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Author(s): Fenyvesi, Kristof; Budinski, Natalija; Lavicza, Zsolt

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Two Solutions to an Unsolvable Problem: Connecting Origami and GeoGebra in a Serbian High School

Kristóf Fenyvesi (†)
University of Jyväskylä, Finland
Ars GEometrica Community, Experience Workshop Math-Art Movement
E-mail: fenyvesi.kristof@gmail.com
Website: www.experienceworkshop.hu

Natalija Budinski
Primary and secondary school “Petro Kuzmjak”
Ruski Krstur, Serbia
E-mail: nbudinski@yahoo.com

Zsolt Lavicza
University of Cambridge, United Kingdom
GeoMaTech Project, Budapest College of Communication & Bussiness, Hungary
E-mail: zl221@cam.ac.uk

Abstract

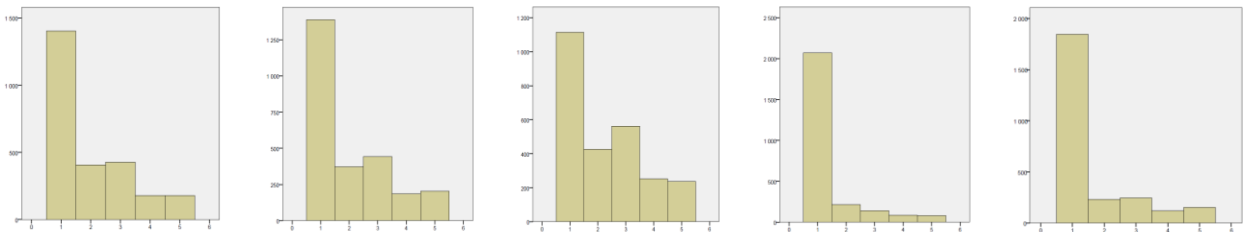
Our “Visuality & Mathematics — Experiential Education of Mathematics Through the Use of Visual Art, Science, and Playful Activities” Tempus Project started in 2012 with the cooperation of eight European universities and scientific institutions, in order to develop Serbian mathematics education with technological equipment and interactive, experience-centered, and art-related content. The general objectives of this two-year-long project are justified by the findings of the PISA 2012 survey as well, which show that 15-year old Serbian students' mathematics performance is significantly below the OECD average. For the improvement of the Serbian students' mathematical literacy and abilities, what we believe is important is research on new approaches in mathematics education and the increase of experience-centered presentations of cultural, interdisciplinary, and artistic embeddedness of mathematical knowledge, such as the creative applications of mathematics by hands-on models, in digital environments and in real-life problems, in the math class. We are not only working on the development of genuinely new content and methods in mathematics education, but are also collecting all of those estimable practices in experience-centered mathematics education in Serbia, which can be disseminated in the wide circles of Serbian mathematics teachers and can be introduced into the education of teachers as well. In this paper, we highlight efficient practice methods, which are already being applied in a Serbian high school and which connect mathematics education with origami and the open-access GeoGebra dynamic geometry software.

1 Introduction

Our *“Visuality & Mathematics — Experiential Education of Mathematics Through the Use of Visual Art, Science, and Playful Activities”* Tempus Project started in 2012 with the cooperation of eight European universities and scientific institutions, in order to develop Serbian mathematics education with technological equipment and interactive, experience-centered, and art-related content (project website: [12]). The general objectives of this two-year-long project are justified by the findings of the PISA 2012 survey as well [18]. PISA 2012 reveals that although Serbia — which scored 449 points — steadily improved in mathematics education from 2003 [18, p. 55], the mathematics performance of 15-year old Serbian students is still statistically significantly below the OECD average. According to PISA's definition of mathematical literacy [18, p. 37-38], Serbian students fall behind the OECD average in their capacity to formulate, employ, and interpret mathematics in various contexts, and also have difficulty recognizing the role that mathematics plays in the world. From the four overarching areas, which the PISA assessment framework for mathematical literacy makes reference to, it is only in *quantity* that the Serbian students score higher than their overall mathematics proficiency scale. Operations in the other three areas of mathematical literacy, i.e., *uncertainity and data*, *change and relationships* [18, p. 101], and *space and shape* [18, p. 104], cause even more difficulties for them. PISA 2012 measured not only the students' performances, but also examined whether and how their exposure to mathematics content is associated with their performance, and this provides a snapshot of the priorities of Serbian mathematics education. The survey has shown that Serbian students' exposure to word problems is under the OECD average [18, p. 147], as is their exposure to applied mathematics [18, p. 149], while they have significantly more opportunities to learn formal mathematics content during their schooling [18, p. 148]. The examination of Serbian students' engagement, drive, and self-beliefs in connection with mathematics learning shows that the index of their mathematics self-efficacy – the extent to which they believe in their own ability to handle mathematical tasks effectively and overcome difficulties – is also relatively low, while their index of openness to problem solving is high, although the latter is not reflected in their mathematics performance [19, p. 11]. Serbian students' intrinsic motivation to learn mathematics is slightly lower than the average as well, but from the survey results it is also obvious that the educational system is not taking full advantage of their positive attitudes and their openness to problem solving. In Serbia, less than 30% — at most — of students enjoy mathematics [19, p. 69].

