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Tytti Saksa

On Modelling and Stability of Axially Moving Viscoelastic Materials

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ABSTRACT

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In many industrial processes e.g. in paper making, there are parts, in which some material is transported through a series of rollers. Spans between the rollers, where the material travels without mechanical support, are called open draws. At high transport velocities, problems with loss of stability may be encountered, especially within the open draws. In this thesis, the axially moving material in an open draw, where the material is supported only at two edges, is modelled using (high order) linear partial differential equations representing axially moving elastic or viscoelastic panels and plates. For the use of linear models, small transverse displacements are assumed. To study the stability of these linear systems, classical dynamic analyses are performed, i.e. the standard time harmonic trial function is inserted to the dynamic equations describing the system and the eigenfrequencies of the system are studied to characterize the system behaviour. Many studied linear models predict axially moving panels or plates to undergo static instability at a high enough transport velocity. If the dynamic characteristics of the system are known, the critical conditions at a possible static instability can be studied in detail by solving an eigenvalue problem for the static form of the equations. The eigenvalue problems are here solved numerically using finite differences, the Fourier–Galerkin method, and analytical expressions when applicable. In this study, some new additions to the models of axially moving materials are presented, and the effects of different material parameters and other physical conditions on the stable behaviour of the moving materials are investigated. It is found, e.g., that material viscosity seems to stabilize the out-of-plane vibrations of a travelling panel but surrounding flowing fluid diminishes this stabilizing effect. The results of this thesis on the stability of axially moving materials give new insight to the behaviour of the production process of paper and other materials. The stability analyses provide a theoretical maximum for the safe transport velocity. As an application of this an optimization problem is presented for seeking desired conditions for production processes. The analytical findings and computationally light models also provide other tools for the use of industry.

Keywords: modelling, solid mechanics, continuum mechanics, out-of-plane vibrations, viscoelasticity, stability

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PREFACE

This thesis is based on the work conducted at the University of Jyväskylä, Department of Mathematical Information Technology during the years 2010 to 2013.

The work was done in a research group lead by professor Pekka Neittaanmäki from the University of Jyväskylä and professor Nikolay Banichuk from the Russian Academy of Sciences. Other researchers in the group include Ph. D. Juha Jeronen, Tech. Lic. Matti Kurki, M. Sc. Maria Tirronen and Ph. D. Tero Tuovinen. Professors Banichuk and Neittaanmäki, professor Raino A. E. Mäkinen from the University of Jyväskylä and Dr. Jeronen supervised the thesis.

The research topic of the group is motivated by a desire for efficient production of paper, in which mechanical stability of the paper machines plays an important role. This thesis extends the previous work of the research group giving new insight to the questions of stable behaviour of the produced paper material, taking into account material viscoelasticity and orthotropicity, and introducing a multi-objective optimization problem of finding optimal conditions for the production process.

ACKNOWLEDGEMENTS

I would like to thank Pekka Neittaanmäki and Raino Mäkinen for their support and help, raising my confidence, and introducing me to the society working within the subject. I thank Nikolay Banichuk, Juha Jeronen, Matti Kurki, Maria Tirronen and Tero Tuovinen for sharing their expertise during the years, for many progressive and broadening discussions, and for excellent research team work during the years. I express my thanks to the reviewers Tetsu Uesaka and Reijo Kouhia for careful reading of my work, and giving clarifying and improving suggestions about it. I would also like to thank Timo Tiihonen for giving valuable comments on my work and thus helping in improving the work. I am thankful to Johan Hoffman for giving a new viewpoint to the problem studied in this work. I would like to express my thanks to Auri Kaihlavirta, Marja-Leena Rantalainen, Kati Valpe, Jaana Markkanen, Outi Hynninen and Sami Kollanus for their help in organisational issues and for their supportive camaraderie. I am thankful to Olli Mali, Svetlana Matculevich, Marjaana Nokka, Hong Wang, Immanuel Anjam and Santtu Salmi for their company and contribution within many (junior) scientist events, and for their friendship during the years.

The Department of Mathematical Information Technology of the University of Jyväskylä served as a workplace and a venue for group meetings. For financial support, I thank the Jenny and Antti Wihuri foundation, the Department, and the Jyväskylä Doctoral Program in Computing and Mathematical Sciences COMAS.

I thank my family and friends for their encouragements in going on this thesis project. I am grateful to you for all the love and joy you give into my life. Special thanks to my beloved Aapo for his help and support whatever the matter.

NOMENCLATURE

Notation

$\frac{\partial u}{\partial x}$, $u_{,x}$	partial derivative of u with respect to x
$\frac{\mathrm{d}u(x,t)}{\mathrm{d}t}$	$= \frac{\partial u(x,t)}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial u(x,t)}{\partial x}$ total (material, Lagrange) derivative of $u(x,t)$ with respect to t, here often $\mathrm{d}x/\mathrm{d}t = V_0$ is constant
$\Delta^2 u$	$= \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}$ bi-harmonic operator acting on u, here used only in **PVI**
ΔT	(small) change of T, variation of T
i	imaginary unit
$\overline{a}$	complex conjugate of a
Re a	real part of a
Im a	imaginary part of a

Symbols

a	length of edge crack. Unit $[a] = \mathrm{m}$
a_0	initial length of edge crack. Unit $[a_0] = \mathrm{m}$
b	width of panel or half-width of plate. Unit $[b] = \mathrm{m}$
c	dimensionless velocity of the panel, $c = V_0\sqrt{m/T_0}$
D	flexural stiffness. For panel, $D = Eh^3/(12(1-\nu^2))$. Unit $[D] = \mathrm{N\,m}$
D_j	(where $j = 1,2,3$) orthotropic flexural stiffness. D_1 is related to the axial direction x, D_2 to the cross (width) direction y, and D_3 to the coupling term between the two
E	Young's modulus of isotropic plate or panel. Unit $[E] = \mathrm{N/m^2}$
E_1	Young's modulus of orthotropic plate in axial direction x. Unit $[E_1] = \mathrm{N/m^2}$
E_2	Young's modulus of orthotropic plate in cross (width) direction y. Unit $[E_2] = \mathrm{N/m^2}$
ε_x	axial component of bending strain. $\varepsilon_x = -z(\partial^2 w/\partial x^2)$
ε_y	width-directional component of bending strain. $\varepsilon_y = -z(\partial^2 w/\partial y^2)$

η	viscous damping coefficient. Unit $[\eta] = \mathrm{N\,s/m^2}$
g	standard gravity. Unit $[g] = \mathrm{m/s^2}$
G_{12}	in-plane shear modulus of orthotropic plate in xy plane. Unit $[G_{12}] = \mathrm{N/m^2}$
γ_{xy}	shear strain due to bending. $\gamma_{xy} = -2z(\partial^2 w/\partial x \partial y)$
h	thickness of panel or plate. Unit $[h] = \mathrm{m}$
K	stress intensity factor. Unit $[K] = \mathrm{Pa}\sqrt{\mathrm{m}}$
K_{C}	fracture toughness. Unit $[K] = \mathrm{Pa}\sqrt{\mathrm{m}}$
k	material constant in Paris law
ℓ	free span length parameter. For the panel submerged in flowing fluid, the free span is taken to be $x \in (-\ell, \ell)$. Otherwise, as $x \in (0, \ell)$. Unit $[\ell] = \mathrm{m}$
M	bending moment per unit length
m	mass per unit area. Unit $[m] = \mathrm{kg/m^2}$
m_j	(where $j = 1, 2, 3$) added mass. m_1 is added mass due to the transverse, m_2 to Coriolis, and m_3 to centripetal acceleration
μ	Poisson ratio for viscosity (dimensionless)
n	number of loading cycles
n^{cr}	critical number of loading cycles due to fatigue fracture
ν	Poisson ratio of isotropic plate or panel (dimensionless)
ν_{12}	Poisson ratio for orthotropic plate. When stretched along axis 1 (x), ν_{12} is the contraction factor along axis 2 (y)
ν_{21}	Poisson ratio for orthotropic plate. When stretched along axis 2 (y), ν_{21} is the contraction factor along axis 1 (x)
Ω	domain of the governing equation (connected open set)
q_{f}	aerodynamic reaction pressure
r	scaled aspect ratio, $r = (1/\pi)(\ell/b)$
s	stability exponent in linear stability analysis (complex-valued)
σ_x	normal stress for plate due to bending in the x direction. Unit $[\sigma_x] = \mathrm{N/m^2}$
σ_y	normal stress for plate due to bending in the y direction. Unit $[\sigma_y] = \mathrm{N/m^2}$
T	tension. Unit $[T] = \mathrm{N/m}$
T_0	constant (homogeneous) tension
T_{xx}	axial tension for plate. Unit $[T_{xx}] = \mathrm{N/m}$

T_{xy}	in-plane shear tension for plate. Unit $[T_{xy}] = \mathrm{N/m}$
T_{yy}	cross-directional (width-directional) tension for plate. Unit $[T_{yy}] = \mathrm{N/m}$
t	time coordinate
t_R	retardation (creep, delay) time constant. Unit $[t_\mathrm{R}] = \mathrm{s}$
τ	scaling constant for nondimensionalization of time coordinate, called the characteristic time. In Section 4.3 and in **PVII**, cycle time period. Unit for both uses $[\tau] = \mathrm{s}$
τ_{xy}	shear stress for the plate due to bending. Unit $[\tau_{xy}] = \mathrm{N/m^2}$
Υ	viscous analogue of flexural stiffness. Unit $[\Upsilon] = \mathrm{N\,m\,s}$
V_0	axial velocity of panel or plate. Unit $[V_0] = \mathrm{m/s}$
V_0^{cr}	critical velocity of elastic instability of travelling panel or plate
v_∞	free-stream velocity of surrounding fluid. Unit $[v_\infty] = \mathrm{m/s}$
W	transverse (z-directional, out-of-plane) displacement (dimensionless). Function of space. Used for the space part of the time-harmonic solution
w	transverse (z-directional, out-of-plane) displacement. Function of space and time. Dimensional or dimensionless depending on context. Unit of dimensional displacement $[w] = \mathrm{m}$

LIST OF FIGURES

LIST OF TABLES

CONTENTS

LIST OF INCLUDED ARTICLES

PI Tytti Saksa, Nikolay Banichuk, Juha Jeronen, Matti Kurki and Tero Tuovinen. Dynamic analysis for axially moving viscoelastic panels. *International Journal of Solids and Structures*, Vol. 49, Issue 23–24, pp. 3355–3366, DOI:10.1016/j.ijsolstr.2012.07.017, 2012.

PII Tytti Saksa, Juha Jeronen and Tero Tuovinen. Stability of moving viscoelastic panels interacting with surrounding fluid. *Rakenteiden mekaniikka (Finnish Journal of Structural Mechanics)*, Vol. 45, No 3, pp. 88–103, 2012.

PIII Tytti Saksa, Juha Jeronen, Nikolay Banichuk and Matti Kurki. On travelling web stability including material viscoelasticity and surrounding air. *Advances in Pulp and Paper Research, Cambridge 2013*, Transactions of the 15th Fundamental Research Symposium held in Cambridge, Ed. S. J. I' Anson, Vol. 1, pp. 449–468, ISBN: 978-0-9926163-0-4, 2013.

PIV Nikolay Banichuk, Juha Jeronen, Tytti Saksa and Tero Tuovinen. Static instability analysis of an elastic band travelling in the gravitational field. *Rakenteiden mekaniikka (Finnish Journal of Structural Mechanics)*, Vol. 44, No 3, pp. 172–185, 2011.

PV Nikolay Banichuk, Juha Jeronen, Matti Kurki, Pekka Neittaanmäki, Tytti Saksa and Tero Tuovinen. On the limit velocity and buckling phenomena of axially moving orthotropic membranes and plates. *International Journal of Solids and Structures*, Vol. 48, Issue 13, pp. 2015–2025, DOI:10.1016/j.ijsolstr.2011.03.010, 2011.

PVI Nikolay Banichuk, Juha Jeronen, Pekka Neittaanmäki, Tytti Saksa and Tero Tuovinen. Theoretical study on travelling web dynamics and instability under non-homogeneous tension. *International Journal of Mechanical Sciences*, Vol. 66, pp. 132–140, DOI:10.1016/j.ijmecsci.2012.10.014, 2013.

PVII Nikolay Banichuk, Matti Kurki, Pekka Neittaanmäki, Tytti Saksa, Maria Tirronen and Tero Tuovinen. Optimization and analysis of processes with moving materials subjected to fatigue fracture and instability. *Mechanics Based Design of Structures and Machines*, Vol. 41, Issue 2, pp. 146–167, DOI:10.1080/15397734.2012.708630, 2013.

The articles **PI** and **PII** were written mainly by the author. In these papers, the author made the literature review regarding (travelling) viscoelastic materials and designed the numerical experiments. The numerical implementations and computations were carried out by the author. The figures in the sections of numerical results were produced by the author. The author derived the fifth boundary condition and the analytical solution for the critical eigenmode of a clamped–simply supported elastic panel, presented in **PI**.

The article **PIII** was written mainly by the author excluding the numerical examples. As in **PI** and **PII**, the literature review on the viscoelastic materials is here due to the author. The author participated in the design of numerical examinations, and carried out a parallel implementation of the presented cases using the finite difference method for comparing the results.

In **PIV**, the author strongly contributed to the writing process. The author designed and implemented the way to solve the critical velocity by minimizing the functional corresponding to energy. The numerical implementations were carried out together by the author and Juha Jeronen. The numerical computations were done mainly by the author.

In **PV**, the author's contribution covers the proof of non-negativeness of the eigenvalues, analysis of when the solutions of the characteristic equation are real-valued, and literature review regarding the orthotropic plates and the measured physical parameters. The author also participated the implementation of the numerics, carried out partly the computations, and checked the technical calculations.

In **PVI**, the author wrote parts of the numerical results, and took part in implementing and computing the numerical results. Some of the figures (Figs. 2 and 3 in **PVI**) were produced by the author.

In **PVII**, the author extended the literature review as for vibrations of cracked materials, added the third study case (Section 6.3) to the optimization problem. The numerical implementations and generating of the illustrations were done together with Maria Tirronen.

The publications by the author related to this thesis also include a monograph Banichuk et al. (2014), two articles in collection Numerical methods for differential equations, optimization, and technological problems (Banichuk et al., 2013; Saksa et al., 2013), six preprints in Reports of the Department of Mathematical Information Technology (Banichuk et al., 2010a, 2011b,c,d, 2012; Saksa and Jeronen, 2012), two non peer reviewed conference articles (Kurki et al., 2012; Saksa, 2011), and one article in Finnish (Kurki et al., 2011). The editorial board of the Finnish Journal of Structural Mechanics awarded **PII** as an article of the year 2012.

Results of this thesis were presented by the author in the following seminars and conferences:

– ECCOMAS Thematic Conference on Computational Analysis and Optimization, Jyväskylä, Finland, June 9, 2011;
– Sixth M.I.T. Conference on Computational Fluid and Solid Mechanics – Focus: Advances in Solids & Structures, Boston, U.S.A., June 17, 2011;
– 4rd Workshop on Advanced Numerical Methods for Partial Differential Equation Analysis, EIMI, St. Petersburg, Russia, August 23, 2011;
– 24th Nordic Seminar on Computational Mechanics, Helsinki, Finland, November 3, 2011;
– The 5th Summer School on Mathematical Analysis, Advanced Methods for Partial Differential Equations and Optimization, Laukaa, Finland, August

3, 2012;

- 6th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2012), Vienna, Austria, September 10, 2012;
- 11th Finnish Mechanics Days (XI Suomen Mekaniikkapäivät), Oulu, Finland, November 30, 2012;
- 6th Workshop on Analysis and Advanced Numerical Methods for Partial Differential Equations, Strobl, Austria, July 8, 2013;

and during a research visit in the Computational Technology Laboratory at the KTH Royal Institute of Technology, April 21 to May 31, 2013.

1 INTRODUCTION

This thesis work is about modelling of out-of-plane behaviour of axially moving materials. Talking about modelling, comparison of the results to measurements or experiments is often involved. In this work, we work on modelling without measured data against which the results could be verified. Modelling, however, can be very useful there, where exact measurements are difficult to be carried out. E.g. Hristopulos and Uesaka (2002) point out that as web breaks in pressrooms are statistically rare, experimental investigations would require lots of effort, which models may complement.

Modelling is an extensive field: almost anything can be modelled. One may use discrete or continuous models depending on the modelled system. In this work, we will use continuous models and apply the theories of the classical continuum mechanics.

Within continuum mechanics, we concentrate on a very specific topic that is out-of-plane behaviour and vibrations of continuous plates, which are travelling between supporting rollers. We study a part of the plate travelling between two supports, such that the plate material flows into the studied region at one edge and flows out at the other edge continuously preserving the material (mass). In the articles by Ulsoy and Mote (1982), Wickert and Mote (1990), Lin (1997) and Lee and Oh (2005), the characteristics of the dynamic out-of-plane behaviour of moving materials have been studied with the help of string, beam, and plate models, investigating the stability of the systems by eigenfrequency analyses. In Figure 1, a schematic figure of a system, in which material travels between two supporting rollers is shown.

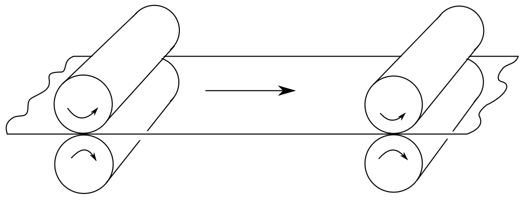
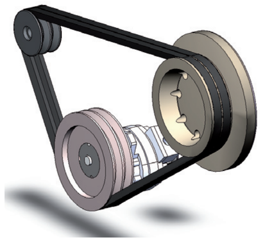
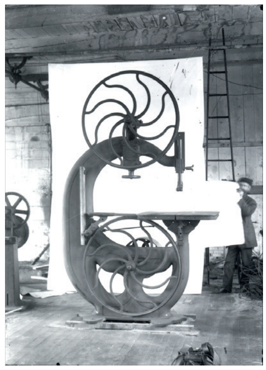


FIGURE 1 Material travelling between two supports.

Axially moving materials have many applications in industry. Examples of systems with axially moving materials are band saws, power transmission belts, paper making processes, printing presses, manufacturing of plastic films and sheets, and extrusion of aluminium foils, textiles and other materials. In Figure 2(a), one can see a mechanical belt. In Figure 2(b), an old band saw is shown. Figure 3 presents a detail of an old paper machine. Dryer cylinders are shown on



(a) A mechanical belt.



(b) A band saw.

FIGURE 2 Examples of applications for systems of axially moving materials.
(Figure 2(a) is a work in the public domain, from
`http://commons.wikimedia.org/wiki/File:Keilriemen-V-Belt.png`
and Figure 2(b) has no known copyright restrictions and is from
`http://www.flickr.com/photos/keenepubliclibrary/7308330284/`)

the left hand side. The applications in Figures 2 and 3 differ from each other in geometry: the path of the material is closed for the mechanical belt and the band saw, and open for the paper machine.

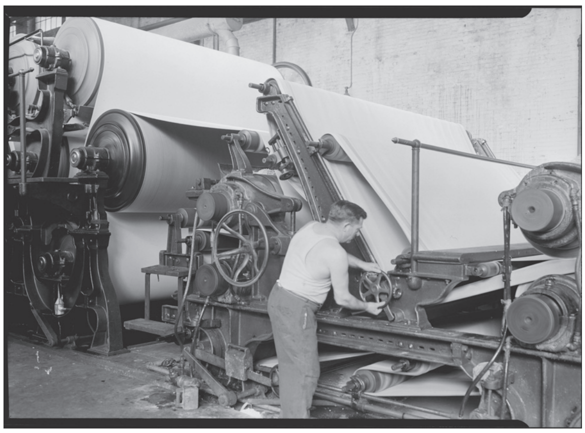


FIGURE 3 A detail of an old paper machine.
(The use of Figure 3 is unrestricted. The figure is from
`http://research.archives.gov/description/518334`)

Application processes, in which the material is transported as a continuous

TABLE 1 Measured values for paper material parameters. E_1, E_2 are the Young's moduli in the machine direction and in the cross-machine direction, respectively, and G_{12} is the in-plane shear modulus, ν_{12} and ν_{21} are the Poisson ratios

	Low basis weight paper[1]	Copy paper[2]	UKP sack paper[2]
E_1 (MPa)	8790	6820	7320
E_2 (MPa)	2740	3160	2680
G_{12} (MPa)	1600	1950	1890
ν_{12}	0.62	0.156	0.164
ν_{21}	0.191	0.098	0.060
basis weight (kg/m^2)	0.060	-	-

[1] Seo (1999); [2] Yokoyama and Nakai (2007).

plate, often require high transport velocities. For example in paper making, the transport velocity of the paper web is today around 100 km/h. Increasing velocity, however, may lead to loss of stability, which from the application viewpoint, may mean breakage of the material. In practice, such material breakage may risk occupational safety or decrease process efficiency, e.g. production.

We mentioned above that from the the applications of axially moving materials, we will mostly concentrate on paper making. Although paper has a fiber network structure, it usually can be regarded as continuous as far as on the centimeter scale (Niskanen, 2012). Let us describe shortly the parts of a paper machine and some important properties of the paper as material.

The paper machine begins with a forming section, in which a mixture of refined pulp and water is spread on a moving continuous screen. The moisture raw paper is then transported through suction units and a press section to remove water from it. A section consisting of dozens of hot cylinders, called a dryer section, follows to evaporate water. Some paper materials are further processed through calendaring or coating sections to make paper smooth or glossy. The final paper is wound to paper rolls. The solids content varies a lot with the process, being about 1 % in the forming section, and about 50 % after the press section (Niskanen, 2012).

Paper web is very thin and wide. For example, the thickness of newspaper is about 0.1 mm, and the width of the produced continuous paper web is usually between 5 m and 10 m (Niskanen, 2012). Table 1 presents some measured values of elastic parameters, basis weight or density for different types of paper. The mass per unit area of paper (basis weight) varies usually between 0.040 kg/m^2 and 0.100 kg/m^2, depending on the type of paper (Niskanen, 2012).

Paper as a material is not purely elastic but viscoelastic, nearly plastic. Thus, the behaviour of the paper is time-dependent. Two phenomena, related to time-dependent behaviour, and examined a lot within paper materials, are creep and relaxation. Creep is defined as the increase in strain over time keeping the state of stress constant, and relaxation is the decrease in stress over time under a constant state of strain. For more details, see Niskanen (2012).

Viscoelastic behaviour of paper is known to be non-linear. Recently, Sorvari et al. (2007) compared different viscoelasticity theories for paper materials and concluded that linear viscoelasticity captured very well the time- and rate-dependent behaviours at practical strain levels of paper. Creep and relaxation test data were compared to predictions by a linear model, Schapery's nonlinear model (Schapery, 1969) and uniaxial multiple integral representation.

In many parts of the paper machine, the processed paper web is transported without mechanical support. Examples of these parts include transportation from the press nip to the drying section, the drying section itself, and the winding of the final paper to rolls. The spans, where the web travels without support are called open draws. Within these open draws the paper web is subject to damage. Stability of these parts is therefore an interesting research question for the applications.

In this study, we will address to questions concerning the stability of axially moving materials involving high transport velocities. The analysis that will be presented can be applied in any processes with axially moving plate-like materials, although our viewpoint is especially paper web handling processes with wide and thin webs, and thus the parameters in the numerical examples are chosen to represent a typical paper material.

1.1 Objectives

Aim of this study is to examine how different factors affect the out-of-plane behaviour of a continuous, axially transported material. These factors include material viscoelasticity, surrounding flowing fluid, Earth's gravity, material orthotropicity and inhomogeneities in tension. To model the axially moving material, the derivation and choice of the models needs to be done carefully, so that the model is able to describe the phenomenon under investigation. Especially, with the help of models, we would like to examine the mechanical stability of travelling material plates, and to find out if the physical factors mentioned above affect the stability. We aim to increase knowledge of the behaviour of axially moving materials from the viewpoint of basic research.

