*Reports of the Department of Mathematical Information Technology*
*Series B. Scientific Computing*
***No. B. 8/2012***

# On Static Instability and Estimates for Critical Velocities of Axially Moving Orthotropic Plates under Inhomogeneous Tension

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ISBN 978-951-39-5002-6
ISSN 1456-436X

# On Static Instability and Estimates for Critical Velocities of Axially Moving Orthotropic Plates under Inhomogeneous Tension*

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**Abstract**

In this study, models for axially moving orthotropic plates are investigated analytically. Linearised Kirchhoff plate theory is used, and the energy forms of steady-state models are considered. With the help of the energy forms, homogeneous and inhomogeneous tension profiles in the moving direction of the web are studied. In the cases of both homogeneous and inhomogeneous tension profiles, some limits for the critical web velocity are found analytically. A numerical example is given about effects of the shear modulus in the case of an inhomogeneous tension profile.

## 1 Introduction

In industrial processes with axially moving materials, such as making of paper, steel or textiles, high transport speed is desired but it also may cause loss of stability. In modeling of such systems, the researchers have generally studied dynamic behavior of strings, membranes, beams and plates taking into account the transverse, Coriolis and centripetal accelerations of the material motion. The first studies on them include Sack (1954), Archibald and Emslie (1958), Miranker (1960), Swope and Ames (1963) and Mote (1968, 1972, 1975).

Sack (1954), Archibald and Emslie (1958) and Simpson (1973) studied the effects of axial motion on the frequency spectrum and eigenfunctions. In their research, it was shown that the natural frequency of each mode decreases as the transport speed is increased, and that the traveling string and beam both experience divergence instability at a sufficiently high speed. Wickert and Mote studied stability of axially moving strings and beams using modal analysis and Green's function method (Wickert and Mote, 1990). They presented the expressions for the critical transport velocities analytically. However recently, Wang et al. (2005) showed analytically that no static instability occurs for the transverse motion of a string at the

*This research was supported by the Jenny and Antti Wihuri Foundation and the Academy of Finland, grant #140221

critical velocity. For axially moving beams with a small flexural stiffness, Kong and Parker (2004) found closed-form expressions for the approximate frequency spectrum by a perturbation analysis.

Stability of travelling rectangular membranes and plates was first studied by Ulsoy and Mote (1982), and Lin and Mote (1995, 1996). Stability of out-of-plane vibrations of axially moving rectangular membranes was studied by Shin et al. (2005). For the behaviour of the membrane, it was found that the motion is stable until a critical speed, at which statical instability occurs. Lin (1997) studied stability of axially moving plates, and numerically showed that loss of stability of the plate occurs in a form of divergence at a sufficiently high speed. Banichuk et al. (2010a) considered stability and studied the critical velocity and the corresponding critical shapes of an axially moving elastic plate.

In the recent studies concerning axially moving plates, material properties such as orthotropicity (Banichuk et al., 2011) or viscoelasticity (Marynowski, 2010) have been taken into consideration and their effects on the plate behaviour have been investigated. In Banichuk et al. (2011), divergence instability for travelling orthotropic rectangular plates, with two opposite edges simply supported and the other two edges free, was studied and an explicit expression for the limit velocity of stable axial motion was found. Hatami et al. (2009) studied free vibration of the moving orthotropic rectangular plate in sub- and super-critical speeds, and flutter and divergence instabilities at supercritical speeds. Their study was limited to simply supported boundary conditions at all edges. Free vibrations of orthotropic rectangular plates, which are not moving, have been studied by Biancolini et al. (2005) including all combinations of simply supported and clamped boundary conditions on the edges. Xing and Liu (2009) obtained exact solutions for free vibrations of stationary rectangular orthotropic plates considering three combinations of simply supported (S) and clamped (C) boundary conditions: SSCC, SCCC and CCCC. Kshirsagar and Bhaskar (2008) studied vibrations and buckling of loaded stationary orthotropic plates. They found critical loads of buckling for all combinations of boundary conditions S, C and F.

Tension inhomogeneities and their effects on the divergence instability of moving plates have been studied in Banichuk et al. (2010b). In their study, a linearly inhomogeneous tension profile was considered in the case of a moving *isotropic* plate. The inhomogeneities in tension were found to change the buckling shapes dramatically compared to the shapes in the case of homogeneous tension.

In this report, we study the energy forms corresponding to (the static form of) a travelling orthotropic plate under homogeneous or inhomogeneous tension. It is shown that the critical velocity for an orthotropic plate under homogeneous tension is greater than the critical velocity of an ideal membrane. The differential form of the equations for a travelling orthotropic plate under an arbitrary tension field are derived from the corresponding energy form. For a linearly inhomogeneous tension profile, we solve the stress field with the help of the (Airy) stress function. For this type of inhomogeneity, we show that the critical velocity is always real-valued and present a numerical example.

