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# On the Critical Velocities and Free Vibrations of Axially Moving Elastic Webs

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## Abstract

Problems of dynamics and stability of a moving web, modelled as an elastic rod or string, and axially travelling between rollers (supports) at a constant velocity, are studied using analytical approaches. Transverse, longitudinal and torsional vibrations of the moving web are described by a hyperbolic second-order partial differential equation, corresponding to the string and rod models. It is shown that in the framework of a quasi-static eigenvalue analysis, for these models, the critical point cannot be unstable. The critical velocities of one-dimensional webs, and the arising non-trivial solution of free vibrations, are studied analytically. The analysis is then extended into the case with damping. The critical points of both static and dynamic types are found analytically. It is shown that if external friction is present, then for mode numbers sufficiently high, dynamic critical points may exist. Graphical examples of eigenvalue spectra are given for both the undamped and damped systems. In the examples, it is seen that external friction leads to stabilization, whereas internal friction in the travelling material will destabilize the system in a dynamic mode at the static critical point. The theory and results summarize and extend theoretical knowledge of the class of models studied, and can be used in various applications of moving materials, such as paper making.

## 1 Introduction

Research into axially moving materials began in the last years of the 19th century, when Skutch (1897) published a paper on the transverse vibrations of a travelling elastic string moving through two pinholes. The English-language literature in the area starts in the 1950s; among the first papers in English was the study by Sack (1954), also on transverse vibrations of travelling strings.

This field of research then gained momentum. Some classical early papers are e.g. Archibald and Emslie (1958), Miranker (1960), Swope and Ames (1963), Ames et al. (1968), Simpson (1973), and Mujumdar and Douglas (1976); and the many papers by C. D. Mote, e.g. Mote (1968a,b, 1972, 1975); and as a coauthor, Thurman and Mote (1969); Ulsoy et al. (1978); Ulsoy and Mote (1980, 1982); Mote and Wu (1985).

Since then, the amount of research has grown tremendously. For a recent review focusing on travelling strings only, see Chen (2005), which alone cites 242 references. The range of topics has also expanded. The interest in moving materials has been fairly continuous over the decades; for example, consider Chonan (1986), Pramila (1986, 1987), Wickert and Mote (1988, 1989, 1990), Yang and Mote (1991); Mote and Wickert (1991); Lee and Mote (1996); Lin and Mote (1995, 1996); Renshaw and Mote (1996); Lin (1997), Shen et al. (1995), Parker (1998), Marynowski (2002), Wang (2003), Kong and Parker (2004), Shin et al. (2005), Wang et al. (2005), Sygulski (2007), Kulachenko et al. (2007a,b), and Vaughan and Raman (2010). The later studies have typically investigated more complex cases, such as two-dimensional moving sheets (membranes and plates), and moving materials with fluid–structure interaction. The latter is especially important for lightweight materials, where the inertial contribution of the surrounding air is significant.

From the viewpoint of analytical approaches, especially interesting of the early papers are Archibald and Emslie (1958), where the relation between co-moving and laboratory coordinates is briefly discussed; Swope and Ames (1963), which goes into detail about the natural vibration frequencies of a travelling string; and Simpson (1973), where an analytical solution based on factoring the characteristic polynomial is presented for the travelling beam (but without external applied tension). Of the more recent ones, one should mention at least Kong and Parker (2004), where an approximate analytical solution is presented for the eigenfrequencies of a travelling beam with small bending rigidity.

The present paper focuses on the fundamental questions of transverse, longitudinal and torsional vibrations of a one-dimensional travelling web, modelled as a continuous elastic element. These are all described by a hyperbolic second-order partial differential equation, corresponding to the string and rod models. The critical velocities of one-dimensional webs, and the arising non-trivial solution of free vibrations, are studied analytically using a classical approach.

It will be shown that, in the framework of a quasi-static eigenvalue analysis based on Bolotin's concept of stability (Bolotin, 1963), for these models the critical point cannot be unstable. Examination of the initial postcritical behaviour based on the explicit solution of the stability exponent is seen to point to the same conclusion. This contrasts classical wisdom (e.g. Wickert and Mote, 1990), but Wang et al. (2005) have obtained a similar result for the ideal string using Hamiltonian mechanics.

The analysis will then be extended into the case where damping is included in the model. It will be shown that if external friction is present, then for mode numbers sufficiently high, dynamic critical points may exist.

Analyses of both the undamped and damped cases are presented with graphical examples of the obtained eigenvalue spectra. It will be seen that while external friction (such as the viscosity of a surrounding medium) causes the vibration to decay, internal friction in the travelling material will destabilize the system at the static critical point, leading to a dynamic type of instability there.

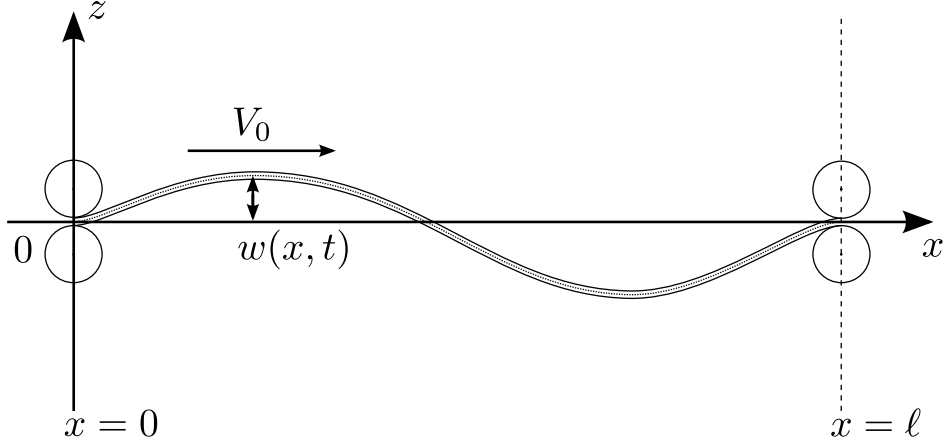


Figure 1: Thin axially moving narrow web modelled as one-dimensional elastic element (rod or string) and supported at  $x = 0$  and  $x = \ell$ . Here the transverse displacement  $w$  is shown.

## 2 Governing equations and reduction into general form

Consider a narrow thin web supported by rollers at  $x = 0$  and  $x = \ell$  (see Figure 1), modelled by a continuous one-dimensional elastic element (rod or string), having rectangular cross-section of given width and thickness.

At first, consider a small amplitude torsional vibration of the web, described by the angle function  $\vartheta(x, t)$ , which represents the angle of torsion per unit length of the rod in the segment  $0 \leq x \leq \ell$ . The dynamics of free torsional small-amplitude vibrations of a classical, stationary (non-travelling) ideal elastic rod is governed by the partial differential equation

$$\rho I_0 \frac{\partial^2 \vartheta}{\partial t^2} - G I_k \frac{\partial^2 \vartheta}{\partial x^2} = 0, \quad 0 < x < \ell \quad (1)$$

with boundary conditions

$$(\vartheta)_{x=0} = 0, \quad (\vartheta)_{x=\ell} = 0, \quad (2)$$

where  $\rho$  is the material density,  $I_0$  the moment of inertia of the web cross-section around the  $x$ -axis,  $G I_k$  is the torsional rigidity,  $G$  is the shear modulus and  $\vartheta = \vartheta(x, t)$  is the angle of torsion.

Consider now an ideal straight rod, which moves in the  $x$  direction and performs torsional vibrations. The rod is travelling at a constant velocity  $V_0$  between rollers, which are fixed at  $x = 0$  and  $x = \ell$ . We will set up the problem for the moving rod using the Euler laboratory coordinate system  $(x, t)$ , see Figure 1, and the co-moving coordinate system  $(\tilde{x}, t)$ , and the notion of the material derivative (also known as the Lagrange derivative or the total derivative). We have

$$x = \tilde{x} + V_0 t, \quad \frac{\partial \vartheta(x(\tilde{x}, t), t)}{\partial \tilde{x}} = \frac{\partial \vartheta}{\partial x} \frac{\partial x}{\partial \tilde{x}} = \frac{\partial \vartheta}{\partial x}, \quad (3)$$

$$\frac{d\vartheta}{dt} \equiv \left( \frac{\partial\vartheta}{\partial t} \right)_{\bar{x}=\text{const.}} = \frac{\partial\vartheta}{\partial t} + V_0 \frac{\partial\vartheta}{\partial x}, \quad (4)$$

$$\frac{d^2\vartheta}{dt^2} \equiv \left( \frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial x} \right) \left( \frac{\partial\vartheta}{\partial t} + V_0 \frac{\partial\vartheta}{\partial x} \right) = \frac{\partial^2\vartheta}{\partial t^2} + 2V_0 \frac{\partial^2\vartheta}{\partial x \partial t} + V_0^2 \frac{\partial^2\vartheta}{\partial x^2}. \quad (5)$$

In (4)–(5) and also in the rest of this paper,  $\partial/\partial t$  without subscript denotes partial differentiation with respect to time in the Euler coordinate system, i.e.  $\partial/\partial t \equiv (\partial/\partial t)_{x=\text{const.}}$ .

Applying transformations (3)–(5), we obtain the equation of small torsional vibrations of an axially travelling rod

$$\rho I_0 \frac{\partial^2\vartheta}{\partial t^2} + 2\rho I_0 V_0 \frac{\partial^2\vartheta}{\partial x \partial t} + \rho I_0 V_0^2 \frac{\partial^2\vartheta}{\partial x^2} - G I_k \frac{\partial^2\vartheta}{\partial x^2} = 0, \quad (6)$$

$$(\vartheta)_{x=0} = 0, \quad (\vartheta)_{x=\ell} = 0.$$

The boundary conditions are (2).

Vibrations can also arise in the longitudinal direction. If the travelling web performs small-amplitude longitudinal vibrations (independent of the torsional vibrations and superposed onto the axial travelling motion), then the dynamics of the rod can be described by the analogous equation and boundary conditions

$$\rho S \frac{\partial^2 u}{\partial t^2} + 2\rho S V_0 \frac{\partial^2 u}{\partial x \partial t} + \rho S V_0^2 \frac{\partial^2 u}{\partial x^2} - E S \frac{\partial^2 u}{\partial x^2} = 0, \quad (7)$$

$$(u)_{x=0} = 0, \quad (u)_{x=\ell} = 0, \quad (8)$$

where  $S$  is the cross-sectional area of the web,  $E$  the Young modulus of the material, and  $u = u(x, t)$  the longitudinal displacement. Defined in the Euler coordinate system, the value  $u(x, t)$  describes the longitudinal displacement, at time instant  $t$ , that the web experiences at laboratory coordinate  $x$ , compared to a corresponding state where the only axial motion is the uniform travelling motion at velocity  $V_0$ .