1.2 Structure of thesis

To study the models for axially moving materials and the questions of their stability, it is reasonable to know the models that have been previously applied and results that have been reported previously. Chapter 2 discusses the previous research first on 1D models, then on 2D models, and finally models for cracked materials. The review is restricted mostly on vibrations of travelling materials.

The same chapter contains a section, in which the contribution of this work to the field is summarized.

The subsequent chapters discuss the models used in this thesis work. Chapter 3 presents a model for an axially moving viscoelastic panel in vacuum or submerged in flowing fluid, and for an axially moving elastic panel in gravitational field, limiting the discussion to 1D models. After presenting the models, we shortly describe the methods used for stability analyses. The papers **PI–PIV** are related to Chapter 3. In Chapter 4, 2D models for an axially moving orthotropic plate under a constant or linear tension profile are described as well as the methods to study their stability and an example of applying the stability analysis on optimization of productivity of some processes. Papers **PV–PVII** are related to Chapter 4.

Chapter 5 summarizes application of the numerical methods for the prescribed models, and in Chapter 6, we discuss the key results, their applicability and outcome of this work.

2 LITERATURE REVIEW AND THIS WORK'S CONTRIBUTION

It is well known that a high transport speed of an axially moving material may lead to loss of stability resulting in damage or even breakage of the processed material. The models used for transverse motion of travelling materials include strings, membranes, beams, panels and plates. Stability of these models has often been studied by dynamic analysis, i.e. by studying the eigenfrequencies of the system. Apart from a loss of stability, damage and breakage of material may result e.g. from arising or growing of cracks.

2.1 Literature review

This section summarizes previous research on axially moving materials, and the models applied for them, concentrating especially on transverse vibrations and questions of stability. The first studies in the area concerned 1D models including Skutch (1897); Sack (1954); Miranker (1960); Swope and Ames (1963); Mote (1968, 1972, 1975); and Simpson (1973), some of which we further discuss below in the context of axially moving elastic and viscoelastic strings and beams. After string and beam models, we continue with the review on 2D models for axially moving elastic and viscoelastic membranes and plates. Models taking into account the surrounding fluid (air) are discussed thereafter. Finally, we look at models for vibrations of thin cracked materials.

Axially moving elastic and viscoelastic strings and beams

Skutch (1897) was the first to investigate axially moving materials, the paper being published originally in German. The first paper on the topic published in English was by Sack (1954). Both of these studies discussed the vibrations of a travelling string.

Archibald and Emslie (1958) and Simpson (1973) studied the effects of axial motion on the frequency spectrum and eigenfunctions. Both the travelling string and beam were shown to experience divergence instability at a sufficiently high velocity. It was also observed that the natural frequency of each eigenmode decreases as the transport velocity is increased. Wickert and Mote (1990) studied stability of axially moving strings and beams presenting the expressions for the critical transport velocities analytically. They used modal analysis and a Green's function method. Recently, Wang et al. (2005a) showed that no static instability occurs at the critical velocity in the case of a string model. Kong and Parker (2004) found closed-form expressions for the approximate frequency spectrum via perturbation analysis in their study concerning axially moving beams with small flexural stiffness.

Industrial materials usually have viscoelastic characteristics (Fung et al., 1997), and consequently, viscoelastic moving materials have been recently studied widely. Viscoelasticity is also a property of paper, playing important part especially for wet paper webs (see e.g. Alava and Niskanen, 2006; Niskanen, 2012).

First studies on the transverse vibrations of a viscoelastic material traveling between two fixed supports were done by Fung et al. (1997), using a string model. Extending the work, they studied the material damping effect in Fung et al. (1998).

Several studies on travelling viscoelastic materials, concerning strings and beams, have been performed during the last decade. Chen and Zhao (2005) presented a modified finite difference method to simplify a non-linear model of an axially moving viscoelastic string. They studied the free transverse vibrations of elastic and viscoelastic strings numerically.

Oh et al. (2004) and Lee and Oh (2005) studied critical speeds, eigenvalues, and natural modes of axially moving viscoelastic beams performing a detailed eigenfrequency analysis, and reported that viscoelasticity did not affect the critical velocity.

Marynowski and Kapitaniak (2002) compared two different internal damping models in the modelling of moving viscoelastic (non-linear) beams. For the linearized Kelvin–Voigt model, it was found that the beam exhibits divergent instability at some critical speed. In the case of non-linear Bürgers model, the critical speed decreased when the internal damping was increased, and the beam was found to experience the first instability in the form of flutter.

A particular question about whether one should use the material time derivative or the partial time derivative in the viscoelastic constitutive relations for moving materials, has recently been discussed especially in the case of the widely used Kelvin–Voigt material model. Using the material derivative instead of the partial time derivative was first suggested by Mockensturm and Guo (2005) and the material derivative has been used in most of the recent studies concerning axially moving viscoelastic beams (see e.g. Chen et al., 2008; Chen and Ding, 2010; Chen and Wang, 2009; Ding and Chen, 2008). Kurki and Lehtinen (2009) suggested, independently, that the material derivative in the constitutive relations should be used in their study concerning the in-plane displacement field of a travelling viscoelastic plate.

Axially moving elastic and viscoelastic membranes and plates

Stability of travelling rectangular membranes and plates was first studied by Ulsoy and Mote (1980, 1982), and Lin and Mote (1995, 1996). Ulsoy and Mote (1980, 1982) studied natural frequencies and stability of the moving plate using the Ritz method and simplified boundary conditions at the free edges. They also compared the analytical results with experimental data reporting a good agreement. Lin and Mote (1995) studied an axially moving membrane in a 2D formulation, predicting the equilibrium displacement and stress distributions under transverse loading. Later, the same authors predicted the wrinkling instability and the corresponding wrinkled shape of a web with small flexural stiffness (Lin and Mote, 1996).

Stability of out-of-plane vibrations of axially moving rectangular membranes was studied by Shin et al. (2005). For the behaviour of the membrane, it was found that the motion is stable until a critical speed, at which static instability occurs.

In recent studies concerning axially moving plates, material properties such as orthotropicity (Wang, 1999) or viscoelasticity (Marynowski, 2010) have been taken into consideration and their effects on the plate behaviour have been investigated. Also such phenomena as winding in the context of axially moving materials has been studied (Garziera and Amabili, 2000).

Lin (1997) studied stability of an axially moving plate, and numerically showed that loss of stability of the plate occurs in a form of divergence at a sufficiently high speed. The critical velocity and the corresponding critical shapes of an axially moving elastic plate were studied, and an analytical expression for the critical velocity was provided e.g. by Banichuk et al. (2010b).

Wang (1999) analysed behaviour of axially moving orthotropic plates via the finite element method and using Reissner–Mindlin plate theory. Wang studied the eigenfrequencies and mode shapes of the moving plate. More recently, Hatami et al. (2009) studied free vibration of the moving orthotropic rectangular plate at sub- and super-critical speeds, and flutter and divergence instabilities at supercritical speeds. Their study was limited to simply supported boundary conditions at all edges. For the solution of equations of orthotropic moving materials, many necessary fundamentals can be found in the book by Marynowski (2008).

Free vibrations of orthotropic rectangular plates, which are not moving, have been studied by Biancolini et al. (2005) including all combinations of simply supported and clamped boundary conditions on the edges. Xing and Liu (2009) obtained exact solutions for free vibrations of stationary rectangular orthotropic plates considering three combinations of simply supported (S) and clamped (C) boundary conditions: SSCC, SCCC and CCCC. Kshirsagar and Bhaskar (2008) studied vibrations and buckling of loaded stationary orthotropic plates. They found critical loads of buckling for all combinations of boundary conditions S, C and F.

A few studies on transverse vibrations of axially moving viscoelastic plates have also been done. Hatami et al. (2008) studied free vibrations of axially moving viscoelastic plates using a finite strip method. They used the standard viscoelastic solid model. They found that, as for an elastic plate, also for a viscoelastic plate the vibration frequency decreases as axial velocity is increased and becomes zero at a critical velocity for the lowest frequency. Zhou and Wang (2007) studied transverse vibration characteristics of axially moving viscoelastic rectangular plates. They assumed the plate to be elastic in dilatation but viscoelastic in distortion, where the viscoelasticity was described by the Kelvin-Voigt law. Two different combinations were used as boundary conditions: all four edges simply supported and two opposite edges simply supported but the remaining edges clamped. It was found that when the dimensionless delay time was very small (plate was almost elastic), the dynamic characteristics and stability of the axially moving viscoelastic plate were nearly the same as for an axially moving elastic plate. It was also reported that the increase of delay time did not alter the critical divergence velocity in the first mode and that the plate did not exhibit a coupled-mode flutter. (Similar behaviour for viscoelastic beams was reported by Lee and Oh, 2005.) Very recently, Yang et al. (2012) studied vibrations, bifurcation and chaos of axially moving viscoelastic plates using finite differences and a non-linear model for transverse displacements. They concentrated on bifurcations and chaos, but also studied the dynamic characteristics of a linearised elastic model with the help of eigenfrequency analysis.

Recently, some studies using the material derivative in the viscoelastic constitutive relations for moving viscoelastic 2D plates have been done. Marynowski (2010) studied free vibrations and stability of Levy-type viscoelastic plates comparing a three-parameter Zener model and a two-parameter Kelvin–Voigt model for the viscoelasticity. It was found that the critical transport velocity predicted by the Zener model was higher than the one predicted by the Kelvin–Voigt model, which in turn was slightly higher than the critical velocity of an elastic plate. Tang and Chen (2013) studied stability in parametric resonance of moving viscoelastic plates with time-dependent velocity of axial motion.

Fluid-structure interaction in the context of moving materials

For thin and wide webs, interaction with surrounding air affects significantly the behaviour of travelling material. In the case of travelling paper webs, the effects of the air are important, see e.g. Kulachenko et al. (2007a,b) and Pramila (1986).

Pramila and Niemi published a series of papers in 1986 to 1987 concerning the interaction between the travelling paper web and the external medium, using at first an analytical added-mass approximation and then the finite element method (Niemi and Pramila, 1986; Pramila, 1986, 1987; Pramila and Niemi, 1987). These studies by Pramila and Niemi are considered the first studies in which fluid-structure interaction has been taken into account in the context of axially moving webs. In their studies, it was found that the presence of air reduces the values of the eigenfrequencies and critical velocities drastically compared to the vacuum case. The presence of air may reduce both down to 15–26 % of the vacuum case (Pramila, 1986). However, the model that was used, was later interpreted by Pramila to mean that the fluid particles move with the travelling web, which probably is not the actual physical case there (Pramila, 1987). Recently, Frondelius et al. (2006) used an added mass model with non-constant coefficients derived from the boundary layer theory. However, for this approach to be applicable, one needs to include a leading edge in the model.

Chang and Moretti (1991) further developed the added mass approach for axially moving webs comparing also their theory to wind-tunnel experiments for stationary webs surrounded by flowing air. They modelled the web as an ideal membrane and the surrounding air was treated using potential flow theory. Kulachenko et al. (2007a) modelled the fluid-structure interaction on the basis of acoustic theory finding also that the presence of air reduces the eigenfrequencies of the web compared to the vacuum case. Recently, Banichuk et al. (2010c, 2011a) studied the interaction between the moving web and flowing air using a panel model (a plate with cylindrical deformation) and an analytical expression for the aerodynamic reaction pressure. The aerodynamic reaction was solved analytically for potential flow in a complex plane from which the panel was cut. Eigenfrequencies and dynamic behaviour of this model were further investigated by Jeronen (2011).

Existing studies on moving viscoelastic materials interacting with surrounding fluid seem to be rather limited, namely to the cases of beams having circular cross-section (Gosselin et al., 2007; Lin and Qiao, 2008; Taleb and Misra, 1981) and to viscoelastic pipes conveying fluid (Drozdov, 1997; Wang et al., 2005b).

Vibrations of cracked materials

Transported materials may also be subject to initial cracks, and these cracks may grow if material is under a load. E.g. in the context of paper making, Wathén (2003) has discussed how damage in paper affect web breaks, and Tryding (1996) has studied crack growth evolution in paper material using experiments and a cohesive crack model with finite element analysis.

In some previous studies on web vibrations, fracture has been included into the problem dynamics and the effects of the cracks on the stability have been studied. Various analyses of vibrations and stability of *stationary* beams and plates exist. An extensive review on fracture of cracked materials and challenges in such models was discussed by Dimarogonas (1996). Finite element analysis has often been applied to analyse the vibrations and stability of cracked rectangular plates, considering centre or edge located cracks. Bachene et al. (2009) used the extended finite element method, and Liew et al. (1994) developed an efficient decomposition method to study vibrations of cracked plates. Brighenti (2005) examined buckling failure of cracked plates for different crack orientations with the help of finite element analysis. Both buckling and vibration analysis were covered in the finite element studies of cracked plates by Prabhakara and Datta (1993, 1997).

Stahl and Keer (1972) studied vibrations and stability of rectangular plates with the help of dual series equations. Vafai et al. (2002) studied parametric instability of plates having one crack at an edge. They considered simply supported rectangular plates under periodic loadings using an integral equation method. Effects of cracks on the eigenfrequencies and eigenmodes of axially moving beams at sub-critical transport speeds were studied by Murphy and Zhang (2000). However, the effect of the cracks on the results was found to be small.

2.2 Contribution of this work to the field

This thesis work discusses models for axially moving viscoelastic panels in vacuum and interacting with surrounding fluid, travelling elastic panels in the gravitational field, and axially moving orthotropic and isotropic plates under homogeneous and non-homogeneous tension profiles. The included articles mainly add new details to the models and concentrate on the dynamic or static stability analyses of axially moving materials.

In **PI**, eigenvalues and stability characteristics of viscoelastic axially moving panels in vacuum were studied using the material derivative in the viscoelastic constitutive relations. A fifth boundary condition for the partial differential equation describing the dynamical out-of-plane behaviour of the system and being of the fifth order in space was derived based on a physical continuity condition for the bending moment. A similar continuity condition had been previously applied by Flügge (1975) for a viscoelastic beam of infinite length and under a moving load. In the numerical studies, it was found that if the viscosity is high enough, all modes behave stably with damped vibrations for all studied values of transport velocity, and no critical speed was detected.

PII presented a model for axially moving thin webs, in which the material viscoelasticity was taken into account by the Kelvin–Voigt type model and the effects of the surrounding air were approximated using an added mass approach. To our knowledge, the study was the first taking into account both (Kelvin–Voigt) material viscoelasticity and aerodynamic effects in the modelling of webs travelling between two fixed supports. As a new result, it was reported that the presence of the flowing air diminished the stabilizing effect of viscosity, i.e. for certain values of the parameters characterizing viscoelasticity, the panel could be stable when surrounded by stationary air but unstable when the air was flowing.

In **PIII**, a model including Kelvin–Voigt viscoelasticity and aerodynamic reaction pressure as solution of potential flow problem around the moving thin web was presented. Predictions of the critical velocities and behaviour at different transport velocities by the model in **PIII** were compared with the predictions by the model in **PII**. As for the added mass model, for the potential flow model it was found that presence of flowing air diminished the stabilizing effect of viscosity. In the case that fluid velocity was equal to the panel velocity, the potential flow model predicted no such stabilizing effect of viscosity at all in the range of values tested.

The effect of gravity has often been neglected within the previous studies concerning axially moving materials. For stationary beams, there exists previous investigations. E.g. Duan and Wang (2008) presented exact solutions for the buckling of columns under self-weight with the help of hypergeometric functions. In **PIV**, the stability analysis of elastic panels was concerned for axially moving panels. It was reported, that the gravitational force had a major effect on the buckled shape of the panel but a minor effect on the critical panel velocity.

The study **PV** provided detailed analytical considerations of static instability (divergence) on the axially moving orthotropic membranes and plates. It was analytically shown that the eigenvalues of the problem determining the buckled shape in the cross-section are non-negative. The buckled shapes were found to significantly depend on the value of the in-plane shear modulus. The effect of the magnitude of the in-plane shear modulus on the critical plate velocity, however, was minor.

PVI presented a study, in which tension inhomogeneities were taken into account in the stability analysis for axially moving isotropic plates. The tension field in the plate with linear tension distribution at the in-flow and out-flow edges was solved analytically and also a lower limit for the critical plate velocity was derived. As a result, it was found that inhomogeneities in a tension profile may significantly decrease the critical velocities, and that even slight inhomogeneities have a dramatic response in the divergence shapes.

In **PVII**, an example of applying the analytical results for the critical velocities in optimization of the productivity of a process, in which the produced material travels as a continuous plate through a system of rollers and under cyclic tension, was presented. As a high plate velocity constrains the value of tension from below, the probability of arising and growing of edge cracks sets an upper limit for the value of tension. A new idea of finding an optimal value for the magnitude of tension was presented by formulating an analytical expression for process effectiveness (productivity) with the help of plate velocity and a number of tension cycles before fracture. Some examples of solving the obtained multi-objective optimization problem of maximizing the transport velocity, the loading cycles before fracture, and process effectiveness were given.

3 1D MODELS FOR MOVING MATERIALS

String and beam models are the usual 1D models used for the modelling of out-of-plane behaviour of thin moving or stationary materials. String and beam models are usually used to describe behaviour objects that are relatively thin and narrow compared to their length, and thus, roughly 1D themselves. The panel model in turn describes a 2D plate but with an assumption that the displacement does not vary in some coordinate direction. Mathematically, the beam and panel models are similar but the physical interpretation of the coefficients is slightly different for these models.

In the case of wide plates, e.g. paper webs, one natural choice is to consider plates under cylindrical deformation. By cylindrical deformation, we mean that the transverse displacement does not vary in the cross direction to the direction of movement of the plate (Figure 4). For the term cylindrical deformation, see

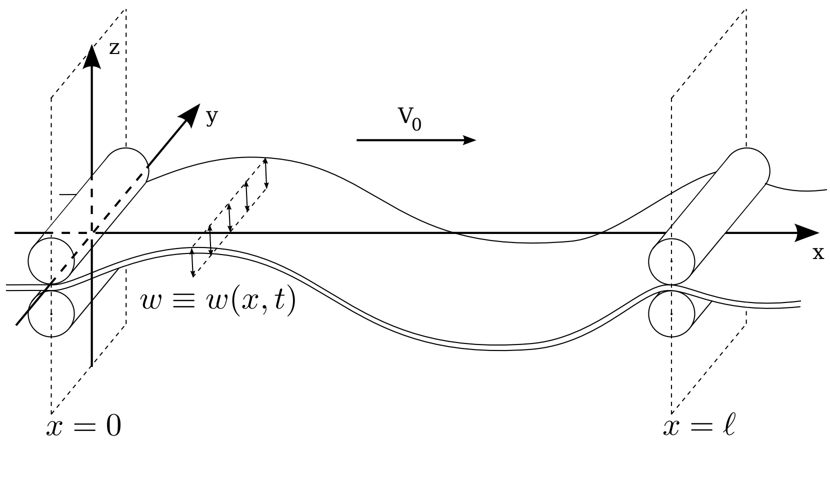


FIGURE 4 Assumption of cylindrical deformation. (From **PI**.)

e.g. Timoshenko and Woinowsky-Krieger (1959). We call the plate model with cylindrical deformation a *panel* model. The term panel has been used e.g. by Bisplinghoff and Ashley (1962) for plates, but since they focused on the case of cylindrical deformations in their examples, the term panel is adapted for this 1D plate model with cylindrical deformation here and in other previous publications (Banichuk et al., 2010c, 2011a, 2014).

Many rheological models to describe material properties such as elasticity, viscosity and plasticity exist. One of the simplest models for viscoelastic solids is the Kelvin–Voigt model. Regardless of its simplicity, the Kelvin–Voigt model is able to describe the most important rheological phenomena related to viscoelasticity such as creep and relaxation. In predicting creep after loading, the Kelvin–Voigt model gives good results, but the model is limited in expressing relaxation. Bearing this limitation in mind, we will use the Kelvin–Voigt model for material viscoelasticity to investigate fundamental qualitative effects of viscosity. As an advantage of simplicity of the Kelvin–Voigt model, as we will see below, we may express material viscosity with a single parameter and directly compare the results with the corresponding purely elastic material model.

3.1 Dynamic equation for an axially moving Kelvin–Voigt panel

We consider dynamic out-of-plane behaviour of an axially moving viscoelastic panel. The panel is assumed to be supported at $x = 0$ and $x = \ell$, and the length of the unsupported interval (span) is ℓ. The axial velocity of the panel is assumed to be constant and is denoted by V_0. The transverse displacement is described by the function $w = w(x, t)$. Assuming small transverse displacements, the dynamic equilibrium for the bending forces affecting the panel (according to Newton's second law) is

$$\frac{\partial^2 M}{\partial x^2} + T_0 \frac{\partial^2 w}{\partial x^2} = m \frac{\partial^2 w}{\partial t^2} + 2mV_0 \frac{\partial^2 w}{\partial x \partial t} + mV_0^2 \frac{\partial^2 w}{\partial x^2} , \qquad 0 < x < \ell , \qquad (1)$$

where m is the mass per unit area, and T_0 is a constant tension at the panel ends, having the unit of force per unit length.

The viscoelasticity of the material is described using the rheological Kelvin–Voigt model. The spring element is described by the elastic parameters E and ν, and the damper by the viscous damping coefficient η and the Poisson ratio for viscosity μ. See Figure 5.

Since we consider a panel, i.e., a 2D plate with the assumption that its transverse displacement does not vary in the y direction, we will first present the stress-strain relations of bending for the plate and then reduce them to the 1D relations to clarify the origin of the elastic and viscoelastic flexural stiffness parameters D and Y of the panel (below in (6)). We assume both volumetric behaviour (dilatation) and response to shear to be viscoelastic, which is a realistic assumption for paper materials (see e.g. Uesaka et al., 1980; Lif et al., 1999).

Under the assumption of plane stress, the constitutive stress-strain relations

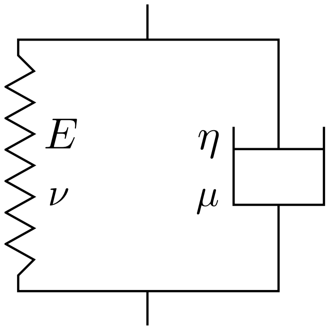


FIGURE 5 The Kelvin–Voigt viscoelastic material model. A 1D scheme of a 2D model. (From **PI**.)

for a viscoelastic Kelvin–Voigt panel are (Sobotka, 1984; Tang and Chen, 2013)

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) + \frac{\eta}{1-\mu^2}\left(\frac{\mathrm{d}\varepsilon_x}{\mathrm{d}t} + \mu\frac{\mathrm{d}\varepsilon_y}{\mathrm{d}t}\right) ,\\
\sigma_y &= \frac{E}{1-\nu^2}\left(\nu\varepsilon_x + \varepsilon_y\right) + \frac{\eta}{1-\mu^2}\left(\mu\frac{\mathrm{d}\varepsilon_x}{\mathrm{d}t} + \frac{\mathrm{d}\varepsilon_y}{\mathrm{d}t}\right) ,\\
\tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy} + \frac{\eta}{2(1+\mu)}\frac{\mathrm{d}\gamma_{xy}}{\mathrm{d}t} ,
\end{aligned} \tag{2}$$

where σ_x and σ_y are the normal stresses, and τ_{xy} is the shear stress due to bending. The bending strains ε_x and ε_y, and the shear strain γ_{xy}, for small deformations, are defined as (Timoshenko and Woinowsky-Krieger, 1959)

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} , \qquad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} , \qquad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} . \tag{3}$$

In (2), the derivative $\mathrm{d}/\mathrm{d}t$ denotes the material derivative $\mathrm{d}/\mathrm{d}t(\cdot) = \partial/\partial t(\cdot) + V_0\partial/\partial x(\cdot)$.

