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Dmitry Petrov

Synthesis and Application of Orthogonal Signal Bases Possessing Enhanced Time-Frequency Localization for Mobile Wireless Networks

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ABSTRACT

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This thesis concentrates on the development and theoretical validation of a framework which can be used on the physical layer of modern and future wireless multi-carrier, in particular OFDM-based, systems. From a practical perspective, mobile networks set a very strict environment for different services, which should be reliably provided even in harsh channel conditions. Thus the study addresses several main problems of OFDM technology, such as poor localization of signal basis in the frequency domain and reduced efficiency due to the cyclic prefix utilization. These problems cannot be efficiently solved because of theoretical limitations of the classical WH basis.

The main approach developed in this thesis is the utilization of modified orthogonal signal bases possessing enhanced time-frequency localization. Their practical application requires overcoming several challenges, which are considered and solved in the dissertation. First, the structure and synthesis of such bases are dealt with. MWH bases have been chosen as the main object of the study. Their generic construction algorithm has been developed, starting from given continuous functions possessing desired localization characteristics. Proven orthogonality criteria allow a considerable increase in the efficiency of the algorithm. Secondly, phase and sampling parameters of the basis are optimized. Finally, computationally efficient signal modulation algorithm is developed. These results form the core of the OFTDM technology, where multiplexing in the frequency domain is complimented with the multiplexing in the time domain.

Keywords: OFDM, Localization, Multiplexing, PHY, Filter bank, Signal processing, Multi-carrier, Pulse-shaping

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GLOSSARY

2G	2nd Generation.
3G	3rd Generation.
3GPP	3rd Generation Partnership Project.
4G	4th Generation.
AWGN	Additive White Gaussian Noise.
BER	Bit Error Rate.
BLER	Block Error Rate.
BS	Base Station.
CDMA	Code Division Multiple Access.
CMT	Cosine modulated MultiTone.
CP	Cyclic Prefix.
DAB	Digital Audio Broadcasting.
DFT	Discrete Fourier Transform.
DVB	Digital Video Broadcasting.
DVB-T2	Digital Video Broadcasting, Second Generation Terrestrial.
EMFB	Exponentially Modulated FB.
FB	Filter Bank.
FBMC	Filter Bank Multiple Carrier.
FDM	Frequency Division Multiplexing.
FDMA	Frequency Division Multiple Access.
FFT	Fast Fourier Transform.
FMT	Filtered MultiTone.
FT	Fourier Transform.
GI	Guard Interval.
GSM	Global System for Mobile.
HetNet	Heterogeneous Network.
ICI	Inter-Carrier Interference.
IDFT	Inverse Discrete Fourier Transform.
IEEE	Institute of Electrical and Electronics Engineers.
IFFT	Inverse Fast Fourier Transform.
ISI	Inter-Symbol Interference.
LTE	Long Term Evolution.

LTEA	LTE Advanced.
MC	Multi-Carrier.
MCS	Modulation and Coding Scheme.
MFB	Modulated Filter Bank.
MIMO	Multiple Input Multiple Output.
MT	Mobile Terminal.
MWH	Modified Weyl-Heisenberg.
ns-2	Network Simulator v.2.
ns-3	Network Simulator v.3.
OFDM	Orthogonal Frequency Division Multiplexing.
OFDM/OQAM	OFDM with offset QAM.
OFDMA	Orthogonal Frequency-Division Multiple Access.
OFTDM	Orthogonal Frequency-Time Division Multiplexing.
PAPR	Peak-to-Average Power Ratio.
PDSL	Power Digital Subscriber Line.
PHY	Physical Level.
PMF	Pulse Matched Filter.
PSD	Power Spectral Density.
PSK	Phase-Shift Keying.
QAM	Quadrature Amplitude Modulation.
QoS	Quality of Service.
RF	Radio Frequency.
S/P	Serial/Parallel transformer.
SINR	Signal to Interference and Noise Ratio.
STFT	Short Time Fourier Transform.
SUI	Stanford University Interim.
SVD	Singular Value Decomposition.
TDMA	Time Division Multiple Access.
UE	User Equipment.
UMTS	Universal Mobile Telecommunications System.
UWB	Ultra Wide Band.
VDSL	Very high rate Digital Subscriber Line.
WFT	Windowed Fourier Transform.
WH	Weyl-Heisenberg.

WiMAX	Worldwide Interoperability for Microwave Access.
WLAN	Wireless Local Area Network.
WRAN	Wireless Regional Area Network.

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PII D. A. Petrov. Algorithms for Construction of Orthogonal Well-Localized Bases. *Journal of Mathematical Models and Computer Simulations 2 (5), 574-581*, 2010.

PIII D. A. Petrov, T. Hämäläinen. Efficient WH-OFTDM Signal Processing. *The International Conference on Ultra-Modern Telecommunications (ICUMT)*, 2010.

PIV D. A. Petrov. Ultrawideband Signals Constructed from Generalized Weyl-Heisenberg Bases. *The International Conference on Ultrawideband and Ultrashort Impulse Signals (UWBUSIS)*, 2010.

PV D. A. Petrov, T. Hämäläinen. Better Performance of Mobile Devices in Time-Frequency Dispersive Channels Using Well-Localized Bases. *The 11th International Conference on Next Generation Wired/Wireless Networking (New2An)*, 2011.

PVI D. A. Petrov, N. V. Gonchukov and T. Hämäläinen. Computationally Efficient Modulation of Well-Localised Signals for OFDM. *The 8th International Conference on Wireless and Mobile Computing, Networking and Communications (WiMob)*, 2012.

1 INTRODUCTION

In recent years, there has been a considerable progress in the development and implementation of high speed wireless telecommunication technologies. Constantly increasing data rates and user demands require guaranteed connection quality even in complicated and noisy conditions. However, it is a usual situation when a real radio channel is the subject to time-frequency dispersion [1, 2], for example, in an urban environment, near strong sources of electromagnetic interference, moving Mobile Terminals (MTs), etc. This highlights the importance of the study of new signal processing algorithms, in general, and the research on new types of signals and signal bases with optimal time-frequency characteristics, in particular.

Time-frequency analysis started its development from the beginning of the 20th century. In 1950's and in the beginning of the current century, its methods and applications were developed in numerous works of such scientists as A. Haar, D. Gabor [3], J. Ville [4], I. Daubechie [5], S. Mallat [6], Y. Meyer [7], F. Hlawatsch [2], H. Bölcskei [8], D. Proakis [9], M. Bellanger [10, 11], T. Strohmer and H. G. Feichtinger [12], P. Siohan, A. V. Oppenheim and R. W. Schafer [13], V. F. Kravchenko [14], W. Kozek, and many others. Theoretical ideas developed in these works have found applications in many areas of science, in particular in signal processing and telecommunications. For example, in many already adopted and still developing 3rd Generation (3G) and 4th Generation (4G) telecommunication systems, Orthogonal Frequency Division Multiplexing (OFDM) is the main technology on the Physical Level (PHY) [15]. An OFDM signal transmitted over the channel is a sequence of symbols, each of which is constructed as a linear combination of bases functions with complex information coefficients, Phase-Shift Keying (PSK) or Quadrature Amplitude Modulation (QAM) symbols [16]. It is well known that modulation and demodulation procedures can be performed efficiently with the help of Inverse Fast Fourier Transform (IFFT) and Fast Fourier Transform (FFT) algorithms correspondingly.

This thesis is mainly aimed to improve interference robustness, as well as, spectral and energy efficiency of wireless Multi-Carrier (MC) devices [17, 18]. During the exploitation of such systems the radio signal usually propagates through dispersive media and experiences Inter-Symbol Interference (ISI) and Inter-Carrier Interference (ICI) [19]. ISI is the result of overlapping of consequent OFDM symbols and of delayed copies of the same symbol. It arises from time selectivity of the channel and from the propagation over multiple paths. These effects can be almost completely removed with the help of Guard Interval (GI) or Cyclic Prefix (CP). ICI is caused by phenomena such as frequency selectivity of the channel, Doppler shifts/spreading, and impulse noise [20], resulting in the destruction of the internal structure of the signal. Interference can even cause breaks in the connection. These kinds of negative effects cannot be mitigated effectively enough because the rectangular form of pulse-shaping function used in OFDM is not optimal from the point of localization in the frequency domain. This causes obstacles in mobile broadband networks. Thus the problem of ICI reduction and adaptation to the time-frequency dispersion does not always have an adequate solution. Moreover, in real applications and computer simulations new problems appear in addition to those related to the enhancement of signal basis localization. The questions of practical realization of basis synthesis and signal processing algorithms become prominent. For example, in modern Digital Video Broadcasting (DVB) systems, the number of sub-carriers can amount to several tens of thousands [21]. Thus the computationally-efficient (or "fast") algorithms of pulse-shaping function generation and signal modulation presented in the thesis have an important practical meaning.

Today, digital video and 3G and 4G wireless telecommunication systems are widely adopted around the world. For example, DVB is the most widely-used transmission standard in the world, and there are about 1.4 billion digital receivers in use for television [22]. WiFi access points can be found almost in every apartment. Long Term Evolution (LTE) networks are spreading very fast as well. To the date of the publication of this thesis, 113 commercial networks in 51 countries have been established and about 500 devices announced by 79 manufacturers [23]. Even though Worldwide Interoperability for Microwave Access (WiMAX) has become mostly niche technology, it still has about 17 million subscribers worldwide [24]. For that reason, the development of validated mathematical methods and algorithms, allowing the enhancement of the parameters of modern telecommunication systems, can be considered as the solution of an important scientific and practical problem.

1.1 Limited Radio Resources

Wireless communication systems are required for higher and higher data rates. This goal is particularly challenging for systems that are power, bandwidth, and complexity limited. In multiple access wireless networks, many users are sharing the same media. This can be achieved by multiplexing, i.e. by the division of an available channel into several sub-channels.

There are several ways to do that [25]. For example, in Time Division Multiple Access (TDMA) systems, a channel is provided to different data streams in different time-slots. In Frequency Division Multiple Access (FDMA) channel access scheme, each user can be assigned to a specific frequency channel. In particular, 2nd Generation (2G) cellar systems like the Global System for Mobile (GSM) are based on a combination of TDMA and FDMA. 3G Universal Mobile Telecommunications System (UMTS) systems are based on the Code Division Multiple Access (CDMA) scheme, where several signals are transferred simultaneously over the same frequency band, utilizing different spreading codes [26]. Orthogonal Frequency-Division Multiple Access (OFDMA), used in particular in 4G cellar communication systems, advances from FDMA. It is a very flexible scheme, where each user can be allocated several sub-carriers dynamically in time, based on the radio channel conditions and load [27].

Space also provides one additional degree of freedom, which increases the efficiency of resource utilization. In cellular systems, each cell needs to have its own group of channels according to some multiplexing principle. Otherwise, inter-cell interference would occur. The first solution to this problem is resource reuse. A tendency favoured currently is to deploy small sells and Heterogeneous Networks (HetNets). Many mobile network operators see small cells, like femtocells, picocells, and microcells as vital to managing LTE Advanced (LTEA) spectrum more efficiently compared to using just macrocells [28]. The second approach for space utilization arises from the idea of parallel data transmission using several transmitting and receiving antennas. This technology is called Multiple Input Multiple Output (MIMO) and is already widely used in modern telecommunication standards, for example in Institute of Electrical and Electronics Engineers (IEEE) 802.11n, IEEE 802.16 and LTE [29].

It can be concluded that, together with all possible enhancements mentioned above, OFDM is currently the most advanced technology from the perspective of spectral efficiency. Nevertheless, there is still room for further enhancements. First of all, it is necessary to take into account that radio spectrum resources are already highly occupied. To ameliorate the situation, cognitive radio has been introduced [30]. This technology detects available channels in the wireless spectrum and allows their use on a concurrent basis. This approach can be applied for licensed spectrum bands (IEEE 802.22 standard for Wireless Regional Area Network (WRAN) operates on unused television channels [31]) and also on unlicensed parts of the Radio Frequency (RF) spectrum (for example, the IEEE 802.15.2 standard [32]). OFDM is affordable but is not the best solution for such kinds of networks. Its low spectral localization may cause unexpected interference for networks functioning in neighboring frequency bands. Guard frequencies, which ensure low interference, result in the waste of valuable frequency resources. Thus, the amelioration of spectrum localization of MC devices is one of the important directions for further enhancement of OFDM-based systems.

Another positive effect that can be achieved, together with better localization is more efficient and robust data transmission over error-prone wireless channels. To ensure high characteristics of Quality of Service (QoS), future wireless networks should be flexible and adaptive. In many applications, the possibility to maintain the connection is even more valuable than the maintenance of constant data rate. The internal structure of the MC signal plays here the essential role. Higher robustness means lower Bit Error Rate (BER) and, consequently, Block Error Rate (BLER) in the same channel conditions. In normal transmission conditions, this allows the usage of higher Modulation and Coding Schemes (MCSs), while in the case of deep fadings helps to maintain connectivity. Additionally, it is desirable to achieve higher robustness without the overheads introduced by CP, which reduces power and spectral efficiency of OFDM systems. From a technical perceptive, these kinds of problems are solved in terms of the Filter Bank (FB) theory considered in the following section.

1.2 Filter Banks

FB is a popular approach and has very efficient processing algorithms, which allow flexible extraction of different spectral components of the signal [33]. The theory of FBs provides a convenient framework for both the study and the implementation of signal decompositions [34]. A digital FB is a collection of digital filters with a common input or a common output. In signal processing, a FB is an array of band-pass filters that separates the input signal into multiple components, each one carrying a single frequency sub-band of the original signal.

FBs are divided in two main groups: analysis FBs and synthesis FBs. The former ones decompose the signal into a number of low-rate signal components and usually consist of a set of filters and a set of down-samplers. The latter ones construct these components back to a single high-rate signal. Synthesis FBs include a set of up-samplers and low-pass, band-pass and high-pass filters. There are many different approaches to the creation of FBs. However, in practical applications the simplicity of realization should be among the foremost concerns. For example, Modulated Filter Banks (MFBs) have easy-derived, efficient realization of sub-band filters [33], and for that reason they became very popular. They can be constructed from single initializing filter using cosine, sine, or exponential modulation [35, 36, 37]. Moreover, exponentially modulated FBs can be synthesized as a combination of cosine and sine MFBs, and thus effective processing algorithms on base of lapped transform can be used [38].

MC modulation is an efficient transmission technique and closely related to the FB theory [39]. The most reputed MC technology is, no doubt, OFDM, where a transmitter should efficiently combine low-rate input signals into a single high-rate signal, which is then transmitted over a channel, enabling a receiver to reconstruct these low-rate signals. Thus OFDM can be considered as a particular case of the Filter Bank Multiple Carrier (FBMC) scheme with rectangular filter shapes. This type of filter does not possess optimal localization in the frequency domain. Therefore FBs provide an alternative way to perform time-frequency signal transforms, providing more frequency selectivity then Discrete Fourier Transform (DFT)/Inverse Discrete Fourier Transform (IDFT). More advanced MC techniques, such as discrete wavelet multitone [40], Filtered MultiTone (FMT) [41], Cosine modulated MultiTone (CMT) [42], the OFDM with offset QAM (OFDM/OQAM) -based technique [43], the modified DFT based technique [36], and the Exponentially Modulated FB (EMFB)-based technique [37], can achieve a much better frequency selectivity.

Improved frequency selectivity can be accomplished by using longer and spectrally well-shaped initializing filters. Because of that, the side lobes of the filters are considerably lower then in the case of OFDM, and a good spectral localization for all the sub-channels can be obtained. This also results in a good resistance against narrowband interference. In practice, any sub-channel overlaps significantly only with its neighboring sub-channels. It is necessary to mention here that the corresponding symbols are highly overlapping in time, which is not true for OFDM. Due to this overlapping, a CP approach need not and even cannot be utilized.

1.3 Computer Simulations

There are several approaches to study wireless mobile networks, from direct measurements on a real network to analytical, mathematical derivations. For the existing networks, the measurements of key indicators and their optimization are performed with the help of traffic analyzers and drive tests. However, when talking about new technologies, it is necessary to take into account that a full-featured network might not even exist. Moreover, the number of devices available on the market can be limited. Until recently, the main method used prior to the implementation of new networks was test zones. Using them, it is possible to test the ease of commissioning, settings and particularities of the equipment in simple scenarios, which include several MTs and one of few Base Stations (BSs). For 2G and even 3G networks with circuit-switched services this approach was reasonable. Nevertheless, in the new generations of mobile networks based on packet switched architecture, BSs have many more functionalities [44]. It is almost impossible to consider radio part of a network separately from its core. Creation of a test zone that could represent a significant cluster of the network is a difficult and very expensive task. On the other hand, purely analytical considerations are also unacceptable for the overall study of such complicated dynamic system because many simplifications should be introduced. Therefore, it can be concluded that the only way to get detailed characteristics of future generation networks is by system level simulations [45, 46]. At the same time, one should not underestimate the role of the theoretical derivations of new features and technologies during the research stage.

System level simulators are used for the modeling of quite big network scenarios. For example, the 3rd Generation Partnership Project (3GPP) macrocell scenario [47] consists of 19 sites (3 sectors each) and, on average, of 10 User Equipments (UEs) per sector. Currently the most popular architecture is discrete-event architecture. In this kind of system, event is one of the most important entities. An event occurs in a defined point of the simulator's time line and causes a change in the system. Examples of such events include the packet transmission, change in the UE's position, Signal to Interference and Noise Ratio (SINR), etc.

System level simulators are usually static, quasi-static or dynamic. Static simulators are usually used to study network planning and interference situation. However, such time-dependent phenomena as fast fading cannot be taken into account. In quasi-static simulators, these effects are included in the model, but UEs still have a constant position during every drop even though the velocity can be assigned to them. Fully dynamic simulators are the most complicated and resource-consuming but enable realistic network behavior simulation: as MTs move, handovers are possible and radio resource management can be performed in real time.

Nowadays, big telecommunication equipment manufacturers usually use their own privately developed simulation tools. However, there are also some quite functional system level simulators with open source code. Among these, it is necessary to mention Network Simulator v.2 (ns-2) [48], which is still quite popular mainly because of the variety of already existing models for it. In Network Simulator v.3 (ns-3) [49], which appeared in 2008, many problems of ns-2 (low performance, lack of support for heterogeneous devices, complicated logging, etc.) had been taken into account and solved. In scientific community, the OmNet++ [50] simulator is widely spread. In addition to these, there are several open source simulators focused mainly on one technology, for example LTE-Sim [51] and OpenWNS [52] for LTE. Taking into account such factors as performance, functionality, architecture, development activity and licensing, the ns-3 simulation environment looks as the most promising.

In order to reduce the computational complexity of system level simulations the PHY is not usually precisely modeled in them. The scale of one or few time slots (1 slot = 1560 chips = 0.6667 ms) is considered to be enough, but the collection of necessary statistics over one scenario can even under this resolution take several days. It is obvious that the use of a higher time resolution will considerably increase the requirements for computational resources. To overcome this problem, one can divide the simulation process in two phases: link level and system level simulation. In link level simulations, bit-resolution is used. Such features as encoding and decoding, MIMO gains, adaptive modulation and coding feedback, realistic channel effects, etc. are precisely modeled. The main result of link level simulations are mapping tables or curves that link SINR and BER or BLER, depending on user modulation, channel model and velocity. Afterwards, when SINR is calculated in a system level simulator, the values are compared to the link level results to get the probability of successful frame transmission.

One of the main goals of this thesis is to provide the initial theoretical and algorithmic input that can be used in link level simulators, which in turn produce the necessary data for the evaluation of the technology on the system level. A more detailed formulation of these kind of problems is presented in the next section.

1.4 Problem Statement

Despite its popularity and efficacy, OFDM suffers from several major drawbacks:

– Sensitivity to the interference in the frequency domain.
– Reduced power and spectral efficiency because of the utilization of CP.
– High Peak-to-Average Power Ratio (PAPR).

This thesis concentrates on the solution of the two problems mentioned above. The main goal is to develop, evaluate and demonstrate an enhanced PHY MC technique and signal processing methods and algorithms that fuction efficiently with it. The main emphasis of the study is on optimally localized MC waveforms that effectively use the available spectrum. Aiming at minimal modifications to the currently existing standards, enhanced OFDM solutions are considered as an alternative. The filter bank theory proposes an promising approach in this direction. This thesis focuses on four main areas:

1. The first problem area is the structure and properties of enhanced signal basis. In engineering, Gabor's idea flourished over the last decade due to the increasing use of OFDM structured communication systems. The main reason for the popularity of OFDM in modern telecommunication systems is that it allows the achievement of high data rates without complicated signal equalization and processing together with the mitigated ISI. However, ICI together with insufficient signal localization in the frequency domain is still an issue. Is it possible to construct a basis with such initializing function, that is still quite well localized in the time domain, so that the energy of its Fourier Transform (FT) would be concentrated in the limited frequency interval? This problem is discussed in Chapter 2.
2. After the structure of the basis is defined, the next step is the construction of its initialization function with the required localization and orthogonality properties. For example, the Gaussian function possessing ideal localization is continuous and defined on the whole real axis. However, in digital signal processing one needs to work with finite discrete signals. Thus it is necessary to propose a theoretically validated approach for the construction of such function, with minimal losses in localization quality. These issues are studied in Chapter 3 and 4.
3. The third problem area is the efficiency of the orthogonalization procedure. It is possible to perform it with the use of Singular Value Decomposition (SVD), but the operation is computationally inefficient. Is it possible to optimize the construction of the basis' initialization function? This question is closely related with the orthogonality conditions which should guarantee the absence of ICI and ISI of the signal transmitted over ideal channel. These problems are introduced and studied in Chapter 5 and in [PII].
4. The fourth and final problem area is signal processing. Because of the more complicated signal structure, it is impossible to use directly the FFT algorithm as in OFDM. For the direct procedure of matrix multiplication, which

can be used for modulation, the number of operations grows like the square function of basis length. Thus this direct approach cannot be used in practical systems and even makes simulations too long. A "fast" signal modulation algorithm is proposed and derived in [PVI].

