inverse Problems 31(4):045001 (2015).

- [II] **Calderón problem for the p -Laplacian: First order derivative of conductivity on the boundary**
T. Brander,
Proceedings of the American Mathematical Society 144(1):177–189 (2016).

- [III] **Superconductive and insulating inclusions for non-linear conductivity equations**
T. Brander, J. Ilmavirta, and M. Kar,
preprint.

The author has participated actively in development of the joint papers [I, III].

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1. Inverse problems

The field of inverse problems is motivated by practical problems, where the objective is to recover information on a medium by indirect measurements; for example:

- To what extent do X-rays penetrate different human tissue and how much of them is absorbed? After measuring the absorption of X-rays, the inverse problem is to calculate which materials the rays passed through.
- How does the composition of the crust of Earth affect the movement of shock waves and sound waves? The inverse problem is to use measurements of the waves to find out the material properties of Earth.
- How does the value of an option (a financial instrument) depend on its volatility? The inverse problem is to determine the volatility of an option from the prices with which it is bought and sold in the free market.
- How does the heat conductivity of an object affect the distribution of heat within it? The inverse problem is to determine the heat conductivity by measuring heat and heat flow on the surface of the object.
- Electrical impedance tomography asks the previous question, but with heat replaced by electricity.

The inverse problems examined in this thesis are related to recovering the conductivity of heat or electricity from surface measurements.

2. Calderón's problem and electrical impedance tomography

We investigate p -Calderón's problem in one dimension before posing the standard Calderón's problem in higher dimensions; in our nomenclature, the standard problem is called 2-Calderón's problem.

We use the following notations throughout the thesis:

- Open bounded set $\Omega \subset \mathbb{R}^d$, where $d \in \mathbb{Z}_+$ is the dimension. We assume $d \geq 2$ except in section 2.2, where $d = 1$.
- Potential $u: \Omega \rightarrow \mathbb{R}$. The potential solves the p -conductivity equation, which will be discussed below. The potential may be electric potential (electric field potential or electrostatic potential), or temperature. We only consider steady state equations, where the potential does not depend on time.
- Conductivity $\sigma: \Omega \rightarrow [0, \infty]$, which we always assume to be measurable. The conductivity may be electrical conductivity or heat conductivity. Sometimes we consider an extension to $\bar{\Omega}$, which we also write as σ .

Also, in the following physical introduction only, we use the current flux or heat flux $j: \Omega \rightarrow \mathbb{R}^d$.

2.1. Physical background. The negative current flux is proportional to differences in potential and to conductivity by Ohm's law or by Fourier's law of thermal

conduction:

$$(2.1) \quad -j = \sigma \nabla u$$

We assume there are no sources or sinks of electricity or heat inside the object Ω , whence by conservation of energy (or Gauss's law or Kirchhoff's law)

$$(2.2) \quad \operatorname{div} j = 0.$$

Thus, by equations (2.1) and (2.2), the conductivity equation

$$(2.3) \quad -\operatorname{div}(\sigma \nabla u) = 0$$

models the steady state of conduction of heat or electricity.

The thesis discusses a non-linear variant of the conductivity equation, where the linear relation (2.1) is replaced with a non-linear power law relation

$$(2.4) \quad -j = \sigma |\nabla u|^{p-2} \nabla u,$$

which leads to the p -conductivity equation

$$(2.5) \quad -\operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0.$$

The materials that do not obey Ohm's law are called non-Ohmic. The power-law behaviour is one possible non-Ohmic behaviour and may be a fine approximation in more complicated non-Ohmic situations. In context of electricity, power-law behaviour has been observed in certain polycrystalline materials near the superconducting-normal transition [16, 11]. Other applications of the p -Laplace equation, which is the p -conductivity equation with $\sigma \equiv 1$, include nonlinear dielectrics [11, 20, 21, 39, 53, 54] and plastic moulding [2]. It is also used to model electro-rheological and thermo-rheological fluids [1, 5, 48], fluids governed by a power law [3], viscous flows in glaciology [23] and some plasticity phenomena [4, 30, 45, 46, 52]. There are further applications to image processing [37] and conformal geometry [41, 35].

2.2. One dimensional case. As a simple special case we investigate a one-dimensional situation, where the domain Ω is an interval, which we write as the open¹ interval $]a, b[$ with $a, b \in \mathbb{R}$. The treatment is somewhat heuristical, as the precise function space where the problem is solved is not defined here, the variational formulation for the forward problem is not justified and the definition of the Dirichlet to Neumann map is not justified. We also use the strong formulation of the forward problem without any worries concerning regularity when doing so is expedient. We give a rigorous treatment in article [III]; see also section 3.

Here and elsewhere the parameter p may always take values strictly between one and infinity, unless otherwise mentioned. The one-dimensional case with $p = 2$ can be found at least in the unpublished manuscript [17].

¹I use the customary Finnish notation for open interval, $]a, b[$, rather than the notation (a, b) , since the first is clearly distinct from a vector in dimension two.

We consider the conductivity to be given by a measurable function $\sigma:]a, b[\rightarrow [0, \infty]$, which we assume to be essentially bounded from above and away from zero in the complement of $D_0 \cup D_\infty$, where and henceforth we write $D_j = \sigma^{-1}(\{j\})$. We also assume that D_0 and D_∞ are unions of finite numbers of open intervals, the closures of which are disjoint from each other and the boundary $\{a, b\}$.

Given some Dirichlet boundary values $(A, B) \in \mathbb{R}^2$, the Dirichlet problem for the one-dimensional p -conductivity equation is

$$(2.6) \quad \begin{cases} (\sigma(x) |u'(x)|^{p-2} u'(x))' = 0 & \text{for } a < x < b \\ u(a) = A \\ u(b) = B. \end{cases}$$

The set of both Dirichlet and Neumann boundary values is \mathbb{R}^2 , so the strong Dirichlet to Neumann map is $\Lambda_\sigma: \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$(2.7) \quad (A, B) \mapsto \left(-\sigma(a) |u'(a)|^{p-2} u'(a), \sigma(b) |u'(b)|^{p-2} u'(b) \right),$$

where u solves the p -conductivity equation (2.6) and when u and σ are smooth enough. The weak definition is $\Lambda_\sigma: \mathbb{R}^2 \rightarrow \{L: \mathbb{R}^2 \rightarrow \mathbb{R}; L \text{ linear}\}$,

$$(2.8) \quad \langle \Lambda_\sigma((A, B)), (\alpha, \beta) \rangle = \int_a^b \sigma |u'|^{p-2} u' h' dx,$$

where $h \in W^{1,p}(\Omega)$, $h(a) = \alpha$, $h(b) = \beta$, $h'|_{D_\infty} = 0$, and u solves the problem (2.6). Supposing the conductivity is somewhat regular near a and b , the strong Dirichlet to Neumann map can be recovered from the weak one by using test functions h with

$$(2.9) \quad \begin{cases} h'(x) = 1/\varepsilon & \text{while } x \geq b - \varepsilon \\ h'(x) = 0 & \text{otherwise} \end{cases}$$

and similar test functions with slope $-1/\varepsilon$ near a that are constant elsewhere.

Before proceeding further with the inverse problem, we consider the issues of existence and uniqueness for the forward problem (2.6). We find out that the correct space for the solutions is a close relative of the space $W^{1,p}(\Omega)$ with the correct Dirichlet boundary values. For rigorous and more precise treatment of the issues see article [III] and section 3.

The solution is discovered by minimizing the energy

$$(2.10) \quad I(v) = \int_a^b \sigma(x) |v'(x)|^p dx.$$

We shall explicitly solve the problem (2.6) by using both the strong formulation of the problem and the energy minimization formulation (2.10).

We first assume that $D_0 \neq \emptyset$. A solution for our problem is

$$(2.11) \quad u(x) = \begin{cases} A & \text{for } x \leq \inf D_0 \\ g(x) & \text{for } \inf D_0 \leq x \leq \sup D_0 \\ B & \text{for } x \geq \sup D_0, \end{cases}$$

where $g(\inf D_0) = A$, $g(\sup D_0) = B$, and $g' = 0$ outside D_0 . We may take g to be smooth. We have $I(u) = 0$, so u minimizes the energy. Any other minimizer must be constant outside D_0 , since its energy must vanish. In particular, $\Lambda_\sigma((A, B)) = 0$.

Next we consider the more interesting situation $D_0 = \emptyset$. By the fundamental theorem of calculus we calculate the solution of problem (2.6) to be

$$(2.12) \quad u(x) = A + \frac{B - A}{\int_a^b \sigma^{1/(1-p)}(t) dt} \int_a^x \sigma^{\frac{1}{1-p}}(t) dt.$$

This makes sense even when $\sigma = \infty$; the integrand is then zero and thence the solution u is locally constant in regions of infinite conductivity.

Now we have explicitly solved the forward problem:

Theorem 2.1. *Suppose $\Omega \subset \mathbb{R}$ is an interval, which we write as the open interval $]a, b[$, and $a, b \in \mathbb{R}$. Let σ be a measurable function mapping $]a, b[$ to $[0, \infty]$. We assume it is essentially bounded from above and away from zero in the complement of $D_0 \cup D_\infty$. We also assume that D_0 and D_∞ are unions of finite numbers of open intervals and $\Omega \setminus D_\infty$ is a set of positive Lebesgue measure.*

Then the function $u \in W^{1,p}(\Omega)$ defined as

$$(2.13) \quad u = \begin{cases} w_0 & \text{if } D_0 \neq \emptyset \\ w_1 & \text{if } D_0 = \emptyset, \end{cases}$$

with

$$(2.14) \quad w_0(x) = \begin{cases} A & \text{for } x \leq \inf D_0 \\ g(x) & \text{for } \inf D_0 \leq x \leq \sup D_0 \\ B & \text{for } x \geq \sup D_0, \end{cases}$$

where $g(\inf D_0) = A$, $g(\sup D_0) = B$, $g'|_{\Omega \setminus D_0} = 0$ and $g \in W^{1,p}(\Omega)$, and

$$(2.15) \quad w_1(x) = A + \frac{B - A}{\int_a^b \sigma^{1/(1-p)}(t) dt} \int_a^x \sigma^{\frac{1}{1-p}}(t) dt,$$

solves the one-dimensional p -conductivity equation (2.6) in the weak sense with Dirichlet boundary values (A, B) . We interpret $\infty^{1/(1-p)}$ as zero.

The strong Dirichlet to Neumann map is

$$(2.16) \quad \Lambda_\sigma(A, B) = \left(\frac{A - B}{\int_a^b \sigma^{1/(1-p)}(t) dt}, \frac{B - A}{\int_a^b \sigma^{1/(1-p)}(t) dt} \right),$$

since

$$(2.17) \quad \sigma |u'|^{p-2} u' = \frac{B - A}{\int_a^b \sigma^{1/(1-p)}(t) dt}.$$

Since $A - B$ is known from the Dirichlet data, all we can learn is the quantity

$$(2.18) \quad \int_a^b \sigma^{1/(1-p)}(t) dt.$$

We can slightly weaken our assumptions by accepting that either $a \in \overline{D_\infty}$ or $b \in \overline{D_\infty}$ with no loss in results, but if both are true, then that is all we can observe.

In a similar way the weak Dirichlet to Neumann map gives

$$(2.19) \quad \begin{aligned} \langle \Lambda_\sigma(A, B), (\alpha, \beta) \rangle &= \int_a^b \sigma |u'|^{p-2} u' h' dx \\ &= \frac{B - A}{\int_a^b \sigma^{1/(1-p)}(t) dt} \int_a^b h' dx \\ &= \frac{(B - A)(\beta - \alpha)}{\int_a^b \sigma^{1/(1-p)}(t) dt} \end{aligned}$$

and thus provides the same information as the strong map.

We have proven the following theorem:

Theorem 2.2 (*p-Calderón's problem in one dimension*). *Suppose the domain $\Omega \subset \mathbb{R}$ is a bounded open interval $]a, b[$. Suppose the conductivity is a measurable function $\sigma:]a, b[\rightarrow [0, \infty]$ that is essentially bounded from above and away from zero in the complement of $D_0 \cup D_\infty$. We also assume that D_0 and $\overline{D_\infty}$ are unions of finite numbers of open intervals. Further, we assume that the sets $\overline{D_0}$, $\overline{D_\infty}$, and $\partial\Omega$ are disjoint.*

We can recover the following information from the Dirichlet to Neumann map:

- (1) *If $D_0 \neq \emptyset$, then we know that this is the case, but can say nothing more.*
- (2) *If $D_0 = \emptyset$, then we recover the quantity*

$$(2.20) \quad \int_a^b \sigma^{1/(1-p)} dt.$$

2.3. Electrical impedance tomography. Alberto Pedro Calderón considered a method for detecting oil wells by electrical measurements in the article ‘On an inverse boundary value problem’ [13], which initiated the modern mathematical study of electrical impedance tomography. The paper was published in 1980 and reprinted in 2006 [14].

The physical principles were outlined in section 2.1. We now consider the equation

$$(2.21) \quad -\operatorname{div}(\sigma \nabla u) = 0$$

in a bounded domain $\Omega \subset \mathbb{R}^d$, where u is the electrical potential (or voltage) and $\sigma \nabla u$ is the current flux. The Dirichlet data $u|_{\partial\Omega}$ corresponds to boundary measurements

of the potential and the Neumann data $\sigma \nabla u \cdot \nu|_{\partial\Omega}$ corresponds to current flux density across the boundary.

The idea of electrical impedance tomography is to prescribe either the Dirichlet or the Neumann data and measure the other. Since both the Dirichlet and the Neumann boundary value problems for the conductivity equation have a unique solution for any applicable boundary data², the Dirichlet to Neumann map Λ_σ and the Neumann to Dirichlet map, which we do not use in this thesis, are well-defined. The strong formulation for the Dirichlet to Neumann map is

$$(2.22) \quad \Lambda_\sigma(f) = \sigma \nabla u \cdot \nu|_{\partial\Omega},$$

where u solves the conductivity equation with Dirichlet boundary data f . For the rigorous weak formulation of the Dirichlet to Neumann map we refer to section 3.2.

At least the following kinds of results are known for Calderón's problem:

- (1) boundary uniqueness
- (2) boundary reconstruction
- (3) higher order derivatives at the boundary
- (4) stability at the boundary
- (5) recovering inclusions; of finite positive, zero or infinite conductivity
- (6) detecting cracks
- (7) interior uniqueness
- (8) interior reconstruction
- (9) interior stability
- (10) numerical algorithms

The unpublished manuscript of Feldman, Salo and Uhlmann [17] is a good introduction. Uhlmann [55] has written a survey of the 2-Calderón's problem. For review of more numerical nature we refer to Borcea [6].

At least the following kinds of variants or generalizations have been considered:

- (1) linearized problem
- (2) partial data
- (3) anisotropic problem
- (4) different base equation or system of equations

This thesis concerns the final kind of generalization; we consider the quasilinear p -conductivity equation or weighted p -Laplace equation, which we discuss in section 3. Most of the results mentioned above are open for p -conductivity equations.

²In case of Neumann data, the solution is unique up to constants.

