Reports of the Department of Mathematical Information Technology
Series B. Scientific Computing
No. B. 16/2008

Computation of Generalized Time-Periodic Waves using Differential Forms, Exact Controllability, Least-Squares Formulation, Conjugate Gradient Method and Discrete Exterior Calculus: Part I - Theoretical Considerations

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ISBN 978-951-39-3343-2
ISSN 1456-436X

Computation of Generalized Time-Periodic Waves using Differential Forms, Exact Controllability, Least-Squares Formulation, Conjugate Gradient Method and Discrete Exterior Calculus: Part I - Theoretical Considerations

Dirk Pauly*   Tuomo Rossi†

Abstract

We present both theory and an algorithm for solving time-harmonic wave problems. The time-harmonic solutions will be achieved by computing time-periodic solutions of the original wave equations. Thus, an exact controllability technique is proposed for the time-dependent wave equations. We discuss a first order Maxwell type system, which will be formulated in the framework of alternating differential forms. This enables us to investigate different kind of classical wave problems at one fell swoop, such as acoustic, electro-magnetic or elastic wave problems. After a sufficient theory is established we formulate our exact controllability problem and suggest a least-squares optimization procedure for its solution, which itself is solved in a natural way by conjugate gradient algorithm operating in a purely L^2-type Hilbert space. Therefore, one of the biggest advances of this approach might be that the conjugate gradient algorithm does not need any preconditioning.

Key words: Wave Equation, Maxwell's equations, Differential Forms, Differential Geometry, Time-Periodic Waves, Time-Harmonic Waves, Controllability, Least-Squares Formulation, Conjugate Gradient Method, Discrete Exterior Calculus, Discrete Differential Forms

AMS MSC-classifications: 35Q60, 78A25, 78A30

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Contents

1 Introduction

Time-harmonic wave propagation is an important phenomenon which has many obvious applications in acoustics, electro-magnetics and elasticity, among others. Traditionally, the numerical solution approaches have been based on finite differences, finite elements or boundary element techniques. As our goal is to consider heterogeneous media as well, we pay attention to methods based on partial differential equations. Hence, some kind of tessellation of the spatial domain is necessary.

To obtain accurate results for wave propagation, the discretization mesh needs to be adjusted to the wavelength. If the time-harmonic case is directly addressed, one is faced with the solution of a large-scale indefinite linear system which is a difficult task.

Instead of solving directly the time-harmonic problem for a given frequency $\omega \in \mathbb{R}_+$, it is possible to find the solution by control techniques. Then the solution is found by searching for an appropriate initial data for the wave equation which minimizes a quadratic functional that measures the difference between the initial state and the final state after one time period $T = 2\pi/\omega$. A natural quadratic error functional is the squared energy norm of the system, allowing to minimize the cost by the conjugate gradient method operating in Hilbert spaces. This approach has been successfully applied to acoustics, electro-magnetics and elasticity [10, 11, 12, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26]. In practice, the method seems to have a good asymptotic computational cost. Even though no theory exists, the computational cost of the method seems to be of order $\mathcal{O}(n)$, where n is the number of spatial degrees of freedom. The drawback of using the traditional (second order in time) formulation of the wave equations is that the energy norm is then of H^1-type, and as such, the minimization problem is badly conditioned. This is handled by applying preconditioning to the conjugate gradient minimization. Unfortunately, this means that a discrete elliptic problem (linear system) still needs to be solved at every conjugate gradient iteration step. In recent papers, the linear system has been solved by an algebraic multi-grid method which still maintains the good asymptotic performance of the solution technique, but makes it quite more difficult to implement the solver to utilize the computing power of modern parallel computers and multi-core processors.

Hence, an alternative approach has recently been proposed in the short paper of Glowinski and Rossi [21]. The idea is to formulate the control method for an equivalent first order system which has an L^2-type energy norm, and hence, a well-conditioned minimization problem results. This eliminates the need for preconditioning the conjugate gradient minimization and, thus, greatly simplifies the parallel implementation of the method. This approach also has drawbacks as the spatial discretization needs to be based, for example, on mixed finite elements like Raviart-Thomas elements, which are more difficult to implement than standard finite elements. Initial numerical experiments (still unpublished) support the hypothesis that the cost of the new approach is also of order $\mathcal{O}(n)$.

In our project, we aim at generalizing the approach of [21] to generalized Maxwell equations formulated in terms of differential forms. The same formulation covers electro-magnetic, acoustic and elastic waves and it can be naturally discretized by so-called discrete differential forms (DDF) or discrete exterior calculus (DEC) which has recently been under very active research [29, 15, 14]. The goal of this is to develop theory and software for efficiently solving the generalized Maxwell equations using a control approach. We are planing to develop a new solution theory for the generalized Cauchy problem at hand, such that we can be sure to have uniquely determined solutions evolving in time. Here the papers [46, 47, 38, 39, 44, 45, 33] and the work [37] as well as the reports [32, 40, 41, 42, 43] will be useful. Moreover, theoretical questions about the domain truncation procedure have to be considered. We are planing to use absorbing boundary conditions (ABC), generalized Dirichlet-to-Neumann operators (DtN), i.e. electric-to-magnetic operators (EtM), as well as perfectly matched layers (PML). All these techniques have to be developed for differential forms. The resulting software is targeted to mid-frequency variable coefficient wave propagation problems in 2D and 3D domains, where the dimension of the computational domain is 10-100 wavelengths. The software is targeted for modern parallel computers and multi-core processors.

In this first report we present and explain the basic ideas of the control approach for wave equations formulated as first order systems using generalized Maxwell equations formulated by differential forms. First in section 3 we investigate the Cauchy problem (CP) and establish a solution theory which meets our needs utilizing the spectral calculus for (unbounded) selfadjoint linear operators in Hilbert space. Then in section 4 we introduce the least squares formulation and discuss the derivative of the least squares functional, which is the essential ingredient in our resulting algorithm, since we plan to use a conjugate gradient method (CGM). In section 5 we discuss the conjugate gradient algorithm (CGA) in some detail. We shortly explain a general CGA in Hilbert space and then present a CGA for our problem at hand. In section 6 we translate our formalism using differential forms to the classical framework of vector analysis and briefly demonstrate, which classical problems are formulated within our generalized theory. Finally in section 7 we outline the ongoing work in this project.

2 Problem formulation and notations

Let $I := (0,T)$ with $T > 0$ be some interval, $\bar{I} = [0,T]$ denote its closure and

$$\Omega \subset \mathbb{R}^N \quad , \quad N \in \mathbb{N} \quad ,$$

be an exterior domain. In [21] Glowinski and Rossi tried to find time-T periodic solutions u of the prototypical scalar linear wave problem

$$\begin{aligned}
(\partial_t^2 - c^2\Delta)u &= 0 && \text{in} \quad \Xi := I \times \Omega \quad , \\
\gamma u &= g && \text{on} \quad \Gamma := I \times \partial\Omega \quad , \\
u(0) = u(T) \quad , \quad \partial_t u(0) &= \partial_t u(T) && \text{in} \quad \Omega \quad ,
\end{aligned} \tag{1}$$

where they utilized a truncation of Ω introducing an artificial boundary (a sphere containing $\mathbb{R}^N \setminus \Omega$) and a homogeneous Neumann-type boundary condition on it, i.e. just setting the translation of Sommerfeld's radiation condition to the time dependent formulation $(c^{-1}\partial_t + \partial_r)u$ to zero. Here c is a positive real number and g is a given right hand side time dependent boundary data. Furthermore, γ denotes the usual scalar trace operator and r the Euclidean norm on $\mathbb{R}^N$.

They transformed the latter system via the well known substitution

$$E := \partial_t u \quad , \quad H := \nabla u$$

into the first order system of 'linear acoustics'

$$\begin{aligned}
\Big(\partial_t - \begin{bmatrix} c^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \nabla\cdot \\ \nabla & 0 \end{bmatrix}\Big)(E,H) &= (0,0) && \text{in} \quad \Xi \quad , \\
\gamma E &= \partial_t g && \text{on} \quad \Gamma \quad , \\
(E,H)(0) &= (E,H)(T) && \text{in} \quad \Omega \quad ,
\end{aligned}$$

which has a 'Maxwell-type flavor', albeit simpler. The advantage is that this second system allows for its solution an algorithm using

$$\mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)^N$$

as control space. In former works there was always at least the first part of the control space a closed subspace of $\mathrm{H}^1(\Omega)$, which makes the corresponding numerics much more difficult due to the need of preconditioning, for instance, in conjugate gradient algorithms. Such preconditioning is not necessary if one uses a purely $\mathrm{L}^2(\Omega)$-control space.

We may generalize this problem using the framework of alternating differential forms of rank q, shortly spoken q-forms. For this purpose we consider our exterior domain Ω a smooth N-dimensional differentiable Riemannian manifold with boundary. Let (U, h) be a chart for Ω. We call a q-form $u = \sum_J u_J \, \mathrm{d}h_J$ (locally) smooth and write $u \in \mathrm{C}^{\infty,q}(U)$, if the corresponding component functions $u_J := u(\partial_J^h)$ are C^∞. (If this holds for one chart h, then it holds true for all charts of the atlas.) If u is locally smooth for all charts of the atlas we call u smooth and write $u \in \mathrm{C}^{\infty,q}(\Omega)$. Here we utilized an obvious multi-index notation, i.e. for ordered multi-indices $J := (j_1, \dots, j_q) \in \{1, \dots, N\}^q$ we define the tuples of chart tangential vectors ∂_J^h and the corresponding special (chart) q-forms $\mathrm{d}h_J$ by

$$\partial_J^h := (\partial_{j_1}^h, \dots, \partial_{j_q}^h) \qquad , \qquad \mathrm{d}h_J := \mathrm{d}h_{j_1} \wedge \dots \wedge \mathrm{d}h_{j_q} \qquad ,$$

where $\wedge$ and d denote the exterior product and derivative, respectively. At most we will use the identity chart, i.e. trivial coordinates, and its differentials $\{\mathrm{d}x_n\}_{n=1}^N$. But also polar-coordinates or others coordinates are of course possible and sometimes useful. In the same way we define the space $\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$ of C^∞-q-forms with compact supports in Ω. This space admits a scalar product

$$(E, H) \mapsto \langle E, H \rangle_\Omega := \int_\Omega E \wedge * \bar{H} \in \mathbb{C} \qquad ,$$

where $*$ denotes the Hodge star operator and the bar complex conjugation. Using this scalar product and its induced norm we may define $\mathrm{L}^{2,q}(\Omega)$ as the closure of $\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$. Then $\mathrm{L}^{2,q}(\Omega)$ equipped with the scalar product $\langle \cdot \, , \cdot \rangle_{\mathrm{L}^{2,q}(\Omega)} := \langle \cdot \, , \cdot \rangle_\Omega$ becomes a Hilbert space, the Hilbert space of square integrable q-forms on Ω.

Following Hermann Weyl [49] and to remind of the electro-magnetic background it has become customary to denote the exterior derivative d by rot (rotation) and the co-derivative δ by div (divergence). Thus we have on q-forms

$$\operatorname{div} = (-1)^{(q-1)N} * \operatorname{rot} * \qquad .$$

We note $\operatorname{rot} \operatorname{div} + \operatorname{div} \operatorname{rot} = \Delta$, where the Laplacian Δ is to be understood componentwise in Euclidean coordinates.

With respect to the latter scalar product the linear operators rot and div are formally skew-adjoint to each other, i.e. for pairs of forms $(E,H)\in\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)\times\overset{\circ}{\mathrm{C}}{}^{\infty,q+1}(\Omega)$ we have by the weak version of Stokes' theorem

$$\begin{aligned}0&=\int_\Omega\operatorname{rot}(E\wedge*\bar{H})=\int_\Omega\operatorname{rot}E\wedge*\bar{H}+(-1)^q\int_\Omega E\wedge\operatorname{rot}*\bar{H}\\&=\int_\Omega\operatorname{rot}E\wedge*\bar{H}+\underbrace{(-1)^{q(N-q+1)}}_{=(-1)^{qN}}\int_\Omega E\wedge*\underbrace{*\operatorname{rot}*\bar{H}}_{=(-1)^{qN}\operatorname{div}\bar{H}}\\&=\langle\operatorname{rot}E,H\rangle_{\mathrm{L}^{2,q+1}(\Omega)}+\langle E,\operatorname{div}H\rangle_{\mathrm{L}^{2,q}(\Omega)}\qquad.\end{aligned}$$

This yields the possibility of weak versions of rot and div using smooth, compactly supported forms as test-forms. Hence, we may define $\operatorname{rot}E$ for a $\mathrm{L}^{2,q}(\Omega)$-form E and say E has weak rotation, if

$$\exists\,G\in\mathrm{L}^{2,q+1}(\Omega)\quad\forall\,\Phi\in\overset{\circ}{\mathrm{C}}{}^{\infty,q+1}(\Omega)\qquad\langle E,\operatorname{div}\Phi\rangle_{\mathrm{L}^{2,q}(\Omega)}=-\langle G,\Phi\rangle_{\mathrm{L}^{2,q+1}(\Omega)}\qquad.$$

Of course, we may define a weak divergence in the same way. Then we put

$$\begin{aligned}\mathrm{R}^q(\Omega)&:=\big\{E\in\mathrm{L}^{2,q}(\Omega):\operatorname{rot}E\in\mathrm{L}^{2,q+1}(\Omega)\big\}\qquad,\\\mathrm{D}^q(\Omega)&:=\big\{H\in\mathrm{L}^{2,q}(\Omega):\operatorname{div}H\in\mathrm{L}^{2,q-1}(\Omega)\big\}\qquad.\end{aligned}$$

Equipped with their natural graph-norms these are Hilbert spaces. Furthermore, we generalize the (electric) homogeneous boundary condition modeling a perfectly conducting obstacle, i.e. vanishing tangential trace ι^*E of a differential form E, where $\iota:\partial\Omega\to\overline{\Omega}$ denotes the natural embedding of the boundary manifold. For this purpose we define $\overset{\circ}{\mathrm{R}}{}^q(\Omega)$ to be the closure of $\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$ in the norm of $\mathrm{R}^q(\Omega)$. Indeed by Stokes' theorem and a density argument one may easily check for sufficiently smooth forms that a vanishing tangential trace is generalized in $\overset{\circ}{\mathrm{R}}{}^q(\Omega)$. Surely $\overset{\circ}{\mathrm{R}}{}^q(\Omega)$ is also a Hilbert space as a closed subspace of $\mathrm{R}^q(\Omega)$. An index 0 at the lower left corners of the spaces $\overset{\circ}{\mathrm{R}}{}^q(\Omega)$, $\mathrm{R}^q(\Omega)$ or $\mathrm{D}^q(\Omega)$ indicates vanishing rotation resp. divergence.

Let us define (formal matrix-) operators

$$M:=\begin{bmatrix}0&\operatorname{div}\\\operatorname{rot}&0\end{bmatrix}\qquad,\qquad\Lambda:=\begin{bmatrix}\varepsilon&0\\0&\mu\end{bmatrix}\qquad,\qquad\mathcal{M}:=\mathrm{i}\,\Lambda^{-1}M\qquad,$$

where ε resp. μ is a real, linear, symmetric, bounded and uniformly positive definite (with respect to the $\mathrm{L}^{2,q}(\Omega)$- resp. $\mathrm{L}^{2,q+1}(\Omega)$-scalar product) transformation on q- resp. $(q+1)$-forms, which is independent of time, and i denotes the imaginary unit. ε and μ model material properties, i.e. in classical electro-magnetic theory ε is the dielectricity and μ the permeability of the underlying medium. We note that ε and

μ are even allowed just to have $\mathrm{L}^{\infty}(\Omega)$-entries in their matrix representations $\nu^h_{J',J}$ given by chart bases

$$\nu E = \sum_{J',J} \nu^h_{J',J} E_J \,\mathrm{d}h_{J'} \qquad \text{if} \qquad E = \sum_J E_J \,\mathrm{d}h_J \qquad .$$

Another way to define these Hilbert spaces is to look at the densely defined linear operator

$$\begin{array}{rcccc} \mathrm{ROT} & : & \overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega) \subset \mathrm{L}^{2,q}(\Omega) & \longrightarrow & \overset{\circ}{\mathrm{C}}{}^{\infty,q+1}(\Omega) \subset \mathrm{L}^{2,q+1}(\Omega) \\ & & E & \longmapsto & \operatorname{rot} E \end{array}$$

and its adjoints, which will be marked by a star. Then $\overline{\mathrm{ROT}} = \mathrm{ROT}^{**}$ is the weak rotation on its domain of definition

$$\overset{\circ}{\mathrm{R}}{}^q(\Omega) := D(\overline{\mathrm{ROT}}) = \overline{\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)}$$

with closure taken in the graph-norm of ROT, i.e. in $\mathrm{R}^q(\Omega)$. The kernel of $\overline{\mathrm{ROT}}$ is just ${}_0\overset{\circ}{\mathrm{R}}{}^q(\Omega)$. Its adjoint operator $\mathrm{ROT}^* = \overline{\mathrm{ROT}}^*$ equals by definition the negative weak divergence $-\,\mathrm{DIV}$ on its domain of definition $\mathrm{D}^{q+1}(\Omega)$, i.e.

$$\begin{array}{rcccc} -\,\mathrm{ROT}^* = \mathrm{DIV} & : & \mathrm{D}^{q+1}(\Omega) \subset \mathrm{L}^{2,q+1}(\Omega) & \longrightarrow & \mathrm{L}^{2,q}(\Omega) \\ & & H & \longmapsto & \operatorname{div} H \end{array} \quad .$$

This is easy to see: Let $H \in D(\mathrm{ROT}^*)$ and $\mathrm{ROT}^* H = F$. Then by definition

$$\forall\, E \in D(\mathrm{ROT}) \qquad \langle \operatorname{rot} E, H \rangle_{\mathrm{L}^{2,q+1}(\Omega)} = \langle E, F \rangle_{\mathrm{L}^{2,q}(\Omega)} \qquad ,$$

which is just the definition of the (negative) weak divergence. Therefore we have $H \in D(\mathrm{ROT}^*) = \mathrm{D}^{q+1}(\Omega)$ and $\mathrm{ROT}^* H = -\operatorname{div} H$.