Yet another type of small-amplitude vibrations of an axially moving elastic web are transverse vibrations, described by the transverse displacement function  $w$ , which is governed by the following second-order partial differential equation and boundary conditions

$$m \frac{\partial^2 w}{\partial t^2} + 2m V_0 \frac{\partial^2 w}{\partial x \partial t} + m V_0^2 \frac{\partial^2 w}{\partial x^2} - T_0 \frac{\partial^2 w}{\partial x^2} = 0, \quad (9)$$

$$(w)_{x=0} = 0, \quad (w)_{x=\ell} = 0. \quad (10)$$

Here  $m \equiv \rho S$  is the mass per unit length of the web. This is the well-known one-dimensional string model of the elastic web, axially moving in the  $x$  direction, subjected to a constant tensile load  $T_0 > 0$ , and having zero bending rigidity.

The first three terms in (9) come from the second material derivative (5), and the term  $T_0 \partial^2 w / \partial x^2$  represents the restoring force of the axial tension. The first three terms physically represent, respectively, the accelerations of local inertia, the Coriolis effect, and the centrifugal effect.

Table 1: Definitions of state variable  $U$ , coefficients  $a$ ,  $b$  and  $c$ , and critical velocities  $C$ , in the three considered cases. The coefficients and critical velocities in each case satisfy the relation  $b^2 - ac = a^2 C^2$ .

Vibration type	Eq.	$U$	$a$	$b$	$c$	$C$
Torsional	(6)	$\vartheta$	$\rho I_0$	$\rho I_0 V_0$	$\rho I_0 V_0^2 - GI_k$	$\sqrt{GI_k / \rho I_0}$
Longitudinal	(7)	$u$	$\rho S$	$\rho S V_0$	$\rho S V_0^2 - ES$	$\sqrt{E / \rho}$
Transverse	(9)	$w$	$m$	$m V_0$	$m V_0^2 - T_0$	$\sqrt{T_0 / m}$

All three cases (6), (7) and (9) are of the same mathematical form. It is convenient to write them as a general second-order constant-coefficient partial differential equation

$$a \frac{\partial^2 U}{\partial t^2} + 2b \frac{\partial^2 U}{\partial x \partial t} + c \frac{\partial^2 U}{\partial x^2} = 0, \quad 0 < x < \ell. \quad (11)$$

See Table 1 for the definitions of  $U$ ,  $a$ ,  $b$  and  $c$  in each particular case.

Because the coefficients are constants, the type of equation (elliptic, parabolic or hyperbolic) holds globally regardless of  $x$  and  $t$ . The discriminants

$$D \equiv b^2 - ac \quad (12)$$

are, respectively,

$$D_\vartheta = \rho I_0 GI_k > 0, \quad D_u = \rho S^2 E > 0, \quad D_w = m T_0 > 0, \quad (13)$$

and hence the equations (6), (7) and (9) are always hyperbolic regardless of the value of axial velocity  $V_0$ .

This observation reflects physical intuition; the introduction of axial velocity should not change the basic vibrational nature of the mechanical response. Indeed, in the co-moving (Lagrange) coordinate system, the mechanics is identical to that of the stationary string or rod subjected to moving boundaries.

Equation (11) requires two initial and two boundary conditions. In the following free vibration analysis, we set only the boundary conditions, and the solution will have free parameters. Free vibration analysis is only concerned with determining possible motions of the unloaded system, i.e. the nontrivial solutions of the homogeneous partial differential equation (11).

The considered boundary conditions (2), (8) and (10) are, in each case, zero Dirichlet:

$$(U)_{x=0} = 0, \quad (U)_{x=\ell} = 0. \quad (14)$$

### 3 Mechanical response at the critical velocity

From Table 1, we observe that for the problems studied in this paper, the coefficients  $a$  and  $b$  in (11) are always positive, but the sign of  $c$  is indeterminate. The special case

where the coefficient  $c$  vanishes is of special interest. If the other problem parameters are considered fixed, this can always be realized by choosing a particular value for  $V_0$  such that the two different contributions to the coefficient  $c$  cancel exactly.

In the following, we will call this special value of  $V_0$  the critical velocity, denoted with the symbol  $C$ . See the corresponding column in Table 1 for the critical velocities in the different cases.

At the critical velocity,  $V_0 = C$ , equation (11) simplifies into (after dividing by  $a$  and noting  $b/a = V_0$ )

$$\frac{\partial^2 U}{\partial t^2} + 2C \frac{\partial^2 U}{\partial x \partial t} = 0. \quad (15)$$

The equation (15) is still globally hyperbolic, since  $C^2 > 0$ . From (15) it follows that

$$\frac{\partial}{\partial t} \left( \frac{\partial U}{\partial t} + 2C \frac{\partial U}{\partial x} \right) = 0, \quad (16)$$

and if we introduce the notation

$$v \equiv \frac{\partial U}{\partial t}, \quad (17)$$

equation (16) can be rewritten as

$$\frac{\partial v}{\partial t} + 2C \frac{\partial v}{\partial x} = 0. \quad (18)$$

Thus, the velocity-like quantity  $v$  obeys the (homogeneous) first-order transport equation along the rod or string, with the transport of this quantity occurring at constant velocity  $2C$ . Hence, we have

$$v(x, t) = g(x - 2Ct)$$

for some differentiable function  $g$ . The function  $U(x, t)$  consists of this moving component  $g(x - 2Ct)$ , and by equation (16), a second component  $f(x)$  which is constant in time:

$$U(x, t) = f(x) + g(x - 2Ct). \quad (19)$$

The boundary conditions (14) imply, by (17), that also

$$(v)_{x=0} = 0, \quad (v)_{x=\ell} = 0 \quad (20)$$

for all  $t$ . Hence, in order to avoid violation of the boundary conditions, we must have  $v(x, t) = g(x - 2Ct) \equiv 0$ , and the considered rod or string, if travelling at the critical velocity, must stay in a steady-state configuration. This very compact way to solve the critical velocity case was discussed by Wang et al. (2005).

It is possible to find the function  $f(x)$  explicitly by using a more direct approach. First, let us integrate equation (16) with respect to  $t$ , obtaining

$$\frac{\partial U}{\partial t} + 2C \frac{\partial U}{\partial x} = h(x). \quad (21)$$



Equation (21) is the standard nonhomogeneous transport equation. Its solution is (Polyanin, 2004a)

$$U(x, t) = \frac{1}{2C} \int h(x) dx + g(x - 2Ct) . \quad (22)$$

As above, this is a linear superposition of two components: a steady-state one, and one being transported toward the  $+x$  direction at velocity  $2C$ . Following the same argument about boundary conditions as above, we find  $g(x - 2Ct) \equiv 0$ . By differentiating (22) with respect to  $x$ , and then setting  $t = 0$ , we can determine

$$h(x) = 2C \frac{\partial U}{\partial x}(x, 0) \quad (23)$$

and finally, by substituting (23) back into (22), obtain

$$U(x, t) = U(x, 0) . \quad (24)$$

Hence, when the axial motion occurs at the critical velocity  $C$  given in Table 1, the initial condition for the position, namely

$$(U)_{t=0} = f_1(x) \quad (25)$$

(where  $f_1(x)$  is a given function), completely determines the solution for all  $t$ . Because the solution (24) does not allow for time-dependent changes, the other initial condition, namely

$$(v)_{t=0} \equiv \left( \frac{\partial U}{\partial t} \right)_{t=0} = f_2(x) , \quad (26)$$

must have  $f_2(x) \equiv 0$  in order to be compatible with the solution (24).

We conclude that if this compatibility holds for our initial conditions, then, upon a quasi-static transition to the limit state  $V_0 = C$ , the state variable profile of a freely vibrating axially travelling elastic element (rod or string) will “freeze” into the shape it had when the limit state was reached.

In practice, the compatibility condition  $(v)_{t=0} \equiv 0$ ,  $0 < x < \ell$  is reasonable for many physically admissible situations for the considered model near  $V_0 = C$ , because as we will see below, all eigenfrequencies of the travelling elastic element tend to zero in the limit  $V_0 \rightarrow C$ .

Nevertheless, the quasi-static analysis has its limitations. For example, consider the case where we would like to initially set  $V_0 = C - \varepsilon$  (with  $\varepsilon > 0$  small),  $(U)_{t=0} \equiv 0$ ,  $(v)_{t=0} = f_2(x) \neq 0$ , and then perform a transition to  $V_0 \rightarrow C$ . The quasi-static analysis is not applicable, because the given initial condition for  $v$  violates the compatibility condition  $f_2(x) \equiv 0$ . If this category of cases is to be analyzed, a more general treatment for the dynamics including the effects of accelerating motion (with constant  $V_0$  replaced by a function  $V = V(x, t)$ ) is required.

## 4 Transformation into canonical form

For the rest of the discussion in this paper, it is assumed that  $V_0 \neq C$ . Let us briefly go through the derivation of an analytical solution for the free vibrations of the considered travelling elastic element. We will proceed in a manner similar to Swope and Ames (1963).

Consider equation (11). A systematic way to derive the solution is to diagonalize the principal part of the operator (see e.g. Polyanin et al., 2008 or references therein). Because (11) contains only second-order derivatives, we have only the principal part to consider.

It is known that the second-order partial differential equation in two variables can always be transformed into one of the canonical forms, depending on its type. We start with the characteristic equation of (11), namely

$$a(dx)^2 - 2bdxdt + c(dt)^2 = 0. \quad (27)$$

Note that the coefficients  $a, b$  and  $c$  are constants. Because the discriminant (12) is positive for the problems under consideration, and  $c \neq 0$  (because  $V_0 \neq C$ ), equation (27) can be understood as a second-order algebraic equation in the variable  $dt/dx$ , with the real-valued solutions

$$\frac{dt}{dx} = \frac{1}{c} \left( b \pm \sqrt{b^2 - ac} \right). \quad (28)$$

From (28), we obtain

$$cdt - \left( b \pm \sqrt{b^2 - ac} \right) dx = 0,$$

and by integrating both sides,

$$\varphi_{\pm}(x, t) \equiv ct - \left( b \pm \sqrt{b^2 - ac} \right) x = \kappa_{\pm} \quad (29)$$

for the  $+$  and  $-$  terms, respectively. Assigning a value to the constants  $\kappa_{\pm}$  picks one specific curve from the family of characteristics; equation (29) represents the whole family. It is seen that the characteristics  $\varphi_{\pm}$  of equation (11) are straight lines, as expected.