3. p -conductivity equation

The Dirichlet problem for the linear conductivity equation, $\operatorname{div}(\sigma \nabla u) = 0$, is solved by minimizing the energy functional

$$(3.1) \quad v \mapsto \int_{\Omega} \sigma |\nabla v|^2 dx$$

in the Sobolev space $W_f^{1,2}(\Omega) = f + W_0^{1,2}(\Omega)$, where the Dirichlet boundary values are f .

The p -conductivity equation generalizes the linear conductivity equation much as the p -Laplace equation generalizes the linear Laplace equation. Let $1 < p < \infty$. The p -conductivity equation is $\operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0$. The Dirichlet problem is solved by minimizing the energy

$$(3.2) \quad v \mapsto \int_{\Omega} \sigma |\nabla v|^p dx$$

in the Sobolev space $W_f^{1,p}(\Omega) = f + W_0^{1,p}(\Omega)$, where the Dirichlet boundary values are f . The quasilinear p -Laplace equation, which is the special case of p -conductivity equation where $\sigma \equiv 1$, has been widely studied. A good introduction is provided by the lecture notes of Peter Lindqvist [42]. The even more general A -harmonic functions and related quasilinear elliptic partial differential equations of second order have been widely covered in various monographs [38, 22, 29].

By the direct method of calculus of variations, the equation is well-defined for measurable $\sigma: \Omega \rightarrow [0, \infty]$ bounded from above and below by positive constants. In article [III] the conductivity can take values zero and infinity in large sets:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\sigma: \Omega \rightarrow [0, \infty]$ be a measurable function. We write $D_0 = \sigma^{-1}(\{0\})$ and $D_\infty = \sigma^{-1}(\{\infty\})$ and suppose the following:*

- *The sets D_0 and D_∞ are open.*
- *The sets $\overline{D_0}$, $\overline{D_\infty}$ and $\partial\Omega$ are disjoint.*
- *The conductivity σ is essentially bounded from below and above by positive constants in $\Omega \setminus (D_0 \cup D_\infty)$.*
- *The set D_0 has Lipschitz boundary.*

Fix $p \in]1, \infty[$. Given any $f \in W^{1,p}(\Omega)$, there is a minimizer $u \in f + W_0^{1,p}(\Omega)$ to the energy

$$(3.3) \quad E(u) = \int_{\Omega} \sigma |\nabla u|^p.$$

The minimal energy is finite and the minimizer is unique up to functions that have zero gradient outside D_0 and zero Dirichlet boundary values on $\partial\Omega$. The minimizer satisfies $\operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0$ in $\Omega \setminus (\overline{D_0} \cup \overline{D_\infty})$ and $\nabla u = 0$ in D_∞ in the weak sense.

The solutions are locally constant in D_∞ and can take essentially arbitrary values in D_0 . Elsewhere they satisfy the usual p -conductivity equation in the weak sense.

The classical interpretation for the problem is as a partial differential equation in $\Omega \setminus (D_0 \cap D_\infty)$ with different boundary conditions on ∂D_0 and ∂D_∞ , as follows:

Remark 3.2. Let the connected components of D_∞ be written as C .

The conductivity equation can be reformulated as

$$(3.4) \quad \begin{cases} \operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0 & \text{in } \Omega \setminus (\overline{D}_0 \cup \overline{D}_\infty) \\ u = f & \text{on } \partial\Omega \\ \sigma |\nabla u|^{p-2} \partial_\nu u = 0 & \text{on } \partial D_0 \\ \text{for each component } C \text{ of } D_\infty & \begin{cases} u|_C = \text{constant} \\ \int_{\partial C} \sigma |\nabla u|^{p-2} \partial_\nu u = 0. \end{cases} \end{cases}$$

The constant may be different for different components and the constant values depend on the boundary data f .

The inverse problem of studying inclusions with finite or zero conductivity is well-known [34, 15, 12, 36, 56, 31, 47, 33, 44, 57]. The region of infinite conductivity has been investigated in a few papers; Gorb and Novikov [24] consider the behaviour of the forward problem when there are two inclusions close to each other for the p -conductivity equation for $2 \leq p \in \mathbb{N}$. We are not aware of any results concerning the inverse problem for $p \neq 2$, but there are results when $p = 2$. Brühl [10, section 4.3.1] and Schmitt [50, section 2.2.2] have used the factorization method to detect perfectly conducting inclusions. Ramdani and Munnier [43] detect the infinitely conductive bodies from the Dirichlet to Neumann map in two dimensional domain for the linear conductivity equation. Their method is based on geometry in the complex plane, in particular Riemann mappings. Friedman and Vogelius [18] have shown that one can recover the location and scale of a finite number of small inclusions with zero or infinite conductivity in an inhomogeneous background from the DN map.

3.1. Solutions of Wolff. We call certain special solutions for the p -Laplace equation the Wolff solutions, as they were originally used by Thomas Wolff [58, section 3] when investigating the boundary behaviour of p -harmonic functions. Wolff only defined the solutions for $2 < p < \infty$, though his proof works when $p > 3/2$. Lewis [40] extended Wolff's work to the case $1 < p < 2$ by duality arguments. Salo and Zhong [49, section 3] provided a unified treatment for all $1 < p < \infty$.

Lemma 3.3. *Let $\rho, \rho^\perp \in \mathbb{R}^d$ satisfy $|\rho| = |\rho^\perp| = 1$ and $\rho \cdot \rho^\perp = 0$. Define $h: \mathbb{R}^d \rightarrow \mathbb{R}$ by $h(x) = e^{-\rho \cdot x} w(\rho^\perp \cdot x)$, where the function w solves the differential equation*

$$(3.5) \quad w''(s) + V(w, w')w = 0$$

with

$$(3.6) \quad V(a, b) = \frac{(2p-3)b^2 + (p-1)a^2}{(p-1)b^2 + a^2}.$$

The function h is then p -harmonic.

Given any initial conditions $(a_0, b_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ there exists a solution $w \in C^\infty(\mathbb{R})$ to the differential equation (3.5) which is periodic with period $\lambda_p > 0$, satisfies the initial conditions $(w(0), w'(0)) = (a_0, b_0)$, satisfies $\int_0^{\lambda_p} w(s) ds = 0$, and furthermore there exist constants c and C depending on a_0, b_0, p such that for all $s \in \mathbb{R}$ we have

$$(3.7) \quad C > w(s)^2 + w'(s)^2 > c > 0.$$

When $p = 2$ the equation (3.5) is simpler: $w + w'' = 0$. The solution in this special case is a linear combination of sine and cosine functions. The solutions for general p behave in a similar way: they oscillate with mean zero and period $\lambda_p > 0$.

3.2. Dirichlet to Neumann map. The strong formulation for the Dirichlet to Neumann map, which we simply call the strong Dirichlet to Neumann map, associates Neumann boundary values to Dirichlet data f :

$$(3.8) \quad \Lambda_\sigma(f) = \sigma |\nabla u|^{p-2} \nabla u \cdot \nu|_{\partial\Omega}$$

The strong Dirichlet to Neumann map is defined pointwise and hence requires the conductivity σ to be extendable pointwise to $\partial\Omega$ and also requires ∇u to be defined pointwise on the boundary. Due to these difficulties the weak formulation of the Dirichlet to Neumann map, or the weak Dirichlet to Neumann map, is used. For properties of the Dirichlet to Neumann map with constant conductivity we refer to Hauer [28]. For p -conductivity equation the weak Dirichlet to Neumann map was introduced by Salo and Zhong [49] and used in the included articles [I, II]. In article [III] we extend the weak Dirichlet to Neumann map to conductivities that include regions of zero or infinite conductivity, as follows.

We assume Ω and σ to be as in theorem 3.1. Let $X = W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$ and X' be its dual. The DN map $\Lambda_\sigma: X \rightarrow X'$ is defined by

$$(3.9) \quad \langle \Lambda_\sigma f, g \rangle = \int_\Omega \sigma |\nabla \bar{f}|^{p-2} \nabla \bar{f} \cdot \nabla \bar{g} \, dx,$$

where $\bar{f} \in W^{1,p}(\Omega)$ is any minimizer of the energy functional E with boundary values $f \in X$ and $\bar{g} \in W^{1,p}(\Omega)$ is an extension of $g \in X$ with $\nabla \bar{g} = 0$ in D_∞ . Since $\bar{D}_\infty \cap \partial\Omega = \emptyset$, there always exists such an extension \bar{g} .

In article [II] we give sufficient conditions for recovering the strong Dirichlet to Neumann map from the weak one. We state the result here with slightly relaxed assumptions:

Lemma 3.4. *Suppose that Ω has C^3 -smooth boundary, boundary values $f \in C^3(\partial\Omega)$ and that $\nabla\sigma$ is Hölder-continuous and the conductivity bounded and strictly positive*

in $\bar{\Omega}$. Then we can recover the pointwise values of the strong Dirichlet to Neumann map

$$(3.10) \quad \sigma(x_0) |\nabla u(x_0)|^{p-2} \nabla u(x_0) \cdot \nu(x_0)$$

from the weak Dirichlet to Neumann map

$$(3.11) \quad \langle \Lambda_\sigma(f), g \rangle = \int_{\Omega} \sigma |\nabla u|^{p-2} \nabla u \cdot \nabla \bar{g} dx.$$

4. p -Calderón's problem

The inverse problem of Calderón was posed in the setting of the p -conductivity equation by Salo and Zhong [49]. See section 5 for their results.

There are some differences between the 2-Calderón's problem and p -Calderón's problem. In particular, many results in the linear case are based on reducing the conductivity equation to Schrödinger's equation

$$(4.1) \quad (-\Delta + q) u = 0$$

with potential

$$(4.2) \quad q = \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}}.$$

It is not clear if a similar reduction is available with the p -conductivity equation.

A standard method in investigating Calderón-type problems for equations with weak non-linearities [51] is taking the Gâteaux derivative of the Dirichlet to Neumann map at constant boundary values. The method does not work in case of the p -Calderón's problem. We follow the presentation of Salo and Zhong [49, appendix]. To see the problem, interpret $a \in \mathbb{R}$ as Dirichlet boundary data, suppose $t > 0$ and $f \in W^{1,p}(\Omega) / W_0^{1,p}(\Omega)$. Write the solution of p -conductivity equation with boundary data g as u_g . Observe that $u_{a+tf} = a + u_{tf} = a + tu_f$. Then the Gâteaux derivative is the limit as $t \rightarrow 0$ of

$$(4.3) \quad \begin{aligned} & \frac{1}{t} (\Lambda_\sigma(a + tf) - \Lambda_\sigma(a)) \\ &= \frac{1}{t} \sigma |\nabla(a + tu_f)|^{p-2} \nabla(a + tu_f) \cdot \nu - 0 \\ &= t^{p-2} \Lambda_\sigma(f). \end{aligned}$$

In particular, the Gâteaux derivative does not even exist when $p < 2$. This calculation does not provide any new information, unlike in the article of Sun [51].

Unique continuation results often play a role in Calderón's problem. The results are much more restricted in the case of the p -conductivity equation [25, 26]. We are not aware of any work on the related Runge approximation property for the p -Laplace equation.

5. Boundary determination

In an article published in 2012, Salo and Zhong [49] showed how to recover conductivity on the boundary of a domain with reasonable regularity assumptions on the conductivity and the boundary of the domain. Their proof is similar to that of Brown [8], and Brown and Salo [9], who only considered the linear situation where $p = 2$.

Theorem 5.1. *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded domain with C^1 boundary, $d \geq 2$, and the conductivity σ is continuous at a point $x_0 \in \partial\Omega$. Then the weak Dirichlet to Neumann map determines $\sigma(x_0)$.*

They proved the theorem by first using complex-valued and then real-valued boundary values – complex geometric optics solutions and Wolff solutions (defined in section 3.1), respectively, multiplied by a cutoff function focused at x_0 . Since results using only real-valued boundary values are stronger than those that use complex-valued functions, further work has used real-valued boundary values exclusively.

In particular, Salo and Zhong use the following lemma with $g = \sigma$. They do not explicitly mention the lemma in their paper.

Lemma 5.2. *Suppose that $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded open set with C^1 -smooth boundary, and $x_0 \in \partial\Omega$.*

Then there exists an explicit sequence of real-valued Dirichlet boundary values f_N such that the corresponding solutions of the p -conductivity equation satisfy

$$(5.1) \quad \lim_{N \rightarrow \infty} q(N) \int_{\Omega} g(x) |\nabla u_N|^p dx = g(x_0),$$

where $q(N)$ is an explicit scaling constant and g is any function that is continuous in a neighbourhood of the boundary point x_0 .

We improve the result in [II] by determining the gradient of conductivity on the boundary:

Theorem 5.3. *Suppose $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded open set with C^3 -smooth boundary. If the conductivity σ is positive and of class $C^2(\overline{\Omega})$, then one can recover $\nabla\sigma|_{\partial\Omega}$ from the weak Dirichlet to Neumann map.*

The extra regularity assumptions on conductivity σ and boundary $\partial\Omega$ are needed to establish a Rellich-type identity (theorem 5.4) and to derive the strong Dirichlet-to-Neumann map from the weak one in lemma 3.4. The assumptions in the theorem 5.3 presented above are stronger than in the paper, as are the assumptions of the Rellich-type identity (theorem 5.4).

A similar Rellich-type identity was used in the linear situation, where $p = 2$, by Brown, García, and Zhang [19, appendix].

Theorem 5.4 (Rellich identity, [II]). *Suppose $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain with C^2 -smooth boundary. Suppose that a function u solves the p -conductivity equation with $1 < p < \infty$ and conductivity $\sigma \in C^1(\overline{\Omega})$ bounded above zero. Let $\alpha \in \mathbb{R}^d$. Then*

$$(5.2) \quad \begin{aligned} & \int_{\Omega} (\alpha \cdot \nabla \sigma) |\nabla u|^p \, dx \\ &= \int_{\partial\Omega} \alpha \cdot \nu \sigma |\nabla u|^p \, dS(x) - p \int_{\partial\Omega} \alpha \cdot \nabla u \sigma |\nabla u|^{p-2} \partial_{\nu} u \, dS(x). \end{aligned}$$

The identity gives the integral $\int_{\Omega} g |\nabla u|^p \, dx$ in lemma 5.2 with $g = \alpha \cdot \nabla \sigma$ a formulation that only depends on known quantities: σ and ∇u on the boundary. The conductivity is known due to results of Salo and Zhong, while the gradient can be recovered from the strong Dirichlet to Neumann map, see lemma 3.4.

6. Detecting inclusions

An inclusion is a set where conductivity is significantly higher or lower than in the rest of the domain. We only consider inclusions embedded in constant background conductivity, which we take to be one.

That is, let $D \subset \Omega$ be an open set so that $\sigma(x) = 1$ whenever $x \notin D$ and let exactly one of the following be true for all $x \in D$:

- (1) $\sigma(x) = 0$
- (2) $0 < c < \sigma(x) < C < 1$
- (3) $1 < c < \sigma(x) < C < \infty$
- (4) $\sigma(x) = \infty$.

Here c and C are constants that do not depend on the spatial variable x .

