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# Bifurcation Method of Stability Analysis and Some Applications

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# Bifurcation Method of Stability Analysis and Some Applications*

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**Abstract**

In this paper a new approach to the analysis of implicitly given functionals is developed in the frame of elastic stability theory. The approach gives an effective procedure to analyse stability behaviour, and to determine the bifurcation points. Examples of application of the proposed approach for analysis of stability are presented, more precisely we consider the stability problem of an axially moving elastic panel, with no external applied tension, performing transverse vibrations. The analysis is applicable for many practical cases, for example, paper making and band saw blades.

## 1 Introduction

Elastic stability analysis comes with a long tradition. The present form of static stability analysis was originally developed by Euler [1766], for a differential equation describing the bending of a beam or column. Dynamic stability analysis for linear elastic systems, extending Euler's method, is due to Bolotin [1963]. According to Mote and Wickert [1991], the stability behaviour of some axially moving materials is mathematically analogous to the buckling of a compressed column, enabling the use of these techniques.

In this article, we propose new approach to the analysis of implicitly given functionals is developed in the frame of elastic stability theory and applied the approach for the the stability problem of an axially moving elastic panel, with no external applied tension.

In previously (see, e.g., Banichuk et al. [2013b,a, 2011a,b]), we have considered many approaches for modelling of the moving materials. Conclusions that have been drawn can be applied, for example, the processing of paper or steel, fabric, rubber or some other continuous material, and looping systems such as band saws and timing belts.

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The most often used models for an axially moving web have been travelling flexible strings, membranes, beams, and plates. The research field of moving materials can be traced back to Skutch [1897]. Among the first English-language papers on moving materials were Sack [1954] and Miranker [1960]. All these studies considered axially moving ideal strings. The analytical solution describing the free vibrations of the axially moving ideal string was derived by Swope and Ames [1963]. Dynamics and stability considerations were first reviewed in the article by Mote [1972].

The effects of axial motion of the web on its frequency spectrum and eigenfunctions were investigated in the classic papers by Archibald and Emslie [1958] and by Simpson [1973]. It was shown that the natural frequency of each mode decreases when the transport speed increases, and that the travelling string and beam both experience divergence instability at a sufficiently high speed. However, in the case of the string, this result was recently contrasted by Wang et al. [2005], who showed using Hamiltonian mechanics that the ideal string remains stable at any speed.

The loss of stability was studied with an application of dynamic and static approaches in the article by Wickert [1992]. It was shown by means of numerical analysis that in the all cases instability occurs when the frequency is zero and the critical velocity coincides with the corresponding velocity obtained from static analysis. Similar results were obtained for travelling plates by Lin [1997].

The dynamical properties of moving plates have been studied by Shen et al. [1995] and by Shin et al. [2005], and the properties of a moving paper web have been studied in the two-part article by Kulachenko et al. [2007a,b]. Critical regimes and other problems of stability analysis have been studied, e.g., by Wang [2003] and Sygulski [2007]. Moreover, in the articles Marynowski [2002, 2004, 2008] the author discusses widely dynamical aspects of the axially moving web. In Yang and Chen [2005] the authors considered transverse vibrations of the axially accelerating viscoelastic beam and in Pellicano and Vestroni [2000] dynamic behavior of a simply supported beam subjected to an axial transport of mass is studied. An extensive literature reviews related to areas presented in this paper, the reader can found, for example, in Ghayesh et al. [2013]. Note also some approaches to bifurcation problems and estimation of critical parameters presented by Nečas et al. [1987] and Neittaanmäki and Ruotsalainen [1985].

In this paper, we will develop a stability analysis technique based on implicitly given functions. The technique will reveal the bifurcation points regardless of their type. Finally, examples of application of the proposed approach will be presented.

## 2 Bifurcation method of stability analysis

Consider the spectral boundary value problem described by the equation

$$\mathcal{L}(\,u(x),\lambda,\gamma\,) = \sum_{k=0}^{m}\sum_{\ell=0}^{n}\lambda^k\gamma^\ell\mathcal{L}_{k\ell}\,(u(x)) = 0\;, \tag{1}$$

where $\gamma$ is a real-valued loading parameter, characterizing the interaction of the structure and external media, $\lambda$ is a spectral parameter, and $\mathcal{L}_{k\ell}\left(u(x)\right)$ are given differential operators applied to the behaviour function $u(x)$, defined in the domain $\Omega$ ($x \in \Omega$). Boundary conditions are considered as included in the differential operator $\mathcal{L}\left(u(x)\right)$.