## 2 A model of an axially moving orthotropic web

Consider an axially moving orthotropic rectangular plate travelling between two supports. The plate is assumed to be subjected to tension $T$. The problem set-up is shown in Figure 1. The plate width is $2b$, its thickness is $h$, and the length of the span is $\ell$. Throughout this study, the plate is assumed to travel at a constant velocity $V_0$. We denote the transverse displacement of the plate by the function $w(x, y)$.

The material parameters for the orthotropic plate are denoted by $m$ (the mass per unit area), $\nu_{12}$ and $\nu_{21}$ (the Poisson ratios in plane), $E_1$ and $E_2$ (the Young's moduli in the $x$ and $y$ directions, respectively), and $G_{12}$ (the shear modulus).

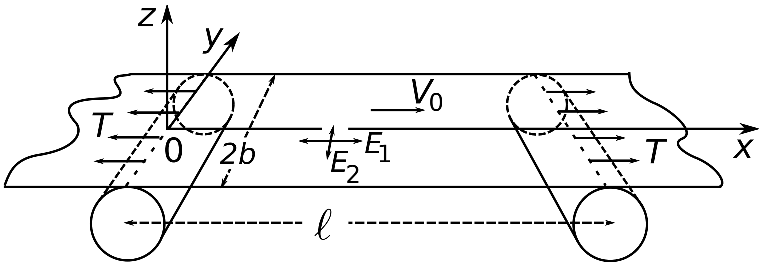


Figure 1: A travelling orthotropic plate.

We consider the case of a travelling orthotropic plate under inhomogeneous tension denoting the in-plane tensions by $T_{xx}$, $T_{yy}$ and $T_{xy} = T_{yx}$. The bilinear form corresponding to the energy of the moving orthotropic thin plate (see also Timoshenko and Woinowsky-Krieger (1959), in which the strain energy is given for a stationary orthotropic plate, and Chen et al. (1998), in which bilinear energy forms, e.g., for stationary isotropic plates are studied) is

$$b(w, v) = \int_\Omega \left[ -m\, V_0^2 \frac{\partial w}{\partial x}\frac{\partial \bar{v}}{\partial x} + T_{xx}\frac{\partial w}{\partial x}\frac{\partial \bar{v}}{\partial x} + T_{yy}\frac{\partial w}{\partial y}\frac{\partial \bar{v}}{\partial y} + T_{xy}(\frac{\partial w}{\partial x}\frac{\partial \bar{v}}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \bar{v}}{\partial x}) \right.$$
$$\left. + D_1 \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \bar{v}}{\partial x^2} + A_1 \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \bar{v}}{\partial y^2} + A_1 \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \bar{v}}{\partial x^2} + D_2 \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \bar{v}}{\partial y^2} + 4\, A_2 \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \bar{v}}{\partial x \partial y} \right] \mathrm{d}\Omega\,, \tag{1}$$

where $\bar{v}$ in the complex conjugate of $v$. In (1), the bending rigidities $D_i$ and $A_i$ are defined as

$$D_1 = \frac{E_1 h^3}{12\,(1 - \nu_{12}\nu_{21})}\,, \qquad D_2 = \frac{E_2 h^3}{12\,(1 - \nu_{12}\nu_{21})}\,, \tag{2}$$

and

$$A_1 = \frac{\nu_{12} E_2 h^3}{12\,(1 - \nu_{12}\nu_{21})} = \frac{\nu_{21} E_1 h^3}{12\,(1 - \nu_{12}\nu_{21})}\,, \qquad A_2 = \frac{G_{12} h^3}{12}\,, \tag{3}$$

where we have assumed the relation

$$E_1 \nu_{21} = E_2 \nu_{12}\,. \tag{4}$$

Let us derive the (partial) differential equation corresponding to the energy (1). Integration by parts of $b(w, v)$ gives (the Rayleigh-Green formula)