1.5 Outline of the Dissertation

The rest of the dissertation is organized as follows.

Chapter 2 installs theoretical connection between the methods of time-frequency analysis and those of MC telecommunication systems, in particular OFDM. It is shown that OFDM signal modulation is a special case of Windowed Fourier Transform (WFT) with a rectangular window. The chapter explains why the localization of the classical OFDM signal basis cannot be enhanced. After the introduction of the main features of the OFDM architecture, the Modified Weyl-Heisenberg (MWH) basis is introduced. Based on its construction, the developed technique can be called Orthogonal Frequency-Time Division Multiplexing (OFTDM): in contrast to OFDM, its basis functions can overlap in the time domain. Even in this case, the OFTDM signal can be demodulated due to the orthogonality of the basis.

Chapter 3 describes the algebraic problem of the search of optimal matrix for the MWH basis. Using the Lagrangian principle the problem of approximation of any symmetrical complex function with discrete N-periodical function from $\mathbb{C}^N$ space is solved. In particular, this allows the construction of an N-periodical approximation of the Gaussian function with ideal time-frequency localization. This is then used for the synthesis of the orthogonal basis. A generic algebraic algorithm of orthogonal basis synthesis close in terms of Frobenius norm to the non-orthogonal Gabor basis constructed from the Gaussian function is briefly considered. The disadvantage of this algorithm is computational inefficiency due to its utilization of SVD.

Chapter 4 is devoted to the optimization of several MWH basis parameters. Firstly, it is shown that the norm between any well-localized MWH basis and constructed orthogonal MWH basis can be additionally reduced by an optimal choice of the phase parameter. For the two types of symmetry of generating functions the optimal values of α are found exactly. The second parameter that can also be optimized is the choice of sampling frequency for the original function with target time-frequency localization. For the Gaussian function, the optimal value is computed. The chapter concludes with the results of computer simulations, which include the comparison of OFDM and OFTDM and the values of the Heisenberg parameter.

Chapter 5 presents the proof of the orthogonality criteria of the MWH basis. These criteria have their own theoretical interest and are also necessary for the derivation of a more efficient basis construction algorithm. Within the derivations, it was necessary to introduce an additional family of functions, which is orthogonal in the terms of regular scalar product together with the orthogonality of the corresponding MWH basis. The connection between the proved orthogonality conditions and Nyquist's criteria and theorem for per-symbol transmission model is shown. Their analogs for the OFTDM signals of a more complicated form are formulated and proved. These results form the necessary basis for the derivation of a "fast" synthesis algorithm. An anitializing function, which is completely identical to the initializing function constructed using the algorithm from Chapter 2, is produced.

Finally **Chapter 6** concludes the thesis. Also the applicability of the derived algorithms in practice and in the simulation process are discussed.

The **appendices** to the thesis contain Matlab realizations of the algorithms developed.

1.6 Main Contribution

A complete mathematical framework for the utilization of bases with enhanced localization in MC telecommunication systems has been developed. The following objectives were set and achieved:

– Study and development of synthesis methods for finite-dimensional signal bases possessing optimal localization in the time and frequency domains.
– Theoretical explanation and optimization of the basis' parameters and pulse-shaping function form.
– Development of a practically implementable and computationally efficient algorithm of signal processing.
– Implementation and verification of derived algorithms for further utilization in link-level simulations.

In the dissertation, the choice of the symmetry type of the initializing function and the phase parameter guaranteeing the optimal time-frequency localization has been justified for the first time ever. For OFTDM signals, the orthogonality criteria ensuring the minimization of ISI and ICI, analogical to the Frequency Division Multiplexing (FDM) Nyquist theorem, has been proved. The advantages of the OFTDM technology over the classical OFDM scheme are explained. This allows the utilization of the received results in future wireless network devices and in computer simulations.

During the work on the subject of the dissertation, the author has produced several publications. The author of this thesis was the first author of all the publications presented and has the main responsibility for the results and themain theoretical derivations. In addition, the author was the main contributor for the analysis and article writing.

Publication [PI] considers the algebraic procedure of signal basis construction originated from SVD. Optimal time-frequency localization together with orthogonality are achieved because of the closeness of the pulse-shaping function to the Gaussian function. The problem of minimization of the Frobenius norm, with its implied orthogonality condition, can be solved analytically. This gives the exact values of the basis' generation function. An important result of the paper is the optimal values of the basis' phase parameter, which are derived for the two types of symmetry of the pulse-shaping function. The modeling results confirm good localization characteristics. However, the synthesis algorithm considered becomes computationally inefficient for high dimensions of the basis, for example, when thousands of sub-carriers are used in DVB.

Publication [PII] addresses the above problem. The sequence of proved orthogonality conditions allows the formulation of a computationally effective algorithm of orthogonalization based on FFT. Moreover, it is shown how the Gaussian function can be converted to a discrete N-periodical function with the same localization properties. The tests performed confirm the identity of the initializing functions obtained as a result of the singular decomposition from [PI] and the fast algorithm.

In publication [PIV], the practical and commercial benefits of OFTDM-based systems are discussed. This technology utilizes the mathematical framework of the developed orthogonal bases with optimal time-frequency localization. The level of interference between sub-carrier channels in the frequency domain is minimized. Orthogonality in the time domain, even when consecutive symbols are overlapping, allows the reduction of the number of utilized GIs or CP. As a result, higher spectral and energy efficiencies for telecommunication systems together with a better robustness against complex realistic channel conditions are achieved.

Publication [PV] presents an additional application of the developed bases in Ultra Wide Band (UWB) communications. One of the UWB modulation types is multi-band-based and created by using multi-carrier. With the OFDM type of modulation, systems can effectively deal with the delay spread or frequency selectivity of UWB channels. In the media, with strong time-frequency dispersion especially, well-localized bases ensure the best signal reconstruction. The paper also presents the optimal sampling frequency which should be selected in the process of initializing function creation.

Publications [PIII] and [PVI] are devoted to the most important issue in the practical application of OFTDM: that is, the signal modulation algorithm. To be competitive with OFDM, the number of its operations required for signal construction should be comparable to that of the FFT algorithms. In [PVI], the whole algorithm derivation process is presented in detail. Firstly, a finite analogue of polyphase decomposition is introduced. After that it is used to reduce the modulation algorithm to the combination of several FFTs and multiplications of sparse matrices. Moreover, the algorithm is also formulated in a matrix form.

In addition to the included articles, the author of this thesis has published a number of conference and journal articles dealing with modern mobile networks. Of these articles, the following were published in English:

– D. Petrov, I. Repo and M. Lampinen. Overview of Single Frequency Multipoint Transmission Concepts for HSDPA and Performance Evaluation of Intra-site Multiflow Aggregation Scheme. *The 75th IEEE Vehicular Technology Conference (VTC)*, 2012.

– D. Petrov, I. Repo and M. Lampinen. Performance of Multiflow Aggregation Scheme for HSDPA with Joint Intra-Site Scheduling and in Presence of CQI Imperfections. *The 12th International Conference on Next Generation Wired/Wireless Networking (NEW2AN)*, 2012.

2 TIME-FREQUENCY ANALYSIS FOR MULTI-CARRIER SYSTEMS

This chapter overviews several main techniques of time-frequency analysis, especially WFT [6] while demostrating the solid theoretical connection between these approaches and OFDM. Furthermore, the methods considered and their limitations pave the way towards mathematical formulation of OFTDM technology.

2.1 Time-Frequency Analysis

Presentation of a function as a linear combination of functions from some predefined set is a popular approach used in different areas of science [6, 53, 54, 55, 56]. In many circumstances, this approach allows simplification of solutions for a large class of problems. FT is one of the most widely-spread instruments of signal processing in linear time-invariant systems. Additionally, there is a constantly growing number of different bases and transforms which can be used in more complicated scenarios when signals are changing rapidly in time.

A family of functions possessing a specially defined structure is used in many signal processing technologies. Such a family can be constructed by utilizing one or several prototype functions (window, mother wavelet, impulse, "atom", etc.)[57, 58]. Then unitary transform groups (shift in time, hift in frequency, spreading, etc.) are applied to them. The group structure is a fundamental requirement because it allows computationally efficient realizations of these transforms. The most popular of these groups are the affine group, which results in wavelet transform, and the Weyl-Heisenberg (WH) group, which results in Short Time Fourier Transform (STFT) or WFT [59, 60]. For that reason, bases constructed by uniform shift in time and frequency are called WH bases.

Historically, the classical signal processing theory paid the most attention to the study of systems and corresponding operators invariant in time (or space). The action of these operators results in the change of signals' stationary parameters. Theoretically, optimal signal processing is usually associated with the solution of the eigenvalue problem, i.e. with the search of eigenvalues and eigenvectors of a liner operators. Furthermore, sinusoidal waves $e^{i\omega t}$ represent the eigenvectors of linear, time-invariant systems L, which can be entirely defined by its eigenvalues $\lambda(\omega)$:

$$For\ \forall\omega \in \mathbb{R}, Le^{\jmath\omega t} = \lambda(\omega)e^{\jmath\omega t}.$$

For the sake of convenience signal $s(t)$ is presented as the combination of eigenvectors $\left\{e^{\jmath\omega t}\right\}_{\omega\in\mathbb{R}}$ with coefficients $S(\omega)$:

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{\jmath\omega t}.$$

If signal $s(t)$ has finite energy

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt,$$

then from the properties of the Fourier integral it follows that amplitude $S(\omega)$ of every sinusoidal wave $e^{\jmath\omega t}$ is the FT of $s(t)$:

$$S(\omega) = \int_{-\infty}^{\infty} s(t)e^{\jmath\omega t}dt.$$

In such a case the impact of the operator L results in the amplification or depletion of the corresponding components $e^{\jmath\omega t}$ of the signal $s(t)$ by the value $\lambda(\omega)$:

$$Ls(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\lambda(\omega)e^{\jmath\omega t}d\omega.$$

Thus until we are working with time-invariant operators, for example with the problems of stationary signal processing, FT remains the most simple and convenient tool. The long predominance of FT in the communication theory, which has left aside other approaches in the area of signal processing, is most probably due to this.

However, the world of short-time, transient processes encompasses more than just the stationary signals. FT becomes a very complicated tool for practical applications when phenomena is localized in time or space. Indeed, Fourier coefficients, calculated as the correlation of the signal with sinusoidal waves $e^{\jmath\omega t}$, which are non-zero on the whole real axis $\mathbb{R}$, are determined by the values of $s(t)$ for all $t \in \mathbb{R}$. Thus $S(\omega)$ contains all information about the signal changing in time. It makes it practically very difficult to get local characteristics of $s(t)$ from $S(\omega)$. Nevertheless this goal can be achieved if the signal is decomposed in wave components, which are well localized in the time and in frequency domains.

In 1946, the physicist Dennis Gabor [3] working on the problems of quantum mechanics introduced, for the first time, elementary time-frequency "atoms" as functions with minimal dispersion on the time-frequency plane. To get time-frequency information about the signals, he proposed to decompose them over these elementary "atoms".

Gabor's "atoms" are constructed by translation in time and frequency of the symmetrical function $g(t) = g(-t)$:

$$g_{\mu,\xi}(t) \triangleq g(t-\mu)e^{\jmath\xi t}.$$

The energy of the functions $g_{\mu,\xi}(t)$ is concentrated in the neighborhood of μ on interval σ_t, which is the standard deviation of $|g(t)|^2$.

It can be easily seen that FT $G(\omega)$ of the function $g(t)$ is shifted for ξ in the frequency domain:

$$G_{\mu,\xi}(t) = G(\omega)e^{\jmath t(\omega-\xi)}.$$

Thus the energy of $g_{\mu,\xi}(t)$ in the frequency domain is concentrated in the interval σ_ω around ξ.

WFT introduced by Gabor in practice defines the degree of correlation of the signal $s(t)$ with every "atom" $g_{\mu,\xi}(t)$:

$$Ws(\mu,\xi) = \int_{-\infty}^{\infty} s(t)g^*_{\mu,\xi}(t)dt = \int_{-\infty}^{\infty} s(t)g(t-\mu)e^{-\jmath\xi t}dt. \tag{2.1}$$

This expression presents a Fourier integral, which is localized around u, with the help of the window $g(t-u)$. Using the Parseval's theorem [61] the integral from 2.1 can be rewritten in the frequency domain:

$$Ws(\mu,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)G^*_{\mu,\xi}(\omega)d\omega.$$

Therefore the transform $Ws(\mu,\xi)$ depends only on $s(t)$ and $S(\omega)$ in the area of the time-frequency plane, where the energies of $g_{\mu,\xi}$ and $G_{\mu,\xi}$ are concentrated. Gabor himself interpreted this fact as "quanta of information" included in the square $\sigma_t \times \sigma_\omega$ of the time-frequency plane.

In general case, one can consider the family of time-frequency "atoms" $\{\phi_\gamma\}_{\gamma\in\Gamma}$, where γ is a multi-index. Assume that $\phi_\gamma \in L^2(\mathbb{R})$ and $\|\phi_\gamma\| = 1$, where $L^2(\mathbb{R})$ is a Hilbert space of square integrable functions with the inner product:

$$\langle f,g\rangle = \int_{-\infty}^{\infty} f(t)g^*(t)dt.$$

The corresponding linear time-frequency transform of the function $f(t) \in L^2(\mathbb{R})$ can be introduced in the following way:

$$Tf(\gamma) = \int_{-\infty}^{\infty} f(t)\phi^*_\gamma(t)dt = \langle f,\phi_\gamma\rangle.$$

Using again Parseval's theorem, one finds that

$$Tf(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\Phi^*_\gamma(\omega)d\omega.$$

From the last two formulas, it follows that the values $Tf(\gamma) = \langle f,\phi_\gamma\rangle$ characterize the function $f(t)$ and its FT $F(\omega)$ in some area of the time-frequency plane defined by the localization of "atoms" ϕ_γ.

Notice, that the "atoms" of WFT correspond to the case, when

$$\phi_\gamma(t) = g_{\mu,\xi}(t) = g(t-\mu)e^{j\xi t}. \tag{2.2}$$

Another important example is that of wavelet "atoms", which are constructed by the scaling and translation of the mother wavelet $\psi(t)$:

$$\phi_\gamma(t) = \psi_{s,u}(t) \triangleq \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right). \tag{2.3}$$

Wavelets 2.3 and window functions 2.2 have the energy localized in time, whereas their FTs are also localised in the limited frequency band. When the parameters (μ, ξ) are changed along the whole real axis $\mathbb{R}$, "atoms" $g_{\mu,\xi}(t)$ cover the whole time-frequency plane. Every function $s(t)$ from $L^2(\mathbb{R})$ can be reconstructed from the known WFT $Ws(\mu, \xi)$:

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ws(\mu,\xi)g(t-u)e^{j\xi t}d\mu d\xi,$$

if the energy is conserved:

$$\int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Ws(\mu,\xi)|d\mu d\xi.$$

It is necessary to notice that because of the redundancy of this transform not every function from $L^2(\mathbb{R}^2)$ will be WFT for some function from $L^2(\mathbb{R})$.

In practice, especially in digital signal processing, it is a usual situation when the values of the function $s(t) \in$ are known only in a number of samples received as a result of discretization with period T. Moreover, signals usually have a finite length. For that reason, we index the components of the vector $\mathbf{s}$ by $J_N \triangleq \{0, 1, \ldots, N-1\}$ and additionally use the periodization over the modulo N, which gives the finite-dimensional space $\mathbb{C}^N$ [62, 63]. The elements of this space are discrete finite signals

$$\mathbf{s} = s[n] = \{s(0), s(1), \ldots, s(N-1)\}^T,$$

where $\mathbf{s}^T$ denotes the transpose of vector $\mathbf{s}$. To emphasize the periodical nature of the space ($s[n+N] = s[N]$) we will use the notation

$$s[(n+k)_N] \triangleq s[(n+k) \mod N]$$

when it will be needed in the further text.

The inner (or scalar, or dot) product in $\mathbb{C}^N$ can be introduced in the following way:

$$\langle f, g\rangle = \sum_{n=0}^{N-1} f[n]g^*[n], \tag{2.4}$$

where $f, g \in \mathbb{C}^N$. The norm in $\mathbb{C}^N$ is induced by this scalar product:

$$\|s[n]\| \triangleq \sqrt{\langle s, s\rangle}.$$

The DFT $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ plays fundamental role in time-frequency analysis. It is given pointwise by

$$\mathcal{F}s[p] \triangleq S[p] = \sum_{n=0}^{N-1} s[n]e^{\frac{-2\pi jnp}{N}}, \quad p \in J_N. \tag{2.5}$$

WFT or STFT [64, 6] in this case can be constructed using discrete "atoms"

$$g_{m,l}[n] \triangleq g[(n-m)_N]e^{\frac{j2\pi ln}{N}}, \tag{2.6}$$

where $\|g[n]\| = 1$. The DFT of the functions from this WH family will look like

$$G_{m,l}[p] = G[(p-l)_N]\frac{-j2\pi(p-l)m}{N},$$

, where $G[n]$ is the DFT (2.5) of the initialization function $g[n]$. Thus the discrete WFT of the signal $s[n]$ is defined as

$$\mathcal{W}s[m,l] \triangleq \langle s, g_{m,l}\rangle = \sum_{n=0}^{N-1} s[n]g[(n-m)_N]e^{\frac{-j2\pi ln}{N}}. \tag{2.7}$$

Signal reconstruction can be made as follows:

$$s[n] = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{l=0}^{N-1} \mathcal{W}s[m,l]g[(n-l)_N]e^{\frac{j2\pi ln}{N}}, \tag{2.8}$$

if the energy is conserved:

$$\sum_{n=0}^{N-1} |s[n]|^2 = \frac{1}{N}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1} |\mathcal{W}s[m,l]|^2.$$

It is necessary to notice, that the transform $\mathcal{W}s[m,l]$ is the N^2 image of the one-dimensional signal $s[n]$ and for that reason it is redundant. The answer to the question of the existence of an inverse transform is given in the frame theory [65].

At this point, it is important to mention that the family of functions described by (2.6) and created by the discrete translations in time and frequency of $g[n]$ can be regarded by construction as the classical WH basis. Its applications in the telecommunication theory will be discussed in the next section.

2.2 Orthogonal Frequency Division Multiplexing

MC modulation was used for the first time in the 1950's [66]. At that time, it did not get popularity because of its complexity for analog devices. In 1966, R. Chang patented the OFDM structure and published the concept of orthogonal signals utilization for telecommunication systems [67]. Several year later, in 1971,

S. B. Weinstein and P. M. Ebert overcame the main problem in the application and published their pioneering work, where they describe the multichannel systems with DFT /IDFT [68]. At the end of the 80's, the work on the first commercial OFDM systems, Digital Audio Broadcasting (DAB) [69], was underway. Later on, the principles of multichannel transmission formed the basis for many main industrial standards.

OFDM is a particular case of information transmission technology with FDM, also called MC modulation. In FDM systems, a high-bit-rate stream is divided into a number of streams with lower bit rate, which are then transmitted over parallel frequency channels (sub-carriers). Assuming M is the number of such sub-carriers, then the bit rate on each of them decreases as a function of M. If the value of M is big enough, each sub-carrier occupies the bandwidth less than the coherence bandwidth of the channel and uniform attenuation is observed. For that reason, at the receiver a simple equalization scheme can be used, which compensates the attenuation and phase shifts caused by a non-ideal channel.

Currently, OFDM technology is widely used in many wired and wireless communication standards, including Very high rate Digital Subscriber Line (VDSL), Power Digital Subscriber Line (PDSL), DAB, Wireless Local Area Network (WLAN) (IEEE 802.11 family of standards) [70], WiMAX (IEEE 802.16 family) [71], DVB [72]. Moreover, many standards which are still under development, among them LTE [73], Digital Video Broadcasting, Second Generation Terrestrial (DVB-T2)[21] are also based on OFDM.

2.2.1 Mathematical Model

In this subsection, the mathematical model describing OFDM signal and its processing will be considered. The conceptual scheme of OFDM transmitter and receiver is presented on the Figure 1.

Usually the transmitter includes a number of blocks, like coding, interleaving, Serial/Parallel transformer (S/P), etc., but for this study it is enough to concentrate on OFDM modulation.