Note that the problems of free harmonic vibrations, and the stability of elastic systems interacting with external fluid (liquid or gas) are reduced to the formulation (1).

Let the function $v(x)$ be the solution of the spectral problem

$$\mathcal{L}^{*}\left(v(x), \lambda, \gamma\right) = 0 \; , \tag{2}$$

which is adjoint to the problem (1). In the case of a self-adjoint problem, $v(x)$ coincides with $u(x)$.

If we multiply equation (1) by $v(x)$ and integrate over the domain, we will have

$$\Phi\left(\lambda, J_{00}, \ldots, J_{mn}, \gamma\right) = \sum_{k=0}^{m}\sum_{\ell=0}^{n}\lambda^{k}\gamma^{\ell}J_{k\ell} = 0 \; , \tag{3}$$

where the functionals $J_{k\ell}$, $k = 1, 2, \ldots m$; $\ell = 1, 2, \ldots, n$ are defined as

$$J_{k\ell} = (v,\, \mathcal{L}_{k\ell}u) = \int_{\Omega} v(x)\mathcal{L}_{k\ell}u(x)\,\mathrm{d}\Omega \; . \tag{4}$$

Here and in the following the relation (3) is considered as an implicit expression for the spectral parameter $\lambda$, which is also considered as a functional. The function $\Phi\left(\lambda, J_{00}, \ldots, J_{mn}, \gamma\right)$ is a polynomial of degree $m$ with respect to $\lambda$, and the solutions of the equation (3) are

$$\lambda_1 = \varphi_1\left(J_{00}, \ldots, J_{mn}, \gamma\right) \; , \quad \ldots \; , \quad \lambda_m = \varphi_m\left(J_{00}, \ldots, J_{mn}, \gamma\right) \; . \tag{5}$$

The values $\lambda_1, \ldots, \lambda_m$ take an extremal meaning for the solutions $u(x)$ and $v(x)$ of the direct and adjoint spectral problems (1), (2); that is,

$$\lambda_k = \varphi_k\left(J_{00}, \ldots, J_{mn}, \gamma\right) \;\rightarrow\; \underset{u,v}{\text{extr}} \; . \tag{6}$$

To show this, suppose that the solutions $u(x)$, $v(x)$ of the problems (1), (2), and the functionals $J_{00}, \ldots, J_{mn}$ defined in accordance with (4), correspond to a fixed value of the parameter $\gamma$. Suppose also that the variations

$$u(x) \rightarrow u(x) + \delta u(x) \; , \quad v(x) \rightarrow v(x) + \delta v(x) \tag{7}$$

of the solutions of (1) and (2) correspond to the variation

$$\lambda \rightarrow \lambda + \delta\lambda \tag{8}$$

of the spectral parameter. Consider the expression for the perturbed value $\widetilde{\Phi}$:

$$\begin{aligned}\widetilde{\Phi} &= \Phi(\lambda + \delta\lambda, J_{00} + \delta J_{00}, \dots, J_{mn} + \delta J_{mn}, \gamma) \\ &= \sum_{k=0}^{m}\sum_{\ell=0}^{n}(\lambda+\delta\lambda)^k\gamma^\ell(J_{k\ell}+\delta J_{k\ell}) \ .\end{aligned} \tag{9}$$

Using equation (1) for $u(x)$ and adjoint equation (2) for $v(x)$, noting definition (4), and performing elementary operations, we will have the following perturbation of the equation (3):

$$\begin{aligned}\widetilde{\Phi} &= \Phi(\lambda, J_{00}, \dots, J_{mn}, \gamma) + \frac{\partial \Phi}{\partial \lambda}\delta\lambda + \sum_{k=0}^{m}\sum_{\ell=0}^{n}\lambda^k\gamma^\ell\delta J_{k\ell} \\ &= \Phi(\lambda, J_{00}, \dots, J_{mn}, \gamma) + \frac{\partial \Phi}{\partial \lambda}\delta\lambda + (\delta v,\, \mathcal{L}u) + (v,\, \delta\mathcal{L}u) \\ &= \Phi(\lambda, J_{00}, \dots, J_{mn}, \gamma) + \frac{\partial \Phi}{\partial \lambda}\delta\lambda + (\delta v,\, \mathcal{L}u) + (\mathcal{L}^*v,\, \delta u) = 0 \ .\end{aligned} \tag{10}$$

The first (unperturbed) term is zero because of (3). The third and fourth terms are also equal to zero because $u(x)$ and $v(x)$ satisfy, respectively, the equations $\mathcal{L}u = 0$, $\mathcal{L}^*v = 0$. Thus, it follows from (10) that

$$\frac{\partial \Phi}{\partial \lambda}\delta\lambda = 0 \ , \tag{11}$$

and if $\partial\Phi/\partial\lambda \neq 0$, then

$$\delta\lambda = 0 \ , \quad \lambda = \lambda_1, \dots, \lambda_m \ . \tag{12}$$

Hence the extremal property.