Performing a change of variables (leaving  $\kappa_{\pm}$  free, taking only the left-hand side)

$$X \equiv \varphi_+(x, t), \quad Y \equiv \varphi_-(x, t), \quad (30)$$

the original equation (11) transforms into the first canonical form of the wave equation (i.e. hyperbolic second-order partial differential equation),

$$\frac{\partial^2 U}{\partial X \partial Y} = 0.$$

Changing variables again, now to  $\xi$  and  $\eta$  such that

$$X = \xi - \eta, \quad Y = \xi + \eta, \quad (31)$$

i.e.,

$$\xi = \frac{1}{2}(X - Y) , \quad \eta = \frac{1}{2}(X + Y) , \quad (32)$$

we arrive at the second canonical form of the wave equation,

$$\frac{\partial^2 U}{\partial \eta^2} - \frac{\partial^2 U}{\partial \xi^2} = 0 . \quad (33)$$

The mixed derivative has been eliminated. The form (33) is particularly convenient, because it admits a separable solution in the form  $U(\xi, \eta) \equiv g_1(\xi)g_2(\eta)$ . The standard separation technique can be used to carry out the rest of the solution process; however, there is a more compact approach that we will use below.

## 5 Analytical solution of free vibrations of travelling elastic element

In the following, we will use dimensionless variables. Let

$$\hat{x} = x/\lambda , \quad \hat{t} = t/\tau , \quad \hat{U} = U/\delta , \quad (34)$$

where the hat indicates a dimensionless quantity. Here  $\lambda$  is a characteristic length,  $\tau$  is a characteristic time, and  $\delta$  is a characteristic value of the state variable. The characteristic values are arbitrary, and are usually chosen in some convenient manner for each problem under discussion. A typical choice for the coordinate scalings is  $\lambda = \ell$ ,  $\tau = \ell/C$ , but for the moment, we will leave these scalings free to be chosen later.

As for the state variable  $\hat{U}$ , we may insert  $U = \hat{U}\delta$  into (11), and cancel  $\delta$ , since the equation is linear in  $U$  and the right-hand side is zero. Hence  $\delta$  can be dropped from further consideration.

Let us insert the dimensionless variables (34) into (11), and multiply the equation by  $\tau^2/a$ . We will omit the hat from the notation. We obtain another equation of the form (11),

$$\alpha \frac{\partial^2 U}{\partial t^2} + 2\beta \frac{\partial^2 U}{\partial x \partial t} + \gamma \frac{\partial^2 U}{\partial x^2} = 0 , \quad 0 < x < 1 , \quad (35)$$

where  $U$ ,  $x$ , and  $t$  are now the dimensionless variables (34), and the constant coefficients are

$$\alpha = 1 , \quad \beta = \frac{\tau b}{\lambda a} = \frac{\tau}{\lambda} V_0 , \quad \gamma = \frac{\tau^2 c}{\lambda^2 a} = \frac{\tau^2}{\lambda^2} (V_0^2 - C^2) . \quad (36)$$

The last forms follow from Table 1. For the dimensionless coordinates  $x$  and  $t$ , we have the coordinate transforms

$$\xi = - \left( \sqrt{\beta^2 - \gamma} \right) x , \quad \eta = \gamma t - \beta x , \quad (37)$$

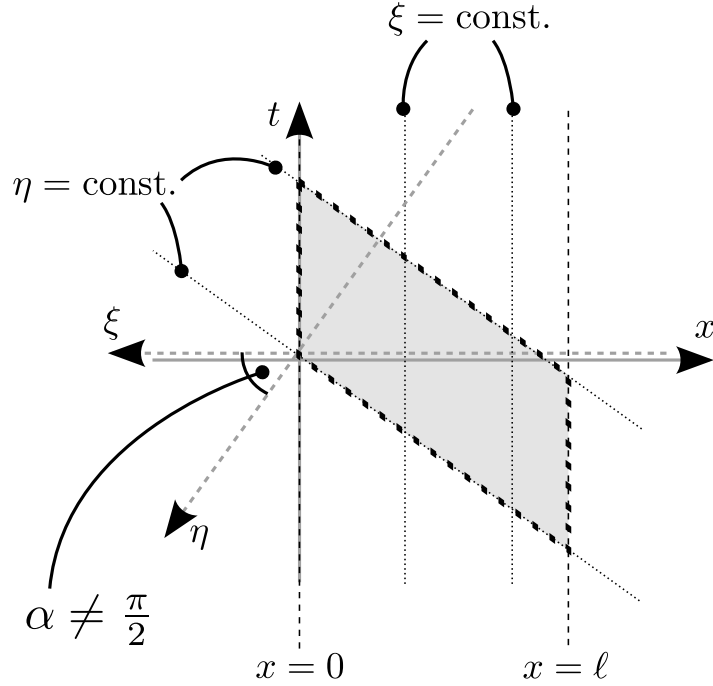


Figure 2: The  $(x, t)$  and  $(\xi, \eta)$  coordinate systems in the plane. The shaded parallelogram indicates an area delimited by constant values for the  $(\xi, \eta)$  coordinates. On the edges of such shapes, boundary conditions (but not initial conditions) can be easily enforced by a solution separable in  $(\xi, \eta)$ .

where (30)–(32) and  $\alpha = 1$  have been used. The transforms (37) bring (35) directly into the form (33). Lines of constant  $x$  transform into lines of constant  $\xi$ , so a suitable separable solution in  $(\xi, \eta)$  coordinates will satisfy boundary conditions at constant  $x$ .

It should be mentioned that this solution approach is only useful for free vibration analysis, because initial conditions cannot be easily enforced. This is due to the fact that  $\eta$  is a linear combination of  $x$  and  $t$ , and hence lines of constant  $\eta$  are not parallel to either of the axes in the original  $(x, t)$  coordinate system. See Figure 2.

A compact way to find the free vibration solution of (33) in the transformed  $(\xi, \eta)$  coordinates is to use the standard complex-valued time-harmonic trial function

$$U(\xi, \eta) = \exp(s\eta)W(\xi) . \quad (38)$$

Substituting (38) into (33) obtains

$$s^2W - W'' = 0 , \quad \text{i.e.,} \quad W'' = s^2W . \quad (39)$$

This is a linear eigenvalue problem for  $(s^2, W)$ , where  $s^2$  is the eigenvalue and  $W \equiv W(\xi)$  is the eigenfunction. Considering (39) and the corresponding boundary conditions, we see that the eigenfunction must be

$$W(\xi) = \sin(\omega\xi) . \quad (40)$$

To obtain  $\omega$ , recall the coordinate transform (37) between  $x$  and  $\xi$ . Observe that

$$\beta^2 - \gamma = \frac{\tau^2}{\lambda^2 a^2} (b^2 - ac) , \quad b^2 - ac = a^2 C^2 ,$$

which leads to

$$\sqrt{\beta^2 - \gamma} = \frac{C\tau}{\lambda} . \quad (41)$$

In order for (40) to satisfy the boundary conditions, for dimensional  $x = \ell$ , i.e. dimensionless  $x = \ell/\lambda$ , we must have  $\omega\xi = k\pi$  for the corresponding  $\xi$ , where  $k$  is a free nonzero integer. In other words, for the  $k$ th vibration mode it must hold that

$$\omega = -\frac{k\pi}{\left(\sqrt{\beta^2 - \gamma}\right) \cdot (\ell/\lambda)} = -\frac{k\pi\lambda}{C\tau} \cdot \frac{\lambda}{\ell} = -\frac{k\pi\ell}{C\tau} . \quad (42)$$

The last form is obtained by choosing the characteristic length as  $\lambda = \ell$ . Furthermore, in order to satisfy the partial differential equation (33), we must have

$$s^2 = -\omega^2 , \quad \text{i.e.,} \quad s = \pm i\omega . \quad (43)$$

Thus, the complex-valued solution for the  $k$ th mode, expressed in  $(\xi, \eta)$  coordinates is

$$U(\xi, \eta) = \exp(s_k \eta) \sin(\omega_k \xi) , \quad (44)$$

where the subscript  $k$  denotes the dependence on the mode number  $k$  of both the exponent  $s = s_k$  and the mode shape parameter  $\omega = \omega_k$ .

By transforming back using (43), (42) and (37), we obtain the solution in the dimensionless  $(x, t)$  coordinates. To make the solution easier to follow, note that by using (36) and  $\lambda = \ell$ , equation (35) can be rewritten as

$$\frac{\partial^2 U}{\partial t^2} + 2V_0 \frac{\tau}{\ell} \frac{\partial^2 U}{\partial x \partial t} + (V_0^2 - C^2) \frac{\tau^2}{\ell^2} \frac{\partial^2 U}{\partial x^2} = 0 , \quad 0 < x < 1 . \quad (45)$$

The complex-valued solution for the  $k$ th vibration mode, with the considered zero Dirichlet boundary conditions at  $x = 0$  and  $x = 1$ , is

$$U(x, t) = E \exp \left( \pm i \frac{k\pi\ell}{C\tau} \left[ \left( V_0^2 - C^2 \right) \frac{\tau^2}{\ell^2} t - V_0 \frac{\tau}{\ell} x \right] \right) \sin(k\pi x) , \quad (46)$$

where the  $\pm$  corresponds to the  $\pm$  in (43),  $i \equiv \sqrt{-1}$ , and the constant coefficient  $E$  is free. As will be observed below, the real and imaginary parts of (46) are real-valued solutions.

It is seen that there is an exp factor in the solution (46), and it admits separation in  $x$  and  $t$ . That is,  $U(x, t) = \chi_1(x)\chi_2(t)$  for some functions  $\chi_1$  and  $\chi_2$ ; specifically, the form of both functions is exp.

Thus, equation (46) represents time-harmonic behaviour, and we can apply Bolotin's approach for determining the dynamic stability (Bolotin, 1963). The stability exponent of the  $k$ th mode is the coefficient of  $t$  in the exp. For the  $k$ th vibration mode, we have

$$s_k^* = \pm i \frac{k\pi\tau}{C\ell} (V_0^2 - C^2) . \quad (47)$$

We see that if  $V_0 \rightarrow C$  (or  $V_0 \rightarrow -C$ ),  $s_k^* \rightarrow 0$  for all  $k$ . Hence the free vibration solution agrees with the steady state solution obtained for the special case.