Denoting $\sigma = \sigma_x$ and $\varepsilon = \varepsilon_x$, and assuming cylindrical deformations (hence $\varepsilon_y = -z\partial^2 w/\partial y^2 = 0$), the relations (2) are reduced to

$$\sigma = \frac{E}{1-\nu^2}\varepsilon + \frac{\eta}{1-\mu^2}\left(\frac{\partial\varepsilon}{\partial t} + V_0\frac{\partial\varepsilon}{\partial x}\right) . \tag{4}$$

For the bending moment per unit length, we have

$$M = \int_{-h/2}^{h/2} z\sigma \,\mathrm{d}z = -\left[D\frac{\partial^2 w}{\partial x^2} + \Upsilon\left(\frac{\partial^3 w}{\partial x^2 \partial t} + V_0\frac{\partial^3 w}{\partial x^3}\right)\right] , \tag{5}$$

where we have used the notations

$$D = \frac{Eh^3}{12(1-\nu^2)} , \qquad \Upsilon = \frac{\eta h^3}{12(1-\mu^2)} . \tag{6}$$

Let us define the parameter t_R as a retardation time constant (see Sobotka, 1984)

$$t_\mathrm{R} = \frac{\eta}{E} . \tag{7}$$

The SI unit of t_{R} is the second. In literature, the parameter t_{R} has been also called a creep time constant (Marynowski, 2008) or a delay time (Zhou and Wang, 2007). With the help of (7) and assuming, for simplicity, that the elastic and viscous Poisson ratios coincide, i.e. $\mu = \nu$, we may write

$$\Upsilon = t_{\mathrm{R}} D \; .$$

Inserting expression (5) for the bending moment per unit length into (1) (see also Ding and Chen, 2008), we have

$$\begin{aligned} & m\frac{\partial^2 w}{\partial t^2} + 2mV_0\frac{\partial^2 w}{\partial x \partial t} + t_{\mathrm{R}} D \frac{\partial^5 w}{\partial x^4 \partial t} + \left(mV_0^2 - T_0\right)\frac{\partial^2 w}{\partial x^2} \\ & + D\frac{\partial^4 w}{\partial x^4} + V_0 t_{\mathrm{R}} D \frac{\partial^5 w}{\partial x^5} = 0 \; , \qquad 0 < x < \ell \; . \end{aligned} \tag{8}$$

In (8), m is mass per unit area, and T_0 is a constant tension at the panel ends, having the unit of force per unit length.

Since (8) is of the fifth order in space, we need five boundary conditions. It is the use of the material derivative in the viscoelastic constitutive relations for a beam or a panel model that leads to a partial differential equation that is fifth-order with respect to the space coordinate. In Ding and Chen (2008), Chen and Wang (2009), and Chen and Ding (2010), the fifth-order dynamic equation was attained but four boundary conditions (in space) were used. We will use five boundary conditions (in space). Mathematically, (8) is an odd order dispersive type partial differential equation, the time-dependence of which is similar to gyroscopic systems. The choice of boundary conditions for the full time-dependent problem to be well-posed is an open question. For boundary value problems of higher order differential equations, see e.g. Agarwal (1986).

Boundary conditions for a panel (or beam) being clamped at both ends can be derived, e.g., by setting clamped boundary conditions for the panel in the reference frame moving with the panel, and then performing an appropriate change of variables. For details, see e.g. Chen and Ding (2010), in which the case of an axially moving beam is considered.

To derive the fifth boundary condition, we assume that the bending moment $M = M(x,t)$ in (5) satisfies the continuity condition

$$\lim_{\delta \to 0} \int_{-\delta}^{+\delta} M(x,t) \, \mathrm{d}x = 0 \; .$$

Similar continuity condition was used by Flügge (1975) for an infinite beam on a continuous support, with a concentrated load moving along the beam at a constant velocity.

In the domain $x < 0$, let w^- denote the displacement of the panel. We have

$$\begin{aligned} \lim_{\delta \to 0} \Bigg\{ & -D\frac{\partial w}{\partial x}(+\delta,t) - \Upsilon\left[\frac{\partial^2 w}{\partial x \partial t}(+\delta,t) + V_0\frac{\partial^2 w}{\partial x^2}(+\delta,t)\right] \\ & + D\frac{\partial w^-}{\partial x}(-\delta,t) + \Upsilon\left[\frac{\partial^2 w^-}{\partial x \partial t}(-\delta,t) + V_0\frac{\partial^2 w^-}{\partial x^2}(-\delta,t)\right]\Bigg\} = 0 \; . \end{aligned} \tag{9}$$

We obtain in the limit $\delta \to 0$

$$- D\frac{\partial w}{\partial x}(0,t) - \Upsilon \left[\frac{\partial^2 w}{\partial x \partial t}(0,t) + V_0 \frac{\partial^2 w}{\partial x^2}(0,t) \right] + D\frac{\partial w^-}{\partial x}(0,t) + \Upsilon \left[\frac{\partial^2 w^-}{\partial x \partial t}(0,t) + V_0 \frac{\partial^2 w^-}{\partial x^2}(0,t) \right] = 0 \ . \quad (10)$$

Since $(\partial w/\partial x)(0,t) = 0$ and thus $(\partial^2 w/\partial x \partial t)(0,t) = 0$, by a kinematic continuity condition for the panel (Flügge, 1975)

$$\lim_{\delta \to 0} \left\{ \frac{\partial w}{\partial x}(+\delta, t) - \frac{\partial w^-}{\partial x}(-\delta, t) \right\} = 0 \ ,$$

we have $(\partial w^- / \partial x)(0,t) = 0$ and thus $(\partial^2 w^- / \partial x \partial t)(0,t) = 0$. Substituting these into (10), we obtain

$$\frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w^-}{\partial x^2}(0,t) \ .$$

That is, the second derivative of the panel deflections before and after the support must coincide at $x = 0$. We set $(\partial^2 w^- / \partial x^2)(0,t) = 0$, and obtain the fifth condition

$$\frac{\partial^2 w}{\partial x^2}(0,t) = 0 \ . \quad (11)$$

The derivation of the fifth boundary condition is also presented in **PI**.

A clamped boundary condition at the out-flow end and three conditions at the in-flow end are

$$w(0,t) = 0 \ , \qquad \frac{\partial w}{\partial x}(0,t) = 0 \ , \quad (12)$$

$$\frac{\partial^2 w}{\partial x^2}(0,t) = 0 \ , \quad (13)$$

$$w(\ell,t) = 0 \ , \qquad \frac{\partial w}{\partial x}(\ell,t) = 0 \ . \quad (14)$$

Equations (12)–(14) will be called the C^+-C conditions. If the condition (13) is removed, we obtain the clamped–clamped (C-C) boundary conditions.

In **PI**, a combination of boundary conditions with an elastic simply supported condition at the out-flow end and three conditions at the in-flow end (12)–(13), was also considered. The elastic simply-supported condition corresponds to zero moment for an elastic panel, but for the viscoelastic panel, it should be considered as a purely kinematical (i.e. displacement-like) condition. Alternatively, if Υ is small, it can be viewed as an approximative condition for the moment given by (5).

3.2 Axially moving viscoelastic panel interacting with surrounding fluid

In this thesis, two different models describing the fluid–structure interaction between the travelling viscoelastic panel and the surrounding, flowing fluid, are considered. In this section, the origin of the coordinate system is chosen in a slightly different manner compared to Section 3.1: the length of the span is 2ℓ, and the panel is supported at $x = -\ell$ and $x = \ell$. See Figure 6. This choice has been used in **PIII** for solving the aerodynamic problem in a convenient form. In **PII**, the coordinate system has been selected as in Section 3.1.

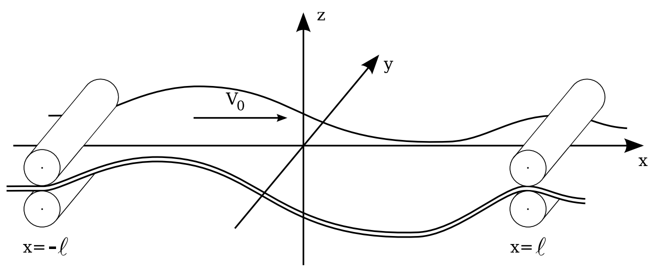


FIGURE 6 A travelling panel with cylindrical deformation. Note the choice of coordinate system. (From **PIII**.)

In **PII**, an added mass approach has been applied. As Chang and Moretti (1991) and Chang et al. (1991), we include added mass due to the transverse, Coriolis and centripetal acceleration, denoted by m_1, m_2 and m_3, respectively. Inserting the added mass terms into (8), we have

$$\begin{aligned} &(m+m_1)\frac{\partial^2 w}{\partial t^2} + 2(m+m_2)V_0\frac{\partial^2 w}{\partial x \partial t} + t_{\mathrm{R}}D\frac{\partial^5 w}{\partial x^4 \partial t} \\ &+ \left[(m+m_3)V_0^2 - T_0\right]\frac{\partial^2 w}{\partial x^2} + D\frac{\partial^4 w}{\partial x^4} + V_0 t_{\mathrm{R}} D\frac{\partial^5 w}{\partial x^5} = 0\,, \qquad -\ell < x < \ell\,, \end{aligned} \tag{15}$$

where

$$\begin{aligned} m_1 &= \frac{\pi}{4}C_{\mathrm{a}}\rho_{\mathrm{f}} b\,, \\ m_2 &= 2\rho_{\mathrm{f}}\frac{1}{V_0}\int_0^\delta U(r)\mathrm{d}r\,, \\ m_3 &= 2\rho_{\mathrm{f}}\frac{1}{V_0^2}\int_0^\delta U^2(r)\mathrm{d}r\,, \end{aligned} \tag{16}$$

and ρ_{f} is the density of the fluid, C_{a} the added mass coefficient depending on the problem geometry, b the width of the panel, $U = U(r)$ the velocity of the fluid with respect to the distance r from the panel, and δ the thickness of the moving fluid layer i.e. the boundary layer.

The second model for an axially moving viscoelastic panel submerged in flowing fluid, presented in **PIII**, accounts for axially flowing fluid as a free stream potential flow. The equation for the out-of-plane motion of the travelling panel is now written with the help of the aerodynamic reaction pressure $q_{\mathrm{f}} = q_{\mathrm{f}}(w)$:

$$\begin{aligned} & m\frac{\partial^2 w}{\partial t^2} + 2mV_0\frac{\partial^2 w}{\partial x \partial t} + t_{\mathrm{R}}D\frac{\partial^5 w}{\partial x^4 \partial t} + \left(mV_0^2 - T_0\right)\frac{\partial^2 w}{\partial x^2} \\ & + D\frac{\partial^4 w}{\partial x^4} + V_0 t_{\mathrm{R}} D\frac{\partial^5 w}{\partial x^5} = q_{\mathrm{f}}(w) \, , \qquad -\ell < x < \ell \, . \end{aligned} \tag{17}$$

We assume non-stationary aerodynamic flow in the xz plane.

Solving the aerodynamic problem via the techniques of complex analysis, we obtain the following analytical expression for $q_{\mathrm{f}}(w)$:

$$q_{\mathrm{f}}(w) = -\rho_{\mathrm{f}}\left(\frac{\ell}{\tau}\frac{\partial}{\partial t'} + v_\infty\frac{\partial}{\partial x'}\right)\int_{-1}^{1} N(\xi, x')\left(\frac{\ell}{\tau}\frac{\partial}{\partial t'} + v_\infty\frac{\partial}{\partial \xi}\right) w'(\xi, t')\mathrm{d}\xi \, , \tag{18}$$

where the kernel function N is

$$N(\xi, x') = \frac{1}{\pi}\ln\left|\frac{1 + \Lambda(\xi, x')}{1 - \Lambda(\xi, x')}\right| \, , \tag{19}$$

$$\Lambda(\xi, x') = \left[\frac{(1 - x')(1 + \xi)}{(1 - \xi)(1 + x')}\right] \, . \tag{20}$$

The solution process of solving the aerodynamic reaction pressure is outlined in **PIII**. The original solution with details is given in Banichuk et al. (2011a). In (18), we have used the dimensionless displacement $w' = w/h$, and the dimensionless coordinates $x' = x/\ell$, $t' = t/\tau$. In (18), v_∞ denotes the free stream velocity of the fluid, τ is called the characteristic time and it can be chosen freely, and ℓ is half the span length.

The boundary conditions for (15) and (17) are

$$w(-\ell, t) = 0 \, , \qquad \frac{\partial w}{\partial x}(-\ell, t) = 0 \, , \tag{21}$$

$$\frac{\partial^2 w}{\partial x^2}(-\ell, t) = 0 \, , \tag{22}$$

$$w(\ell, t) = 0 \, , \qquad \frac{\partial w}{\partial x}(\ell, t) = 0 \, . \tag{23}$$

3.3 Elastic panel travelling in gravitational field

The effect of gravity on the behaviour of axially moving materials has usually been considered relatively small and thus neglected from the models. Some recent studies on the topic exist, e.g. by Luo and Mote (2000). They studied equilibrium of travelling, elastic, sagged cables under uniformly distributed loading using a three-dimensional model.

Investigating the behaviour of an axially moving elastic panel in the gravitational field, the value of tension T varies within the longitudinal direction of the panel. Choosing the coordinate system such that the panel is supported at $x = 0$ and at $x = \ell$ and travelling in parallel with the x axis, tension depends on x, i.e. $T = T(x)$. The dynamic equation for an elastic plate with x-dependent tension reads

$$\begin{aligned} & m\frac{\partial^2 w}{\partial t^2} + 2mV_0\frac{\partial^2 w}{\partial x \partial t} + \frac{\partial T}{\partial x}\frac{\partial w}{\partial x} \\ & + \left(mV_0^2 - T\right)\frac{\partial^2 w}{\partial x^2} + D\frac{\partial^4 w}{\partial x^4} = 0\,, \qquad 0 < x < \ell\,. \end{aligned} \tag{24}$$

Note the additional term $\partial T/\partial x$, which is zero if T is a constant.

The value of tension T depends also on the angle between the direction of the gravitational force and the direction of motion of the panel. If the angle is zero, the value of tension T is given as follows:

$$T(x) = T_0 + mgx\,, \tag{25}$$

where T_0 is constant tension at the lower edge of the panel and g is the standard gravity. See Figure 7. The dynamic equation for the transverse displacement w

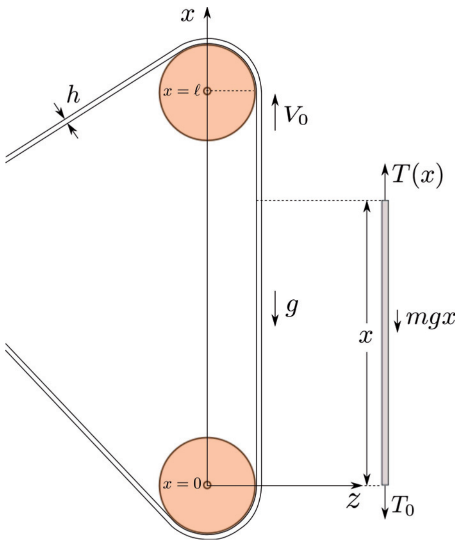


FIGURE 7 A travelling panel in a gravitational field. (From **PIV**.)

under tension expressed in (25) is

$$\begin{aligned} & m\frac{\partial^2 w}{\partial t^2} + 2mV_0\frac{\partial^2 w}{\partial x \partial t} + mg\frac{\partial w}{\partial x} \\ & + \left(mV_0^2 - T_0 - mgx\right)\frac{\partial^2 w}{\partial x^2} + D\frac{\partial^4 w}{\partial x^4} = 0\,, \qquad 0 < x < \ell\,. \end{aligned} \tag{26}$$

For (26), we set the simply supported boundary conditions

$$w(0,t) = 0\,, \qquad \frac{\partial^2 w}{\partial x^2}(0,t) = 0\,, \tag{27}$$

$$w(\ell,t)=0\,,\qquad \frac{\partial^2 w}{\partial x^2}(\ell,t)=0\,. \tag{28}$$

The static form of (26) is of the same form as the equation for stationary beams under self-weight, see e.g. Duan and Wang (2008).

In **PIV**, the static form of (26) with (27), (28) is analysed. Analytical estimates for the critical velocity V_0^{cr} can be derived. For details, see **PIV**.

3.4 Stability of travelling strings, beams, and panels

A natural choice to study the stability of systems, the behaviour of which can be expressed by a linear partial differential equation, is to use linear stability analysis. This analysis is often called dynamic (stability) analysis or Bolotin type of stability analysis after Bolotin (1963).

It is known that the normal vibrations of an elastic linear system are time-harmonic. This is noted e.g. by Xing and Liu (2009). For the stability analysis of such systems, it is standard to use the trial function

$$w(x,t)=e^{st}\,W(x)\,, \tag{29}$$

where s is complex, and $W(x)$ is an unknown eigenmode to be determined. The use of this trial function removes the time dependence from the partial differential equation, making it sufficient to solve a steady-state problem including the unknown scalar s. The resulting equation will be a partial differential equation in space, but polynomial with respect to s.

The trial function (29) produces a complex-valued solution $w(x,t)$. The space component $W(x)$ is typically real-valued for stationary materials, and complex-valued for moving materials. It is easy to see that in the case of linear partial differential equations with real-valued coefficients, the real and imaginary components of $w(x,t)$ will also be solutions of the original problem. Let $\mathcal{L}$ be a linear differential operator. For example, for the real part, we have

$$\begin{aligned}\mathrm{Re}\,(\mathcal{L}(w)) &= \mathrm{Re}\,[\mathcal{L}(\,\mathrm{Re}\,(w)+i\,\mathrm{Im}\,(w)\,)]\\ &= \mathrm{Re}\,[\mathcal{L}(\mathrm{Re}\,(w))+i\,\mathcal{L}(\mathrm{Im}\,(w))]\\ &= \mathcal{L}(\mathrm{Re}\,(w))\,,\end{aligned} \tag{30}$$

where the last equality holds only if the coefficients of $\mathcal{L}$ are real. The same observation holds for the imaginary part. Thus, both Re $w(x,t)$ and Im $w(x,t)$ are real-valued solutions of the original problem.

For moving materials, the real and imaginary components of $W(x)$ are typically not solutions of the auxiliary steady-state problem: using the trial function (29), only the full complex-valued solution $W(x)$ is valid for the auxiliary problem. It is only the complete solution $w(x,t)$ whose real and imaginary components satisfy the original problem separately. The reason is that the stability exponent s is complex.

Considering the system behaviour, the stability exponent s characterizes it in the following manner:

- If the imaginary part of s is non-zero, and
  - the real part of s is zero, the system vibrates harmonically with a small amplitude.
  - the real part of s is positive, the amplitude of transverse vibrations grows exponentially (flutter).
  - the real part of s is negative, the transverse vibrations are damped exponentially.
- If the imaginary part of s is zero, and
  - the real part of s is zero, the system has a critical point.
  - the real part of s is positive, the displacement of the system grows exponentially (divergence, buckling).
  - the real part of s is negative, the displacement of the system decreases exponentially.

The sign of the real part of s characterizes the stability of the system: if $\text{Re}\, s > 0$, the behaviour is unstable, and otherwise it is stable.

Depending on whether $\text{Im}\, s = 0$ or not in the critical state, at which the real part of s changes sign from negative to positive, we may classify the type of instability. If $\text{Im}\, s = 0$, we talk about static instability, and otherwise about dynamic instability. This classification is due to Bolotin (1963).

4 2D MODELS FOR MOVING MATERIALS

In the modelling of axially moving wide materials is reasonable to use 2D models. The membrane or plate models have been used in this context. The first studies on the stability of axially moving plates include Ulsoy and Mote (1980, 1982), and Lin (1997). Different models for axially moving elastic and viscoelastic orthotropic plates are reviewed or presented in the book of Marynowski (2008). In this section, models for axially moving elastic plates are concerned.

4.1 A travelling orthotropic plate

In this section, we discuss the models for axially moving orthotropic plates under a constant or a skewed tension profile at the fixed edges. Consider a rectangular part

$$\Omega = \{0 < x < \ell\,, \quad -b < y < b\} \tag{31}$$

of a moving plate in the cartesian coordinate system. The plate is supported at $x = 0$ and $x = \ell$. The length of the span is ℓ, and the width of the plate is $2b$. The constant axial velocity of the plate is denoted by V_0. The mass per unit area is denoted by m. The setup with a skewed tension profile is presented in Figure 8.

For an axially moving orthotropic plate, the dynamic equation for the transverse displacement $w(x, y, t)$ has the form (see e.g. Marynowski, 2008; for a stationary orthotropic plate Timoshenko and Woinowsky-Krieger, 1959)

$$\begin{aligned} m\frac{\partial^2 w}{\partial t^2} + 2mV_0\frac{\partial^2 w}{\partial x \partial t} + mV_0^2\frac{\partial^2 w}{\partial x^2} - T_{xx}\frac{\partial^2 w}{\partial x^2} - 2T_{xy}\frac{\partial^2 w}{\partial x \partial y} - T_{yy}\frac{\partial^2 w}{\partial y^2} \\ + D_1\frac{\partial^4 w}{\partial x^4} + 2D_3\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2\frac{\partial^4 w}{\partial y^4} = 0 \qquad \text{in } \Omega\,, \end{aligned} \tag{32}$$

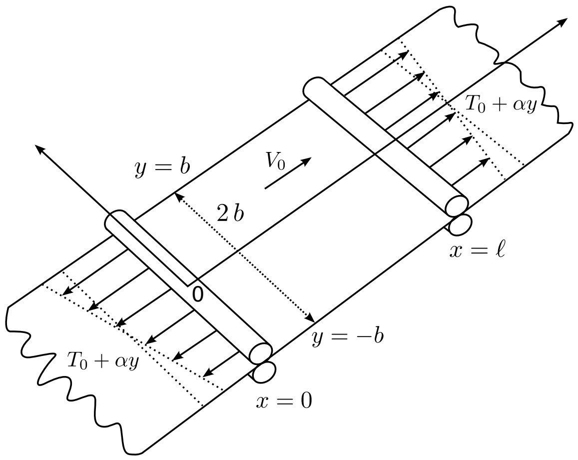


FIGURE 8 An axially moving orthotropic plate under a skewed tension profile. (From **PVI**.)

where the flexural stiffness parameters are

$$
\begin{aligned}
D_1 &= \frac{h^3}{12} \frac{E_1}{1 - \nu_{12}\nu_{21}} , \\
D_2 &= \frac{h^3}{12} \frac{E_2}{1 - \nu_{12}\nu_{21}} , \\
D_3 &= \frac{h^3}{12} \left(\frac{\nu_{12} E_2}{1 - \nu_{12}\nu_{21}} + 2G_{12} \right) .
\end{aligned}
\tag{33}
$$

In (33), the parameters E_1 and E_2 are the Young's moduli in the x and y directions, respectively, G_{12} is the shear modulus in the xy plane, ν_{12} and ν_{21} are the Poisson ratios in the xy plane, and T_{xx}, T_{yy}, and T_{xy} are the components of in-plane tension (xx and yy corresponding to uniaxial stress components and xy to shear stress). The in-plane tension components are assumed to satisfy the following equilibria:

$$
\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} = 0 , \qquad \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} = 0 , \qquad \text{in } \Omega . \tag{34}
$$

In the case of an orthotropic plate, the boundary conditions with two opposite edges simply supported and the remaining two edges free of traction read

$$
w(0,y,t) = w(\ell,y,t) = 0 , \qquad \frac{\partial^2 w}{\partial x^2}(0,y,t) = \frac{\partial^2 w}{\partial x^2}(\ell,y,t) = 0 , \qquad -b \le y \le b , \tag{35}
$$

$$
\left(\frac{\partial^2 w}{\partial y^2} + \beta_1 \frac{\partial^2 w}{\partial x^2} \right)(x,-b,t) = \left(\frac{\partial^2 w}{\partial y^2} + \beta_1 \frac{\partial^2 w}{\partial x^2} \right)(x,b,t) = 0 , \qquad 0 \le x \le \ell , \tag{36}
$$

$$
\left(\frac{\partial^3 w}{\partial y^3} + \beta_2 \frac{\partial^3 w}{\partial x^2 \partial y} \right)(x,-b,t) = \left(\frac{\partial^3 w}{\partial y^3} + \beta_2 \frac{\partial^3 w}{\partial x^2 \partial y} \right)(x,b,t) = 0 , \qquad 0 \le x \le \ell , \tag{37}
$$

where β_1 and β_2 are mechanical parameters defined as

$$\beta_1 = \nu_{12} , \qquad \beta_2 = \nu_{12} + \frac{4\,G_{12}}{E_2}(1 - \nu_{12}\nu_{21}) . \tag{38}$$

Equation (32) reduces to a dynamic equation for a membrane if we set the flexural stiffnesses D_1, D_2, and D_3 equal to zero. Setting two opposite edges fixed and two edges free, the boundary conditions for a (travelling) membrane are written as

$$w(0,y,t) = 0 , \quad w(\ell,y,t) = 0 , \quad -b \le y \le b , \tag{39}$$

$$\frac{\partial w}{\partial y}(x,-b,t) = 0 , \quad \frac{\partial w}{\partial y}(x,b,t) = 0 , \quad 0 \le x \le \ell . \tag{40}$$

Equation (32) becomes the dynamic equation of an isotropic plate if

$$\begin{aligned} E_1 &= E_2 = E , \\ \nu_{12} &= \nu_{21} = \nu , \\ G_{12} &= G = \frac{E}{2(1+\nu)} . \end{aligned} \tag{41}$$

Using a geometric average in-plane shear modulus introduced by Huber (1923),

$$G_{12} = G_{\mathrm{H}} = \frac{\sqrt{E_1 E_2}}{2(1 + \sqrt{\nu_{12}\nu_{21}})} , \tag{42}$$

the dynamic equation (32) can be represented mathematically in the same form as the dynamic equation for an axially moving isotropic plate.