$$\begin{aligned}
b(w,v) = & \int_\Omega \Bigg[ m\,V_0^2 \frac{\partial^2 w}{\partial x^2} - T_{xx}\frac{\partial^2 w}{\partial x^2} - 2T_{xy}\frac{\partial^2 w}{\partial x \partial y} - T_{yy}\frac{\partial^2 w}{\partial y^2} \\
& - \left( \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} \right) \frac{\partial w}{\partial x} - \left( \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} \right) \frac{\partial w}{\partial y} \\
& + D_1 \frac{\partial^4 w}{\partial x^4} + 2(A_1 + 2A_2)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} \Bigg] \bar{v}\, \mathrm{d}\Omega \\
& + \int_{-b}^{b} \left[ \left( -m\,V_0^2 \frac{\partial w}{\partial x} + T_{xx}\frac{\partial w}{\partial x} + T_{xy}\frac{\partial w}{\partial y} \right) \bar{v} \right]_{x=0}^{x=\ell} \mathrm{d}y \\
& + \int_0^\ell \left[ \left( T_{yy}\frac{\partial w}{\partial y} + T_{xy}\frac{\partial w}{\partial x} \right) \bar{v} \right]_{y=-b}^{y=b} \mathrm{d}x \\
& + \int_{-b}^{b} \left[ D_1 \left( \frac{\partial^2 w}{\partial x^2} + \frac{A_1}{D_1}\frac{\partial^2 w}{\partial y^2} \right) \frac{\partial \bar{v}}{\partial x} + 4\,A_2 \frac{\partial^2 w}{\partial x \partial y}\frac{\partial \bar{v}}{\partial y} \right]_{x=0}^{x=\ell} \mathrm{d}y \\
& + \int_0^\ell \left[ D_2 \left( \frac{\partial^2 w}{\partial y^2} + \frac{A_1}{D_2}\frac{\partial^2 w}{\partial x^2} \right) \frac{\partial \bar{v}}{\partial y} \right]_{y=-b}^{y=b} \mathrm{d}x \\
& - \int_{-b}^{b} \left[ D_1 \left( \frac{\partial^3 w}{\partial x^3} + \frac{A_1}{D_1}\frac{\partial^3 w}{\partial x \partial y^2} \right) \bar{v} \right]_{x=0}^{x=\ell} \mathrm{d}y \\
& - \int_{x=0}^{x=\ell} \left[ D_2 \left( \frac{\partial^3 w}{\partial y^3} + \frac{A_1 + 4A_2}{D_2}\frac{\partial^3 w}{\partial x^2 \partial y} \right) \bar{v} \right]_{y=-b}^{y=b} \mathrm{d}x\,. 
\end{aligned} \tag{5}$$

We take into account the tension equilibria

$$\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} = 0\,, \quad \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} = 0\,. \tag{6}$$

and tension boundary conditions on the free boundaries

$$(T_{yy})_{y=\pm b} = 0\,, \qquad (T_{xy})_{y=\pm b} = 0\,, \qquad 0 \le x \le \ell\,. \tag{7}$$

For $w$, we introduce the (simply supported and free) boundary conditions

$$(w)_{x=0,\,\ell} = 0\,, \qquad \left( \frac{\partial^2 w}{\partial x^2} \right)_{x=0,\,\ell} = 0\,, \qquad -b \le y \le b\,, \tag{8}$$

$$\left( \frac{\partial^2 w}{\partial y^2} + \beta_1 \frac{\partial^2 w}{\partial x^2} \right)_{y=\pm b} = 0\,, \qquad 0 \le x \le \ell\,, \tag{9}$$

$$\left( \frac{\partial^3 w}{\partial y^3} + \beta_2 \frac{\partial^3 w}{\partial x^2 \partial y} \right)_{y=\pm b} = 0\,, \qquad 0 \le x \le \ell\,, \tag{10}$$

where

$$\beta_1 = \frac{A_1}{D_2} = \nu_{12}\,, \qquad \beta_2 = \frac{A_1 + 4A_2}{D_2} = \nu_{12} + \frac{4G_{12}}{E_2}(1 - \nu_{12}\nu_{21})\,. \tag{11}$$

In (5), the terms on the second row zero out after inserting (6). We use the boundary conditions for tension in (7), and for the displacement $w$ in (8), (9) and (10). Due to linearity, $\bar{v}$ satisfies the boundary conditions if and only if $v$ does. We take into account that for $v$, we have $v = \bar{v} = 0$ at $x = 0,\ \ell$, which implies that also the derivative of $v$ in the $y$ direction vanishes, i.e. $\partial v/\partial y = \partial \bar{v}/\partial y = 0$. All the boundary terms (on the last six lines) vanish after these insertions.

The partial differential equation corresponding to (5) is

$$m\,V_0^2\frac{\partial^2 w}{\partial x^2} - T_{xx}\frac{\partial^2 w}{\partial x^2} - 2\,T_{xy}\frac{\partial^2 w}{\partial x \partial y} - T_{yy}\frac{\partial^2 w}{\partial y^2} + D_1\frac{\partial^4 w}{\partial x^4} + 2(A_1 + 2A_2)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2\frac{\partial^4 w}{\partial y^4} = 0\,, \quad (12)$$

being subjected to the boundary conditions in (8)–(10).