For all  $V_0 \neq C$ , the stability exponent  $s_k^*$  is always pure imaginary. Thus, the axially travelling elastic element will undergo undamped harmonic vibrations, as expected. The fact that this is the case also for any  $|V_0| > C$  suggests that, at least when judged by the initial postcritical behaviour, the critical points  $V_0 = \pm C$  for the considered models are neutrally stable.

The star in the symbol  $s_k^*$  is a reminder of the fact that (47) is expressed in the original (dimensionless)  $(x, t)$  coordinate system, instead of the transformed  $(\xi, \eta)$  coordinates used in (43). The exponent  $s_k^*$  is the one that determines the elastic stability behaviour. Because  $\eta = \gamma t - \beta x$ , and no additional transformations were needed to produce the solution  $U(x, t)$ , the relation between the stability exponents in the two coordinate systems is simply  $s_k^* = \gamma s_k$ .

Now consider the problem (11), (14), augmented with some initial conditions matching a free vibration solution, as an initial boundary value problem. Because a purely time-harmonic vibration ( $\text{Re } s_k^* \equiv 0$ ) never exceeds its original maximum amplitude (defined as  $M \equiv \max |U|$  taken over one period), we conclude that if the original maximum amplitude at  $V_0 = 0$  was "small" ( $M = \varepsilon$  for  $\varepsilon > 0$  small), so is the maximum amplitude at every  $V_0$ , as  $V_0$  is increased quasi-statically. This holds especially in the limit of the dynamic free vibration solution as  $V_0 \rightarrow C$  (and also as  $V_0 \rightarrow -C$ ). On the other hand, we have seen that the limit solution is a steady state. Because the limit solution does not depend on time, it cannot become unstable in the Bolotin sense.

Gathering these facts together, we conclude that by the performed quasi-static analysis, the critical point of the presently considered model at  $V_0 = \pm C$  cannot be unstable. This contrasts classical wisdom (e.g. Wickert and Mote, 1990), but Wang et al. (2005) have obtained a similar result for the ideal string via a completely different route; their argument was based on Hamiltonian mechanics.

This result is of fundamental theoretical importance. However, since no practical physical system has exactly zero bending rigidity, its main practical relevance is as an illuminating example underlining the significance of including in the model all relevant aspects of system behaviour; in this case, bending rigidity, however small.

If one is interested only in an approximate value for the critical velocity for materials with small bending rigidity, the string model considered here is already sufficient. The introduction of small bending rigidity has no major effect on the value of the critical velocity (see e.g. Kong and Parker, 2004 for an analytical approach), even though it completely changes the qualitative behaviour of the model around the critical point.

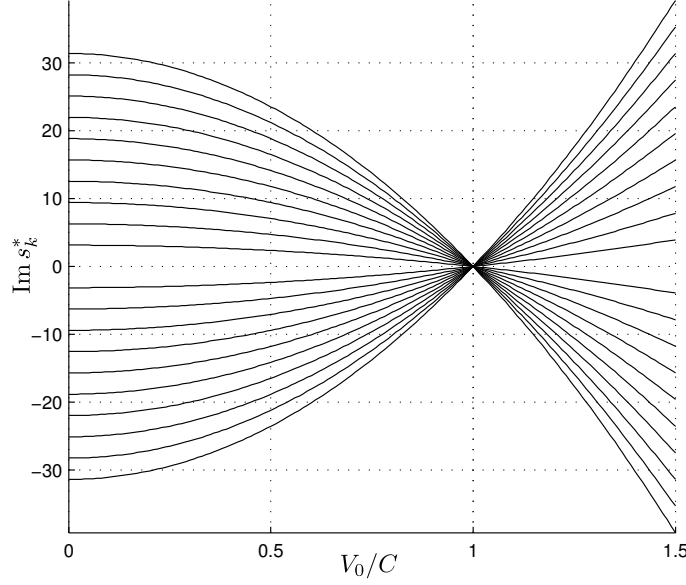


Figure 3: Eigenvalues of the axially moving ideal one-dimensional web, up to vibration mode  $k = 10$ . Analytical result, equation (47).  $\text{Re } s_k^* \equiv 0$  for all  $k$  and  $V_0$ . At the critical point  $V_0 = C$ ,  $\text{Im } s_k^* = 0$  for all  $k$ .

## 6 Example and discussion

A plot of the eigenfrequency spectrum described by equation (47), as a function of the axial velocity  $V_0$ , is provided in Figure 3 up to mode number  $k = 10$ . The parameter values correspond to the model of a narrow panel in the membrane limit with  $T_0 = 500 \text{ N/m}$ ,  $m = 0.08 \text{ kg/m}^2$ , and  $\ell = 1 \text{ m}$ . The characteristic time  $\tau$  was chosen as  $\tau = \ell/C$ , which leads to  $\sqrt{\beta^2 - \gamma} = 1$ , see equation (41).

We observe that the eigenvalues (47) come in pairs. This is a common feature of undamped gyroscopic systems, and is due to the structure of the characteristic equation. If  $s$  is an eigenvalue of such a system, then  $-s$  is also an eigenvalue. For the relation of axially moving materials to classical gyroscopic systems, see Wickert and Mote (1990).

From (46), the real and imaginary parts of the solution follow easily, by using the fact  $\exp(a + bi) = \exp(a) \exp(bi)$  and de Moivre's formula  $\exp(i\phi) = \cos \phi + i \sin \phi$ . We have

$$\text{Re } U(x, t) = E \cos \left( \pm \frac{k\pi\ell}{C\tau} \left[ (V_0^2 - C^2) \frac{\tau^2}{\ell^2} t - V_0 \frac{\tau}{\ell} x \right] \right) \sin(k\pi x), \quad (48)$$

$$\text{Im } U(x, t) = E \sin \left( \pm \frac{k\pi\ell}{C\tau} \left[ (V_0^2 - C^2) \frac{\tau^2}{\ell^2} t - V_0 \frac{\tau}{\ell} x \right] \right) \sin(k\pi x). \quad (49)$$

That these are indeed real-valued solutions of (45) can be easily verified by substitution. By comparing (48) and (49), it is obvious that their only difference is a phase

shift with regard to the variable  $\eta = \gamma t - \beta x$ , because  $\cos \phi = \sin(\phi + \pi/2)$  for all  $\phi \in \mathbb{R}$ . Observe that while the original complex-valued solution (46) is separable in  $x$  and  $t$ , i.e.  $U(x, t) = \chi_1(x)\chi_2(t)$ , its real and imaginary parts are not.

To finish this section, let us list some alternative representations for the solution, and observe certain properties from them. A classical way to solve the free vibration problem with constant coefficients is to directly substitute  $U(x, t) = \exp(st)W(x)$  into (35), leading to a homogeneous ordinary differential equation in  $W(x)$ ,

$$s^2W + 2\beta sW' + \gamma W'' = 0. \quad (50)$$

Equation (50) with the boundary conditions  $W(0) = W(1) = 0$  is then solved. One way is to substitute  $W(x) = \exp(qx)Z(x)$  with constant  $q$ , and then choose the value of  $q$  such that the substitution eliminates the first-order term. The resulting equation for the new unknown  $Z(x)$  then becomes similar to (39), which is easily solved.

Once the general solution is obtained, the admissible values of  $s$  are determined from the condition that a nontrivial solution  $W(x) \neq 0$  must be possible. If the two linearly independent solutions of (50) are denoted  $W_1(x)$  and  $W_2(x)$ , the zero Dirichlet boundary conditions require that

$$\begin{aligned} W(0) = a_1W_1(0) + a_2W_2(0) = 0, \\ W(1) = a_1W_1(1) + a_2W_2(1) = 0 \end{aligned} \quad \text{i.e.} \quad \begin{bmatrix} W_1(0) & W_2(0) \\ W_1(1) & W_2(1) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0,$$

which, as is well known, can have nontrivial solutions  $(a_1, a_2) \neq (0, 0)$  if and only if the determinant vanishes,

$$W_1(0)W_2(1) - W_2(0)W_1(1) = 0.$$

The end result is

$$W(x) = E [\exp(-ik\pi [1 + \nu] x) - \exp(+ik\pi [1 - \nu] x)] , \quad 0 < x < 1 , \quad (51)$$

where

$$\nu \equiv \frac{V_0}{C}, \quad (52)$$

the complex-valued amplitude coefficient  $E$  is free, and  $x$  is the dimensionless space coordinate. From (51), it is readily apparent that if  $V_0 = 0$ , the space component reduces to  $\sin(k\pi x)$ , by choosing  $E = -1/2i$  and again applying de Moivre's formula. Nonzero  $V_0$  introduces a shift into the exponent in both terms, and in this general case, the space component  $W(x)$  can no longer be reduced into a real-valued form.

After applying some trigonometric identities, (51) can be rewritten as

$$W(x) = F \sin(k\pi x) \exp(-ik\pi \nu x) , \quad 0 < x < 1 , \quad (53)$$

where the coefficient  $F = -2iE$ . From this representation, a velocity-dependent phase shift along the web (as reported e.g. by Wickert and Mote, 1990) is apparent.



When the time component is included into (51), the full solution is

$$\begin{aligned}
U(x, t) &= E \exp(iC \frac{\tau}{\ell} [\nu^2 - 1] k\pi t) [\exp(-ik\pi [1 + \nu] x) - \exp(+ik\pi [1 - \nu] x)] \\
&\equiv E \exp(i\tilde{C}t) [\exp(i\tilde{A}x) - \exp(i\tilde{B}x)] \\
&= E [\exp(i[\tilde{A}x + \tilde{C}t]) - \exp(i[\tilde{B}x + \tilde{C}t])] \\
&= E [\exp(i \langle (x, t), (\tilde{A}, \tilde{C}) \rangle) - \exp(i \langle (x, t), (\tilde{B}, \tilde{C}) \rangle)] , \tag{54}
\end{aligned}$$

where we have defined the constants

$$\tilde{A} = -k\pi [1 + \nu] , \quad \tilde{B} = +k\pi [1 - \nu] , \quad \tilde{C} = C[\tau/\ell][\nu^2 - 1]k\pi ,$$

and  $\langle \cdot, \cdot \rangle$  denotes the usual scalar product. From this form it is easily seen that when considered in the space-time plane, the complex-valued solution is a superposition of two plane waves with different directions of propagation.