In case of finite non-zero conductivity we show that for $1 < p < \infty$ we can detect the convex hull of the inclusion; this is done in article [I]. In case of zero or infinite conductivity and $1 < p < \infty$ we can detect some convex set K , which is a superset of the convex hull of the inclusion; this is done in article [III]. We use the enclosure method of Ikehata [32]. We next formulate the results as theorems:

Theorem 6.1. *Suppose that $\Omega \subset \mathbb{R}^d$ is open and bounded with a priori known constant conductivity σ outside an obstacle $D = D_0 \cup D_{\infty}$. Suppose $\sigma : \Omega \rightarrow [0, \infty]$ is measurable. Suppose either D_0 or D_{∞} is empty, the sets $\partial\Omega$, $\overline{D_0}$ and $\overline{D_{\infty}}$ are pairwise disjoint, and the inclusion D has Lipschitz boundary.*

Then we can, from knowledge of the Dirichlet to Neumann map, find a set D' , which is a superset of the convex hull of D . Furthermore, we can detect whether D_0 or D_{∞} is non-empty.

The proof of the previous theorem can be found in the article [III]. By the boundary reconstruction result of Salo and Zhong, theorem 5.1, if the domain Ω is of class C^1 , then we don't need to know the value of the background conductivity a priori.

Theorem 6.2. *Suppose $\Omega \subset \mathbb{R}^d, d \geq 2$, is a bounded domain and the inclusion $D \subset \Omega$ is a bounded open set with Lipschitz boundary. Furthermore assume that the measurable conductivity σ has a jump discontinuity along the interface ∂D ; that is, $\sigma(x) := 1 + \sigma_D(x)\chi_D(x)$, where σ_D is bounded away from zero and either positive and bounded from above or negative and bounded from below by a constant greater than minus one, and where χ_D is the characteristic function of D .*

Then the convex hull of D can be recovered from the Dirichlet to Neumann map. Further, we know if σ_D is positive or negative.

The previous theorem is proven in article [I].

The enclosure method relies on an indicator function, which is a difference of Dirichlet to Neumann maps that take particular Wolff solutions as input.

Definition 6.3. The indicator function I is defined as

$$(6.1) \quad I(t, \rho, \rho^\perp, \tau) = \tau^{d-p} \langle (\Lambda_\sigma - \Lambda_1) f, f \rangle,$$

where $\tau > 0$,

$$(6.2) \quad f(x, t, \rho, \rho^\perp, \tau) = e^{\tau(\rho \cdot x - t)} w(\tau \rho^\perp \cdot x)$$

is a Wolff solution (considered in the space of boundary values) and $(\Lambda_\sigma - \Lambda_1) f = \Lambda_\sigma(f) - \Lambda_1(f)$.

Note that $|\nabla u(x)| \rightarrow 0$ as $\tau \rightarrow \infty$ when $\rho \cdot x < t$ and $|\nabla u(x)|$ blows up as $\tau \rightarrow \infty$ when $\rho \cdot x > t$. Here u is the Wolff solution with boundary values f . Hence, at least heuristically, the parameters t and ρ divide the space so that one half-space has plenty of energy while the other has very little. The idea is that the indicator function reveals if the inclusion intersects the high-energy half space or is disjoint from it. Combining this information for all parameters t and ρ yields the convex hull of the inclusion.

When the inclusion is of finite non-zero conductivity (as investigated in article [I]) we have the following theorem:

Theorem 6.4. *There exist $c, C > 0$ such that*

$$(6.3) \quad c < \left| I \left(\sup_{x \in D} (x \cdot \rho), \rho, \rho^\perp, \tau \right) \right| < C \tau^d$$

for $\tau \gg 1$.

We only need to consider the case $t = \sup_{x \in D} x \cdot \rho$, since the identity

$$(6.4) \quad I(t, \rho, \rho^\perp, \tau) = e^{2\tau(t_0 - t)} I(t_0, \rho, \rho^\perp, \tau)$$

with $t_0 = \sup_{x \in D} (x \cdot \rho)$ shows that the behaviour of the indicator function is exponentially increasing or decreasing as a function of τ for other values of t . The previous theorem is proven with the help of a monotonicity inequality:

Theorem 6.5. *If $0 < c < \sigma_0, \sigma_1 \in L^\infty(\Omega)$ and $1 < p < \infty$, and if $f \in W^{1,p}(\Omega)$, then*

$$(6.5) \quad (p-1) \int_{\Omega} \frac{\sigma_0}{\sigma_1^{1/(p-1)}} \left(\sigma_1^{\frac{1}{p-1}} - \sigma_0^{\frac{1}{p-1}} \right) |\nabla u_0|^p \, dx$$

$$(6.6) \quad \leq ((\Lambda_{\sigma_1} - \Lambda_{\sigma_0})f, f) \leq \int_{\Omega} (\sigma_1 - \sigma_0) |\nabla u_0|^p \, dx,$$

where $u_0 \in W^{1,p}(\Omega)$ solves $\operatorname{div}(\sigma_0 |\nabla u_0|^{p-2} \nabla u_0) = 0$ in Ω with $u_0|_{\partial\Omega} = f$.

Monotonicity inequalities have been used in the linear case [57].

In case of inclusion with zero or perfect conductivity, as discussed in article [III], we only detect when the inclusion intersects the half-space of high energy, hence possibly detecting too large a set. This means that we only have the lower bound in theorem 6.4.

7. Other interior results

Very recently, Guo, Kar and Salo [27] used the monotonicity inequality, theorem 6.5, to show injectivity for the Dirichlet to Neumann map under a monotonicity assumption. Their result show injectivity in two dimensions for Lipschitz conductivities. In higher dimensions they need an additional assumption; one of the conductivities must be almost constant.

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CALDERÓN PROBLEM FOR THE p -LAPLACIAN: FIRST ORDER DERIVATIVE OF CONDUCTIVITY ON THE BOUNDARY

TOMMI BRANDER

(Communicated by Joachim Krieger)

ABSTRACT. We recover the gradient of a scalar conductivity defined on a smooth bounded open set in \mathbb{R}^d from the Dirichlet to Neumann map arising from the p -Laplace equation. For any boundary point we recover the gradient using Dirichlet data supported on an arbitrarily small neighbourhood of the boundary point. We use a Rellich-type identity in the proof. Our results are new when $p \neq 2$. In the $p = 2$ case boundary determination plays a role in several methods for recovering the conductivity in the interior.

1. INTRODUCTION

Throughout the article we assume $1 < p < \infty$, unless explicitly written otherwise. We investigate a generalisation of the Calderón problem to the case of the p -Laplace equation: Given a bounded open set $\Omega \subset \mathbb{R}^d$ and a bounded conductivity $\gamma > 0$ on $\bar{\Omega}$ we have the Dirichlet problem

$$(1.1) \quad \begin{cases} \Delta_p^\gamma(u) = \operatorname{div}(\gamma(x)|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega. \end{cases}$$

For $v \in W^{1,p}(\Omega)$ there exists a unique weak solution that satisfies $\|u\|_{W^{1,p}} \leq C\|v\|_{W^{1,p}}$, where the constant C does not depend on v . This is completely standard; see e.g. [43, Proposition A.1]. The equation arises from the variational problem of minimising the energy $\int_\Omega \gamma|\nabla u|^p dx$.

We next define the Dirichlet to Neumann map (hereafter DN map). We use the notation X' for the dual space of continuous linear functionals on a Banach space X . In the $p = 2$ case the DN map gives the stationary measurements of electrical current for given voltage v .

Definition 1.1 (Dirichlet to Neumann map). Suppose $\Omega \subset \mathbb{R}^d$ is a bounded open C^1 -set and $0 < \gamma_0 < \gamma \in L^\infty(\bar{\Omega})$ for some constant γ_0 . The weak DN map

$$\Lambda_\gamma^w: W^{1,p}(\Omega)/W_0^{1,p}(\Omega) \rightarrow \left(W^{1,p}(\Omega)/W_0^{1,p}(\Omega)\right)'$$

is defined by

$$(1.2) \quad \langle \Lambda_\gamma^w(v), g \rangle = \int_\Omega \gamma|\nabla u|^{p-2}\nabla u \cdot \nabla \tilde{g} dx,$$

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where u solves the boundary value problem (1.1) with boundary values v , and $\bar{g} \in W^{1,p}(\Omega)$ with trace g on the boundary. The strong DN map is defined pointwise on $\partial\Omega$ by the expression

$$(1.3) \quad \Lambda_\gamma^s(v)(x_0) = \gamma(x_0)|\nabla u(x_0)|^{p-2}\nabla u(x_0) \cdot \nu(x_0)$$

when it is well-defined.

We also write $\Lambda_\gamma(v)$ when the function v is continuous and defined on a superset of $\partial\Omega$. With sufficient regularity we can recover the values of the strong DN map from the weak DN map (see Lemma 4.4).

Given knowledge of the DN map we determine $\nabla\gamma|_{\partial\Omega}$, which in the $p = 2$ case stands for the gradient of conductivity on the boundary.

Theorem 1.2. *Suppose $\Omega \subset \mathbb{R}^d$, $d > 1$, is a bounded open $C^{2,\beta}$ set for some $0 < \beta < 1$ and the conductivities γ_1 and γ_2 are bounded from below by a positive constant and are continuously differentiable with Hölder-continuous derivatives in $\bar{\Omega}$. If $\Lambda_{\gamma_1}^w = \Lambda_{\gamma_2}^w$, then $\nabla\gamma_1|_{\partial\Omega} = \nabla\gamma_2|_{\partial\Omega}$.*

We use an explicit sequence of boundary values to reconstruct $\nabla\gamma(x_0)$ at a boundary point x_0 . The sequence is supported in an arbitrarily small neighbourhood of x_0 . The boundary values we use were first introduced by Wolff [49] and used by Salo and Zhong [43] to recover the conductivity on the boundary.

Salo and Zhong [43] show that for all boundary points $x_0 \in \partial\Omega$ there is a sequence of solutions $(u_M)_M$ such that

$$(1.4) \quad M^{n-1}N^{1-p} \int_{\Omega} g(x)|\nabla u_M|^p dx \rightarrow c_p g(x_0)$$

as $M, N(M) \rightarrow \infty$, with explicitly computable constant c_p (see (3.6)). They recover the integrals with $g = \gamma$ from the DN map. We recover the integrals with $g = \alpha \cdot \nabla\gamma$ from a Rellich-type identity, Theorem 4.1 for arbitrary direction $\alpha \in \mathbb{R}^d$.

The Calderón problem was first introduced in [13]. For a review see [48].

There have been several boundary determination results in the $p = 2$ case. Boundary uniqueness for derivatives of the conductivity was first proved by Kohn and Vogelius [28]. Sylvester and Uhlmann in [47] recovered, for smooth conductivity in a smooth domain, conductivity and all its derivatives on the boundary by considering Λ_γ as a pseudodifferential operator. Nachman [36] recovered $\gamma|_{\partial\Omega}$ in the Lipschitz domain when $\gamma \in W^{1,q}$, $q > d$, and its first derivative when $\gamma \in W^{2,q}$, $q > d/2$. Alessandrini [1] used singular solutions with singularity near the boundary to recover all derivatives of $\gamma|_{\partial\Omega}$ with less regularity assumptions on γ . There have been several local boundary determination results; e.g. [25, 38]. Other reasonably recent boundary determination results include [2, 12].

A Rellich identity was used by Brown, Garcia and Zhang [16, appendix] to recover the gradient of conductivity on the boundary in the $p = 2$ case. The Rellich identity was, to the best of our knowledge, introduced in [40].

Several results (e.g. [1]) rely on investigating the difference $\Lambda_{\gamma_1} - \Lambda_{\gamma_2}$ of DN maps at different conductivities. This is difficult in the present setting due to non-linearity of the p -Laplace equation. We use the Rellich identity, Theorem 4.1 to avoid this problem.

Electrical impedance tomography has applications in medical (e.g. [11]) and industrial imaging (e.g. [26]), and geophysics and environmental sciences; see e.g. the review [9] and references therein. There are practical numerical algorithms for

boundary determination in the $p = 2$ case; e.g. [37]. Boundary determination is used in recovering the conductivity in the interior; e.g. [19]. For example, the algorithm in [45] uses the values of conductivity and its first derivative on the boundary to extend the conductivity, thence applying to conductivities that are not constant on the boundary.

For more on the p -Laplace equation see e.g. [15][22][33]. The equation has applications in e.g. image processing [29], fluid mechanics [6], plastic moulding [4], and modelling of sand-piles [5].

The Calderón problem for the p -Laplace equation was first introduced in [43]. The authors consider the real and the complex case separately. In the complex case they define p -harmonic versions of complex geometrical optics solutions to create highly oscillating functions focused around a given boundary point and use them as Dirichlet data, thus recovering conductivity at the boundary point in question. In the real case they replace the p -harmonic CGO solutions with real-valued functions having similar behaviour. The real-valued functions were originally introduced by Wolff [49].

Further progress on the p -Calderón problem was made by the author, Kar and Salo [10]. In the article we show that we can detect the convex hulls of inclusions, which are regions of significantly higher or lower conductivity, from the DN map. Our main tools are the p -harmonic functions of Wolff and a monotonicity inequality.

Hauer [21] has investigated the DN map related to the p -Laplace equation.

One method of investigating the Calderón-type inverse problems for non-linear equations is based on studying the Gâteaux derivatives of the map Λ_γ at constant boundary values a . In our case this does not work [43 appendix]:

$$(1.5) \quad \Lambda_\gamma(a + tf) = t^{p-1}\Lambda_\gamma(f)$$

for positive t . For $p < 2$ the Gâteaux derivatives do not exist and for $p > 2$ the higher derivatives fail to exist, or vanish, although one of them might equal $\Lambda_\gamma(f)$. Hence non-linear methods are necessary.

Other inverse problems related to non-linear equations similar to the p -Laplace equation have been investigated before; the 1-Laplace equation is used in current density imaging (e.g. [23][24][27][35][44]), and the 0-Laplace equation is related to ultrasound modulated electrical impedance tomography (e.g. [3][7][8][17]).

In section 2 we introduce our notation and state lemmata. In section 3 we restate boundary determination results we use in this article. In the final section 4 we reconstruct $\nabla\gamma|_{\partial\Omega}$ from the DN map.

2. PRELIMINARIES

We use the following notation: We write $\partial_\alpha = \alpha \cdot \nabla$ for vectors α . On the C^1 boundary $\partial\Omega$ of an open set Ω we decompose the gradient of a function f defined in $\bar{\Omega}$ as $\nabla f(x_0) = \nabla_\nu f(x_0) + \nabla_T f(x_0)$, where ∇_ν is the normal component of the derivative, i.e. the orthogonal projection of the gradient to the normal space of the boundary, and ∇_T is the tangential derivative, i.e. the orthogonal projection of the gradient on the tangent space of the boundary. We have the following identity: $|\nabla_\nu f|^2 = (\partial_\nu f)^2$. We denote the outer unit normal by ν .

We write the Hölder seminorm of order k as $|f|_{k,\beta}$ and the characteristic function of a set A as

$$\chi_A(x) = \begin{cases} 1 & \text{when } x \in A, \\ 0 & \text{when } x \notin A. \end{cases}$$

We need the following elementary inequality:

Lemma 2.1. *Suppose $0 \leq A, B \in \mathbb{R}$ and $2 \leq q < \infty$. Then for some $C > 0$ we have*

$$(2.1) \quad |A^{q-1} - B^{q-1}| \leq C(A+B)^{q-2} |A-B|.$$

Proof. By the fundamental theorem of calculus

$$|A^{q-1} - B^{q-1}| = (q-1)|A-B| \left| \int_0^1 (A+t(B-A))^{q-2} dt \right|,$$

from which the estimate follows by observing that $0 \leq A+t(B-A) \leq A+B$. \square

It is well-known that the solutions of the p -Laplace equation are in $C^{1,\beta}$. For a proof of the following lemma see e.g. [31].