Let us study the dependencies of $\lambda_k$, $k = 1, 2, \dots, m$, on the parameter $\gamma$ in more detail. There is an important peculiarity in equation (3). Let us show that the functionals $J_{00}, \dots, J_{mn}$ can be considered as constant when the function $\Phi(\lambda, J_{00}, \dots, J_{mn}, \gamma)$ is differentiated with respect to $\gamma$. To do this, we write the total derivative

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\gamma} = \frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\gamma} + \frac{\partial\Phi}{\partial\gamma} + \sum_{k=0}^{m}\sum_{\ell=0}^{n}\frac{\partial\Phi}{\partial J_{k\ell}}\frac{\mathrm{d}J_{k\ell}}{\mathrm{d}\gamma} \ . \tag{13}$$

The double sum in (13) is evaluated as

$$\begin{aligned}&\sum_{k=0}^{m}\sum_{\ell=0}^{n}\frac{\partial\Phi}{\partial J_{k\ell}}\frac{\mathrm{d}J_{k\ell}}{\mathrm{d}\gamma} \\ =&\sum_{k=0}^{m}\sum_{\ell=0}^{n}\lambda^k\gamma^\ell\left[\left(\frac{\mathrm{d}v}{\mathrm{d}\gamma},\, \mathcal{L}_{k\ell}u\right) + \left(v, \mathcal{L}_{k\ell}\frac{\mathrm{d}u}{\mathrm{d}\gamma}\right)\right] \\ =&\left[\left(\frac{\mathrm{d}v}{\mathrm{d}\gamma},\, \mathcal{L}u\right) + \left(v, \mathcal{L}\frac{\mathrm{d}u}{\mathrm{d}\gamma}\right)\right] \\ =&\left[\left(\frac{\mathrm{d}v}{\mathrm{d}\gamma},\, \mathcal{L}u\right) + \left(\mathcal{L}^*v, \frac{\mathrm{d}u}{\mathrm{d}\gamma}\right)\right] = 0 \ ,\end{aligned} \tag{14}$$

by taking into account the equalities $\mathcal{L}u = 0$ and $\mathcal{L}^* v = 0$. Thus the function $\Phi = \Phi(\lambda, J_{00}, \ldots, J_{mn}, \gamma)$ can be considered as a function of two variables $\lambda$ and $\gamma$, and denoted as $F(\lambda, \gamma)$, i.e.

$$F(\lambda, \gamma) = \sum_{k=0}^{m} \sum_{\ell=0}^{n} \lambda^k \gamma^n J_{k\ell} = 0 \; . \tag{15}$$

This equation can be taken for determination of the dependence $\lambda = \lambda(\gamma)$. From the mathematical point of view, it determines a set of functions $\lambda_1(\gamma), \ldots, \lambda_m(\gamma)$.

In correspondence with the fundamental theorem on implicit functions (see, e.g., Rektorys, 1969), a unique solution of (15) exists in a small vicinity of the fixed values $\lambda = \widetilde{\lambda}$, $\gamma = \widetilde{\gamma}$, if $\partial F / \partial \lambda \neq 0$.

Thus nonuniqueness of the solution of (15), or in other words, bifurcation of the considered system, can be realized for some values $\lambda = \lambda^*$, $\gamma = \gamma^*$ when the condition of the theorem on implicit functions is violated. Hence the bifurcation values $\lambda^*$ and $\gamma^*$ are found with the help of the equations

$$F(\lambda^*, \gamma^*) = 0 \; , \quad \frac{\partial F(\lambda^*, \gamma^*)}{\partial \lambda} = 0 \; . \tag{16}$$

Denote by $(\lambda_1^*, \gamma_1^*), (\lambda_2^*, \gamma_2^*), \ldots$ the solutions of the nonlinear system of equations (16), representing the points on the $\lambda, \gamma$ plane, and investigate the behaviour of functions $\lambda_i = \lambda_i(\gamma)$ in a small vicinity of the bifurcation points $(\lambda_k^*, \gamma_k^*)$. For simplicity, the subscript indices of the considered functions and points will be omitted.