Note, furthermore, that (32) reduces to a dynamic equation of a travelling elastic panel if the displacement in the y direction is assumed not to vary. In this case, all the derivatives with respect to y are equal to zero.

Consider the case where the axial tension is not constant but has a linear profile at the fixed edges. The boundary conditions for T_{xx}, T_{xy} and T_{yy} are

$$T_{xx}(0,y) = T_{xx}(\ell,y) = T_0 + \alpha y , \quad T_{xy}(0,y) = T_{xy}(\ell,y) = 0 , \quad -b \le y \le b , \tag{43}$$

$$T_{yy}(x,-b) = T_{yy}(x,b) = 0 , \quad T_{xy}(x,-b) = T_{xy}(x,b) = 0 , \quad 0 \le x \le \ell , \tag{44}$$

where T_0 is a constant tension and α,

$$0 < \alpha < T_0/b ,$$

is a given constant that is called here the tension profile skew parameter. The in-plane tensions satisfying (34), (43) and (44) are

$$T_{xx}(x,y) = T_0 + \alpha y , \quad T_{xy}(x,y) = 0 , \quad T_{yy}(x,y) = 0 , \qquad \text{in } \Omega . \tag{45}$$

The dynamic equation (32) then reads as

$$\begin{aligned} & m\frac{\partial^2 w}{\partial t^2} + 2mV_0\frac{\partial^2 w}{\partial x \partial t} + (mV_0^2 - T_0 - \alpha y)\frac{\partial^2 w}{\partial x^2} \\ & + D_1\frac{\partial^4 w}{\partial x^4} + 2D_3\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2\frac{\partial^4 w}{\partial y^4} = 0 \qquad \text{in } \Omega . \end{aligned} \tag{46}$$

The boundary conditions for (46) with two opposite edges simply supported and the two remaining edges free are given in (35)–(37). In **PVI**, these equations were analysed in the case of an elastic isotropic plate. For the case of an orthotropic plate under constant tension, (46) and (35)–(37) were analysed in **PV**.

4.2 Stability of travelling membranes and plates

In Section 3.4, stability of moving materials in the case of 1D models (strings, beams, and panels) was discussed. The same analysis can be applied for 2D models. As discussed in Section 2.1, stability of axially moving membranes and plates have been studied to some extent. The previous analyses predict that an axially moving elastic plate undergoes instability at a sufficiently high transport velocity and that the instability is of the divergence type. See e.g. Lin (1997) for axially moving isotropic plates, and Marynowski (2008) for axially moving orthotropic plates.

In this thesis, the analysis of travelling membranes and plates is limited to static analysis, in which we assume that a critical velocity, at which the membrane or plate undergoes divergence instability, exists. In such a case, the critical velocity and the corresponding divergence shape of the plate can be found by static analysis: the time-independent equation with boundary conditions and in homogeneous form is solved as an eigenvalue problem such that the smallest eigenvalue corresponds to the critical velocity and the corresponding eigenmode to the critical divergence shape. This is conducted by inserting the trial function

$$w(x,t) = e^{st}\, W(x) \,, \tag{47}$$

to the time-dependent equations and solving the case $s = 0$ for the critical value of the velocity V_0.

In the case of a constant or a linear tension profile at the fixed edges of the travelling (isotropic or orthotropic) plate, the eigenvalue problem can be solved in a simple manner using the trial function

$$W(x,y) = \sin\left(\frac{\pi x}{\ell}\right) f\left(\frac{y}{b}\right) \,, \tag{48}$$

where $f(y/b)$ is an unknown function, and $\sin(\pi x/\ell)$ is known to be a solution in x direction in the case of an isotropic case (see e.g. Lin, 1997). The same form can be seen to be applicaple also for an orthotropic plate. With this trial function, the x dependence is removed from the equations, and it is sufficient to solve an eigenvalue problem, in which the eigenfunctions depend only on the y coordinate.

Let us outline the procedure for an axially moving orthotropic plate under a linear tension profile (Saksa and Jeronen, 2012). We present the solution of (46) and (35)–(37) in the form of (47) and set $s = 0$.

Introducing a new variable $\eta = y/b$ and inserting (47) (with $s = 0$) and (48) into (46), we obtain

$$r^4 H_2 \frac{\mathrm{d}^4 f}{\mathrm{d}\eta^4} - 2\, r^2 H_3 \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + (H_1 + \tilde{\alpha}\eta) f = \lambda f \,, \qquad -1 < \eta < 1 \,, \tag{49}$$

where

$$r = \frac{\ell}{\pi b} \,, \qquad \tilde{\alpha} = \frac{b \ell^2}{\pi^2 D_0} \alpha \,, \tag{50}$$

the eigenvalue λ is defined as

$$\lambda = \frac{\ell^2}{\pi^2 D_0} \left(m V_0^2 - T_0 \right) \,, \tag{51}$$

and the dimensionless bending rigidities are

$$H_1 = \frac{D_1}{D_0} \,, \qquad H_2 = \frac{D_2}{D_0} \,, \qquad H_3 = \frac{D_3}{D_0} \,. \tag{52}$$

In (52), D_0 can be chosen freely, e.g., $D_0 = D_1$.

The boundary conditions (36)–(37) at $\eta = \pm 1$ become

$$\left(r^2 \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} - \beta_1 f \right)(-1) = 0 \,, \qquad \left(r^2 \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} - \beta_1 f \right)(1) = 0 \,, \tag{53}$$

$$\left(r^2 \frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} - \beta_2 \frac{\mathrm{d} f}{\mathrm{d}\eta} \right)(-1) = 0 \,, \qquad \left(r^2 \frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} - \beta_2 \frac{\mathrm{d} f}{\mathrm{d}\eta} \right)(1) = 0 \,. \tag{54}$$

The parameters β_1 and β_2 are explained above in equation (38). Note that for an isotropic material $H_1 = H_2 = H_3 = 1$ with $D_0 = D$, and $\beta_1 = \nu$ and $\beta_2 = 2 - \nu$. For comparison, see **PVI**.

For the case of an orthotropic plate under constant tension, the details of the solution process are presented in **PV**. Details for the case of an isotropic plate under a linear tension profile are given in **PVI**.

4.3 Application of stability analysis on productivity optimization

Considering out-of-plane vibrations of axially moving materials, it is known that an increase in tension has a stabilizing effect but a decrease in tension may lead to a loss of stability. From the viewpoint of fracture, tension has the opposite effect: high tension may lead to the growth or appearance of cracks, and tension low enough then guarantees safe operation. Seeking the optimal value of tension but having opposing objectives, we encounter a multi-objective optimization problem, which usually has no unique optimal solution but a set of "equally optimal", Pareto optimal solutions. In this optimization analysis, we may apply the solutions from stability analysis for the critical velocities.

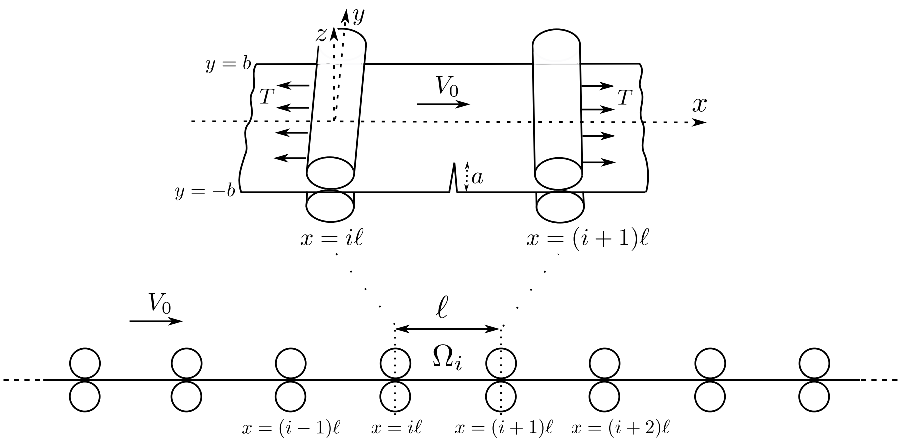


FIGURE 9 An axially moving plate having an initial crack and supported by a system of rollers. (From **PVII**.)

Consider an axially moving elastic isotropic plate, travelling at a constant velocity V_0 and having an initial crack, between a system of subsequent supports (rollers). The plate undergoes open draws between the supports, the open draws being assumed to be equal in length. For the problem setup, see Figure 9.

The plate is assumed to be tensioned and subject to small cyclic tension variations during the process. For one cycle, tension increases from $T = T_{\min}$ up to $T = T_{\max}$ (the loading process) and then decreases from $T = T_{\max}$ down to $T = T_{\min}$ (the unloading process). We suppose quasi-static processes meaning that the dynamic effects are excluded.

We define parameters T_0 (average tension) and ΔT (small tension variation) such that

$$T_{\min} = T_0 - \Delta T \quad \text{and} \quad T_{\max} = T_0 + \Delta T \,, \tag{55}$$

$$T_{\min} \leq T \leq T_{\max} \,, \tag{56}$$

and

$$T_0 - \Delta T > 0 \quad \text{and} \quad \frac{\Delta T}{T_0} \ll 1 \,. \tag{57}$$

Note that $T_{\max} - T_{\min} = 2\Delta T$.

The critical velocity at which the plate undergoes divergence instability can be found analytically as presented for an orthotropic plate in **PV**. The critical velocity can be solved for a plate under constant tension T_0, e.g., from Eq. (35) in **PV**:

$$V_0^{\text{cr}} = \sqrt{\frac{T_0}{m} + \frac{\gamma_*^2}{m}\frac{\pi^2 D_0}{\ell^2}} \,. \tag{58}$$

In the case of an isotropic plate,

$$D_0 = D = \frac{h^3}{12}\frac{E}{1-\nu^2} \,.$$

The parameter $\gamma = \gamma_*$ is found as the root of the equation

$$\Phi(\gamma, r) - \Psi(\gamma, \nu) = 0 \,, \tag{59}$$

where

$$\begin{aligned} \Phi(\gamma, r) &= \tanh\left(\frac{\sqrt{1-\gamma}}{r}\right) \coth\left(\frac{\sqrt{1+\gamma}}{r}\right) \,, \\ \Psi(\gamma, \nu) &= \frac{\sqrt{1+\gamma}}{\sqrt{1-\gamma}} \frac{(\gamma+\nu-1)^2}{(\gamma-\nu+1)^2} \,, \end{aligned} \tag{60}$$

and r is the scaled aspect ratio defined in (50). Note that in place of the analytical expression (58), one could use a more complicated model for computing the critical velocity with respect to tension e.g. taking into account surrounding air. Here, we limit the discussion to the vacuum case with provided analytical solution.

For predicting the critical number cycles before fracture, we use Paris' law

$$\frac{\mathrm{d}a}{\mathrm{d}n} = C(\Delta K)^k \,, \tag{61}$$

where $a = a(n)$ is the length of the crack, n is number of loading cycles, C and k are constants depending on the material, and ΔK is the difference between the maximum stress intensity factor $K_{\max}$ (at maximum loading) and the minimum stress intensity factor $K_{\min}$ (at minimum loading). Here, ΔK can be expressed as

$$\Delta K = \frac{2\beta\sqrt{\pi a}}{h} \Delta T \,, \tag{62}$$

where β is a parameter depending on the crack geometry and ΔT is the small tension variation defined in (56) and (57). The critical number of cycles can be solved from (61). Using an initial condition $a(0) = a_0$ and the assumption that $\Delta T / T \ll 1$, i.e., the variation of tension is small compared to the average tension T, we obtain in the case that $k \neq 2$:

$$n^{\mathrm{cr}} = A \left[a_0^{-\frac{k-2}{2}} - (\zeta T_0)^{k-2} \right] \,, \tag{63}$$

where

$$A = \frac{2}{(k-2)C\kappa^k} \,, \qquad \kappa = \frac{2\beta\sqrt{\pi}}{h} \Delta T \,, \qquad \zeta = \frac{\beta\sqrt{\pi}}{K_{\mathrm{C}} h} \,, \tag{64}$$

and K_{C} is the fracture toughness of material.

To find an optimal value of tension, we would like to maximize the transport velocity V_0^{cr} in (58), maximize the number of cycles before fracture n^{cr} in (63), and maximize the process effectiveness. We estimate the process effectiveness or production as the total mass of the material that is being processed:

$$M = m_0 V_0 n \tau \,, \tag{65}$$

where $m_0 = 2bm$, and τ is a cycle time period, which is assumed to be small as the number of cycles n is assumed to be large, so that $n\tau$ gives approximately the process time.

Using the critical velocity V_0^{cr}, longevity n^{cr} and process effectiveness M^{cr} and noticing that these values depend on the value of in-plane average tension T_0, we define the following vector function:

$$J(T_0) = \left\{ \begin{array}{c} J_{\mathrm{V}}(T_0) \\ J_{\mathrm{N}}(T_0) \\ J_{\mathrm{M}}(T_0) \end{array} \right\} \equiv \left\{ \begin{array}{c} V_0^{\mathrm{cr}}(T_0) \\ n^{\mathrm{cr}}(T_0) \\ M^{\mathrm{cr}}(T_0) \end{array} \right\} , \tag{66}$$

where M^{cr} is given by (65) with critical parameter values.

Now, we formulate the multi-objective optimization problem. It is required to determine the optimal value T_0^* of in-plane tension T_0 that gives a maximum of the considered vector function, i.e.

$$J^* = J(T_0^*) = \max_{T_0} \ J(T_0) \ , \tag{67}$$

where the *max* operation is considered in the Pareto sense.

To solve this multi-objective optimization problem, we apply the weighting method, which is applicable for concave object functions in seeking Pareto optimal results. The multi-objective optimization problem of finding the optimal in-plane tension T_0^* separately for different particular cases is analysed in **PVII**.

5 NUMERICAL SOLUTION PROCESS

For numerical computations, it is often convenient to express the problem to be solved in a dimensionless form. This helps the investigator to better understand how the solution depends on different parameters and also simplifies the mathematical formulation and the implementation.

Consider the axially moving viscoelastic panel discussed in Section 3.1 and the problem (8), (12)–(14). We first introduce the dimensionless coordinates

$$x' = \frac{x}{\ell}\,, \quad t' = \frac{t}{\tau}\,, \tag{68}$$

and the dimensionless out-of-plane displacement

$$w'(x', t') = \frac{w(x, t)}{h}\,. \tag{69}$$

Inserting (68) and (69) into (8), omitting the primes, multiplying by $\ell^2/(T_0 h)$, and choosing

$$\tau = \ell\sqrt{\frac{m}{T_0}}\,, \tag{70}$$

we obtain

$$\frac{\partial^2 w}{\partial t^2} + 2c\frac{\partial^2 w}{\partial x \partial t} + \gamma\alpha\frac{\partial^5 w}{\partial x^4 \partial t} + (c^2 - 1)\frac{\partial^2 w}{\partial x^2} + \alpha\frac{\partial^4 w}{\partial x^4} + \gamma\alpha c\frac{\partial^5 w}{\partial x^5} = 0\,, \qquad 0 < x < 1\,, \tag{71}$$

where

$$c = \frac{V_0}{\sqrt{T_0/m}}\,, \qquad \alpha = \frac{D}{\ell^2 T_0}\,, \qquad \gamma = \frac{\lambda}{\tau} = \frac{\eta}{E}\frac{\sqrt{T_0}}{\ell\sqrt{m}}\,. \tag{72}$$

Inserting the time-harmonic trial function (29) into (71) and dividing by e^{st}, we get

$$s^2 W + s\left(2c\frac{\partial W}{\partial x} + \gamma\alpha\frac{\partial^4 W}{\partial x^4}\right) + (c^2 - 1)\frac{\partial^2 W}{\partial x^2} + \alpha\frac{\partial^4 W}{\partial x^4} + \gamma\alpha c\frac{\partial^5 W}{\partial x^5} = 0\,, \tag{73}$$

for $0 < x < 1$. The C$^+$-C boundary conditions for the dimensionless displacement function W are

$$W(0) = 0\,, \qquad \frac{\partial W}{\partial x}(0) = 0\,, \tag{74}$$

$$\frac{\partial^2 W}{\partial x^2}(0) = 0\,, \tag{75}$$

$$W(1) = 0\,, \qquad \frac{\partial W}{\partial x}(1) = 0\,. \tag{76}$$

Problem (73)–(76) is homogeneous and can be considered as an eigenvalue problem with respect to eigenvalues s and eigenfunctions W. To solve the eigenvalue problem numerically, it is to be discretized. We outline here the finite difference discretization approach, which is a relevant choice, as we have a simple geometry and the method adapts to five boundary conditions.

We seek $\mathbf{w} = (w_1, \ldots, w_n)$ satisfying the discretised form of (73)–(76). We use central differences of second-order asymptotic accuracy:

$$\begin{gathered}
\frac{\partial w_j}{\partial x} = \frac{w_{j+1} - w_{j-1}}{2\Delta x}\,, \qquad \frac{\partial^2 w_j}{\partial x^2} = \frac{w_{j+1} - 2w_j + w_{j-1}}{(\Delta x)^2}\,, \\
\frac{\partial^4 w_j}{\partial x^4} = \frac{w_{j+2} - 4\,w_{j+1} + 6\,w_j - 4\,w_{j-1} + w_{j-2}}{(\Delta x)^4}\,, \\
\frac{\partial^5 w_j}{\partial x^5} = \frac{w_{j+3} - 4\,w_{j+2} + 5\,w_{j+1} - 5\,w_{j-1} + 4\,w_{j-2} - w_{j-3}}{2(\Delta x)^5}\,.
\end{gathered} \tag{77}$$

The interval $[0, \ell]$ is divided to $n+1$ subintervals equal in length. The end points of the subintervals are labelled as $0 = x_0, x_1, x_2, \ldots, x_n, x_{n+1} = \ell$. We use two virtual points (w_{-2} and w_{-1}) at the in-flow end and one virtual point (w_{n+2}) at the out-flow end. From the boundary conditions (74)–(76), we get at the in-flow end:

$$\begin{aligned}
w_{-2} &= -w_2 && \left(\text{from } \frac{\partial^2 W}{\partial x^2}(0) = 0\right), \\
w_{-1} &= w_1 && \left(\text{from } \frac{\partial W}{\partial x}(0) = 0\right), \\
w_0 &= 0\,,
\end{aligned}$$

and at the out-flow end:

$$\begin{aligned}
w_{n+1} &= 0\,, \\
w_{n+2} &= w_n\,.
\end{aligned}$$

In (77), the grid spacing $\Delta x = 1/(n+1)$. We use the following backward difference scheme to calculate the fifth-order derivative at the out-flow end ($j = n$):

$$\frac{\partial^5 w_j}{\partial x^5} = \frac{3\,w_{j+2} - 16\,w_{j+1} + 35\,w_j - 40\,w_{j-1} + 25\,w_{j-2} - 8\,w_{j-3} + w_{j-4}}{2(\Delta x)^5}\,.$$

We denote the derivative matrices by $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_4, \mathbf{K}_5$, built up with the help of (77) with the following correspondence:

$$\mathbf{K}_1 : \frac{\partial W}{\partial x} \,, \qquad \mathbf{K}_2 : \frac{\partial^2 W}{\partial x^2} \,, \qquad \mathbf{K}_4 : \frac{\partial^4 W}{\partial x^4} \,, \qquad \mathbf{K}_5 : \frac{\partial^5 W}{\partial x^5} \,.$$

Inserting the matrices $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_4, \mathbf{K}_5$ into (73), we obtain the matrix equation

$$s^2 \mathbf{w} + s \left[2c\mathbf{K}_1 + \gamma\alpha\mathbf{K}_4 \right] \mathbf{w} + \left[(c^2 - 1)\mathbf{K}_2 + \alpha\mathbf{K}_4 + \gamma\alpha c\mathbf{K}_5 \right] \mathbf{w} = 0 \,. \tag{78}$$

Note that in the case $\alpha = 0$ or $c = 0$, we obtain a fourth-order equation needing only four boundary conditions. This has been taken into account: the virtual point w_{-2} is needed only by the matrix $\mathbf{K}_5$. When $\mathbf{K}_5$ is removed from the matrix equation (78), the boundary condition (75) is simultaneously removed from the discretised problem.

It was numerically confirmed that when we decrease the value of α, the solution of (78) with the boundary conditions C^+-C approaches the solution of the corresponding elastic problem with the boundary conditions C-C. This was the case even if we selected $w_{-2} = w_2$ from $(\partial W / \partial x)(0) = 0$, and $w_{-1} = -w_1$ from $(\partial^2 W / \partial x^2)(0) = 0$.

The matrix equation (78), which is a quadratic eigenvalue problem with respect to s, can be rewritten as

$$\begin{bmatrix} -\mathbf{M}_1 & -\mathbf{M}_0 \\ \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} s\mathbf{w} \\ \mathbf{w} \end{bmatrix} = s \begin{bmatrix} s\mathbf{w} \\ \mathbf{w} \end{bmatrix} \,, \tag{79}$$

where

$$\begin{aligned} \mathbf{M}_0 &= (c^2 - 1)\mathbf{K}_2 + \alpha\mathbf{K}_4 + \gamma\alpha c\mathbf{K}_5 \,, \\ \mathbf{M}_1 &= 2c\mathbf{K}_1 + \gamma\alpha\mathbf{K}_4 \,. \end{aligned} \tag{80}$$

The matrix equation (79) is a linear eigenvalue problem of the standard form

$$\mathbf{A}\mathbf{y} = s\mathbf{y} \tag{81}$$

with

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}_1 & -\mathbf{M}_0 \\ \mathbf{I} & 0 \end{bmatrix} \,, \qquad \mathbf{y} = \begin{bmatrix} s\mathbf{w} \\ \mathbf{w} \end{bmatrix} \,. \tag{82}$$

The technique of linearising a quadratic eigenvalue problem in the way presented in (79), (81), and (82) is standard (see e.g. Tisseur and Meerbergen, 2001).