In-plane tensions and stresses $\sigma_{xx}$, $\sigma_{xy}$ and $\sigma_{yy}$ are related by

$$T_{xx} = h\;\sigma_{xx}\,, \quad T_{xy} = h\;\sigma_{xy}\,, \quad T_{yy} = h\;\sigma_{yy}\,. \quad (13)$$

Referring to (6), the in-plane stresses $\sigma_{xx}$, $\sigma_{xy}$ and $\sigma_{yy}$ satisfy the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0\,, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0\,. \quad (14)$$

In the following, we will present the Airy stress equilibrium for an orthotropic plate. With the assumption of small deflections, the strains $\varepsilon_{xx}$, $\varepsilon_{yy}$ and $\gamma_{xy}$ are defined with the help of the in-plane displacement $u$ and $v$ (in the $x$ and $y$ directions, respectively) as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}\,, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}\,, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\,,$$

for which the following equation holds

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0\,. \quad (15)$$

Stresses and strains are related to each other by Hooke's law (inverse relation):

$$\varepsilon_{xx} = \frac{1}{E_1}\sigma_{xx} - \frac{\nu_{21}}{E_2}\sigma_{yy}\,, \quad \varepsilon_{yy} = \frac{1}{E_2}\sigma_{yy} - \frac{\nu_{12}}{E_1}\sigma_{xx}\,, \quad \gamma_{xy} = \frac{1}{G_{12}}\sigma_{xy}\,. \quad (16)$$

We introduce the stress function $F$:

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2}\,, \quad \sigma_{yy} = \frac{\partial^2 F}{\partial x^2}\,, \quad \sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y}\,. \quad (17)$$

The stresses in (17) satisfy automatically (14). Inserting (17) and (16) into (15), we obtain (with the help of the relation in (4)):

$$\frac{\partial^4 F}{\partial x^4} + \left(\frac{E_2}{G_{12}} - 2\nu_{21}\right)\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{E_2}{E_1}\frac{\partial^4 F}{\partial y^4} = 0\,. \quad (18)$$

Note that the orthotropic model can be reduced to an isotropic model by choosing $G_{12} = G_{\mathrm{H}}$ (Huber, 1923; Timoshenko and Woinowsky-Krieger, 1959), where

$$G_{\mathrm{H}} = \frac{\sqrt{E_1 E_2}}{2(1 + \sqrt{\nu_{12}\nu_{21}})} . \tag{19}$$

In practice, the measured values for the shear modulus $G_{12}$ may significantly differ from this ideal value (Seo, 1999; Yokoyama and Nakai, 2007). In such cases, the full orthotropic model must be used.

## 2.1 A travelling orthotropic plate under homogeneous tension

The dynamic equation for an axially moving orthotropic plate under homogeneous tension is (Marynowski, 2008; Banichuk et al., 2011)

$$m\left(\frac{\partial^2 w}{\partial t^2} + 2V_0\frac{\partial^2 w}{\partial x \partial t} + V_0^2\frac{\partial^2 w}{\partial x^2}\right) - T_0\frac{\partial^2 w}{\partial x^2} + D_1\frac{\partial^4 w}{\partial x^4} + 2D_3\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2\frac{\partial^4 w}{\partial y^4} = 0 , \tag{20}$$

where

$$D_3 = A_1 + 2A_2 .$$

For the edges at the supports, we set simply supported boundary conditions and for the non-supported edges we set free boundary conditions. See equations (8)–(10).

We consider the steady-state form of equation (20):

$$(m\,V_0^2 - T_0)\frac{\partial^2 w}{\partial x^2} + D_1\frac{\partial^4 w}{\partial x^4} + 2D_3\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2\frac{\partial^4 w}{\partial y^4} = 0 . \tag{21}$$

The corresponding energy form is (compare with (1))

$$\begin{aligned} b_1(w,v) = \int_\Omega \Bigg[ &-(m\,V_0^2 - T_0)\frac{\partial w}{\partial x}\frac{\partial \bar{v}}{\partial x} + D_1\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \bar{v}}{\partial x^2} + A_1\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \bar{v}}{\partial y^2} \\ &+ A_1\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \bar{v}}{\partial x^2} + D_2\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \bar{v}}{\partial y^2} + 4\,A_2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \bar{v}}{\partial x \partial y}\Bigg]\,\mathrm{d}\Omega , \end{aligned} \tag{22}$$