Let us represent the function $F(\lambda, \gamma)$ in a small vicinity of the point $(\lambda^*, \gamma^*)$ as a series expansion,

$$\begin{aligned} F(\lambda, \gamma) = {} & F(\lambda^*, \gamma^*) + \frac{\partial F(\lambda^*, \gamma^*)}{\partial \lambda} [\lambda - \lambda^*] + \frac{\partial F(\lambda^*, \gamma^*)}{\partial \gamma} [\gamma - \gamma^*] \\ & + \frac{1}{2} \frac{\partial^2 F(\lambda^*, \gamma^*)}{\partial \lambda^2} [\lambda - \lambda^*]^2 + \frac{\partial^2 F(\lambda^*, \gamma^*)}{\partial \lambda \partial \gamma} [\lambda - \lambda^*] [\gamma - \gamma^*] + \frac{1}{2} \frac{\partial^2 F(\lambda^*, \gamma^*)}{\partial \gamma^2} [\gamma - \gamma^*]^2 \\ & + \ldots \end{aligned} \tag{17}$$

Taking into account (16), $F(\lambda^*, \gamma^*) = 0$ and $\partial F / \partial \lambda = 0$. The latter implies also $\partial^2 F / \partial \lambda \partial \gamma = 0$. Considering only the lowest-order nonzero terms, we have

$$F(\lambda, \gamma) = \frac{\partial F(\lambda^*, \gamma^*)}{\partial \gamma} [\gamma - \gamma^*] + \frac{1}{2} \frac{\partial^2 F(\lambda^*, \gamma^*)}{\partial \lambda^2} [\lambda - \lambda^*]^2 + \ldots \tag{18}$$

Let us now represent the behaviour of the function $\lambda = \lambda(\gamma)$ in the vicinity of the bifurcation point $(\lambda^*, \gamma^*)$ as

$$\lambda(\gamma) = \lambda^* + \alpha \, [\gamma - \gamma^*]^{\varepsilon} + \ldots \; , \tag{19}$$

where $\alpha$ and $\varepsilon$ are determined with the help of the condition $F(\lambda, \gamma) = 0$. By substituting (19) into (18), equation (18) is transformed into

$$\widetilde{F} = \widetilde{F}(\gamma - \gamma^*) \equiv 0 \; ,$$

which must be satisfied identically. Here $\widetilde{F}(\gamma-\gamma^*)$ is a series expansion with respect to $(\gamma - \gamma^*)$. As a result we have

$$F(\lambda,\gamma) = \frac{\partial F(\lambda^*,\gamma^*)}{\partial \gamma}\,[\gamma-\gamma^*] + \frac{\alpha^2}{2}\frac{\partial^2 F(\lambda^*,\gamma^*)}{\partial \lambda^2}\,[\gamma-\gamma^*]^{2\varepsilon} + \cdots \equiv 0. \tag{20}$$

Suppose that

$$\frac{\partial F(\lambda^*,\gamma^*)}{\partial \gamma} \neq 0\ , \quad \frac{\partial^2 F(\lambda^*,\gamma^*)}{\partial \lambda^2} \neq 0\ . \tag{21}$$

Now there are three cases:

$$2\varepsilon < 1\ , \quad 2\varepsilon = 1\ , \quad 2\varepsilon > 1\ .$$

In the case $2\varepsilon < 1$, the first term in (20) is of a higher order in $\gamma$ than the second (and hence can be omitted), and the equality (20) is satisfied if $\partial^2 F(\lambda^*,\gamma^*)/\partial\lambda^2 = 0$. This contradicts the second inequality in (21). Similarly, the case $2\varepsilon > 1$ contradicts the first inequality in (21). As a result of this asymptotic analysis, we find $2\varepsilon = 1$, and consequently we must have

$$\alpha^2 = -2\left(\frac{\partial F(\lambda^*,\gamma^*)}{\partial \gamma}\right)\left(\frac{\partial^2 F(\lambda^*,\gamma^*)}{\partial \lambda^2}\right)^{-1}\ . \tag{22}$$

Thus $\alpha$ is either real or pure imaginary , and we have (using (19))

$$\lambda(\gamma) = \lambda^* + \alpha\sqrt{\gamma-\gamma^*}\ , \quad |\gamma-\gamma^*| \ll 1\ , \tag{23}$$

provided that the inequalities (21) are satisfied.