The size of the eigenvalue problem (81), depending on the number of subintervals in the finite difference discretization, is $2n$. Thus, the number of eigenvalues s of (81) is $2n$. The larger the value of n is, the more accurate are the values of the lowest eigenvalues. In Figure 10, we present three lowest eigenvalue pairs with respect to the dimensionless panel velocity c. The critical velocity is denoted by c^{cr}, marking the point at which the real part of s becomes positive for the first eigenmode.

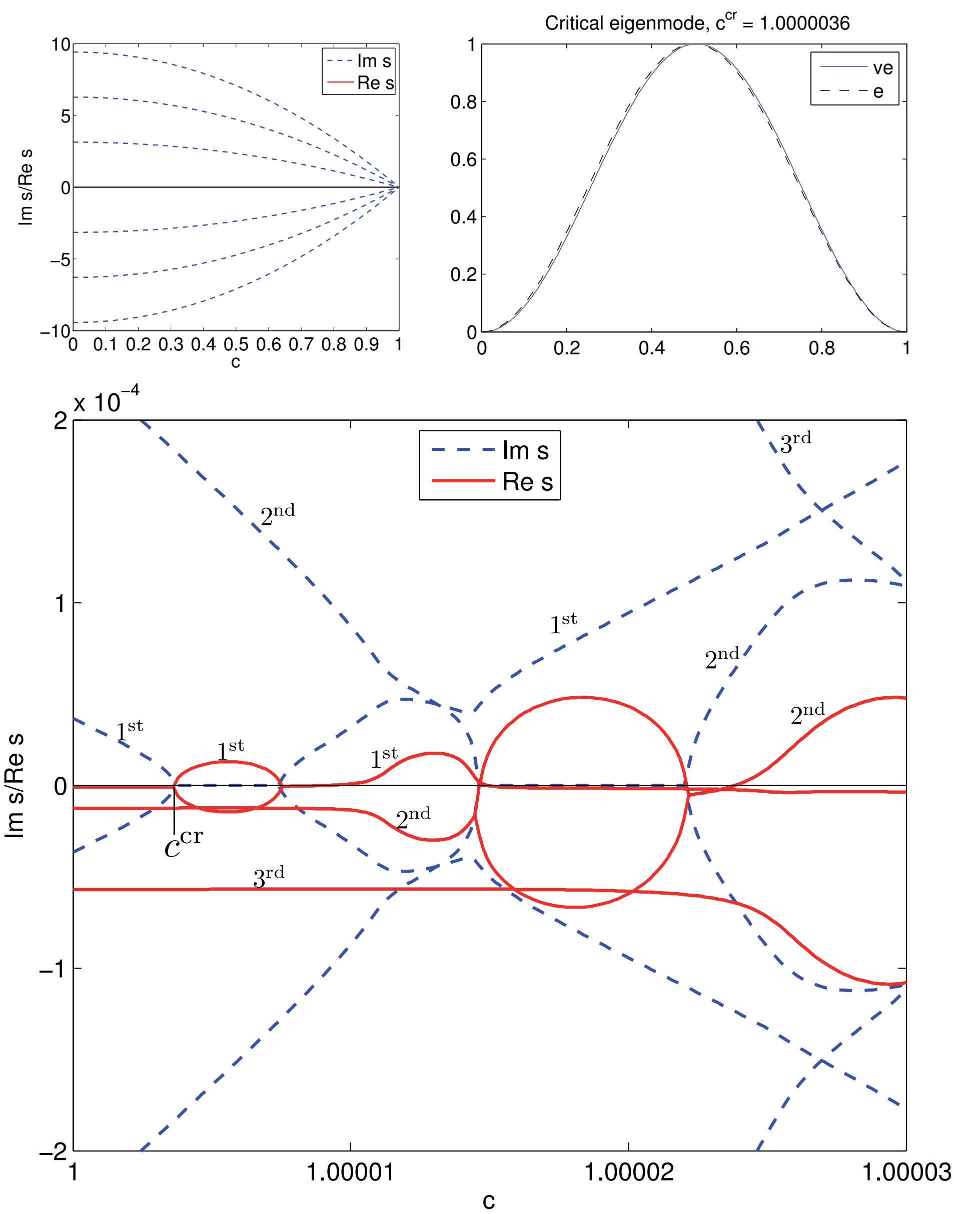


FIGURE 10 Three lowest eigenvalue pairs for a travelling viscoelastic panel. The parameter c is the dimensionless panel velocity. In the upper right subfigure, the eigenmode corresponding to the critical velocity is plotted with a solid line. The dashed line shows the corresponding critical eigenmode for an elastic clamped-clamped panel. (From **PI**.)

Also in **PII**, the finite difference discretization method is used. Due to the added mass terms, there will be some additional multipliers ζ, ζ_2 and ζ_3 in the dimensionless forms, and the matrices $\mathbf{M}_0$ and $\mathbf{M}_1$ are in that case

$$\begin{aligned}\mathbf{M}_0 &= (\zeta_3 c^2 - \zeta)\mathbf{K}_2 + \alpha\zeta\mathbf{K}_4 + \gamma\alpha c\zeta\mathbf{K}_5 \,, \\ \mathbf{M}_1 &= 2c\zeta_2\mathbf{K}_1 + \gamma\alpha\zeta\mathbf{K}_4 \,, \end{aligned} \tag{83}$$

where

$$\zeta = \frac{m}{m+m_1} \,, \qquad \zeta_2 = \frac{m+m_2}{m+m_1} \,, \qquad \zeta_3 = \frac{m+m_3}{m+m_1} \,. \tag{84}$$

In **PIII**, the results presented are computed via the finite element method using a Hermite basis with C^2 continuity, consisting of fifth-degree polynomials. The author solved the problem (17), (12)–(14) via the finite difference method outlined above. Comparison was made with the finite element results presented in the article: both methods gave the same results.

If, by analysing the eigenvalue spectra, a velocity exists, at which the panel (or plate) loses stability in a divergence form, the critical velocity and the corresponding mode can be solved from (78) in the case that $s = 0$. Again, we end up with a quadratic eigenvalue problem, but this time, with respect to c:

$$c^2(\mathbf{K}_2\mathbf{w}) + c(\gamma\alpha\mathbf{K}_5\mathbf{w}) + (-\mathbf{K}_2 + \alpha\mathbf{K}_4)\,\mathbf{w} = 0 \,. \tag{85}$$

If we neglect viscosity and consider the elastic equation corresponding to (85), the parameter $\gamma = 0$. By choosing e.g. $\lambda = c^2$, the eigenvalue problem is reduced to a linear eigenvalue problem with respect to the eigenvalue λ.

In **PIV–PVII**, indeed, the critical velocity and the corresponding critical mode are solved in the case that $s = 0$. The discretization for a travelling elastic panel in a gravitational field with simply supported boundary conditions is done using the Fourier–Galerkin method in **PIV**. The Fourier–Galerkin method is related to the finite element method, the both methods belonging to the Galerkin methods, and in the Fourier–Galerkin method we have global basis functions. The unknown function W was presented as a Galerkin series in basis

$$\varphi_j(x) = \sin(j\pi x) \,, \qquad 0 < x < 1 \,, \quad j = 1, 2, 3, \ldots \,, \tag{86}$$

in which the basis functions satisfy the boundary conditions

$$W(0) = 0 \,, \quad \frac{\partial^2 W}{\partial x^2}(0) = 0 \,, \quad W(1) = 0 \,, \quad \frac{\partial^2 W}{\partial x^2}(1) = 0 \,.$$

In **PVI**, stability of an axially moving isotropic plate under a linear tension profile was investigated via static analysis. In this case, the use of the trial function (48) lead to an eigenvalue problem that was solved via central finite differences.

In **PV** and **PVII**, the problems were practically solved analytically and numerical tools were used for simply solving transcendental equations and for visualisations. In Figure 11, the critical eigenmodes are plotted for a travelling orthotropic plate under a skewed tension profile summarizing the effects of in-plane shear modulus and tension inhomogeneities on the shape of the mode. For the separate analyses, see **PV** and **PVI**.

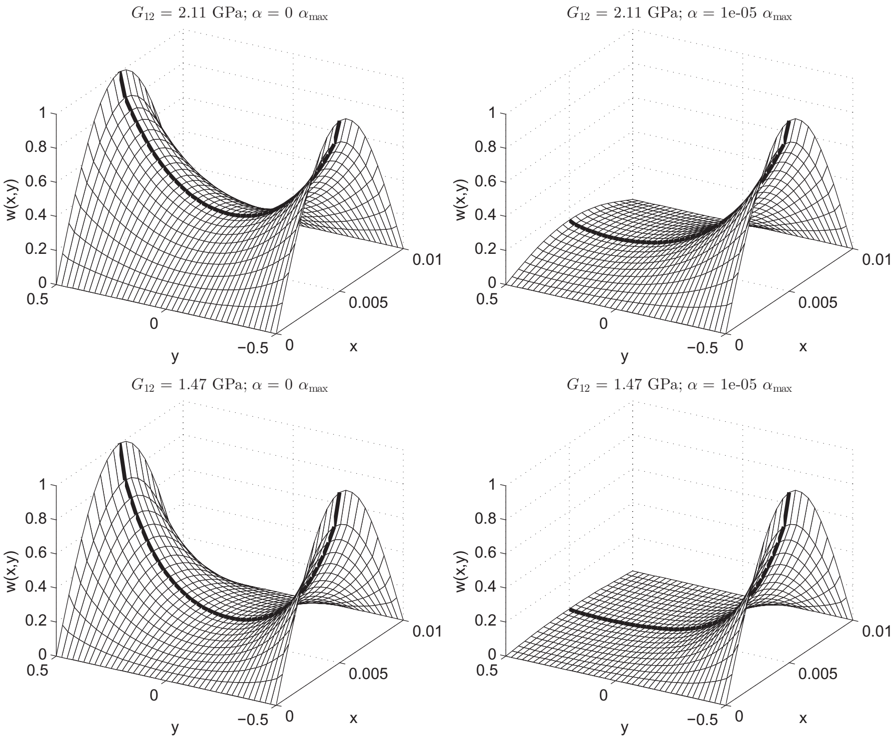


FIGURE 11 Effects of the tension inhomogeneities and the value of the shear modulus on the critical eigenmode. The tension profile skew parameter α increases from left to right, and the shear modulus decreases from top to bottom. (From Saksa and Jeronen, 2012.)

6 CONCLUSION

Out-of-plane vibrations of axially moving materials have been studied previously to some extent and the results often have given insight to the limits inside which the vibrations are predicted to be stable. These models are often based on continuum mechanics, where equilibrium equations describe the dynamic (or static) behaviour.

This thesis presented new additions to the models for travelling panels and plates. We presented a derivation for a fifth physical boundary condition for the differential equation of a travelling viscoelastic Kelvin–Voigt panel, introducing a new set of boundary conditions for this equation. Furthermore, we combined the vacuum equations for travelling viscoelastic panel with the aerodynamic reaction pressure term accounting for surrounding flow. Tension variation due to Earth's gravity was taken into account in a model of an axially moving panel for the first time. A new application for stability results was presented as an optimization problem to find an optimal value of tension average for an axially moving brittle plate undergoing cyclic tension variations, instability limiting the value from below and fatigue fracture from above. For the models studied previously in this field, some new properties were found.

To analyse the models and the mechanical stability of the systems they present, classical tools (e.g. Bolotin, 1963) were applied. In numerical studies, the appropriate methods were chosen taking into account problem geometry, boundary conditions, partial solutions etc.

As results of stability analyses, it was found that, for a travelling Kelvin–Voigt panel, the introduced viscosity may make the panel behave stably at a velocity at which the elastic panel undergoes divergence instability. If fluid is introduced to the model, the stabilizing effect of panel viscosity is diminished. E.g. for certain values of the parameters characterizing viscoelasticity, the panel behaviour was stable when surrounded by stationary air but unstable when the air was flowing. The effect of the gravitational force on the critical panel velocity was found to be minor, though the buckled shape corresponding to the critical velocity of the elastic panel notably depended on the angle between the directions of transport and the gravitational force.

For a travelling orthotropic plate, the effects of the value of the in-plane shear modulus and the ratio of Young's moduli on the buckled shape were investigated. Both the ratio of Young's moduli and the value of the in-plane shear modulus affected notably the buckled shapes and slightly the critical plate velocity. The skewness of tension profile at the edges was reported to significantly affect the critical velocity of an axially moving (isotropic) plate.

The models presented or applied in this thesis were limited to linear equations. If e.g. large deformations or the behaviour of the panel or the plate at a super-critical velocity are to be analyzed, non-linear models are needed. From the application point of view, it is important that the models describe the real situation as precisely as possible in order to give realistic predictions. E.g. by excluding the surrounding fluid from the model for an axially moving thin paper web, one may overestimate the critical transport velocity of the web by a factor of four (Pramila, 1986). Surrounding fluid can be seen as destabilizing factor for the out-of-plane vibrations of a travelling web. Tension inhomogeneities were also found to have a destabilizing effect. Material viscosity, in turn, seemed to slightly stabilize the out-of-plane vibrations. In computations, the studied parameter ranges were limited to the values describing paper material and paper production. For other materials, the importance of the factors affecting the predictions of the critical velocity may appear in a different manner. In this thesis, we also limited the discussion to the cases, in which the web velocity is constant. In paper machines, there is actually a velocity difference between each two consecutive rollers. Taking this into account will be a topic of future research.

The results in this thesis extend knowledge about models for axially moving panels and plates. The research itself is basic research, but closely related to applications in industry. The models studied in this thesis are relatively light as for numerical computations. This allows for comprehensive parametric studies and separate analyses of effects of different physical factors. Fast solvers provide us also with applicability for real time computations and tools for the use of industry.

YHTEENVETO (FINNISH SUMMARY)

Työn otsikko: Aksiaalisesti liikkuvien viskoelastisten materiaalien mallinnuksesta ja stabiiliudesta

Tämän työn esittelemän tutkimuksen innoittajana ovat olleet teollisuuden prosessit, joissa jokin materiaali kulkee nauhana tai leveänä jatkuvana levynä rullien tai telojen ohjaamaa rataa pitkin. Näissä prosesseissa esiintyy usein kohtia, joissa kulkevan materiaalin rata ei ole tuettu rullien välisellä osalla. Esimerkiksi paperinvalmistuksessa tällainen kohta on useissa paperikoneissa märän pään ja kuivatusosan välissä, missä paperiraina kulkee ilman tukea puristinosasta kuivatusosan teloille.

Kuten paperinvalmistuksessa myös monissa muissa näiden tasossa liikkuviksi materiaaleiksi kutsuttujen kohteiden sovelluksissa havitellaan suuria radankuljetusnopeuksia. Suuret nopeudet näissä kuljetuksissa saattavat kuitenkin radan tukemattomissa kohdissa johtaa epästabiiliin käyttäytymiseen. Tämä tosiasia onkin varmasti osasyynä tasossa liikkuvien materiaalien mallintamisen jo yli puoli vuosisataa kestäneelle tieteelliselle suosiolle.

Tässä väitöskirjassa mallintaminen tehdään osittaisdifferentiaaliyhtälömalleja käyttäen ja rajoittuen lineaarisiin malleihin, joiden käyttämiseksi joudutaan olettamaan, että radan poikkisuuntaiset värähtelyt ovat poikkeamaltaan pieniä. Tässä tai aiemmin esitettyjen mallien avulla analysoidaan mallin kuvaaman kohteen stabiiliutta eli vakavuutta aikaharmonisen yritefunktion avulla. Yritteen sijoittaminen kohdetta kuvaaviin aikariippuviin yhtälöihin tuottaa ominaisarvotehtävän. Ominaisarvoista suoraan laskettavien ominaistaajuuksien (tai suoraan ominaisarvojen) avulla voidaan luonnehtia systeemin käyttäytymistä.

Tasossa liikkuvien materiaalien stabiiliutta on tutkittu paljon ennestäänkin. Tunnettu ilmiö monille tutkituille malleille on se, että radan käyttäytyminen on stabiilia alhaisilla nopeuksilla, mutta jos radan kuljetusnopeus kasvatetaan kyllin suureksi, niin stabiilius menetetään. Useiden lineaaristen mallien ennusteet esittävät radan käyttäytymisen muuttuvan stabiilista epästabiiliksi tietyllä nopeuden arvolla, jota kutsutaan kriittiseksi nopeudeksi. Tätä suuremmilla nopeuden arvoilla materiaaliradan poikkeama kasvaa ennusteen mukaan rajatta.

Kriittisen nopeuden olemassaoloa ja mahdollisen epästabiiliuden luonnetta voidaan tutkia mm. laskemalla materiaaliradan ominaistaajuudet nopeusparametrin suhteen. Tällaista joko matemaattista tai numeerista analyysia kutsutaan tässä dynamiikan analyysiksi. Jos esimerkiksi yllä mainitun dynamiikan analyysin avulla tiedetään, että kriittinen nopeus on olemassa ja vastaava ominaismuoto ei ole aikariippuva, niin kriittisen nopeuden arvo voidaan ratkaista yhtälöiden aikariippumattomasta muodosta edelleen ominaisarvotehtävän avulla. Ominaisarvotehtävät ratkaistaan tässä työssä numeerisesti käyttäen apuna analyyttisiä osaratkaisuja, differenssimenetelmää sekä Fourierin ja Galerkinin menetelmää.

Malleja koskevia uusia tuloksia esitetään muutamia. Tasossa liikkuvan viskoelastisen paneelin poikkivärähtelyjä kuvaavalle osittaisdifferentiaaliyhtälölle johdetaan viides reunaehto perustuen fysikaaliseen paneelin vääntömomentin jatkuvuusoletukseen. Edellä mainitun liikkuvan viskoelastisen paneelin tyhjiömalliin liitetään ympäröivän fluidin mukaan tuomat paneelin reaktiotermit. Gravitaatio liitetään elastisen liikkuvan paneelin yhtälöihin. Liikkuvan elastisen laatan mallia tarkastellaan tilanteessa, jossa laatan päihin kohdistuvan jännityksen profiili on vino. Stabiiliusanalyysin tulosten uutena sovelluksena esitetään tuotantofunktio, joka antaa prosessoidun materiaalin massan, liikkuvan materiaalin kuljetusnopeuden funktion ja syklisten jännitysvaihtelujen lukumääräfunktion avulla.

Numeerisina uusina tuloksina havaitaan, että Kelvinin ja Voigtin viskoelastisten relaatioiden avulla mallinnetun liikkuvan paneelin ominaistaajuudet ennustavat tutkitulla nopeuden arvojen alueella, että kriittistä nopeutta ei ole olemassa, kunhan viskositeettia kuvaava kerroin on tarpeeksi suuri. Pienillä viskositeettikertoimen arvoilla kriittinen nopeus on olemassa ja sen arvo on hieman suurempi kuin vastaavalla elastisella paneelilla, mikä on käyttäytymisenä yhdenmukaista aiempien mm. liikkuvien viskoelastisten laattojen malleja koskevien tutkimusten tulosten kanssa.

Ympäröivän fluidin virtauksen ottaminen mukaan liikkuvan viskoelastisen paneelin malliin vaikuttaa häivyttävän tyhjiömallille havaittua viskositeetin stabiloivaa vaikutusta. Esimerkiksi tietyillä materiaaliparametrien arvoilla paikallaan olevan fluidin ympäröivä paneeli käyttäytyy stabiilisti, ja fluidin virtausnopeuden kasvattaminen nollasta johonkin positiiviseen arvoon riittää muuttamaan käyttäytymisen epästabiiliksi.

Maan gravitaation mukaan ottaminen tasossa liikkuvan elastisen paneelin malliin vaikuttaa vain vähän paneelin kriittisen nopeuden ennusteeseen verrattaessa malliin, jossa gravitaatio jätettiin huomiotta. Sen sijaan kriittistä nopeutta vastaava paneelin poikkeama muuttuu ennusteessa muodoltaan huomattavasti.

Analyyttisesti osoitetaan, että liikkuvan ortotrooppisen laatan kriittinen nopeus on suurempi kuin vastaavan kalvomallin ennustama kriittinen nopeus. Numeerisissa tutkimuksissa nähdään, että laatan leikkausmoduulin arvo vaikuttaa hieman laatan kriittiseen kuljetusnopeuteen ja jonkin verran vastaavaan poikkeaman muotoon. Vinon jännitysprofiilin alla kuljetettavan laatan tapauksessa kriittisen nopeuden havaitaan alenevan huomattavasti vinoutta kasvatettaessa.

Työssä todetaan myös, että käytetyttyjen lineaaristen mallien soveltamisessa on rajoitteita. Epälineaarisia malleja tarvitaan, jos halutaan analysoida esimerkiksi poikkeamaltaan suuria radan värähtelyjä tai liikkuvien materiaalien käyttäytymistä kriittistä nopeutta suuremmilla nopeuden arvoilla. Huomionarvoista on myös se, että leveiden ja ohuiden levyjen tapauksessa ympäröivän ilman huomioimatta jättäminen mallintamisessa voi tuottaa kriittiselle nopeudelle jopa neljä kertaa liian suuren arvion.

Tässä väitöskirjassa esitetty tutkimus laajentaa tietämystä tasossa liikkuvien materiaalien malleista, erityisesti paneeli- ja laattamallien osalta. Esitetyt perustutkimukselliset tulokset ja laskennallisesti kevyet mallit tarjoavat myös työkaluja sovellusten kohteina oleviin teollisuuden radanhallintakysymyksiin.

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PII

STABILITY OF MOVING VISCOELASTIC PANELS INTERACTING WITH SURROUNDING FLUID

by

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Stability of moving viscoelastic panels interacting with surrounding fluid

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Summary. We study a model describing the out-of-plane vibrations of an axially moving viscoelastic panel submerged in flowing fluid. The panel is assumed to travel at a constant velocity between two fixed supports, and it is modeled as a flat panel made of viscoelastic Kelvin-Voigt material. The fluid flow is modeled with the help of the added mass coefficients. The resulting dynamic equation is a partial differential equation of fifth order in space. Five boundary conditions are set for the studied problem. The behavior of the panel is analyzed with the help of its eigenvalues (eigenfrequencies). These characteristics are studied with respect to the velocity of the panel. In our study, we have included the material (total) derivative in the viscoelastic relations. We study the effects of the surrounding flowing fluid on the behavior of the moving viscoelastic panel. It was found that, in presence of flowing fluid, the critical panel velocity was significantly lower than in the vacuum case. Secondly, for high enough values of viscosity, the panel did not experience instability detected at low values of viscosity in the form of divergence. The flowing fluid was found to diminish the stabilizing effects brought about by material viscosity.

Key words: moving panel, viscoelasticity, eigenvalues, FSI, axial flow, stability, paper industry

Introduction

In industrial processes with axially moving materials, such as making of paper, steel or textiles, high transport speed is desired but it also may cause loss of stability. In modeling of such systems, the researchers have generally studied dynamic behavior of strings, membranes, beams and plates taking into account the transverse, Coriolis and centripetal accelerations of the material motion. For materials with low density, interaction with surrounding fluid affects significantly the behavior of traveling material. For example for traveling paper webs, the effect of the surrounding air is important [20, 21, 36].

Industrial materials usually have viscoelastic characteristics [14], and consequently, viscoelastic moving materials have been recently studied widely. In paper making, wet paper webs are highly viscous, and therefore, viscoelasticity should be taken into account in the model [1]. Both fluid-structure interaction and material viscosity belong to fields of research, which are challenging and remain many open questions.