From (54) it is also directly obvious why the real and imaginary parts of the solution can satisfy equation (35) separately. The differentiation in the Coriolis term is of the second order when considered in the  $(x, t)$  plane, which conveniently eliminates the imaginary unit that would result from first-order differentiation of (54) with respect to only either  $x$  or  $t$ .

An alternative and more general way to make this final observation is as follows. Let us write the complex-valued function as  $U = \text{Re}(U) + i\text{Im}(U)$ . For any linear differential operator  $\mathcal{L}$  with real coefficients, observe that

$$\begin{aligned}
\text{Re}(\mathcal{L}(U)) &= \text{Re}[\mathcal{L}(\text{Re}(U) + i\text{Im}(U))] \\
&= \text{Re}[\mathcal{L}(\text{Re}(U)) + i\mathcal{L}(\text{Im}(U))] \\
&= \mathcal{L}(\text{Re } U) \tag{55}
\end{aligned}$$

and similarly for the imaginary part. For example, for (35),  $\mathcal{L}$  is given by

$$\mathcal{L}(U) \equiv \alpha \frac{\partial^2 U}{\partial t^2} + 2\beta \frac{\partial^2 U}{\partial x \partial t} + \gamma \frac{\partial^2 U}{\partial x^2} ,$$

and we observe that in this case the requirement of real-valued coefficients holds. Similarly, the requirement cannot hold for (50), where  $\mathcal{L}$  is given by

$$\mathcal{L}(W) \equiv s^2 W + 2\beta s \frac{\partial W}{\partial x} + \gamma \frac{\partial^2 W}{\partial x^2} ,$$

because  $s_k^*$  is imaginary by equation (47).

## 7 Travelling elastic element with friction

Internal or external friction is often modelled by adding first-order damping terms to equation (35). See e.g. Vaughan and Raman (2010) for the case of a travelling

plate, where the external friction caused by the viscosity of the surrounding medium plays a role.

It is fairly simple to generalize the above analysis to cover the case with damping. We will also add a linear reaction term (elastic foundation of the classical Winkler type) to the equation, because as we will see, the solution process will in any case generate one. It is instructive to see how the generated reaction term relates to the original one, if one was originally present. Consider the partial differential equation

$$a \frac{\partial^2 U}{\partial t^2} + 2b \frac{\partial^2 U}{\partial x \partial t} + c \frac{\partial^2 U}{\partial x^2} + A_1 \frac{\partial U}{\partial t} + A_2 \frac{\partial U}{\partial x} + BU = 0, \quad 0 < x < \ell. \quad (56)$$

Equation (56) follows the form of (35), but with damping and linear reaction terms added. It is the general constant-coefficient second-order autonomous homogeneous partial differential equation in two variables.

External friction in its most basic form appears as a nonzero value for  $A_1$ . This occurs, for example, when the travelling elastic element vibrates in a stationary surrounding medium. If the surrounding medium is in axial motion, independent of the axial velocity  $V_0$  of the travelling material, we have the case where both  $A_1$  and  $A_2$  are nonzero.

If the travelling material experiences internal friction, we will have  $A_2 = V_0 A_1$ , because internal phenomena operate in the co-moving coordinate system. When we look at the travelling material in the Euler coordinate system, the  $d/dt \equiv (\partial/\partial t)_{\bar{x}=\text{const}}$  (related to internal friction) will be transformed into the material derivative, as was discussed at the beginning (equation (4)).

As above, we will use the zero Dirichlet boundary conditions for  $U$ , (14). Again, let us introduce dimensionless variables and multiply (56) by  $\tau^2/a$  to obtain

$$\alpha \frac{\partial^2 U}{\partial t^2} + 2\beta \frac{\partial^2 U}{\partial x \partial t} + \gamma \frac{\partial^2 U}{\partial x^2} + \psi_1 \frac{\partial U}{\partial t} + \psi_2 \frac{\partial U}{\partial x} + \psi_3 U = 0, \quad 0 < x < 1, \quad (57)$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are given by (36) and

$$\psi_1 = A_1 \frac{\tau}{a}, \quad \psi_2 = A_2 \frac{\tau^2}{\lambda a}, \quad \psi_3 = B \frac{\tau^2}{a}. \quad (58)$$

Equation (57) can be reduced to the Klein–Gordon equation, which can be solved using the same techniques as above. We begin by eliminating the mixed second derivative, as was done for (35) earlier. It is easy to confirm that the coordinate transformation (37) works also here. The first-order terms become, by the chain rule,

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial \eta} \frac{\partial \eta}{\partial t} + \frac{\partial U}{\partial \xi} \frac{\partial \xi}{\partial t} = \gamma \frac{\partial U}{\partial \eta}, \quad (59)$$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial U}{\partial \xi} \frac{\partial \xi}{\partial x} = -\beta \frac{\partial U}{\partial \eta} - \sqrt{\beta^2 - \gamma} \frac{\partial U}{\partial \xi}. \quad (60)$$

Note that  $\partial U/\partial x$  will contribute to both  $\partial U/\partial \eta$  and  $\partial U/\partial \xi$ . By defining

$$\varkappa_1 \equiv \psi_1 \gamma - \psi_2 \beta, \quad \varkappa_2 \equiv \psi_2 \sqrt{\beta^2 - \gamma}, \quad (61)$$

we have

$$\frac{\partial^2 U}{\partial \eta^2} - \frac{\partial^2 U}{\partial \xi^2} + \varkappa_1 \frac{\partial U}{\partial \eta} - \varkappa_2 \frac{\partial U}{\partial \xi} + \psi_3 U = 0. \quad (62)$$

The signs of  $\varkappa_1$  and  $\varkappa_2$  have been chosen so that the signs of the first-order terms match the second-order ones. Next, to remove the first-order terms, we use the technique that reduces the telegraph equation into the Klein–Gordon equation (Polyanin, 2004b). Let

$$U(\xi, \eta) \equiv \exp(p\eta)Y(\xi, \eta)$$

with  $p = -\varkappa_1/2$ , and  $Y(\xi, \eta)$  an unknown function to be determined. This eliminates  $\partial U/\partial \eta$ . A similar substitution can be used to eliminate  $\partial U/\partial \xi$ . The result is

$$U(\xi, \eta) \equiv \exp\left(-\frac{1}{2}[\varkappa_1 \eta + \varkappa_2 \xi]\right) Z(\xi, \eta), \quad (63)$$

and (62) becomes the Klein–Gordon equation for the unknown function  $Z(\xi, \eta)$ ,

$$\frac{\partial^2 Z}{\partial \eta^2} - \frac{\partial^2 Z}{\partial \xi^2} = rZ, \quad (64)$$

with

$$r \equiv \frac{1}{4}(\varkappa_1^2 - \varkappa_2^2 - 4\psi_3). \quad (65)$$

If  $r = 0$ , the rest of the solution process reduces to the undamped case analyzed further above. We will assume that  $r \neq 0$ . The final task is to find a free vibration solution for (64). We will again use the classical time-harmonic trial function

$$Z(\xi, \eta) = \exp(s\eta)G(\xi), \quad (66)$$

obtaining

$$s^2 G - G'' = rG, \quad \text{i.e.} \quad G'' = (s^2 - r)G. \quad (67)$$

Equation (67) differs from the previous eigenvalue problem only in that the eigenvalue has shifted from  $s^2$  to  $s^2 - r$ . Thus, by (67) and the boundary conditions (14) (which are inherited by  $G(\xi)$ ) we again have

$$G = \sin(\omega \xi), \quad (68)$$

where  $\omega$  is to be determined using (14). By the original argument, and again choosing  $\lambda = \ell$ , we have

$$\omega = -\frac{k\pi\ell}{C\tau},$$

where  $k$  is any nonzero integer. Substituting (68) into (67) produces

$$-\omega^2 G = (s^2 - r)G,$$

and hence

$$s^2 = r - \omega^2, \quad \text{i.e.} \quad s = \pm \sqrt{r - \omega^2}. \quad (69)$$

Similarly as before,

$$s^2 = r - \frac{k^2 \pi^2 \ell^2}{C^2 \tau^2}, \quad (70)$$

and

$$Z(\xi, \eta) = \exp(s\eta) \sin(\omega\xi). \quad (71)$$

Before we can make any elastic stability conclusions, we must transform back to  $(x, t)$  coordinates by using (71), (70), (65), (63), (61), (42) and the coordinate transformations (37).

Let us extract the stability exponent only. We start by combining (71) and (63),

$$U(\xi, \eta) = \exp\left(-\frac{1}{2}[\varkappa_1 \eta + \varkappa_2 \xi]\right) \exp(s\eta) \sin(\omega\xi),$$

which separates as

$$U(\xi, \eta) = \exp\left(\left[s - \frac{1}{2}\varkappa_1\right] \eta\right) \cdot \exp\left(-\frac{1}{2}\varkappa_2 \xi\right) \sin(\omega\xi).$$

Due to the fact that we use the coordinate transformations (37), only the part involving  $\eta$  will contribute to the component that will eventually involve  $t$ . Because again we have  $\eta = \gamma t - \beta x$ , we see that for this problem the coefficient of  $t$  in the exp in the final complex-valued solution in  $(x, t)$  coordinates is

$$s_* = \gamma \left[s - \frac{1}{2}\varkappa_1\right] = \gamma \left[\pm \sqrt{r - \omega^2} - \frac{1}{2}\varkappa_1\right], \quad (72)$$

where the last form follows from (69),  $r$  is given by (65),  $\varkappa_1$  by (61),  $\omega$  by (42), and  $\gamma$  by (36).

Again, the star indicates that (72) refers to the stability behaviour in  $(x, t)$  coordinates instead of the transformed  $(\xi, \eta)$  coordinates. Due to the  $-\varkappa_1/2$  term, we see a global (but  $V_0$ -dependent) shift along the real axis for all eigenvalues. This is typical for damped systems (see Bolotin, 1963). As before, the eigenvalues come in pairs, corresponding to the choice of sign in the  $\pm$ , but due to the shift term  $-\varkappa_1/2$ , the pairs are no longer of the type  $\pm s$ . Instead, each pair is centered on the shift value, with a pure real or pure imaginary offset depending on the sign of  $r - \omega^2$ .