Lemma 2.2 (Regularity result). *Suppose Ω is a bounded open C^{1,β_1} set with $0 < \beta_1 \leq 1$, and suppose the conductivity $0 < \gamma \in C^{0,\beta_2}(\overline{\Omega})$ is bounded from above and away from zero. Consider the weighted p -Laplace equation (1.1) with boundary values $v \in C^{1,\beta_1}(\partial\Omega)$. Then the solution u of the weighted p -Laplace equation (1.1) is in $C^{1,\beta_3}(\overline{\Omega})$ for some $\beta_3 > 0$.*

In general the solutions are not twice continuously differentiable [34].

To establish the Rellich identity, Theorem 4.1 we use the following interpolation lemma similar to [32] lemma A.1]:

Lemma 2.3 (Interpolation lemma). *Suppose Ω is a bounded open set in \mathbb{R}^d satisfying the measure density condition; that is, suppose there exist $\delta_0, C > 0$ such that for all $0 < \delta < \delta_0$ and for all $y \in \overline{\Omega}$ we have*

$$(2.2) \quad |B(y, \delta)| \leq C|B(y, \delta) \cap \overline{\Omega}|.$$

Also suppose $f \in C^\beta(\overline{\Omega})$ for some $\beta > 0$. Take $1 \leq p < \infty$. Let $M > 0$ be a constant such that $|f|_{0,\beta} \leq M$. Then

$$(2.3) \quad \|f\|_{L^\infty(\overline{\Omega})} \leq C_{d,p,\beta,\Omega} M^{\frac{d}{d+\beta p}} \|f\|_{L^p(\Omega)}^{\frac{\beta p}{d+\beta p}}.$$

The measure density condition (2.2) is satisfied in a wide variety of situations [20]. In particular, domains that admit a Sobolev extension theorem satisfy the condition [20] proposition 1]. Domains of class $C^{1,\alpha}$, and more generally Lipschitz domains, certainly admit a Sobolev extension theorem [14] theorem 12], [46] chapter VI, section 3.1, theorem 5]. A bounded Lipschitz set has a finite number of connected components, and thus also satisfies the condition.

The proof of the interpolation lemma is almost the same as in [32].

Proof. Let $y \in \bar{\Omega}$ and write $B_\delta = B(y, \delta)$. For $\delta > 0$ we have

$$\begin{aligned} \|f\|_{L^p(\bar{\Omega} \cap B_\delta)} &\geq \|f(y)\|_{L^p(\bar{\Omega} \cap B_\delta)} - \|f(x) - f(y)\|_{L^p(\bar{\Omega} \cap B_\delta)} \\ &= |f(y)| |\bar{\Omega} \cap B_\delta|^{1/p} - \left(\int_{\bar{\Omega} \cap B_\delta} |f(x) - f(y)|^p dx \right)^{1/p} \\ &\geq c_{d,p,\Omega} \delta^{d/p} |f(y)| - |f|_{0,\beta} \left(\int_{\bar{\Omega} \cap B_\delta} |x - y|^{\beta p} dx \right)^{1/p} \\ &\geq c_{d,p,\Omega} \delta^{d/p} |f(y)| - C_{d,p,\beta} \delta^{\beta+d/p} M, \end{aligned}$$

whereby it follows that

$$(2.4) \quad \|f\|_{L^\infty(\bar{\Omega})} \leq C_{d,p,\Omega} \delta^{-d/p} \|f\|_{L^p(\bar{\Omega})} + C_{d,p,\beta} \delta^\beta M.$$

Choose

$$(2.5) \quad \delta = \left(\frac{\|f\|_{L^p(\bar{\Omega})}}{M} \right)^{\frac{p}{\beta+dp}}. \quad \square$$

In our proofs we use the ε -perturbed p -Laplace equation. Let $\varepsilon > 0$ and define u_ε as the solution of the boundary value problem

$$(2.6) \quad \begin{cases} \operatorname{div} \left(\gamma(x) (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \right) = 0 & \text{in } \Omega, \\ u_\varepsilon = v & \text{on } \partial\Omega. \end{cases}$$

By calculus of variations there exists a unique solution to (2.6) when Ω is a bounded open set and $v \in W^{1,p}(\Omega)$. The energy corresponding to (2.6) is

$$(2.7) \quad \int_{\Omega} \gamma (|\nabla u_\varepsilon|^2 + \varepsilon)^{p/2} dx.$$

Lemma 2.4 (Regularity result for perturbed equation). *Suppose Ω is a bounded open set with C^{1,β_1} boundary. Suppose that boundary values $v \in C^{1,\beta_1}(\partial\Omega)$ and conductivity γ is Hölder-continuous. Then $u_\varepsilon \in C^{1,\beta_2}(\bar{\Omega})$ with some $0 < \beta_2 < 1$ independent of ε and the norm $|u_\varepsilon|_{1,\beta_2}$ has an upper bound independent of ε .*

Furthermore, suppose that Ω has C^{2,β_3} boundary, $v \in C^{2,\beta_3}(\partial\Omega)$ and that $\nabla\gamma$ is Hölder-continuous. Then $u_\varepsilon \in C^{2,\beta_4}(\bar{\Omega})$ for some $0 < \beta_4 < 1$, which depends on ε .

Proof. That ∇u_ε are Hölder-continuous with β_2 and the corresponding norm independent of ε follows from [31]. The C^2 -regularity is also standard; see for example [30] part 4, section 8] or [18] section 15.5]. □

Lemma 2.5 (Convergence of the perturbed equations). *Suppose Ω is a bounded open set with $C^{1,\beta}$ boundary, $0 < \beta < 1$, and boundary values v are in $C^{1,\beta}$. Also assume that the conductivity γ is Hölder continuous. Then $u_\varepsilon \rightarrow u$ in $C^1(\bar{\Omega})$.*

Proof. We first show that the energy (2.7) of the perturbed equation (2.6) converges to the energy of the non-perturbed equation (1.1): The solution u minimises energy,

and so we have

$$\begin{aligned}
& \int_{\Omega} \gamma |\nabla u|^p dx \\
& \leq \int_{\Omega} \gamma |\nabla u_{\varepsilon}|^p dx \\
& \leq \int_{\Omega} \gamma (|\nabla u_{\varepsilon}|^2 + \varepsilon)^{p/2} dx \\
& \leq \int_{\Omega} \gamma (|\nabla u|^2 + \varepsilon)^{p/2} dx \\
& \leq \int_{\Omega} \gamma |\nabla u|^p dx + C \begin{cases} \int_{\Omega} \varepsilon (2|\nabla u|^2 + \varepsilon)^{-1+p/2} dx & \text{when } p > 2, \\ \int_{\Omega} \varepsilon^{p/2} dx & \text{when } p \leq 2, \end{cases}
\end{aligned}$$

where the latter integrals vanish as $\varepsilon \rightarrow 0$ since $|\nabla u|$ is bounded (Lemma 2.2). We used the fact that u_{ε} minimises the perturbed energy (2.7), and also Lemma 2.1 with $q = 1 + p/2 > 2$ when $p > 2$, and the fact that for non-negative numbers $(a + b)^q \leq a^q + b^q$ when $0 \leq q = p/2 \leq 1$, in the final inequality.

We established that the energy of u_{ε} converges to the energy of u as $\varepsilon \rightarrow 0$. Since $\|u_{\varepsilon}\|_{L^{\infty}}$ are uniformly bounded, it follows that u_{ε} are bounded in $W^{1,p}$ with weight. This implies that the sequence has a weakly converging subsequence with limit that has lesser or equal energy than u . But u uniquely minimises energy, so $u_{\varepsilon} \rightarrow u$ weakly in weighted $W^{1,p}$. Since the weighted L^p norms of ∇u_{ε} converge to weighted L^p norm of ∇u , and the sequence converges weakly, we get that $\nabla u_{\varepsilon} \rightarrow \nabla u$ strongly in weighted L^p space; this is the Radon-Riesz property proved in [39, 41, 42]. Since the weighted and standard norms are equivalent, $\nabla u_{\varepsilon} \rightarrow \nabla u$ strongly in standard L^p .

Since ∇u_{ε} are uniformly Hölder in $\bar{\Omega}$ (Lemma 2.4) we can use the interpolation Lemma 2.3 to deduce convergence of ∇u_{ε} to ∇u in $C(\bar{\Omega})$. The same argument holds for u_{ε} , since by the Friedrichs-Poincaré inequality $u_{\varepsilon} \rightarrow u$ strongly in L^p . \square

3. PREVIOUS RESULTS ON BOUNDARY DETERMINATION FOR p -LAPLACIAN

We use several results in the paper [43] and restate them here for convenience.

Suppose $\Omega \subset \mathbb{R}^d$ is an open bounded C^1 set and suppose $\rho \in C^1(\mathbb{R}^d)$ is its boundary defining function; that is,

$$\Omega = \{x \in \mathbb{R}^d; \rho(x) > 0\}, \quad \partial\Omega = \{x \in \mathbb{R}^d; \rho(x) = 0\}$$

and $\nabla\rho \neq 0$ on $\partial\Omega$. By translation we may assume that we are trying to recover $\nabla\gamma$ at the origin $0 \in \partial\Omega$, and by rotation and scaling we may assume that

$$-\nabla\rho(0) = \nu(0) = -e_d,$$

the d th coordinate vector. We define the map $f: \Omega \rightarrow \mathbb{R}_+^d$ by

$$f(x', x_d) = (x', \rho(x))$$

for $x = (x', x_d)$. Since ρ is a C^1 function, the map is close to the identity near the origin, and hence invertible in some neighbourhood of the origin.

We now introduce the p -harmonic oscillating functions of Wolff [49].

Lemma 3.1. *Define the function $h: \mathbb{R}^d \rightarrow \mathbb{R}$ by $h(x) = e^{-x_d}a(x_1)$, where $a: \mathbb{R} \rightarrow \mathbb{R}$ is the solution to the differential equation*

$$(3.1) \quad a''(x_1) + V(a, a')a = 0$$

with

$$(3.2) \quad V(a, a') = \frac{(2p-3)(a')^2 + (p-1)a^2}{(p-1)(a')^2 + a^2}.$$

Then $\Delta_p^1(h) = 0$ and the function a is smooth and periodic with period $\lambda(p)$, so that $\int_0^\lambda a(x_1) dx_1 = 0$.

For a proof of the lemma see [43] lemma 3.1].

We define a smooth positive cutoff function $\zeta \in C_0^\infty(\mathbb{R}^d)$, such that $\zeta(x) = 1$ when $|x| < 1/2$ and $\text{supp } \zeta \subset B(0, 1)$. We write

$$(3.3) \quad v_M(x) = h(Nf(x))\zeta(Mx),$$

where M and $N = N(M)$ are large positive numbers, $M = o(N)$. We define u_M to be the solutions of the initial value problem

$$(3.4) \quad \begin{cases} \Delta_p^\gamma(u_M) = 0 & \text{in } \Omega, \\ u_M = v_M & \text{on } \partial\Omega. \end{cases}$$

In [43] lemma 3.3] Salo and Zhong show that

$$(3.5) \quad M^{d-1}N^{1-p} \int_\Omega \gamma |\nabla v_M|^p dx \rightarrow c_p \gamma(0)$$

as $M \rightarrow \infty$. The constant c_p is explicit:

$$(3.6) \quad c_p = \frac{K}{p} \int_{\mathbb{R}^{d-1}} (\eta(x', 0))^p dx'; \text{ with } K = \lambda^{-1} \int_0^\lambda (a^2(t) + (a'(t))^2)^{p/2} dt.$$

The proof also holds when γ is replaced by any other function continuous at 0, so as $M \rightarrow \infty$ we have

$$(3.7) \quad M^{d-1}N^{1-p} \int_\Omega \partial_\alpha \gamma |\nabla v_M|^p dx \rightarrow c_p \partial_\alpha \gamma(0).$$

Using [43] lemma 3.4] we get the following result:

Lemma 3.2. *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded open set with C^1 boundary, $x_0 \in \partial\Omega$, $d \geq 2$, and g is continuous. Then*

$$(3.8) \quad M^{d-1}N^{1-p} \int_\Omega g(x) |\nabla u_M|^p dx \rightarrow c_p g(x_0)$$

as $M \rightarrow \infty$.

4. PROOF OF THE MAIN RESULT

In this section we establish that the quantity

$$(4.1) \quad \int_\Omega \partial_\alpha \gamma(x) |\nabla u|^p dx$$

can be calculated from the measurements represented by the DN map, with the function u solving the weighted p -Laplace equation (1.1) with sufficiently smooth boundary values. Then it follows from Lemma 3.2 that we can recover $\partial_\alpha \gamma(x_0)$ at any boundary point $x_0 \in \partial\Omega$. The main result, Theorem 1.2 immediately follows.

Theorem 4.1 (Rellich identity). *Suppose $1 < p < \infty$ and that $\Omega \subset \mathbb{R}^d$ is a bounded open set with $C^{2,\beta}$ boundary for some $0 < \beta < 1$. Let $0 < \gamma \in C^1(\overline{\Omega})$ and let u be a weak solution of the equation $\Delta_p^\gamma u = 0$. Let $\alpha \in \mathbb{R}^d$.*

Then

$$(4.2) \quad \int_{\Omega} (\partial_\alpha \gamma) |\nabla u|^p dx = \int_{\partial\Omega} \gamma (\alpha \cdot \nu) |\nabla u|^p dS - p \int_{\partial\Omega} (\partial_\alpha u) \gamma |\nabla u|^{p-2} \partial_\nu u dS.$$

Proof. Suppose we have the following identity for the solution u_ε of the perturbed p -Laplace equation (2.6):

$$\begin{aligned} & \int_{\Omega} (\partial_\alpha \gamma) (|\nabla u_\varepsilon|^2 + \varepsilon)^{p/2} dx \\ &= \int_{\partial\Omega} \gamma (\alpha \cdot \nu) (|\nabla u_\varepsilon|^2 + \varepsilon)^{p/2} dS - p \int_{\partial\Omega} \partial_\alpha u_\varepsilon \gamma (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \partial_\nu u_\varepsilon dS. \end{aligned}$$

Then, when $\varepsilon \rightarrow 0$, $\nabla u_\varepsilon \rightarrow \nabla u$ uniformly (by Lemma 2.5), so we get the claimed identity.

We now prove the perturbed identity, integrating by parts twice:

$$\begin{aligned} & \int_{\partial\Omega} (\alpha \cdot \nu) \gamma (|\nabla u_\varepsilon|^2 + \varepsilon)^{p/2} dS - \int_{\Omega} (\partial_\alpha \gamma) (|\nabla u_\varepsilon|^2 + \varepsilon)^{p/2} dx \\ &= \int_{\Omega} \gamma \partial_\alpha ((|\nabla u_\varepsilon|^2 + \varepsilon)^{p/2}) dx \\ &= p \int_{\Omega} \gamma (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla \partial_\alpha u_\varepsilon dx \\ &= p \int_{\partial\Omega} \partial_\alpha u_\varepsilon \gamma (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \partial_\nu u_\varepsilon dS \\ &\quad - p \int_{\Omega} \operatorname{div} \left(\gamma (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \right) \partial_\alpha u_\varepsilon dx \\ &= p \int_{\partial\Omega} \partial_\alpha u_\varepsilon \gamma (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \partial_\nu u_\varepsilon dS. \quad \square \end{aligned}$$

We need to recover $\gamma|_{\partial\Omega}$ and $\nabla u|_{\partial\Omega}$ to recover $\int_{\Omega} \partial_\alpha \gamma |\nabla u|^p dx$ from the DN map.