We note since $\operatorname{div}\operatorname{div} = 0$ and $\operatorname{rot}\operatorname{rot} = 0$ in the smooth case,

$$\operatorname{rot}\operatorname{rot} = 0 \qquad , \qquad \operatorname{div}\operatorname{div} = 0$$

still hold true in the weak sense and we also have

$$\operatorname{rot}\operatorname{div} + \operatorname{div}\operatorname{rot} = \Delta \qquad ,$$

where the Laplacian acts on each Euclidean component. Moreover, we get with closures taken in $\mathrm{L}^{2,q}(\Omega)$

$$\overline{\operatorname{rot} \overset{(\circ)}{\mathrm{R}}{}^{q-1}(\Omega)} \subset {}_0\overset{(\circ)}{\mathrm{R}}{}^q(\Omega) \qquad , \qquad \overline{\operatorname{div} \overset{(\circ)}{\mathrm{D}}{}^{q+1}(\Omega)} \subset {}_0\overset{(\circ)}{\mathrm{D}}{}^q(\Omega)$$

and for sufficiently smooth functions, i.e. C^1, and suitable q-forms E we obtain

$$\begin{aligned} \operatorname{rot}(\varphi E) &= (\operatorname{rot}\varphi) \wedge E + \varphi \operatorname{rot} E \qquad , \\ \operatorname{div}(\varphi E) &= (-1)^{(q-1)N} * \big((\operatorname{rot}\varphi) \wedge *E\big) + \varphi \operatorname{div} E \qquad . \end{aligned}$$

Since $\overline{\mathrm{ROT}}$ and DIV are skewadjoint to each other in this setting the (unbounded) linear operator

$$\begin{array}{rcccc} \mathcal{M} & : & D(\mathcal{M}) \subset \mathrm{L}^{2,q,q+1}_{\Lambda}(\Omega) & \longrightarrow & \mathrm{L}^{2,q,q+1}_{\Lambda}(\Omega) \\ & & (E,H) & \longmapsto & \mathcal{M}(E,H) = \mathrm{i}(\varepsilon^{-1}\operatorname{div} H, \mu^{-1}\operatorname{rot} E) \end{array} \quad , \tag{2}$$

where

$$\mathrm{L}^{2,q,q+1}_{\Lambda}(\Omega) := \mathrm{L}^{2,q,q+1}(\Omega) := \mathrm{L}^{2,q}(\Omega) \times \mathrm{L}^{2,q+1}(\Omega)$$

as a set equipped with the weighted scalar product $\langle \,\cdot\,,\,\cdot\,\rangle_{\mathrm{L}^{2,q,q+1}_{\Lambda}(\Omega)} := \langle \Lambda\,\cdot\,,\,\cdot\,\rangle_{\mathrm{L}^{2,q,q+1}(\Omega)}$ and

$$D(\mathcal{M}) := \overset{\circ}{\mathrm{R}}{}^{q}(\Omega) \times \mathrm{D}^{q+1}(\Omega) \quad ,$$

is selfadjoint and its spectrum equals the entire real axis. For more details please see [37, 38, 40] or in the (classical) case of vector analysis [36].

In this paper we investigate T-periodic solutions in time of the following generalized Maxwell controllability problem (GMCP)

$$\begin{aligned} (\partial_t + \mathrm{i}\,\mathcal{M})(E,H) &= (F,G) && \text{in} \quad \Xi \quad , \\ \gamma_\tau E &= \lambda && \text{in} \quad \Gamma \quad , \\ (E,H)(0) &= (E_0,H_0) && \text{in} \quad \Omega \quad , \\ (E,H)(T) &\overset{!}{=} (E,H)(0) && \text{in} \quad \Omega \quad , \end{aligned} \tag{3}$$

where γ_τ denotes the tangential trace, i.e. $\gamma_\tau = \iota^*$ in the smooth case with the natural embedding of the boundary $\iota : \partial\Omega \hookrightarrow \overline{\Omega}$ regarding $\partial\Omega$ as a $(N-1)$-dimensional Riemannian submanifold of $\overline{\Omega}$. Of course, the first equation may be written more explicitly

$$\begin{aligned} \partial_t E - \varepsilon^{-1}\operatorname{div} H &= F && \text{in} \quad \Xi \quad , \\ \partial_t H - \mu^{-1}\operatorname{rot} E &= G && \text{in} \quad \Xi \quad . \end{aligned}$$

Let us outline some heuristics: Assuming enough smoothness on the data we may compute by the differential equation

$$\begin{aligned} \big(\partial_t^2 - (\Lambda^{-1}M)^2\big)(E,H) &= (\partial_t^2 + \mathcal{M}^2)(E,H) = (\partial_t - \mathrm{i}\,\mathcal{M})(\partial_t + \mathrm{i}\,\mathcal{M})(E,H) \\ &= (\partial_t + \Lambda^{-1}M)(F,G) =: (\tilde{F},\tilde{G}) \end{aligned}$$

and since the tangential trace and the exterior derivative commute we get

$$\gamma_\tau\,\partial_t\,\mu H = \gamma_\tau \mu G + \gamma_\tau \operatorname{rot} E = \gamma_\tau \mu G + \operatorname{Rot}\lambda =: \tilde{\lambda} \quad ,$$

where Rot denotes the exterior derivative d on $\partial\Omega$. Moreover, if the right hand side (F,G) is time-T periodic as well, i.e. $(F,G)(T) = (F,G)(0)$, we have by the differential equation

$$\partial_t(E,H)(T) = -\mathrm{i}\,\mathcal{M}\underbrace{(E,H)(T)}_{=(E,H)(0)} + \underbrace{(F,G)(T)}_{=(F,G)(0)} = \partial_t(E,H)(0) \quad .$$

Thus, (E,H) solves also the (vector) wave equation-type controllability problem

$$\begin{aligned}
\Big(\partial_t^2 - \begin{bmatrix}\varepsilon^{-1}\operatorname{div}\mu^{-1}\operatorname{rot} & 0\\ 0 & \mu^{-1}\operatorname{rot}\varepsilon^{-1}\operatorname{div}\end{bmatrix}\Big)(E,H) &= (\tilde{F},\tilde{G}) && \text{in} \quad \Xi \quad ,\\
(\gamma_\tau E, \gamma_\tau\, \partial_t\, \mu H) &= (\lambda, \tilde{\lambda}) && \text{in} \quad \Gamma \quad ,\\
(E,H)(T) &= (E,H)(0) && \text{in} \quad \Omega \quad ,\\
\partial_t (E,H)(T) &= \partial_t (E,H)(0) && \text{in} \quad \Omega \quad .
\end{aligned} \tag{4}$$

Furthermore, because $\operatorname{rot}\operatorname{rot} = 0$ and $\operatorname{div}\operatorname{div} = 0$ we obtain by differentiation of the first equation in (3)

$$\partial_t \operatorname{div}\varepsilon E = \operatorname{div}\varepsilon F \qquad , \qquad \partial_t \operatorname{rot}\mu H = \operatorname{rot}\mu H \qquad .$$

Hence, $\operatorname{div}\varepsilon E$ and $\operatorname{rot}\mu H$ must be constant (in time) for solenoidal data εF and irrotational data μG. Therefore, in this case $\operatorname{div}\varepsilon E$ and $\operatorname{rot}\mu H$ vanish, if and only if the initial data $\operatorname{div}\varepsilon E_0$ and $\operatorname{rot}\mu H_0$ vanish. Since $\operatorname{div}\operatorname{rot} + \operatorname{rot}\operatorname{div} = \Delta$ we get for constant $\Lambda = \Lambda_0 = \begin{bmatrix}\varepsilon_0 & 0\\ 0 & \mu_0\end{bmatrix}$, $\varepsilon_0, \mu_0 \in \mathbb{R}_+$, as well as solenoidal F, E_0 and irrotational G, H_0 the (vector) wave equation controllability problem

$$\begin{aligned}
(\partial_t^2 - \varepsilon_0^{-1}\mu_0^{-1}\Delta)(E,H) &= (\tilde{F},\tilde{G}) && \text{in} \quad \Xi \quad ,\\
(\gamma_\tau E, \mu_0 \gamma_\tau\, \partial_t\, H) &= (\lambda, \tilde{\lambda}) && \text{in} \quad \Gamma \quad ,\\
(E,H)(T) &= (E,H)(0) && \text{in} \quad \Omega \quad ,\\
\partial_t (E,H)(T) &= \partial_t (E,H)(0) && \text{in} \quad \Omega \quad .
\end{aligned} \tag{5}$$

This means that for $q := 0$, $(F,G) := (0,0)$, $\lambda := g$, $\varepsilon_0 := \mu_0 := 1/c$ and $u := E$ the original problem (1) is recovered. Clearly in this case the assumption $\operatorname{div}\varepsilon E = 0$ is trivial and always fulfilled.

On the other hand for a solution (E,H) of (4) we may set $(\tilde{E},\hat{H}) := \partial_t(E,H)$ and $(\hat{E},\tilde{H}) := -\,\mathrm{i}\,\mathcal{M}(E,H) = \Lambda^{-1}M(E,H)$. Then

$$\partial_t(\tilde{E},\hat{H}) + \mathrm{i}\,\mathcal{M}(\hat{E},\tilde{H}) = (\tilde{F},\tilde{G}) \qquad , \qquad \partial_t(\hat{E},\tilde{H}) + \mathrm{i}\,\mathcal{M}(\tilde{E},\hat{H}) = (0,0)$$

or equivalently

$$(\partial_t + \mathrm{i}\,\mathcal{M})(\tilde{E},\tilde{H}) = (\tilde{F},0) \qquad , \qquad (\partial_t + \mathrm{i}\,\mathcal{M})(\hat{E},\hat{H}) = (0,\tilde{G}) \qquad .$$

Moreover, we obtain

$$\begin{aligned}
\gamma_\tau \tilde{E} &= \partial_t\,\lambda \qquad ,\\
\gamma_\tau \mu \tilde{H} &= \operatorname{Rot}\lambda \qquad , \qquad\qquad \gamma_\tau \mu \hat{H} = \tilde{\lambda}
\end{aligned}$$

and hence for $(\tilde{E},\tilde{H})$ a Maxwell controllability problem is recovered.

To start our analysis we first have to establish a solution theory for the boundary value Cauchy problem (CP)

$$\begin{aligned} (\partial_t + \mathrm{i}\,\mathcal{M})(E,H) &= (F,G) && \text{in} && \Xi && , \\ \gamma_\tau E &= \lambda && \text{in} && \Gamma && , \\ (E,H)(0) &= (E_0,H_0) && \text{in} && \Omega && \end{aligned} \tag{6}$$

with given right hand sides F , G and λ as well as initial data (E_0, H_0) belonging to our control (Hilbert) space

$$\mathbb{H} := \mathrm{L}^{2,q,q+1}_{\Lambda}(\Omega) \qquad .$$

3 Solution theory for the Cauchy problem

Let us try to solve (6) for some exterior domain $\Omega \subset \mathbb{R}^N$ with a Lipschitz boundary $\partial\,\Omega$, which will be fixed from now on. First we want to extend the boundary data from Γ to Ξ .

3.1 Traces and extensions

Recently Weck [47] showed, how to obtain traces of differential forms on Lipschitz boundaries. Let Ω_{b} be a bounded Lipschitz domain in $\mathbb{R}^N$. Then by [47, Theorem 3] there exists a linear and continuous tangential trace operator

$$\gamma_\tau : \mathrm{R}^q(\Omega_{\mathrm{b}}) \longrightarrow \mathcal{R}^q(\partial\,\Omega_{\mathrm{b}}) := \left\{\lambda \in \mathrm{H}^{-1/2,q}_{\rho}(\partial\,\Omega_{\mathrm{b}}) : \ \mathrm{Rot}\,\lambda \in \mathrm{H}^{-1/2,q+1}_{\rho}(\partial\,\Omega_{\mathrm{b}})\right\} \qquad .$$

Moreover, by [47, Theorem 4] γ_τ is surjective. Hence, there exists a corresponding linear and continuous tangential extension operator (a right inverse of γ_τ)

$$\check{\gamma}_\tau : \mathcal{R}^q(\partial\,\Omega_{\mathrm{b}}) \longrightarrow \mathrm{R}^q(\Omega_{\mathrm{b}}) \qquad .$$

Applying the usual Helmholtz decomposition

$$\mathrm{L}^{2,q}(\Omega_{\mathrm{b}}) = \mathrm{rot}\,\mathring{\mathrm{R}}^{q-1}(\Omega_{\mathrm{b}}) \oplus_\varepsilon {}_\varepsilon\mathcal{H}^q(\Omega_{\mathrm{b}}) \oplus_\varepsilon \varepsilon^{-1}\,\mathrm{div}\,\mathrm{D}^{q+1}(\Omega_{\mathrm{b}}) \qquad ,$$

where we introduce the Dirichlet forms

$${}_\varepsilon\mathcal{H}^q(\Omega_{\mathrm{b}}) := {}_0\mathring{\mathrm{R}}^q(\Omega_{\mathrm{b}}) \cap \varepsilon^{-1}{}_0\mathrm{D}^q(\Omega_{\mathrm{b}}) \qquad ,$$

and using $\gamma_\tau \mathring{\mathrm{R}}^q(\Omega_{\mathrm{b}}) = \{0\}$ we receive a linear and continuous tangential extension operator

$$\check{\gamma}_\tau : \mathcal{R}^q(\partial\,\Omega_{\mathrm{b}}) \longrightarrow \mathrm{R}^q(\Omega_{\mathrm{b}}) \cap \varepsilon^{-1}\,\mathrm{div}\,\mathrm{D}^{q+1}(\Omega_{\mathrm{b}}) \subset \mathrm{R}^q(\Omega_{\mathrm{b}}) \cap \varepsilon^{-1}{}_0\mathrm{D}^q(\Omega_{\mathrm{b}}) \qquad .$$

Now we turn to our exterior Lipschitz domain $\Omega \subset \mathbb{R}^N$. Using an usual cut-off-technique we obtain the following

Lemma 3.1 *There exists a linear and continuous tangential trace operator*

$$\gamma_\tau : \mathrm{R}^q(\Omega) \longrightarrow \mathcal{R}^q(\partial\,\Omega)$$

and a corresponding linear and continuous tangential extension operator (right inverse)

$$\check{\gamma}_\tau : \mathcal{R}^q(\partial\,\Omega) \longrightarrow \mathrm{R}^q(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q(\Omega) \qquad ,$$

which maps even to forms with compact supports and satisfies on $\mathcal{R}^q(\partial\,\Omega)$

$$\gamma_\tau \check{\gamma}_\tau = \mathrm{Id} \qquad .$$

The kernel of γ_τ equals $\overset{\circ}{\mathrm{R}}{}^q(\Omega)$ and γ_τ may also be defined even on $\mathrm{R}^q_{\mathrm{loc}}(\overline{\Omega})$. $\check{\gamma}_\tau$ may be chosen, such that $\operatorname{supp} \check{\gamma}_\tau\lambda \subset \overline{\Omega \cap U_r}$ holds for all $\lambda \in \mathcal{R}^q(\partial\,\Omega)$ and for a fixed $r > 0$ with $\mathbb{R}^N \setminus \Omega \subset U_r$. Here $U_r \subset \mathbb{R}^N$ denotes the open Euclidean ball with radius $r > 0$ centered at the origin.

Remark 3.2 *If the boundary is sufficiently smooth, i.e. C^{m+1}, then even*

$$\gamma_\tau E \in \mathrm{H}^{m-1/2,q}(\partial\,\Omega)$$

holds for all forms $E \in \mathrm{H}^{m,q}(\Omega)$ or $E \in \mathrm{H}^{m,q}_{\mathrm{loc}}(\overline{\Omega})$. Moreover, applied to smooth forms from $\mathrm{C}^{\infty,q}(\overline{\Omega})$ we have $\gamma_\tau = \iota^*$ and of course γ_τ commutates with the exterior derivative. Contrarily if $\lambda \in \mathrm{H}^{m-1/2,q}(\partial\,\Omega)$ we may choose an extension, such that $\check{\gamma}_\tau\lambda \in \mathrm{H}^{m,q}(\Omega)$ holds and $\check{\gamma}_\tau\lambda$ is supported in $\overline{\Omega \cap U_r}$. For details see [32, 33].