The picture provided by PISA 2012 on Serbian students' mathematics education and attitudes is further refined by our own Tempus Attitude Survey 2013 (TAS 2013). We succeeded in identifying a number of features of pedagogical practices applied to the mathematics education of 11–18-year-old Serbian students, which could be recommended to be developed or changed to improve and build more efficiently on students' attitudes towards mathematics and thus support them in achieving better results. We found that the majority of the 2,607 11–18-year-old Serbian students who participated in TAS 2013 do not feel personally addressed to by the content of their math classes: most of them feel math classes are boring and think that the content could be taught in a much more engaging way. The majority of the students agree that they would learn more mathematics if the content of the classes would be more interesting to them. TAS 2013 has also shown that the presentation methods of the mathematics education content in school does not provide an account of the cultural embeddedness of mathematical knowledge and does not connect to the students' real-life world. According to our results, most Serbian mathematics teachers also do not apply all of the methodologies, tools, and equipment for experience-centered mathematics education [9], which could be effectively implemented to support their students' creative and imaginative abilities in the comprehension of complex and difficult mathematical problems and would make mathematics classes more engaging. Although at least a third of the students rarely used computers in mathematics classes, more than half of them never used a computer in their maths class. This demonstrates that not only do students not rely on the support of computer applications in math classes, the teachers also use them only very rarely for the illustration of teaching content (e.g., in the form of PowerPoint presentations). The situation is not significantly better in the case of using hands-on tools, physical models, and other visualization equipment: almost half of the students have never had an opportunity to work with these kinds of physical materials in their math classes. The situation is rather unfavorable in connection to the school presentation of the cultural embeddedness of mathematics. The general mathematics education practice in Serbia almost entirely excludes all accounts of art connections to mathematics (Figure 1).

For improving Serbian students' mathematical literacy and abilities, what we believe is important is research on new, experience-centered, art-related approaches in mathematics education and the increase of presentations of cultural, interdisciplinary, and artistic embeddedness of mathematical knowledge, leading to creative applications of mathematics using hands-on models, the use of digital environments and the incorporation real-life problems into math classes. We are not only working on the development of genuinely new content and methods in mathematics education, but also on collecting all of those good practices in experience-centered mathematics education in Serbia, which can be disseminated in the wide circles of Serbian mathematics teachers and can be introduced in the education of teachers as well. For this purpose, annually we are organizing the *European Summer School for Visual Mathematics and Education*, which provides the opportunity for practicing mathematics teachers and university students from Serbia to come together and exchange ideas with specialists of experience-centered mathematics education from all over the world.



(a) *(b)* *(c)* *(d)* *(e)*

Figure 1: *TAS 2013 results. Students rate how often their mathematics teachers (a) use computers; (b) computer-aided presentations, such as PowerPoint; (c) real physical objects or models for visualization; (d) references to artworks, like paintings or sculpture, etc.; (e) or how often they visited art or science museums to support the understanding of mathematical content. The vertical line shows the number of students; the horizontal line: 1 = never; 2 = a few times; 3 = sometimes; 4 = often; 5 = many times.*

In 2013 July, the Summer School program took place in Eger, Hungary [13] where the authors of this paper first met. They studied each other's approaches and decided to further develop and publish their method, which connects mathematics education with origami and the open-access GeoGebra dynamic geometry software. The method is already being applied by mathematics teacher Natalija Budinski (Figure 2) in her workplace, the Petro Kuzmjak School (Ruski Krstur, Serbia).