The contribution of the damping term is always purely real. Let us investigate the external friction case, where the damping coefficients  $A_1$  and  $A_2$  do not depend on  $V_0$ .

In the subcritical regime,  $\gamma < 0$ , and if  $A_1 > 0$  and  $A_2 \geq 0$ , then  $\varkappa_1 < 0$  for positive  $V_0$  (see equations (61) and (36)). If  $A_2 = 0$ , this holds also for negative  $V_0$ . If  $A_2 > 0$ , negative  $V_0$  will reduce the magnitude of  $\varkappa_1$ , but for  $V_0$  sufficiently near the origin,  $\varkappa_1$  will remain negative. Hence  $\gamma\varkappa_1 > 0$ , and  $-\gamma\varkappa_1 < 0$ , at least near  $V_0 = 0$ . Thus for many practically interesting cases,  $-\gamma\varkappa_1 < 0$ . If, in addition,  $r < \omega^2$  (we

will investigate this condition below), then the square root in (72) is imaginary, and (72) represents exponentially decaying time-harmonic vibrations.

In the limit  $V_0 \rightarrow C$  (or  $V_0 \rightarrow -C$ ), we have  $s_* \rightarrow 0$  just as for the undamped system, which suggests the existence of a steady state there. However, the present analysis is only applicable if  $\gamma \neq 0$ , i.e.,  $|V_0| \neq C$ , because at that point, the  $\partial^2 U / \partial x^2$  term vanishes from (57) and the coordinate transformation (37) becomes invalid. To determine the situation at  $|V_0| = C$ , a separate analysis for the case  $\gamma = 0$  would be needed, as was done above for the undamped case.

Note the dependence of the quantities in (72) on the axial velocity:  $\gamma = \gamma(V_0)$ ,  $\varkappa_1 = \varkappa_1(\beta, \gamma) = \varkappa_1(V_0)$ , and  $r = r(V_0)$ . Using (65), (61) and (36), the function  $r(V_0)$  can be explicitly expanded as

$$\begin{aligned} r(V_0) &= \frac{1}{4} \left( \varkappa_1^2 - \varkappa_2^2 - 4\psi_3 \right) \\ &= \frac{\tau^2}{4\ell^2} \left[ V_0^2 - C^2 \right] \left[ \psi_1^2 \left( V_0^2 - C^2 \right) \frac{\tau^2}{\ell^2} - 2\psi_1\psi_2 V_0 \frac{\tau}{\ell} + \psi_2^2 \right] - \psi_3, \end{aligned} \quad (73)$$

which is a fourth-order polynomial in  $V_0$ . Expression (73) is written for the case of external friction, where  $A_1$  and  $A_2$  (hence also  $\psi_1$  and  $\psi_2$ ) do not depend on  $V_0$ . In the case of internal friction, substituting  $\psi_2 = V_0 [\tau / \ell] \psi_1$  obtains the final form.

The quantity  $\omega = \omega(k)$  does not depend on the velocity, but it depends on the vibration mode number  $k$ . Thus each mode  $k$  has a different base frequency at  $V_0 = 0$ , as expected. These frequencies also differ from the ones of the undamped system; if  $r > 0$ , the frequencies of the damped system are lower than those of the undamped one; and if  $r < 0$ , higher.

If  $r(V_0) < \omega(k)^2$ , the quantity under the square root in (72) is negative, and hence the contribution of the square root is purely imaginary (representing time-harmonic vibrations). From (42), we see that  $|\omega(k)|$  increases monotonically as the mode number  $k$  increases. Thus at  $V_0 = 0$ , the criterion  $r(V_0 = 0) < \omega(k)^2$  is most easily violated for the first few modes. If such an integer  $k_0$  exists that for all  $k = 1, \dots, k_0$  it occurs that  $\omega(k)^2 < r(V_0 = 0)$ , these modes will not vibrate, but will either just decay or grow exponentially, depending on the sign of the resulting  $s_*$ . The sign of  $s_*$  is also affected by the sign and relative magnitude of  $\varkappa_1$ .

The other possibility for the criterion  $r(V_0) < \omega(k)^2$  to be violated is when  $|V_0|$  is large, due to the  $\varkappa_1^2$  term in  $r(V_0)$ , which (in the case of external friction) leads to a fourth-degree polynomial with a positive leading coefficient. This suggests that asymptotically, we will have  $\text{Im } s_* = 0$  for large  $|V_0|$ .

## 8 Stability analysis of the system with damping

Let us first show that the damped system may have steady-state solutions only at  $V_0 = \pm C$ . Upon expanding (72) by using (65), (61), (42) and (41), we have

$$s_* = \gamma f^\pm(V_0; k), \quad (74)$$

where

$$f^\pm(V_0; k) \equiv \pm \sqrt{\frac{1}{4} \left[ (\psi_1 \gamma - \psi_2 \beta)^2 - (\psi_2 C \frac{\tau}{\ell})^2 - 4\psi_3 \right] - \frac{k^2 \pi^2 \ell^2}{\tau^2 C^2} - \frac{1}{2} (\psi_1 \gamma - \psi_2 \beta)} . \quad (75)$$

Expression (74) may be zero if  $\gamma = 0$  ( $V_0 = \pm C$ ), or if  $f^\pm = 0$ . Equation (75) defines a set of two functions, each belonging to a different eigenvalue  $s_*$ .

Because the term outside the square root is always real-valued for real  $V_0$ , we see that  $f^\pm = 0$  is possible only if the square root term is real. Thus, a steady-state solution can only exist if the square root term is nonnegative,  $r(V_0) - \omega(k)^2 \geq 0$ . The sign of the square root term depends only on the sign chosen in the  $\pm$ .

We observe that (75) is of the form

$$f^\pm(a_1, a_2) = \pm \sqrt{a_1^2 + a_2} - a_1 , \quad (76)$$

where  $a_1 \equiv \varkappa_1/2 = (\psi_1 \gamma - \psi_2 \beta)/2$ , and we have collected the rest of the terms under the square root into  $a_2$ . If  $a_1 > 0$ , then  $f^\pm = 0$  is possible only for  $f^+$ , and additionally then it must hold that  $a_2 = 0$ , which (in general) is not the case here. Similarly, if  $a_1 < 0$ , then  $f^\pm = 0$  is possible only for  $f^-$ , and again we must have  $a_2 = 0$ , which (in general) does not occur. In either case,  $f^\pm = 0$  has no solution.

Finally, if  $r(V_0) - \omega(k)^2 < 0$ , the square root becomes imaginary and (75) has no (real) solution. Therefore, in either case, we conclude that equation (75) has no (real) solutions at all in terms of  $V_0$ . We have thus shown that for the damped system,  $s_*$  may pass through the origin only at  $V_0 = \pm C$ .

This also completes the stability analysis of the case  $r(V_0) - \omega(k)^2 \geq 0$ , where the whole expression (74) is real-valued. At parameter values where  $r(V_0) - \omega(k)^2 < 0$ , dynamic critical points may still exist. The question of finding such critical points becomes that of determining the critical values of  $V_0$  where the real part  $\text{Re } s_*$  becomes zero.

When  $r(V_0) - \omega(k)^2 < 0$ , the square root term in (75) is purely imaginary, and it will not affect the real part. We have

$$\begin{aligned} \text{Re } s_* &= -\frac{\gamma}{2} (\psi_1 \gamma - \psi_2 \beta) \\ &= -\frac{\tau^2}{2\ell^2} [V_0^2 - C^2] \left[ \left( \psi_1 \frac{\tau^2}{\ell^2} \right) V_0^2 + \left( -\psi_2 \frac{\tau}{\ell} \right) V_0 + \left( -\psi_1 C^2 \frac{\tau^2}{\ell^2} \right) \right] , \end{aligned} \quad (77)$$

where we have used (36) and  $\lambda = \ell$ . We have obtained a fourth-order polynomial in  $V_0$ . Two of its zeroes are, obviously,  $V_0 = \pm C$ . This is as expected; if  $s_* = 0$ , then also  $\text{Re } s_* = 0$ .

For the case with external friction ( $\psi_2$  independent of  $V_0$ ), solving for the zeroes of the second expression in brackets produces

$$V_0 = V_0^{\text{dyn}\pm} \equiv M \pm \sqrt{M^2 + C^2} , \quad (78)$$

where

$$M \equiv \frac{\psi_2 \ell}{2\psi_1 \tau} = \frac{A_2}{2A_1} . \quad (79)$$

The values of  $V_0$  given by (78) are the dynamic critical points. These are the only other points, in addition to the static critical points  $V_0 = \pm C$ , where  $\text{Re } s_* = 0$  may occur for the damped system with external friction.

If  $A_2 = 0$ , as is the case for friction caused by a stationary external medium, we observe from (78) that then the dynamic critical points coincide with the static ones at  $V_0 = \pm C$ . If  $A_2 \neq 0$ , they will be distinct.

The existence of any distinct dynamic critical points is conditional on the validity of (77) as a representation for  $\text{Re } s_*$  at the particular values of  $V_0$  given by (78). The only exception is if the dynamic critical points coincide with the static ones, because the static critical points always exist by the analysis above.

If the dynamic critical points exist, their location on the  $V_0$  axis depends on the ratio of the damping coefficients, and on those problem parameters which affect the value of  $C$ . It does not depend on the mode number  $k$ , because (77) does not involve  $\omega(k)$ .

However, it may occur that the dynamic critical points do not exist for all modes, but only for mode numbers higher than some limiting number. Because  $|\omega(k)|$  increases monotonically with increasing  $|k|$ , it follows that for any given value of  $V_0$ , we can always choose  $|k_{\min}|$  such that the condition  $r(V_0) - \omega(k)^2 < 0$  (with the restriction  $|k| \geq |k_{\min}|$ ) is satisfied at the given  $V_0$ .

Summarizing, the system with external friction has two kinds of critical points: the static ones at  $V_0 = \pm C$ , which are independent of mode number, and up to two dynamic ones. The dynamic critical points may only exist for sufficiently high mode numbers, but their location on the  $V_0$  axis is the same for all modes for which they exist.

Now let us consider internal friction. We insert  $\psi_2 = V_0 [\tau/\ell] \psi_1$  into (77), obtaining (after cancellation)

$$\text{Re } s_* = \frac{\psi_1 C^2 \tau^4}{2\ell^4} \left[ V_0^2 - C^2 \right], \quad (80)$$

which is valid whenever  $r(V_0) - \omega(k)^2 < 0$ . When this condition holds, we see that the real parts of all modes coincide, because (80) does not depend on  $k$ . It is seen that for the system with internal friction, there are no dynamic critical points; the only zeroes of (80) are  $V_0 = \pm C$ .