γ_τ and $\check{\gamma}_\tau$ may also be defined on time dependent forms. We get bounded linear operators

$$\gamma_\tau : \mathrm{S}\big(I, \mathrm{R}^q(\Omega)\big) \longrightarrow \mathrm{S}\big(I, \mathcal{R}^q(\partial\,\Omega)\big)$$

and

$$\check{\gamma}_\tau : \mathrm{S}\big(I, \mathcal{R}^q(\partial\,\Omega)\big) \longrightarrow \mathrm{S}\big(I, \mathrm{R}^q(\Omega) \cap \varepsilon^{-1}\mathrm{D}^q(\Omega)\big)$$

with similar properties as mentioned above, where the function space S could be, for instance,

$$\mathrm{C}^0 \quad , \quad \mathrm{C}^1 \quad , \quad \mathrm{L}^2 \quad , \quad \mathrm{H}^1 \quad .$$

Finally we also need the corresponding normal trace and extension operators

$$\gamma_\nu := (-1)^{qN} \circledast \gamma_\tau \, * \qquad , \qquad \check{\gamma}_\nu := (-1)^{q(N-q)} * \check{\gamma}_\tau \circledast$$

defined on $(q+1)$- resp. $(q-1)$-forms, where $\circledast$ denotes Hodge's star operator on the $(N-1)$-dimensional submanifold $\partial\,\Omega$ of $\overline{\Omega}$.

To get more information about traces of differential forms we refer, for example, to [32].

3.2 Solution theory

Now we return to the Cauchy Problem (6). Let $\lambda \in \mathrm{C}^1\big(\bar{I}, \mathcal{R}^q(\partial\,\Omega)\big)$. Then the ansatz

$$(\tilde{E}, \tilde{H}) := (E, H) - (\check{\gamma}_\tau\lambda, 0)$$

leads to a problem with homogeneous boundary condition

$$\begin{aligned} (\partial_t + \mathrm{i}\,\mathcal{M})(\tilde{E}, \tilde{H}) &= (\tilde{F}, \tilde{G}) && \text{in} \quad \Xi \quad , \\ \gamma_\tau \tilde{E} &= 0 && \text{in} \quad \Gamma \quad , \\ (\tilde{E}, \tilde{H})(0) &= (\tilde{E}_0, \tilde{H}_0) && \text{in} \quad \Omega \quad , \end{aligned} \tag{7}$$

where

$$(\tilde{F}, \tilde{G}) := (F, G) + (-\,\partial_t\,\check{\gamma}_\tau\lambda, \mu^{-1}\,\mathrm{rot}\,\check{\gamma}_\tau\lambda) \quad , \quad (\tilde{E}_0, \tilde{H}_0) := (E_0, H_0) - \big(\check{\gamma}_\tau\lambda(0), 0\big) \quad .$$

Since $\mathcal{M}$ from (2) is linear and selfadjoint the spectral theory suggests the solution $(\tilde{E}, \tilde{H})$ of (7) defined by

$$\begin{aligned} (\tilde{E}, \tilde{H})(t) &:= \exp(-\,\mathrm{i}\,t\mathcal{M})(\tilde{E}_0, \tilde{H}_0) + \int_0^t \exp\big(-\,\mathrm{i}(t-s)\mathcal{M}\big)(\tilde{F}, \tilde{G})(s)\,ds \\ &= \exp(-\,\mathrm{i}\,t\mathcal{M})\Big((\tilde{E}_0, \tilde{H}_0) + \int_0^t \exp(\mathrm{i}\,s\mathcal{M})(\tilde{F}, \tilde{G})(s)\,ds\Big) \quad , \quad t \in [0, \infty) \quad . \end{aligned}$$

Let us analyze this solution more thoroughly. For instance, for $(\tilde{E}_0, \tilde{H}_0) \in \mathbb{H}$ and $(\tilde{F}, \tilde{G}) \in \mathrm{L}^2(I, \mathbb{H})$ we obtain $(\tilde{E}, \tilde{H}) \in \mathrm{C}^0(\bar{I}, \mathbb{H})$ and thus a solution

$$(E, H) \in \mathrm{C}^0(\bar{I}, \mathbb{H}) \quad , \tag{8}$$

if $(E_0, H_0) \in \mathbb{H}$ and $(F, G) \in \mathrm{L}^2(I, \mathbb{H})$. Assuming stronger assumptions on the data, i.e. $(\tilde{E}_0, \tilde{H}_0) \in D(\mathcal{M})$ and $(\tilde{F}, \tilde{G}) \in \mathrm{C}^0(\bar{I}, \mathbb{H}) \cap \mathrm{L}^2\big(I, D(\mathcal{M})\big)$, we even get

$$(\tilde{E}, \tilde{H}) \in \mathrm{C}^1(\bar{I}, \mathbb{H}) \cap \mathrm{C}^0\big(\bar{I}, D(\mathcal{M})\big) \quad .$$

Hence, we achieve a solution

$$(E, H) \in \mathrm{C}^1(\bar{I}, \mathbb{H}) \cap \mathrm{C}^0\big(\bar{I}, \mathrm{R}^q(\Omega) \times \mathrm{D}^{q+1}(\Omega)\big) \quad ,$$

if, for instance,

$$\begin{aligned} (E_0, H_0) &\in \mathrm{R}^q(\Omega) \times \mathrm{D}^{q+1}(\Omega) && \text{with} \quad \gamma_\tau E_0 = \lambda(0) \quad , \\ (F, G) &\in \mathrm{C}^0(\bar{I}, \mathbb{H}) \cap \mathrm{L}^2\big(I, \mathrm{R}^q(\Omega) \times \mathrm{D}^{q+1}(\Omega)\big) && \text{with} \quad \gamma_\tau F(t) = \partial_t\,\lambda(t) \\ & && \text{and} \quad \mu^{-1}\,\mathrm{rot}\,\check{\gamma}_\tau\lambda(t) \in \mathrm{D}^{q+1}(\Omega) \end{aligned} \tag{9}$$

hold for all t . Then of course (E, H) is a solution of the CP (6) in the strong sense.

Summing up we obtain:

Theorem 3.3 *Let* $\lambda \in \mathrm{C}^1\big(\bar{I}, \mathcal{R}^q(\partial\,\Omega)\big)$ *as well as* (E_0, H_0) *and* (F, G) *satisfy* (9). *Then the Cauchy problem* (6) *is uniquely solved in* $\mathrm{C}^1(\bar{I}, \mathbb{H}) \cap \mathrm{C}^0\big(\bar{I}, \mathrm{R}^q(\Omega) \times \mathrm{D}^{q+1}(\Omega)\big)$ *by*

$$\begin{aligned}(E,H)(t) &= (\check{\gamma}_\tau\lambda, 0)(t) + \exp(-\,\mathrm{i}\, t\mathcal{M})\big(E_0 - \check{\gamma}_\tau\lambda(0), H_0\big)\\ &\quad + \int_0^t \exp\big(-\mathrm{i}(t-s)\mathcal{M}\big)(F - \partial_s\,\check{\gamma}_\tau\lambda, G + \mu^{-1}\operatorname{rot}\check{\gamma}_\tau\lambda)(s)\,ds \quad , \quad t \in \bar{I} \quad .\end{aligned}$$

We call (E,H) *a* 'strong solution of the Cauchy problem (6) with data $(F, G, \lambda, E_0, H_0)$'.

Proof: Existence is clear from the latter considerations. However, for the convenience of the reader we have by definition

$$\partial_t(\tilde{E}, \tilde{H})(t) = -\,\mathrm{i}\, M_\Lambda(\tilde{E}, \tilde{H})(t) + \underbrace{\exp(-\,\mathrm{i}\, t\mathcal{M}_\Lambda)\exp(\mathrm{i}\, t\mathcal{M}_\Lambda)}_{=\mathrm{Id}}(\tilde{F}, \tilde{G})(t)$$

and thus

$$\begin{aligned}\partial_t(E,H) &= \partial_t(\check{\gamma}_\tau\lambda, 0) + \partial_t(\tilde{E}, \tilde{H})\\ &= \partial_t(\check{\gamma}_\tau\lambda, 0) - \mathrm{i}\, M_\Lambda(\tilde{E}, \tilde{H}) + (\tilde{F}, \tilde{G})\\ &= -\,\mathrm{i}\, M_\Lambda(\tilde{E}, \tilde{H}) + (F, G) + \underbrace{(0, \mu^{-1}\operatorname{rot}\check{\gamma}_\tau\lambda)}_{=-\,\mathrm{i}\, M_\Lambda(\check{\gamma}_\tau\lambda, 0)}\\ &= -\,\mathrm{i}\, M_\Lambda(E,H) + (F,G) \quad .\end{aligned}$$

By linearity to prove uniqueness we look at (6) with homogeneous data $(0,0,0,0,0)$, i.e.

$$\begin{aligned}(\partial_t + \mathrm{i}\,\mathcal{M})(E,H) &= (0,0) && \text{in} && \Xi \quad ,\\ \gamma_\tau E &= 0 && \text{in} && \Gamma \quad ,\\ (E,H)(0) &= (0,0) && \text{in} && \Omega \quad .\end{aligned}$$

By the second equation we have $E(t) \in \overset{\circ}{\mathrm{R}}{}^q(\Omega)$ for all t, i.e. $(E,H)(t) \in D(\mathcal{M})$ holds for all t. We compute for all t using the first equation

$$\begin{aligned}\partial_t \left\|(E,H)(t)\right\|_{\mathbb{H}}^2 &= 2\Re\big\langle\, \partial_t(E,H)(t), (E,H)(t)\big\rangle_{\mathbb{H}}\\ &= -2\Re\,\mathrm{i}\,\big\langle\, \underbrace{\mathcal{M}(E,H)(t)}_{=\mathcal{M}(E,H)(t)}, (E,H)(t)\big\rangle_{\mathbb{H}} = 0\end{aligned}$$

since the scalar product is real because $\mathcal{M} : D(\mathcal{M}) \subset \mathbb{H} \to \mathbb{H}$ is selfadjoint. (Of course the interested reader may calculate this once again by hand using partial integration.) Hence, the norm $\left\|(E,H)(t)\right\|_{\mathbb{H}}$ is constant in time, i.e.

$$\left\|(E,H)(t)\right\|_{\mathbb{H}} = \left\|(E,H)(0)\right\|_{\mathbb{H}} = 0$$

for all t by the third equation. Thus, $(E,H) = (0,0)$. ■

Actually we are interested in the Hilbert space $\mathbb{H}$ as control space and even not in $D(\mathcal{M})$ or $\mathrm{R}^q(\Omega)\times\mathrm{D}^{q+1}(\Omega)$. Moreover, the constraints (9) are too complicated and the assumptions on the data much too strong. Thus, we have to weaken our solution concept. To approach weak solutions we first have to define test forms.

Definition 3.4 *For $(\Phi_0,\Psi_0)\in D(\mathcal{M})$ and $t\in\mathbb{R}$ the family*

$$(\Phi,\Psi)(t):=\exp(-\,\mathrm{i}\,t\mathcal{M})(\Phi_0,\Psi_0)$$

defines a strong solution of the homogeneous Cauchy problem

$$\begin{aligned}(\partial_t+\mathrm{i}\,\mathcal{M})(\Phi,\Psi)&=(0,0) && \text{in} && \mathbb{R}\times\Omega \quad ,\\ \gamma_\tau\Phi&=0 && \text{in} && \mathbb{R}\times\partial\,\Omega \quad ,\\ (\Phi,\Psi)(0)&=(\Phi_0,\Psi_0) && \text{in} && \Omega \quad .\end{aligned}$$

These solutions (Φ,Ψ) are elements of $\mathrm{C}^1(\mathbb{R},\mathbb{H})\cap\mathrm{C}^0\big(\mathbb{R},D(\mathcal{M})\big)$ and we will call them 'test forms with initial values (Φ_0,Ψ_0)'.

Next we present the idea of the definition of weak solutions. Thus, let (E,H) be a strong solution of (6) and (Φ,Ψ) be a test form with initial value $(\Phi_0,\Psi_0)\in D(\mathcal{M})$. Then we may compute

$$\begin{aligned}\big\langle(F,G),(\Phi,\Psi)\big\rangle_{\mathbb{H}}&=\big\langle(\partial_t+\mathrm{i}\,\mathcal{M})(E,H),(\Phi,\Psi)\big\rangle_{\mathbb{H}}\\ &=\big\langle\,\partial_t(E,H),(\Phi,\Psi)\big\rangle_{\mathbb{H}}-\big\langle M(E,H),(\Phi,\Psi)\big\rangle_{\mathrm{L}^{2,q,q+1}(\Omega)}\\ &=\partial_t\big\langle(E,H),(\Phi,\Psi)\big\rangle_{\mathbb{H}}-\big\langle(E,H),\partial_t(\Phi,\Psi)\big\rangle_{\mathbb{H}}\\ &\qquad-\langle\operatorname{rot}E,\Psi\rangle_{\mathrm{L}^{2,q+1}(\Omega)}-\langle\operatorname{div}H,\Phi\rangle_{\mathrm{L}^{2,q}(\Omega)}\quad .\end{aligned}$$

Since $\Phi\in\mathring{\mathrm{R}}^q(\Omega)$ we obtain

$$\langle\operatorname{div}H,\Phi\rangle_{\mathrm{L}^{2,q}(\Omega)}=-\langle H,\operatorname{rot}\Phi\rangle_{\mathrm{L}^{2,q+1}(\Omega)}$$

and assuming for these heuristic arguments that E , Ψ and $\partial\,\Omega$ are sufficiently smooth we get by Stokes' theorem

$$\begin{aligned}&\int_\Omega\operatorname{rot}(E\wedge*\bar\Psi)=\langle\operatorname{rot}E,\Psi\rangle_{\mathrm{L}^{2,q+1}(\Omega)}+\langle E,\operatorname{div}\Psi\rangle_{\mathrm{L}^{2,q}(\Omega)}\\ &=\int_{\partial\,\Omega}\iota^*(E\wedge*\bar\Psi)=(-1)^{qN}\int_{\partial\,\Omega}\iota^*E\wedge\circledast\circledast\iota^**\bar\Psi=\langle\gamma_\tau E,\gamma_\nu\Psi\rangle_{\mathrm{L}^{2,q}(\partial\,\Omega)}\quad .\end{aligned}$$

Putting all together yields

$$\begin{aligned}\big\langle(F,G),(\Phi,\Psi)\big\rangle_{\mathbb{H}}&=\partial_t\big\langle(E,H),(\Phi,\Psi)\big\rangle_{\mathbb{H}}-\big\langle(E,H),\underbrace{(\partial_t+\mathrm{i}\,\mathcal{M})(\Phi,\Psi)}_{=(0,0)}\big\rangle_{\mathbb{H}}\\ &\qquad-\langle\lambda,\gamma_\nu\Psi\rangle_{\mathrm{L}^{2,q}(\partial\,\Omega)}\quad .\end{aligned}$$

Hence, we only have to remove the time derivative from the forms (E,H) to get our weak solutions. (Please compare to Weck [46].)