At the input of the transmitter comes the binary information flow with bit rate $R_b = \frac{1}{T_b}$. If the symbol period of the OFDM system equals T, then at the exit of S/P in the q-th symbol period we will get vector of $B = R_b T$ bits:

$$\mathbf{d}_q = \{d_{0,q}, d_{1,q}, \ldots, d_{B-1,q}\}^T,$$

where $s_{i,q} = d(i + qB)$ is the i-th bit transmitted in the q-th symbol period.

The structure and properties of the OFDM signal are determined by the corresponding finite-dimensional WH basis. The signal is constructed as a linear combination of basis' functions with the coefficients, which contain transmitted information. These coefficients can be real or complex. That is determined by the selected signal constellation (for example, QAM or PSK). As a result of modulation, bit vector $\mathbf{d}_q$ is identically associated with the vector of information symbols of length M:

$$\mathbf{a}_q = \{a_{0,q}, a_{1,q}, \ldots a_{M-1,q}\}^T.$$

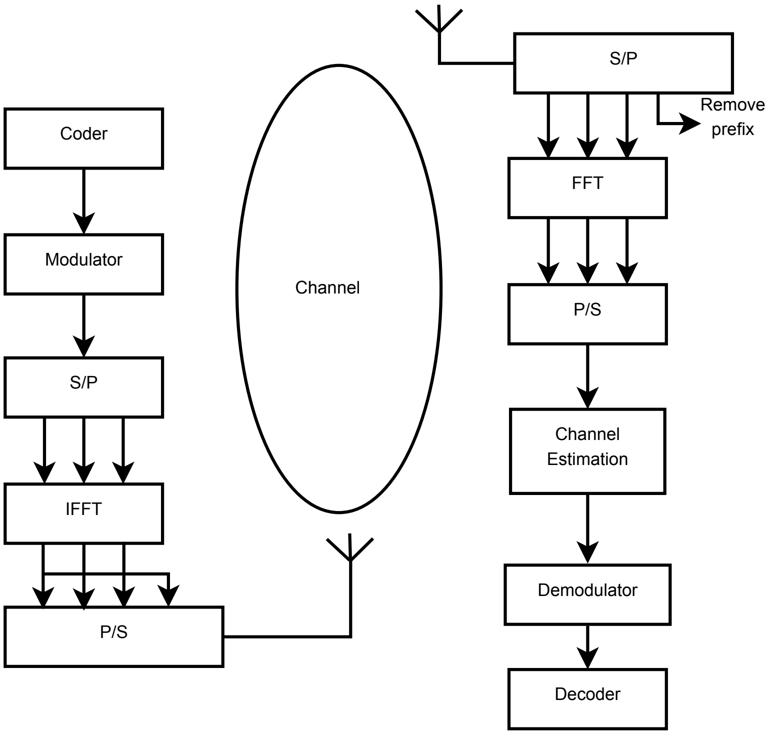


FIGURE 1 Structural scheme of OFDM-base communication.

Thus $a_{m,q}$ are used as the coefficients (modulating symbols) in the linear combination of sub-carriers $g_{m,q}(t)$. The signal transmitted in the q-th symbol period $[qT, (q+1)T]$ is constructed as follows:

$$s_q(t) = \sum_{m=0}^{M-1} a_{m,q} g_{m,q}(t).$$

Furthermore, the OFDM signal transmitted in consecutive periods can be written as

$$s(t) = \sum_{q=-\infty}^{\infty} \sum_{m=0}^{M-1} a_{m,q} g_{m,q}(t).$$

Every information symbol $a_{m,q}$ is transmitted by sub-carrier m in symbol interval q.

Therefore every function $g_{m,q}(t)$ represents the element of synthesis basis with indexes $q\, in\mathbb{Z}$, $m \in J_M$. It is received with the help of time-frequency transform of function $g(t)$:

$$g_{m,q}(t) \triangleq e^{j2\pi Fmt} g(t - qT), \tag{2.9}$$

where F is the distance between subcarriers in the frequency domain (frequency step). For the reason which will be shown a bit later, it is set equal to $\frac{1}{MT}$. Thus the basis of functions $g_{m,q}(t)$ is the sequence of impulses, located in time intervals of length T and shifted in frequency for the bands divisible by F. The density of the time-frequency lattice of such OFDM system equals $\frac{1}{TF}$.

In classical OFDM systems, prototype function is selected as a rectangular impulse

$$g(t) = \begin{cases} \dfrac{1}{\sqrt{T}} & \text{for } 0 \leq t \leq T \\ 0 & \text{for all other } t. \end{cases} \tag{2.10}$$

The scalar product of any two functions of the signal basis with $g(t)$ from (2.10) equals

$$\begin{aligned} \langle g_{q,m}, g_{q',m'} \rangle &= \int_{-\infty}^{\infty} g_{q,m}(t) g^*_{q',m'}(t) dt = \int_{-\infty}^{\infty} e^{\jmath 2\pi (m-m')Ft} g^*(t-qT) g(t-q'T) dt = \\ &= \frac{1}{T} \int_{qT}^{(q+1)T} e^{\jmath 2\pi (m-m')Ft} dt = \frac{e^{\jmath 2\pi (m-m')T)} - 1}{\jmath 2\pi (m-m')T}. \end{aligned}$$

Only if $TF = 1$

$$\langle g_{q,m}, g_{q',m'} \rangle = \delta_{m,m'} \delta_{q,q'}, \tag{2.11}$$

where $\delta_{m,m'}$ is the Kronecker symbol, it follows that the functions $g_{q,m}$ and $g_{q',m'}$ are orthogonal. This explains the density of the time-frequency lattice mentioned above.

The OFDM signal $r(t)$ at the receiver will look like

$$r(t) = h * s(t) + n(t) = \sum_{q=-\infty}^{+\infty} \sum_{m=0}^{M} h_{m,q} g_{m,q}(t) + n(t),$$

where h is the channel impulse response ($h_{m,q}$ are the characteristic of every sub-channel on every sub-carrier, varying in time), $n(t)$ - additive nose, which is usually modeled as Additive White Gaussian Noise (AWGN), $*$ - symbolizes the operation of convolution.

At the receiver, the signal $r(t)$ passes through M parallel correlator demodulators from the analyszing basis. This basis is identical to the synthesis basis (2.9), however it is not an obligatory requirement. At the l-th exit in time interval $qT \leq t \leq (q+1)T$, it will be the value

$$\begin{aligned} \tilde{a}_q(l) &= \langle g_{l,q}, r \rangle = \sum_{k=-\infty}^{+\infty} \sum_{m=0}^{M-1} h_{m,k} a_{m,k} \langle g_{m,n}, g_{m,k} \rangle + \{g_{l,q}, n\} = \\ &= \sum_{k=-\infty}^{+\infty} \sum_{m=0}^{M-1} h_{m,k} a_{m,k} \delta_{l,m} \delta_{q,k} + n_q(l) = \sum_{m=0}^{M-1} h_{m,k} a_{m,k} \delta_{l,q} + n_q(l) = \\ &= h_{l,q} a_{l,q} + n_q(l). \end{aligned} \tag{2.12}$$

In the detector, this signal is multiplied by the channel coefficients $\frac{1}{h_{l,n}}$. Thus transmitted symbols are recovered after the demodulation with the presence of noise, which can be filtered from the signal.

The spectral efficiency of the OFDM system is

$$\eta = \frac{\beta}{TF} = \frac{\log_2 N}{TF} = log_2 N[\text{bit/s/Hz}], \tag{2.13}$$

where $\beta = log_2 N$ is the amount of bits per QAM symbol.

2.2.2 Spectrum of OFDM Signal

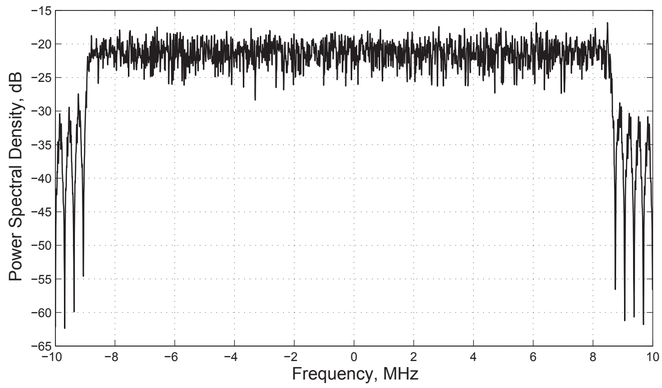


FIGURE 2 PSD of OFDM signal.

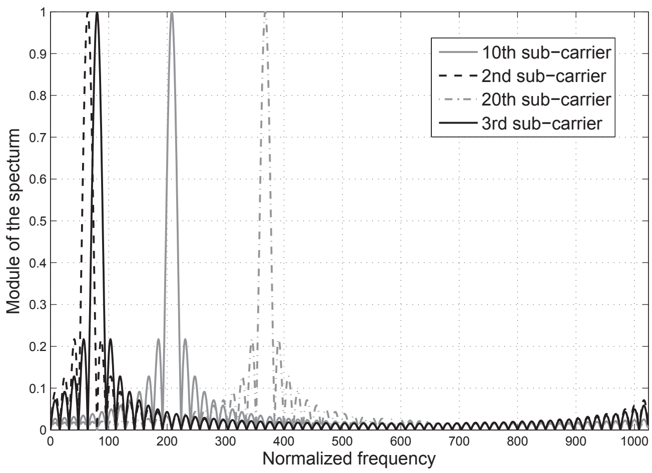


FIGURE 3 Spectrum of OFDM sub-carriers.

Let the bits $d_{q,i}$, $i \in J_M$, $q \in \mathbb{Z}$ be independent equally distributed random variables, thus:

– Complex modulation symbols $a_{q,i}$ from the same q-th OFDM symbol are equally distributed random variables;

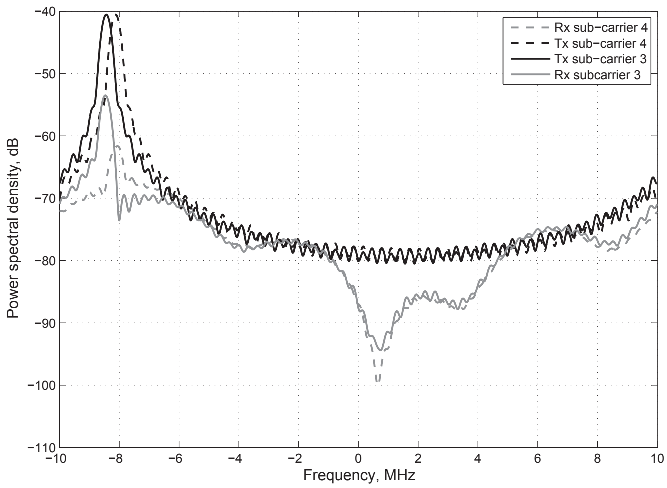


FIGURE 4 Example of the ICI in OFDM signal.

- modulation symbols from the different OFDM symbols are also equally distributed random variables.

Assuming additionally that symbols $a_{q,i}$ has zero mean and dispersion $E[|a_{q,i}|^2] = \sigma_i^2$, it can be shown [74] that Power Spectral Density (PSD) of the OFDM signal is

$$P_s(f) = \sum_{i=0}^{M-1} \left(\sigma_i \operatorname{sinc} \left(\left(f - \frac{i}{T} \right) T \right) \right)^2 .$$

Therefore it is a combination of sinc functions in the frequency domain. The total bandwidth occupied by the signal is

$$W_{\mathrm{OFDM}} = \frac{2}{T} + \frac{M-1}{T} \approx \frac{M}{T}.$$

Figure 2 shows the PSD of OFDM signal received from Matlab simulations. Several OFDM frames have been used to construct it. Figure 3 demonstrates that OFDM sub-carriers have an overlapping spectrum, which is in accordance with the idea of frequency multiplexing. However, the sub-carriers or corresponding sub-channel filters are not spectrally well isolated. They consist of a main lobe overlapping with immediately adjacent sub-channels and high side lobes that extend over a wide frequency band. The first side lobe of a sub-channel is only 13 dB below the main lobe. These side lobes interfere with the main lobes across the entire band and this can lead to an extensive ICI problem in case of fast fading or carrier frequency offsets. Also, the system performance may degrade significantly in the presence of narrowband interference. The energy of a narrowband interference spreads into many adjacent sub-channels and this cannot be avoided by simply switching-off the sub-channel in which the interference lies. Figure 4 shows the effect of strong frequency fading on the sub-carriers.

2.2.3 Discrete realization

In practice, OFDM is used in digital communication systems where a discrete realization of the approach considered above is utilized. It was shown in the previous subsection that the signal is band-limited and thus can be made discrete with sampling period $T_s = \frac{1}{W_{\text{OFDM}}} = \frac{T}{M}$ (corresponding sampling frequency $\frac{1}{T_s}$) in each time interval $qT \le t \le (q+1)T$:

$$s_q[k] \triangleq s(qT + kT_s) = \sum_{m=0}^{M-1} a_{m,n} e^{\jmath 2\pi m F k T_s} = \sum_{m=0}^{M-1} a_{m,n} e^{\jmath 2\pi \frac{mk}{M}}, \tag{2.14}$$

where $k \in J_M$, $q \in \mathbb{Z}$.

It is easy to notice that discrete signal $s_q[k]$ is in fact the IDFT of the modulating symbols $a_{m,n}$ within one symbol period. Thus, the OFDM modulator can be changed with the block, which makes IFFT.

Analogically, on the receiver side the discretization of the signal $r(t)$ is made with frequency $\frac{1}{T_s}$. From (2.9) it follows that

$$\begin{aligned}\tilde{a}_q(l) = \langle g_{l,q}, r \rangle = \int_{qT}^{(q+1)T} g^*_{q,l}(t) r(t) dt \backsimeq \\ \backsimeq \sum_{m=0}^{M-1} r(qT + mT_s) e^{-\jmath 2\pi \frac{ml}{2}} = \sum_{m=0}^{M-1} r_q[m] e^{-\jmath 2\pi \frac{ml}{M}}.\end{aligned} \tag{2.15}$$

Symbols $\tilde{a}_q(l)$, $q \in \mathbb{Z}$, $l = 0, 1, \ldots, M-1$ also overcome the DFT of discrete signal $r_q[m]$, $m = 0, 1, \ldots, M-1$.

Let's introduce notation $\mathbf{s}_q = \{s_q[0], s_q[1], \ldots, s_q[M-1]\}^T$, $\mathbf{a}_q = \{a_{0,q}, a_{1,q}, \ldots, a_{M-1,q}\}^T$, $\mathbf{r}_q = \{r_q[0], r_q[1], \ldots, r_q[M-1]\}^T$. Using these notations, the modulation and demodulation procedures can be rewritten

$$\mathbf{s}_q = \text{IDFT}(\mathbf{a}_q); \quad \tilde{\mathbf{a}}_q = \text{DFT}(\mathbf{r}_q).$$

This shows that for modulation and demodulation in OFDM systems the algorithms of FFT can be used and makes this scheme of modulation attractive when high data rates are needed.

Moreover, it can be easily noticed that formula (2.15) is in fact a variant of the inverse WFT (2.8) in the case of the rectangular window form (2.10). The modulation of the signal in(2.15) is the analogue of (2.7).

2.2.4 Cyclic Prefix

In high-bit-rate wireless distributed systems, the channel possesses time-frequency dispersion. The signal comes to the receiver by several paths after multiple reflections from unstationary inhomogeneities of the media (city buildings, moving objects, ionosphere layers, etc.). In addition, such effect as amplitude-phase fading, Doppler shift and spreading are observed. These phenomena result in ICI and ISI which can considerably worsen the receiving characteristics. ISI appears mainly as a result of multipath propagation, when copies of the same OFDM propagated by different paths are overlapping in the receiver. This results in the decrease of the system performance because of the higher number of errors. In practice, a quite simple but effective approach is used to cope with this problem. GI or CP are added to the initializing function $g(t)$. Its length T_c should be longer than the dispersion time of the channel T_d. In this case, ISI will be removed almost completely.

After the addition of CP, the forming function of the basis will be

$$p(t) = \begin{cases} \dfrac{1}{\sqrt{T_0}} & \text{for } -T_g \leq t < T \\ 0 & \text{for all other } t, \end{cases}$$

where $T_0 = T_g + T$ is the length of the OFDM frame. According to that, the synthesis basis will look like

$$p_{m,n} = e^{j2\pi mFt} p(t - qT_0).$$

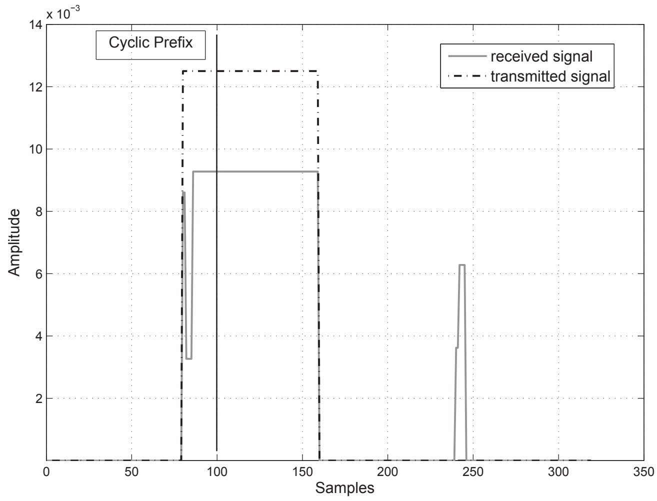


FIGURE 5 Amplitude of the OFDM basis function in time domain after propagation through SUI channel.

On the receiver side, the analyzing basis stays without changes, but with time step T_0 and interval of integration $[qT_0; qT_0 + T]$. The orthogonality condition (2.11) in this case will have the form

$$\begin{aligned}\langle g_{m,q}, p_{m',q'}\rangle &= \int_{-\infty}^{\infty} e^{\jmath 2\pi(m'-m)Ft} g^*(t-qT_0)p(t-q'T_0)dt = \\ &= \frac{1}{T}\int_{qT_0}^{qT_0+T} e^{\jmath 2\pi(m'-m)Ft} p(t-q'T_0)dt = \\ &= \begin{cases} \sqrt{\dfrac{T}{T_0}} & \text{, for } m=m' \text{ and } q=q' \\ 0 & \text{, for all other } m \text{ and } q. \end{cases}.\end{aligned}$$

Setting the length of the GI $T_g = \tau T_s$, $\tau \in \mathbb{N}$ and sampling $p(t)$ with previously used frequency $\frac{1}{T_s}$ inside the interval $[qT_0 - T_g; qT_0 + T_0]$ we get

$$\tilde{s}_q[k] \triangleq s(qT_0 + kT_s) = \sum_{m=0}^{M-1} a_{m,q} e^{\jmath 2\pi \frac{mk}{N}},$$

where $k = -\tau, -\tau+1, \ldots, M-1$, $q \in \mathbb{Z}$.

This expression can be rewritten in a vector from

$$\begin{aligned}\tilde{\mathbf{s}}_q &= \left[s_q[-\tau], s_q[-\tau+1], \ldots, s_q[-1], s_q[0], \ldots, s_q[N-1]\right]^T = \\ &= \left[\underbrace{s_q[N-\tau], s_q[N-\tau+1], \ldots, s_q[N-1]}_{\text{last } \tau \text{ elements of } \mathbf{s}_q}, \underbrace{s_q[0], \ldots, s_q[N-1]}_{\mathbf{s}_q}\right]^T,\end{aligned}$$

where the last equality follows from the periodicity of DFT and the first τ elements are referenced to the CP. Therefore, the addition of GI to the $g(t)$ is equivalent to the addition of CP to the OFDM symbol after modulation.

Figure 5 demonstrates the effect of multipath propagation on the OFDM symbol. The Stanford University Interim (SUI) channel model was used to produce this result [75]. On the receiver side the first τ elements, which contain ISI, are simply removed.

Because of the utilization of the CP, the spectral efficiency of the system (2.13) reduces

$$\eta_{\text{CP}} = \frac{\beta}{TF} = \frac{\log_2 M}{(T+T_g)F} = \left(1 - \frac{T_g}{T_0}\right)\log_2 M[\text{bit/s/Hz}]$$

in $\frac{T_g}{T_0}$ times. In practice, the size of the CP $\frac{T_g}{T_0} \leq \frac{1}{4}$ and thus the loss of spectral efficiency is up to 25%.

After the analysis of the structure of OFDM signals presented in the section above, it is logical to deal with the problems of localization enhancement of signal basis and of the elimination of the CP. Some of the limitations to this will be discussed in the next section.

2.3 Theoretical localization Constraints

Is it possible to construct such a function f, which is well localized in the time domain so that the energy of its FT F would be concentrated in the limited frequency interval? For example, Dirac impulse $\delta(t-u)$ has the support concentrated in only one point $t = u$, but its FT $e^{-\jmath u \omega}$ has energy equally distributed over all frequencies. It is well known that $|F(\omega)|$ decays fast on high frequencies only when f changes periodically in time. In other words, the energy of f should be distributed in a quite wide area.