Vibrations of traveling elastic strings, beams, and bands in vacuum have been studied extensively. The first studies on them include Sack [39], Archibald and Emslie [2], Miranker [27], Swope and Ames [42], and Mote [29, 30, 31].

Archibald and Emslie [2] and Simpson [41] studied the effects of axial motion on the frequency spectrum and eigenfunctions. In their research, it was shown that the natural frequency of each mode decreases as the transport speed is increased, and that the traveling string and beam both experience divergence instability at a sufficiently high speed. Wickert and Mote studied stability of axially moving strings and beams using modal analysis and Green's function method [49]. They presented the expressions for the critical transport velocities analytically. However recently, Wang et al. [46] showed analytically that no static instability occurs for the transverse motion of a string at the critical velocity. For axially moving beams with a small flexural stiffness, Kong and Parker [19] found closed-form expressions for the approximate frequency spectrum by a perturbation analysis.

First studies on modeling the effects of the surrounding air on the moving web behavior by the analytic added mass approximation include the research by Pramila, and Niemi and Pramila [32, 36, 37, 38]. In all of these studies, the surrounding air was found to reduce the eigenfrequencies and critical transport velocities significantly compared to the vacuum case. According to Pramila's study from the year 1986, the presence of air may reduce both about 15–26 % of the vacuum case. However, the model that was used, was later interpreted by Pramila to mean that the fluid particles move with the traveling web, which probably is not the actual physical case there. Recently, Frondelius et al. used an added mass model with non-constant coefficients computed from the boundary-layer theory [13]. However, if the boundary-layer theory is used, one needs to include a leading edge in the model.

The added mass approach has been further used and developed, e.g., by Chang and Moretti in their study on out-of-plane vibrations of a moving web [7]. They developed a method for computing the effect of surrounding enclosure on the aerodynamic inertia coefficient and presented an example calculation for a web translating through a drying oven. They also compared their theory with wind-tunnel experiments for stationary webs surrounded by flowing air.

Recently, Lin and Qiao [24] studied vibrations and stability of axially moving beams taking into account both the material viscoelasticity and the effects of surrounding fluid. They investigated a beam with uniform circular cross-section using similar approach to Gosselin et al. [16]. Gosselin et al. studied extruding of a cantilevered beam with circular cross-section, in which case the formulations for axial tension are different from that of beams with both ends being supported. The problem of extruding of cantilevered beams immersed in fluid was first studied by Taleb and Misra [43]. Their study was corrected by Gosselin et al. [16] and Païdoussis [35]. In all these studies on extruding of cantilevered beams, material viscoelasticity was taken into account with the help of Kelvin-Voigt model.

Lin and Qiao found that moving beams with circular cross-section undergo buckling-type instability at a sufficiently high speed [24]. At higher values of traveling speed, the beam may undergo flutter instability.

Fluid surrounding the moving web has been modeled also as potential flow [3, 4, 6, 7, 17, 45, 48], by acoustic elements placed on one side of the web [18], by utilizing fluid-solid interaction based on acoustic theory [20] and by using a Navier–Stokes code [48].

First studies on transverse vibration of viscoelastic material traveling between two fixed supports was done by Fung et al. [14] using a string model. Extending their work, they studied the material damping effect in their later research [15].

Oh et al. [33] and Lee and Oh [23] studied critical speeds, eigenvalues, and natural modes of axially moving viscoelastic beams using the spectral element model. They analyzed dynamic behavior of axially moving viscoelastic beams using modal analysis, performed a detailed eigenfrequency analysis, and reported that viscoelasticity did not affect the critical moving speed.

Marynowski and Kapitaniak compared two different internal damping models in modeling of moving viscoelastic (non-linear) beams [26]. For the linearized Kelvin–Voigt model, it was found that the beam exhibits divergent instability at some critical speed. In the case of non-linear Bürgers model, the critical speed decreased when the internal damping was increased, and the beam was found to experience the first instability in the form of flutter.

In the discussed studies above, a partial time derivative has been used instead of a material derivative in the viscoelastic constitutive relations. Mockensturn and Guo suggested that the material derivative should be used [28]. They studied non-linear vibrations and dynamic response of axially moving viscoelastic strings, and found significant discrepancy in the frequencies at which non-trivial limit cycles exist, comparing the models with the partial time derivative or the material time derivative. Recently, the material derivative has been used in most of the studies concerning axially moving viscoelastic beams (see e.g. [8, 9, 10, 11]). Kurki and Lehtinen [22] suggested, independently, that the material derivative in the constitutive relations should be used in their study concerning the in-plane displacement field of a traveling viscoelastic plate.

In a recent study by Saksa et al., eigenvalues and stability characteristics of viscoelastic axially moving panels in *vacuum* were studied [40]. They used the material derivative in the viscoelastic constitutive relations, which leads to a partial differential equation of fifth order in space. The similar equation was also obtained by the other researchers who used the material derivative but usually the problem was solved setting only four boundary conditions. Saksa et al. derived a fifth boundary condition for the studied problem. In their study, it was also found in the numerical studies that if the viscosity is high enough, all the modes behave stable with damping vibrations for any value of transport velocity and no critical speed was detected.

Models for pipes conveying fluid often share similarities with the models for axially moving materials [34, 35]. In the study by Drozdov [12], a pipe filled with a moving fluid was studied modeling the pipe as a viscoelastic beam driven by the forces caused by the fluid. Drozdov investigated stability of the system under a periodic flow. It was found that for some parameter values, an increase in viscoelasticity resulted in a decrease in the critical fluid velocity while for other choices of parameters, an increase in viscoelasticity resulted in an increase in the critical velocity. Recently, Wang et al. [47] derived a sixth order model for a curved viscoelastic pipe conveying fluid based on Hamilton's principle. Viscoelasticity of the pipe was modeled with the help of the Kelvin–Voigt model. The viscoelastic pipe was found to undergo divergent instability in the first and second order modes and, for greater values of fluid velocity, single-mode flutter took place in the first order mode.

Existing studies on moving viscoelastic materials interacting with surrounding fluid seems to be limited to the cases of beams having circular cross-section [16, 24, 43] and to viscoelastic pipes conveying fluid [12, 47]. These models do not fit to the case in which we tackle a problem with thin and wide webs traveling between supports and having low density and high viscosity.

In this study, we take both material viscosity and interaction with fluid into account in the model for thin panels, moving axially at a high speed. We use the term *panel* for a two-dimensional web with the assumption that the transverse displacement of the web does not vary in the direction perpendicular to the moving direction of the web. Term *flat panel* has been used e.g. by Bisplinghoff and Ashley in their classical book on aeroelasticity [5].

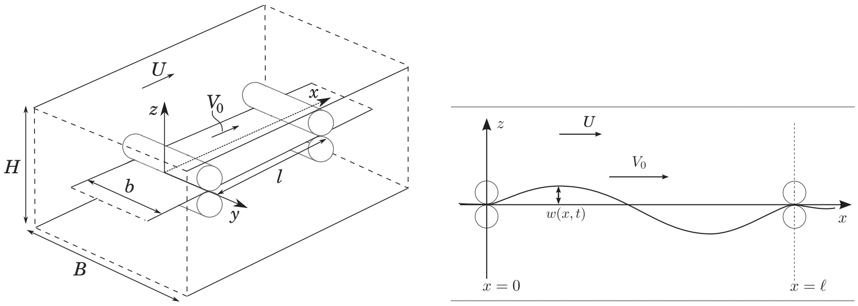

Figure 1. An axially moving panel submerged in flowing fluid.

An axially moving panel traveling through an enclosure

Consider an axially moving panel, traveling between two fixed supports at a constant velocity. We assume that the transverse displacement does not vary in the y direction, i.e. the transverse deformation of the panel is cylindrical [5, 44]. The panel is supported at $x = 0$ and $x = \ell$, and the length of the span is ℓ. The transport velocity of the panel is assumed to be constant and denoted by V_0. The transverse displacement of the panel is denoted by the function $w = w(x, t)$. The width of the panel is denoted by b, and the thickness of the panel by h (assumed to be constants).

The panel is assumed to travel through a long enclosure with rectangular cross-section to model a web traveling through a drying oven. The height of the enclosure is H and the width of it is B. The velocity field of fluid is denoted by U (not necessarily constant). See Figure 1.

A traveling viscoelastic panel in vacuum

We study a panel be made of viscoelastic material. Viscoelasticity is taken into account with the help of the Kelvin–Voigt model consisting of an elastic spring and a viscous damper connected in parallel. The spring element is described by the parameters E (the Young's modulus) and ν (the elastic Poisson ratio), and the damper by η (the viscous damping coefficient) and μ (the Poisson ratio for viscosity). See Fig. 2.

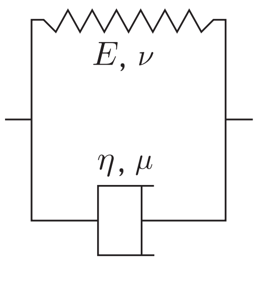

Figure 2. The rheological Kelvin–Voigt model.

We denote stress and strain in the x direction by σ and ε, respectively. Assuming the cylindrical deformation, the stress-strain relation for the Kelvin–Voigt panel is described as [40]

$$\sigma = \frac{E}{1-\nu^2}\varepsilon + \frac{\eta}{1-\mu^2}(\varepsilon_{,t} + V_0\varepsilon_{,x})\,. \tag{1}$$

Using the stress-strain relation in (1), the dynamic equilibrium for the transverse displacement w can be written as [11, 40]

$$mw_{,tt} + 2V_0 mw_{,xt} + \lambda D w_{,xxxxt} + (mV_0^2 - T_0)w_{,xx} + Dw_{,xxxx} + V_0\lambda D w_{,xxxxx} = 0\,. \tag{2}$$

In Eq. (2), m is the mass per unit area, T_0 is constant tension at the panel ends, D is the bending rigidity of the panel defined as

$$D = \frac{Eh^3}{12(1-\nu^2)}\,, \tag{3}$$

and λ is the creep time constant defined as

$$\lambda = \frac{\eta}{E}\,, \tag{4}$$

the unit of which is the second. We have assumed that the Poisson ratios ν and μ coincide.

Traveling panel interacting with flowing fluid

In this section, we consider the model for an axially moving viscoelastic panel that was introduced in the previous section, but we further take into account the aerodynamic effects. As Chang and Moretti [7] and Chang et al. [6], we include added mass due to the transverse, Coriolis and centripetal acceleration (in all inertia terms) denoted by m_1, m_2, and m_3, respectively.

We insert the added mass terms into Eq. (2), and have the following final equation for the out-of-plane displacement w:

$$\begin{aligned}(m+m_1)w_{,tt} &+ 2V_0(m+m_2)w_{,xt} + \lambda D w_{,xxxxt} \\ &+ [(m+m_3)V_0^2 - T_0]w_{,xx} + Dw_{,xxxx} + V_0\lambda D w_{,xxxxx} = 0\,.\end{aligned} \tag{5}$$

The added mass terms in (5) can be calculated as [6, 7]

$$\begin{aligned} m_1 &= \frac{\pi}{4}C_{\mathrm{a}}\rho b\,, \\ m_2 &= 2\rho\delta^*\,, \\ m_3 &= 2\rho\theta\,, \end{aligned} \tag{6}$$

where C_{a} is the added mass coefficient depending on the problem geometry, ρ is the density of air, δ^* is the displacement thickness of the boundary layer and θ is the momentum thickness of the boundary layer.

If $U = U(r)$ is the velocity of the fluid flow with respect to the distance r from the panel, δ^* can be calculated as

$$\delta^* = \frac{1}{V_0}\int_0^{\delta} U(r)\,\mathrm{d}r\,, \tag{7}$$

where δ is the thickness of the moving fluid layer.

Similarly, the momentum thickness θ is

$$\theta = \frac{1}{V_0^2} \int_0^\delta U^2(r)\,\mathrm{d}r\,. \tag{8}$$

In the case of stationary air, terms m_2 and m_3 are negligible compared to the mass m of the panel [35]. In that case, the dynamic equation reads

$$\begin{aligned}(m+m_1)w_{,tt} &+ 2V_0 m w_{,xt} + \lambda D w_{,xxxxt} \\ &+ [mV_0^2 - T_0]w_{,xx} + Dw_{,xxxx} + V_0\lambda D w_{,xxxxx} = 0\,.\end{aligned} \tag{9}$$

As boundary conditions, we use clamped-clamped conditions at both ends and an additional boundary condition at the in-flow end indicating that we have more information there than at the out-flow end. The fifth condition can be derived with the help of continuity of the panel [40]. The boundary conditions are

$$w(0,t) = w_{,x}(0,t) = w_{,xx}(0,t) = 0\,, \qquad w(\ell,t) = w_{,x}(\ell,t) = 0\,. \tag{10}$$

We transform the problem (5) and (10) into a dimensionless form. We perform the following transformations

$$x \to \frac{x}{\ell}\,, \qquad t \to \frac{t}{\tau}\,, \qquad w(x,t) \to \frac{w(x,t)}{h}\,, \tag{11}$$

choose

$$\tau = \ell\sqrt{\frac{m}{T_0}}$$

as a characteristic time, and introduce the dimensionless problem parameters

$$\zeta = \frac{m}{m+m_1}\,, \qquad \zeta_2 = \frac{m_2}{m+m_1}\,, \qquad \zeta_3 = \frac{m_3}{m+m_1}\,, \tag{12}$$

and

$$c = \frac{V_0}{\sqrt{T_0/m}}\,, \qquad \alpha = \frac{D}{\ell^2 T_0}\,, \qquad \gamma\alpha = \frac{\lambda D}{\ell^3\sqrt{mT_0}}\,, \tag{13}$$

where

$$\gamma = \frac{\lambda}{\tau} = \frac{\eta}{E}\frac{\sqrt{T_0}}{\ell\sqrt{m}} \tag{14}$$

is here called the dimensionless creep time constant. After transformations in (11) and insertion of (12) – (14), we obtain

$$w_{,tt} + 2c\zeta_2 w_{,xt} + \gamma\alpha\zeta w_{,xxxxt} + (c^2\zeta_3 - \zeta)w_{,xx} + \alpha\zeta w_{,xxxx} + \gamma\alpha c\zeta w_{,xxxxx} = 0\,, \tag{15}$$

with the boundary conditions

$$w(0,t) = w_{,x}(0,t) = w_{,xx}(0,t) = 0\,, \quad w(1,t) = w_{,x}(1,t) = 0\,. \tag{16}$$

Dynamic analysis

To study stability of the problem (15) and (16), we perform classical dynamic analysis by inserting the standard harmonic trial function

$$w(x,t) = W(x)e^{st}\,, \tag{17}$$

into (15) and (16).

In (17),

$$s = i\omega\,, \tag{18}$$

and ω is the dimensionless angular frequency of small transverse vibrations. The sign of the real part of s characterizes the stability of the panel: if Re $s > 0$, the behavior is unstable, and otherwise it is stable.

Insert (17) into (15), and obtain

$$s^2 W + s(2c\,\zeta_2 W_{,x} + \gamma\alpha\zeta W_{,xxxx}) + (c^2\zeta_3 - \zeta)W_{,xx} + \alpha\zeta W_{,xxxx} + \gamma\alpha c\zeta W_{,xxxxx} = 0\,. \tag{19}$$

The boundary conditions for W are

$$W(0) = W_{,x}(0) = W_{,xx}(0) = 0\,, \qquad W(1) = W_{,x}(1) = 0\,, \tag{20}$$

We study the stability behavior of the traveling viscoelastic panel by solving Eqs. (19)–(20) with respect to the transport velocity.

The problem (19)–(20) was discretized via the finite difference method. We used central differences of second-order asymptotic accuracy but at the out-flow edge for the fifth order term, a backward difference scheme of second order asymptotic accuracy was used. The finite differences schemes are given, e.g., in [40]. The interval $[0,\ell]$ is divided to $n+1$ sub-intervals equal in length. The end points of the sub-intervals are labeled as $0 = x_0, x_1, x_2, \ldots, x_n, x_{n+1} = \ell$. We use two virtual points (x_{-2} and x_{-1}) at the in-flow end and one virtual (x_{n+2}) point at the out-flow end. From the boundary conditions (20), we get at the in-flow end:

$$w_{-2} = -w_2\,, \quad w_{-1} = w_1\,, \quad w_0 = 0\,,$$

and at the out-flow end:

$$w_{n+1} = 0\,, \quad w_{n+2} = w_n\,.$$

We denote the derivative matrices by $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_4, \mathbf{K}_5$ built up with the help of the finite difference schemes with the following correspondence:

$$\mathbf{K}_1 : W_{,x}\,, \qquad \mathbf{K}_2 : W_{,xx}\,, \qquad \mathbf{K}_4 : W_{,xxxx}\,, \qquad \mathbf{K}_5 : W_{,xxxxx}\,.$$

Inserting the matrices $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_4, \mathbf{K}_5$ into (19), we obtain the matrix equation

$$s^2\mathbf{w} + s\left[2c\zeta_2\mathbf{K}_1 + \gamma\alpha\zeta\mathbf{K}_4\right]\mathbf{w} + \left[(c^2\zeta_3 - \zeta)\mathbf{K}_2 + \alpha\zeta\mathbf{K}_4 + \gamma\alpha c\zeta\mathbf{K}_5\right]\mathbf{w} = 0\,. \tag{21}$$

Note that in the case $\alpha = 0$ or $c = 0$, we obtain a fourth-order equation needing only four boundary conditions. This has been taken into account: the virtual point w_{-2} is needed only by the matrix $\mathbf{K}_5$. When $\mathbf{K}_5$ is removed from the matrix equation (21), the boundary condition $w_{,xx}(0) = 0$ is simultaneously removed from the discretized problem.

The matrix equation (21), which is a quadratic eigenvalue problem with respect to s, can be rewritten as

$$\begin{bmatrix} -\mathbf{M}_1 & -\mathbf{M}_0 \\ \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} s\mathbf{w} \\ \mathbf{w} \end{bmatrix} = s \begin{bmatrix} s\mathbf{w} \\ \mathbf{w} \end{bmatrix}, \tag{22}$$

where

$$\begin{aligned} \mathbf{M}_0 &= (\zeta_3 c^2 - \zeta)\mathbf{K}_2 + \alpha\zeta\mathbf{K}_4 + \gamma\alpha c\zeta\mathbf{K}_5 \,, \\ \mathbf{M}_1 &= 2c\,\zeta_2\mathbf{K}_1 + \gamma\alpha\zeta\mathbf{K}_4 \,. \end{aligned} \tag{23}$$

The matrix equation (22) is now an eigenvalue problem of the standard form

$$\mathbf{A}\mathbf{y} = s\mathbf{y} \tag{24}$$

with

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}_1 & -\mathbf{M}_0 \\ \mathbf{I} & 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} s\mathbf{w} \\ \mathbf{w} \end{bmatrix}.$$

Some example studies

As an example, we consider simple flow through an enclosure with a rectangular cross-section. We assume a Couette type flow such that the fluid velocity coincides with the panel velocity on the panel surface and is equal to zero at the surface of the enclosure. See Figure 3.

Similar example was considered by Chang and Moretti [7]. They computed also the added mass coefficient C_a for different simple problem geometries assuming potential flow in the cross-direction plane, obtaining the stream-function by a finite difference method, summing up the kinetic energy in the flow field, and referring it to the web velocity. In such conditions that $H/B = 0.4$ and $b/B = 0.8$, they found that $C_\mathrm{a} = 1.66$. If b/B was small, the added mass coefficient was close to 1 as would be expected.

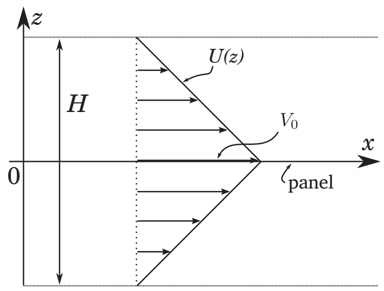

Figure 3. Simple flow through a drying oven.

The parameters that were used were the following:

$$\begin{array}{lllll} T_0 = 500\ \mathrm{N/m} & m = 0.08\ \mathrm{kg/m^2} & E = 10^9\ \mathrm{N/m^2} & \nu = 0.3 & \rho = 1.225\ \mathrm{kg/m^3} \\ \ell = 1\ \mathrm{m} & b = 0.6\ \mathrm{m} & h = 10^{-4}\ \mathrm{m} & H = 0.3\ \mathrm{m} & B = 0.75\ \mathrm{m} \end{array} \tag{25}$$

Using the physical parameters in (25), the dimensionless parameter α in Eq. (13) gets the value $\alpha = 1.8315 \cdot 10^{-7}$. Creep time constant λ was given the values $\lambda =$

$5 \cdot 10^{-5}$ s, $5 \cdot 10^{-4}$ s, and $5 \cdot 10^{-3}$ s, the dimensionless creep time constant γ getting the values $\gamma = 3.953 \cdot 10^{-3}$, $3.953 \cdot 10^{-2}$, and 0.3953, respectively.

For the example flow, the added masses calculated from Eqs. (6) are

$$m_1 = \frac{\pi}{4} C_{\mathrm{a}} \rho b \approx 0.9583 \text{ kg/m}^2 , \tag{26}$$

$$m_2 = \frac{1}{2} \rho H \approx 0.1838 \text{ kg/m}^2 , \tag{27}$$

$$m_3 = \frac{1}{3} \rho H \approx 0.1225 \text{ kg/m}^2 , . \tag{28}$$

Three different cases were studied:

1. traveling viscoelastic panel in vacuum ($m_1 = m_2 = m_3 = 0$),
2. traveling viscoelastic panel surrounded by stationary fluid in an enclosure ($m_2 = m_3 = 0$), and
3. traveling viscoelastic panel surrounded by laminar fluid flow in an enclosure.

The dimensionless frequency F was calculated with the help of the dimensionless angular frequency $\omega = \mathrm{Im}\ s$. The dimensional frequency f is

$$f = \frac{\omega}{2\pi\tau} = \frac{\omega}{2\pi\ell}\sqrt{\frac{T_0}{m}} .$$

We define F by dividing it by the natural frequency of a non-moving panel in vacuum, that is, by $1/(2\ell)\sqrt{T_0/m}$:

$$F = f 2\ell \sqrt{\frac{m}{T_0}} = \frac{\omega}{\pi} = \frac{\mathrm{Im}\ s}{\pi} . \tag{29}$$

The behavior of the dimensionless frequency F was studied with respect to the dimensionless panel velocity c. Computations were carried out for all the three cases. In Figure 4 on the left hand side, the lowest dimensionless frequencies are plotted in the case of elastic material. The results coincide with the previous investigations [4, 7, 37]: the presence of fluid decreases the natural frequencies, and the effect of the flowing fluid is that the critical panel velocity in decreased notably.

In Figure 4 on the right hand side, the effect of the material viscosity on the eigenfrequencies can be seen. The greater the creep time constant λ, the greater the are the values of the eigenfrequencies. That is, the effect of the material viscosity is opposite to that of the fluid.