Definition 3.5 *Let* $(E_0, H_0) \in \mathbb{H}$ *,* $(F, G) \in \mathrm{L}^2(I, \mathbb{H})$ *and* $\lambda \in \mathrm{L}^2\big(I, \mathcal{R}^q(\partial\,\Omega)\big)$ *. Then the pair of forms* (E, H) *is said to be a* 'weak solution of the Cauchy problem (6) with data $(F, G, \lambda, E_0, H_0)$'*, if and only if* $(E, H) \in \mathrm{C}^0(\bar{I}, \mathbb{H})$ *and*

$$\big\langle (E, H), (\Phi, \Psi) \big\rangle_{\mathbb{H}}(t) - \big\langle (E_0, H_0), (\Phi_0, \Psi_0) \big\rangle_{\mathbb{H}}$$
$$= \int_0^t \Big(\big\langle (F, G), (\Phi, \Psi) \big\rangle_{\mathbb{H}}(s) + \langle \lambda, \gamma_\nu \Psi \rangle_{\mathrm{L}^{2,q}(\partial\,\Omega)}(s) \Big)\, ds$$

holds for all $t \in \bar{I}$ *and all test forms* (Φ, Ψ) *with initial values* $(\Phi_0, \Psi_0) \in D(\mathcal{M})$ *.*

Remark 3.6

(i) *Since the integrand belongs to* $\mathrm{L}^1(I)$ *the scalar product* $\big\langle (E, H), (\Phi, \Psi) \big\rangle_{\mathbb{H}}$ *is weakly differentiable, i.e. an element of* $\mathrm{W}^{1,1}(I)$ *and then, of course, even of* $\mathrm{W}^{1,2}(I)$ *. Hence, we obtain an equivalent formulation in* $\mathrm{W}^{1,1}(I)$

$$\begin{aligned} \big\langle (E, H), (\Phi, \Psi) \big\rangle'_{\mathbb{H}}(t) &= \big\langle (F, G), (\Phi, \Psi) \big\rangle_{\mathbb{H}}(t) + \langle \lambda, \gamma_\nu \Psi \rangle_{\mathrm{L}^{2,q}(\partial\,\Omega)}(t) \quad , \\ \big\langle (E, H), (\Phi, \Psi) \big\rangle_{\mathbb{H}}(0) &= \big\langle (E_0, H_0), (\Phi_0, \Psi_0) \big\rangle_{\mathbb{H}} \end{aligned} \tag{10}$$

for all test forms (Φ, Ψ) *and almost all* t *. If the integrand is even continuous, which holds, for example, if* $(F, G) \in \mathrm{C}^0(\bar{I}, \mathbb{H})$ *and* $\lambda \in \mathrm{C}^0\big(\bar{I}, \mathcal{R}^q(\partial\,\Omega)\big)$ *, then the scalar product* $\big\langle (E, H), (\Phi, \Psi) \big\rangle_{\mathbb{H}}$ *is even an element of* $\mathrm{C}^1(I)$ *and* (10) *holds for all* t *.*

(ii) *The term* $\langle \lambda, \gamma_\nu \Psi \rangle_{\mathrm{L}^{2,q}(\partial\,\Omega)}$ *needs some detailed interpretation. The normal trace of a* $(q+1)$*-form from* $\mathrm{D}^{q+1}(\Omega)$ *is only an element of*

$$\mathcal{D}^q(\partial\,\Omega) := \big\{ \lambda \in \mathrm{H}_\pi^{-1/2,q}(\partial\,\Omega) : \operatorname{Div} \lambda \in \mathrm{H}_\pi^{-1/2,q-1}(\partial\,\Omega) \big\} \quad ,$$

where $\operatorname{Div} = (-1)^{(q-1)(N-1)} \circledast \operatorname{Rot} \circledast$ *denotes the co-derivative on* $\partial\,\Omega$ *applied to* q*-forms. Please see again [47] for details. Hence, at first sight the scalar product*

$$\langle \lambda, \gamma_\nu \Psi \rangle_{\mathrm{L}^{2,q}(\partial\,\Omega)}(s) = \big\langle \lambda(s), \gamma_\nu \Psi(s) \big\rangle_{\mathrm{L}^{2,q}(\partial\,\Omega)} \tag{11}$$

for almost all s *makes only sense as an usual dual pairing*

$$\gamma_\nu \Psi(s) \lambda(s) = \big\langle \lambda(s), \gamma_\nu \Psi(s) \big\rangle_{\mathrm{H}_\pi^{1/2,q}(\partial\,\Omega), \mathrm{H}_\pi^{-1/2,q}(\partial\,\Omega)} \quad .$$

Thus, $\lambda(s)$ *should be an element of* $\mathrm{H}_\pi^{1/2,q}(\partial\,\Omega)$ *for almost all* s *. But since for (almost) all* s *the boundary forms* $\lambda(s) \in \mathcal{R}^q(\partial\,\Omega)$ *and* $\gamma_\nu \Psi(s) \in \mathcal{D}^q(\partial\,\Omega)$ *have more regularity than just* $\mathrm{H}_{\rho/\pi}^{-1/2,q}(\partial\,\Omega)$ *, the scalar product* (11) *still makes sense for almost all* s *. The exact meaning of this will be clarified in the next lemma.*

Lemma 3.7 *The* $\mathrm{L}^{2,q}(\partial\,\Omega)$*-scalar product may be extended as a continuous bilinear form to* $\mathcal{R}^q(\partial\,\Omega) \times \mathcal{D}^q(\partial\,\Omega)$ *(using Stokes' theorem) by the following mapping:*

$$\begin{aligned} b \;:\; \mathcal{R}^q(\partial\,\Omega) \times \mathcal{D}^q(\partial\,\Omega) &\longrightarrow \mathbb{C} \\ (\alpha, \beta) &\longmapsto \langle \operatorname{rot} \check{\gamma}_\tau \alpha, \check{\gamma}_\nu \beta \rangle_{\mathrm{L}^{2,q+1}(\Omega)} + \langle \check{\gamma}_\tau \alpha, \operatorname{div} \check{\gamma}_\nu \beta \rangle_{\mathrm{L}^{2,q}(\Omega)} \end{aligned}$$

Moreover, for all $(E,H)\in\mathrm{R}^q(\Omega)\times\mathrm{D}^{q+1}(\Omega)$ *Stokes' theorem*

$$\langle\operatorname{rot}E,H\rangle_{\mathrm{L}^{2,q+1}(\Omega)}+\langle E,\operatorname{div}H\rangle_{\mathrm{L}^{2,q}(\Omega)}=b(\gamma_\tau E,\gamma_\nu H)$$

remains valid. Further on we will denote b *as usual by* $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{L}^{2,q}(\partial\Omega)}$ *.*

Proof: For $\alpha\in\mathcal{R}^q(\partial\Omega)$ and $\beta\in\mathcal{D}^q(\partial\Omega)$ the respective extensions $\check{\gamma}_\tau\alpha$ and $\check{\gamma}_\nu\beta$ to Ω are elements of $\mathrm{R}^q(\Omega)$ and $\mathrm{D}^{q+1}(\Omega)$. Therefore, the definition of b makes sense. To show that b is well defined, i.e. does not depend on the extensions, we pick some $(E,H)\in\mathrm{R}^q(\Omega)\times\mathrm{D}^{q+1}(\Omega)$ with $\gamma_\tau E=\alpha$ and $\gamma_\nu H=\beta$. Since $\gamma_\tau(E-\check{\gamma}_\tau\alpha)=0$ and $\gamma_\nu(H-\check{\gamma}_\nu\beta)=0$ we have $E-\check{\gamma}_\tau\alpha\in\overset{\circ}{\mathrm{R}}{}^q(\Omega)$ and $H-\check{\gamma}_\nu\beta\in\overset{\circ}{\mathrm{D}}{}^{q+1}(\Omega)$. Thus, by definition

$$\begin{aligned}0&=\big\langle\operatorname{rot}(E-\check{\gamma}_\tau\alpha),H\big\rangle_{\mathrm{L}^{2,q+1}(\Omega)}+\langle E-\check{\gamma}_\tau\alpha,\operatorname{div}H\rangle_{\mathrm{L}^{2,q}(\Omega)}\qquad,\\0&=\langle\operatorname{rot}\check{\gamma}_\tau\alpha,H-\check{\gamma}_\nu\beta\rangle_{\mathrm{L}^{2,q+1}(\Omega)}+\big\langle\check{\gamma}_\tau\alpha,\operatorname{div}(H-\check{\gamma}_\nu\beta)\big\rangle_{\mathrm{L}^{2,q}(\Omega)}\end{aligned}$$

hold and addition gives $\langle\operatorname{rot}E,H\rangle_{\mathrm{L}^{2,q+1}(\Omega)}+\langle E,\operatorname{div}H\rangle_{\mathrm{L}^{2,q}(\Omega)}=b(\alpha,\beta)$, which proves also the asserted formula. Finally the continuity of b follows from the Cauchy-Scharz inequality and the continuity of the extensions, i.e.

$$\big|b(\alpha,\beta)\big|\leq 2\|\check{\gamma}_\tau\alpha\|_{\mathrm{R}^q(\Omega)}\|\check{\gamma}_\nu\beta\|_{\mathrm{D}^{q+1}(\Omega)}\leq c\|\alpha\|_{\mathcal{R}^q(\partial\Omega)}\|\beta\|_{\mathcal{D}^q(\partial\Omega)}\qquad.$$

∎

Remark 3.8 *From the latter lemma it is clear that the two mappings*

$$\alpha\mapsto\langle\alpha,\beta\rangle_{\mathrm{L}^{2,q}(\partial\Omega)}\qquad,\qquad\beta\mapsto\overline{\langle\alpha,\beta\rangle_{\mathrm{L}^{2,q}(\partial\Omega)}}$$

are elements of the dual spaces $\mathcal{R}^q(\partial\Omega)'$ *and* $\mathcal{D}^q(\partial\Omega)'$ *, respectively, since for the norms*

$$\sup_{\alpha\in\mathcal{R}^q(\partial\Omega)\setminus\{0\}}\frac{\big|\langle\alpha,\beta\rangle_{\mathrm{L}^{2,q}(\partial\Omega)}\big|}{\|\alpha\|_{\mathcal{R}^q(\partial\Omega)}}\leq c\|\beta\|_{\mathcal{D}^q(\partial\Omega)}\quad,\quad\sup_{\beta\in\mathcal{D}^q(\partial\Omega)\setminus\{0\}}\frac{\big|\langle\alpha,\beta\rangle_{\mathrm{L}^{2,q}(\partial\Omega)}\big|}{\|\beta\|_{\mathcal{D}^q(\partial\Omega)}}\leq c\|\alpha\|_{\mathcal{R}^q(\partial\Omega)}$$

hold. Therefore, identifying β *and* α *with these two respective mappings we have*

$$\mathcal{D}^q(\partial\Omega)\subset\mathcal{R}^q(\partial\Omega)'\qquad,\qquad\mathcal{R}^q(\partial\Omega)\subset\mathcal{D}^q(\partial\Omega)'\qquad.$$

Of course, in the case of a smooth boundary these inclusions are improved by the well known formulas

$$\mathcal{D}^q(\partial\Omega)=\mathcal{R}^q(\partial\Omega)'\qquad,\qquad\mathcal{R}^q(\partial\Omega)=\mathcal{D}^q(\partial\Omega)'\qquad.$$

We are ready to prove the main result of this section.

Theorem 3.9 *There exists at most one weak solution of* (6). *If additionally*

$$\lambda\in\mathrm{H}^1\big(I,\mathcal{R}^q(\partial\Omega)\big)$$

then there exists always a unique weak solution of (6). *In this case (since* T *is arbitrary) there exists a unique weak solution in* $\mathrm{C}^0\big([0,\infty),\mathbb{H}\big)$ *.*

Proof: The difference (E,H) of two solutions satisfies $\big\langle (E,H),(\Phi,\Psi)\big\rangle_{\mathbb{H}}(t)=0$ for all t and all test forms (Φ,Ψ). Since $\exp(\mathrm{i}\,t\mathcal{M})$ is an unitary operator and $D(\mathcal{M})$ dense in $\mathbb{H}$ we obtain $\exp(\mathrm{i}\,t\mathcal{M})(E,H)(t)=(0,0)$ and thus $(E,H)(t)$ vanishes for all t, which proves uniqueness. To show existence we use the solution (E,H) from Theorem 3.3 suggested by spectral theory, which is still well defined and still belongs to $\mathrm{C}^0(\bar{I},\mathbb{H})$ by (8) even for our weak assumptions. Note that we have replaced the stronger condition $\lambda\in\mathrm{C}^1\big(\bar{I},\mathcal{R}^q(\partial\,\Omega)\big)$ by $\lambda\in\mathrm{H}^1\big(I,\mathcal{R}^q(\partial\,\Omega)\big)\subset\mathrm{C}^0\big(\bar{I},\mathcal{R}^q(\partial\,\Omega)\big)$. So it remains to check, if (E,H) satisfies the integral equation of Definition 3.5. Let

$$(\Phi,\Psi)(t)=\exp(-\,\mathrm{i}\,t\mathcal{M})(\Phi_0,\Psi_0)\qquad,\qquad t\in\mathbb{R}$$

be a test form with $(\Phi_0,\Psi_0)\in D(\mathcal{M})$. We start with the second term in the sum of the representation of (E,H):

$$\begin{aligned}&\Big\langle \exp(-\,\mathrm{i}\,t\mathcal{M})\big(E_0-\check{\gamma}_\tau\lambda(0),H_0\big),(\Phi,\Psi)(t)\Big\rangle_{\mathbb{H}}\\ &=\Big\langle\big(E_0-\check{\gamma}_\tau\lambda(0),H_0\big),(\Phi_0,\Psi_0)\Big\rangle_{\mathbb{H}}=\big\langle(E_0,H_0),(\Phi_0,\Psi_0)\big\rangle_{\mathbb{H}}-\big\langle\varepsilon\check{\gamma}_\tau\lambda(0),\Phi_0\big\rangle_{\mathrm{L}^{2,q}(\Omega)}\end{aligned}$$

The third term may be handled utilizing Fubini's theorem as follows:

$$\begin{aligned}&\Big\langle\int_0^t\exp\big(-\,\mathrm{i}(t-s)\mathcal{M}\big)(F-\partial_s\,\check{\gamma}_\tau\lambda,G+\mu^{-1}\operatorname{rot}\check{\gamma}_\tau\lambda)(s)\,ds,(\Phi,\Psi)(t)\Big\rangle_{\mathbb{H}}\\ &=\int_0^t\big\langle\exp(\mathrm{i}\,s\mathcal{M})(F-\partial_s\,\check{\gamma}_\tau\lambda,G+\mu^{-1}\operatorname{rot}\check{\gamma}_\tau\lambda)(s),(\Phi_0,\Psi_0)\big\rangle_{\mathbb{H}}\,ds\\ &=\int_0^t\big\langle(F-\partial_s\,\check{\gamma}_\tau\lambda,G+\mu^{-1}\operatorname{rot}\check{\gamma}_\tau\lambda),(\Phi,\Psi)\big\rangle_{\mathbb{H}}(s)\,ds\\ &=\int_0^t\big\langle(F,G),(\Phi,\Psi)\big\rangle_{\mathbb{H}}(s)\,ds+\int_0^t\big\langle(-\,\partial_s\,\check{\gamma}_\tau\lambda,\mu^{-1}\operatorname{rot}\check{\gamma}_\tau\lambda),(\Phi,\Psi)\big\rangle_{\mathbb{H}}(s)\,ds\end{aligned}$$

We proceed with calculating the last integral.

$$\begin{aligned}&-\int_0^t\langle\partial_s\,\check{\gamma}_\tau\lambda,\varepsilon\Phi\rangle_{\mathrm{L}^{2,q}(\Omega)}(s)\,ds\\ &=-\int_0^t\partial_s\langle\check{\gamma}_\tau\lambda,\varepsilon\Phi\rangle_{\mathrm{L}^{2,q}(\Omega)}(s)\,ds+\int_0^t\langle\check{\gamma}_\tau\lambda,\varepsilon\,\partial_s\,\Phi\rangle_{\mathrm{L}^{2,q}(\Omega)}(s)\,ds\\ &=-\langle\check{\gamma}_\tau\lambda,\varepsilon\Phi\rangle_{\mathrm{L}^{2,q}(\Omega)}(t)+\big\langle\check{\gamma}_\tau\lambda(0),\varepsilon\Phi_0\big\rangle_{\mathrm{L}^{2,q}(\Omega)}+\int_0^t\langle\check{\gamma}_\tau\lambda,\operatorname{div}\Psi\rangle_{\mathrm{L}^{2,q}(\Omega)}(s)\,ds\end{aligned}$$

Hence, we get

$$\begin{aligned}&\int_0^t\big\langle(-\,\partial_s\,\check{\gamma}_\tau\lambda,\mu^{-1}\operatorname{rot}\check{\gamma}_\tau\lambda),(\Phi,\Psi)\big\rangle_{\mathbb{H}}(s)\,ds\\ &=-\langle\check{\gamma}_\tau\lambda,\varepsilon\Phi\rangle_{\mathrm{L}^{2,q}(\Omega)}(t)+\big\langle\check{\gamma}_\tau\lambda(0),\varepsilon\Phi_0\big\rangle_{\mathrm{L}^{2,q}(\Omega)}\\ &\quad+\int_0^t\big(\underbrace{\langle\check{\gamma}_\tau\lambda,\operatorname{div}\Psi\rangle_{\mathrm{L}^{2,q}(\Omega)}(s)+\langle\operatorname{rot}\check{\gamma}_\tau\lambda,\Psi\rangle_{\mathrm{L}^{2,q+1}(\Omega)}(s)}_{=\,\langle\lambda,\gamma_\nu\Psi\rangle_{\mathrm{L}^{2,q}(\partial\,\Omega)}(s)}\big)\,ds\end{aligned}$$

by Lemma 3.7. Putting all together completes the proof. ■

Remark 3.10 *We have to be sure that Fubini's theorem can be applied in the first step of the calculation of the third term. Introducing the pointwise scalar product on q-forms* $\langle\varphi,\psi\rangle_q := \eta\wedge *\bar{\psi}$ *and on pairs* $\big\langle(\varphi,\alpha),(\psi,\beta)\big\rangle_{q,p} := \langle\varphi,\psi\rangle_q + \langle\alpha,\beta\rangle_p$*, respectively, we have so show by Tonelli's theorem that the absolute value of the measurable function*

$$(s,x)\mapsto \Big\langle \exp\big(-\mathrm{i}(t-s)\mathcal{M}\big)(\tilde{F},\tilde{G})(s),\Lambda(\Phi,\Psi)(t)\Big\rangle_{q,q+1}(x)$$

is an element of $\mathrm{L}^1\big((0,t)\times\Omega\big)$*. Hence, we compute (denoting the Lebesgue measure on* $\mathbb{R}^N$ *by* μ*)*

$$\begin{aligned}&\int_0^t\int_\Omega\Big|\Big\langle \exp\big(-\mathrm{i}(t-s)\mathcal{M}\big)(\tilde{F},\tilde{G})(s),\Lambda(\Phi,\Psi)(t)\Big\rangle_{q,q+1}(x)\Big|\,d\mu(x)\,ds\\ \leq\ & c\int_0^t\Big\|\exp\big(-\mathrm{i}(t-s)\mathcal{M}\big)(\tilde{F},\tilde{G})(s)\Big\|_{\mathbb{H}}\big\|(\Phi,\Psi)(t)\big\|_{\mathbb{H}}\,ds\\ \leq\ & c\int_0^t\big\|(\tilde{F},\tilde{G})(s)\big\|_{\mathbb{H}}\big\|(\Phi_0,\Psi_0)\big\|_{\mathbb{H}}\,ds\leq c\sqrt{T}\big\|(\Phi_0,\Psi_0)\big\|_{\mathbb{H}}\big\|(\tilde{F},\tilde{G})\big\|_{\mathrm{L}^2(I,\mathbb{H})}\quad.\end{aligned}$$

A shorter justification is the following: Since the scalar product of $\mathbb{H}$ *is (clearly) continuous and the integral over* I *is a limes of elements in* $\mathbb{H}$ *(Bochner's integral) we can, of course, interchange the integration over* I *and the scalar product.*

3.3 A new notation

Let us change to a new and shorter notation, which enables us to follow the forthcoming arguments and basic ideas more easily.