As is seen from (15) and (23), the value $\alpha$ is expressed in terms of (derivatives of) the functional $F$, and does not require the analytical solution of the behavioural equation in an explicit manner.

# 3 Applications of bifurcation method

As an example of using the presented bifurcation analysis, let us consider the stability problem of an axially moving elastic panel, with no external applied tension, performing transverse vibrations. In the fixed (laboratory, Euler) coordinate system the equation of small transverse vibrations and the corresponding boundary conditions can be written as

$$\begin{gathered}\frac{\partial^2 w}{\partial t^2} + 2V_0\frac{\partial^2 w}{\partial x\partial t} + V_0^2\frac{\partial^2 w}{\partial x^2} + \frac{D}{\rho S}\frac{\partial^4 w}{\partial x^4} = 0\ ,\\ w(0,t) = w(\ell,t) = 0\ , \quad D\frac{\partial^2 w(0,t)}{\partial x^2} = D\frac{\partial^2 w(\ell,t)}{\partial x^2} = 0\ ,\end{gathered} \tag{24}$$

where $w = w(x,t)$ describes the transverse displacement, $\rho$ is the density of the material, $S$ the cross-sectional area of the panel, $t$ time and $x \in [0,\ell]$.

In what follows the time-harmonic transverse vibrations of the panel are represented as

$$w(x,t) = e^{i\omega t} u(x) \ , \tag{25}$$

and the dimensionless variables

$$x = \ell \widetilde{x} \ , \quad \widetilde{\omega}^2 = \frac{\rho S \omega^2 \ell^4}{D} \ , \quad \widetilde{V_0}^2 = \frac{\rho S \ell^2}{D} V_0^2 \tag{26}$$

will be used. The tilde will be omitted.

We obtain

$$\begin{aligned} &\omega^2 u - 2i\omega V_0 \frac{\mathrm{d}u}{\mathrm{d}x} - V_0^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} = 0 \ , \\ &u(0) = u(1) = 0 \ , \quad \left( \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)_{x=0} = \left( \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)_{x=1} = 0 \ . \end{aligned} \tag{27}$$

In (25)–(27) $\omega$ is a frequency, $u = u(x)$ is the amplitude function, and $i$ the imaginary unit.

After multiplication of the equation (27) by the complex conjugate amplitude function $u^*(x)$ and performing integration, taking into account the boundary conditions (27), we obtain

$$\Phi = a\omega^2 + 2bV_0\omega + V_0^2 c - d = 0 \ , \tag{28}$$

where

$$\begin{aligned} a &= \int_0^1 u u^* \,\mathrm{d}x > 0 \ , \\ ib &= \int_0^1 u^* \frac{\mathrm{d}u}{\mathrm{d}x} \,\mathrm{d}x = -\int_0^1 u \frac{\mathrm{d}u^*}{\mathrm{d}x} \,\mathrm{d}x > 0 \ , \quad (b \text{ real}) \\ c &= -\int_0^1 u^* \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}u}{\mathrm{d}x} \frac{\mathrm{d}u^*}{\mathrm{d}x} \,\mathrm{d}x > 0 \ , \\ d &= \int_0^1 u^* \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}^2 u^*}{\mathrm{d}x^2} \,\mathrm{d}x > 0 \ . \end{aligned} \tag{29}$$

Using the notation $a$, $b$, $c$, $d$ for the considered functionals, determined by the expressions (29), it is possible to find the coefficient $\alpha$ in the asymptotic representation of the function $\lambda(\gamma)$. We have ($\Phi = F$)

$$\frac{\partial F}{\partial V_0} = 2\left(b\omega + cV_0\right) \ , \quad \frac{\partial^2 F}{\partial \omega^2} = 2a \ , \tag{30}$$

and consequently,

$$\alpha^2 = -2 \frac{b\omega + cV_0}{a} \ . \tag{31}$$

Thus we find the following asymptotic representation for the dependence $\omega(V_0)$ in the vicinity of the bifurcation point $(\omega_k^*, V_0^*)$:

$$\begin{aligned} \omega(V_0) &\approx \omega^* \pm \sqrt{-2\frac{b\omega^* + cV_0^*}{a}} \sqrt{V_0 - V_0^*} \\ &= \omega^* \pm \sqrt{2\frac{b^2 - ac}{a^2} V_0^*} \sqrt{V_0 - V_0^*} \ . \end{aligned} \tag{32}$$