Supposing Ω is an open set with C^1 boundary and γ is continuous, we can recover $\gamma(x_0)$ for all boundary points x_0 . This follows directly from 4.3.

To calculate $\partial_\nu u$ we use $C^{1,\beta}$ boundary values v so that the solution u of the boundary value problem (1.1) is continuously differentiable up to the boundary (by Lemma 2.2), so $\partial_\nu u$ is defined pointwise. Recovering $\partial_\nu u$ from the strong definition of the DN map is straightforward, and we do so in Lemma 4.2. However, we do not a priori have access to the strong DN map, and must recover it from the weak Definition 1.1. This we do in Lemma 4.4 which requires more smoothness on Ω and boundary values.

We have

$$(4.3) \quad \Lambda_\gamma^s(v) = \gamma |\nabla u|^{p-2} \partial_\nu u,$$

from which by expanding ∇u we get the equation

$$\gamma |\nabla_T u + \nabla_\nu u|^{p-2} \partial_\nu u = \Lambda_\gamma^s(v).$$

Taking absolute values and reorganising we have

$$(|\nabla_T u|^2 + |\nabla_\nu u|^2)^{\frac{p-2}{2}} |\partial_\nu u| = \frac{1}{\gamma} |\Lambda_\gamma^s(v)|.$$

We write the left-hand side as a function of $|\partial_\nu u| = |\nabla_\nu u| = t$:

$$(4.4) \quad F(t) = (|\nabla_T u|^2 + t^2)^{\frac{p-2}{2}} t = \frac{1}{\gamma} |\Lambda_\gamma^s(v)|.$$

There is a unique $t_0 \in [0, \infty[$ with $F(t_0) = \frac{1}{\gamma} |\Lambda_\gamma^s(v)|$. To see this, observe that $\lim_{t \rightarrow 0} F(t) = 0$, $\lim_{t \rightarrow \infty} F(t) = \infty$, and that the function F is strictly increasing and continuous.

Returning to the original formulation and expanding $|\nabla u|^2$ we therefore get

$$(4.5) \quad \gamma(|\nabla_T u|^2 + t_0^2)^{\frac{p-2}{2}} \partial_\nu u = \Lambda_\gamma^s(v),$$

from which we can solve $\partial_\nu u$. We have proven the following lemma:

Lemma 4.2. *Suppose Ω is a bounded open set with $C^{1,\beta}$ boundary. Let the boundary voltage $v \in C^{1,\beta}$ be known. Then, for any boundary point $x_0 \in \partial\Omega$, we can compute $\partial_\nu u(x_0)$ from the strong DN map with the following algorithm:*

(1) *Solve t from the equation*

$$(|\nabla_T u(x_0)|^2 + t^2)^{\frac{p-2}{2}} t = \frac{1}{\gamma(x_0)} |\Lambda_\gamma^s(v)(x_0)|.$$

(2) *Set $\partial_\nu u(x_0) = t \operatorname{sign}(\Lambda_\gamma^s(v)(x_0))$.*

The following remark is not used in this paper, but in general locating the points where $\nabla u = 0$ is interesting for p -harmonic functions u .

Remark 4.3. We know the boundary points x_0 where $\nabla u(x_0) = 0$, since $\partial_\nu u(x_0) = 0$ if and only if $\Lambda_\gamma^s(v)(x_0) = 0$, and the Dirichlet data determines $\nabla_T u$ at boundary points.

We still need to recover the strong DN map from the weak one.

Lemma 4.4. *Suppose that Ω has $C^{2,\beta}$ boundary, boundary values $v \in C^{2,\beta}(\partial\Omega)$ and that $\nabla\gamma$ is Hölder-continuous. Then we can recover the pointwise values of the strong DN map*

$$(4.6) \quad \gamma(x_0) |\nabla u(x_0)|^{p-2} \partial_\nu u(x_0)$$

from the weak DN map (see Definition [1.1](#))

$$(4.7) \quad \langle \Lambda_\gamma^w(v), g \rangle = \int_\Omega \gamma |\nabla u|^{p-2} \nabla u \cdot \nabla \bar{g} \, dx.$$

Proof. For $x_0 \in \partial\Omega$ define test functions $\eta_\delta: \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$(4.8) \quad \eta_\delta(x) = c_d \delta^{-d} (\delta - |x - x_0|) \chi_{B(x_0, \delta)}$$

where $c_d > 0$ is selected so that $\int_L \eta_\delta \, dx = 1$, where L is the tangent space of $\partial\Omega$ at x_0 . Then η_δ lies in all first order Sobolev spaces with $|\nabla \eta_\delta(x)| = c_d \delta^{-d}$ in $B(x_0, \delta)$, and 0 elsewhere.

Suppose $\nabla u(x_0) = 0$. By Hölder continuity of $|\nabla u|^{p-1}$ we get

$$\begin{aligned} |\langle \Lambda_\gamma^w(v), \eta_\delta \rangle| &\lesssim \| |\nabla u|^{p-2} \nabla u \cdot \nabla \eta_\delta \|_{L^1(\Omega \cap B(x_0, \delta))} \\ &\lesssim \| |\nabla u|^{p-1} |\nabla \eta_\delta| \|_{L^1(B(x_0, \delta))} \lesssim \| \delta^{\beta_1} \delta^{-d} \|_{L^1(\Omega \cap B(x_0, \delta))} \\ &\lesssim \delta^{\beta_1 - d} \delta^d = \delta^{\beta_1} \rightarrow 0 = \Lambda_\gamma^s(x_0) \end{aligned}$$

as $\delta \rightarrow 0$.

Suppose, then, that $\nabla u(x_0) \neq 0$. Since $\nabla u_\varepsilon \rightarrow \nabla u$ as $\varepsilon \rightarrow 0$, there is $a > 0$ such that

$$(4.9) \quad |\nabla u_\varepsilon(x_0)| \geq |\nabla u(x_0)|/2$$

holds whenever $a > \varepsilon \geq 0$.

We next prove that the equality

$$(4.10) \quad \gamma(x_0) \left(|\nabla u_\varepsilon(x_0)|^2 + \varepsilon \right)^{\frac{p-2}{2}} \partial_\nu u_\varepsilon(x_0) = \lim_{\delta \rightarrow 0} \int_\Omega \gamma \left(|\nabla u_\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla \eta_\delta \, dx$$

holds. By taking the limit $\varepsilon \rightarrow 0$ we get the original claim if the rate of convergence in (4.10) is uniform in ε . By integrating by parts we get

$$(4.11) \quad \begin{aligned} & \int_\Omega \gamma \left(|\nabla u_\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla \eta_\delta \, dx \\ &= - \int_\Omega \operatorname{div} \left(\gamma \left(|\nabla u_\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla u_\varepsilon \right) \eta_\delta \, dx \\ & \quad + \int_{\partial\Omega} \eta_\delta \gamma \left(|\nabla u_\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} \partial_\nu u_\varepsilon \, dS \\ &= \int_{\partial\Omega \cap B(x_0, \delta)} \eta_\delta \gamma \left(|\nabla u_\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} \partial_\nu u_\varepsilon \, dS, \end{aligned}$$

since u_ε solves the perturbed equation, and the integral (4.11) equals

$$\begin{aligned} & \int_{\partial\Omega \cap B(x_0, \delta)} \eta_\delta \gamma(x_0) \left(|\nabla u_\varepsilon(x_0)|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nu(x_0) \cdot \nabla u_\varepsilon(x_0) \, dS \\ & \quad + \int_{\partial\Omega \cap B(x_0, \delta)} \eta_\delta \left(\gamma \left(|\nabla u_\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} \partial_\nu u_\varepsilon \right. \\ & \quad \left. - \gamma(x_0) \left(|\nabla u_\varepsilon(x_0)|^2 + \varepsilon \right)^{\frac{p-2}{2}} \partial_\nu u_\varepsilon(x_0) \right) \, dS \\ &= \left(\gamma(x_0) \left(|\nabla u_\varepsilon(x_0)|^2 + \varepsilon \right)^{\frac{p-2}{2}} \partial_\nu u_\varepsilon(x_0) + o_{\delta \rightarrow 0}(1) \right) \int_{\partial\Omega} \eta_\delta \, dS, \end{aligned}$$

since γ and ∇u_ε are continuous. By normalisation of the test function this proves the equality (4.10).

The rate of convergence is uniform in ε when δ is so small that $|\nabla u_\varepsilon(x)|$ is bounded away from zero for $x \in B(x_0, \delta)$, since then the following ε -indexed family of functions (4.12) are equicontinuous with parameter ε :

$$(4.12) \quad x \mapsto \gamma(x) \left(|\nabla u_\varepsilon(x)|^2 + \varepsilon \right)^{\frac{p-2}{2}} \partial_\nu u_\varepsilon(x).$$

The family is Hölder-continuous on $\partial\Omega$ when $x - x_0$ is so small that $|\nabla u_\varepsilon(x)| \geq m > 0$ for some m and all $\varepsilon < a$: By Lemma 2.4 the functions $x \mapsto \nabla u_\varepsilon(x)$ are equicontinuous with parameter ε , and the functions $w \mapsto \gamma(|w|^2 + \varepsilon)^{\frac{p-2}{2}} w \cdot \nu$ are equicontinuous when $|w|$ is bounded from above and away from zero. \square

Remark 4.5. We have shown that

$$\int_{\Omega} (\partial_{\alpha}\gamma)|\nabla u|^p dx = \int_{\partial\Omega} \gamma(\alpha \cdot \nu)|\nabla u|^p dS - p \int_{\partial\Omega} (\partial_{\alpha}u)\Lambda_{\gamma}^s(u) dS.$$

Since we know $\gamma|_{\partial\Omega}$ by [43] theorem 1.1], $|\nabla u|_{\partial\Omega}$ by Lemma 4.2 and the strong DN map by Lemma 4.4 we also know $\int_{\Omega} (\partial_{\alpha}\gamma)|\nabla u|^p dx$.

Remark 4.6. In using the Rellich identity we need to know the values of the conductivity γ on the entire boundary $\partial\Omega$. The gradient ∇u_M should be large near x_0 and small elsewhere (we have not made this precise), so our proof is essentially local in nature.

The main Theorem 1.2 now follows, since the value of the integral

$$(4.13) \quad \int_{\Omega} \partial_{\alpha}\gamma|\nabla u_M|^p dx$$

is known (see Remark 4.5) and it converges to $\partial_{\alpha}\gamma(x_0)$ by Lemma 3.2

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Superconductive and insulating inclusions for non-linear conductivity equations,

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SUPERCONDUCTIVE AND INSULATING INCLUSIONS FOR NON-LINEAR CONDUCTIVITY EQUATIONS

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ABSTRACT. We rigorously treat the forward problem for the partial differential equation $\operatorname{div}(\sigma|\nabla u|^{p-2}\nabla u) = 0$, where the measurable conductivity $\sigma: \Omega \rightarrow [0, \infty]$ is zero or infinity in large sets and $1 < p < \infty$. We use the enclosure method to find an upper bound for the convex hull of an inclusion with zero or infinite conductivity.

1. INTRODUCTION

We study inverse boundary value problems for the partial differential equation

$$(1.1) \quad \operatorname{div}(\sigma(x) |\nabla u(x)|^{p-2} \nabla u(x)) = 0,$$

where the measurable coefficient $\sigma \geq 0$ is allowed to take the values 0 and ∞ in large sets and the exponent is in the range $1 < p < \infty$. This includes the case $p = 2$ where our PDE becomes the linear conductivity equation $\operatorname{div}(\sigma \nabla u) = 0$ which appears in Calderón's inverse problem [16].

To arrive at the PDE from a physical starting point, we consider an electric potential (voltage) $u: \Omega \rightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$. By Ohm's law current density is given by $J(x) = -\sigma(x)\nabla u(x)$ and Kirchhoff's law entails $\operatorname{div}(J(x)) = 0$. To arrive at the non-linear equation (1.1) instead of $\operatorname{div}(\sigma \nabla u) = 0$, we replace Ohm's law with the non-linear law $J(x) = -\sigma(x) |\nabla u(x)|^{p-2} \nabla u(x)$. Physically such non-linear laws can occur in dielectrics [12, 21, 22, 39, 52, 53], plastic moulding [3, 35], electro-rheological and thermo-rheological fluids [2, 7, 48], viscous flows in glaciology [25] and plasticity phenomena [6, 30, 45, 46, 51].

Kirchhoff's law retains its linear form and consequently our PDE is of divergence form. This is convenient for the study of weak solutions and calculus of variations. Weak solutions of (1.1) in a domain $\Omega \subset \mathbb{R}^n$ are minimizers of the energy functional

$$(1.2) \quad E(u) = \int_{\Omega} \sigma(x) |\nabla u(x)|^p dx$$

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in the space $W^{1,p}(\Omega)$ with some prescribed boundary values. This is true even if σ takes the values 0 and ∞ in non-empty open sets. This is our main result for the direct problem, and the details will be discussed in sections 1.1 and 2.

Our PDE can be classified as a quasilinear elliptic equation. We do not, however, bound the coefficient σ away from zero or infinity, so ellipticity holds in a weaker sense than usual. If $\sigma \equiv 1$, the solutions of (1.1) are known as p -harmonic functions. They have been studied extensively (see for instance [38, 24, 29, 41] and the references therein), but inverse problems for elliptic equations of this type have received considerably less attention.

We assume our potential to be real-valued since this is physically most relevant. For complex-valued functions u one can obtain essentially the same results with the same tools, but we restrict our attention to the real case.

Our goal is, given Dirichlet and Neumann boundary values of all solutions of (1.1), to reconstruct the shape of an unknown obstacle having zero or infinite conductivity. This data is encoded in the so-called Dirichlet-to-Neumann map which we will describe in section 1.1 and in more detail in section 2.2. When $p \neq 2$, there are very few results in this direction. When $p = 2$, this is Calderón's famous inverse boundary value problem. We will summarize earlier results in section 1.2 and new results proven in this article in section 1.3.

1.1. The direct problem. Before embarking on a study of inverse problems, it is good to show that the direct problem is well-posed. The well-posedness result we present is, to the best of our knowledge, new in its generality.

Let Ω be a bounded domain and $\sigma: \Omega \rightarrow [0, \infty]$ a measurable map. For simplicity, we make the standing assumption that the sets $D_0 = \sigma^{-1}(0)$ and $D_\infty = \sigma^{-1}(\infty)$ are open and the three sets $\partial\Omega$, \bar{D}_0 and \bar{D}_∞ are disjoint. We also assume that outside the sets D_0 and D_∞ the function σ is bounded away from both zero and infinity.