We set $\mathbf{0} := (0,0)$ as well as

$$\begin{aligned}\mathbf{u} &:= (E,H) \quad&,\quad \mathbf{f} &:= (F,G) \quad&,\\ \mathbf{u}_0 &:= \mathbf{u}(0) = (E_0,H_0) \quad&,\quad \mathbf{e}_\lambda &:= \mathbf{e}(\lambda) := (\check{\gamma}_\tau\lambda, 0) \quad&,\\ \mathbf{u}_T &:= \mathbf{u}(T) \quad&,\quad \mathbf{g}_\lambda &:= \mathbf{g}(\lambda) := (-\partial_s\,\check{\gamma}_\tau\lambda, \mu^{-1}\operatorname{rot}\check{\gamma}_\tau\lambda) \quad&.\end{aligned}$$

With this notation our Cauchy problem (6) reads as

$$\begin{aligned}(\partial_t + \mathrm{i}\,\mathcal{M})\mathbf{u} &= \mathbf{f} &\text{in}\quad &\Xi \quad,\\ \gamma_\tau\pi\mathbf{u} &= \lambda &\text{in}\quad &\Gamma \quad,\\ \mathbf{u}(0) &= \mathbf{u}_0 &\text{in}\quad &\Omega \quad,\end{aligned}\tag{12}$$

where for a pair of forms π denotes the projection onto the first component. Moreover, $\mathbf{u}$ may be decomposed into $\mathbf{u} = \mathbf{u}^l + \mathbf{u}^c$, where $\mathbf{u}^l$ and $\mathbf{u}^c$ are the unique weak solutions of the Cauchy problems

$$\begin{aligned}(\partial_t + \mathrm{i}\,\mathcal{M})\mathbf{u}^l &= \mathbf{0} \quad,\quad & (\partial_t + \mathrm{i}\,\mathcal{M})\mathbf{u}^c &= \mathbf{f} &\text{in}\quad &\Xi \quad,\\ \gamma_\tau\pi\mathbf{u}^l &= 0 \quad,\quad & \gamma_\tau\pi\mathbf{u}^c &= \lambda &\text{in}\quad &\Gamma \quad,\\ \mathbf{u}^l(0) &= \mathbf{u}_0 \quad,\quad & \mathbf{u}^c(0) &= \mathbf{0} &\text{in}\quad &\Omega \quad.\end{aligned}\tag{13}$$

$\mathbf{u}^l$ depends linearly and continuously on the initial data $\mathbf{u}_0$ and $\mathbf{u}^c$ is independent of the initial data $\mathbf{u}_0$, i.e. constant with respect to $\mathbf{u}_0$. The unique weak solutions of (12) and (13) exist by Theorem 3.9 in $\mathrm{C}^0\big(\bar{I},\mathbb{H}\big)$ for all T and all

$$\mathbf{u}_0 \in \mathbb{H} \quad , \quad \mathbf{f} \in \mathrm{L}^2(I,\mathbb{H}) \quad , \quad \lambda \in \mathrm{H}^1\big(I,\mathcal{R}^q(\partial\,\Omega)\big) \tag{14}$$

and are given by the formulas

$$\begin{aligned} \mathbf{u}(t) &= \mathbf{e}_\lambda(t) + \mathrm{e}^{-\,\mathrm{i}\,t\mathcal{M}}\big(\mathbf{u}_0 - \mathbf{e}_\lambda(0)\big) + \int_0^t \mathrm{e}^{-\,\mathrm{i}(t-s)\mathcal{M}}(\mathbf{f}+\mathbf{g}_\lambda)(s)\,ds \quad , \\ \mathbf{u}^l(t) &= \mathrm{e}^{-\,\mathrm{i}\,t\mathcal{M}}\,\mathbf{u}_0 \quad , \\ \mathbf{u}^c(t) &= \mathbf{e}_\lambda(t) - \mathrm{e}^{-\,\mathrm{i}\,t\mathcal{M}}\,\mathbf{e}_\lambda(0) + \int_0^t \mathrm{e}^{-\,\mathrm{i}(t-s)\mathcal{M}}(\mathbf{f}+\mathbf{g}_\lambda)(s)\,ds \quad . \end{aligned} \tag{15}$$

4 Least-squares formulation of the controllability problem

From now on let the right hand side data $\mathbf{f}$ and λ satisfying (14) as well as the time $T > 0$ be given and fixed.

In order to solve the controllability problem (3), which reads now as

$$\text{‘find } \mathbf{u}_0 \in \mathbb{H}\text{, such that } \mathbf{u} \text{ satisfies (12) and } \mathbf{u}_T = \mathbf{u}_0\text{‘,} \tag{16}$$

we investigate the equation

$$\mathbf{u}_T - \mathbf{u}_0 = 0 \tag{17}$$

more thoroughly. With the help of (15) we obtain

$$\begin{aligned} \mathbf{u}_T = \mathbf{u}(T) = \mathbf{u}^l(T) + \mathbf{u}^c(T) &= \mathrm{e}^{-\,\mathrm{i}\,T\mathcal{M}}\,\mathbf{u}_0 + \mathbf{u}_T^c \quad , \\ \mathbf{u}_T - \mathbf{u}_0 &= (\mathrm{e}^{-\,\mathrm{i}\,T\mathcal{M}} - 1)\mathbf{u}_0 + \mathbf{u}_T^c \quad . \end{aligned}$$

Consequently, with the continuous linear operator in $\mathbb{H}$

$$\mathcal{C}_t := \mathcal{C}(t) := \mathrm{e}^{-\,\mathrm{i}\,t\mathcal{M}} - 1 \quad ,$$

which satisfies $\|\mathcal{C}_t\| \leq 2$ for all t and we will call ‘control operator’, we get

$$\mathbf{u}_T - \mathbf{u}_0 = \mathcal{C}_T\mathbf{u}_0 + \mathbf{u}_T^c \quad . \tag{18}$$

Hence, we have to solve the linear equation

$$\mathcal{C}_T\mathbf{u}_0 + \mathbf{u}_T^c = 0$$

in the Hilbert space $\mathbb{H}$. Since, of course, $\mathcal{C}_T$ is neither symmetric nor selfadjoint we cannot apply the usual conjugate gradient algorithm for its solution. Thus, we may consider the corresponding normal equation

$$\mathcal{C}_T^*\mathcal{C}_T\mathbf{u}_0 + \mathcal{C}_T^*\mathbf{u}_T^c = 0 \quad , \tag{19}$$

where $\mathcal{C}_t^* = e^{i t \mathcal{M}} - 1$ denotes the adjoint operator of $\mathcal{C}_t$. We note $\mathcal{C}_t^{**} = \mathcal{C}_t$. Now clearly $\mathcal{C}_T^* \mathcal{C}_T$ is selfadjoint and the usual ideas of conjugate gradient methods may by applied to our problem. Consequently, we are forced to consider and to minimize the quadratic functional

$$\begin{aligned} \mathbf{u}_0 \mapsto \frac{1}{2} \langle \mathcal{C}_T^* \mathcal{C}_T \mathbf{u}_0, \mathbf{u}_0 \rangle_{\mathbb{H}} + \Re \langle \mathcal{C}_T^* \mathbf{u}_T^c, \mathbf{u}_0 \rangle_{\mathbb{H}} &= \frac{1}{2} \langle \mathcal{C}_T \mathbf{u}_0, \mathcal{C}_T \mathbf{u}_0 \rangle_{\mathbb{H}} + \Re \langle \mathbf{u}_T^c, \mathcal{C}_T \mathbf{u}_0 \rangle_{\mathbb{H}} \\ &= \frac{1}{2} \| \mathcal{C}_T \mathbf{u}_0 + \mathbf{u}_T^c \|_{\mathbb{H}}^2 - \frac{1}{2} \| \mathbf{u}_T^c \|_{\mathbb{H}}^2 \qquad , \end{aligned}$$

which, of course, is minimized, if and only if the quadratic functional

$$\begin{array}{rcccl} \mathcal{F} & : & \mathbb{H} & \longrightarrow & [0, \infty) \\ & & \mathbf{u}_0 & \longmapsto & \frac{1}{2} \| \mathcal{C}_T \mathbf{u}_0 + \mathbf{u}_T^c \|_{\mathbb{H}}^2 = \frac{1}{2} \| \mathbf{u}_T - \mathbf{u}_0 \|_{\mathbb{H}}^2 \end{array} \tag{20}$$

is minimized.

4.1 Least-squares formulation and calculation of the derivative of the least-squares functional

We will investigate the following least-squares formulation:
Find initial data $\mathbf{u}_0 \in \mathbb{H}$, such that

$$\forall \quad \mathbf{v}_0 \in \mathbb{H} \qquad : \qquad \mathcal{F}(\mathbf{u}_0) \leq \mathcal{F}(\mathbf{v}_0) \qquad . \tag{21}$$

Here $\mathbf{u}$ resp. $\mathbf{v}$ is the unique weak solution of the Cauchy problem (12) with initial data $\mathbf{u}_0$ resp. $\mathbf{v}_0$. A solution $\mathbf{u}_0$ of our controllability problem (16) would clearly satisfy $\mathcal{F}(\mathbf{u}_0) = 0$ and thus solve our least-squares problem. As a minimum $\mathbf{u}_0$ would then clearly satisfy $\mathcal{F}'(\mathbf{u}_0) = 0$, provided that $\mathcal{F}$ is differentiable. Clearly this holds true for every local extremum as long as $\mathcal{F}$ is differentiable. Moreover, in order to solve the least-squares problem we will use as indicated a conjugate gradient algorithm operating in the Hilbert space $\mathbb{H}$. The implementation of such an algorithm is greatly facilitated by the knowledge of the derivative $\mathcal{F}'$. Thus, having now two good reasons we may compute the derivative of $\mathcal{F}$. But this is quite easy since $\mathcal{F}$ is a quadratic functional. To do so, we pick some $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{H}$ and derive

$$\mathcal{F}(\mathbf{u}_0 + \mathbf{v}_0) = \mathcal{F}(\mathbf{u}_0) + \Re \langle \mathcal{C}_T \mathbf{v}_0, \mathcal{C}_T \mathbf{u}_0 + \mathbf{u}_T^c \rangle_{\mathbb{H}} + \frac{1}{2} \| \mathcal{C}_T \mathbf{v}_0 \|_{\mathbb{H}}^2 \qquad . \tag{22}$$

Thus, $\mathcal{F}$ is differentiable in $\mathbf{u}_0$ with derivative

$$\mathcal{F}'(\mathbf{u}_0)\mathbf{v}_0 = \Re \big\langle \mathbf{v}_0, \mathcal{C}_T^* (\mathcal{C}_T \mathbf{u}_0 + \mathbf{u}_T^c) \big\rangle_{\mathbb{H}} = \Re \langle \mathbf{v}_0, \mathcal{C}_T^* \mathcal{C}_T \mathbf{u}_0 + \mathcal{C}_T^* \mathbf{u}_T^c \rangle_{\mathbb{H}} \tag{23}$$

and, of course, the normal equation is recovered. In this sense we may identify

$$\mathcal{F}'(\mathbf{u}_0) \qquad \text{with} \qquad \mathcal{C}_T^* \mathcal{C}_T \mathbf{u}_0 + \mathcal{C}_T^* \mathbf{u}_T^c \qquad .$$

Furthermore, we receive the representations

$$\mathcal{D}_t := \mathcal{D}(t) := \mathcal{C}_t^* \mathcal{C}_t = (\mathrm{e}^{\mathrm{i}\, t\mathcal{M}} - 1)(\mathrm{e}^{-\mathrm{i}\, t\mathcal{M}} - 1) = 2\big(1 - \cos(t\mathcal{M})\big) \quad , \tag{24}$$

$$\begin{aligned} \hat{\mathbf{u}}_t := \hat{\mathbf{u}}(t) &:= \mathcal{C}_t^* \mathbf{u}^c(t) = (\mathrm{e}^{\mathrm{i}\, t\mathcal{M}} - 1)\mathbf{u}^c(t) \\ &= (\mathrm{e}^{\mathrm{i}\, t\mathcal{M}} - 1)\mathbf{e}_\lambda(t) + (\mathrm{e}^{-\mathrm{i}\, t\mathcal{M}} - 1)\mathbf{e}_\lambda(0) + \int_0^t (1 - \mathrm{e}^{-\mathrm{i}\, t\mathcal{M}})\, \mathrm{e}^{\mathrm{i}\, s\mathcal{M}} (\mathbf{f} + \mathbf{g}_\lambda)(s)\, ds \quad , \end{aligned}$$

where we will call the continuous linear operator $\mathcal{D}_t$ in $\mathbb{H}$ the 'derivative operator'. We have $\|\mathcal{D}_t\| \leq 4$ for all t. Finally we obtain

$$\mathcal{F}'(\mathbf{u}_0)\mathbf{v}_0 = \Re\langle \mathbf{v}_0, \mathcal{D}_T \mathbf{u}_0 + \hat{\mathbf{u}}_T \rangle_{\mathbb{H}} \quad . \tag{25}$$

By (22) we get also

$$\mathcal{F}(\mathbf{u}_0) \leq \mathcal{F}(\mathbf{u}_0 + \mathbf{v}_0) - \mathcal{F}'(\mathbf{u}_0)\mathbf{v}_0$$

for all $\mathbf{v}_0 \in \mathbb{H}$ and thus

Lemma 4.1 *For $\mathbf{u}_0 \in \mathbb{H}$ the following assertions are equivalent:*

- **(i)** $\mathbf{u}_0$ *is a solution of the least squares problem* (21).
- **(ii)** $\mathcal{F}'(\mathbf{u}_0) = 0$
- **(iii)** $\mathcal{D}_T \mathbf{u}_0 + \hat{\mathbf{u}}_T = 0$

We note that **(iii)** *is the normal equation* (19).

Using (18) let us interpret the derivative vector

$$\mathcal{D}_T \mathbf{u}_0 + \hat{\mathbf{u}}_T = \mathcal{C}_T^* \mathbf{u}_0^* = (\mathrm{e}^{\mathrm{i}\, T\mathcal{M}} - 1)\mathbf{u}_0^* \in \mathbb{H} \quad , \quad \mathbf{u}_0^* := \mathbf{u}_T - \mathbf{u}_0 \in \mathbb{H}$$

little more thoroughly. Clearly the forms $\mathbf{u}^{*,+}$ and $\mathbf{u}^{*,-}$ defined by

$$\mathbf{u}^{*,+}(t) := \mathrm{e}^{\mathrm{i}\, t\mathcal{M}} \mathbf{u}_0^* \quad , \quad \mathbf{u}^{*,-}(t) := \mathrm{e}^{\mathrm{i}(T-t)\mathcal{M}} \mathbf{u}_0^*$$

are the unique weak solutions of the (homogeneous) adjoint Cauchy problems

$$\begin{aligned} (\partial_t - \mathrm{i}\, \mathcal{M})\mathbf{u}^{*,+} &= \mathbf{0} \quad , & (\partial_t + \mathrm{i}\, \mathcal{M})\mathbf{u}^{*,-} &= \mathbf{0} & \text{in} \quad \Xi \quad , \\ \gamma_\tau \pi \mathbf{u}^{*,+} &= 0 \quad , & \gamma_\tau \pi \mathbf{u}^{*,-} &= 0 & \text{in} \quad \Gamma \quad , \\ \mathbf{u}^{*,+}(0) &= \mathbf{u}_0^* \quad , & \mathbf{u}^{*,-}(T) &= \mathbf{u}_T^* := \mathbf{u}_0^* & \text{in} \quad \Omega \end{aligned} \tag{26}$$

and we have $\mathbf{u}^{*,+}(T) = \mathbf{u}^{*,-}(0) = \mathrm{e}^{\mathrm{i}\, T\mathcal{M}} \mathbf{u}_0^*$, i.e.

$$\begin{aligned} \mathcal{C}_T^* \mathbf{u}_0^* &= \mathbf{u}_T^{*,+} - \mathbf{u}_0^{*,+} = \mathbf{u}_0^{*,-} - \mathbf{u}_T^{*,-} \\ &= \mathbf{u}_T^{*,+} - \mathbf{u}_0^* = \mathbf{u}_0^{*,-} - \mathbf{u}_0^* = \mathbf{u}_0^{*,-} - \mathbf{u}_T^* \quad . \end{aligned}$$

Here the signs $\pm$ indicate that the wave $\mathbf{u}^{*,+}$ evolves <u>forward</u> in time, whereas the wave $\mathbf{u}^{*,-}$ evolves <u>backward</u> in time. Of course, this implies a change in the ∂_t-term. We note that we define the weak solutions of the adjoint Cauchy problems analogously to Definition 3.5. We do not want to write down these definitions in detail here. Finally we obtain two more nice representations of our derivative vector utilizing the solutions of the adjoint Cauchy problems (26)

$$\mathcal{D}_T \mathbf{u}_0 + \hat{\mathbf{u}}_T = \mathbf{u}_T^{*,+} - \mathbf{u}_0^{*,+} = \mathbf{u}_0^{*,-} - \mathbf{u}_T^{*,-} \quad . \tag{27}$$

4.2 Classical calculation of the derivative

Just for the sake of completeness let us shortly calculate the derivative via the 'classical' procedure. The idea is to compute the derivative and then to determine the vector in Riesz' representation theorem. This procedure is usually called perturbation analysis.