In Figure 5, the dependence of the dimensionless critical speed c_{cr} on the dimensionless creep time constant γ is shown for two different cases: on the left hand side, the behavior in vacuum and in presence of stationary air are shown, and the right hand side presents the behavior in the case of flowing air. As seen, the presence of stationary air does not alter the value of the critical velocity independent of the value for the dimensionless creep time constant. The effect of the viscosity is very small but visible. In the case of flowing air, the critical speed of the viscoelastic panel with $\gamma = 0.1$ is about $1.6 \cdot 10^{-4}$ % greater than that of the elastic panel. In the case of stationary air and vacuum, the critical speed of the viscoelastic panel with $\gamma = 0.1$ is about $4.4 \cdot 10^{-4}$ % greater than that of the elastic panel. In these cases, the effect of the viscosity is almost three times bigger than in the

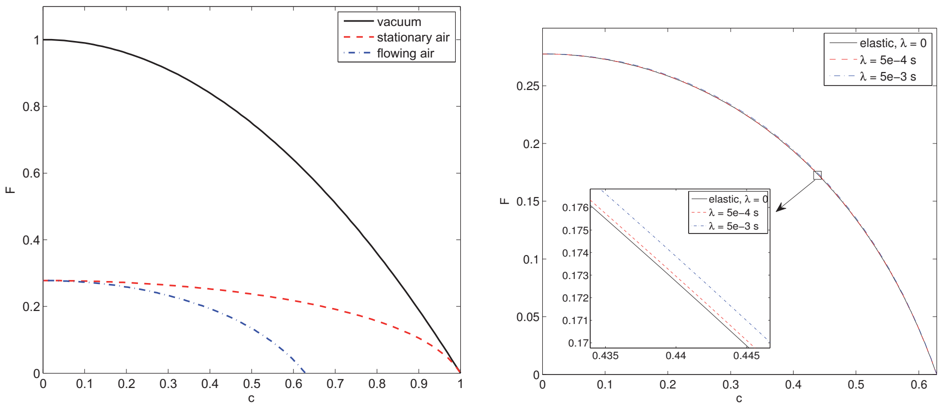

Figure 4. Dimensionless eigenfrequency F with respect to the dimensionless critical speed c.

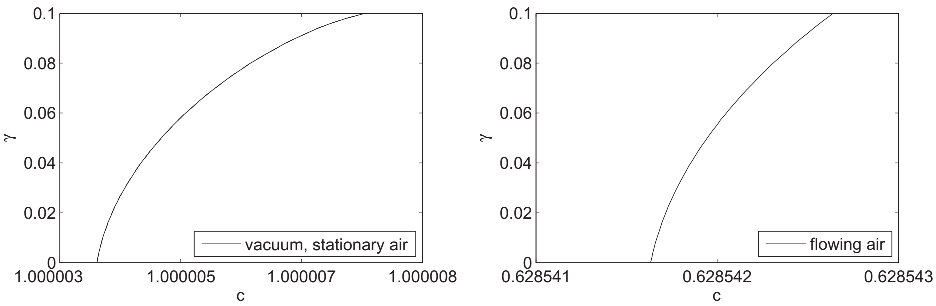

Figure 5. Dependence of the dimensionless critical speed c on the dimensionless creep time constant γ.

case of flowing fluid. This suggests that the effect of the viscosity is diminished in presence of flowing fluid.

In Figures 6 and 7, the three lowest eigenvalues s are given for an elastic panel and for three viscoelastic panels having different creep time constants. Figure 6 presents the case for a stationary fluid (air). In the upper left corner, the eigenvalues for an elastic panel are shown. In the sub-figures from left to right, from top to bottom, the viscosity increases (the creep time constant increases). It can be seen that the real parts of the eigenvalues before the critical velocity become negative when the viscosity is inserted to the model. This means damping vibrations in the behavior of the panel. In a sub-figure in the lower left corner, one may see that the critical velocity becomes slightly after the point at which the imaginary part of the lowest eigenvalue becomes zero. The computed critical velocity is also slightly greater than that of an elastic panel, see Figure 5. In the lower right corner of Figure 6, critical velocity can not be detected and all the three lowest eigenvalue stay negative, which means stable behavior at any value of velocity. The limit value of the dimensionless creep time constant, after which no instability can be detected, was calculated via the bisection method, and it was $\gamma = 0.1022$ which it exactly same as in vacuum case [40].

In Figure 7, we see the three lowest eigenvalues in the case of flowing air. As above, we have four cases: elastic panel, and viscoelastic panels with the three different creep time constants. The behavior seems qualitatively similar to that of the case with stationary fluid. However, the absolute value of the real parts of the eigenvalues are significantly smaller suggesting that the damping of the vibrations before the critical velocity is weaker than in the case of stationary fluid. The limit value of the dimensionless creep time constant after which all the modes stay stable was calculated to be $\gamma = 0.1625$, which is 59 % greater than the one in the case of stationary fluid (and vacuum). This suggests that in presence of flowing fluid, the viscoelastic panel is more unstable than in the case of stationary fluid or vacuum, since e.g. for $\gamma = 0.11$ the panel surrounded by stationary air is stable while the panel surrounded by flowing air still undergoes divergence instability at some sufficiently high speed.

Typical behavior of moving viscoelastic materials were seen to remain even if the fluid was inserted to the model with the added mass approach. For example such a characteristic as removal of the coupled mode flutter typical of moving elastic materials, was detected in the eigenvalue spectra of the viscoelastic moving panels [23, 47].

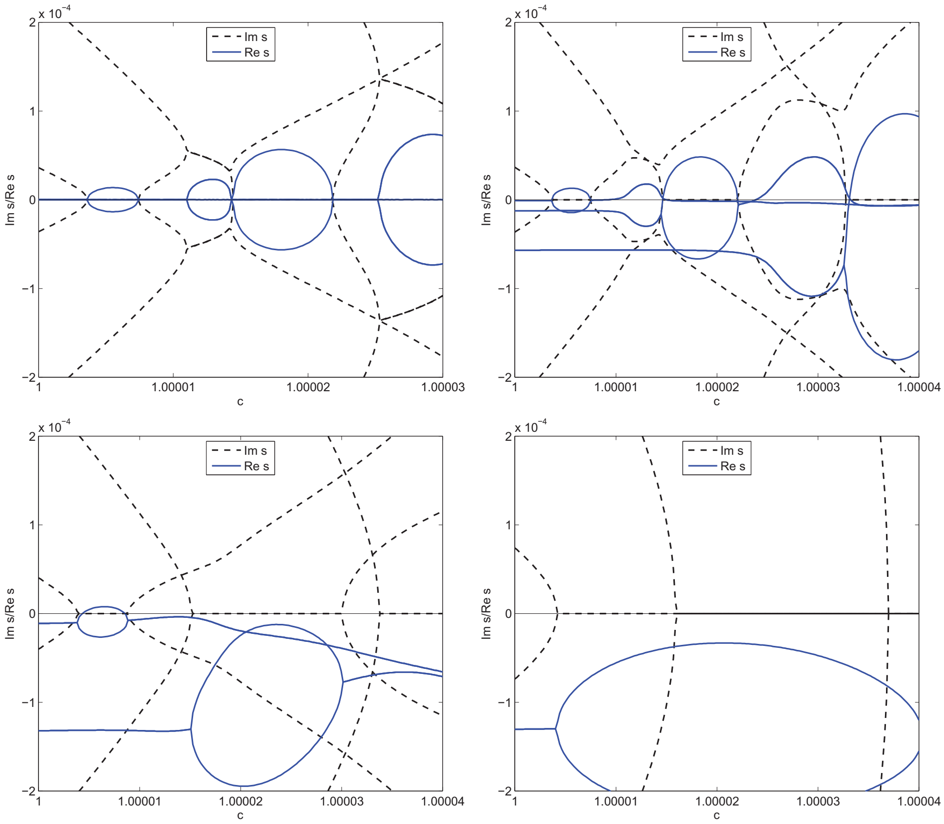

Figure 6. Behavior of the eigenvalues for the stationary fluid case. The values of the creep time constants in the figures from left to right, from top to bottom are $\lambda = 0$, $\lambda = 5 \cdot 10^{-5}$ s, $\lambda = 5 \cdot 10^{-4}$ s, and $\lambda = 5 \cdot 10^{-3}$ s, in that order.

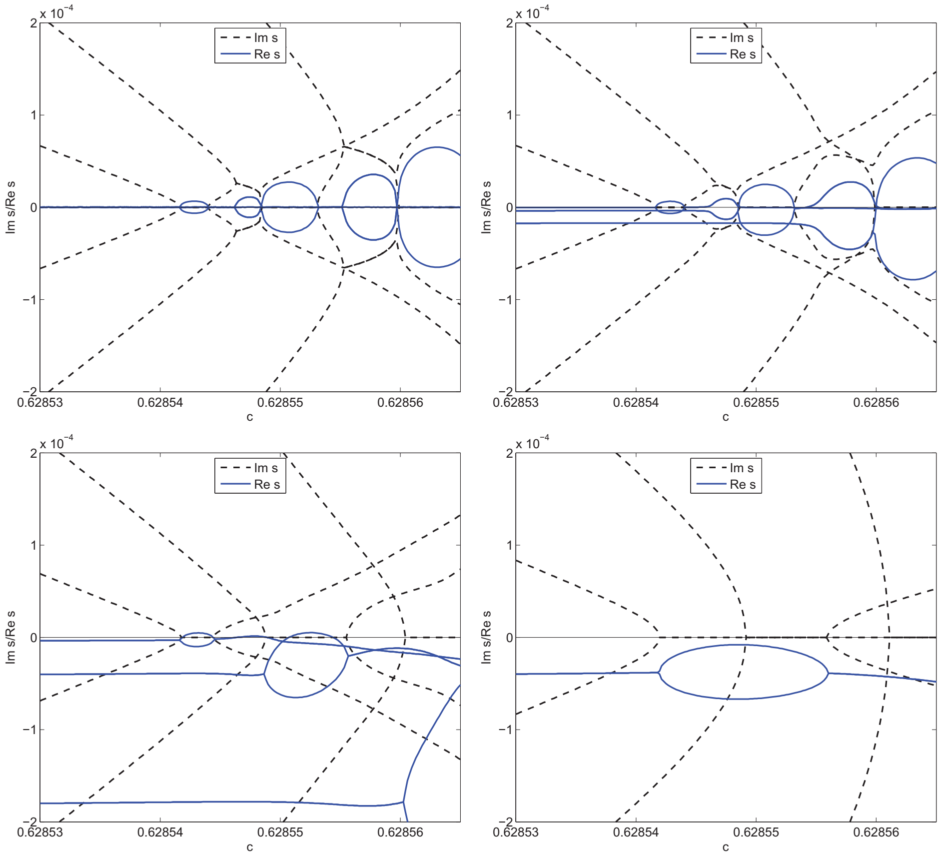

Figure 7. Behavior of the eigenvalues for the flowing fluid case. The values of the creep time constants in the figures from left to right, from top to bottom are $\lambda = 0$, $\lambda = 5 \cdot 10^{-5}$ s, $\lambda = 5 \cdot 10^{-4}$ s, and $\lambda = 5 \cdot 10^{-3}$ s, in that order.

Conclusions

Stability characteristics of an axially moving viscoelastic web interacting with surrounding fluid were studied. The material viscoelasticity was modeled with the help of the Kelvin-Voigt model. Interaction with the fluid was taken into account by the added mass terms based on potential flow theory. To our knowledge, this is the first study in which both material viscoelasticity and aerodynamic effects were taken into account in modeling of moving webs traveling between two supports.

Two different kinds of flow models were investigated in the numerical part. They both concerned the case, in which a panel is traveling through a rectangular enclosure. The first study concerned the case with assumption that the surrounding air is stationary or that the effect of the boundary layer is negligible. In the second study, a laminar flow around the moving panel was taken into account resulting in added mass terms containing the displacement and momentum thicknesses of the boundary layer.

As expected, the presence of fluid decreased the value of the critical speed, and the viscoelasticity had a stabilizing effect on the web behavior: the viscosity increased the critical speed and for high enough values of viscosity, no instability occurred. These results are known from the studies were either the effects of the fluid or the effects of material viscosity have been studied [25, 35].

As a new result, it was found that the presence of flowing fluid diminished the stabilizing effect of viscosity. In other words, the viscoelastic panel with certain creep time constant was stable when surrounded by stationary air but could be unstable when fluid was flowing.

The presented model has an application in modeling the behavior of fast moving wide webs in industry, e.g. in paper making. For more accurate predictions than in this paper, one should notice that viscoelasticity in paper does not behave linearly and that, to take account the complicated flows inside the machine, the added mass approach is probably not accurate enough.

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STATIC INSTABILITY ANALYSIS OF AN ELASTIC BAND TRAVELLING IN THE GRAVITATIONAL FIELD

by

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Static instability analysis of an elastic band travelling in the gravitational field

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Summary. Static instability analysis is performed for an axially moving elastic band, which is travelling at a constant velocity in a uniform gravitational field between two supports. The buckling of the band is investigated with the help of admitting small transverse deflections. The model of a thin elastic beam (panel) subjected to bending, centrifugal forces and nonhomogeneous tension (including a gravitational term) is used. Buckling analysis and estimation of the critical velocities of elastic instability are based on variational principles and variational inequalities. As a result, explicit formulas for upper and lower limits for critical velocities are found. It is shown analytically that a critical velocity always exists. The critical buckling modes are found, first, by solving the original differential equation directly, and, secondly, by energy minimization. The buckling modes and corresponding critical velocities are found and illustrated with some numerical examples. The gravitational force is shown to have a major effect on the buckled shape, but a minor effect on the critical velocity.

Key words: stability, paper industry, paper, elasticity, gravitation, partial differential equations, optimization

Introduction

Vibrations and stability of axially moving materials, such as strings, beams, membranes and plates, have been studied widely, since such models have various applications in industry, e.g., in paper making or in transmission cables. Stability studies of travelling materials are important, since they yield information on critical transport velocities for machine operation.

In previous studies concerning vibrations of (moving) materials, the effect of gravity has usually been neglected, since its magnitude is minor compared to the magnitude of the axial tension. Recently, Luo and Mote have studied equilibrium of travelling elastic, sagged cables under uniformly distributed loading [10]. For a three-dimensional model, they derived exact, closed-form solutions for the equilibrium configuration and tension distribution of the cables. The present study concentrates on stability of thin, elastic panels and critical velocity estimations.

Vibrations of travelling strings, beams and bands have first been studied by Archibald and Emslie [1], Miranker [11], Swope and Ames [17] Mote [12, 13, 14], Simpson [16], Ulsoy and Mote [18], Chonan [5], Wickert and Mote [21]. These studies focused on free and forced vibrations including the nature of wave propagation in moving media and the effects of axial motion on the eigenfrequencies and eigenmodes. Stability of travelling two-dimensional rectangular membranes and plates have been studied by Ulsoy and Mote [19], Lin and Mote [9], Lin [8], Banichuk et al. [2].

Archibald and Emslie [1] and Simpson [16] studied effects of the axial motion on the eigenfrequencies and eigenfunctions. It was shown that the natural frequency of each mode decreases when the transport speed increases, and that the travelling string and beam both experince divergence instability at a sufficiently high speed. Stability considerations were reviewed by Mote [13].

Recent studies on travelling materials have been performed, e.g., by Shin et al. [15], Wang et al. [20], Frondelius et al. [7], Banichuk et al. [3]. In Shin et al., the out-of-plane vibrations of an axially moving membrane were studied, and it was shown numerically that membrane is stable until a critical velocity, at which statical instability occurs [15]. Wang et al. studied transverse vibrations of axially moving strings [20]. In their study, it was shown using a Hamiltonian approach that for the transverse motion of the string, no instability occurs at the critical velocity, where a steady-state solution exists.

Frondelius et al. [7] and Banichuk et al. [3] studied a travelling band interacting with surrounding fluid and the fluid-web-contact influence on the critical velocity. Banichuk et al. studied the static instability of the travelling band submerged in flowing ideal fluid, and it was found that in presence of fluid the critical divergence speed is much lower than the vacuum solution and the divergence shape also differs from the vacuum case [3].

In the present study, the static instability of an axially moving thin, elastic band is investigated, when the band is travelling in the gravitational field. The investigation is performed in the case where the band is moving parallel to the gravity but it can also be applied to cases in which the direction of motion with respect to the gravity varies, of which several numerical examples are given. Based on variational principles and variational inequalities, explicit estimations for the critical velocity are derived. In the numerical examples, the effects of the gravity are visualised.

The critical transport velocity and the corresponding buckled shape are found numerically by solving directly the original differential equation via the Fourier–Galerkin method. Secondly, they are solved by minimizing a functional corresponding to the energy. The obtained nonlinear optimization problem is solved numerically using the Rayleigh–Ritz method.

Basic relations

We consider an elastic band, travelling at a constant velocity between two supports parallel to a uniform gravitational field (Earth's) in a rectangular coordinate system. We study the transverse displacement w of the band assuming that the displacement is cylindrical, i.e., the displacement does not vary in the cross section (in the y direction). The distance between the two supports is constant, denoted by ℓ.

The transport velocity of the panel is assumed to be a constant, V_0, and the panel is moving along the x axis. The panel is tensioned at the edges, and the tension T depends on the space coordinate x, $T = T(x)$. The standard gravity constant is denoted by g, and the thickness of the panel by h. The panel is assumed to have a constant mass per unit area, m.

The dynamic equation for the transverse displacement of the panel can be written as

$$m\, w_{tt} + 2mV_0\, w_{xt} + mV_0^2\, w_{xx} = T\, w_{xx} + T_x\, w_x - D\, w_{xxxx}\,. \tag{1}$$

Here, D describes the bending rigidity of the panel.

Tension T varies due to the gravity as

$$T(x) = T_0 + mgx\,, \tag{2}$$

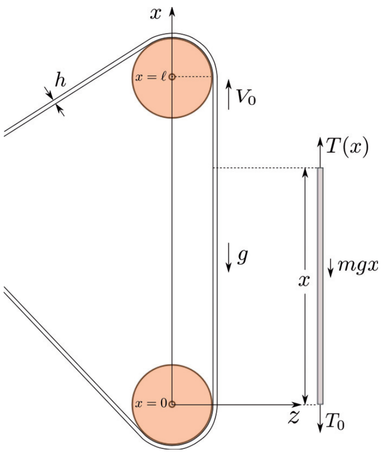

Figure 1. Panel moving in the gravitational field.

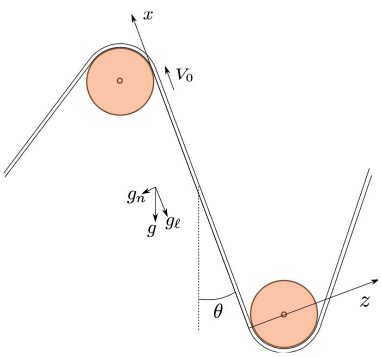

Figure 2. The elastic band moving in a nonvertical direction (inclined with respect to the gravity).

where T_0 is a constant tension at the lower edge of the panel (see Figure 1). Substituting (2) into (1), we obtain

$$m\,w_{tt} + 2mV_0\,w_{xt} + mV_0^2\,w_{xx} = (T_0 + mgx)\,w_{xx} + mg\,w_x - D\,w_{xxxx}\,. \tag{3}$$

In the more general case, where the direction of motion of the band has some nonzero angle θ with respect to the uniform gravitational field, we can perform analogous analysis using the expression

$$T(x) = mgx\cos\theta + T_0 \tag{4}$$

instead of (2), and include the term $mg\sin\theta$ in the right-hand side of (3). See Figure 2.

Buckling analysis

To study the instability of the panel, we perform a buckling analysis. The static form of the dynamic equation (3) is studied as a spectral boundary value problem using simply supported (also known as pinned or hinged) boundary conditions.

Equilibrium for the transverse displacement w is described by the following differential equation

$$mV_0^2\,w_{xx} - (T_0 + mgx)\,w_{xx} - mg\,w_x + D\,w_{xxxx} = 0\,. \tag{5}$$

The boundary conditions are

$$w(0) = w_{xx}(0) = 0, \quad w(\ell) = w_{xx}(\ell) = 0. \tag{6}$$

We represent (5)–(6) in a dimensionless form

$$\begin{gathered}\left[\frac{V_0^2}{T_0/m} - 1 - \frac{mg\ell}{T_0}x\right] w_{xx} - \frac{mg\ell}{T_0}\, w_x + \frac{D}{\ell^2 T_0}\, w_{xxxx} = 0\,, \\ w(0) = w_{xx}(0) = 0\,, \quad w(1) = w_{xx}(1) = 0\,, \end{gathered} \tag{7}$$

where $x \in [0,1]$ is the dimensionless coordinate. By defining

$$c_0 := V_0/\sqrt{T_0/m}\,, \tag{8}$$

$$\alpha := \frac{D}{\ell^2 T_0}\,, \tag{9}$$

$$\beta := \frac{mg\ell}{T_0}\,, \tag{10}$$

and substituting (8)–(10) into (7), we obtain

$$\alpha\, w_{xxxx} + (c_0^2 - 1 - \beta x)\, w_{xx} - \beta\, w_x = 0\,, \tag{11}$$

$$w(0) = w_{xx}(0) = 0\,, \quad w(1) = w_{xx}(1) = 0\,. \tag{12}$$

Boundary value problem and variational principle

Consider the boundary value problem (11)–(12). Multiply (11) by a test function v, satisfying the boundary conditions (12) and integrate the obtained equation over $[0,1]$:

$$\alpha \int_0^1 w_{xxxx} v \,\mathrm{d}x + (c_0^2 - 1) \int_0^1 w_{xx} v \,\mathrm{d}x - \beta \int_0^1 x w_{xx} v \,\mathrm{d}x - \beta \int_0^1 w_x v \,\mathrm{d}x = 0\,. \tag{13}$$

Integration by parts yields

$$\begin{gathered}\int_0^1 w_{xx} v \,\mathrm{d}x = -\int_0^1 w_x v_x \,\mathrm{d}x\,, \qquad \int_0^1 x w_{xx} v \,\mathrm{d}x = -\int_0^1 x w_x v_x \,\mathrm{d}x - \int_0^1 w_x v \,\mathrm{d}x\,, \\ \int_0^1 w_{xxxx} v \,\mathrm{d}x = \int_0^1 w_{xx} v_{xx} \,\mathrm{d}x\,. \end{gathered} \tag{14}$$

Substitute (14) into (13) to obtain

$$\alpha \int_0^1 w_{xx} v_{xx} \,\mathrm{d}x + (1 - c_0^2) \int_0^1 w_x v_x \,\mathrm{d}x + \beta \int_0^1 x w_x v_x \,\mathrm{d}x = 0\,. \tag{15}$$

Substituting $v = w$ into (15), we finally obtain

$$\alpha \int_0^1 w_{xx}^2 \,\mathrm{d}x + (1 - c_0^2) \int_0^1 w_x^2 \,\mathrm{d}x + \beta \int_0^1 x w_x^2 \,\mathrm{d}x = 0\,. \tag{16}$$

The left hand side of the equation (16) corresponds to the energy of the system. We want to find the minimal velocity c_0, at which equation (16) has a non-trivial solution.

Thus, we resolve c_0^2 from (16), and minimize the other side of the obtained equation. This minimal velocity is the critical velocity for the buckling problem [6], and is obtained from

$$(c_0^*)^2 = \min_{w\in K} \left(1 + \alpha \frac{\int_0^1 (w_{xx})^2 \, \mathrm{d}x}{\int_0^1 (w_x)^2 \, \mathrm{d}x} + \beta \frac{\int_0^1 x\,(w_x)^2 \, \mathrm{d}x}{\int_0^1 (w_x)^2 \, \mathrm{d}x} \right) , \tag{17}$$

where

$$K = \{ w \in C([0,1]) : w(0) = 0 \,, \quad w(1) = 0 \} \,. \tag{18}$$

The function w minimizing (17)–(18) is the buckling mode corresponding to the critical velocity c_0^*.