In this setting we look for minimizers of the energy (1.2) in $W^{1,p}(\Omega)$, given boundary values $f \in W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$. Minimization of this energy corresponds to solving an Euler–Lagrange equation.

Usually such problems are posed on the domain $\Omega \setminus (\bar{D}_0 \cup \bar{D}_\infty)$ and one would impose suitable boundary conditions on ∂D_0 and ∂D_∞ . We prefer to work with all of Ω when possible, for in the inverse problem the function σ and therefore the sets D_0 and D_∞ are unknown. See remark 2.10 for the usual formulation.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set. Let $\sigma: \Omega \rightarrow [0, \infty]$ be a measurable function, and denote $D_0 = \sigma^{-1}(0)$ and $D_\infty = \sigma^{-1}(\infty)$. Assume that both D_0 and D_∞ are open and Lipschitz, and the three sets $\partial\Omega$, \bar{D}_0 and \bar{D}_∞ are disjoint. Assume furthermore*

that σ is bounded away from zero and infinity outside the sets D_0 and D_∞ .

Let $p \in (1, \infty)$. Fix any boundary value $f \in W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$. The energy (1.2) has a minimizer $u \in W^{1,p}(\Omega)$ with boundary values $u|_{\partial\Omega} = f$. The minimizer is unique up to functions that have vanishing gradient outside D_0 and that vanish on $\partial\Omega$.

A function $u \in W^{1,p}(\Omega)$ is such a minimizer if and only if it satisfies the Euler–Lagrange equation $\operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0$ weakly in the sense that

$$(1.3) \quad \int_{\Omega} \sigma |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = 0$$

for all $\phi \in W_0^{1,p}(\Omega)$ with $\nabla \phi = 0$ in D_∞ .

For a proof, see theorems 2.4 and 2.7 and their proofs.

This theorem is true, in particular, in the linear case $p = 2$. Then the PDE is $\operatorname{div}(\sigma \nabla u) = 0$.

We point out that even though the minimizer u of the energy is not unique, the vector-valued function $\sigma |\nabla u|^{p-2} \nabla u$ is unique. Since the exceptional sets D_0 and D_∞ do not reach the boundary, the boundary values can be interpreted in the Sobolev sense as usual.

The information about boundary values of solutions is encoded in the Dirichlet-to-Neumann map (DN map) Λ_σ from the quotient space $W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$ to its dual. Let $f, g \in W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$ be any functions and let $\bar{f}, \bar{g} \in W^{1,p}(\Omega)$ be their extensions so that $\operatorname{div}(\sigma |\nabla \bar{f}|^{p-2} \nabla \bar{f}) = 0$, while the extension \bar{g} satisfies $\nabla \bar{g} = 0$ in D_∞ . We define Λ_σ so that

$$(1.4) \quad \langle \Lambda_\sigma f, g \rangle = \int_{\Omega} \sigma |\nabla \bar{f}|^{p-2} \nabla \bar{f} \cdot \nabla \bar{g}.$$

The Dirichlet-to-Neumann map is linear if and only if $p = 2$.

1.2. Known results. Let us first fix $p = 2$. If σ is assumed to be bounded away from zero and infinity, the DN map Λ_σ determines σ in two dimensions [5]. In higher dimensions this is true if σ is additionally assumed to be Lipschitz [18]. For more information about Calderón’s problem, we refer to [16, 54]. Astala, Lassas and Päiväranta [4] have investigated anisotropic conductivities that are not bounded from above or away from zero.

A different kind of problem is to reconstruct an inclusion — a subdomain of Ω with zero, infinite or non-zero finite conductivity in a constant background conductivity — from the DN map. Problems of this kind are our object of study.

There are several methods for recovering inclusions with zero or finite conductivity by making boundary measurements of both voltages and currents [33, 19, 13, 36, 55, 31, 47, 32, 44, 56]. For regions of infinite

conductivity a recent article by Ramdani and Munnier [43] shows how to detect infinitely conductive bodies from the Dirichlet to Neumann map in two dimensional domain for the linear conductivity equation. Their method is based on geometry in the complex plane, in particular Riemann mappings. Brühl [11, section 4.3.1] and Schmitt [50, section 2.2.2] have used the factorization method to detect perfectly conducting inclusions. Alessandrini and Valenzuela [1] detect perfectly conducting or insulating cracks with two boundary measurements. Friedman and Vogelius [20] have shown that one can recover the location and scale of a finite number of small inclusions with zero or infinite conductivity in an inhomogeneous background from the DN map. Superconductive but grounded inclusions have been detected in, for example, [37, 8]. Moradifard, Nachman and Tamakan [42] consider a single interior measurement for conductivity equation and detect insulating or perfectly conducting inclusions, though their approach to the direct problem is not variational. Perfectly conducting inclusions in the context of the Maxwell equations have been detected by sampling methods [23, 14].

Other results relating to infinitely conducting obstacles concern the situation of two such obstacles being close to each other and the main concern is the blow-up of the solutions [34, 26].

For other values of p much less is known. We assume $1 < p < \infty$ throughout this article. The p -conductivity equation (1.1) with infinite conductivity has been considered by Gorb and Novikov [26] for $2 \leq p \in \mathbb{N}$ in dimensions two and three. We are not aware of a rigorous treatment of the forward problem in the literature, so we provide one in section 2 as summarized above in section 1.1.

The p -Calderón's problem, or Calderón's problem related to the p -conductivity equation, was introduced by Salo and Zhong [49]. They recover the conductivity on the boundary of the domain. Brander [9] improved the result to first order derivative of conductivity on the boundary, but with increased regularity assumptions. A recent result by Brander, Kar and Salo [10] shows that one can detect the convex hull of an inclusion with conductivity bounded away from zero and infinity. Very recently Guo, Kar and Salo [27] proved that under a monotonicity assumption the DN map is injective for Lipschitz conductivities when $n = 2$ for general p , and when $n \geq 3$ when one of the conductivities is almost constant. Their results assume the conductivity to be bounded from above and away from zero.

1.3. New results. We use the enclosure method to detect the convex hull of an inclusion. We assume that one of D_0 and D_∞ is empty and the other domain (called D) has Lipschitz boundary. We are unable to detect the convex hull of D , but we can recover a larger set, giving an estimate from above for the inclusion. We can also determine whether $D = \emptyset$ or not. See corollary 3.9 for more details.

If we allow σ to take the values 0 or ∞ in large sets, the DN map Λ_σ no longer determines σ uniquely. For example, consider the domain $\Omega = B(0, 3) \subset \mathbb{R}^n$ and a function $\sigma: \Omega \rightarrow [0, \infty]$ which is zero or infinity on $B(0, 2) \setminus \bar{B}(0, 1)$. Then the values of σ in $B(0, 1)$ have no effect on the DN map. We can only ever hope to recover σ up to the “outermost boundaries of D_0 and D_∞ ” and whether or not these boundaries belong to D_0 or D_∞ .

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2. THE DIRECT PROBLEM

2.1. Well-posedness. Let us now carefully formulate the direct problem and see that it is well-posed.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a finite number of connected components, such that each connected component is a Sobolev extension domain (with exponent p) and the closures of the components are disjoint. Let $\Gamma \subset \partial\Omega$ be open and meet all connected components of Ω . There exists a constant C so that any function $u \in W^{1,p}(\Omega)$ satisfying $u|_\Gamma = 0$ in the Sobolev sense satisfies*

$$(2.1) \quad \|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

Proof. Let $W_{0'}^{1,p}(\Omega)$ be the closure under the $W^{1,p}(\Omega)$ norm of the space of smooth functions in $\bar{\Omega}$ supported away from Γ . The prime reminds that the zero boundary value in the Sobolev sense is only assumed on Γ which may be a proper subset of $\partial\Omega$.

Suppose there was no such constant C . Then there is a sequence of functions $u_k \in W_{0'}^{1,p}(\Omega)$ so that

$$(2.2) \quad \|u_k\|_{L^p(\Omega)} \geq k \|\nabla u_k\|_{L^p(\Omega)} > 0.$$

We may normalize this sequence so that $\|u_k\|_{L^p(\Omega)} = 1$ and $\|\nabla u_k\|_{L^p(\Omega)} < 1/k$. By the Rellich–Kondrachov theorem there is a subsequence converging in $L^p(\Omega)$. The Rellich–Kondrachov theorem holds in Sobolev extension domains, and a finite union of Sobolev extension domains with positive distance from each other also admits an extension operator. We denote the subsequence by (u_k) and the limit function by u .

For any test function $\eta \in C_0^\infty(\Omega)$ we have

$$(2.3) \quad \int_{\Omega} u \nabla \eta = \lim_{k \rightarrow \infty} \int_{\Omega} u_k \nabla \eta = - \lim_{k \rightarrow \infty} \int_{\Omega} \eta \nabla u_k = 0,$$

using $W^{1,p}$ regularity of each u_k and the norm bound on ∇u_k . Therefore u is weakly differentiable and its weak gradient is identically zero.

We have in fact $u \in W^{1,p}(\Omega)$ and the convergence $u_k \rightarrow u$ happens also in $W^{1,p}(\Omega)$, so $u \in W_0^{1,p}(\Omega)$. Since u has zero weak gradient, it must be constant on each connected component. The only possible constant value is zero due to the zero boundary value on Γ and the connectedness assumption. This contradicts the normalization $\|u_k\|_{L^p(\Omega)} = 1$ for all k . \square

Lemma 2.2. *Let $\omega, \Omega \subset \mathbb{R}^n$ be bounded domains with Lipschitz boundaries. Assume that the closure of each connected component of $\Omega \setminus \bar{\omega}$ meets $\partial\Omega$. (This happens, in particular, if $\bar{\omega} \subset \Omega$ and $\Omega \setminus \bar{\omega}$ is connected.) There exists a constant C so that*

$$(2.4) \quad \|u\|_{L^p(\Omega \setminus \bar{\omega})} \leq C \|\nabla u\|_{L^p(\Omega \setminus \bar{\omega})}$$

for all $u \in W^{1,p}(\Omega \setminus \bar{\omega})$ that vanish (in the Sobolev sense) on $\partial\Omega$.

Proof. This is a special case of lemma 2.1. In particular, Lipschitz domains are Sobolev extension domains [15, theorem 12]. \square

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Any p -harmonic function $u \in W^{1,p}(\Omega)$ satisfies*

$$(2.5) \quad \|\nabla u\|_{L^p(\Omega)} \leq \|u|_{\partial\Omega}\|_{W^{1,p}(\Omega)/W_0^{1,p}(\Omega)}.$$

Proof. We simply observe that

$$(2.6) \quad \begin{aligned} \|u|_{\partial\Omega}\|_{W^{1,p}(\Omega)/W_0^{1,p}(\Omega)} &= \inf_{v-u \in W_0^{1,p}(\Omega)} \|v\|_{W^{1,p}(\Omega)} \\ &\geq \inf_{v-u \in W_0^{1,p}(\Omega)} \|\nabla v\|_{L^p(\Omega)} \\ &= \|\nabla u\|_{L^p(\Omega)}, \end{aligned}$$

since a p -harmonic function minimizes, by definition, the L^p norm of the gradient. \square

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\sigma: \Omega \rightarrow [0, \infty]$ be a measurable function. Denote $D_0 = \sigma^{-1}(0)$ and $D_\infty = \sigma^{-1}(\infty)$. Suppose the following:*

- The sets D_0 and D_∞ are open.
- The sets \bar{D}_0 , \bar{D}_∞ and $\partial\Omega$ are disjoint.
- The quantity $\log \sigma$ is essentially bounded in $\Omega \setminus (D_0 \cup D_\infty)$.
- The set D_0 has Lipschitz boundary.

Fix $p \in (1, \infty)$. Given any $f \in W^{1,p}(\Omega)$, there is a minimizer $u \in f + W_0^{1,p}(\Omega)$ to the energy

$$(2.7) \quad E(u) = \int_{\Omega} \sigma |\nabla u|^p dx.$$

The minimal energy is finite and the minimizer is unique modulo functions that have zero gradient outside D_0 but still satisfy the Dirichlet boundary conditions on $\partial\Omega$. The minimizer satisfies $\operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0$ in $\Omega \setminus (\bar{D}_0 \cup \bar{D}_\infty)$ and $\nabla u = 0$ in D_∞ in the weak sense.

Note that the essential boundedness of $\log \sigma$ is equivalent to the existence of $c > 0$ such that for almost all x we have $1/c < \sigma(x) < c$.

Remark 2.5. The result which states that the minimizer is unique modulo functions that have zero gradient outside D_0 is simplified when the closure of each connected component of $\Omega \setminus D_0$ intersects $\partial\Omega$ and in particular when $\Omega \setminus D_0$ is connected. Under this assumption the uniqueness holds modulo $W_0^{1,p}(D_0)$.

The minimizer can be made unique with a small number of adjustments.

Remark 2.6. Suppose u and v are any two minimizers of the energy in theorem 2.4. Define u_0 to equal u in all connected components of $\Omega \setminus D_0$ that touch the boundary $\partial\Omega$, set $u_0 = 0$ in connected components of $\Omega \setminus D_0$ that do not touch the boundary $\partial\Omega$ and let u_0 be p -harmonic in D_0 with Dirichlet boundary values determined by the previous conditions. Define v_0 in a similar way, but based on v . Then $u_0 = v_0$ in Ω and $\nabla(u - u_0) = \nabla(v - v_0) = 0$ in $\Omega \setminus D_0$.

Proof of theorem 2.4. First of all, the energy $E(u)$ is finite if and only if $\nabla u = 0$ in D_∞ ; in fact, for such functions we have

$$(2.8) \quad E(u) = \int_{\Omega \setminus D_\infty} \sigma |\nabla u|^p dx.$$

(We use the convention $0 \cdot \infty = 0$.) Since \bar{D}_∞ is disjoint from $\partial\Omega$, there are such functions with the prescribed boundary values. The space $A = \{u \in W^{1,p}(\Omega); \nabla u|_{D_\infty} = 0\}$ is a closed subspace of $W^{1,p}(\Omega)$, and so is $B = \{u \in W_0^{1,p}(\Omega); \nabla u|_{\Omega \setminus D_0} = 0\}$.

It is clear that changing the function u in D_0 does not change $E(u)$. Therefore we consider the quotient space

$$(2.9) \quad S = A/B.$$

The energy functional E is well defined on this quotient space and $E(u) < \infty$ for all $u \in S$.

Since the only thing that matters about f are its boundary values and the sets $\partial\Omega$, \bar{D}_0 , and \bar{D}_∞ are disjoint, we may assume that f vanishes on D_0 and D_∞ .

We denote the equivalence of $u \in A$ by $[u] = u + B$. Let us define $S_f = \{[u] \in S; u - f \in W_0^{1,p}(\Omega)\}$. Notice that the truth value of $u - f \in W_0^{1,p}(\Omega)$ does not depend on the choice of the representative of the equivalence class $[u] \subset A$, so S_f is well-defined. We will also use the space S_0 where the boundary value is assumed to be zero instead of that of f . We shall show that there is a unique minimizer of E in S_f .

Let U be a connected component of $\Omega \setminus \bar{D}_0$ so that $U \cap \partial\Omega = \emptyset$. Then one can shift the values of $u \in S$ in U by a constant without changing u as an element of S . A minimizer must clearly have vanishing gradient in U , so we may assume that the minimizer (and all functions in a minimizing sequence) vanishes in U . We therefore assume, to the end of simplifying presentation, from now on that there are no such components U .