From (22) and (18) we may assume, that we already know the representation of the derivative, which is by the way only heuristically clear without using the spectral theory,

$$\mathcal{F}'(\mathbf{u}_0)\mathbf{v}_0 = \Re\langle \mathbf{v}_T - \mathbf{v}_0, \mathbf{u}_T - \mathbf{u}_0\rangle_{\mathbb{H}} = \Re\langle \mathbf{v}_T - \mathbf{v}_0, \mathbf{u}_T^*\rangle_{\mathbb{H}} \qquad ,$$

where $\mathbf{u}_T^* = \mathbf{u}_T - \mathbf{u}_0$ as well as $\mathbf{u}$ and $\mathbf{v}$ are the unique weak solutions of the Cauchy problems

$$\begin{aligned}
(\partial_t + \mathrm{i}\,\mathcal{M})\mathbf{u} &= \mathbf{f} \quad , & (\partial_t + \mathrm{i}\,\mathcal{M})\mathbf{v} &= \mathbf{0} \quad \text{in} \quad \Xi \quad ,\\
\gamma_\tau \pi \mathbf{u} &= \lambda \quad , & \gamma_\tau \pi \mathbf{v} &= 0 \quad \text{in} \quad \Gamma \quad ,\\
\mathbf{u}(0) &= \mathbf{u}_0 \quad , & \mathbf{v}(0) &= \mathbf{v}_0 \quad \text{in} \quad \Omega \quad .
\end{aligned}$$

All we have to do is to find a vector $\tilde{\mathbf{u}} \in \mathbb{H}$ with

$$\Re\langle \mathbf{v}_T - \mathbf{v}_0, \mathbf{u}_T^*\rangle_{\mathbb{H}} = \Re\langle \mathbf{v}_0, \tilde{\mathbf{u}}\rangle_{\mathbb{H}} \qquad .$$

Now the classical method proceeds as follows: Putting $\mathbf{u}_T^{*,-} := \mathbf{u}_T^*$ we compute for the solution $\mathbf{u}^{*,-}$ of some adjoint problem, which has to be determined,

$$\Re\langle \mathbf{v}_T - \mathbf{v}_0, \mathbf{u}_T^{*,-}\rangle_{\mathbb{H}} = \Re\langle \mathbf{v}_0, \mathbf{u}_0^{*,-} - \mathbf{u}_T^{*,-}\rangle_{\mathbb{H}} + \Re\langle \mathbf{v}, \mathbf{u}^{*,-}\rangle_{\mathbb{H}}(T) - \Re\langle \mathbf{v}, \mathbf{u}^{*,-}\rangle_{\mathbb{H}}(0)$$

and show $\Re\langle \mathbf{v}, \mathbf{u}^{*,-}\rangle_{\mathbb{H}}(T) - \Re\langle \mathbf{v}, \mathbf{u}^{*,-}\rangle_{\mathbb{H}}(0) = 0$. But if we choose $\mathbf{u}^{*,-}$ as the unique weak solution of the backward in time homogeneous adjoint Cauchy problem (26) with initial condition $\mathbf{u}_T^{*,-} = \mathbf{u}_T^*$, i.e.

$$\mathbf{u}^{*,-}(t) := \mathrm{e}^{\mathrm{i}(T-t)\mathcal{M}}\,\mathbf{u}_T^* \qquad ,$$

we get trivially $\langle \mathbf{v}, \mathbf{u}^{*,-}\rangle_{\mathbb{H}}(T) = \langle \mathbf{v}, \mathbf{u}^{*,-}\rangle_{\mathbb{H}}(0)$ since $\mathrm{e}^{-\mathrm{i}\,t\mathcal{M}}$ is unitary. Actually we have for all $t \in \bar{I}$

$$\langle \mathbf{v}, \mathbf{u}^{*,-}\rangle_{\mathbb{H}}(t) = \langle \mathrm{e}^{-\mathrm{i}\,t\mathcal{M}}\,\mathbf{v}_0, \mathrm{e}^{\mathrm{i}(T-t)\mathcal{M}}\,\mathbf{u}_T^*\rangle_{\mathbb{H}} = \langle \mathbf{v}_0, \underbrace{\mathrm{e}^{\mathrm{i}\,T\mathcal{M}}\,\mathbf{u}_T^*}_{=\mathbf{u}_0^{*,-}}\rangle_{\mathbb{H}} = \langle \mathbf{v}, \mathbf{u}^{*,-}\rangle_{\mathbb{H}}(0) \qquad .$$

Consequently,

$$\Re\langle \mathbf{v}_T - \mathbf{v}_0, \mathbf{u}_T^{*,-}\rangle_{\mathbb{H}} = \Re\langle \mathbf{v}_0, \mathbf{u}_0^{*,-} - \mathbf{u}_T^{*,-}\rangle_{\mathbb{H}}$$

and we get like in (25), (27)

$$\mathcal{F}'(\mathbf{u}_0)\mathbf{v}_0 = \Re\langle \mathbf{v}_0, \mathbf{u}_0^{*,-} - \mathbf{u}_T^*\rangle_{\mathbb{H}} \qquad .$$

Thus, we obtain the representation of the derivative using the solution of the adjoint Cauchy problem (26) evolving backward in time. We note that in the classical computations no weak solutions were considered. Thus, one was forced to calculate

$$\begin{aligned}\langle \mathbf{v}, \mathbf{u}^{*,-} \rangle_{\mathbb{H}}(T) - \langle \mathbf{v}, \mathbf{u}^{*,-} \rangle_{\mathbb{H}}(0) &= \int_0^T \partial_t \langle \mathbf{v}, \mathbf{u}^{*,-} \rangle_{\mathbb{H}}(t)\, dt \\ &= \int_0^T \big\langle \mathbf{v}, \underbrace{(\partial_t + \mathrm{i}\,\mathcal{M})\mathbf{u}^{*,-}}_{=\mathbf{0}} \big\rangle_{\mathbb{H}}(t)\, dt = 0\end{aligned}$$

using $\partial_t \mathbf{v} = -\mathrm{i}\,\mathcal{M}\mathbf{v}$ as well as $\gamma_\tau \pi \mathbf{v} = \gamma_\tau \pi \mathbf{u}^{*,-} = 0$ and partial integration (or the selfadjointness of $\mathcal{M}$) to see that this term vanishes, if $\mathbf{u}^{*,-}$ is chosen as the unique backward in time solution of the adjoint Cauchy problem (26).

We finally note, if one wants to use the control space $\mathbb{H} = \mathrm{L}^{2,q,q+1}_{\Lambda}(\Omega)$, like we do, in general the forms possess neither strong time- nor spatial-derivatives. Thus, this method is not applicable in our weak framework.

4.3 Further discussion of the derivative

As already pointed out the derivative vector

$$\begin{aligned}\mathcal{D}_T \mathbf{u}_0 + \hat{\mathbf{u}}_T = \mathcal{C}_T^* \mathbf{u}_0^* &= \mathrm{e}^{\mathrm{i}\,T\mathcal{M}} \mathbf{u}_0^* - \mathbf{u}_0^* \quad , \\ \mathbf{u}_0^* &= \mathcal{C}_T \mathbf{u}_0 + \mathbf{u}_T^c = \mathrm{e}^{-\mathrm{i}\,T\mathcal{M}} \mathbf{u}_0 - \mathbf{u}_0 + \mathbf{u}_T^c \\ &= \mathbf{u}_T - \mathbf{u}_0 = \mathbf{u}_T^l - \mathbf{u}_0 + \mathbf{u}_T^c\end{aligned}$$

depends on the initial condition $\mathbf{u}_0$ both directly and indirectly through the solution $\mathbf{u}$ of the (Maxwell) wave equation (12) and one of the solutions $\mathbf{u}^{*,\pm}$ of the adjoint (Maxwell) wave equations (26). Moreover, we saw in (15) that $\mathbf{u} = \mathbf{u}^l + \mathbf{u}^c$ splits up in a linear (and continuous) and a constant part (with respect to $\mathbf{u}_0$). Of course, the same holds true for the solutions of the adjoint equations. Let us pick, for instance, the forward in time solution $\mathbf{u}^* := \mathbf{u}^{*,+}$. Then $\mathbf{u}^*$ depends linearly (and continuously) on the initial data $\mathbf{u}_0^*$ and may be decomposed into $\mathbf{u}^* = \mathbf{u}^{*,l} + \mathbf{u}^{*,c}$, where $\mathbf{u}^{*,l}$ and $\mathbf{u}^{*,c}$ are the unique weak solutions of the Cauchy problems

$$\begin{aligned}(\partial_t - \mathrm{i}\,\mathcal{M})\mathbf{u}^{*,l} &= \mathbf{0} \quad , & (\partial_t - \mathrm{i}\,\mathcal{M})\mathbf{u}^{*,c} &= \mathbf{0} && \text{in } \Xi \ , \\ \gamma_\tau \pi \mathbf{u}^{*,l} &= 0 \quad , & \gamma_\tau \pi \mathbf{u}^{*,c} &= 0 && \text{in } \Gamma \ , \\ \mathbf{u}^{*,l}(0) = \mathbf{u}_0^{*,l} := \mathbf{u}_T^l - \mathbf{u}_0 &= \mathcal{C}_T \mathbf{u}_0 \quad , & \mathbf{u}^{*,c}(0) = \mathbf{u}_0^{*,c} &:= \mathbf{u}_T^c && \text{in } \Omega\end{aligned}$$

with $\mathbf{u}_0^* = \mathbf{u}_0^{*,l} + \mathbf{u}_0^{*,c}$. Again $\mathbf{u}^{*,l}$ depends linearly (and continuously) on $\mathbf{u}_0$, whereas $\mathbf{u}^{*,c}$ does not depend on $\mathbf{u}_0$. Of course, we have

$$\mathbf{u}^{*,l}(t) = \mathrm{e}^{\mathrm{i}\,t\mathcal{M}} \mathbf{u}_0^{*,l} \quad , \quad \mathbf{u}^{*,c}(t) = \mathrm{e}^{\mathrm{i}\,t\mathcal{M}} \mathbf{u}_0^{*,c} \quad .$$

Putting all together we see

$$\mathcal{D}_T \mathbf{u}_0 = \mathbf{u}_T^{*,l} - \mathbf{u}_T^l + \mathbf{u}_0 \quad , \quad \hat{\mathbf{u}}_T = \mathbf{u}_T^{*,c} - \mathbf{u}_T^c \quad .$$

Finally by the representation (24) of $\mathcal{D}_t$ we observe that $\tilde{\mathbf{u}}(t) := \mathcal{D}_t \mathbf{u}_0$ solves in the weak sense

$$(\partial_t^2 + \mathcal{M}^2)\tilde{\mathbf{u}} = 2\mathcal{M}^2\mathbf{u}_0 \quad , \quad \tilde{\mathbf{u}}(0) = \mathbf{0} \quad , \quad \partial_t\,\tilde{\mathbf{u}}(0) = \mathbf{0}$$

resp.

$$(\partial_t^2 + \mathcal{M}^2)\tilde{\mathbf{u}} = 2\mathcal{M}^2\mathbf{u}_0 \quad , \quad \tilde{\mathbf{u}}(0) = \mathbf{0} \quad , \quad \partial_t\,\tilde{\mathbf{u}}(0) = \mathbf{0} \quad , \quad \gamma_\tau \pi \tilde{\mathbf{u}} = 0 \quad .$$

5 Conjugate gradient method for the least-squares formulation

Although it has become quite customary to use the conjugate gradient algorithm in Hilbert spaces we want to repeat the main ideas and the algorithm here. For a solid foundation see [13] and for applications one may have a look at [16, 17, 18, 19, 20] and of course at [21].

5.1 Conjugate gradient method in Hilbert space

Let $\mathbf{H}$ be a Hilbert space with scalar product $\langle \cdot \,, \cdot \rangle$ and norm $\| \cdot \|$. Moreover, let A be a continuous linear operator in $\mathbf{H}$ and let A^* denote its adjoint operator.

We want to solve the problem

$$Au + f = 0$$

for given $f \in \mathbf{H}$, i.e. determine some $u \in \mathbf{H}$.

Since in general A might not be symmetric or selfadjoint, we turn to the normal equation

$$A^*Au + A^*f = 0 \quad .$$

Now, of course, A^*A is selfadjoint and we try for the solution of the problem the usual conjugate gradient ansatz. Thus, we want to minimize the quadratic functional

$$u \mapsto \frac{1}{2}\langle A^*Au, u\rangle + \Re\langle A^*f, u\rangle = \frac{1}{2}\|Au\|^2 + \Re\langle Au, f\rangle = \frac{1}{2}\|Au + f\|^2 - \frac{1}{2}\|f\|^2 \quad .$$

Consequently, we will try to minimize the quadratic functional

$$F := F_f \; : \; \begin{array}{ccc} \mathbf{H} & \longrightarrow & [0,\infty) \\ u & \longmapsto & \frac{1}{2}\|Au + f\|^2 \end{array} \quad . \tag{28}$$

From the formula

$$F(u+h) = F(u) + \Re\langle Ah, Au + f\rangle + \frac{1}{2}\|Ah\|^2 \geq F(u) + \Re\langle Ah, Au + f\rangle$$

for all $h \in \mathbf{H}$ we see two things. First, F is differentiable with derivative

$$F'(u)h = \Re\langle Ah, Au + f\rangle = \Re\langle h, A^*Au + A^*f\rangle$$

and, second, for some $u \in \mathbf{H}$

$$F(u) = \min_{v\in\mathbf{H}} F(v) \qquad \Leftrightarrow \qquad F'(u) = 0 \qquad \Leftrightarrow \qquad A^*Au + A^*f = 0 \qquad .$$

Hence, our minimization problem and our normal equation are equivalent.

We note the following useful formulas, which we be frequently used during the next two subsections:

$$F(u+h) = F(u) + F'(u)h + \frac{1}{2}\|Ah\|^2 \geq F(u) + F'(u)h$$
$$F'(u+v)h = F'(u)h + \Re\langle Ah, Av\rangle = F'(u)h + \Re\langle h, A^*Av\rangle$$

Moreover, again $F'(u) = F'_f(u)$ and $F'_0(u)$ may be identified with $A^*Au + A^*f$ and A^*Au respectively.

5.1.1 Method of steepest descent

Now let us assume that an approximation u_{n-1} of u is given and let us define the residual

$$r_{n-1} := A^*Au_{n-1} + A^*f \qquad ,$$

where we can identify $F'(u_{n-1})$ with r_{n-1}. Then we try to find a better approximation u_n with hopefully $F(u_n) < F(u_{n-1})$ by a one dimensional line search

$$u_n = u_{n-1} + \alpha_n d_n \qquad ,$$

where $\alpha_n \in \mathbb{R}$ and $d_n \in \mathbf{H}$ have to be determined. The vector d_n should be the direction of steepest descent. Since we search for the minimum

$$F(u_n) = \min_{\alpha_n\in\mathbb{R}} F(u_{n-1} + \alpha_n d_n)$$

we set the derivative of the real function

$$\alpha_n \mapsto F(u_{n-1} + \alpha_n d_n) = F(u_{n-1}) + \alpha_n F'(u_{n-1})d_n + \frac{1}{2}\alpha_n^2\|Ad_n\|^2$$

to zero and get

$$0 = F'(u_{n-1} + \alpha_n d_n)d_n = F'(u_{n-1})d_n + \alpha_n\|Ad_n\|^2 \quad \Leftrightarrow \quad \alpha_n = -\frac{F'(u_{n-1})d_n}{\|Ad_n\|^2} \qquad .$$

Then we derive

$$F(u_n) - F(u_{n-1}) = \alpha_n F'(u_{n-1})d_n + \frac{\alpha_n^2}{2}\|Ad_n\|^2 = -\frac{1}{2}\frac{\left(F'(u_{n-1})d_n\right)^2}{\|Ad_n\|^2} < 0 \qquad , \tag{29}$$

if and only if $F'(u_{n-1})d_n = \Re\langle d_n, A^*Au_{n-1} + A^*f\rangle = \Re\langle d_n, r_{n-1}\rangle \neq 0$. This shows that the best choice for d_n is the direction of steepest descent $d_n = r_{n-1}$, the residual. Thus, we put

$$d_n := r_{n-1} \qquad , \qquad \alpha_n := -\frac{\|r_{n-1}\|^2}{\|Ar_{n-1}\|^2}$$

and consequently

$$u_n := u_{n-1} - \frac{\|r_{n-1}\|^2}{\|Ar_{n-1}\|^2} r_{n-1} \qquad .$$

By the equation

$$r_n = A^*Au_n + A^*f = r_{n-1} + \alpha_n A^*Ar_{n-1}$$

we get a simplification in the determination of r_n without using f. Note that A^*Ar_{n-1} may be identified with $F_0'(r_{n-1})$.