Let us show that the minimization problem (17)–(18) corresponds to the original problem (11)–(12). Denote

$$I(w) = 1 + \alpha \frac{\int_0^1 (w_{xx})^2 \, \mathrm{d}x}{\int_0^1 (w_x)^2 \, \mathrm{d}x} + \beta \frac{\int_0^1 x\,(w_x)^2 \, \mathrm{d}x}{\int_0^1 (w_x)^2 \, \mathrm{d}x} ,$$

and

$$I_1(w) = \int_0^1 (w_{xx})^2 \, \mathrm{d}x \,, \quad I_2(w) = \int_0^1 (w_x)^2 \, \mathrm{d}x \,, \quad I_3(w) = \int_0^1 x(w_x)^2 \, \mathrm{d}x \,. \tag{19}$$

It can be easily seen that if $w \not\equiv 0$, then

$$I_1(w) > 0 \,, \quad I_2(w) > 0 \,, \quad I_3(w) > 0 \,.$$

Let w^* be the minimizer of (17)–(18). To prove that w^* satisfies the necessary extremum condition, we derive the Euler equation for w^*. Since w^* is the minimizer, the first variation of $I(w^*)$ is zero, that is

$$\delta I(w^*) = 0 \,.$$

Notice that

$$\delta I = \frac{1}{I_2}(\alpha\, \delta I_1 + \beta\, \delta I_3) - \frac{1}{I_2^2}(\alpha\, I_1 + \beta\, I_3)\, \delta I_2 \,. \tag{20}$$

As a result from the variation in (20), we obtain

$$\frac{2}{I_2(w^*)} \int_0^1 \left(\alpha w^*_{xxxx} - \beta x w^*_{xx} - \beta w^*_x + (c_0^2 - 1) w^*_{xx} \right) \delta w \, \mathrm{d}x = 0 \,, \tag{21}$$

for any function $\delta w \in K$. In addition, w must satisfy the boundary conditions (12). Equation (21) holds for all $\delta w \in K$. Thus, the resulting Euler equation is

$$\alpha w^*_{xxxx} - \beta x w^*_{xx} - \beta w^*_x + (c_0^2 - 1) w^*_{xx} = 0 \,,$$

which is (11). In other words, the solution w^* of the minimization problem (17)–(18) is a weak solution of the original problem (11)–(12).

Estimations for the critical velocity

In this section, we derive analytical lower and upper bounds for the critical velocity. We estimate the critical velocity c_0^* corresponding to (11)–(12) given by the relation

$$(c_0^*)^2 = I(w^*) = 1 + \alpha\,\frac{I_1(w^*)}{I_2(w^*)} + \beta\,\frac{I_3(w^*)}{I_2(w^*)}\,. \tag{22}$$

For the lower bound of (22), we obtain

$$(c_0^*)^2 \geq 1 + \alpha\,\min_{w\in K}\frac{I_1(w)}{I_2(w)}\,. \tag{23}$$

For the upper bound of (22), we have

$$(c_0^*)^2 \leq 1 + \alpha\,\frac{I_1(w^a)}{I_2(w^a)} + \beta\,\frac{I_3(w^a)}{I_2(w^a)}\,, \tag{24}$$

where w^a is any function in K.

Combining (23) and (24), we can estimate c_0^* as follows:

$$1 + \alpha\,\min_{w\in K}\frac{I_1(w)}{I_2(w)} \leq (c_0^*)^2 \leq 1 + \alpha\,\frac{I_1(w^a)}{I_2(w^a)} + \beta\,\frac{I_3(w^a)}{I_2(w^a)}\,. \tag{25}$$

To find the lower bound accurately, we solve the minimisation problem

$$\min_{w\in K}\frac{I_1(w)}{I_2(w)}\,. \tag{26}$$

Problem (26) can be transformed to the following boundary value problem (Euler equation with boundary conditions):

$$\begin{aligned} &w_{xxxx} + \lambda\,w_{xx} = 0 \\ &w(0) = w_{xx} = 0\,, \quad w(1) = w_{xx}(1) = 0\,. \end{aligned} \tag{27}$$

Solutions to eigenvalue problem (27) are known to be

$$\begin{aligned} w_k(x) &= A\sin(k\pi x)\,, \quad k = 1, 2, 3, \ldots\,, \\ \lambda_k &= (k\pi)^2\,. \end{aligned}$$

The normalised solution ($A = 1$) corresponding to the minimum in (26) can be shown to be

$$\begin{aligned} w_{\min}(x) &= w_1(x) = \sin(\pi x)\,, \\ \lambda_{\min} &= \lambda_1 = \pi^2\,. \end{aligned}$$

Inserting w_1 into (19), we obtain

$$\begin{aligned} I_1(w_1) = \pi^2\int_0^1 \cos^2(\pi x)\,\mathrm{d}x = \frac{\pi^2}{2}\,, \qquad I_2(w_1) = \int_0^1 \sin^2(\pi x)\,\mathrm{d}x = \frac{1}{2} \\ I_3(w_1) = \int_0^1 x\,\sin^2(\pi x)\,\mathrm{d}x = \frac{1}{4}\,. \end{aligned} \tag{28}$$

Choosing $w^a = w_1 \in K$ in (24) and inserting (28) into (25), we obtain the estimate

$$1 + \alpha\,\pi^2 \leq (c_0^*)^2 \leq 1 + \alpha\,\pi^2 + \frac{\beta}{2}\,. \tag{29}$$

Estimate (29) gives lower and upper bounds for $(c_0^*)^2$. Here, c_0, α and β are defined in (8), (9)–(10).

Numerical solution by the Rayleigh–Ritz and the Fourier–Galerkin methods

The minimization problem was discretized using the Rayleigh-Ritz method and solved using the interior point method. The differential equation was solved via the Fourier-Galerkin method. Both methods were realized in Matlab.

Numerical solution for the minimization problem

Consider the minimization problem (17). Constant one does not effect the location of the optimum and it can be omitted. Divide (17) by constant α. The obtained minimization problem is equivalent to (17):

$$\min_{w \in K} \left[\frac{\int_0^1 (w_{xx})^2 \, \mathrm{d}x + a \int_0^1 x \, (w_x)^2 \, \mathrm{d}x}{\int_0^1 (w_x)^2 \, \mathrm{d}x} \right] , \tag{30}$$

where

$$a = \frac{\beta}{\alpha} = \frac{g m \ell^3}{D} .$$

The problem (30) has infinite amount of solutions: If w is a solution, then also cw, c is a constant, is a solution. Problem (30) can be solved as [6, p. 273]

$$\begin{aligned} &\min_{w \in K} \int_0^1 (w_{xx})^2 \, \mathrm{d}x + a \int_0^1 x \, (w_x)^2 \, \mathrm{d}x \\ &\text{subject to } \int_0^1 (w_x)^2 \, \mathrm{d}x = 1 . \end{aligned} \tag{31}$$

In addition, w must satisfy boundary conditions $w(0) = w(1) = 0$. Now, the condition $\int_0^1 (w_x)^2 \mathrm{d}x = 1$ sets the absolute value of the constant c mentioned above. Note that the normalization constant, which is now chosen to be one, can be chosen freely.

We discretize (31) using the Rayleigh-Ritz method. We present function w as a series

$$w(x) = \sum_{j=1}^{\infty} v_j \varphi_j(x) \tag{32}$$

in the basis

$$\varphi_j(x) \equiv \sin(j \pi x) , \quad x \in [0, 1]. \tag{33}$$

The basis (33) fulfills the boundary conditions $w(0) = w(1) = 0$ naturally. We fix a finite positive integer n_0 (number of modes), and approximate (32) with its finite analog

$$w(x) = \sum_{j=1}^{n_0} v_j \varphi_j(x) . \tag{34}$$

Inserting (33)–(34) into (31), we obtain

$$\begin{aligned} &\min \quad \mathbf{v}^{\mathrm{T}} \mathbf{A} \mathbf{v} + a \, \mathbf{v}^{\mathrm{T}} \mathbf{B} \mathbf{v} \\ &\text{subject to} \quad \mathbf{v}^{\mathrm{T}} \mathbf{C} \mathbf{v} - 1 = 0 , \end{aligned} \tag{35}$$

where $\mathbf{v} = (v_1, \ldots, v_{n_0})^{\mathrm{T}} \in \mathbb{R}^{n_0}$.

The elements of the matrices in (35) can be found analytically. They are

$$
\begin{aligned}
A_{ij} &:= \int_0^1 (\varphi_j)_{xx}\,(\varphi_i)_{xx}\ \mathrm{d}x = \frac{j^4\pi^4}{2}\delta_{ij}\,,\\
B_{ij} &:= \int_0^1 x\,(\varphi_j)_x\,(\varphi_i)_x\ \mathrm{d}x = -\int_0^1 x\,(\varphi_j)_{xx}\,\varphi_i\ \mathrm{d}x - \int_0^1 (\varphi_j)_x\,\varphi_i\ \mathrm{d}x = -E_{ij} - D_{ij}\,,\\
C_{ij} &:= \int_0^1 (\varphi_j)_x\,(\varphi_i)_x\ \mathrm{d}x = \frac{j^2\pi^2}{2}\delta_{ij}\,,\\
D_{ij} &:= \int_0^1 (\varphi_j)_x\,\varphi_i\ \mathrm{d}x = \begin{cases} 0 & , \quad i = j \\ \frac{ij\left[(-1)^{i+j}-1\right]}{j^2-i^2} & , \quad i \neq j \end{cases}\,,\\
E_{ij} &:= \int_0^1 x\,(\varphi_j)_{xx}\,\varphi_i\ \mathrm{d}x = \begin{cases} -\frac{j^2\pi^2}{4} & , \quad i = j \\ -\frac{2ij^3\left[(-1)^{i+j}-1\right]}{[i-j]^2[i+j]^2} & , \quad i \neq j \end{cases}\,,
\end{aligned}
\tag{36}
$$

where δ_{ij} is the Kronecker delta.

The discretized problem (35) is a nonlinear optimization problem with a nonlinear objective function and a nonlinear constraint. The optimization problem variable is vector $\mathbf{v}$. Note that the size of the problem depends on n_0. Note also that the boundary conditions are included in matrices $\mathbf{A}$, $\mathbf{B}$ ja $\mathbf{C}$.

Numerical solution of the differential equation

The numerical solution of the original boundary value problem (5)–(6) was performed by using the Fourier–Galerkin method.

Define

$$\lambda := 1 - c_0^2\,. \tag{37}$$

Consider the dimensionless problem (11)–(12). Eigenvalue problem for the eigenvalue-eigenfunction pair (λ, w) is

$$\alpha\, w_{xxxx} - \beta\,(x w_{xx} + w_x) = \lambda\, w_{xx}\,, \tag{38}$$

$$w(0) = w_{xx}(0) = 0\,, \quad w(1) = w_{xx}(1) = 0\,. \tag{39}$$

By solving c_0 from (37), we see that the largest eigenvalue $\lambda_{\max}$ corresponds to the minimal critical velocity.

It is easy to show that all eigenvalues $\lambda \leq 0$. Inserting λ in (37) into (16), we obtain

$$-\alpha \int_0^1 w_{xx} v_{xx}\ \mathrm{d}x - \beta \int_0^1 x w_x v_x\ \mathrm{d}x = \lambda \int_0^1 w_x v_x\ \mathrm{d}x\,. \tag{40}$$

Denote $a(w,v) = -\alpha \int_0^1 w_{xx} v_{xx}\ \mathrm{d}x - \beta \int_0^1 x w v\ \mathrm{d}x$. Choosing $v = w$, we have $a(w,w) \leq 0$, but on the other hand $a(w,w) = \lambda \int_0^1 w_x^2\ \mathrm{d}x$. Thus $\lambda \leq 0$, and a physically meaningful solution ($c_0^2 > 0$, by (37)) always exists. Furthermore, we see that $c_0^2 \geq 1$, i.e., the critical velocity cannot be smaller than that of a travelling string (for which it is $\sqrt{T_0/m}$).

We apply the Fourier–Galerkin method to problem (38)–(39). Once we have the eigenvalue λ, we can obtain V_0^* from (37) and (8).

We present function w as a Galerkin series (32) in the basis (33). The basis (33) fulfills the boundary conditions (39) naturally. Inserting (33)–(34) into (40), we obtain the matrix equation

$$(-\alpha\,\mathbf{A} - \beta\,\mathbf{B})\,\mathbf{v} = \lambda\,\mathbf{C}\mathbf{v}. \tag{41}$$

Table 1. Physical parameters for the reference case.

g	T_0	m	ℓ	h	E	ν
$9.81\ \mathrm{m/s^2}$	$500\ \mathrm{N/m}$	$0.08\ \mathrm{kg/m^2}$	$1\ \mathrm{m}$	$10^{-4}\ \mathrm{m}$	$10^9\ \mathrm{N/m^2}$	0.3

$\Rightarrow$

$D = Eh^3/(12 \cdot (1 - \nu^2))$
$9.1575 \cdot 10^{-5}\ \mathrm{Nm}$

The elements of matrices in (41) are given in (36).

Equation (41) now becomes a standard generalized linear eigenvalue problem for the pair $(\lambda, \mathbf{v})$, to which any standard solver may be applied.

Numerical results

The physical parameters used are given in Table 1. The number of basis functions used was $n_0 = 200$. By equations (9) and (10), the given values lead to the nondimensional parameter values $\alpha = 1.8315 \cdot 10^{-7}$, $\beta = 1.5696 \cdot 10^{-3}$ and $a_0 = \beta/\alpha = 8.5700 \cdot 10^3$, which was used as a reference case. Problem (35) was solved using the Matlab Optimization Toolbox. Optimization method was chosen to be the interior point method. The results given by optimizer were validated by comparing to solutions given by the direct solver (Fourier–Galerkin method).

Figure 3 presents the buckling modes and the corresponding critical velocities for the different values of a in the case of the Rayleigh–Ritz and the Fourier–Galerkin methods. In each figure, a solid line corresponds to the solution given by the optimizer, and a dash-dot line corresponds to the critical mode given by the differential equation solver. The buckling modes and the critical velocities (in figure titles) were the same in the results given by the optimizer and the direct solver. (The solution plots for the calculated buckling modes overlap.)

Some qualitative observations were made. First, the effect of gravity on the eigenmode is very large, see Figures 3 and 4. The extremum of the eigenmode concentrates toward the start of the span in both studied models. This result is as expected, because positive x axis was chosen to point up in the gravitational field.

The effect of the gravity on the critical velocity is very minor, typically less than 0.01%. See Figure 5. In this respect, the results resemble those from our earlier study, where the effect of a linear tension profile to the buckling behaviour of an axially moving plate was investigated [4].

Furthermore, the strength of the concentration effect depends on the ratio of the dimensionless parameters β and α, i.e. on the quantity $a = \beta/\alpha = mg\ell^3/D$. The larger this parameter is, the stronger is the effect. In the limit $a \to 0$, the effect vanishes. In the other limit $a \to \infty$, the eigenmode approaches a characteristic shape that resembles a sawtooth function (but starts smoothly from 0). See Figure 4.

In the case of nonvertical direction of motion with respect to the gravity, the buckling problem was solved for different values of the angle θ in equation (4). In Figure 6 on the left, the graphs of the buckled shapes are shown for the values $\theta = 0,\ \pi/8,\ ,\pi/4,\ 3\pi/8,\ \pi/2$, and the displacement maxima are marked by $\star$.

In Figure 6 on the right, the buckling modes are represented for the values $[0, \pi/2]$ of θ as a colour sheet. It is seen that the buckling mode rapidly becomes nonsymmetric, when

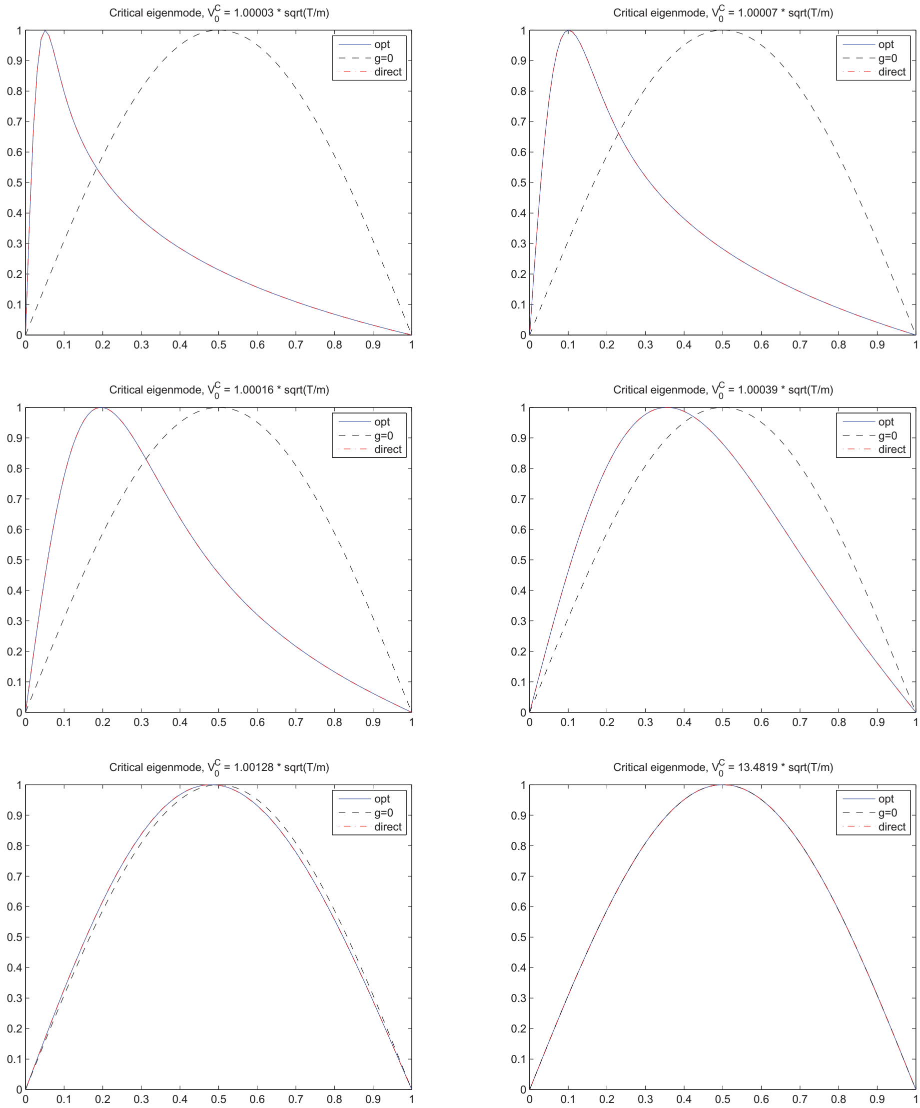

Figure 3. Critical buckling mode for some values of the parameters. The band moves toward the right and gravity points toward the left. Comparison of the solutions given by the energy minimization (”opt”, solid line) and by the direct solving of a differential equation (”direct”, dash-dot line). The dashed line corresponds to a reference solution with no gravity ($g = 0$).
The top right picture represents the reference case given in the text, for which $a = 8.57 \cdot 10^3$.
In the pictures, $a/a_0 = 10$, 1, 0.1, 0.01, 0.001, 10^{-8} (from left to right, top to bottom, in that order).

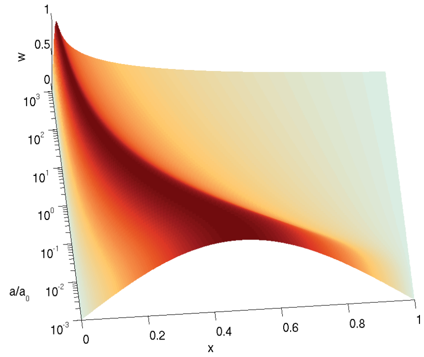

Figure 4. Critical buckling mode for different parameter values. The band moves toward the right and gravity points toward the left.
The reference value is $a_0 = 8.57 \cdot 10^3$. Note the logarithmic scale of a/a_0.

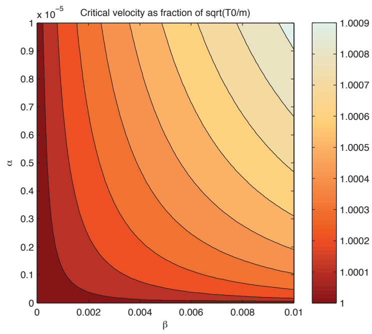

Figure 5. Effect of the dimensionless parameters α and β (defined by (9) and (10)) on the critical velocity. Note that the reference case is at the lower edge of the figure at $\beta \approx 0.0016$. The scaling for the axes was chosen to show the structure of the data.

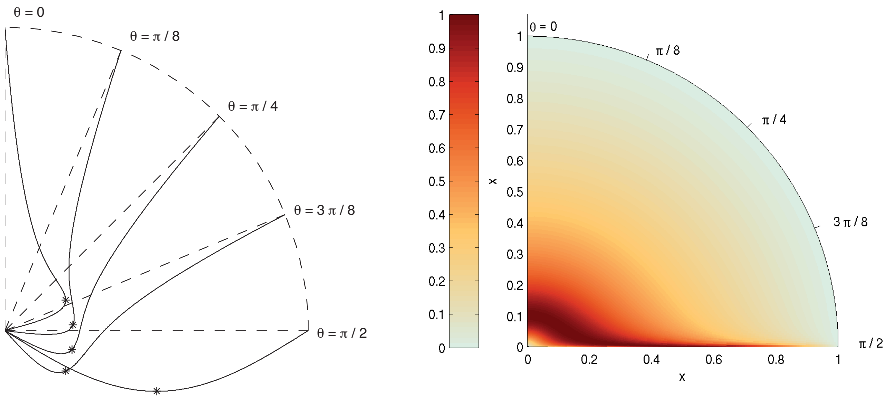

Figure 6. Buckling modes, when the direction of motion of the band is at an angle to the gravity. Left: Graphs of buckling modes for some selected cases. Displacement maxima are marked by $\star$. Right: Colour sheet of the buckling mode for values of θ between 0 and $\pi/2$.

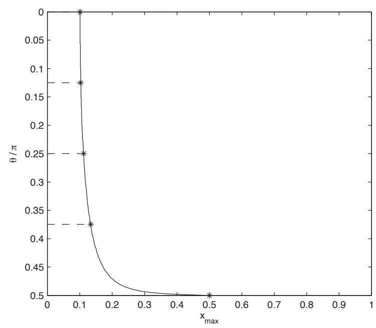

Figure 7. The location of the maximum displacement for θ in $[0, \pi/2]$. The stars ($\star$) correspond to those shown in Figure 6.

the moving direction of the panel non-orthogonal to the gravity, and the nonsymmetric modes are quite similar for small values of θ. This behaviour is illustrated more clearly in Figure 7.

Note that the sign of the axial velocity V_0 does not affect the buckling problem (5)–(6). Therefore, the range $\theta \in [0, \pi/2]$ covers all possible band orientations with respect to the gravity.

Conclusion

The loss of elastic stability of an axially moving band (panel) was investigated, taking the gravitational force into account. The studies performed were mainly based on analytical approaches. The onset of instability in a divergence (static) form for some critical value of the transport velocity was estimated using a variational principle to develop variational inequalities. Analytical lower and upper bounds for the critical velocity were derived. Additionally, it was shown that a critical velocity always exists.

Nonsymmetric solutions of the buckling problem (divergence forms and corresponding critical eigenvalues) were illustrated with the help of numerical examples. As a result, a large influence of the gravity force on the buckling mode, and a small effect on the critical transport velocity, were established and discussed.

Optimization methods were shown to give reliable results for solving this type of differential equations. However, the used interior point method was seen to be slow compared to the direct solver.

The effect of the angle of the motion of the band with respect to the gravity was numerically illustrated. This was done by solving the buckling problem, and the obtained buckling modes were illustrated. Compared to the case where the gravity is orthogonal to the axial motion, and does not affect the buckling mode, it was seen that even a small angle is enough to produce a notably nonsymmetric shape.

Comparing to the effect of introducing a linear tension profile in the noncylidrical deformation case (see [4]), it seems that very high sensitivity of the buckling mode, and very low sensitivity of the critical velocity, to various aspects of the problem setup is a phenomenon typical to this class of problems.

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