Figure 2: *Natalija Budinski's origami workshop at the European Summer School of Visual Mathematics and Education in Eger, 2013 (left); Budinski's origami-mathematics class in her school in Serbia (right).*

2 The Problem: Thinking Out of the Box

PISA measures not only the extent to which students can reproduce mathematical knowledge, but also how they can apply their knowledge in unfamiliar situations, in real-life contexts, and how well they can use tools, such as rulers, calculators, dynamic geometry software, etc., each of which require a degree of mathematical reasoning: *"Strong mathematics performance in PISA is not only related to opportunities to learn formal mathematics, such as solving a quadratic equation, using complex numbers, or calculating the volume of a box, but is also related to opportunities for learning how to apply mathematics (using mathematics in a real-world context)."* [18, p. 146] In pursuit of the idea of opportunities for learning different kinds of mathematics in an engaging way, we were looking for a mathematical problem, whose solution requires real-world context application of formal mathematics knowledge, experience-centered approaches, and the genuine ability of "thinking out of the box". The famous Delian problem of doubling the cube we considered an appropriate example for illustrating certain characteristics of our approach, because (a) the mathematical knowledge needed to understand and solve this problem is available to 15–

18-year-old Serbian high school students; (b) of its historical origin as well as the fact that famous solution; (c) its origami solution is able to call the students' attention to some unexpected aspects of the mathematical potential of the art of paper-folding; (d) its solution with the involvement of two conic curves can be studied in the GeoGebra dynamic geometry software environment; (e) both the origami and the GeoGebra solutions of the problem requires that the students recognize the epistemological limits of Euclidean geometry, to change perspective and experiment with new approaches.

(a) The Delian problem is based on the mathematical knowledge of calculating the volume of geometric solids, although it is impossible to solve it within the constraints of Euclidean geometry. For the solution, students need some prerequisite mathematical knowledge, which is covered by the Serbian secondary and high school curriculum, such as: basic geometric shapes, the Pythagorean Theorem, similarity of the triangles, simple quadratic and cubic equations, curves like circles, parabolas, hyperbolas, and ellipses in an analytical sense. Students should also be familiar with origami and GeoGebra basics.

(b) There is a plenty of information available on the Delian problem's significance in the history of science and its philological background [21, pp. 82-88], as well as about its mathematical details [4, pp. 122-134]. Teachers can efficiently make use of these resources according to the students' interests and their own demands to illustrate the cultural embeddedness of this mathematical problem. In our class, we were content with the most widely prevalent, to some extent romantic version of the ancient story [2, p. 98], which also contextualizes the otherwise formal mathematical problem of doubling a cube as a real-world application of mathematical knowledge. According to the version we used in our class (cf. T. Sundara Row quotes [10, p. 82, 207] in [20, p. 55]), ancient Athenians were struck by a plague and consulted the oracle at Delos. The oracle advised the Athenians to double the size of Apollo's perfect cube-shaped altar. They constructed the new altar where the sides of the cube were double the length of those of the original altar. But Apollo then made the pestilence worse, as the Athenians, by doubling the side of the original cube, increased its volume not to double that of the original, but falsely, by two cubed, or eight. This narrative might catch students' interest and support their imagination, compelling them to visualize the problem. It also makes possible for the teacher to link the mathematical information with the students' knowledge of Greek culture, and Euclid and Plato, in an interdisciplinary way.