We obtain our first method, the method of steepest descent:

Algorithm 1 Steepest Descent Algorithm (SDA) in $\mathbf{H}$

---

initialization
set $n = 0$
set $u_n \in \mathbf{H}$
set $r_n = A^*Au_n + A^*f$
if r_n small **then**
  goto exit
end if
loop {for $n \geq 1$ assuming u_{n-1} and $r_{n-1} \neq 0$ are known}
  set $d = Ar_{n-1}$
  set $\alpha = -\|r_{n-1}\|^2/\|d\|^2$
  set $u_n = u_{n-1} + \alpha r_{n-1}$
  set $r_n = r_{n-1} + \alpha A^*d$
  if r_n small **then**
    goto exit
  end if
  set $n = n + 1$
end loop
exit
take u_n as solution

5.1.2 Method of conjugate gradients

Suppose now that approximate solutions $u_0, \ldots, u_{n-1}$ with corresponding search directions $d_1, \ldots, d_{n-1}$ are constructed. One may ask, if it is possible to construct this sequence in a way, such that

$$u_n = u_{n-1} + \alpha_n d_n$$

even minimizes F over the space

$$U_n := \{u_0\} + \mathrm{Lin}\{d_1, \dots, d_n\}$$

and not only over $\{u_{n-1}\} + \mathrm{Lin}\{d_n\}$? To give a positive answer to this question we look at

$$u_n = u_0 + \sum_{m=1}^{n} \alpha_m d_m = u_\ell + \sum_{m=\ell+1}^{n} \alpha_m d_m \qquad , \qquad \ell = 0, \dots, n-1$$

and calculate

$$F(u_n) = F(u_0) + \sum_{m=1}^{n} \alpha_m F'(u_0) d_m + \frac{1}{2} \Big\| \sum_{m=1}^{n} \alpha_m A d_m \Big\|^2 \qquad .$$

Now we want to use Pythagoras' theorem and hence wish on our search directions the orthogonality property

$$\Re\langle Ad_m, Ad_n\rangle = \delta_{n,m} \|Ad_n\|^2 \qquad ,$$

which we will call the 'A-orthogonal property'. Assuming and utilizing this property we obtain

$$F(u_n) = F(u_0) + \sum_{m=1}^{n} \alpha_m F'(u_0) d_m + \frac{1}{2} \sum_{m=1}^{n} \alpha_m^2 \|Ad_m\|^2 \qquad .$$

Consequently, our optimization problem over U_n decouples completely into n one dimensional line search minimization problems. Of course, again by setting the derivative of the real function

$$\alpha_m \mapsto \alpha_m F'(u_0) d_m + \frac{\alpha_m^2}{2} \|Ad_m\|^2 = F(u_0 + \alpha_m d_m) - F(u_0)$$

to zero we get

$$0 = F'(u_0) d_m + \alpha_m \|Ad_m\|^2 \qquad \Leftrightarrow \qquad \alpha_m = -\frac{F'(u_0) d_m}{\|Ad_m\|^2} \qquad .$$

Furthermore, we note for all n, m

$$\begin{aligned} F'(u_n) d_m &= F'(u_0) d_m + \sum_{\ell=1}^{n} \alpha_\ell \Re\langle Ad_\ell, Ad_m\rangle = F'(u_0) d_m + \sum_{\ell=1}^{n} \alpha_\ell \delta_{\ell,m} \|Ad_\ell\|^2 \\ &= \begin{cases} F'(u_0) d_m & \text{, if } n < m \\ F'(u_0) d_m + \alpha_m \|Ad_m\|^2 = 0 & \text{, if } n \geq m \end{cases} \end{aligned} \tag{30}$$

and hence for all $n = 0, \dots, m-1$

$$\alpha_m = -\frac{F'(u_n) d_m}{\|Ad_m\|^2} \qquad .$$

At this point for all n we may remind of the equation

$$F'(u_n)h = \Re\langle h, r_n\rangle \qquad .$$

Now we try to find A-orthogonal directions of steepest descent d_n. By (29) we see that again something like $d_n = r_{n-1}$ would be the best choice. Hence, to ensure also the A-orthogonality we try an ansatz

$$d_n = r_{n-1} + \sum_{\ell=1}^{n-1} \beta_\ell d_\ell \qquad .$$

Then by (30) we have

$$F'(u_{n-1})d_n = F'(u_{n-1})r_{n-1} = \|r_{n-1}\|^2 \qquad , \tag{31}$$

which is the best one can achieve, and for $m = 1, \ldots, n-1$

$$\begin{aligned} 0 \stackrel{!}{=} \Re\langle Ad_n, Ad_m\rangle &= \Re\langle Ar_{n-1}, Ad_m\rangle + \sum_{\ell=1}^{n-1} \beta_\ell \Re\langle Ad_\ell, Ad_m\rangle \\ &= \Re\langle Ar_{n-1}, Ad_m\rangle + \beta_m \|Ad_m\|^2 \qquad , \end{aligned}$$

if we assume that the directions $d_1, \ldots, d_{n-1}$ are already A-orthogonal. Thus, our choice must be

$$\beta_\ell := -\frac{\Re\langle Ar_{n-1}, Ad_\ell\rangle}{\|Ad_\ell\|^2} = -\frac{F'(u_{n-1})A^*Ad_\ell}{\|Ad_\ell\|^2} \qquad , \qquad \ell = 1, \ldots, n-1 \qquad .$$

Let us note two more properties: For $m = 0, \ldots, n-1$ we have by (30)

$$\Re\langle r_n, r_m\rangle = F'(u_n)r_m = F'(u_n)d_{m+1} - \sum_{\ell=1}^{m} \beta_\ell F'(u_n)d_\ell = 0 \qquad ,$$

i.e.

$$\Re\langle r_n, r_m\rangle = \delta_{n,m}\|r_n\|^2 \qquad ,$$

and thus

$$\begin{aligned} F'(u_n)A^*Ad_m &= \Re\langle r_n, A^*Ad_m\rangle \qquad , \\ \Re\langle Ar_n, Ad_m\rangle &= \Re\langle r_n, A^*Ad_m\rangle = \frac{1}{\alpha_m}\big(\Re\langle r_n, r_m\rangle - \Re\langle r_n, r_{m-1}\rangle\big) = 0 \end{aligned}$$

with the help of

$$r_n = r_{n-1} + \alpha_n A^*Ad_n \qquad . \tag{32}$$

Of course, we also have

$$F'(u_n)r_n = \|r_n^2\| \qquad , \qquad F'(u_n)A^*Ad_n = \Re\langle Ar_n, Ad_n\rangle = \frac{\|r_n^2\|}{\alpha_n} \qquad .$$

But this shows $\beta_\ell = 0$ for $\ell = 0, \ldots, n-2$ and hence our current search direction is simply

$$d_n = r_{n-1} + \beta_{n-1}d_{n-1} \qquad .$$

Summing up we get the following procedure: For some given approximation u_{n-1} and last search direction d_{n-1} we choose

$$u_n := u_{n-1} + \alpha_n d_n \qquad , \qquad d_n := r_{n-1} + \beta_{n-1}d_{n-1}$$

with

$$\alpha_n := -\frac{F'(u_{n-1})d_n}{\|Ad_n\|^2} \qquad , \qquad \beta_{n-1} := -\frac{F'(u_{n-1})A^*Ad_{n-1}}{\|Ad_{n-1}\|^2} \qquad .$$

Utilizing (31) we finally have

$$\alpha_n = -\frac{\|r_{n-1}\|^2}{\|Ad_n\|^2} \qquad , \qquad \beta_{n-1} = -\frac{\|r_{n-1}\|^2}{\alpha_{n-1}\|Ad_{n-1}\|^2} = \frac{\|r_{n-1}\|^2}{\|r_{n-2}\|^2}$$

and we note that the update of the residual r_n is given by (32).

We obtain the following conjugate gradient method:

Algorithm 2 Conjugate Gradient Algorithm (CGA) in $\mathbf{H}$

initialization
set $n = 0$
set $u_n \in \mathbf{H}$
set $r_n = A^*Au_n + A^*f$
set $\rho_n = \|r_n\|^2$
if ρ_n small **then**
  goto exit
end if
set $d_{n+1} = r_n$
loop {for $n \geq 1$ assuming u_{n-1} and $r_{n-1} \neq 0$, ρ_{n-1} as well as $d_n \neq 0$ are known}
  set $d = Ad_n$
  set $\alpha = -\rho_{n-1}/\|d\|^2$
  set $u_n = u_{n-1} + \alpha d_n$
  set $r_n = r_{n-1} + \alpha A^*d$
  set $\rho_n = \|r_n\|^2$
  if ρ_n small **then**
    goto exit
  end if
  set $\rho = 1/\rho_{n-1}$
  set $\rho = \rho_n\rho$
  set $d_{n+1} = r_n + \rho d_n$
  set $n = n + 1$
end loop
exit
take u_n as solution

Using $\|Ad_n\|^2 = \langle A^*Ad_n, d_n\rangle$ we note that a variant in the loop would be replacing the sequence

set $d = Ad_n$		**set** $d = A^*Ad_n$	
set $\alpha = -\rho_{n-1}/\|d\|^2$	by	**set** $\alpha = -\rho_{n-1}/\langle d, d_n\rangle$	.
set $u_n = u_{n-1} + \alpha d_n$		**set** $u_n = u_{n-1} + \alpha d_n$	
set $r_n = r_{n-1} + \alpha A^*d$		**set** $r_n = r_{n-1} + \alpha d$	

Of course, the similar modification is possible in Algorithm 1.

Let us remark that the procedures $r_0 = A^*Au_0 + A^*f$ and $d = A^*Ad_n$ may be regarded as the computations of the derivatives $F'(u_0) = F'_f(u_0)$ and $F'_0(d_n)$ respectively.

5.2 Conjugate gradient algorithm for the problem at hand

To solve our least squares problem (LSP) (21) or by Lemma 4.1 equivalently our normal equation (19)

$$\mathcal{D}_T \mathbf{u}_0 + \hat{\mathbf{u}}_T = \mathcal{C}_T^* \mathcal{C}_T \mathbf{u}_0 + \mathcal{C}_T^* \mathbf{u}_T^c = 0$$

approximately we propose to apply the latter CGA with the indicated variant. Hence, we set

$$F = F_f := \mathcal{F} \quad , \quad \mathbf{H} := \mathbb{H} \quad , \quad A := \mathcal{C}_T \quad , \quad f := \mathbf{u}_T^c \quad , \quad u := \mathbf{u}_0 \qquad .$$

We note that the procedure

$$r_0 = A^* A u_0 + A^* f$$

in the CGA, which reads as

$$\mathbf{r}^0 = \mathcal{C}_T^* \mathcal{C}_T \mathbf{u}_0^0 + \mathcal{C}_T^* \mathbf{u}_T^c \qquad , \tag{33}$$

i.e.

$$\mathbf{r}^0 = \mathcal{C}_T^* \mathbf{u}_0^* = \mathbf{u}_T^* - \mathbf{u}_0^* \qquad , \qquad \mathbf{u}_0^* = \mathcal{C}_T \mathbf{u}_0^0 + \mathbf{u}_T^c = \mathbf{u}_T - \mathbf{u}_0^0 \qquad ,$$

where we picked the forward in time solution $\mathbf{u}^* := \mathbf{u}^{*,+}$, needs the solution $\mathbf{u}$ at time T of the inhomogeneous Cauchy problem (ICP) (12) with initial data $\mathbf{u}_0^0$ as well as the forward in time solution $\mathbf{u}^*$ at time T of the homogeneous adjoint Cauchy problem (HACP+) (26) with initial data $\mathbf{u}_0^*$. Analogously the procedure

$$d = A^* A d_n$$

in the CGA, which reads as

$$\mathbf{d} = \mathcal{C}_T^* \mathcal{C}_T \mathbf{d}^n \qquad , \tag{34}$$

i.e.

$$\mathbf{d} = \mathcal{C}_T^* \mathbf{u}_0^* = \mathbf{u}_T^* - \mathbf{u}_0^* \qquad , \qquad \mathbf{u}_0^* = \mathcal{C}_T \mathbf{d}^n = \mathbf{u}_T - \mathbf{d}^n \qquad ,$$

where we once again used the forward in time solution $\mathbf{u}^* := \mathbf{u}^{*,+}$, needs the solution $\mathbf{u} := \mathbf{u}^l$ at time T of the homogeneous Cauchy problem (HCP) (13) with initial data $\mathbf{d}^n$ as well as the forward in time solution $\mathbf{u}^*$ at time T of the homogeneous adjoint Cauchy problem (HACP+) (26) with initial data $\mathbf{u}_0^*$.

We recall that the procedure (33) resp. (34) may be identified with the calculation of the derivative or 'gradient' $\mathcal{F}'(\mathbf{u}_0^0) = \mathcal{F}'_{\mathbf{u}_T^c}(\mathbf{u}_0^0)$ resp. $\mathcal{F}'_0(\mathbf{d}^n)$ of the least squares functional

$$\mathcal{F} = \mathcal{F}_{\mathbf{u}_T^c} \; : \; \begin{array}{ccc} \mathbb{H} & \longrightarrow & [0,\infty) \\ \mathbf{u}_0 & \longmapsto & \frac{1}{2}\|\mathcal{C}_T \mathbf{u}_0 + \mathbf{u}_T^c\|_{\mathbb{H}}^2 \end{array} \quad \text{resp.} \quad \mathcal{F}_0 \; : \; \begin{array}{ccc} \mathbb{H} & \longrightarrow & [0,\infty) \\ \mathbf{u}_0 & \longmapsto & \frac{1}{2}\|\mathcal{C}_T \mathbf{u}_0\|_{\mathbb{H}}^2 \end{array} \quad .$$

We will present the CGA for the approximate solution of the LSP as our Algorithm 3. In the beginning of the algorithm, before entering the iteration loop, we choose an initial control vector $\mathbf{u}_0^0 \in \mathbb{H}$ and compute the first residual vector $\mathbf{r}^0$, i.e.

the 'gradient' of the functional $\mathcal{F}_{\mathbf{u}_T^c}$ at the point $\mathbf{u}_0^0$, which gives the first minimizing direction $\mathbf{d}^1 = \mathbf{r}^0$. The computation of this residual requires the solutions of the inhomogeneous Cauchy problem ICP (12) with initial control vector $\mathbf{u}_0^0$ and of the homogeneous adjoint Cauchy problem HACP+ (26). Then on each CG iteration we calculate the solutions of the homogeneous Cauchy problem HCP (13) with initial vector $\mathbf{d}^n$ and of the homogeneous adjoint Cauchy problem HACP+ (26). This gives the 'gradient' of the functional $\mathcal{F}_0$ at the point $\mathbf{d}^n$, which is needed to update the new residual vector $\mathbf{r}^n$ and the new control vector $\mathbf{u}_0^n$. Finally we set the new minimizing direction $\mathbf{d}^{n+1}$.

Algorithm 3 CGA in $\mathbb{H}$ with variant for LSP (21)

```
initialization
set n = 0
set initial control vector u_0^n ∈ H
solve ICP (12) with initial vector u_0^n and get u
solve HACP+ (26) with initial vector u_0^* = u_T − u_0^n and get u^*
compute residual vector (gradient F'_{u_T^c}(u_0^0)) r^n = u_T^* − u_0^*
compute norm ρ^n = ‖r^n‖_H^2
if ρ^n small then
   goto exit
end if
set first minimizing direction d^{n+1} = r^n
loop {for n ≥ 1 assuming u_0^{n−1} and r^{n−1} ≠ 0, ρ^{n−1} as well as d^n ≠ 0 are known}
   solve HCP (13) with initial vector d^n and get u
   solve HACP+ (26) with initial vector u_0^* = u_T − d^n and get u^*
   compute gradient (F'_0(d^n)) d = u_T^* − u_0^*
   compute parameter α = −ρ^{n−1}/⟨d, d^n⟩_H
   update control vector u_0^n = u_0^{n−1} + αd^n
   update residual vector r^n = r^{n−1} + αd
   compute norm ρ^n = ‖r^n‖_H^2
   if ρ^n small or n = N then
      goto exit
   end if
   compute parameter ρ = 1/ρ^{n−1}
   compute parameter ρ = ρ^n ρ
   update minimizing direction d^{n+1} = r^n + ρd^n
   set n = n + 1
end loop
exit
take u_0^n as solution
```

We note that we may use the backward in time system HACP- (26) instead of HACP+ as well. Then in this variant by (27) we have to replace the computation of the residual or gradient vector $\mathbf{u}_T^* - \mathbf{u}_0^* = \mathbf{u}_T^{*,+} - \mathbf{u}_0^*$ by $\mathbf{u}_0^{*,-} - \mathbf{u}_T^*$ with $\mathbf{u}_T^* = \mathbf{u}_0^*$.