From (22), it can be easily shown for the divergence velocity $V_0^{\mathrm{div}}$ that $m(V_0^{\mathrm{div}})^2 - T_0 \geq 0$. For $v = w$, we obtain (for a complex number $a$, $\|a\|^2 = a\bar{a}$)

$$\begin{aligned} b_1(w,w) = \int_\Omega \Bigg[ &-(m\,V_0^2 - T_0)\left\|\frac{\partial w}{\partial x}\right\|^2 + D_1\left\|\frac{\partial^2 w}{\partial x^2} + \frac{A_1}{D_1}\frac{\partial^2 w}{\partial y^2}\right\|^2 \\ &+ (D_2 - \frac{A_1^2}{D_1})\left\|\frac{\partial^2 w}{\partial y^2}\right\|^2 + 4A_2\left\|\frac{\partial^2 w}{\partial x \partial y}\right\|^2\Bigg]\,\mathrm{d}\Omega . \end{aligned}$$

The constant $D_2 - A_1^2/D_1 \geq 0$, since

$$D_2 - \frac{A_1^2}{D_1} = D_2(1 - \frac{A_1^2}{D_1 D_2}) = D_2(1 - \nu_{12}\nu_{21}) ,$$

and $0 \le \sqrt{\nu_{12}\nu_{21}} \le 1/2$ and $D_2 > 0$. For $b_1(w, w) = 0$, we obtain

$$(m\,(V_0^{\mathrm{div}})^2 - T_0) \underbrace{\int_\Omega \left\| \frac{\partial w}{\partial x} \right\|^2 \mathrm{d}\Omega}_{\ge 0}$$
$$= \underbrace{\int_\Omega \left[ D_1 \left\| \frac{\partial^2 w}{\partial x^2} + \frac{A_1}{D_1} \frac{\partial^2 w}{\partial y^2} \right\|^2 + (D_2 - \frac{A_1^2}{D_1}) \left\| \frac{\partial^2 w}{\partial y^2} \right\|^2 + 4A_2 \left\| \frac{\partial^2 w}{\partial x \partial y} \right\|^2 \right] \mathrm{d}\Omega}_{\ge 0} \,.$$

This implies that $m(V_0^{\mathrm{div}})^2 - T_0 \ge 0$, and finally for the critical velocity: $V_0^{\mathrm{div}} \ge \sqrt{T_0/m}$. This is to say that in the case of homogeneous tension, the divergence velocity of an orthotropic plate is always greater than the divergence velocity of an ideal membrane, which is $\sqrt{T_0/m}$.

## 2.2 A travelling orthotropic plate under a linear tension profile

Consider now a case in which the tension profile is linear at the web edges $x = 0$ and $x = \ell$. That is, $(T_{xx})_{x=0,\,\ell} = T_0 + \alpha y$ where $T_0$ and $\alpha$ are positive constants such that $T_{xx}$ is non-negative along the edge. The parameter $\alpha$ will be called the tension profile skew parameter. See Figure 2. We introduce boundary conditions for the stresses:

$$\begin{aligned} (\sigma_{xx})_{x=0,\,\ell} = \frac{1}{h}(T_0 + \alpha y)\,, \qquad (\sigma_{xy})_{x=0,\,\ell} = 0\,, \qquad -b \le y \le b\,, \\ (\sigma_{yy})_{y=\pm b} = 0\,, \qquad (\sigma_{xy})_{y=\pm b} = 0\,, \qquad 0 \le x \le \ell\,. \end{aligned} \tag{23}$$

With the help of the boundary conditions in (23) and the relations in (17), we find

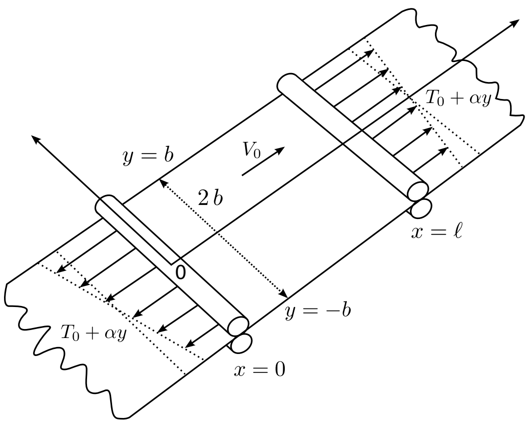


Figure 2: A moving plate subjected to tension with a linear profile at the edges.

the boundary conditions for $F$. Note that above the last two boundary conditions in (23) guarantee that (7) are satisfied. The boundary conditions for the stress function