For instance, it is possible to reduce the spread of the signal f in the time domain using the scaling coefficient $m < 1$ without the change in its energy:

$$f_m(t) = \frac{1}{\sqrt{m}} f\left(\frac{t}{m}\right), \quad \|f_m\|^2 = \|f\|^2.$$

However, the FT $F_m(\omega) = \sqrt{m}F(m\omega)$ will spread in $\frac{1}{m}$ times, thus reducing the localization in the frequency domain. Therefore, it is necessary to keep the balance between time and frequency localization.

The concentration of energy is regulated by the Heisenberg's uncertainty principle [76]. This principle has an important interpretation in quantum mechanics, similar to that of uncertainty in the position of the free particle[77]. The state of one-dimensional particle is described by wave function $f \in L^2(\mathbb{R})$. The probability density function that the particle is in point t has the form $\frac{1}{\|f\|^2}|f(t)|^2$. The probability density function that the impulse of the particle equals ω is $\frac{1}{2\pi\|f\|^2}|F(\omega)|^2$. The averaged position of the particle is

$$u = \frac{1}{\|f\|^2}\int_{-\infty}^{\infty} t|f(t)|^2 dt$$

and the averaged value of the momentum is

$$\xi = \frac{1}{2\pi\|f\|^2}\int_{-\infty}^{\infty} \omega|F(\omega)|^2 d\omega. \tag{2.16}$$

Dispersion around these average value is correspondingly

$$\sigma_t^2 = \frac{1}{\|f\|^2}\int_{-\infty}^{\infty} (t-u)^2|f(t)|^2 dt,$$
$$\sigma_\omega^2 = \frac{1}{2\pi\|f\|^2}\int_{-\infty}^{\infty} (\omega-\xi)^2|F(\omega)| d\omega.$$

The larger σ_t is, the bigger the uncertainty in the particle's position. Analogically, the larger σ_ω is, the higher the uncertainty in its momentum.

Theorem 2.1 (Heisenberg's uncertainty principle)**.** *Time and frequency dispersion of function $f \in L^2(\mathbb{R})$ meets the following inequality*

$$\sigma_t^2\sigma_\omega^2 \geq \frac{1}{4}. \tag{2.17}$$

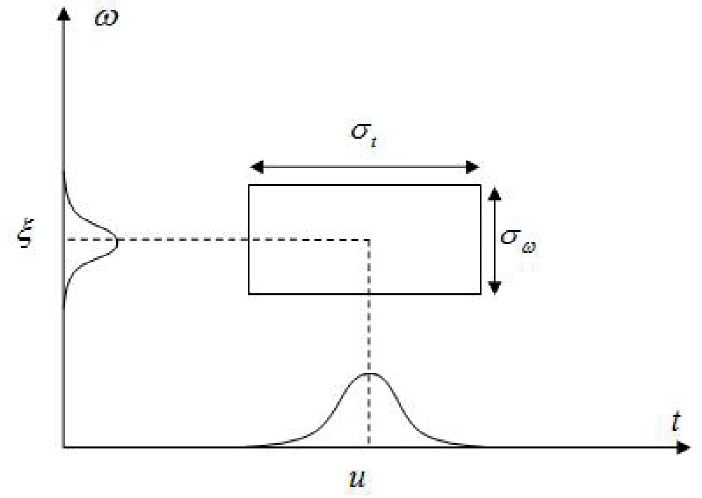


FIGURE 6 Heisenberg's rectangle on time-frequency plane.

In particular, this inequality becomes equality if and only if $\exists(u, \xi, a, b) \in \mathbb{R}^2 \times \mathbb{C}^2$ *so that*

$$f(t) = a e^{j\xi t} e^{-b(t-u)^2}.$$

In quantum mechanics, it means that it is impossible to simultaneously reduce the uncertainty in the position of the particle and in its impulse.

In signal processing, this principle means that it is impossible to improve the localization in the time domain without the loss in the frequency localization of the function, and vice versa (see Figure 6). Moreover, modulated Gaussian functions have minimal joint time-frequency dispersion because they are continuous functions with asymptotically decaying "tails". Regardless the limitation introduced by the Heisenberg uncertainty function, there is still a potential possibility to create such a function with compact support simultaneously in time and frequency. From the presented theorem it follows that it is not possible:

Theorem 2.2. *If* $f \neq 0$ *has compact support, then* $F(\omega)$ *cannot be equal to zero on any closed interval. Analogically, if* $F(\omega)$ *has compact support than* $f(t)$ *cannot be equal to zero on any closed interval.*

The frame theory continues and develops the theoretical analysis of redundant linear signal presentations and their properties such as completeness and stability [65]. Initially, this theory was proposed by R. J. Duffin and A.C. Schaeffer upon the construction of band-limited signals from the values of non-periodical discrete functions [78]. After that, these authors tried to define some generic conditions according to which it would be possible to reconstruct element f from the Hilbert space $\mathcal{H}$ if its scalar products with the family of elements $\{\phi_m\}_{m\in\Gamma}$ are known. The set of indexes Γ might be either finite or infinite. In fact, the family of functions $\{\phi_m\}_{m\in\Gamma}$, which describes any signal through the scalar products $\langle f, \phi_m\rangle$, can be considered a frame, or more exactly:

Definition 2.1 (Frame)**.** The set $\{\phi_m\}_{m\in\Gamma}$ is the frame in $\mathcal{H}$, if there are such constants $A > 0$ and $B > 0$ that for $\forall f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{m\in\Gamma} |\langle f, \phi_m\rangle|^2 \leq B\|f\|^2. \tag{2.18}$$

Moreover, if $A = B$, the frame is called tight.

The most important is that equation (2.18) is the necessary and sufficient condition of the invertibility of the operation that presents f as the set of scalar products $\langle f, \phi_m\rangle_{m\in\Gamma}$. This is true even though this presentation is redundant. When $\|\phi_m\| = 1$ this redundancy is defined by the borders of the frame A and B. If functions $\{\phi_m\}$ are linearly independent, then it is possible to prove [6] that $A \leq 1 \leq B$. Frame is an orthonormal basis if and only if $A = B = 1$.

Let's consider again the family of functions $\{g_{\mu_n,\xi_k}(t)\}$ received by translations of function $g(t)$ in time and frequency

$$g_{\mu_n,\xi_k}(t) = g(t - \mu_n)e^{j\xi_k t},$$

where $n, k \in \mathbb{Z}$. In this case, the discrete form of WFT for function $s(t)$ will look like

$$\{\mathcal{W}s(\mu_n, \xi_k) = \langle s(t), g_{\mu_n,\xi_k}(t)\rangle\}.$$

This presentation of $s(t)$ is complete and stable if the family $\{g_{\mu_n,\xi_k}(t)\}$ is frame in $L^2(\mathbb{R})$.

Setting the time-frequency lattice to be uniform, i.e. shifts in time and frequency are constant and equal to μ_0 and ξ_0, notations

$$g_{m,l}(t) = g(t - m\mu_0)e^{jk\xi_0 t}$$

can be used.

I. Daubechies [5] proved several necessary conditions on $g(t), \mu_0$ and ξ_0 guaranteeing that $\{g_{m,l}(t)\}$ is a frame in $L^2(\mathbb{R})$:

Theorem 2.3 (Daubechies)**.** *The family of Fourier window functions $\{g_{m,l}(t)\}$ is the frame only if the condition*

$$\frac{2\pi}{\mu_0\xi_0} \geq 1 \tag{2.19}$$

is fulfilled. Moreover, the borders of the frame A and B should be so that

$$A \leq \frac{2\pi}{\mu_0\xi_0} \leq B.$$

Inequality (2.19) is the measure of the density of window Fourier "atoms" on the time-frequency plane. Frame $\{g_{\mu_n,\xi_k}(t)\}$ becomes the orthonormal basis only when $A = B = 1$, i.e. in the case the critical time-frequency lattice $\mu_0\xi_0 = 2\pi$. It is important to notice here that this density corresponds exactly to the one used in OFDM systems ($\frac{1}{TF} = 1$).

Unfortunately, the Balian-Low theorem [79] says that in the case of critical time-frequency density $g(t)$ can be either non-continuous or slowly decaying (slower than linear):

Theorem 2.4 (Balian-Low)**.** *If $\{g_{\mu_n,\xi_k}(t)\}_{n,k\in\mathbb{Z}}$ is window Fourier frame with $\mu_0\xi_0 = 2\pi$, then either $\int_{-\infty}^{\infty} t^2|g(t)|^2dt = \infty$ or $\int_{-\infty}^{\infty} \omega^2|G(\omega)|^2d\omega = \infty$.*

For that reason, in practice it is necessary to use oversampled decompositions with density $\mu_0\xi_0 < 2\pi$. For example, when$1 \leq \frac{2\pi}{\mu_0\xi_0} \leq 2$, only neighboring windows are overlapping. This limitation became a considerable obstacle in the development of WFT with good time-frequency localization and WH bases.

2.4 Modified WH Basis

As it was shown in the previous section, it is impossible to construct an orthogonal WH basis with good frequency localization, on one hand, and with a dense time-frequency lattice, on the other. However such bases can be synthesized with the extension of the class of considered bases:

- Let's pass from the Hilbert space with regular dot product to the space with so-called real scalar product;
- Instead of one initializing function, a family of initializing functions can be utulized.

Definition 2.2. The family of function $\mathcal{B}[\mathbb{R}] \triangleq \{\Psi^R_{k,l}(t), \Psi^I_{k,l}(t)\}$ which satisfies the assumptions mentioned above where functions

$$\begin{aligned}\Psi^R_{k,l}(t) &= g(t - lT)e^{2\pi \jmath Fk\left(t - \frac{\alpha T}{2M}\right)},\\ \Psi^I_{k,l}(t) &= -\jmath g(t + \frac{T}{2} - lT)e^{2\pi \jmath Fk\left(t - \frac{\alpha T}{2M}\right)},\end{aligned}$$

for $t \in \mathbb{R}$, $k \in J_M$ and $l \in \mathbb{Z}$ are normed and orthogonal

$$\langle \Psi^R_{k,l}, \Psi^R_{k',l'}\rangle_R = \delta_{k,k'}\delta_{l,l'}, \quad \langle \Psi^I_{k,l}, \Psi^I_{k',l'}\rangle_R = \delta_{k,k'}\delta_{l,l'}, \quad \langle \Psi^R_{k,l}, \Psi^I_{k',l'}\rangle_R = 0, \tag{2.20}$$

in terms of real scalar product ($\delta_{l,l'}$ is Kronecker's symbol)

$$\langle x(t), y(t)\rangle_R = \Re\left[\int_{-\infty}^{\infty} x(t)y^*(t)dt\right]$$

will be called orthogonal MWH basis.

Functions $\Psi^R_{k,l}$ and $\Psi^I_{k,l}$ result from uniform shifts in time and frequency of the tow families of functions $\{g(t)e^{-2\pi \jmath Fk\frac{\alpha T}{2M}}\}_{k\in J_L}$ and $\{g(t+\frac{T}{2})e^{-2\pi \jmath Fk\frac{\alpha T}{2M}}\}_{k\in J_L}$ correspondingly. Physically the parameters of the basis can be interpreted in the following way:

- $M > 2$ is the number of sub-carriers;
- $F = \frac{1}{T}$ inter-carrier distance;
- T - symbol period;

– $\alpha \in \mathbb{R}$ - phase parameter.

Upon that, the decomposition of a complex function or signal over the basis $\mathcal{B}[\mathbb{R}]$ can be written like

$$s(t) = \sum_{k=0}^{M-1} \left[\sum_{l=-\infty}^{\infty} c_{k,l}^R \Psi_{k,l}^R(t) - \sum_{l=-\infty}^{\infty} c_{k,l}^I \Psi_{k,l}^I(t) \right], \tag{2.21}$$

where $c_{k,l}^R = \langle s(t), \Psi_{k,l}^R(t) \rangle$ and $c_{k,l}^I = \langle s(t), \Psi_{k,l}^I(t) \rangle$ are real-valued decomposition coefficients. One of the first proposals for this construction can be found in [80].

Algorithm (2.21) can be used for the modulation of the signal, where $c_{k,l}^R = \Re(a_{k,l})$ and $c_{k,l}^I = \Im(a_{k,l})$ are the real and imaginary parts of complex QAM information symbols $a_{k,l}$.

As it was discussed earlier, in digital signal processing we deal with discrete signals of finite length. Thus, a discrete realization of the modified WH basis should be introduced, i.e. it is necessary to pass from the space $L^(\mathbb{R})$ to the space $\mathbb{C}^N$ with introduced the real scalar product

$$\langle x[n], y[n] \rangle_R = \Re\{ \sum_{n=0}^{N-1} x[n] y^*[n] \}, \tag{2.22}$$

where $x[n], y[n] \in \mathbb{C}^N$.

If the forming window function $g(t)$ has the bandwidth $F = \frac{1}{T}$ then with the account of M shifts in the frequency domain the signal's bandwidth will be $W = \frac{M}{T}$. Assuming M to be even, the signal (2.21) can be easily sampled on the finite interval $[0, NT]$ with the frequency $f_s = W$. The discrete signal and the corresponding modified WH bases will now have the following form:

$$s[n] = \sum_{k=0}^{M-1} \left(c_{k,l}^R \Psi_{k,l}^R[n] - c_{k,l}^I \Psi_{k,l}^I[n] \right), \tag{2.23}$$

$$\Psi_{k,l}^R[n] = g[(n - lM)_N] e^{\jmath \frac{2\pi}{M} k (n - \frac{\alpha}{2})}, \tag{2.24}$$

$$\Psi_{k,l}^I[n] = -\jmath g[(n + \frac{M}{2} - lM)_N] e^{\jmath \frac{2\pi}{M} k (n - \frac{\alpha}{2})}, \tag{2.25}$$

$$\mathcal{B}[J_N] \triangleq \{ \Psi_{k,l}^R[n], \Psi_{k,l}^I[n] \}, \tag{2.26}$$

where $s[n] = s(\frac{n}{f_s}) = s(\frac{nT}{M})$ and $N = ML$, L - is the number of shifts in the time domain.

Thus the system of basis' functions $\mathcal{B}[J_N]$ is the discrete analogue of system $\mathcal{B}[\mathbb{R}]$ and orthogonal in terms of real scalar product (2.22).

In the following text bases (2.26) will be called MWH bases. The corresponding signal processing technology will be called OFTDM.

2.5 Conclusions of OFDM Time-Frequency Analysis

From the analysis presented above, several important conclusions can be made. First of all, WH bases and WFT are powerful and useful tools in time-frequency analysis. OFDM technology is based on them and uses a rectangular initializing function. This allows the utilization of efficient FFT algorithms for modulation and demodulation. The addition of CP solves the problem with ISI but reduces the efficiency of the OFDM systems.

With the introduction of MWH bases, it is possible to overcome the limitation of the Balian-Low theorem and to come closer to the fundamental limit introduced by the Heisenberg uncertainty relation. The OFTDM technology, which rises from the MWH bases, posseses better signal localization, in particular lower out-of-band emission. Thus the system becomes more robust in complicated channel conditions and more efficient because of the absence of CP and smaller GIs in the frequency domain.

3 SYNTHESIS OF MWH BASES

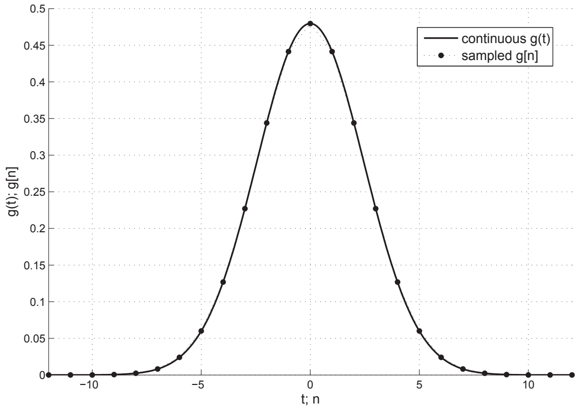
As it was mentioned in the previous chapter, the Gaussian function possesses optimal time-frequency localization. However, the MWH basis constructed out of it will not be orthogonal. Thus, the following generic problem can be posed: it is necessary to find such basis that, firstly, will be orthogonal and, secondly, will have the initializing function close to some given function. If the last condition is fulfilled, then it is possible to have an orthogonal basis with required localization characteristics. This property cannot be achieved with standard orthogonalization procedures. For example, the Gram-Schmidt procedure [81] might result in a considerable change in the shape of the initial function. To avoid this effect, several authors [82, 53, 83] have proposed to form the basis with the help of special digital filters. The disadvantage of this approach is due to the structural limitations of these filters, which create obstacles during their application. Later in this chapter, a generic algebraic method of basis synthesis free from these limitations will be overviewed. However, before that it is necessary to know how to create discrete initial function which afterwards can be orthogonalized.

3.1 $\mathbb{C}^N$ Approximation of Symmetrical Continuous Function

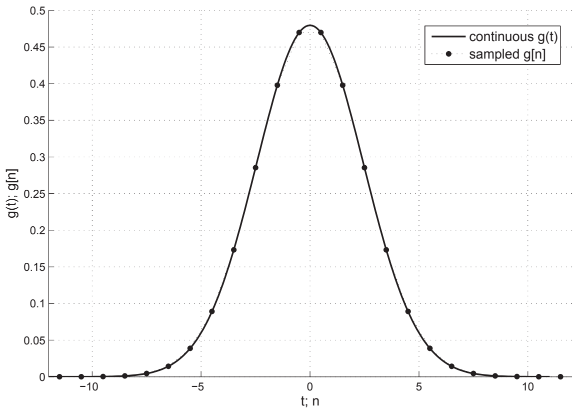
For example, the Gaussian function discussed above is defined and continuous on the whole real axis, thus it is impossible to use it directly as an initializing function for discrete MWH basis. First of all, it is necessary to solve the problem of function search from $\mathbb{C}^N$ that will approximate a given continuous function. In this section, this problem will be considered in the most general case, when the only applied constraint is the function's symmetry, i.e. $g_0(t) = g_0(-t)$ [84].

After sampling of $g_0(t)$ with period T_s, the original property of symmetry can be kept in two ways:

1. if $g_0(t)$ is sampled so that $g_0[n] = g_0(nT_s)$, $n \in \mathbb{Z}$ and $g_0[n] =_0 [-n]$;
2. if $g_0(t)$ is sampled so that $g_0[n] = g_0((n + \frac{1}{2})T_s)$, $n \in \mathbb{Z}$ and $g_0[n] = g_0[-n - 1]$.



(a) symmetry option (1)



(b) symmetry option (2)

FIGURE 7 Sampling of symmetrical continuous function.

The first variant corresponds to the case when discrete function $g_0[n]$ is symmetrical relative to the center of interval $[-N; N]$. In the second case, $g_0[n]$ overcomes symmetrical relatively to the center of interval $[-N; N-1]$. The examples of such sampling are given in Figure 7 for a real function $g_0(t)$. In the case when it is complex-defined, the same considerations are also true but there may be several more variants of symmetry, for instance conjugated symmetry: $g_0[n] = g_0^*[-n]$ [13].

In the N-periodical discrete space $\mathbb{C}^N$ it is possible to introduce symmetry in the two following ways:

Definition 3.1 (($N-1$)-symmetry)**.** Function $g[n]$ from $\mathbb{C}^N$ is $(N-1)$-symmetrical if

$$g[N-1-n] = g^*[n], \quad n \in J_N.$$

Definition 3.2 (((N)-symmetry). Function $g[n]$ from $\mathbb{C}^N$ is (N)-symmetrical if

$$g[n] = g^*[(-n)_N], \quad n \in J_N.$$

Using Lagrange's method of undetermined coefficients, it is possible to find the exact expression for the approximating functions from $\mathbb{C}^N$. Moreover, in the case of the symmetry of the given initial function $g_0[n]$ the approximating functions will also be symmetrical.

It is necessary to introduce several new notations:

– Discrete interval interval $\{-N, -N+1, \ldots, N-1, N\} \triangleq J_{-N,N}$ equals to two periodicity intervals N from $\mathbb{C}^N$.
– Let $g^{(N)}[n] \triangleq g[(n) \mod N] = g[(n)_N]$ be N-periodical function defined by N values of any complex function $g[n]$ from J_N.