(c) In the 19th century, Friedrich Fröbel, the German educationist and founder of the Kindergarten concept, encouraged the use of paper-folding in education with the aim of conveying mathematical concepts to children [8, pp. 214–225]. The Indian mathematician T. Sundara Row, in his Fröbel-inspired *Geometric Exercises of Paper Folding* (first published in 1901), already mentioned the Delian problem under the chapter Arithmetic Series and provided Menaechmus, a pupil of Plato's, solution to the problem [20, p. 55]. But it was Margherita Piazzolla Beloch in 1936 who proved that starting with a length L on a piece of paper, she could fold a length that was the cube root of L, and although she might have not recognized it, this way she solved the Delian problem by using origami. Her paper was rediscovered decades later, when the mathematical world started to take origami seriously (see [7, 14]. Implementing origami in mathematics education was systematically researched from the 1950s [15] and it has been found that origami has several pedagogical benefits. Origami allows the students to feel the objects created, rather than just imagine them or see them in pictures. A teacher can blend mathematical vocabulary and content within the steps s/he must go through to teach the folding of a particular model [3]. Additionally, it is well-suited to working with a classroom of 30 or more students, supports community building, encourages cooperative learning, develops planar and spatial reasoning, allows students to create and manipulate basic geometric shapes such as squares, rectangles, and triangles, and in many ways contributes to the students' cognitive development (on several mathematics educational benefits of origami, see: [17]).

(d) The experience-centered process of exploratory introduction to geometry problems and proofs by paper-folding can be successfully supported by using Dynamic Geometry Softwares (DGS) such as the free-access GeoGebra (www.geogebra.org) to extend investigations and foster deeper understanding of a proof [5]. GeoGebra is accessible, engaging, encourages students to further explore the geometrical situation and provides opportunities for making and evaluating conjectures of geometrical results.

(e) The experience-centered study of the Delian problem effectively shows what kind of potential and limitation certain mathematical frameworks have, such as Euclidean geometry. Students can make thought experiments with the application of an origami based geometry and they can be introduced to origami axioms, including Geretschlager's [11], who notes that his last procedure (7*) makes origami different from Euclidean geometry. Euclidean constructions are equivalent to origami built from Geretschlager's axioms nr. 1 to 7, but 7* amounts to the solution of a cubic problem, which is not achievable using Euclidean methods. It is the axiom that allows paper-folding methods to solve the classic problems of doubling the cube and trisecting the angle [5, p. 10].

3 An Origami-GeoGebra Mathematics Lesson Plan

Our lesson is planned for 45 minutes. To conduct the lesson, we need at least one computer per every two students, a computer for the teacher, equipped with an LCD projector, and colored paper for the origami experiment. As software support, we need GeoGebra and PowerPoint. Our lesson follows the PISA 2012 mathematics frameworks' structure as it is explained in Figure 3.

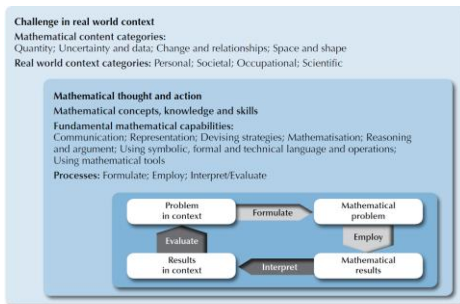


Figure 3: *Main features of the PISA 2012 mathematics framework* [18, p. 37]

Introduction to the lesson (duration: 5 minutes). In a PowerPoint presentation with several illustrations on ancient Greek cultural artefacts, the teacher provides the basic information on the Delian problem by presenting the advice of the oracle as it is explained in (b) of section 2; and shows visualizations of the mathematical content. In the presentation the teacher underlines with a brief explanation that in the framework of Euclidean geometry the construction of figures and lengths are restricted to the use of only an unmarked straight edge and compass. At the end of the introduction, the teacher starts a dialogue with the students and refreshes their preliminary knowledge on area and volume of geometrical solids.

In search of the solution (duration: 5 minutes). Students are encouraged to share their views and opinions on the Delian problem. The teacher orients the discussion with questions and the students are encouraged to devise different assumptions. After some thinking, students most likely come up with the wrong assumption that *"the size of the altar is x, and if we double it, we obtain the doubled volume."* In this case, the teacher reminds the students of the formula of calculating the volume of the cube and they prove together that the assumption is not correct. The formula shows that when we double the cube's volume, the side length of the new cube should be $\sqrt[3]{2}$ times larger. It is important to underline the significance of this assumption and to call attention on the importance of the analysis of the number $\sqrt[3]{2}$, which is not a constructible length in the Euclidean framework [6]. Students might be reminded that, for example, the number $\sqrt[2]{2}$ is a constructible number and during their education they previously learned how to construct it by only using an unmarked straightedge and compass. Students understand that because of the unconstructibility of $\sqrt[3]{2}$, the Delian problem is not solvable in the Euclidean framework. The teacher briefly recounts the most famous attempts to approach and solve this problem from a non-Eucledian framework including Menaechmus, Galois, and the origami solution developed in 1986 by Peter Messer [16].