$F$ read as

$$\left(\frac{\partial^2 F}{\partial y^2}\right)_{x=0,\,\ell} = \frac{1}{h}(T_0 + \alpha y)\,, \qquad \left(\frac{\partial^2 F}{\partial x \partial y}\right)_{x=0,\,\ell} = 0\,, \qquad -b \le y \le b\,,$$
$$\left(\frac{\partial^2 F}{\partial x^2}\right)_{y=\pm b} = 0\,, \qquad \left(\frac{\partial^2 F}{\partial x \partial y}\right)_{y=\pm b} = 0\,, \qquad 0 \le x \le \ell\,. \tag{24}$$

The solution to the boundary value problem (18) and (24) is

$$F(x,y) = \frac{1}{h}\left(T_0 \frac{y^2}{2} + \alpha \frac{y^3}{6} + c_1 x + c_2 y + c_0\right)$$

where $c_0$, $c_1$ and $c_2$ are arbitrary real constants. Now the tensions are

$$T_{xx} = h\frac{\partial^2 F}{\partial y^2} = T_0 + \alpha y\,, \qquad T_{yy} = h\frac{\partial^2 F}{\partial x^2} = 0\,, \qquad T_{xy} = h\frac{\partial^2 F}{\partial x \partial y} = 0\,. \tag{25}$$

The partial differential equation for an axially moving orthotropic plate under a linear tension profile can now be written:

$$m\,V_0^2 \frac{\partial^2 w}{\partial x^2} - (T_0 + \alpha y)\frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = 0\,. \tag{26}$$

Assuming that $T_0 + \alpha y \ge 0$ and proceeding similarly as in the previous section, we may show that the divergence velocity $V_0^{\mathrm{div}}$ gets always real values:

$$m\,(V_0^{\mathrm{div}})^2 \underbrace{\int_\Omega \left\|\frac{\partial w}{\partial x}\right\|^2 \mathrm{d}\Omega}_{\ge 0} = \int_\Omega \Bigg[\underbrace{(T_0 + \alpha y)\left\|\frac{\partial w}{\partial x}\right\|^2}_{\ge 0}$$
$$+ \underbrace{D_1 \left\|\frac{\partial^2 w}{\partial x^2} + \frac{A_1}{D_1}\frac{\partial^2 w}{\partial y^2}\right\|^2 + (D_2 - \frac{A_1^2}{D_1})\left\|\frac{\partial^2 w}{\partial y^2}\right\|^2 + 4A_2 \left\|\frac{\partial^2 w}{\partial x \partial y}\right\|^2}_{\ge 0}\Bigg]\,\mathrm{d}\Omega\,.$$

Thus, $m(V_0^{\mathrm{div}})^2 \ge 0$ and $V_0^{\mathrm{div}}$ is real-valued.

# 3 Static analysis of stability loss

We present the solution of (26) and (8)–(10) in the following form:

$$w(x,y) = \sin\left(\frac{\pi x}{\ell}\right) f\left(\frac{y}{b}\right)\,, \tag{27}$$

where $f(y/b)$ is an unknown function. Introducing a new variable $\eta = y/b$ and inserting (27) into (26), we obtain

$$\mu^4 H_2 \frac{\mathrm{d}^4 f}{\mathrm{d}\eta^4} - 2\,\mu^2 H_3 \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + (H_1 + \tilde{\alpha}\eta) f = \lambda f\,, \qquad -1 \le \eta \le 1\,, \tag{28}$$

where

$$\mu = \frac{\ell}{\pi b}\,, \qquad \tilde{\alpha} = \frac{b\ell^2}{\pi^2 D_0}\alpha\,, \tag{29}$$

the eigenvalue $\lambda$ is defined as

$$\lambda = \frac{\ell^2}{\pi^2 D_0}\left(mV_0^2 - T_0\right)\,, \tag{30}$$

and the dimensionless bending rigidities are

$$H_1 = \frac{D_1}{D_0}\,, \qquad H_2 = \frac{D_2}{D_0}\,, \qquad H_3 = \frac{D_3}{D_0}\,. \tag{31}$$

In (31), $D_0$ can be chosen freely, e.g., $D_0 = D_1$.

The boundary conditions (9)–(10) become

$$\left(\mu^2 \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} - \beta_1 f\right)_{\eta=\pm 1} = 0\,, \tag{32}$$

$$\left(\mu^2 \frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} - \beta_2 \frac{\mathrm{d} f}{\mathrm{d}\eta}\right)_{\eta=\pm 1} = 0\,. \tag{33}$$

The parameters $\beta_1$ and $\beta_2$ are explained above in equation (11). Note that for an isotropic material $H_1 = H_2 = H_3 = 1$ with $D_0 = D$, and $\beta_1 = \nu$ and $\beta_2 = 2 - \nu$. For comparison, see Banichuk et al. (2010b).