Now it is more convenient to formulate the main theorem:

Theorem 3.1. *Let $g_0[n]$ be any conjugate-symmetrical function ($g_0[n] = g_0^*[-n]$) defined for all $n \in J_{-N,N}$.*

Then the best N-periodical approximation $\tilde{g}_0^{(N)} \in \mathbb{C}^N$ for the function $g_0[n]$, which brings the minimum in the following extremal problem:

$$\tilde{g}_0^{(N)}[n] : \quad \sum_{n=-N}^{N-1} |g_0[n] - g^{(N)}[(n)_N]|^2 \to \min_{g^{(N)}[n] \in \mathbb{C}^N} \tag{3.1}$$

with constraint (conservation of energy or normalization)

$$\sum_{n=-N}^{N-1} |g_0[n]|^2 = \sum_{n=0}^{N-1} |g^{(N)}[n]|^2 \tag{3.2}$$

has the following form:

$$\tilde{g}_0^{(N)}[n] = \frac{1}{\sqrt{A}} \left(g_0^{(N)}[n] + g_0^{(N)^*}[N-n] \right), \quad n = 1, 2, \ldots, N-1;$$
$$\tilde{g}_0^{(N)}[0] = \frac{1}{\sqrt{A}} \left(g_0[0] + g_0^*[0] \right),$$

where normalizing coefficient

$$A = \frac{\sum_{n=1}^{N-1} |g_0^{(N)}[n] + g_0^{(N)^*}[N-n]|^2 + |g_0[0] + g_0^*[N]|^2}{\sum_{n=-N}^{N-1} |g_0[n]|^2}. \tag{3.3}$$

Proof. Let's define two functionals on the linear space $\mathbb{C}^N$:

$$X\left(g^{(N)}[n]\right) \triangleq \sum_{n=-N}^{N-1} \left| g_0[n] - g^{(N)}[n] \right|^2,$$

$$\Phi\left(g^{(N)}[n]\right) \triangleq \sum_{n=-N}^{N-1} |g_0[n]|^2 - \sum_{n=0}^{N-1} \left| g^{(N)}[n] \right|^2.$$

It is not difficult to check that for these functionals Jensen's inequality [85]

$$F\left(\alpha g_1^{(N)}[n] + (1-\alpha) g_2^{(N)}[n]\right) \leq \alpha F\left(g_1^{(N)}[n]\right) + (1-\alpha) F\left(g_2^{(N)}[n]\right)$$

is true and for that reason they are convex.

Therefore, the initial problem (3.1) can be changed with the following problem of convex programming:

$$X\left(g^{(N)}[n]\right) \to \min, \quad \Phi\left(g^{(N)}[n]\right) = 0, \quad g^{(N)}[n] \in \mathbb{C}^N. \tag{3.4}$$

To solve this problem, the Lagrange's method of undetermined coefficients can be used [86]. Firstly, it is necessary to construct the Lagrangian function

$$L\left(g^{(N)}[n], \xi_0, \xi\right) \triangleq \xi_0 X\left(g^{(N)}[n]\right) + \xi\,\Phi\left(g^{(N)}[n]\right). \tag{3.5}$$

It is obvious that for any given $g_0[n]$ from the hypotheses of the theorem it is possible to find such function $g_0^{(N)}[n] \in \mathbb{C}^N$, that $\Phi\left(g_0^{(N)}[n]\right) < 0$. Thus the Slater's condition [87] is fulfilled and $\xi_0 = 1$.

Function $g^{(N)}[n] \in \mathbb{C}^N$ is N-periodical and can be presented in the form of Fourier series

$$g^{(N)}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \lambda_k e^{j\frac{2\pi}{N}kn},\ n \in J_N. \tag{3.6}$$

Moreover $g^{(N)}[n]$ is defined uniquely by its Fourier coefficients λ_k, which are complex numbers

$$\lambda_k = \lambda_k^R + j\lambda_k^I, \quad k \in J_N.$$

In order to reduce the complexity of formulas, notations

$$\boldsymbol{\lambda} \triangleq \left\{\lambda_0^R, \ldots, \lambda_{N-1}^R, \lambda_0^I, \ldots, \lambda_{N-1}^I\right\}$$

will be used.

Next, it is necessary to introduce functions

$$X_R(\boldsymbol{\lambda}) \triangleq \sum_{n=-N}^{N-1}\left(\Re\left(g_0[n] - g^{(N)}[n]\right)\right)^2 =$$
$$= \sum_{n=-N}^{N-1}\left\{\Re(g_0[n]) - \frac{1}{N}\sum_{k=0}^{N-1}\left(\lambda_k^R \cos\left(\frac{2\pi}{N}kn\right) - \lambda_k^I \sin\left(\frac{2\pi}{N}kn\right)\right)\right\}^2,$$

$$X_I(\boldsymbol{\lambda}) \triangleq \sum_{n=-N}^{N-1}\left(\Im\left(g_0[n] - g^{(N)}[n]\right)\right)^2 =$$
$$= \sum_{n=-N}^{N-1}\left\{\Im(g_0[n]) - \frac{1}{N}\sum_{k=0}^{N-1}\left(\lambda_k^I \cos\left(\frac{2\pi}{N}kn\right) + \lambda_k^R \sin\left(\frac{2\pi}{N}kn\right)\right)\right\}^2.$$

With the account of the presentation (3.6), the problem (3.4) will look like

$$X(\boldsymbol{\lambda}) = X\left(g^{(N)}[n]\right) = X_R(\boldsymbol{\lambda}) + X_I(\boldsymbol{\lambda}) \to \min, \quad \Phi(\boldsymbol{\lambda}) = 0, \quad \boldsymbol{\lambda} \in \mathbb{R}^{2N}. \tag{3.7}$$

Upon that the Lagrangian function (3.5) will be

$$L(\boldsymbol{\lambda},\ \xi) \triangleq X(\boldsymbol{\lambda}) + \xi\,\Phi(\boldsymbol{\lambda}).$$

In accordance with the Lagrange's method of undetermined coefficients the necessary condition of local extremum in problem (3.7) is the equality to zero of all partial derivatives of the Lagrangian function:

$$\left.\frac{\partial L}{\partial \lambda_0^R}\right|_{\lambda_0^R=\tilde{\lambda}_0^R} = 0, \ldots, \left.\frac{\partial L}{\partial \lambda_{N-1}^R}\right|_{\lambda_{N-1}^R=\tilde{\lambda}_{N-1}^R} = 0,\ \left.\frac{\partial L}{\partial \lambda_0^I}\right|_{\lambda_0^I=\tilde{\lambda}_0^I} = 0, \ldots, \tag{3.8}$$
$$\ldots, \left.\frac{\partial L}{\partial \lambda_{N-1}^I}\right|_{\lambda_{N-1}^I=\tilde{\lambda}_{N-1}^I} = 0,\ \left.\frac{\partial L}{\partial \xi}\right|_{\xi=\tilde{\xi}} = 0.$$

Let's consider, in more detail, the expression of partial derivatives of functional $\Phi(\boldsymbol{\lambda})$ and components $X_R(\boldsymbol{\lambda})$ and $X_I(\boldsymbol{\lambda})$ of functional $X(\boldsymbol{\lambda})$:

$$\frac{\partial X_R}{\partial \lambda_p^R} = 2\sum_{n=-N}^{N-1}\left\{\Re\left(g_0[n] - g^{(N)}[n]\right)\left(-\frac{1}{N}\cos\left(\frac{2\pi}{N}np\right)\right)\right\},$$
$$\frac{\partial X_R}{\partial \lambda_p^I} = 2\sum_{n=-N}^{N-1}\left\{\Re\left(g_0[n] - g^{(N)}[n]\right)\left(\frac{1}{N}\sin\left(\frac{2\pi}{N}np\right)\right)\right\},$$
$$\frac{\partial X_I}{\partial \lambda_p^R} = 2\sum_{n=-N}^{N-1}\left\{\Im\left(g_0[n] - g^{(N)}[n]\right)\left(-\frac{1}{N}\sin\left(\frac{2\pi}{N}np\right)\right)\right\},$$
$$\frac{\partial X_I}{\partial \lambda_p^I} = 2\sum_{n=-N}^{N-1}\left\{\Im\left(g_0[n] - g^{(N)}[n]\right)\left(-\frac{1}{N}\cos\left(\frac{2\pi}{N}np\right)\right)\right\},$$
$$\frac{\partial \Phi}{\partial \lambda_p^R} = -\xi\sum_{n=0}^{N-1}\left\{2\Re\left(g^{(N)}[n]\right)\frac{1}{N}\cos\left(\frac{2\pi}{N}np\right)\right\} -$$
$$-\,\xi\sum_{n=0}^{N-1}\left\{2\Im\left(g^{(N)}[n]\right)\frac{1}{N}\sin\left(\frac{2\pi}{N}np\right)\right\},$$
$$\frac{\partial \Phi}{\partial \lambda_p^I} = \xi\sum_{n=0}^{N-1}\left\{2\Re\left(g^{(N)}[n]\right)\frac{1}{N}\sin\left(\frac{2\pi}{N}np\right)\right\} -$$
$$-\,\xi\sum_{n=0}^{N-1}\left\{2\Im\left(g^{(N)}[n]\right)\frac{1}{N}\cos\left(\frac{2\pi}{N}np\right)\right\}$$

for any $p \in J_N$.

Then the extremum conditions (3.8) for any $p \in J_N$ can be rewritten like

$$\begin{aligned}
\left.\frac{\partial L}{\partial \lambda_p^R}\right|_{\lambda_p^R=\lambda_p^R} &= \Re\left\{\sum_{n=-N}^{N-1}\left(g_0[n]-\tilde{g}_0^{(N)}[n]\right)e^{-j\frac{2\pi}{N}pn}+\right.\\
&\left.+\xi\sum_{n=0}^{N-1}\tilde{g}_0^{(N)}[n]e^{-j\frac{2\pi}{N}pn}\right\}=0,\\
\left.\frac{\partial L}{\partial \lambda_p^I}\right|_{\lambda_p^R=\lambda_p^R} &= \Im\left\{\sum_{n=-N}^{N-1}\left(g_0[n]-\tilde{g}_0^{(N)}[n]\right)e^{-j\frac{2\pi}{N}pn}+\right.\\
&\left.+\xi\sum_{n=0}^{N-1}\tilde{g}_0^{(N)}[n]e^{-j\frac{2\pi}{N}pn}\right\}=0,\\
\left.\frac{\partial L}{\partial \xi}\right|_{\xi=\tilde{\xi}} &= \sum_{n=-N}^{N-1}|g_0[n]|^2-\sum_{N=0}^{N-1}\left|\tilde{g}_0^{(N)}[n]\right|^2=0.
\end{aligned}$$

From these expressions it directly follows that the extremum conditions are

$$\begin{aligned}
&\sum_{n=-N}^{N-1}\left(g_0[n]-\tilde{g}_0^{(N)}[n]\right)e^{-j\frac{2\pi}{N}pn}+ \qquad (3.9)\\
&\qquad +\xi\sum_{n=0}^{N-1}\tilde{g}_0^{(N)}[n]e^{-j\frac{2\pi}{N}pn}=0,\\
&\sum_{n=-N}^{N-1}|g_0[n]|^2-\sum_{N=0}^{N-1}\left|\tilde{g}_0^{(N)}[n]\right|^2=0. \qquad (3.10)
\end{aligned}$$

for $\forall p \in J_N$.

Taking into account that ${g_0}^{(N)}[n] \triangleq g_0[n]$, $n \in J_N$ and $g_0[n] = {g_0}^*[-n]$, $n \in J_{-N,N}$, one gets that

$$\sum_{n=0}^{N-1} g_0[n]e^{-j\frac{2\pi}{N}pn} = \sum_{n=0}^{N-1} {g_0}^{(N)}[n]e^{-j\frac{2\pi}{N}pn}$$

and

$$\begin{aligned}
\sum_{n=-N}^{-1} g_0[n]e^{j\frac{2\pi}{N}pn} &= \sum_{n=-(N-1)}^{-1} {g_0}^*[-n]e^{j\frac{2\pi}{N}pn}+g_0[-N]=\\
&= \sum_{n=-(N-1)}^{-1} {g_0}^{(N)*}[-n]e^{j\frac{2\pi}{N}pn}+{g_0}^*[N]=\\
&= \sum_{n'=1}^{N-1} {g_0}^{(N)*}\left[N-n'\right]e^{j\frac{2\pi}{N}p(n')}+{g_0}^*[N],
\end{aligned}$$

where $n' = n + N$.

Therefore, from the first condition (3.9) it follows that for $\forall p \in J_N$

$$\sum_{n=0}^{N-1} (2-\xi)\, \tilde{g}_0^{(N)}[n]\, e^{-j\frac{2\pi}{N}pn} =$$
$$= \sum_{n=1}^{N-1} \left({g_0}^{(N)}[n] + g_0^{(N)*}[N-n] \right) e^{-j\frac{2\pi}{N}pn} + {g_0}^{(N)}[0] + {g_0}^{*}[N] .$$

This conditions means the equality of the Fourier coefficients of two functions, which in turn should be the same. That is why

$$\tilde{g}_0^{(N)}[n] = \frac{1}{(2-\xi)} \left({g_0}^{(N)}[n] + g_0^{(N)*}[N-n] \right),\ n = 1, 2, ..., (N-1), \tag{3.11}$$
$$\tilde{g}_0^{(N)}[0] = \frac{1}{(2-\xi)} \left({g_0}^{(N)}[0] + {g_0}^{*}[N] \right).$$

In order to find the Lagrangian coefficient, ξ the second condition (3.10) can be used:

$$\sum_{n=-N}^{N-1} |g_0[n]|^2 - \frac{1}{(2-\xi)^2} \left\{ \sum_{n=1}^{N-1} \left| \left({g_0}^{(N)}[n] + g_0^{(N)*}[N-n] \right) \right|^2 + \right.$$
$$\left. + \left| {g_0}^{(N)}[0] + {g_0}^{*}[N] \right|^2 \right\} = 0,$$

which is, in fact, a square equation in respect to ξ. Consequently,

$$\xi = 2 \pm \sqrt{A},$$

where A is defined by (3.3).

Taking into account that in the case, when $\xi > 0$ functions $\tilde{g}_0^{(N)}[0]$ and $\tilde{g}^{(N)}[0]$ have different signs, $\xi = 2 - \sqrt{A}$ should be selected.

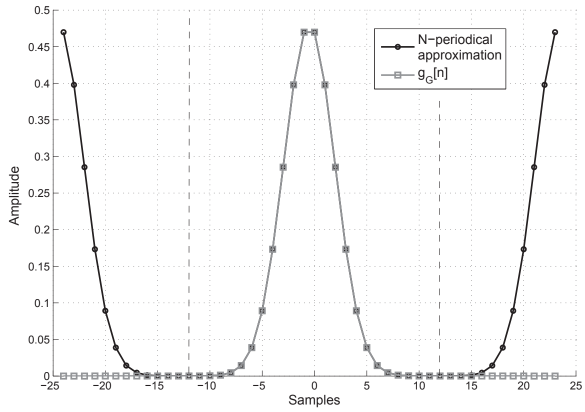
In accordance with the Kuhn–Tucker theorem, [86] conditions (3.7) are not only necessary but also sufficient conditions for the function $\tilde{g}_0^{(N)}[n]$ from (3.11) to be the solution of the posed extremal problem (3.1) with constraint (3.2). □

Additionally, in the case when function $g_0[n]$ has a real value in point N, i.e. $g_0[N] = {g_0}^{*}[-N] = g_0[-N]$, then the constructed function $\tilde{g}_0^{(N)}[n]$ is conjugated N-symmetrical. Indeed,

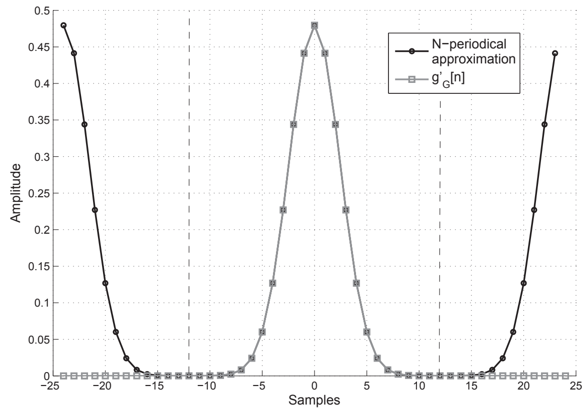
$$\tilde{g}_0^{(N)*}[-n] = \frac{1}{\sqrt{A}} \left(g_0^{(N)} * [-n] + g_0^{(N)}[N+n] \right) =$$
$$= \frac{1}{\sqrt{A}} \left(g_0^{(N)*}[N-n] + g_0^{(N)}[n] \right) = \tilde{g}_0^{(N)}[n],$$

and

$$\tilde{g}_0^{(N)*}[0] = \frac{1}{\sqrt{A}} \left({g_0}^{*}[0] + g_0[N] \right) = \frac{1}{\sqrt{A}} \left(g_0[0] + {g_0}^{*}[N] \right) = \tilde{g}_0^{(N)}[0].$$



(a) N-symmetry



(b) (N-1)-symmetry

FIGURE 8 Examples of N-periodical approximation.

It is also possible to prove the same theorem but in the case when the given function is sampled so that $g_0[n] = g_0^*[-n-1]$, $n \in J_{-N,N} \triangleq \{-N, \ldots, N-1$. Then the optimal N-periodical function will resemble

$$\tilde{g}_0^{(N)}[n] = \frac{1}{\sqrt{A}} \left({g_0}^{(N)}[n] + g_0^{(N)*}[N-1-n] \right), \; n \in J_N, \tag{3.12}$$

where the normalizing coefficient

$$A = \frac{\sum\limits_{n=0}^{N-1} \left| \left({g_0}^{(N)}[n] + {g_0}^{(N)*}[N-1-n] \right) \right|^2}{\sum\limits_{n=-N}^{N-1} |g_0[n]|^2}.$$

Moreover, if function $g_0[n]$ has a real value in the point N, i.e. $g_0[N] =$

${g_0}^*\,[-N] = g_0\,[-N]$,then the received function $\tilde{g}_0^{(N)}\,[n]$ is conjugated N-symmetrical. Indeed,

$$\tilde{g}_0^{(N)*}\,[n] = \frac{1}{\sqrt{A}}\left(g_0^{(N)*}\,[n] + g_0^{(N)}\,[N-1-n]\right) = \tilde{g}_0^{(N)}\,[N-1-n]\,.$$

The algorithms of N-periodical approximation for two types of symmetry were realized in Matlab and can be found in the Appendix.

Figure 8 present the simulation results, when the initial function is Gaussian and is sampled in two different ways that keep symmetry:

$$g_0[n] = g_G[n] = (2\sigma)^{\frac{1}{4}} e^{-\pi\sigma n^2},$$
$$g_0'[n] = g_G'[n] = (2\sigma)^{\frac{1}{4}} e^{-\pi\sigma(n+0.5)^2}.$$

Function $g_G[n]$ corresponds to the conditions of Theorem 3.1 and is plotted on Figure 8a together with its N-periodical approximation. Figure 8b shows function $g_G'[n]$ and its approximation defined by (3.12).

3.2 Algebraic Basis Synthesis Algorithm

In the previous section, it was shown how to construct the initial discrete function from $\mathbb{C}^{\mathbb{N}}$ out of any given complex symmetrical continuous function. There is no guarantee that the modified WH basis constructed from it will be orthogonal. For that reason special orthogonalization algorithm is needed.

For further considerations, it is important to introduce matrix presentation of the MWH basis. Basis $\mathcal{B}[J_N]$ can be rewritten in the form of bloc rectangular matrix $\mathbf{U} = [\mathbf{U_R}, \mathbf{U_I}]$ of size $N \times 2N$. Blocs $\mathbf{U_R}$ and $\mathbf{U_I}$ are square complex matrices of size $N \times N$ constructed from the basis' functions $\Psi^R_{k,l}[n]$ and $\Psi^I_{k,l}[n]$ taken as the columns of length N for all $k = 0, 1, \ldots, M-1$; $l = 0, 1, \ldots, L-1$; $n \in J_N$. Thus matrices $\mathbf{U_R}$ and $\mathbf{U_I}$ consist from L blocs of M columns and their elements are

$$\mathbf{U_R}[n, lM+k] = \Psi^R_{k,l}[n];\ \mathbf{U_I}[n, lM+k] = \Psi^I_{k,l}[n].$$

Instead of the complex matrix $\mathbf{U}$ it is also possible to consider a real square matrix of size $2N \times 2N$:

$$\mathbf{U_B} = \begin{bmatrix} \Re(\mathbf{U_R}) & \Re(\mathbf{U_I}) \\ \Im(\mathbf{U_R}) & \Im(\mathbf{U_I}) \end{bmatrix}. \tag{3.13}$$

Let $\mathbf{G}$ be the matrix of some non-orthogonal basis constructed in the same way as the MWH basis, using some symmetrical function from $\mathbb{C}^N$ with some desired localization properties. In particular, if the initial function is Gaussian then $\mathbf{G}$ represents the Gabor basis. The idea is to find the orthogonal MWH basis with matrix $\mathbf{U}_{\text{opt}}$ close to matrix $\mathbb{G}$. The closeness of matrices (and consequently of two bases) will be understood in terms of root-mean-square Frobenius norm of difference between the corresponding basis matrices $\mathbf{U^1}$ and $\mathbf{U^2}$:

$$\| \triangle \mathbf{U} \|_E = \| \mathbf{U}_{\text{opt}} - \mathbf{G} \|_E, \tag{3.14}$$

where $\|\mathbf{A}\|_E \triangleq \operatorname{tr}(\mathbf{A}\mathbf{A}^*)$.