6 Translation to classical vector analysis

We shortly turn out, which classical problems of vector analysis are covered by our Cauchy problem (6). Hence, in this heuristic section we assume sufficient smoothness of the boundary $\partial\,\Omega$ and of the differential forms. We may write down (6) slightly more detailed:

$$\begin{aligned} \partial_t\, E - \varepsilon^{-1} \operatorname{div} H = F \qquad & \text{in} \qquad \Xi \\ \partial_t\, H - \mu^{-1} \operatorname{rot} E = G \qquad & \text{in} \qquad \Xi \\ \gamma_\tau E = \lambda \qquad & \text{in} \qquad \Gamma \end{aligned} \tag{35}$$

The condition $(E, H)(0) = (E_0, H_0)$ in Ω stays always the same, so we do not write it down every time. Since the exterior derivative and the tangential trace operator commutate the second equation always contains a boundary condition for H as well, if we assume slightly more regularity on the data, i.e. $G \in \mu^{-1}\mathrm{R}^{q+1}(\Omega)$. Applying γ_τ we obtain in Γ

$$\gamma_\tau\, \partial_t\, \mu H = \gamma_\tau \mu G + \operatorname{Rot} \lambda$$

or after integration

$$\gamma_\tau \mu H(t) = \int_0^t (\gamma_\tau \mu G + \operatorname{Rot} \lambda)(s)\, ds + \gamma_\tau \mu H_0 \qquad .$$

Furthermore, we may also discuss Neumann problems. Again with little more regularity on the data we get in Γ by applying γ_τ

$$\gamma_\tau \varepsilon^{-1} \operatorname{div} H = \partial_t\, \lambda - \gamma_\tau F \qquad .$$

In the classical framework, where we use Euclidean coordinates $\{x_1, \ldots, x_N\}$, we identify tangential vectors $\partial_n = \partial_n^{\mathrm{Id}}$ with unit vectors $\partial_n^{\mathrm{Id}}(\mathrm{Id}) = e^n \in \mathbb{R}^N$. Moreover, we then identify 0-forms E with functions E and N-forms $E\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_N$ via the Hodge star operator with 0-forms E and hence also with functions. Furthermore, we identify 1-forms

$$E = \sum_{n=1}^N E_n\, \mathrm{d}x_n$$

by Riesz' representation theorem with vector fields $(E_1, \ldots, E_N)^t$ in $\mathbb{R}^N$ and also $(N-1)$-forms

$$E = (-1)^{N-1} \sum_{n=1}^N E_n * \mathrm{d}x_n$$

(again via the Hodge star operator) with 1-forms and thus also with vector fields $(E_1, \ldots, E_N)^t$ in $\mathbb{R}^N$.

Let $N = 3$. On the surface $\partial\Omega$ we identify 0-forms and 2-forms with functions as well as 1-forms with tangential vector fields. Since we have generally $** = (-1)^{q(N-q)}$ and $\circledast\circledast = (-1)^{q(N-1-q)}$ we obtain $** = \mathrm{Id}$ and $\circledast\circledast = (-1)^q$. Moreover, $*$ acts like Id for functions and vector fields, whereas we have

	$q=0$	$q=1$	$q=2$
$\circledast$	Id	$\nu\times$	Id

,

where ν denotes the exterior unit normal on $\partial\Omega$. The wedge product $\wedge$ in $\mathbb{R}^3$ is just the scalar or vector multiplication, i.e.

$E \wedge H$	$q=0$	$q=1$	$q=2$	$q=3$
$q=0$	EH	EH	EH	EH
$q=1$	EH	$E \times H$	$E \cdot H$	0
$q=2$	EH	$E \cdot H$	0	0
$q=3$	EH	0	0	0

.

On the boundary $\partial\Omega$ we have for $\wedge$

$E \wedge H$	$q=0$	$q=1$	$q=2$
$q=0$	EH	EH	EH
$q=1$	EH	$\nu \cdot (E \times H)$	0
$q=2$	EH	0	0

.

The exterior derivative $\mathrm{rot} = \mathrm{d}$ and co-derivative $\mathrm{div} = \delta = (-1)^{q+1} * \mathrm{d} *$ in $\mathbb{R}^3$ turn to the classical differential operators from vector analysis

$$\mathrm{grad} = \nabla \quad , \quad \mathrm{curl} = \nabla \times \quad , \quad \mathrm{div} = \nabla \cdot$$

and, therefore, our Sobolev spaces for forms may be identified with the well known Sobolev spaces. Moreover, the tangential $\gamma_\tau = \iota^*$ and normal trace $\gamma_\nu = (-1)^{q+1} \circledast \iota^* *$ need to be translated. We get the following identification table:

	$q=0$	$q=1$	$q=2$	$q=3$
rot	grad	curl	div	0
div	0	div	$-$ curl	grad
$\overset{\circ}{\mathrm{R}}{}^q(\Omega)$	$\overset{\circ}{\mathrm{H}}(\mathrm{grad}, \Omega)$	$\overset{\circ}{\mathrm{H}}(\mathrm{curl}, \Omega)$	$\overset{\circ}{\mathrm{H}}(\mathrm{div}, \Omega)$	$\mathrm{L}^2(\Omega)$
$\mathrm{D}^q(\Omega)$	$\mathrm{L}^2(\Omega)$	$\mathrm{H}(\mathrm{div}, \Omega)$	$\mathrm{H}(\mathrm{curl}, \Omega)$	$\mathrm{H}(\mathrm{grad}, \Omega)$
$\gamma_\tau E$	$E\|_{\partial\Omega}$	$\nu \times E\|_{\partial\Omega}$	$\nu \cdot E\|_{\partial\Omega}$	0
$\gamma_\nu H$	0	$\nu \cdot H\|_{\partial\Omega}$	$-\nu \times (\nu \times H\|_{\partial\Omega})$	$H\|_{\partial\Omega}$

We note $\overset{\circ}{\mathrm{H}}(\mathrm{grad}, \Omega) = \overset{\circ}{\mathrm{H}}{}^1(\Omega)$ and $\mathrm{H}(\mathrm{grad}, \Omega) = \mathrm{H}^1(\Omega)$.

On the boundary $\partial\Omega$ we have for $\mathrm{Rot} = \mathrm{d}$ and $\mathrm{Div} = \delta = \circledast\,\mathrm{d}\,\circledast$:

	$q=0$	$q=1$	$q=2$
Rot	$\mathrm{grad}_{\partial\Omega}$	$\mathrm{curl}_{\partial\Omega}$	0
Div	0	$\mathrm{div}_{\partial\Omega}$	$-\,\mathrm{cograd}_{\partial\Omega}$

Here $\mathrm{grad}_{\partial\Omega}$, $\mathrm{curl}_{\partial\Omega}$, $\mathrm{div}_{\partial\Omega}$ resp. $\mathrm{cograd}_{\partial\Omega}$ denotes the surface gradient, (scalar) rotation, divergence resp. co-gradient. These boundary differential operators may be defined by extending a function f or a tangential vector field v defined on $\partial\Omega$ arbitrarily to a small neighborhood in $\mathbb{R}^3$ and then applying the usual differential operators to the extensions $\tilde{f}$ or $\tilde{v}$ as well as restricting them back to $\partial\Omega$. If necessary we take the tangential or normal part of the vector. We have

$$\mathrm{grad}_{\partial\Omega} f := -\nu\times(\nu\times\mathrm{grad}\,\tilde{f}\big|_{\partial\Omega}) \quad , \quad \mathrm{cograd}_{\partial\Omega} f := -\nu\times\mathrm{grad}\,\tilde{f}\big|_{\partial\Omega} \quad ,$$
$$\mathrm{curl}_{\partial\Omega} v := \nu\cdot\mathrm{curl}\,\tilde{v}|_{\partial\Omega} \quad , \quad \mathrm{div}_{\partial\Omega} v := \mathrm{div}\,\tilde{v}|_{\partial\Omega}$$

and note

$$\mathrm{cograd}_{\partial\Omega} f = -\nu\times\mathrm{grad}_{\partial\Omega} f \quad ,$$
$$\mathrm{curl}_{\partial\Omega}(\nu\times v) = \mathrm{div}_{\partial\Omega} v \quad , \quad \mathrm{div}_{\partial\Omega}(\nu\times v) = -\,\mathrm{curl}_{\partial\Omega} v \quad .$$

Furthermore, the rules of partial integration

$$\langle\mathrm{grad}_{\partial\Omega} f, v\rangle_{\partial\Omega} = -\langle f, \mathrm{div}_{\partial\Omega} v\rangle_{\partial\Omega} \quad , \quad \langle\mathrm{cograd}_{\partial\Omega} f, v\rangle_{\partial\Omega} = \langle f, \mathrm{curl}_{\partial\Omega} v\rangle_{\partial\Omega}$$

hold for suitable functions resp. tangential vector fields f, v. As long as the vector product $\times$ or curl are not used all operations still hold true for arbitrary dimensions N.

Now we obtain the following problems:

$q = 0$:

$$\begin{aligned}
\partial_t E - \varepsilon^{-1}\,\mathrm{div}\,H &= F && \text{in} \quad \Xi\\
\partial_t H - \mu^{-1}\,\mathrm{grad}\,E &= G && \text{in} \quad \Xi\\
E|_{\partial\Omega} &= \lambda && \text{on} \quad \Gamma\\
\nu\times(\partial_t\,\mu H)|_{\partial\Omega} &= \nu\times\mu G|_{\partial\Omega} + \mathrm{grad}_{\partial\Omega}\,\lambda && \text{on} \quad \Gamma\\
\varepsilon^{-1}\,\mathrm{div}\,H\big|_{\partial\Omega} &= \partial_t\,\lambda - F|_{\partial\Omega} && \text{on} \quad \Gamma
\end{aligned}$$

$q = N-1$:

$$\begin{aligned}
\partial_t E - \varepsilon^{-1}\,\mathrm{grad}\,H &= F && \text{in} \quad \Xi\\
\partial_t H - \mu^{-1}\,\mathrm{div}\,E &= G && \text{in} \quad \Xi\\
\nu\cdot E|_{\partial\Omega} &= \lambda && \text{on} \quad \Gamma\\
\nu\cdot\varepsilon^{-1}\,\mathrm{grad}\,H\big|_{\partial\Omega} &= \partial_t\,\lambda - \nu\cdot F|_{\partial\Omega} && \text{on} \quad \Gamma
\end{aligned}$$

$q = N$ (trivial case): $\partial_t E = F$ and $E(0) = E_0$. Thus, $E = E_0 + \int_0^t F(s)\, ds$.

$q = 1$ and $N = 3$:

$$
\begin{aligned}
\partial_t E + \varepsilon^{-1} \operatorname{curl} H &= F && \text{in} \quad \Xi \\
\partial_t H - \mu^{-1} \operatorname{curl} E &= G && \text{in} \quad \Xi \\
\nu \times E|_{\partial\Omega} &= \lambda && \text{on} \quad \Gamma \\
\nu \cdot (\partial_t \mu H)|_{\partial\Omega} &= \nu \cdot \mu G|_{\partial\Omega} + \operatorname{curl}_{\partial\Omega} \lambda && \text{on} \quad \Gamma \\
\nu \times \varepsilon^{-1} \operatorname{curl} H\big|_{\partial\Omega} &= -\partial_t \lambda + \nu \times F|_{\partial\Omega} && \text{on} \quad \Gamma
\end{aligned}
$$

We would like to note that the equations of linear elasticity are also covered by our approach, if we change the tangential boundary condition into the more simple ones of componentwise scalar Dirichlet boundary conditions in Euclidean coordinates. In detail the system (4) with the corresponding divergence equations, where we may assume εF to be solenoidal, reads for $N = 3$, $q = 1$ using our isomorphisms and, for instance, only the equations for E

$$
\begin{aligned}
\partial_t^2 E + \varepsilon^{-1} \operatorname{curl} \mu^{-1} \operatorname{curl} E &= \tilde{F} && \text{in} \quad \Xi \quad , \\
\operatorname{div} \varepsilon E &= \operatorname{div} \varepsilon E_0 && \text{in} \quad \Xi \quad , \\
\nu \times E|_{\partial\Omega} &= \lambda && \text{on} \quad \Gamma \quad .
\end{aligned}
$$

If we now replace the space $\overset{\circ}{\mathrm{R}}{}^q(\Omega)$ by $\overset{\circ}{\mathrm{H}}{}^{1,q}(\Omega)$, the closure of $\overset{\circ}{\mathrm{C}}{}^{\infty,q}(\Omega)$ in the norm of $\mathrm{H}^{1,q}(\Omega)$, in our formulations and after taking the gradient insert the second equation into the first one, we get

$$
\begin{aligned}
\partial_t^2 E + \varepsilon^{-1} \operatorname{curl} \mu^{-1} \operatorname{curl} E + \operatorname{grad} \operatorname{div} \varepsilon E &= \tilde{\tilde{F}} && \text{in} \quad \Xi \quad , \\
E|_{\partial\Omega} &= \lambda && \text{on} \quad \Gamma \quad .
\end{aligned}
$$

Choosing $\varepsilon := -b \operatorname{Id}$ and $\mu := -\frac{1}{ab} \operatorname{Id}$ with $a, b > 0$ we obtain the homogeneous isotropic linear elasticity operator

$$
\partial_t^2 E + a \operatorname{curl} \operatorname{curl} E - b \operatorname{grad} \operatorname{div} E = \tilde{\tilde{F}} \qquad \text{in} \quad \Xi \quad ,
$$

where in classical language $a = \mu$, $b = 2\mu + \kappa$ and μ , κ are the Lamé constants.

7 Conclusion and outlook

Of course, it is also possible in a very similar way to establish our theory using the domain of definition $\mathrm{R}^q(\Omega) \times \overset{\circ}{\mathrm{D}}{}^{q+1}(\Omega)$ instead of $D(\mathcal{M})$ for $\mathcal{M}$. Then we always would have prescribed normal traces.

This report represents the first part of an ongoing longer research project. It contains most of the theoretical work. Anyhow, in a forthcoming second report we wish to investigate the 'equivalence' between the time-harmonic and the time-periodic problem. In a third report we will try to explain some domain truncation procedures, which are needed for the implementation of the CGA, and study their approximation properties thoroughly. There we will discuss approximate radiation conditions (ARC), perfectly matched layers (PML), infinite elements (IFE) and in particular tangential to normal operators (TtN) for differential forms, which are the counterparts of the classical Dirichlet to Neumann (DtN) or electric to magnetic (EtM) operators for Helmholtz' or (time-harmonic) Maxwell's equations, respectively. For the TtN method we plan to work out a new theory.

Since we formulate our differential equations within the framework of differential calculus there are two methods for their discretization. One idea is to use the finite element method (FEM) for differential forms, i.e. the finite element exterior calculus (FEEC), [28, 1], which is the more common method. However, we intend to use the discrete exterior calculus (DEC) since the DEC utilizes in a very natural way the properties and the calculus of differential forms. In fact, this method uses another approach to discretize the forms. The idea is not to use a variational formulation of the problem and approximate the forms by polynomial forms, which then would be evaluated at some points, but to measure the action of a differential q-form E on a q-dimensional volume element C, called q-cell. This mapping from q-cells to real (or complex) numbers will be denotes by $\hat{E}$ and equals in fact the integral of the q-form E over the q-cell. This is even the more physical way of viewing Maxwell's equations. The electric field may then be regarded as a 1-form and the magnetic field as a 2-form. Due to Stokes' theorem for a differential q-form E we have on $(q+1)$-cells C

$$\langle \mathrm{d}\,E, C\rangle := \widehat{\mathrm{d}\,E}(C) = \int_C \mathrm{d}\,E = \int_{\partial C} \iota^* E = \widehat{\iota^* E}(\partial\, C) =: \langle \iota^* E, \partial\, C\rangle \qquad .$$

(Here again $\iota : \partial\, C \hookrightarrow \overline{C}$ denotes the natural embedding of the boundary.) Hence, the exterior derivative d, i.e. a partial differential operator, is transformed into the 'simple' geometric boundary operation ∂ of computing the boundary $\partial\, C$ of C. Therefore, the boundary operator ∂ is often called the 'dual' or 'adjoint' operator to the exterior derivative d. Thus, roughly spoken, all one needs are operators mapping for a given mesh volumes to faces, faces to edges, edges to vertices. These boundary operators are naturally sparse and consist of entries ± 1. Of course, also the star operator needs to be discretized which results in a diagonal matrix. The proper inner product is then the corresponding quadratic form. For a more detailed discussion of the DEC we refer to the recent papers [29, 15, 14]. We note that in special cases the classical Yee-scheme [50] and Yee-like schemes [8, 9] are recovered.

We plan to publish the corresponding numerical results in further forthcoming reports. First promising numerical experiments are already done.

Acknowledgements

The first author is grateful to the *Department of Mathematical Information Technology* (MIT) of the University of Jyväskylä, in particular to Prof. Dr. Pekka Neittaanmäki and Prof. Dr. Tuomo Rossi, where he had the opportunity to spend two sabbaticals of three month during spring 2007 and two month during spring 2008 as a postdoc fellow and this report could have been worked out.

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