Note that for the considered problem, the equation $\Phi(\omega, a, b, c, d, V_0)$ can be solved with respect to the variable $\omega$. As a result, we have

$$\omega_{1,2}(V_0) = \frac{-bV_0 \pm \sqrt{(b^2 - ac)V_0^2 + ad}}{a} \ . \tag{33}$$

It is possible now to analyze the dependence $\omega(V_0)$, determined by expression (33) in the small vicinity of the bifurcation point $(\omega^*, V_0^*)$. Taking into account the representations for the bifurcation values of harmonic vibration frequency and velocity of axial motion,

$$\omega^* = -\frac{b}{a}V_0^* \ , \quad (ac - b^2)(V_0^*)^2 = acV_0^2 + ad \ , \tag{34}$$

and the asymptotic expression

$$V_0^2 \approx (V_0^*)^2 + 2V_0^* \left[V - V_0^*\right] \ , \quad \left|V_0 - V_0^2\right| \ll 1 \ , \tag{35}$$

we obtain an important asymptotic result

$$\omega_{1,2} \approx \omega^* \pm \sqrt{2\frac{b^2 - ac}{a^2}V_0^*}\sqrt{V_0 - V_0^*} \ , \quad |V_0 - V_0^*| \ll 1 \ , \tag{36}$$

which completely coincides with the asymptotic representation (32).

As a second example of application of the bifurcation method, we consider the problem of harmonic vibrations of a (stationary) panel compressed by the force $\gamma$ ($\gamma > 0$). The following relations will be used for the amplitude functions $u(x)$ ($x \in [0, 1]$):

$$\begin{aligned} &\frac{\mathrm{d}^4u}{\mathrm{d}x^4} + \gamma\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \omega^2 u = 0, \\ &u(0) = u(1) = 0 \ , \quad \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)_{x=0} = \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)_{x=1} = 0 \ . \end{aligned} \tag{37}$$

Let us investigate the asymptotic behaviour of the frequency $\omega$ as a function of the loading parameter $\gamma$, i.e. $\omega = \omega(\gamma)$, using the discussed perturbation method. To do this, we multiply the equation (37) by the function $u(x)$, which coincides in the considered case with $u^*(x)$ (because the problem (37) is self-adjoint), and perform integration.

As a result, we will find the following expression for $\Phi$ as a function of the functionals $a$, $c$ and $d$ (as defined in (29)). We have

$$\Phi(\omega, a, c, d, \gamma) = -a\omega^2 - \gamma c + d = 0 \ . \tag{38}$$

The functionals $a$, $c$ and $d$ can be expressed with the help of eigenmodes of vibrations

$$u_k(x) = B_k \sin(k\pi x) \ .$$

We find

$$a_k = \int_0^1 (u_k(x))^2 \,\mathrm{d}x = \frac{B_k^2}{2} ,$$
$$c_k = \int_0^1 \left(\frac{\mathrm{d}u_k}{\mathrm{d}x}\right)^2 \mathrm{d}x = \frac{k^2\pi^2}{2} B_k^2 , \tag{39}$$
$$d_k = \int_0^1 \left(\frac{\mathrm{d}^2 u_k}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = \frac{k^4\pi^4}{2} B_k^2 .$$

In correspondence with the general formulas (22)–(23), the asymptotic behaviour of the frequencies in the vicinity of the bifurcation points

$$\omega_k^* = 0 , \quad \gamma_k^* = k^2\pi^2 \tag{40}$$

will be described by the expressions

$$\omega_k = \omega_k(\gamma) = \pm\alpha\sqrt{\gamma - k^2\pi^2} , \quad \left|\gamma - k^2\pi^2\right| \ll 1 , \tag{41}$$

and the value of the coefficient $\alpha$ will be given by

$$\alpha^2 = -2\left(\frac{\partial F(\omega_k^*, \gamma_k^*)}{\partial\gamma}\right)\left(\frac{\partial^2 F(\omega_k^*, \gamma_k^*)}{\partial\omega^2}\right)^{-1} = -k^2\pi^2 . \tag{42}$$

# 4 Conclusions

In this paper we presented new version of bifurcation analysis, based on introduction of adjoint spectral problem and implicitly given functionals, and applied to some stability problem of an axially moving elastic panel with no external applied tension, performing transverse vibration. Using the bifurcation analysis of the considered functional we determined effectively asymptotic behaviour in the vicinity of the bifurcations points.

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