3.2.1 Difference Norm of Two MWH Bases

It is possible to show that in order to calculate the difference norm (3.14) between two MWH bases it is enough to consider the norm of difference between their initializing functions. The following theorem proves that:

Theorem 3.2. *Let $\mathcal{B}_1[J_N]$ and $\mathcal{B}_1[J_N]$ be modified WH bases constructed from functions $g_1[n]$ and $g_2[n]$, $n \in J_N$. $\mathbf{U^1}$ and $\mathbf{U^2}$ are their matrices.*

Then the difference norm (3.14) equals

$$\| \triangle \mathbf{U} \|_E^2 = \| \mathbf{U^1_B} - \mathbf{U^2_B} \|_E^2 = 2N \| g_1[n] - g_2[n] \|^2 = 2N \sum_{n=0}^{N-1} (g_1[n] - g_2[n])^2. \tag{3.15}$$

Proof. Firstly, it is necessary to prove that the norm of the complex matrix of MWH basis $\mathbf{U^1}$ and its real matrix

$$\mathbf{U^1_B} = \begin{bmatrix} \Re\left\{\mathbf{U^1}\right\} \\ \Im\left\{\mathbf{U^1}\right\} \end{bmatrix}$$

are equal, i.e.

$$\left\| \mathbf{U^1} \right\|_E^2 = \left\| \mathbf{U^1_B} \right\|_E^2. \tag{3.16}$$

Taking into account that matrix $\mathbf{U^1}$ is the block-matrix

$$\begin{aligned} \|\mathbf{U^1}\|_E^2 &= \operatorname{tr}\left(\begin{bmatrix} \mathbf{U^1_R} & \mathbf{U^1_I} \end{bmatrix} \begin{bmatrix} \mathbf{U^1_R} & \mathbf{U^1_I} \end{bmatrix}^* \right) = \\ &= \operatorname{tr}\left(\begin{bmatrix} \mathbf{U^1_R} & \mathbf{U^1_I} \end{bmatrix} \begin{bmatrix} \mathbf{U^1_R}^* \\ \mathbf{U^1_I}^* \end{bmatrix} \right) = \operatorname{tr}\left(\mathbf{U^1_R}\mathbf{U^1_R}^* + \mathbf{U^1_I}\mathbf{U^1_I}^* \right) = \\ &= \operatorname{tr}\left(\left(\Re\left\{\mathbf{U^1_R}\right\} + \jmath \Im\left\{\mathbf{U^1_R}\right\} \right) \left(\left(\Re\left\{\mathbf{U^1_R}\right\}\right)^T - \jmath \left(\Im\left\{\mathbf{U^1_R}\right\}\right)^T \right) + \right. \\ &\quad \left. + \left(\Re\left\{\mathbf{U^1_I}\right\} + \jmath \Im\left\{\mathbf{U^1_I}\right\} \right) \left(\left(\Re\left\{\mathbf{U^1_I}\right\}\right)^T - \jmath \left(\Im\left\{\mathbf{U^1_I}\right\}\right)^T \right) \right) = \\ &= \operatorname{tr}\left(\Re\left\{\mathbf{U^1_R}\right\} \Re\left\{ \left(\mathbf{U^1_R}\right)^T \right\} \right) + \operatorname{tr}\left(\Re\left\{\mathbf{U^1_I}\right\} \Re\left\{ \left(\mathbf{U^1_I}\right)^T \right\} \right) + \\ &\quad + \operatorname{tr}\left(\Im\left\{\mathbf{U^1_R}\right\} \Im\left\{ \left(\mathbf{U^1_R}\right)^T \right\} \right) + \operatorname{tr}\left(\Im\left\{\mathbf{U^1_I}\right\} \Im\left\{ \left(\mathbf{U^1_I}\right)^T \right\} \right). \end{aligned}$$

On the other hand

$$\|\mathbf{U^1_B}\|_E^2 = \operatorname{tr}\left(\begin{bmatrix} \Re\left\{\mathbf{U^1_{\mathit{R}}}\right\} & \Re\left\{\mathbf{U^1_{\mathit{I}}}\right\} \\ \Im\left\{\mathbf{U^1_{\mathit{R}}}\right\} & \Im\left\{\mathbf{U^1_{\mathit{I}}}\right\} \end{bmatrix} \times \begin{bmatrix} \left(\Re\left\{\mathbf{U^1_{\mathit{R}}}\right\}\right)^T & \left(\Im\left\{\mathbf{U^1_{\mathit{R}}}\right\}\right)^T \\ \left(\Re\left\{\mathbf{U^1_{\mathit{I}}}\right\}\right)^T & \left(\Im\left\{\mathbf{U^1_{\mathit{I}}}\right\}\right)^T \end{bmatrix} \right),$$

which is equivalent to the last expression in the previous formula. That proves the equality (3.16).

For a new notation $\mathbf{\Delta U} = \begin{bmatrix}\mathbf{\Delta U}_R & \mathbf{\Delta U}_I\end{bmatrix} \triangleq \mathbf{U^1} - \mathbf{U}^2$, it can be shown that

$$\begin{gathered}\|\mathbf{\Delta U}\|_E^2 = \left\|\mathbf{U^1} - \mathbf{U^2}\right\|_E^2 = \\ = \operatorname{tr}\left(\Re\left\{\mathbf{\Delta U}_R\right\}\Re\left\{\left(\mathbf{\Delta U}_R\right)^T\right\}\right) + \operatorname{tr}\left(\Re\left\{\mathbf{\Delta U}_I\right\}\Re\left\{\left(\mathbf{\Delta U}_I\right)^T\right\}\right) + \\ + \operatorname{tr}\left(\Im\left\{\mathbf{\Delta U}_R\right\}\Im\left\{\left(\mathbf{\Delta U}_R\right)^T\right\}\right) + \operatorname{tr}\left(\Im\left\{\mathbf{\Delta U}_I\right\}\Im\left\{\left(\mathbf{\Delta U}_I\right)^T\right\}\right).\end{gathered}$$

it is enough to consider only the first member of this sum because the derivations for all the other members will be the same.

The elements of square $(N \times N)$ matrix $\Re\{\mathbf{\Delta U_R}\}$ are all possible multiplications of difference in initializing functions $\Delta g\left[(n - lM)_N\right] \triangleq g_1\left[(n - lM)_N\right] - g_2\left[(n - lM)_N\right]$ and $\cos\left(\frac{2\pi}{M}k(n - \frac{\alpha}{2})\right)$. The columns of this matrix are vectors $\Delta g\left[n\right],\ n = 0, 1, ..., N - 1$, which are multiplied by cosine functions (M columns). Then these vectors are cyclically shifted for M points and again multiplied by cosines (next M columns). In this way, we get L blocks with M columns each.

Transposed matrix $\Re\left\{\left(\mathbf{\Delta U}_R\right)^T\right\}$ has a similar structure, but it consists from L blocks out of M rows each.

Matrix $\Re\left\{\mathbf{\Delta U}_R\right\}\Re\left\{\left(\mathbf{\Delta U}_R\right)^T\right\}$ of size $(N \times N)$ has N diagonal elements. Each of them is equal to the sum of all elements of the matrix $\Re\left\{\mathbf{\Delta U}_R\right\}$ (or of matrix $\Re\left\{\left(\mathbf{\Delta U}_R\right)^T\right\}$) corresponding to some fixed n and squared:

$$\begin{gathered}\operatorname{tr}\left(\Re\left\{\mathbf{\Delta U}_R\right\}\Re\left\{\left(\mathbf{\Delta U}_R\right)^T\right\}\right) = \\ = \sum_{n=0}^{N-1}\left\{\sum_{l=0}^{L-1}\sum_{k=0}^{M-1}\left(\Delta g^2\left[(n - lM)_N\right]\cos^2\left(\frac{2\pi}{M}k(n - \frac{\alpha}{2})\right)\right)\right\}.\end{gathered}$$

Analogically,

$$\begin{gathered}\operatorname{tr}\left(\Im\left\{\mathbf{\Delta U}_R\right\}\Im\left\{\left(\mathbf{\Delta U}_R\right)^T\right\}\right) = \\ = \sum_{n=0}^{N-1}\left\{\sum_{l=0}^{L-1}\sum_{k=0}^{M-1}\left(\Delta g^2\left[(n - lM)_N\right]\sin^2\left(\frac{2\pi}{M}k(n - \frac{\alpha}{2})\right)\right)\right\},\end{gathered}$$

thus

$$\begin{gathered}\operatorname{tr}\left(\Re\left\{\mathbf{\Delta U}_R\right\}\Re\left\{\left(\mathbf{\Delta U}_R\right)^T\right\}\right) + \operatorname{tr}\left(\Re\left\{\mathbf{\Delta U}_I\right\}\Re\left\{\left(\mathbf{\Delta U}_I\right)^T\right\}\right) = \\ = M\sum_{n=0}^{N-1}\sum_{l=0}^{L-1}\Delta g^2\left[(n - lM)_N\right],\end{gathered}$$

and

$$\begin{gathered}\operatorname{tr}\left(\Im\left\{\mathbf{\Delta U}_R\right\}\Im\left\{\left(\mathbf{\Delta U}_R\right)^T\right\}\right) + \operatorname{tr}\left(\Im\left\{\mathbf{\Delta U}_I\right\}\Im\left\{\left(\mathbf{\Delta U}_I\right)^T\right\}\right) = \\ = M\sum_{n=0}^{N-1}\sum_{l=0}^{L-1}\Delta g^2\left[\left(n - lM + \frac{M}{2}\right)_N\right].\end{gathered}$$

Notice, that functions $g_1[n]$ and $g_2[n]$ are N-periodical. For that reason function $\Delta g[n]$ keeps this property. Consequently, the shift of $\Delta g^2[n]$ on any integer number of samples in the sum $\sum_{n=0}^{N-1} \Delta g^2[n]$ will not change it. Thus

$$\sum_{n=0}^{N-1} \sum_{l=0}^{L-1} \Delta g^2 \left[(n - lM)_N\right] = L \sum_{n=0}^{N-1} \Delta g^2[n],$$

$$\sum_{n=0}^{N-1} \sum_{l=0}^{L-1} \Delta g^2 \left[\left(n - lM + \frac{M}{2}\right)_N\right] = L \sum_{n=0}^{N-1} \Delta g^2[n].$$

From here it directly follows, that

$$\left\|\mathbf{U^1} - \mathbf{U^2}\right\|_E^2 = 2N \sum_{n=0}^{N-1} \Delta g^2[n].$$

It means that the theorem is proved. □

3.2.2 Main Extremal Problem

Unfortunately, to construct the wanted optimal bases it is not enough just to minimize the difference norm between the two bases because the orthogonality property is required in practice. Therefore, it is necessary to solve the extremal problem with an orthogonality constraint:

Problem 3.1 (Main)**.** *On the subset $\mathcal{U} = \{\mathbf{U} \in M_{N,2N}(\mathbb{C}) : \Re(\mathbf{U}^*\mathbf{U}) = \mathbf{I}_{2N}\}$ of complex rectangular matrix of size $N \times 2N$ with the orthogonality condition $\Re(\mathbf{U}^*\mathbf{U}) = \mathbf{I}_{2N}$ it is necessary to find optimal matrix $\mathbf{U}_{opt}$ which minimises the difference norm*

$$\mathbf{U}_{opt} : \min_{\mathbf{U} \in \mathcal{U}} \|\mathbf{G} - \mathbf{U}\|_E^2, \tag{3.17}$$

where $\mathbf{G} \in M_{N,2N}$ is the matrix of initial non-orthogonal basis.

This problem can be solved theoretically with the help of SVD [88]. The real matrix of optimal basis is calculated from the SVD components of real matrix $\mathbf{G_B}$:

$$\mathbf{U}_{\mathbf{B},\mathrm{opt}} = \mathbf{S}\mathbf{W}^T,$$

where $\mathbf{G_B} = \mathbf{V}\mathbf{\Sigma}\mathbf{W}^T$. Matrices $\mathbf{S}$ and $\mathbf{S}$ are real, square and orthogonal. They are composed from the eigenvalues of matrices $\mathbf{G_B}\mathbf{G_B}^T$ and $\mathbf{G_B}^T\mathbf{G_B}$ correspondingly. Matrix $\mathbf{\Sigma}$ is square diagonal and contains singular values of matrix $\mathbf{G_B}$ in a descending order on the main diagonal. The complex matrix $\mathbf{U}_{\mathrm{opt}}$ of optimal MWH basis can be found from upper $\mathbf{V}_{1,\mathrm{opt}}$ and lower blocks $\mathbf{U}_{2,\mathrm{opt}}$ of matrix $\mathbf{U}_{\mathbf{B},\mathrm{opt}}$:

$$\mathbf{U}_{\mathrm{opt}} = \mathbf{V}_{1,\mathrm{opt}} + \jmath \mathbf{V}_{2,\mathrm{opt}},$$

where $\mathbf{U}_{\mathbf{B},\mathrm{opt}} = \begin{bmatrix} \mathbf{V}_{1,\mathrm{opt}} \\ \mathbf{V}_{2,\mathrm{opt}} \end{bmatrix}$. Moreover the optimal initializing function $g_{\mathrm{opt}}[n]$ of the MWH basis is defined by the first column of matrix $\mathbf{U}_{\mathrm{opt}}$, i.e.

$$g_{\mathrm{opt}}[n] = \mathbf{U}_{\mathrm{opt}}[n, 1].$$

A computer representation of this algorithm can be found in the Appendix to the thesis.

The actual value of the minimum in problem (3.17) can also be found exactly and equals

$$\min_{\mathbf{U}\in\mathcal{U}} \|\mathbf{G}-\mathbf{U}\|_E^2 = \|\mathbf{G}-\mathbf{U}_{\mathrm{opt}}\|_E^2 =$$
$$= \|\mathbf{G}_{\mathbf{B}}-\mathbf{U}_{\mathbf{B},\mathrm{opt}}\|_E^2 = 2N\sum_{n=0}^{N-1}\Delta g^2\left[n\right] = \sum_{i=1}^{2N}(\sigma_i-1)^2, \tag{3.18}$$

where $\Delta g[n]$ is the difference between the given initializing function and found optimal $g_{\mathrm{opt}}[n]$; σ_i are the singular values of matrix $\mathbf{G}_{\mathbf{B}}$.

3.3 Conclusions of MWH Basis Synthesis

The study conducted in this chapter allows the formulation of several important conclusions.

It is necessary to have a construction algorithm of the functions from space $\mathbb{C}^N$ for given continuous functions. For symmetrical functions, the optimal presentation has been found in the direct form. However, it is not enough just to orthogonalize the given function to construct an optimal MWH basis because required localization properties can be lost. A special extremal problem should be posed. This problem has an analytical solution. The disadvantage is that the basis construction algorithm, which has SVD in its core, is not optimal from the computational point of view and should be modified

Additionally, the MWH basis can be presented in a matrix form. The distance between any two such bases can be estimated by the distance between its initializing functions.

4 PARAMETERS OPTIMIZATION

Even though the initializing function $g_{\text{opt}}[n]$ found in the previous chapter is the solution of the optimization problem (3.17), there are several ways to adjust the parameters of the MWH basis, which define its localization on the time-frequency plane. Firstly, one could use the phase parameter α of the basis (2.24),(2.25). Secondly, it is the sampling frequency of the initial function which is used for basis synthesis. These problems are studied in this chapter.

4.1 Phase Parameter

It is possible to make the optimal orthogonal MWH bases even closer to the initial non-orthogonal bases with a desired localization, i.e. to reduce additionally the norm (3.2.2) between matrices $\mathbf{G}$ and $\mathbf{U}_{\text{opt}}$. This can be achieved because the phase parameter α of the basis can be considered as the subject for extra optimization. It means that the main extremal problem depends on α as on a parameter:

$$F(\alpha) = \min_{\mathbf{U_B} \in \mathcal{U}_{\mathcal{B}}} \|\mathbf{G_B}(\alpha) - \mathbf{U_B}\|_E^2 = \|\mathbf{G_B}(\alpha) - \mathbf{U_{B,opt}}\|_E^2.$$

Thus it is possible to reduce additionally the value of the norm by the optimal choice of α. To do that, it is necessary to solve a new extremal problem

$$F(\alpha) \rightarrow \min_{\alpha \in \mathbb{R}},$$

or in the equivalent form

$$\alpha_{opt} : \quad \min_{\alpha} \left(\|\mathbf{G_B} - \mathbf{U_{B,opt}}\|_E^2 \right). \tag{4.1}$$

Unfortunately, this problem is too complicated for analytical solution, but the introduction of the following norm can solve this problem:

$$X(\alpha) = \|\mathbf{G_B}(\alpha)\mathbf{G_B}^T(\alpha) - \mathbf{I}\|_E^2. \tag{4.2}$$

Theorem 4.1. *Norm $X(\alpha)$ majorizes the norm $F(\alpha)$, i.e. for all $\alpha \in \mathbb{R}$*

$$\|\mathbf{G_B}(\alpha) - \mathbf{U_{B,opt}}\|_E^2 \leq \|\mathbf{G_B}(\alpha)\mathbf{G_B}^T(\alpha) - \mathbf{I}\|_E^2.$$

Proof. Let's modify expression (4.2), skipping α in the derivation for the sake of simplicity

$$\left\|\mathbf{G_B}(\alpha)\,\mathbf{G_B}^T(\alpha) - \mathbf{I}\right\|_E^2 = \left\|\mathbf{G_B}\mathbf{G_B}^T - \mathbf{I}\right\|_E^2 = \operatorname{tr}\left[\left(\mathbf{G_B}\mathbf{G_B}^T - \mathbf{I}\right)\left(\mathbf{G_B}\mathbf{G_B}^T - \mathbf{I}\right)^T\right] =$$
$$= \operatorname{tr}\left[\left(\mathbf{G_B}\mathbf{G_B}^T - \mathbf{I}\right)\left(\mathbf{G_B}\mathbf{G_B}^T - \mathbf{I}\right)\right] = \operatorname{tr}\left[\mathbf{G_B}\mathbf{G_B}^T\mathbf{G_B}\mathbf{G_B}^T - 2\mathbf{G_B}\mathbf{G_B}^T + \mathbf{I}\right].$$

Taking into account the SVD of matrix $\mathbf{G_B}$ and the orthogonality of matrices $\mathbf{S}$ and $\mathbf{W}$ one tends up with

$$\operatorname{tr}\left[\mathbf{G_B}\mathbf{G_B}^T\right] = \operatorname{tr}\left[\mathbf{S\Sigma W^T W \Sigma S^T}\right] = \operatorname{tr}\left[\mathbf{\Sigma I \Sigma S^T S}\right] = \operatorname{tr}\left[\mathbf{\Sigma}^2\right],$$
$$\operatorname{tr}\left[\mathbf{G_B}\mathbf{G_B}^T\mathbf{G_B}\mathbf{G_B}^T\right] = \operatorname{tr}\left[\mathbf{S\Sigma W}^T\mathbf{W\Sigma S}^T\mathbf{S\Sigma W}^T\mathbf{W\Sigma S}^T\right] = \operatorname{tr}\left[\mathbf{\Sigma}^4\right].$$

Consequently

$$\left\|\mathbf{G_B}\cdot\mathbf{G_B}^T - \mathbf{I}\right\|_E^2 = \operatorname{tr}\left[\mathbf{\Sigma}^4 - 2\mathbf{\Sigma}^4 + \mathbf{I}\right] =$$
$$= \sum_{i=1}^{2N}\left((\sigma_i(\alpha))^4 - 2(\sigma_i(\alpha))^2 + 1\right) = \sum_{i=1}^{2N}\left((\sigma_i(\alpha))^2 - 1\right)^2.$$

Now it is possible to consider the difference between the old (3.2.2) and the new norm (4.2):

$$\|\mathbf{G_B}(\alpha) - \mathbf{U_{B,opt}}\|_E^2 - \left\|\mathbf{G_B}(\alpha)\,\mathbf{G_B}^T(\alpha) - \mathbf{I}\right\|_E^2 =$$
$$= \sum_{i=1}^{2N}\left((\sigma_i(\alpha))^2 - 1\right)^2 - \sum_{i=1}^{2N}(\sigma_i(\alpha) - 1)^2 =$$
$$= \sum_{i=1}^{2N}\left[\left((\sigma_i(\alpha))^2 - 1\right)^2 - (\sigma_i(\alpha) - 1)^2\right] = \sum_{i=1}^{2N}\sigma_i(\sigma_i - 1)\left(\sigma_i^2 + \sigma_i - 2\right).$$

For the reason that singular numbers of matrix $\mathbf{G_B}$ are non-negative ($\sigma_i \geq 0$)

$$\text{if } \sigma_i \in [0,1]: \quad \sigma_i \geq 0,\ (\sigma_i - 1) \leq 0,\ \left(\sigma_i^2 + \sigma_i - 2\right) \leq 0;$$
$$\text{if } \sigma_i \in [1,+\infty): \quad \sigma_i \geq 0,\ (\sigma_i - 1) \geq 0,\ \left(\sigma_i^2 + \sigma_i - 2\right) \geq 0.$$

Therefore,

$$\|\mathbf{G_B}(\alpha) - \mathbf{U_{B,opt}}\|_E^2 - \left\|\mathbf{G_B}(\alpha)\cdot\mathbf{G_B}^T(\alpha) - \mathbf{I}\right\|_E^2 \geq 0,$$

which was to be proved. □

Thus, instead of the problem $F(\alpha) \to \min_{\alpha\in\mathbb{R}}$ the problem $X(\alpha) \to \min_{\alpha\in\mathbb{R}}$ can be considered. It can be solved analytically for symmetrical functions from $\mathbb{C}^N$.