We observe from (80) that for  $V_0^2 < C^2$ , the real parts are all negative, while for  $V_0^2 > C^2$  (provided that the validity condition  $r(V_0) - \omega(k)^2 < 0$  still holds), the real parts are positive. This strongly suggests that for the system with internal friction, the static critical points  $V_0 = \pm C$  are unstable.

Finally, even if  $r(V_0) - \omega(k)^2 > 0$ , we see that the real parts  $\text{Re } s_*$  in the internal friction case will be centered on the value (80), separated from it by plus and minus the contribution of the square root term that has become real-valued.

One final remark concerning both external and internal friction cases is in order. We have determined all zeroes of the real part  $\text{Re } s_*$  for the damped system. It is obvious that  $s_*$  is continuous with respect to  $V_0$ . Hence,  $\text{Re } s_*$  can change sign only at its zeroes.

Therefore, if we determine  $s_*$  at  $V_0 = 0$  (obtaining the sign of  $\text{Re } s_*$  in the initial

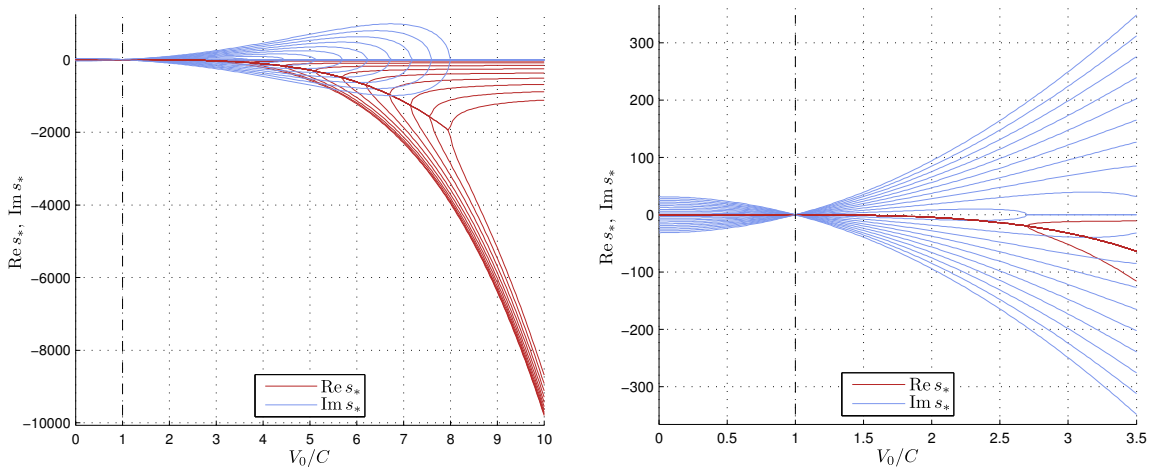


Figure 4: Eigenvalues of the axially moving one-dimensional web with damping, up to vibration mode  $k = 10$ . Analytical result, equation (74). Typical basic case with only time-dependent damping term present; damping parameters  $\psi_1 = 1$ ,  $\psi_2 = 0$ . Physical interpretation is external friction in a stationary surrounding medium. *Left and right*: different zoom levels.

state), and examine all the critical points, we will know the sign of  $\text{Re } s_*$  for any  $V_0 \in \mathbb{R}$ . This implies that the graphical examples below are conclusive; for the modes visualized, there cannot be any stability surprises outside the plotting range.

## 9 Examples and discussion of the system with damping

To finish, we will show some examples of eigenvalues (74) for various types of damping. The problem parameter values again correspond to the model of a narrow panel in the membrane limit with  $T_0 = 500 \text{ N/m}$ ,  $m = 0.08 \text{ kg/m}^2$ , and  $\ell = 1 \text{ m}$ . The characteristic time  $\tau$  was chosen as  $\tau = \ell/C$ , which leads to  $\sqrt{\beta^2 - \gamma} = 1$ , see equation (41).

In the examples presented here, the reaction coefficient  $B = \psi_3 = 0$ . It was found that the elastic foundation (reaction term) has only a minor effect on the eigenvalues. If  $B > 0$ , the imaginary parts pack much closer together, and there is more space between the smallest imaginary part and the real axis. Otherwise the results with  $B > 0$  are very similar to the ones shown with  $B = 0$ , and are omitted for brevity.

Figure 4 represents the typical basic case of a moderate amount of purely time-dependent damping, such as that generated by external friction when the elastic element vibrates in a stationary surrounding medium. We see that  $\text{Re } s_* \leq 0$  for all  $V_0$ . When  $\text{Im } s_*$  vanishes, each eigenvalue pair settles onto a pair of purely real values diverging from each other.

In Figure 5, we have a heavy amount of purely time-dependent damping. The



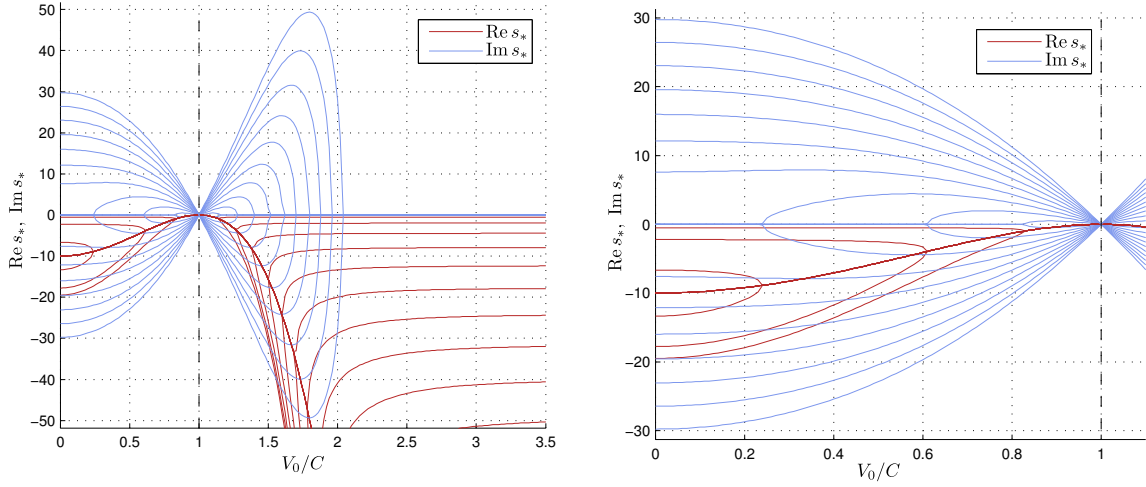


Figure 5: Eigenvalues of the axially moving one-dimensional web with damping, up to vibration mode  $k = 10$ . Analytical result, equation (74). Typical case with heavy but only time-dependent damping;  $\psi_1 = 20$ ,  $\psi_2 = 0$ . Note purely real eigenvalues near  $V_0 = 0$ . *Left and right*: different zoom levels.

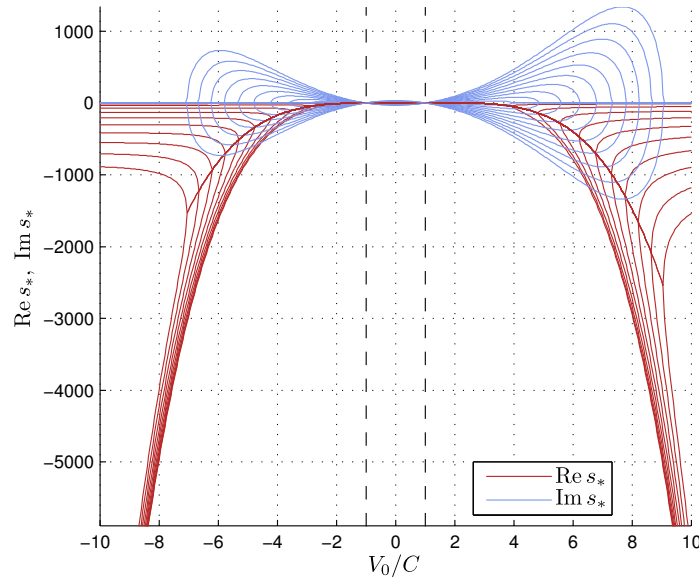


Figure 6: Eigenvalues of the axially moving one-dimensional web with damping, up to vibration mode  $k = 10$ . Analytical result, equation (74). Typical case with time- and space-dependent damping;  $\psi_1 = 1$ ,  $\psi_2 = 2$ . Nonzero  $\psi_2$  makes the spectrum asymmetric with respect to positive and negative values for  $V_0$ . Physical interpretation is external friction in a surrounding medium which is in axial motion, independent of the axial velocity  $V_0$  of the travelling elastic material.

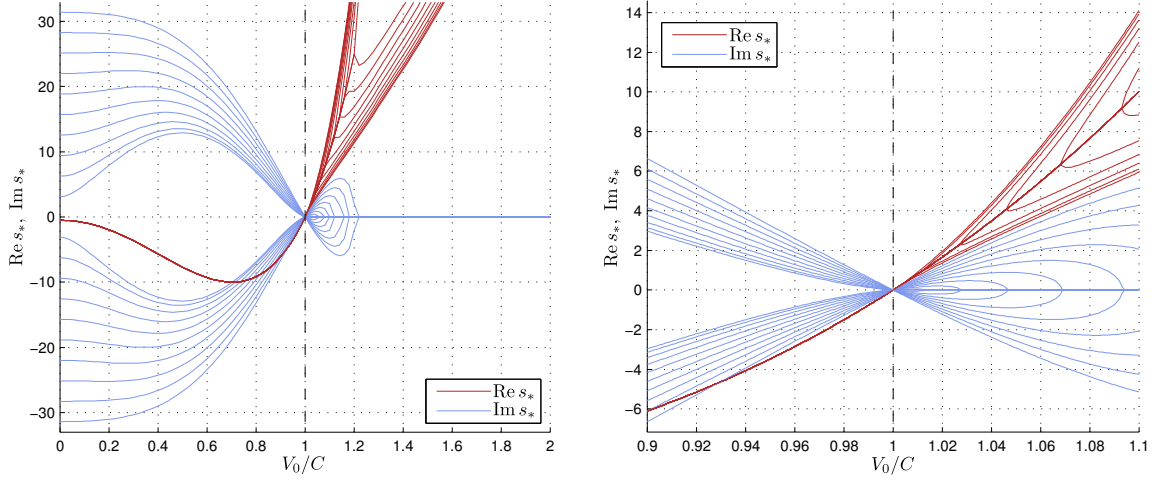


Figure 7: Eigenvalues of the axially moving one-dimensional web with damping, up to vibration mode  $k = 10$ . Analytical result, equation (74). Case with  $\psi_1 = 1$ ,  $\psi_2 = V_0\psi_1$ . Physical interpretation is internal friction in the travelling material itself. *Left and right*: different zoom levels.