# 4 Numerical analysis and results

The problem (28), (32)–(33) was solved numerically via the finite difference method. The solution process is reported in details in Banichuk et al. (2010b).

The strong form (28), (32)–(33) was discretized directly, with classical central differences of second-order asymptotic accuracy. To account for the boundary conditions, the method of virtual points was used. Because the boundary conditions are homogeneous, it is possible to add them to the discrete system by rewriting the original discrete problem as a generalized linear eigenvalue problem

$$\mathbf{A}\mathbf{f} = \lambda \mathbf{B}\mathbf{f}\,, \tag{34}$$

where $\mathbf{B}$ is an identity matrix with the first two and last two rows zeroed out, $\mathbf{A}$ contains the differential operators (in (28) on the left hand side), $\mathbf{f}$ is the discretized form of $f$. In (34), the first two and the last two rows of $\mathbf{A}$ contain the discretized boundary conditions.

In the computations, the geometric parameters for the plate were $\ell = 0.01$ m, $2b = 1$m, $h = 10^{-4}$m. The used material parameters were $m = 0.08$ kg/m$^2$, $E_1 = 6.8$ GPa, $E_2 = 3.4$ GPa, $\nu_{12} = 0.2$ (and $\nu_{21} = 0.1$ from equation (4)) and $G_{12} = 0.7G_{\mathrm{H}},\ G_{\mathrm{H}},$ or $1.3G_{\mathrm{H}}$ (where $G_{\mathrm{H}} \approx 2.11$ GPa is calculated with the help of other material parameters by equation (19)). The tension profile skew parameter was $\alpha = 10^{-5}\alpha_{\max}$, where $\alpha_{\max} = T_0/b$. That is, the tension at the egdes $y = \pm b$ differs $0.001$ % from the average tension $T_0$. (For homogeneous tension, $\alpha = 0$.) For the finite difference method, $600$ computation nodes were used.

For different values of the shear modulus, the buckling mode and the critical velocity were studied.

In Table 1, the results for the critical velocities are shown for three different values of the shear modulus $G_{12}$. It can be seen that the smaller the value of the shear modulus the lower the critical velocity $V_0^{\mathrm{cr}}$. However, the effect on the critical velocity is relatively small. The critical velocity for an orthotropic plate with $G_{12} = G_{\mathrm{H}}$ and homogeneous tension ($\alpha = 0$) is $V_0^{\mathrm{cr}} = 83.4460$ m/s (by keeping other parameter values the same). We may see that for a large value of the shear modulus ($1.3G_{\mathrm{H}}$), the critical velocity for a plate with an inhomogeneous tension profile can be larger than that of a plate with homogeneous tension but with a smaller value of the shear modulus.

Table 1: Critical velocities $V_0^{\mathrm{cr}}$ for three different shear moduli $G_{12}$.

| $G_{12}$ | $0.7G_{\mathrm{H}}$ | $G_{\mathrm{H}} \approx 2.11$ GPa | $1.3G_{\mathrm{H}}$ |
|---|---|---|---|
| $V_0^{\mathrm{cr}}$ (m/s) | 83.4452 | 83.4458 | 83.4461 |

In Figure 3, the effect of the shear modulus variation on the buckling mode is visualised by presenting the displacement $w$ at $x = \ell/2$. As expected, the larger the values of the shear modulus the smaller the change in the buckling mode when compared to the case with a homogeneous tension profile.

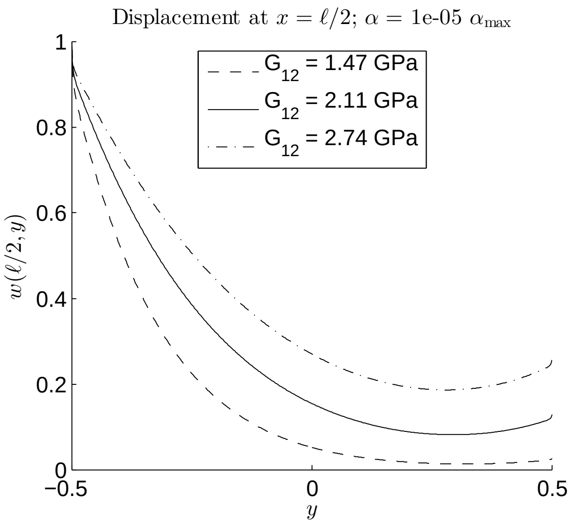


Figure 3: Effect of the value of the shear modulus $G_{12}$ on the buckling mode. The plate is under a linear tension profile, the tension profile skew parameter having the value $\alpha = 10^{-5}\alpha_{\max}$.