4.1.1 Phase Parameter Derivation

Theorem 4.1 proved in the previous subsection allows the transform of the problem (4.1) into the problem

$$\alpha_{\text{opt}}: \quad \min_{\alpha\in\mathbb{R}} \left(\|\mathbf{G_B}\mathbf{G_B}^T - \mathbf{I}\|_E^2 \right) \tag{4.3}$$

The solution of this optimization problem can be formulated in the form of the theorem:

Theorem 4.2. *Let function $g[n] \in \mathbb{C}^N$ be $(N-1)$-symmetrical and be used as the initialization function of MWH basis $\mathcal{B}[J_N]$. Then the best localization of the basis upon the criteria*

$$\|\mathbf{G}(\alpha) - \mathbf{U_{opt}}\|_E^2 \to \min_{\alpha\in\mathbb{R}} (\min_{\mathbf{U}\in\mathcal{U}})$$

is achieved for the following optimal values

$$\alpha_{opt} = \frac{M}{2} - 1 + q\frac{M}{2}, \quad q \in \mathbb{Z}. \tag{4.4}$$

Proof. For the sake of simplicity and compactness here it will be presented only the scheme of the proof. Some of the longest derivations will be omitted.

Notice that

$$\|\mathbf{G_B}\mathbf{G_B}^T - \mathbf{I}\|_E^2 = \operatorname{tr}\left[\left(\mathbf{G_B}\mathbf{G_B}^T\right)^2\right] - 2\operatorname{tr}\left[\mathbf{G_B}\mathbf{G_B}^T\right] + 2N.$$

$2N$ does not depend on α. For that reason the problem 4.3 can be rewritten like

$$\operatorname{tr}\left[\left(\mathbf{G_B}\mathbf{G_B}^T\right)^2\right] - 2\operatorname{tr}\left[\mathbf{G_B}\mathbf{G_B}^T\right] \to \min \tag{4.5}$$

It overcomes that $\operatorname{tr}\left[\mathbf{G_B}\mathbf{G_B}^T\right]$ also does not depend on α. Thus, the optimization problem deduces to

$$\operatorname{tr}\left[\left(\mathbf{B}(\alpha)^2\right)\right] \to \min, \tag{4.6}$$

where $\mathbf{B} \triangleq \mathbf{G_B}\mathbf{G_B}^T$.

It is possible to prove that permutation of any two rows in matrix $\mathbf{G_B}$ will not change $\operatorname{tr}\left[\left(\mathbf{G_B}\mathbf{G_B}^T\right)^2\right]$.

The block structure of matrix $\mathbf{G_B}$ is given by (3.13). Let's change it so that, firstly, the row from block $[\Re(\mathbf{G}_R)\,\Re(\mathbf{G}_I)]$ goes, then the row from block $[(\Im\mathbf{G}_R)\,(\Im\mathbf{G}_I)]$, after that next row from $[(\Re\mathbf{G}_R)\,\Re(\mathbf{G}_I)]$ and so on. The matrix received in such a way will be denoted as $\tilde{\mathbf{G}}_\mathbf{B}$. Correspondingly, $\tilde{\mathbf{B}} = \tilde{\mathbf{G}}_\mathbf{B}\tilde{\mathbf{G}}_\mathbf{B}^T$.

According to the previous statement

$$\operatorname{tr}\left[(\tilde{\mathbf{G_B}}\tilde{\mathbf{G_B}}^T)\right] = \operatorname{tr}\left[\tilde{\mathbf{B}}\right] = \operatorname{tr}\left[(\mathbf{G_B}\mathbf{G_B}^T)\right] = \operatorname{tr}\left[\mathbf{B}\right].$$

Therefore in further derivation matrix $\tilde{\mathbf{B}}$ will be used instead of $\mathbf{B}$. At that, problem (4.6) can be changed with the problem

$$\mathrm{tr}\left[\left(\tilde{\mathbf{G}}_B\tilde{\mathbf{G}}_B^T\right)^2\right] = tr\left[\tilde{\mathbf{B}}^2\right] \xrightarrow{\alpha} \min \tag{4.7}$$

Let's decompose matrix $\tilde{\mathbf{B}}$ in the following way:

$$\tilde{\mathbf{B}} = \sum_{m=0}^{M-1}\mathbf{B}_m = \sum_{m=0}^{M-1}\left(\tilde{\mathbf{U}}\left(\alpha\right)\right)^m\mathbf{A}\left(\tilde{\mathbf{U}}^T\left(\alpha\right)\right)^m, \tag{4.8}$$

where it can be checked directly that matrix $\mathbf{A}$ is symmetrical and has size $(2N \times 2N)$. It does not depend on α and has the following structure:

$$\mathbf{A} = \sum_{l=0}^{L}\mathbf{G}_l\mathbf{G}_l^T.$$

$\mathbf{G}_l$ are block matrices of size $(2N \times 2)$:

$$\mathbf{G}_l = \begin{bmatrix}\mathbf{G}_l^{(1)} & \mathbf{G}_l^{(2)} & ... & \mathbf{G}_l^{(N)}\end{bmatrix}^T,$$

where $\mathbf{G}_l^{(n)}$ are diagonal matrices of size (2×2):

$$\mathbf{G}_l^{(n)} = \begin{bmatrix}\mathbf{G}_{l,R}^{(n)} & 0 \\ 0 & \mathbf{G}_{l,I}^{(n)}\end{bmatrix}.$$

Thus

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_{11} & ... & \mathbf{A}_{1N} \\ ... & ... & ... \\ \mathbf{A}_{N1} & ... & \mathbf{A}_{NN}\end{bmatrix},$$

where $\mathbf{A}_{k,n} = \sum\limits_{l=0}^{L-1}\mathbf{G}_l^{(k)}\mathbf{G}_l^{(n)}$ are matrices of size (2×2), or in more details:

$$\mathbf{A}_{k,n} = \begin{bmatrix}\sum\limits_{l=0}^{L-1}\mathbf{G}_{l,R}^{(k)}\mathbf{G}_{l,R}^{(n)} & 0 \\ 0 & \sum\limits_{l=0}^{L-1}\mathbf{G}_{l,I}^{(k)}\mathbf{G}_{l,I}^{(n)}\end{bmatrix},$$

where $\mathbf{G}_{l,R}^{(n+1)} = g\left[(n-lM)_N\right]$, $\mathbf{G}_{l,I}^{(n+1)} = g\left[\left(n-lM+\frac{M}{2}\right)_N\right]$ and $n \in J_N$. Notice, that matrices $\mathbf{A}_{k,n}$ are symmetrical, i.e. $\mathbf{A}_{k,n} = \mathbf{A}_{n,k}$.

Matrix $\tilde{\mathbf{U}}\left(\alpha\right)$ is block-diagonal of size $(2N \times 2N)$:

$$\tilde{\mathbf{U}}\left(\alpha\right) = \begin{bmatrix}\mathbf{U}_1\left(\alpha\right) & 0 & ... & 0 \\ 0 & \mathbf{U}_2\left(\alpha\right) & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & \mathbf{U}_N\left(\alpha\right)\end{bmatrix},$$

which has matrices of rotation of size (2×2) on its diagonal

$$\mathbf{U}_{n+1}(\alpha) = \begin{bmatrix} \cos\left(\frac{2\pi}{M}\left(n - \frac{\alpha}{2}\right)\right) & -\sin\left(\frac{2\pi}{M}\left(n - \frac{\alpha}{2}\right)\right) \\ \sin\left(\frac{2\pi}{M}\left(n - \frac{\alpha}{2}\right)\right) & \cos\left(\frac{2\pi}{M}\left(n - \frac{\alpha}{2}\right)\right) \end{bmatrix},$$

Finally, the structure of matrix $\mathbf{B}_m$ looks like

$$\mathbf{B}_m = \begin{bmatrix} (\mathbf{U}_1)^m \mathbf{A}_{1,1} \left(\mathbf{U}_1^T\right)^m & (\mathbf{U}_1)^m \mathbf{A}_{1,2} \left(\mathbf{U}_2^T\right)^m & \dots & (\mathbf{U}_1)^m \mathbf{A}_{1,N} \left(\mathbf{U}_N^T\right)^m \\ (\mathbf{U}_2)^m \mathbf{A}_{2,1} \left(\mathbf{U}_1^T\right)^m & (\mathbf{U}_2)^m \mathbf{A}_{2,2} \left(\mathbf{U}_2^T\right)^m & \dots & (\mathbf{U}_2)^m \mathbf{A}_{2,N} \left(\mathbf{U}_N^T\right)^m \\ \dots & \dots & \dots & \dots \\ (\mathbf{U}_N)^m \mathbf{A}_{N,1} \left(\mathbf{U}_1^T\right)^m & (\mathbf{U}_N)^m \mathbf{A}_{N,2} \left(\mathbf{U}_2^T\right)^m & \dots & (\mathbf{U}_N)^m \mathbf{A}_{N,N} \left(\mathbf{U}_N^T\right)^m \end{bmatrix}$$

Using these notations and decomposition (4.8) one gets that

$$\operatorname{tr}\left[\tilde{\mathbf{B}}^2\right] = \operatorname{tr}\left[\left(\sum_{m=0}^{M-1} \mathbf{B}_m\right)^2\right] = \sum_{m=0}^{M-1} \operatorname{tr}\left[(\mathbf{B}_m)^2\right] + \sum_{m_1=0}^{M-1} \sum_{m_2=0, m_1 \neq m_2}^{M-1} \operatorname{tr}\left[(\mathbf{B}_{m_1} \mathbf{B}_{m_2})\right].$$

$\sum\limits_{m=0}^{M-1} tr\left((\mathbf{B}_m)^2\right)$ does not depend on α. Therefore minimisation problem (4.7) again can be reduced to

$$\sum_{m_1=0}^{M-1} \sum_{m_2=0\; m_1 \neq m_2}^{M-1} tr\left[\mathbf{B}_{m_1} \mathbf{B}_{m_2}\right] \xrightarrow{\alpha} \min. \tag{4.9}$$

Thus it is necessary to consider the diagonal elements of matrices $[\mathbf{B}_{m_1} \mathbf{B}_{m_2}]$:

$$[\mathbf{B}_{m_1} \mathbf{B}_{m_2}](n, n) = \sum_{k=1}^{N} (\mathbf{U}_n)^{m_1} \mathbf{A}_{n,k} \left(\mathbf{U}_k^T\right)^{m_1} (\mathbf{U}_k)^{m_2} \mathbf{A}_{k,n} \left(\mathbf{U}_n^T\right)^{m_2}.$$

Consequently,

$$\operatorname{tr}\left[\mathbf{B}_{m_1} \mathbf{B}_{m_2}\right] = \sum_{n=1}^{N} \operatorname{tr}\left[[\mathbf{B}_{m_1} \mathbf{B}_{m_2}](n, n)\right] =$$

$$= \sum_{n=1}^{N} \sum_{k=1}^{N} tr\left[\mathbf{A}_{n,k} \left(\mathbf{U}_k^T\right)^{m_1} (\mathbf{U}_k)^{m_2} \mathbf{A}_{k,n} \left(\mathbf{U}_n^T\right)^{m_2} (\mathbf{U}_n)^{m_1}\right].$$

Matrices $\mathbf{U}_n$ are the matrices of rotation and for that reason $\mathbf{U}_n^T = \mathbf{U}_n^{-1}$, so

$$\operatorname{tr}\left[\mathbf{B}_{m_1} \mathbf{B}_{m_2}\right] = \sum_{n=1}^{N} \sum_{k=1}^{N} \operatorname{tr}\left[\mathbf{A}_{n,k} (\mathbf{U}_k)^{m_2 - m_1} \mathbf{A}_{k,n} (\mathbf{U}_n)^{m_1 - m_2}\right] =$$
$$= \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \operatorname{tr}\left[\mathbf{A}_{n+1,k+1} (\mathbf{U}_{k+1})^{m_2 - m_1} \mathbf{A}_{k+1,n+1} (\mathbf{U}_{n+1})^{m_1 - m_2}\right].$$

It can be shown that

$$\operatorname{tr}\left[\mathbf{A}_{n+1,k+1} (\mathbf{U}_{k+1})^{m_2 - m_1} \mathbf{A}_{k+1,n+1} (\mathbf{U}_{n+1})^{m_1 - m_2}\right] =$$

$$= \Gamma_{k,n}^2 \cos\left(\Delta m \frac{2\pi}{M}(k + n - \alpha)\right) + X_{k,n}^2 \cos\left(\Delta m \frac{2\pi}{M}(k - n)\right),$$

where

$$\Gamma^2_{k,n} \triangleq \frac{1}{2}\left(\left(\sum_{l=0}^{L-1} \mathbf{G}^{(k+1)}_{l,R}\mathbf{G}^{(n+1)}_{l,R}\right) - \left(\sum_{l=0}^{L-1} \mathbf{G}^{(k+1)}_{l,I}\mathbf{G}^{(n+1)}_{l,I}\right)\right)^2,$$

$$X^2_{k,n} \triangleq \frac{1}{2}\left(\left(\sum_{l=0}^{L-1} \mathbf{G}^{(k+1)}_{l,R}\mathbf{G}^{(n+1)}_{l,R}\right) + \left(\sum_{l=0}^{L-1} \mathbf{G}^{(k+1)}_{l,I}\mathbf{G}^{(n+1)}_{l,I}\right)\right)^2$$

and $\Delta m \triangleq (m_1 - m_2)$.

The summand $X^2_{k,n} \cos\left(\Delta m \frac{2\pi}{M}(k-n)\right)$ does not depend on α and can be excluded from the minimisation problem (4.9), which becomes

$$f(\alpha) = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\left\{\Gamma^2_{n,k}\left[\sum_{m_1=0}^{M-1}\ \sum_{m_2=0;m_1\neq m_2}^{M-1} \cos\left[(\Delta m)\frac{2\pi}{M}(k+n-\alpha)\right]\right]\right\} \xrightarrow{\alpha} \min.$$

The necessary condition of minimum for function $f(\alpha)$ is the equality of its derivative $f'(\alpha)$ to zero. This condition is fulfilled, if

$$\sum_{m_1=0}^{M-1}\ \sum_{m_2=0;m_1\neq m_2}^{M-1}\left\{(\Delta m)\frac{2\pi}{M}\left[\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\left[\Gamma^2_{n,k}\sin\left((\Delta m)\frac{2\pi}{M}(k+n-\alpha)\right)\right]\right]\right\} = 0.$$

It is enough to consider only internal sums over k and k for all fixed $m_1 = 0, 1, ..., M-1;\ m_2 = 0, 1, ..., M-1;\ m_1 \neq m_2$, i.e. for all fixed $\Delta m = 1, 2, ..., M-1$:

$$\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\left[\Gamma^2_{n,k}\sin\left((\Delta m)\frac{2\pi}{M}(k+n-\alpha)\right)\right].$$

This expression can be rewritten like

$$\sum_{p=0}^{N-1}\left[\sum_{n=p}^{N-1}\Gamma^2_p(n)\cdot\sin\left((\Delta m)\frac{2\pi}{M}(2n-p-\alpha)\right)\right], \tag{4.10}$$

where $p = n-k = 0, 1, \ldots, N-1$ and functions $\Gamma^2_p(n) = \Gamma^2_{n-p,n}$ are fully defined by initializing function $g[n]$.

After that it is necessary to take into account the symmetry of function $g[n]$ ($g[N-1-n] = g[n],\ n \in J_N$). In this case it overcomes that function $\Gamma^2_P(n)$ and $\sin\left((\Delta m)\frac{2\pi}{M}(2n-p-\alpha)\right)$ has the same period $\frac{M}{2}$. Moreover, when α is selected in the optimal way (4.4) these two components follow in opposite phase and makes the internal sum (4.10) equal to 0. Consequently $f'(\alpha) = 0$ for found values of α, what proves the theorem. □

An analogical study can be performed in the case where function $g[n] \in \mathbb{C}^N$ is N-symmetrical. The corresponding optimal values of phase parameter are

$$\alpha_{opt} = \frac{M}{2} + q\frac{M}{2}, \quad q \in \mathbb{Z}. \tag{4.11}$$

As it was shown in [PI], from simulations it follows that the optimal choice of phase parameter ensures that the constructed orthogonal initializing function keeps the initial symmetry. Moreover, a non-optimal choice of α results in big non-symmetrical side lobes, and the synthesized function is not that close to the original any more.

4.2 localization Control and Sampling of the Gaussian Function

Construction of an optimal initialization function for the MWH basis is accomplished by the orthogonalization of some given function. Thus the parameters of these functions can be considered as an additional tuning factor. In particular, if the Gaussian function is taken as the initial one for the synthesis algorithm, then its dispersion is the parameter which defines localization properties in time and frequency.

As it was mentioned in Chapter 2, the Gaussian function is particular in a sense that it turns the Heisenberg uncertainty relation into equality. Let's take the Gaussian function in the following form:

$$g_G \triangleq (2\sigma_0)^{\frac{1}{4}} e^{-\pi\sigma_0 t^2}, \tag{4.12}$$

where $t \in \mathbb{R}$.

It is easy to show that the energy of such function

$$\int_{-\infty}^{\infty} g_G^2(t)dt = (2\sigma_0)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-2\pi\sigma_0 t^2} dt = (2\sigma_0)^{\frac{1}{2}} \frac{\sqrt{\pi}}{\sqrt{2\pi\sigma_0}} = 1$$

is normalized.

The mean value of the signal equals 0. Its dispersion in the time domain (or "width")

$$\tau_s^2 = \sigma_\tau^2 = \int_{-\infty}^{\infty} t^2 (2\sigma_0)^{\frac{1}{2}} e^{-2\pi\sigma_0 t^2} dt = \frac{\sqrt{2\sigma_0\pi}}{2\left(\sqrt{2\pi\sigma_0}\right)^3} = \frac{1}{4\pi\sigma_0}.$$

Thus the "width" of the Gaussian can be expressed though its dispersion

$$\sigma_\tau = \frac{1}{\sqrt{4\pi\sigma_0}}.$$

In the frequency domain, function (4.12) has the form

$$\begin{aligned} G_G(\omega) &= (2\sigma_0)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-\pi\sigma_0 t^2 - \jmath\omega t} dt = (2\sigma_0)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-\pi\sigma_0\left(t^2 + \frac{\jmath\omega}{\pi\sigma_0} t\right)} dt = \\ &= (2\sigma_0)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-\pi\sigma_0\left(t + \frac{\jmath\omega}{\pi\sigma_0}\right)^2 - \frac{\omega^2}{4\pi\sigma_0}} dt = \\ &= (2\sigma_0)^{\frac{1}{4}} e^{\left(\frac{-\omega^2}{4\pi\sigma_0}\right)} \frac{\sqrt{\pi}}{\sqrt{\pi\sigma_0}} = \left(\frac{2}{\sigma_0}\right)^{\frac{1}{4}} e^{\left(\frac{-\omega^2}{4\pi\sigma_0}\right)}, \end{aligned}$$

$$\int_{-\infty}^{\infty} |G_G(\omega)|^2 d\omega = 2\pi.$$

The dispersion of $g_G(t)$ in the frequency domain (or its bandwidth) is

$$\triangle\omega_s^2 = \sigma_\omega^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 \left(\frac{2}{\sigma_0}\right)^{\frac{1}{2}} e^{\left(\frac{-\omega^2}{2\pi\sigma_0}\right)} = \left(\frac{2}{\sigma_0}\right)^{\frac{1}{2}} \sqrt{pi} \frac{(2\pi\sigma_0)^{\frac{3}{2}}}{4\pi} = \pi\sigma_0$$

and

$$\sigma_\omega = \sqrt{\pi}\sqrt{\sigma_0}.$$

It can be easily seen that function (4.12) has the best possible localization. Indeed, taking the Heisenberg uncertainty relation, one gets

$$\sigma_\omega \sigma_\tau = \frac{1}{\sqrt{4\pi}} \frac{1}{\sqrt{\sigma_0}} \sqrt{\pi \sigma_0} = \frac{1}{2}.$$

With the introduction of

$$\sigma_f = \frac{\sigma_\omega}{2\pi} = \frac{\sqrt{\sigma_0}}{2\sqrt{\pi}}$$

The Gaussian function can be rewritten like

$$g_G(t) = (2\pi)^{\frac{1}{4}} \sqrt{\sigma_\omega} e^{-\sigma_\omega^2 t^2} = 2^{\frac{3}{4}} \pi^{\frac{1}{4}} \sqrt{\sigma_f} e^{-4\pi^2 \sigma_f^2 t^2}.$$

For the construction of a discrete MWH, it is necessary to use a discrete N-periodical approximation of the Gaussian function. First of all, it is necessary to sample function $g_G(t)$ with the sampling interval

$$\Lambda = \frac{1}{MK\sigma_f} = \frac{T}{M},$$

where $T = \frac{1}{K\sigma_f}$, M is the number of sub-carriers, K - is the scaling sampling coefficient. Thus sampling frequency

$$f_d = \frac{1}{\Lambda} = KM\sigma_f.$$

The sampled Gaussian function is

$$\begin{aligned} g_G[n] = g_G(\Lambda n) = 2^{\frac{3}{4}} \pi^{\frac{1}{4}} \sqrt{\sigma_f} e^{-4\pi\sigma_f^2 \frac{n^2}{M^2 K^2 \sigma_f^2}} = \\ = 2^{\frac{3}{4}} \pi^{\frac{1}{4}} \sqrt{\sigma_f} e^{-\left(\frac{2\pi n}{MK}\right)^2}. \end{aligned}$$

In so-called isotropic case, the resolutions in the frequency and time domains are equivalent, i.e. $\sigma_\tau = \sigma_f = \frac{1}{\sqrt{4\pi}}$, thus

$$g_G^{\text{iso}}[n] = 2^{\frac{1}{4}} e^{-\left(\frac{2\pi n}{MK}\right)^2}.$$

This function is symmetrical because

$$g_G^{\text{iso}}[-n] = g_G^{\text{iso}}[n].$$

It means that its N-periodical approximation $g_0[n]$ can be found according to the results of the previous chapter:

$$\begin{aligned} g_0[n] &= \frac{1}{\sqrt{A}} \left(g_G^{\text{iso}}[n] + g_G^{\text{iso}}[n] \right), \quad n = 1, 2, \ldots, (N-1), \\ g_0[0] &= \frac{1}{\sqrt{A}} \left(g_G^{\text{iso}}[0] + g_G^{\text{iso}}[N] \right). \end{aligned}$$

The value of K, which defines the sampling frequency, is the parameter of this function. Thus it can be used to minimize the norm of difference between the initial basis constructed from $g_0[n]$ and the optimal orthogonal MWH basis. It was shown in Chapter 2 that this norm is defined by the difference between the initializing functions of the bases. The optimal value of K can be found as the solution of the following extremal problem:

$$K_{\text{opt}} : \min_{K \in \mathbb{R}^+} \left(\|g_0[n](K) - g_{\text{ort}}[n]\|_E^2 \right), \tag{4.13}$$

where $g_{\text{ort}}[n]$ is the initializing function of the optimal orthogonal MWH basis.