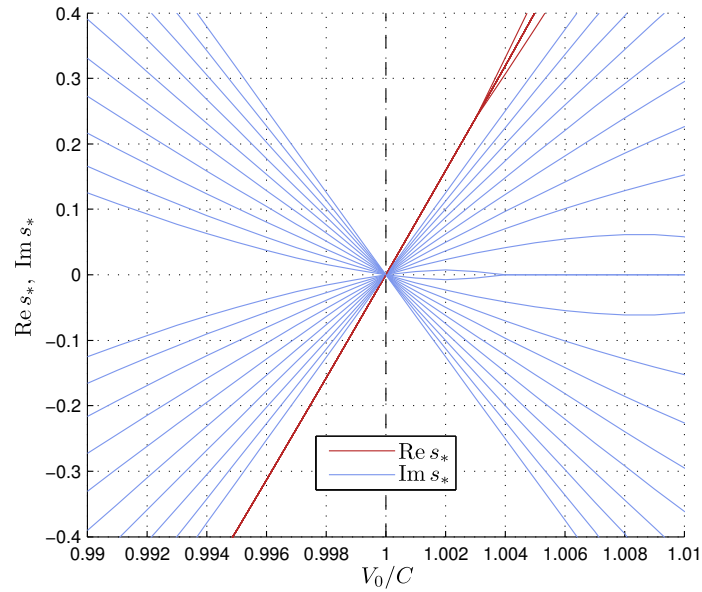


Figure 8: Close-up view centered on the static critical point in Figure 7. Because the mode  $k = 1$  is nearest the imaginary axis, it is seen that immediately after the critical point, the imaginary part  $\text{Im } s_* \neq 0$  for all  $k$ .

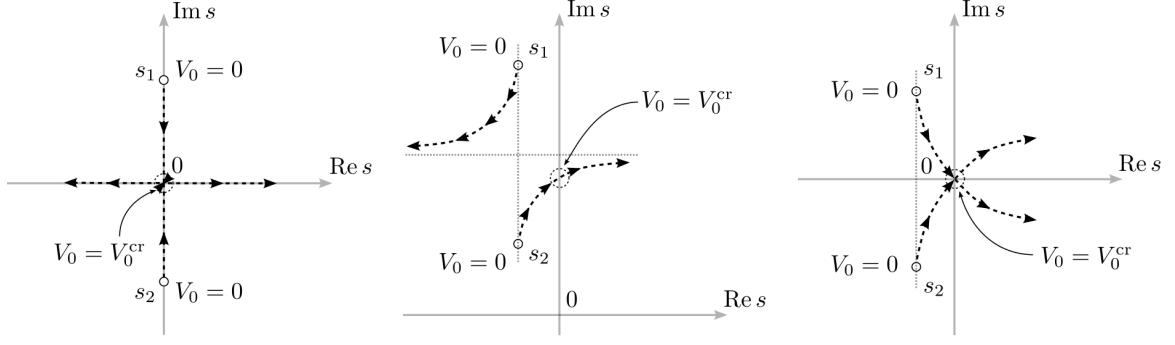


Figure 9: *Left and center*: Behaviour of the stability exponent  $s$  for the two types of instability, after Bolotin (1963). *Left*: static (or divergence) instability. *Center*: dynamic (or flutter) instability. *Right*: schematic of present result. Dynamic instability at a static critical point; eigenvalues pass through each other without interacting. Both obtain a positive real part as they pass the critical point. Compare Figures 7–8.

physical interpretation is the same as in the first case, only the amount of external friction is higher. Also here  $\text{Re } s_* \leq 0$  for all  $V_0$ . This figure illustrates  $\omega(k)^2 < r(V_0)$  for the first few modes near  $V_0 = 0$ . Because the condition  $r(V_0) - \omega(k)^2 < 0$  is not fulfilled, the imaginary part is zero, and these first few modes simply decay exponentially without vibrating.

Constant time- and space-dependent damping is illustrated in Figure 6. This type may arise due to external friction in an axially moving surrounding medium, where the axial velocity of the medium is independent of the axial velocity  $V_0$  of the travelling elastic element.

In the final example, Figures 7–8, we have  $\psi_2 = V_0\psi_1$ , which is of the form that is generated by the material derivative. As was discussed at the beginning of the analysis of the damped system, this case arises if the travelling material experiences internal friction.

In the case of internal friction, we observe an interesting phenomenon. Figures 7–8 indicate that the real parts  $\text{Re } s_*$  of the eigenvalues cross the real axis at  $V_0 = C$ , moving into the unstable region  $\text{Re } s_* > 0$ . The presence of internal friction has introduced an instability, which did not exist in the undamped system! This general effect of dissipation-induced destabilization is well-known; for its history, see Kirillov and Verhulst (2010). The mathematics behind the topic, and the importance of including damping effects (however small) in the model being analyzed, is discussed in several monographs by Bolotin; for example, see Bolotin (1963).

The type of this particular instability is curious in that it does not follow either of the classical typical behaviours as described by Bolotin (1963); see Figure 9. Often the kind of instability which is produced by  $s_*$  passing through the origin leads to purely real values of  $s_*$  slightly above the critical point. This is called the divergence instability or static instability. Slightly above the critical point, the state variable of the system will grow exponentially without oscillating.

The other classical category is the flutter instability, also known as dynamic insta-

bility, where  $s_*$  crosses the imaginary axis at some nonzero value of  $\text{Im } s_*$ , and  $\text{Re } s_*$  becomes positive. Slightly above the critical point, the state variable of the system will oscillate at an exponentially increasing amplitude.

In this system, the critical point is at the origin, but the initial postcritical behaviour is oscillatory. Unlike in the usual static or dynamic instability types, the eigenvalues simply pass through each other at this point without interacting. This behaviour is similar to that of the undamped system, except that now the real parts of the eigenvalues are nonzero. Compare behaviour near the static critical point in Figures 3 and 8.

In a sense, the instability is of the dynamic type, because of the type of its initial postcritical behaviour. However, it is possible to find the critical value for  $V_0$  by a static stability analysis (in which we insert the time-harmonic trial function, set  $s_* = 0$ , and solve the resulting steady-state problem), because at the critical point  $s_* = 0$ .

In conclusion, for the system considered, the presence of internal friction destabilizes the static critical point, leading to a dynamic instability there; whereas external friction causes the vibrations to decay as expected.

## 10 Conclusion

We considered classical axially moving ideal one-dimensional elastic elements with zero Dirichlet (pinhole) boundary conditions. Both undamped and damped cases were studied.

For both cases, at the critical velocity  $|V_0| = C$ , one of the terms in the partial differential equation is eliminated, requiring a separate analysis. This analysis was performed for the undamped case, and it was shown that at the critical velocity, a steady state occurs, for any state function satisfying the boundary conditions is a solution.

The free vibrations for  $|V_0| \neq C$  were solved following a classical approach. The eigenfrequencies were found explicitly for both systems, and for the undamped system also the eigenfunctions were determined. It was seen that for both systems, the eigenvalues come in pairs, which agrees with existing knowledge.

Based on quasi-static transition and the maximum-amplitude preserving property of time-harmonic vibration, it was argued that the ideal elastic web experiences no instability at the critical velocity. The initial postcritical behaviour of the stability exponent was seen to point to the same conclusion. This contrasts classical wisdom, but agrees with the result of Wang et al. (2005), obtained via a completely different approach.

This result is of fundamental theoretical importance. However, since no practical physical system has exactly zero bending rigidity, its main practical relevance is as an illuminating example underlining the significance of including in the model all relevant aspects of system behaviour; in this case, bending rigidity, however small.

But if one is interested only in an approximate value for the critical velocity for materials with small bending rigidity, the string model considered here is already

sufficient. The introduction of small bending rigidity has no major effect on the value of the critical velocity (see e.g. Kong and Parker, 2004 for an analytical approach), even though it completely changes the qualitative behaviour of the model around the critical point.

The present results lend themselves to the following interpretation. After a quasi-static transition from an initial state ( $V_0 = 0$ ) to the present state ( $V_0$  nonzero), if the assumption of small displacement has not been violated anywhere up to that point, the model considered determines the behaviour of free vibrations in the present state.

Especially, the system is seen to be stable at  $V_0 = 0$ . Hence, the governing partial differential equation is valid at least for all  $|V_0| < C$ . Whether the equation is valid at supercritical velocities,  $|V_0| > C$ , is therefore a question of whether there exists an instability at the critical velocity, where the stability exponents  $s_k^*$  vanish.

The argument given in the present paper leaves open two possibilities for instability at the critical point. First, it is easy to construct such dynamic free vibration solutions that are not compatible with the quasi-static transition into the steady state at  $V_0 = \pm C$ . An analysis accounting for axial acceleration — in order to perform a fully dynamic transition — may be needed to remove this limitation. The second open question is the behaviour of the global coefficient multiplying the solution as  $V_0$  is changed; determining this requires techniques beyond eigenvalue analysis.

Damping can be used to model both internal and external friction. It was found that also for the damped system, static critical points always exist at  $V_0 = \pm C$ . In the case of the system with external friction, it was additionally shown that for mode numbers  $k$  sufficiently high, dynamic critical points may exist. It was also observed, and demonstrated in the examples, that if the travelling material is subjected to heavy external friction, the first few modes  $k$ , near  $V_0 = 0$ , only decay exponentially without vibrating.

It was seen that the presence of internal friction in the travelling material itself destabilizes the static critical point, leading to a dynamic instability there, while external friction (such as caused by the viscosity of an external medium in which the travelling elastic element vibrates) exhibits no such effect.

The theory and results presented in this paper systematically summarize and extend theoretical knowledge of the class of models studied. The models can be used as simplified but powerful analysis tools in various applications of moving materials, for example paper making.

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