For any $u \in S$ we pick a preferred representative $\bar{u} \in W^{1,p}(\Omega)$ by demanding \bar{u} to be p -harmonic in D_0 . The boundary values of \bar{u} on ∂D_0 are determined by u .

From lemma 2.3 we obtain

$$(2.10) \quad \|\nabla \bar{u}\|_{L^p(D_0)} \leq \|\bar{u}|_{\partial D_0}\|_{W^{1,p}(D_0)/W_0^{1,p}(D_0)}.$$

By continuity of the quotient map and lemma 2.2 we have

$$(2.11) \quad \|u|_{\partial D_0}\|_{W^{1,p}(\Omega \setminus D_0)/W_0^{1,p}(\Omega \setminus D_0)} \leq C \|\nabla u\|_{L^p(\Omega \setminus D_0)}$$

for all $u \in W_0^{1,p}(\Omega)$. Since the boundary norms on ∂D_0 from different sides are comparable — in fact both are comparable to the Besov norm on $B_{p,p}^{1-1/p}(\partial D_0)$ — we have

$$(2.12) \quad \|\nabla \bar{u}\|_{L^p(D_0)} \leq C \|\nabla u\|_{L^p(\Omega \setminus D_0)}$$

for all $u \in W_0^{1,p}(\Omega)$.

Let now (u_k) be a minimizing sequence of E in S_f . Using the estimate (2.12) and $u_k - f \in S_0$, we get

$$(2.13) \quad \|\nabla(u_k - f)\|_{L^p(\Omega \setminus D_0)} \geq C \left\| \nabla(\overline{u_k - f}) \right\|_{L^p(D_0)},$$

so for some other constant C' we have

$$(2.14) \quad \|\nabla(u_k - f)\|_{L^p(\Omega \setminus D_0)} \geq C' \left\| \nabla(\overline{u_k - f}) \right\|_{L^p(\Omega)}.$$

It follows from the assumption on $\log \sigma$ that $\sigma|_{\Omega \setminus D_0} \geq \lambda$ for some $\lambda > 0$. Therefore

$$(2.15) \quad \begin{aligned} \lambda^{-1/p} E(u_k)^{1/p} &\geq \|\nabla u_k\|_{L^p(\Omega \setminus D_0)} \\ &\geq \|\nabla(u_k - f)\|_{L^p(\Omega \setminus D_0)} - \|\nabla f\|_{L^p(\Omega \setminus D_0)} \\ &\geq C' \left\| \nabla(\overline{u_k - f}) \right\|_{L^p(D_0)} - \|\nabla f\|_{L^p(\Omega \setminus D_0)}. \end{aligned}$$

Since $E(u_k)$ is bounded, this estimate guarantees that also the sequence $(\overline{u_k - f})$ is bounded in $W_0^{1,p}(\Omega)$. Therefore there is a subsequence (which

we denote by the sequence itself) which converges weakly in $W^{1,p}(\Omega)$. Let $u_0 - f$ be the limit function.

The energy functional $E(u)$ is just a weighted Dirichlet energy of $u|_{\Omega \setminus (D_0 \cup D_\infty)}$ with weight bounded away from zero and infinity. Such functionals are weakly lower semicontinuous and $\overline{u_k - f}$ converges weakly to $u_0 - f$ also in $W^{1,p}(\Omega \setminus (D_0 \cup D_\infty))$ (with the functions restricted appropriately). Therefore

$$(2.16) \quad E(u_0) \leq \lim_{k \rightarrow \infty} E(u_k),$$

so u_0 indeed minimizes the energy.

Suppose there are two minimizers u_0 and v_0 of E in S_f . Since the functions are distinct, they must differ in $\Omega \setminus (D_0 \cup D_\infty)$, and so the set $\{x \in \Omega \setminus (D_0 \cup D_\infty); \nabla u_0(x) \neq \nabla v_0(x)\}$ must have positive measure. By strict convexity of $t \mapsto t^p$ this implies $E(\frac{1}{2}u_0 + \frac{1}{2}v_0) < \frac{1}{2}E(u_0) + \frac{1}{2}E(v_0)$. But this is impossible because u_0 and v_0 minimize the energy, so $u_0 = v_0$ as elements in S_f .

The fact that our unique minimizer solves the PDE follows from standard techniques in the calculus of variations. \square

We remark that the weak limit of a locally uniformly bounded sequence of p -harmonic functions is p -harmonic.¹ Therefore the minimizer constructed in the proof above is p -harmonic in D_0 at least if f is bounded. It is also p -harmonic (weighted with σ) in $\Omega \setminus (D_0 \cup D_\infty)$, but it need not be p -harmonic in any sense across ∂D_0 . Using p -harmonic extensions to D_0 was just a matter of convenience; the values in D_0 are irrelevant for the energy. All minimizers are p -harmonic in D_∞ since the gradient must vanish identically there.

The variational problem also leads to a PDE in the whole domain Ω despite σ being zero or infinite as we shall see next, as we provide a weak formulation for the equation $\operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0$.

Theorem 2.7. *Let Ω and σ be as in theorem 2.4. Fix any $f \in W^{1,p}(\Omega)$ and only consider the functions satisfying the constraint $u - f \in W_0^{1,p}(\Omega)$. Such a function $u \in W^{1,p}(\Omega)$ minimizes the energy functional E if and only if $\nabla u = 0$ in D_∞ and*

$$(2.17) \quad \int_{\Omega} \sigma |\nabla u|^{p-2} \nabla u \cdot \nabla \phi = 0$$

for all $\phi \in W_0^{1,p}(\Omega)$ satisfying $\nabla \phi = 0$ on D_∞ .

Proof. Let us again denote $A = \{u \in W^{1,p}(\Omega); \nabla u|_{D_\infty} = 0\}$ and in addition $A_0 = A \cap W_0^{1,p}(\Omega)$. Suppose u is a minimizer of E in $\tilde{f} + A_0$,

¹Every locally uniformly bounded family of weak solutions to the p -Laplace equation is equicontinuous [29, Theorem 6.12] and thus by the Ascoli-Arzelà theorem has a uniformly convergent subsequence. A locally uniform limit of solutions is still a solution [29, Theorem 3.78]. A uniform limit is also a weak $W^{1,p}$ limit.

where $f - \tilde{f} \in W_0^{1,p}(\Omega)$ and $\nabla \tilde{f} = 0$ in D_∞ . Take any $\phi \in A_0$. Now

$$\begin{aligned}
 (2.18) \quad 0 &= \left. \frac{d}{dt} E(u + t\phi) \right|_{t=0} \\
 &= \left. \frac{d}{dt} \int_{\Omega} \sigma |\nabla(u + t\phi)|^p \right|_{t=0} \\
 &= \int_{\Omega} \sigma \left. \frac{d}{dt} |\nabla(u + t\phi)|^p \right|_{t=0} \\
 &= p \int_{\Omega} \sigma |\nabla u|^{p-2} \nabla u \cdot \nabla \phi.
 \end{aligned}$$

Commuting differentiation and integration is possible by the dominated convergence theorem and the mean value theorem for the map $t \mapsto |\nabla(u + t\phi)|^p$.

Conversely, suppose (2.17) holds for all $\phi \in A_0$. The map $\mathbb{R}^n \ni \xi \mapsto |\xi|^p \in \mathbb{R}$ is convex and smooth outside the origin, so

$$(2.19) \quad |\nabla u(x)|^p + p |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \phi(x) \leq |\nabla(u + \phi)(x)|^p$$

for all $x \in \Omega$. Integrating this with weight σ and using (2.17), we have $E(u) \leq E(u + \phi)$ for all $\phi \in A_0$. \square

We also have a result for general ϕ – not just $\phi \in A_0$ – by formal integration by parts.

Remark 2.8. The minimizer of theorem 2.7 also satisfies²

$$(2.20) \quad \int_{\Omega} \sigma |\nabla u|^{p-2} \nabla u \cdot \nabla \phi = \int_{\partial D_\infty} \sigma |\nabla u|^{p-2} (\partial_\nu u) \phi$$

for any $\phi \in W_0^{1,p}(\Omega)$, provided that the integral over the boundary is well-defined. This is true, for example, when $\sigma \in C(\Omega \setminus (D_0 \cup D_\infty))$.

Remark 2.9. Given the boundary values, a minimizer u of the energy E is not unique, but the function $\sigma |\nabla u|^{p-2} \nabla u$ is unique. This is the only quantity appearing in the weak Euler–Lagrange equation (2.17). In fact, this function is in $L^{p'}(\Omega; \mathbb{R}^n)$ because $\nabla u = 0$ whenever $\sigma = \infty$, with the conjugate exponent p' defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Remark 2.10. We can also formulate the partial differential equation $\operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0$ (or in weak form (2.17)) in the domain $\Omega \setminus (\bar{D}_0 \cup \bar{D}_\infty)$. We only need to find the correct boundary conditions on ∂D_0 and ∂D_∞ .

Let C be a connected component of D_∞ . Since ∇u must vanish in C , u is constant on C , but there is also another condition. We can

²The unit normal ν is pointing outward from the perspective of $\Omega \setminus D_\infty$. This means towards the interior of D_∞ .

choose a function $\phi \in C_0^\infty$ so that $\phi \equiv 1$ on C and $\phi \equiv 0$ in $D_\infty \setminus C$. Comparing the two equations in theorem 2.7 (or integrating by parts), we observe that $\int_{\partial C} \sigma |\nabla u|^{p-2} \partial_\nu u = 0$.

Let us then find the boundary conditions on ∂D_0 . To that end, we take an arbitrary test function $\phi \in C_0^\infty(\Omega)$ vanishing in D_∞ for weak Euler–Lagrange equation (2.17). Since σ vanishes in D_0 , the integral over Ω in (2.17) is in fact an integral over $\Omega \setminus D_0$. Integration by parts gives $\int_{\partial D_0} \sigma |\nabla u|^{p-2} (\partial_\nu u) \phi = 0$. If this is to hold for all such ϕ , we obtain the Neumann boundary condition $\sigma |\nabla u|^{p-2} \partial_\nu u = 0$ on ∂D_0 .

Therefore the equation of theorem 2.7 can be reformulated as

$$(2.21) \quad \begin{cases} \operatorname{div}(\sigma |\nabla u|^{p-2} \nabla u) = 0 & \text{in } \Omega \setminus (\bar{D}_0 \cup \bar{D}_\infty) \\ u = f & \text{on } \partial\Omega \\ \sigma |\nabla u|^{p-2} \partial_\nu u = 0 & \text{on } \partial D_0 \\ \text{for each component } C \text{ of } D_\infty & \begin{cases} u|_C = \text{constant} \\ \int_{\partial C} \sigma |\nabla u|^{p-2} \partial_\nu u = 0. \end{cases} \end{cases}$$

The constant may be different for different components and the constant values depend on the boundary data f .

The next lemma shows that the minimal energy depends monotonically on the conductivity σ . We remind the reader that the functions u_σ and u_γ in the lemma are not unique but the minimal energy is.

Lemma 2.11. *Suppose σ and γ are conductivities satisfying the assumptions of theorem 2.4 with $\sigma \leq \gamma$ almost everywhere. Fix some $f \in W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$ and let u_σ and u_γ solve the p -conductivity equation in the sense of theorem 2.4 with the boundary values f and conductivities σ and γ , respectively. Then*

$$(2.22) \quad \int_{\Omega} \sigma(x) |\nabla u_\sigma|^p dx \leq \int_{\Omega} \gamma(x) |\nabla u_\gamma|^p dx.$$

Proof. We define $E_\gamma: W^{1,p}(\Omega) \rightarrow [0, \infty]$ by

$$(2.23) \quad E_\gamma(v) = \int_{\Omega} \gamma(x) |\nabla v|^p dx$$

and E_σ similarly. For every $v \in f + W_0^{1,p}(\Omega)$ we have $E_\sigma(v) \leq E_\gamma(v)$. Taking the infimum over $v \in f + W_0^{1,p}(\Omega)$ gives $E_\sigma(u_\sigma) \leq E_\gamma(u_\gamma)$ as desired. The minimizers exist by theorem 2.4. \square

2.2. The weak Dirichlet-to-Neumann map. We can now define the weak Dirichlet-to-Neumann map (DN map). We assume Ω and σ to be as in theorems 2.4 and 2.7 above. The simpler case where $D_0 = D_\infty = \emptyset$ was treated in [49, 28]. Let $X = W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$ and X' be its dual. The DN map $\Lambda_\sigma: X \rightarrow X'$ is defined by

$$(2.24) \quad \langle \Lambda_\sigma f, g \rangle = \int_{\Omega} \sigma |\nabla \bar{f}|^{p-2} \nabla \bar{f} \cdot \nabla \bar{g},$$

where $\bar{f} \in W_0^{1,p}(\Omega)$ is any minimizer of the energy functional E with boundary values $f \in X$ and $\bar{g} \in W^{1,p}(\Omega)$ is an extension of $g \in X$ with $\nabla \bar{g} = 0$ in D_∞ . Since $\bar{D}_\infty \cap \partial\Omega = \emptyset$, there always exists such an extension \bar{g} , and the existence of a minimizer \bar{f} follows from theorem 2.4.

Under the same assumptions we also have

$$(2.25) \quad \langle \Lambda_\sigma f, g \rangle = \int_{\Omega \setminus D_\infty} \sigma |\nabla \bar{f}|^{p-2} \nabla \bar{f} \cdot \nabla \bar{g}.$$

For a sufficiently nice conductivity σ we can use arbitrary extensions $\bar{g} \in W^{1,p}(\Omega)$; their gradient does not have to vanish in D_∞ . See remark 2.8 for more details. The DN map $\Lambda_\sigma: X \rightarrow X'$ has an extra term in such cases:

$$(2.26) \quad \langle \Lambda_\sigma f, g \rangle = \int_{\Omega} \sigma |\nabla \bar{f}|^{p-2} \nabla \bar{f} \cdot \nabla \bar{g} - \int_{\partial D_\infty} \sigma |\nabla \bar{f}|^{p-2} (\partial_\nu \bar{f}) \bar{g},$$

where $\bar{f} \in W_0^{1,p}(\Omega)$ is any minimizer of the energy functional E with boundary values $f \in X$ and $\bar{g} \in W^{1,p}(\Omega)$ is any extension of $g \in X$.

Let us see why the DN map is well-defined. As pointed out in remark 2.9, the minimizer \bar{f} is not unique, but $\sigma |\nabla \bar{f}|^{p-2} \nabla \bar{f}$ is. To see that the definition is independent of the choice of \bar{g} , we need to show that the above expression for $\langle \Lambda_\sigma f, g \rangle$ vanishes when $\bar{g} \in W_0^{1,p}(\Omega)$ and $\nabla \bar{g} = 0$ in D_∞ or the extra integral over ∂D_∞ is present. But this is just the claim of theorem 2.7 and remark 2.8.