Solving the unsolvable (duration: 30 minutes). The teacher divides the students into two groups. One group works with origami on the Messer-solution, and the other works with GeoGebra on the Menaechmus-solution of the Delian problem. Groups can be formed by teacher choice or by student preference. Both groups get adequate printed instructions, previously prepared by the teacher. Following that, Origami and GeoGebra students will be matched in pairs and work collaboratively. Origami students create an origami solution of the problem from a squared piece of paper, while GeoGebra students create a dynamic worksheet based on Menaechmus' solution, a sketch of intersecting parabolas.

Origami instructions:
1. Gently fold the squared paper in half;
2. Fold paper as is shown to match the line segments *AC* and *BE*;
3. Fold paper in three equal parts;
4. Fold the angle *C* to match the line *AB*, and match the point *I* with line *FG*;
5. Point *C* will make the line segments *AC* and *CB* in proportion to $1:\sqrt[3]{2}$;
6. Analyze the proportion! Why is it correct?

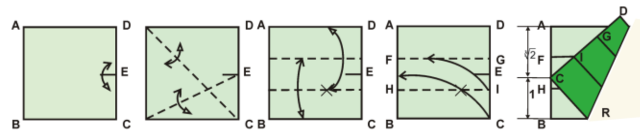


Figure 4: *Messer's solution.*

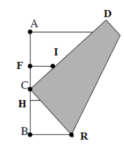


Figure 5: *The origami solution.*

GeoGebra instructions:
1. Open the GeoGebra file;
2. Insert two parabolas: one with the focal length twice of the other. In case you do not know, their analytical formulas: $x^2 = ky \;\; y^2 = 2kx$;
3. Use a slider;
4. Observe the intersection, while changing the slider's values;
5. Analyze if there is $\sqrt[3]{2}$. How did you obtain it?

The students work approximately 10 minutes individually on finding the answer to the question. Then each team comes together and for 10 minutes discusses and makes drafts or completes proofs. At this phase it is not required of the students that they prove the solutions; rather, they must make several assumptions and experiment. The teacher is observing their progress and helps if there is a question or the students need support. When the groups obtain some solutions, or maybe even a proof (see Figure 6), the teacher starts an open discussion with the participation of all of the students. Each delegates one student to present the solution origami and the GeoGebra solution. The teacher and students support the presenters and the teacher gently leads the students through the proof. For studying the GeoGebra proof, the teacher may ask questions like: Why do parabolas intersect? How do we formulate that intersection analytically? Do you know how to solve the equation? What about dividing by zero while solving the equation? What is the solution? etc.

The intersections of the parabolas occur at two points. One of them is the origin *A(0,0)* and the second one is the point *B* with coordinates $B(\sqrt[3]{2}\,k, \sqrt[3]{4}\,k)$. The teacher may ask the student about what is obtained when the value of a slider is changed? If we estimate the slider value on 1, we obtain the solution to the Delian problem as the length of segment *AC*. The length is $\sqrt[3]{2}$. The possible solution of the student in GeoGebra is shown in Figure 6.

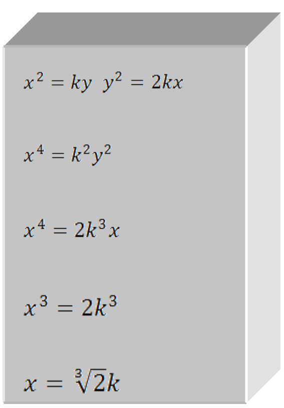


Figure 6: *Proof of the problem.*

For the study of the origami proof, the teacher may ask questions and may give instructions like: Would you make a sketch? Note the main points. Use the Pythagorean Theorem. Are there similar triangles? What does that imply? When you simplify the equation, what is the solution? etc.