In Figure 4, both the effect of the value of the shear modulus $G_{12}$ and the effect of tension inhomogeneity are illustrated.

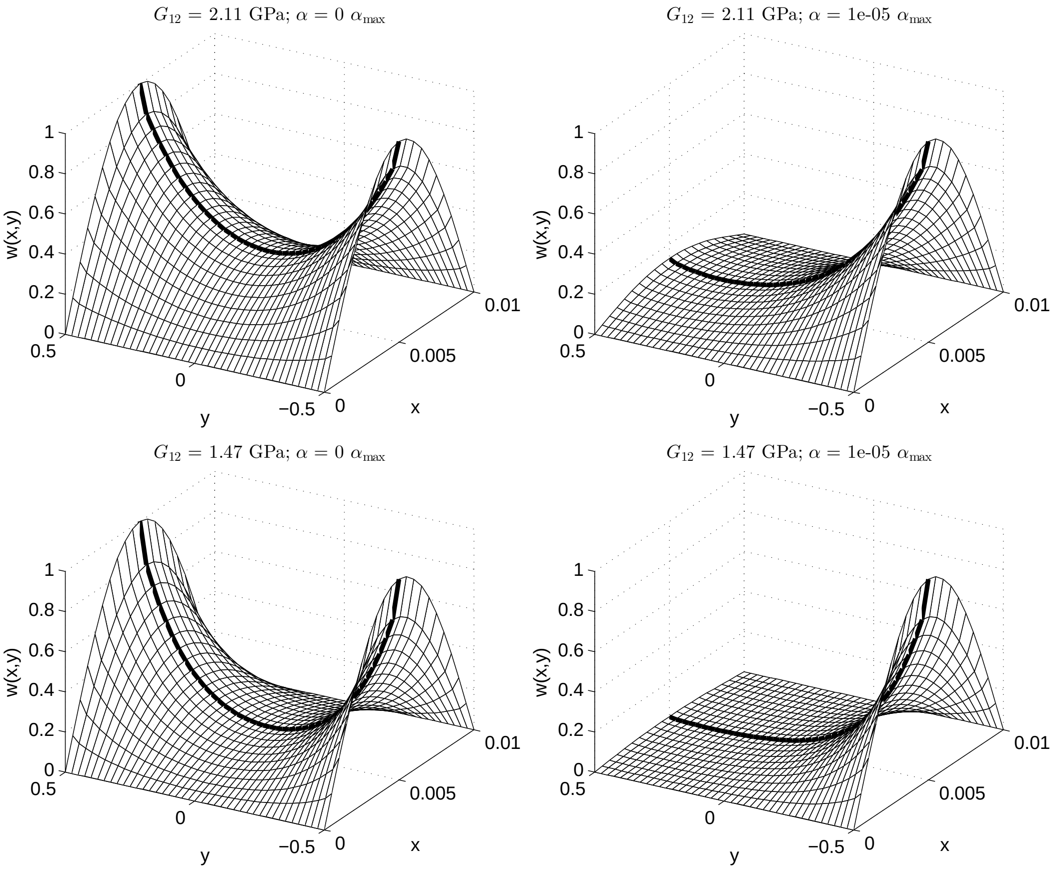


Figure 4: Effects of the tension inhomogeneities and the value of the shear modulus on the buckling mode. The tension profile skew parameter $\alpha$ increases from left to right, and the shear modulus decreases from top to bottom.

# 5 Conclusion

Models for axially moving orthotropic plates under an inhomogeneous tension profile were studied. For the critical velocity of a moving orthotropic plate under homogeneous tension, it was shown that the critical velocity is higher than that of an ideal membrane. A partial differential equation corresponding to the energy form of a moving orthotropic plate under any tension field was derived. The case of a linearly inhomogeneous tension profile was studied in details. It was shown analytically that the values of the critical valocity for the buckling problem are real-valued. A numerical example about the effects of the shear modulus on the critical speeds and buckling modes was given. It was seen that the greater the value of the shear modulus the smaller the change in the buckling mode when compared to the case with a homogeneous tension profile. The effect of the shear modulus on the critical speed was minor but for greater values of the shear modulus the plate was found to be more stable.

The analytical results found can be seen helpful for future studies in the area and for implementing numerical algorithms. For example, the limits for the critical velocities guarantee that the results given by a numerical algorithm are physically meaningful.

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