Linearity of $\Lambda_\sigma f$ as a functional on X is evident (although $f \mapsto \Lambda_\sigma f$ is linear if and only if $p = 2$). To see that $\Lambda_\sigma f$ is continuous consider the estimate

$$(2.27) \quad \left| \int_{\Omega} \sigma |\nabla \bar{f}|^{p-2} \nabla \bar{f} \cdot \nabla \bar{g} \right| \leq C \left\| \nabla \bar{f} \right\|_{L^p(\Omega)}^{p-1} \|\bar{g}\|_{W^{1,p}(\Omega)},$$

which gives

$$(2.28) \quad \|\Lambda_\sigma f\|_{X'} \leq C \left\| \nabla \bar{f} \right\|_{L^p(\Omega)}^{p-1}.$$

The constant C depends on σ but not on f .

3. ENCLOSURE METHOD FOR p -CALDERÓN PROBLEM

For any subset $D \subseteq \mathbb{R}^n$ we define the (convex) support function $h_D: \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ as

$$(3.1) \quad h_D(\rho) = \sup_{x \in D} x \cdot \rho.$$

We will use specific solutions that are oscillating in one direction and have exponential behaviour in a perpendicular direction. The solutions were first introduced by Wolff [57, section 3] (see also [40]) and later applied to inverse problems by Salo and Zhong [49, section 3].

Definition 3.1 (Wolff solutions). For directions $\rho, \rho^\perp \in \mathbb{R}^n$, parameters $t \in \mathbb{R}, \tau > 0$, and for points $x \in \mathbb{R}^n$ we define the functions

$$(3.2) \quad u(x, \tau, t, \rho, \rho^\perp) = \exp(\tau(x \cdot \rho - t)) w(\tau x \cdot \rho^\perp),$$

where w is defined in lemma 3.2. When $\rho, \rho^\perp \in \mathbb{R}^n$ satisfy $|\rho| = |\rho^\perp| = 1$ and $\rho \cdot \rho^\perp = 0$, we call them the Wolff solutions to the p -Laplace equation. We also write $f = u|_{\partial\Omega}$.

The solutions are p -harmonic:

Lemma 3.2. *Let $\rho, \rho^\perp \in \mathbb{R}^n$ satisfy $|\rho| = |\rho^\perp| = 1$ and $\rho \cdot \rho^\perp = 0$. Define $h: \mathbb{R}^n \rightarrow \mathbb{R}$ by $h(x) = e^{-\rho \cdot x} w(\rho^\perp \cdot x)$, where the function w solves the differential equation*

$$(3.3) \quad w''(s) + V(w, w')w = 0$$

with

$$(3.4) \quad V(w, w') = \frac{(2p-3)(w')^2 + (p-1)w^2}{(p-1)(w')^2 + w^2},$$

The function h is then p -harmonic.

Given any initial conditions $(a_0, b_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ there exists a solution $w \in C^\infty(\mathbb{R})$ to the differential equation (3.3) which is periodic with period $\lambda_p > 0$, satisfies the initial conditions $(w(0), w'(0)) = (a_0, b_0)$, satisfies $\int_0^{\lambda_p} w(s) ds = 0$, and furthermore there exist constants c and C depending on a_0, b_0, p such that for all $s \in \mathbb{R}$ we have

$$(3.5) \quad C > w(s)^2 + w'(s)^2 > c > 0.$$

For proof see [49, Lemma 3.1] and [10, Lemma 3.1]. In particular the gradient of the Wolff solutions is

$$(3.6) \quad \nabla u = \tau \exp(\tau(x \cdot \rho - t)) (\rho w(\tau x \cdot \rho^\perp) + \rho^\perp w'(\tau x \cdot \rho^\perp)).$$

In this section we assume that the conductivity σ is constant 1 outside the possibly empty open Lipschitz sets $D_0 = \sigma^{-1}(\{0\})$ and $D_\infty = \sigma^{-1}(\{\infty\})$. Note that this conductivity function satisfies the assumptions of theorem 2.4, so the forward problem is well-posed. For definition of the DN map Λ_σ , see section 2.2.

Definition 3.3 (Indicator function). Let f denote the Wolff solutions defined in definition 3.1. Then we define the indicator function I by

$$(3.7) \quad I(t, \tau, \rho, \rho^\perp) = \tau^{n-p} \langle (\Lambda_\sigma - \Lambda_\emptyset) f, f \rangle,$$

where Λ_\emptyset is the DN map associated with no obstacle and conductivity 1. We use the shorthand notation $(\Lambda_\sigma - \Lambda_\emptyset)f := \Lambda_\sigma f - \Lambda_\emptyset f$ although the DN maps are non-linear. As we keep the directions ρ and ρ^\perp fixed, we often omit them from our notation and write the indicator function simply as $I(t, \tau)$.

We record the following equality, which follows from the definition of the Wolff solutions (definition 3.1) and the indicator function (definition 3.3).

Lemma 3.4. $I(t, \tau) = \exp(2\tau(h_D(\rho) - t))I(h_D(\rho), \tau)$.

Recall that $h_D(\rho)$ is the convex support function.

The following lemma is crucial for the proof of the lower bound:

Lemma 3.5. *Suppose $1 < p < \infty$ and $D \subset \Omega$ has Lipschitz boundary. Then, for sufficiently large $\tau > 0$, we have*

$$(3.8) \quad \int_D \exp(-p\tau(h_D(\rho) - x \cdot \rho)) dx \geq C\tau^{-n}.$$

For proof, see [10, Lemma 4.7].

Lemma 3.6. *Suppose $D_\infty = \emptyset$ and D_0 has Lipschitz boundary. Then*

$$(3.9) \quad |I(h_{D_0}(\rho), \tau)| > C > 0$$

for sufficiently large τ .

Proof. We write $D = D_0$.

Note that $\partial D_\infty = \emptyset$ and so the indicator function can be rewritten as

$$(3.10) \quad I(t, \tau) = \tau^{n-p} \left(\int_{\Omega \setminus \overline{D}} |\nabla u_z|^p dx - \int_{\Omega} |\nabla u|^p dx \right),$$

where u_z solves the Zaremba problem (see equation (2.21))

$$(3.11) \quad \begin{cases} \operatorname{div}(|\nabla u_z|^{p-2} \nabla u_z) = 0 & \text{in } \Omega \setminus \overline{D} \\ u_z = f & \text{on } \partial\Omega \\ |\nabla u_z|^{p-2} \nabla u_z \cdot \nu = 0 & \text{on } \partial D \end{cases}$$

and u are the Wolff solutions.

By theorem 2.7 we get

$$(3.12) \quad \int_{\Omega \setminus \overline{D}} |\nabla u_z|^{p-2} \nabla u_z \cdot \nabla(u_z - u) dx = 0,$$

since $u|_{\partial\Omega} = u_z|_{\partial\Omega} = f$. For $1 < p < \infty$, we now recall the following inequality

$$(3.13) \quad |\eta|^p \geq |\zeta|^p + p|\zeta|^{p-2} \zeta \cdot (\eta - \zeta),$$

for all $\zeta, \eta \in \mathbb{R}^n$. Replacing η by ∇u and ζ by ∇u_z in the above inequality and then integrating over $\Omega \setminus \overline{D}$, we obtain

$$(3.14) \quad \begin{aligned} \int_{\Omega \setminus \overline{D}} |\nabla u|^p dx &\geq \int_{\Omega \setminus \overline{D}} |\nabla u_z|^p dx \\ &+ p \int_{\Omega \setminus \overline{D}} |\nabla u_z|^{p-2} \nabla u_z \cdot \nabla(u - u_z) dx. \end{aligned}$$

Finally, using (3.12) we have

$$(3.15) \quad \int_{\Omega \setminus \bar{D}} |\nabla u_z|^p dx \leq \int_{\Omega \setminus \bar{D}} |\nabla u|^p dx.$$

Therefore, the estimate for the indicator function becomes

$$(3.16) \quad \begin{aligned} I(t, \tau) &= \tau^{n-p} \left(\int_{\Omega \setminus \bar{D}} |\nabla u_z|^p dx - \int_{\Omega} |\nabla u|^p dx \right) \\ &\leq \tau^{n-p} \left(\int_{\Omega \setminus \bar{D}} |\nabla u|^p dx - \int_{\Omega} |\nabla u|^p dx \right) \\ &= -\tau^{n-p} \int_D |\nabla u|^p dx, \end{aligned}$$

that is,

$$(3.17) \quad -I(t, \tau) \geq \tau^{n-p} \int_D |\nabla u|^p dx.$$

Hence, combining the gradient of Wolff functions (3.6) and lemma 3.5 we obtain at $t = h_D(\rho)$

$$(3.18) \quad \begin{aligned} |I(h_D(\rho), \tau)| &\geq C \tau^{n-p} \int_D \tau^p e^{-p\tau(h_D(\rho) - x \cdot \rho)} dx \\ &\geq C \int_D \tau^n e^{-p\tau(h_D(\rho) - x \cdot \rho)} dx \\ &\geq C > 0. \end{aligned}$$

This estimate concludes the proof. \square

Lemma 3.7. *Suppose $D_0 = \emptyset$ and D_∞ has Lipschitz boundary. Then*

$$(3.19) \quad |I(h_{D_\infty}(\rho), \tau)| > C > 0$$

for sufficiently large τ .

Proof. We write $D = D_\infty$. Recall that

$$(3.20) \quad \langle \Lambda_D f, g \rangle = \int_{\Omega \setminus \bar{D}} |\nabla u_\infty|^{p-2} \nabla u_\infty \cdot \nabla \phi dx$$

where u_∞ satisfies (see equation (2.21))

$$(3.21) \quad \begin{cases} \operatorname{div}(|\nabla u_\infty|^{p-2} \nabla u_\infty) = 0 & \text{in } \Omega \setminus \bar{D} \\ u_\infty = \text{constant} & \text{in each component of } D \\ \int_{\partial D} |\nabla u_\infty|^{p-2} \frac{\partial u_\infty}{\partial \nu} = 0 \\ u_\infty = f & \text{on } \partial \Omega \end{cases}$$

and $\phi \in W^{1,p}(\Omega)$ is an extension of $g \in W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$ with $\phi|_{\partial \Omega} = g$ and $\nabla \phi = 0$ in D_∞ . Recall that u is the Wolff solution for the p -Laplacian and $u = f$ on $\partial \Omega$. Replacing ϕ in (3.20) by u_∞ we obtain

$$(3.22) \quad \langle \Lambda_D f, f \rangle = \int_{\Omega \setminus \bar{D}} |\nabla u_\infty|^p dx.$$

On the other hand, for the free DN map we write

$$(3.23) \quad \langle \Lambda_\emptyset f, g \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx$$

where $\phi \in W^{1,p}(\Omega)$ is an extension of g with $g = \phi|_{\partial\Omega} \in W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$. Since $u = u_\infty = f$ on $\partial\Omega$, by replacing ϕ in (3.23) by u or u_∞ we get

$$(3.24) \quad \begin{aligned} \langle \Lambda_\emptyset f, f \rangle &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla u_\infty \, dx \\ &= \int_{\Omega} |\nabla u|^p \, dx. \end{aligned}$$

Therefore,

$$(3.25) \quad \begin{aligned} \langle (\Lambda_D - \Lambda_\emptyset) f, f \rangle &= \int_{\Omega \setminus \bar{D}} |\nabla u_\infty|^p \, dx - \int_{\Omega} |\nabla u|^p \, dx \\ &= (p-1) \int_D |\nabla u|^p \, dx \\ &\quad - \left[p \int_D |\nabla u|^p \, dx + \int_{\Omega \setminus \bar{D}} |\nabla u|^p \, dx - \int_{\Omega \setminus \bar{D}} |\nabla u_\infty|^p \, dx \right] \\ &= (p-1) \int_D |\nabla u|^p \, dx - \mathcal{I}. \end{aligned}$$

Now, by using $\nabla u_\infty = 0$ in D and the identity (3.24), we get

$$(3.26) \quad \begin{aligned} \mathcal{I} &= p \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega \setminus \bar{D}} |\nabla u_\infty|^p \, dx + (1-p) \int_{\Omega \setminus \bar{D}} |\nabla u|^p \, dx \\ &= p \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla u_\infty \, dx - \int_{\Omega \setminus \bar{D}} |\nabla u_\infty|^p \, dx - (p-1) \int_{\Omega \setminus \bar{D}} |\nabla u|^p \, dx \\ &= p \int_{\Omega \setminus \bar{D}} |\nabla u|^{p-2} \nabla u \cdot \nabla u_\infty \, dx - \int_{\Omega \setminus \bar{D}} |\nabla u_\infty|^p \, dx - (p-1) \int_{\Omega \setminus \bar{D}} |\nabla u|^p \, dx. \end{aligned}$$

Replacing η by ∇u_∞ and ζ by ∇u in the inequality

$$(3.27) \quad |\eta|^p \geq |\zeta|^p + p |\zeta|^{p-2} \zeta \cdot (\eta - \zeta)$$

we obtain that $\mathcal{I} \leq 0$, i.e.,

$$(3.28) \quad I_\rho(t, \tau) = \tau^{n-p} \langle (\Lambda_D - \Lambda_\emptyset) f, f \rangle \geq (p-1) \tau^{n-p} \int_D |\nabla u|^p \, dx.$$

By the properties of the Wolff solutions (3.6) we get

$$(3.29) \quad I_\rho(h_D(\rho), \tau) \geq C \tau^n \int_D \exp(-p\tau(h_D(\rho) - x \cdot \rho)) \, dx.$$

Lemma 3.5 suffices to finish the proof. \square

Theorem 3.8 (Lower bound for the indicator function). *When $t < h_D(\rho)$ and either $D_0 = \emptyset$ or $D_\infty = \emptyset$, and the non-empty D has Lipschitz boundary, then there exist positive constants C_1 and C_2 such that for sufficiently large τ we have*

$$(3.30) \quad |I(t, \tau)| \geq C_1 \exp(C_2 \tau).$$

Furthermore, the sign of the indicator function depends on the non-empty D ; if D_0 is non-empty, then the indicator function is negative, and if D_∞ is non-empty, then the indicator function is positive.

Proof. By lemmata 3.7 and 3.6 the indicator function at time h_D is bounded from below. By lemma 3.4 the main part of the theorem holds.

The sign of the indicator function agrees with the sign of

$$(3.31) \quad \langle (\Lambda_D - \Lambda_\emptyset) f, f \rangle.$$

where the f are the Wolff solutions. That is, we want to prove

$$(3.32) \quad \langle \Lambda_D f, f \rangle \geq \langle \Lambda_\emptyset f, f \rangle$$

when $D = D_\infty$ and the opposite inequality when $D = D_0$. On the left-hand side, let the extension of f solve the equation with the inclusion D and on the right-hand side without it. The desired estimate then follows from lemma 2.11. \square

Corollary 3.9. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with a priori known constant conductivity σ outside an obstacle $D = D_0 \cup D_\infty$. Suppose $\sigma: \Omega \rightarrow \mathbb{R}_+ \cup \{0\} \cup \{\infty\}$ is measurable with $D_0 = \sigma^{-1}(\{0\})$ and $D_\infty = \sigma^{-1}(\{\infty\})$. Suppose either D_0 or D_∞ is empty, the sets $\partial\Omega$, $\overline{D_0}$ and $\overline{D_\infty}$ are pairwise disjoint, the set D has Lipschitz boundary, and $\sigma|_{\Omega \setminus \overline{D}}$ is constant.*

Then we can find a set D' , which is a superset of the convex hull of D . Furthermore, we can detect whether D_0 or D_∞ is non-empty.

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