Consider the square of paper shown in Figure 5 with the noted points *A*, *B*, *C*, *R*, *F* and *I*. Let us assume *BC*=1 and let x=*AC* and y=*BR*. From this it follows *AB*=x+1 and *CR*=1+x-y. If we apply the Pythagorean theorem $CR^2=BR^2+BC^2$ and use the notation we get $(1+x-y)^2=y^2+1^2$. Simplifying the equation we get:

$$y = \frac{x^2+2x}{2+2x} \quad (1)$$

Let us consider the triangles Δ*IFC* and Δ*CBR*. They are similar. This implies: $\frac{BR}{CR} = \frac{FC}{IC}$ (2)

The segment *FC* is part of segment *AB* which satisfies the following equations:

AB=*AF*+*FC*+*CB*

$$1+x=\frac{1}{3}\cdot(1+x)+FC+1$$

$$FC=\frac{2x-1}{3}$$

If we substitute this in (2) we get:

$$\frac{y}{1+x-y} = \frac{\frac{2x-1}{3}}{\frac{1+x}{3}}$$

Simplifying the expression we get: $y = \frac{(2x-1)(x+1)}{3x}$ (3)

As a result of equating expressions (1) and (3) we get $x^3+3x^2+2x=2x^3+3x^2+2x-2$ from whence it follows: $x^3=2$.

Closing the lesson (duration: 5 minutes). The teacher and students summarize the content of the lesson and the teacher asks the students for their impressions on the content of the lesson and the introduced approaches (Figure 7). As a homework task, the teacher may provide students materials for experimenting with the solution of the two other famous unsolvable problem of Greek mathematics, the trisection of an angle and the squaring a circle.

4 Conclusion

Our combined approach connected hands-on activity with computer-based learning. By the combination of different experience-centered approaches, students can improve their skills in reasoning in various contexts. While making origami requires following certain procedures in paper folding, GeoGebra allows the students to create a set of procedures that will lead to the solution. Both approaches illustrate real-world application of mathematical knowledge and bring abstract and difficult geometrical notions closer to the students. The fold diagrams in origami and the interactive visualizations in GeoGebra might give an opportunity to make transitions from visual to formal statements. Due to their visual and interactive nature, both origami and GeoGebra might enable students to see connections in the geometric statements more easily compared to students who receive standard instruction.[1] The origami-based instruction might contribute to students performing higher in achievement tests like PISA as well [1, p. 18].

[1] The GeoMatech Project, started in 2014 by the University of Applied Sciences, Budapest is working on special GeoGebra curricula applied to the Hungarian mathematics education. Numerous results of the project are expected to be implemented in various areas of mathematics education also in other countries. Website: http://geomatech.hu/

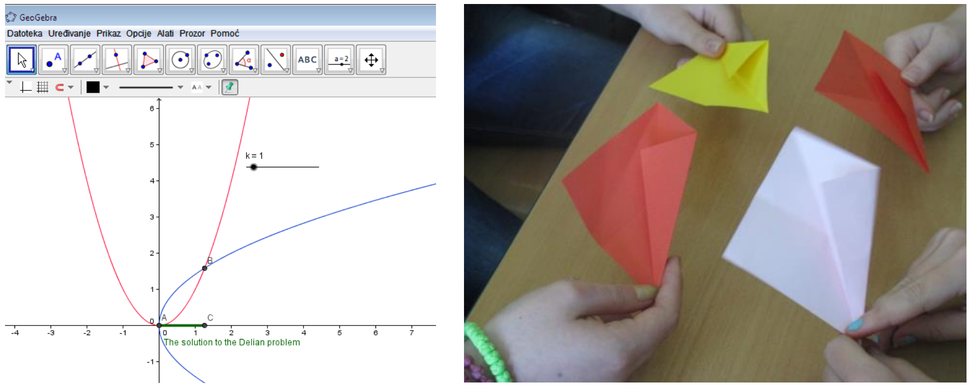


Figure 7: *The GeoGebra (left) and the origami (right) solution of the Delian problem.*

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