This problem can be solved numerically. These results together with some other simulation results will be discussed in the next section.

4.3 Simulation Results

All simulations in this thesis were performed with the help of Matlab. The realization of the main algorithms can be found at the end of the dissertation, in Appendix 1. In the following subsections, it will be shown how optimization can influence and improve the localization characteristics of the MWH basis.

4.3.1 Sampling Parameter K

A number of simulations for different values of K were introduced in the previous section. Some of these results are presented in Figure 9.

It was decided to solve problem (4.13) numerically, i.e. with the help of Matlab's internal function `fminsearch`. It turns out (Figure 10) that the value of K is constant in a wide range of main parameters M and L. This means that the found value $K = 2.5066$ should be taken for the sampling of the Gaussian function to keep the good localization of the constructed orthogonal optimal function $g_{\text{ort}}[n]$. This value will be used for the simulations presented in the subsection below.

4.3.2 localization and Comparison with OFDM

There are many ways to characterize the localization of the function in the time or frequency domain, starting from it dispersion, which can be measured with the help of an ambiguity function

$$A_g(\tau, \gamma) = \int_{-\infty}^{\infty} g(t + \frac{\tau}{2}) g(t - \frac{\tau}{2}) e^{-\jmath\gamma t} dt.$$

In the frequency domain, after the utilization of Parseval's theorem this function is like

$$A_g(\tau, \gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega + \frac{\gamma}{2}) G(\omega - \frac{\gamma}{2}) e^{-\jmath\tau\omega} dt.$$

Thus the shape of the ambiguity function is defined by the spreading of function $g(t)$ in the time domain and its FT $G(\omega)$ in the frequency domain.

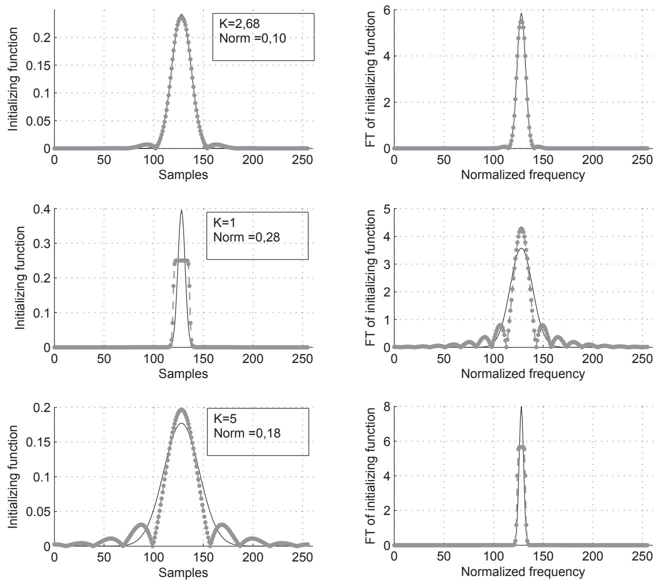


FIGURE 9 The localization quality depending on different values of K (black curves indicate discrete Gaussian function, dotted grey curves show constructed orthogonal initializing functions).

Figure 11 demonstrates the ambiguity function of the optimal initializing function of the MWH bases constructed by the orthogonalization of the Gaussian function.

Other popular figures that characterize localization are:

– Mean-square frequency band $\Delta\omega$, which is defined from the equality

$$\frac{|G\left(\frac{\Delta\omega}{2}\right)|^2}{|G(0)|^2} = \frac{1}{2}.$$

– Maximal amplitude A of the first side-lobe, which is achieved in the points $\pm\omega_0$:

$$A = 10\log_{10}\frac{|G(\omega_0)|^2}{|G(0)|^2}.$$

– The power of polynomial function, which characterizes the decay rate of $G(\omega)$ on high frequencies

$$|G(\omega)| = O(\omega^{-p-1}).$$

– The uncertainty constant $H \triangleq \sigma_t\sigma_\omega$ defined by the Heisenberg uncertainty theorem.

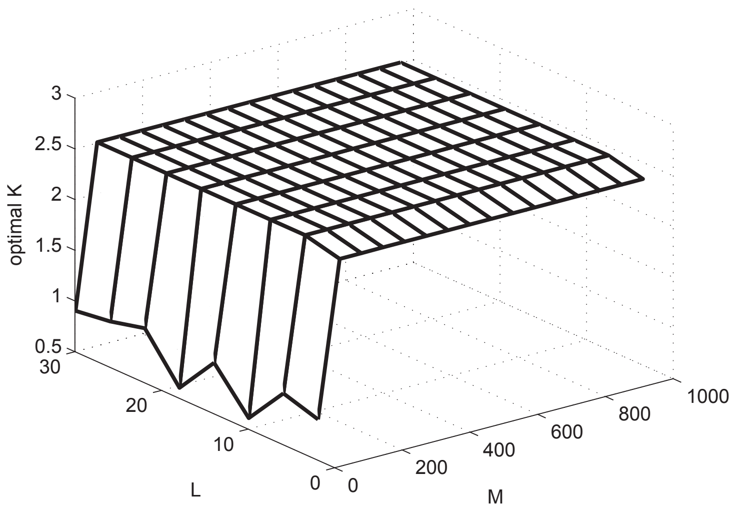


FIGURE 10 Dependence of optimal values of K on M and L.

In the case of discrete function $g[n] \in \mathbb{C}^N$ dispersion in time and frequency domains can be calculated in the following way:

$$\sigma_t^2 = \frac{1}{\|g[n]\|_E^2}\frac{1}{N}\sum_{n=0}^{N-1}(n-\tilde{n})^2|g[n]|^2, \quad \tilde{n} = \frac{1}{\|g[n]\|_E^2}\sum_{n=0}^{N-1} n|g[n]|^2;$$
$$\sigma_\omega^2 = \frac{1}{\|G[k]\|_E^2}\frac{1}{4\pi^2 N}\sum_{k=0}^{N-1}(k-\tilde{k})^2|G[k]|^2, \quad \tilde{k} = \frac{1}{\|G[k]\|_E^2}\sum_{n=0}^{N-1} k|G[k]|^2,$$

where $G[k]$ is DFT of $g[n]$.

Hereby is the comparison of H values for used/constructed functions and several known wavelets:

1. For sampled Gaussian function on the interval $[-N; N]$, where $M = 1024$, $L = 10$, $N = ML$ and sampling parameter $K = 2.5066$ $H = 0.5000$ with high precision.
2. Constructed N-periodical approximation of the Gaussian function does not reduce the quality of localization, because again $H = 0.5000$.
3. In the result of optimal orthogonalization localization decreases but not considerably: for $g_{\text{opt}}[n]$ $H = 0.5118$
4. A rectangular impulse on width M has much worse localization: $H = 15.3831$.
5. Uncertainty constants for several popular wavelets:
   (a) Daubechies wavelets: $H = 0.635$
   (b) Kravchnko wavelets ($a = 4$): $H = 0.99270$
   (c) Meyer wavelets: $H = 1.1149$

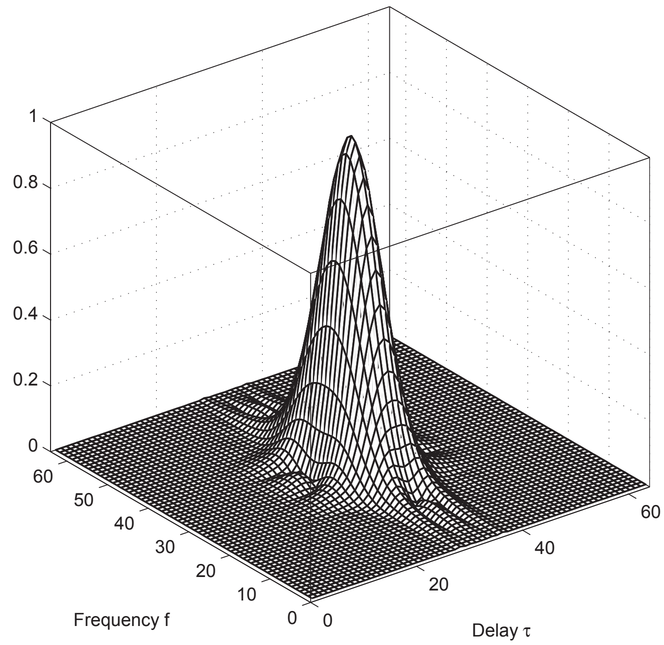


FIGURE 11 The ambiguity function of the optimal initializing function of MWH basis.

(d) Kotelnikov-Shenon wavelets: $H = 1.6854$.

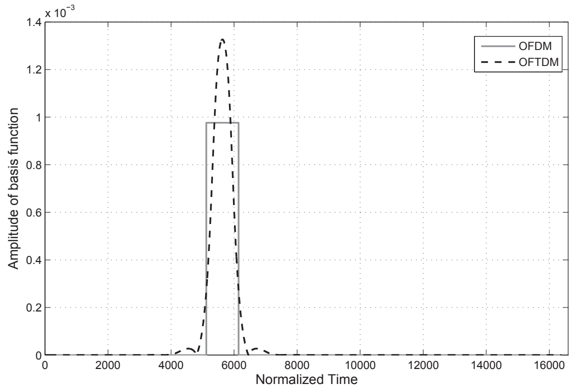
Finally, Figure 12 presents the comparison of the time-frequency localization of rectangular impulse used in the wg basis of OFDM and the optimal initialization function of MWH for OFTDM.

4.4 Conclusions of Parameters Optimization

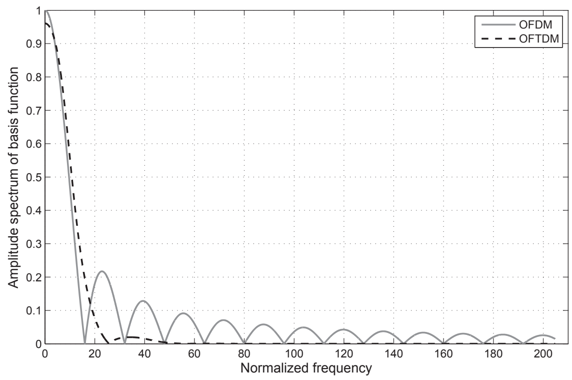
In this chapter, it was shown that the value of the phase parameter has a considerable influence on the quality of basis localization. Unfortunately, in the most general case of initializing function, the problem of analytical optimal α search is too complicated. However, in two particular but common cases of conjugated N and $(N-1)$ symmetry the optimal phase parameter values can be found exactly.

An other important parameter that can influence the localization of the basis on the construction phase is the sampling frequency. This effect is studied in the a particular case, when the initial function is Gaussian. The optimal value of sampling parameter K, which defines sampling frequency, was found from simulations.

Finally, the actual comparison of OFTDM with OFDM shows that MWH basis functions possess better frequency localization in compare to rectangular



(a) Time domain



(b) Frequency domain

FIGURE 12 Localization comparison of basis functions: OFDM/OFTDM.

impulses. Moreover, the presented initializing function outperforms several popular wavelets, from the point of view of the uncertainty parameter, and is quite close to the Gaussian function.

5 ORTHOGONALITY AND NYQUIST CRITERIA

So far, one of the implementation problems of well-localised bases has been the complexity of their constructing algorithm. In particular, it can be a considerable limitation for the devices for which it is necessary to correct or tune the signal bases dynamically. Regardless of the high theoretical importance of the SVD based synthesis algorithm considered in the previous chapter it is difficult to use it in practice. In modern telecommunication systems, the number of sub-carriers can be up to several tens of thousands and consequently the size of the corresponding MWH basis is the same. The overviewed algorithm includes the search of the eigenvalues of the matrix of size $(2N \times 2N)$, highly resource-consuming operation.

The basis synthesis algorithm with much higher computational efficiency is presented in publication [PII]. In this chapter, orthogonality conditions, which form the theoretical basis of this algorithm, are derived. An additional but still important result of this chapter is the formulation of Nyquist-type criteria and theorem ensuring the absence of ISI and ICI in OFTDM signals .

5.1 Orthogonality Conditions

Finite-dimensional discrete modified WH basis $\mathcal{B}[J_N]$ is orthogonal by definition, if the following conditions are true:

$$\langle \Psi^R_{k,l}, \Psi^R_{k',l'} \rangle_R = \delta_{k,k'} \delta_{l,l'}, \tag{5.1}$$

$$\langle \Psi^R_{k,l}, \Psi^I_{k',l'} \rangle_R = 0, \tag{5.2}$$

$$\langle \Psi^I_{k,l}, \Psi^I_{k',l'} \rangle_R = \delta_{k,k'} \delta_{l,l'} \tag{5.3}$$

for all $\forall k, k' \in J_M$ and $\forall l, l' \in J_L$, where $\delta_{k,k'}$ is a Kronecker's symbol.

5.1.1 Conditions for WH Basis

Conditions (5.1 - 5.3) can be rewritten in a more compact form:

Theorem 5.1. *Necessary and sufficient conditions of the orthogonality of the basis $\mathcal{B}[J_N]$ are equalities*

$$\langle \Psi^R_{k,l}, \Psi^R_{0,0} \rangle_R = \delta_{k,0}\delta_{l,0}, \tag{5.4}$$

$$\langle \Psi^R_{k,l}, \Psi^I_{0,0} \rangle_R = 0 \tag{5.5}$$

for $\forall k \in J_M$ and $\forall l \in J_L$.

Proof. 1. Necessity Equalities (5.1 - 5.3) are true for any $k, k' \in J_M$ and any $l, l' \in J_L$, thus taking $k' = 0$ and $l' = 0$ one gets equalities (5.4) and (5.4).

2. Sufficiency Let's introduce indexes $k_0 \triangleq k - k'$ and $l_0 \triangleq l - l'$, then conditions (5.1 - 5.3) will be

$$\left\langle \psi^R_{k'+k_0,l'+l_0}[n], \psi^R_{k',l'}[n] \right\rangle_R = \delta_{k_0,0}\delta_{l_0,0}, \tag{5.6}$$

$$\left\langle \psi^R_{k'+k_0,l'+l_0}[n], \psi^I_{k',l'}[n] \right\rangle_R = 0, \tag{5.7}$$

$$\left\langle \psi^I_{k'+k_0,l'+l_0}[n], \psi^I_{k',l'}[n] \right\rangle_R = \delta_{k_0,0}\delta_{l_0,0}. \tag{5.8}$$

Using the properties of DFT, N-periodicity of functions and Parseval's theorem for the scalar product in (5.6) one can get

$$\begin{gathered}
\left\langle \psi^R_{k'+k_0,l'+l_0}[n], \psi^R_{k',l'}[n] \right\rangle_R = \\
= \Re\left(\sum_{n=0}^{N-1} \left\{ g\left[(n - (l'+l_0)M)_N\right] \exp\left(j\tfrac{2\pi}{M}(k'+k_0)\left(n - \tfrac{\alpha}{2}\right)\right) \right.\right. \\
\left.\left. g^*\left[(n - l'M)_N\right] \exp\left(-j\tfrac{2\pi}{M}k'\left(n - \tfrac{\alpha}{2}\right)\right)\right\}\right) = \\
= \Re\left\{ \sum_{n=0}^{N-1} g\left[(n - (l'+l_0)M)_N\right] \exp\left(j\tfrac{2\pi}{M}k_0\left(n - \tfrac{\alpha}{2}\right)\right) g^*\left[(n - l'M)_N\right] \right\} = \\
= \Re\left\{ \exp\left(-j\tfrac{2\pi}{M}k_0\tfrac{\alpha}{2}\right) \sum_{k=0}^{N-1} G\left[(k-k_0)_N\right] \exp\left(-j\tfrac{2\pi}{N}(l'-l_0)Mk\right) \right. \\
\left. G^*\left[(-k)_N\right] \exp\left(j\tfrac{2\pi}{N}l'Mk\right) \right\} = \\
= \Re\left\{ \exp\left(-j\tfrac{2\pi}{M}k_0\tfrac{\alpha}{2}\right) \sum_{k=0}^{N-1} G\left[(k-k_0)_N\right] \exp\left(j\tfrac{2\pi}{N}l_0Mk\right) G^*\left[(-k)_N\right] \right\} = \\
= \Re\left\{ \sum_{n=0}^{N-1} g\left[(n - l_0M)_N\right] g^*[n] \exp\left(j\tfrac{2\pi}{M}k_0\left(n - \tfrac{\alpha}{2}\right)\right) \right\} = \left\langle \psi^R_{k,l}[n], \psi^R_{0,0}[n] \right\rangle_R,
\end{gathered}$$

where $G[k] \triangleq \sum_{p=0}^{N-1} g[n] \exp\left(\jmath\frac{2\pi}{N}kn\right)$ is the DFT of function $g[n]$.

From here is directly follows that condition (5.6) is sufficient for the equality (5.1) to be true.

Analogical considerations result in that condition

$$\begin{gathered}
\left\langle \psi^I_{k,l}[n], \psi^I_{0,0}[n] \right\rangle_R = \\
= \Re\left\{ \sum_{n=0}^{N-1} (-j)\, g\left[\left(n - lM + \tfrac{M}{2}\right)_N\right] \exp\left(j\tfrac{2\pi}{M}k\left(n - \tfrac{\alpha}{2}\right)\right) \right. \\
\left. jg^*\left[\left(n + \tfrac{M}{2}\right)_N\right] \right\} = \delta_{k,0}\delta_{l,0}
\end{gathered} \tag{5.9}$$

is sufficient for the equality (5.8) and condition (5.5) - for (5.7).

Moreover, making the change of the variables $n = n + \frac{M}{2}$ in (5.9) and taking into account that $\exp\left(\jmath \frac{2\pi}{M} k(n \pm \frac{M}{2})\right) = \exp\left(\jmath \frac{2\pi}{M} kn\right)$ and $g\left[(n)_N\right]$ is N-periodical function, we will get that conditions (5.9) and (5.5) are the same. That what was to be demonstrated. □

5.1.2 Supplementary Family of Functions $\mathcal{E}[J_N]$

Let's address the practical case of the conjugated N-symmetrical initializing function $g[n] = g^*[(-n)_N]$ considered in the previous chapter. It was shown that for such modified WH bases the optimal value of the phase parameter can be found analytically and equals $\frac{M}{2}$.

Based upon the conditions presented above, it can be proved that the family of $\frac{N}{2}$ functions

$$\mathcal{E}[J_N] = \{g_{l,m}[n]\}, \tag{5.10}$$

$$g_{l,m}[n] = g[(n - lM)_N] e^{\jmath \frac{4\pi}{M} mn}, \quad m \in J_{\frac{M}{2}}, \quad l \in J_L \tag{5.11}$$

is orthogonal in terms of regular scalar product (2.4) when the modified WH constructed from the same function $g[n]$ is orthogonal in terms of real scalar product (2.22).

Firstly, the following theorem should be proved:

Theorem 5.2. *Let the initializing function of the modified WH basis $\mathcal{B}[J_N]$ be real and N-symmetrical. The phase parameter is selected in an optimal way: $\alpha = \frac{M}{2}$.*

Then the necessary and sufficient condition of $\mathcal{B}[J_N]$ is the equality

$$\sum_{n=0}^{N-1} g[n] g[(n - lM)_N] e^{\pm \jmath \frac{2\pi mn}{\frac{M}{2}}} = \delta_{l,0} \delta_{m,0} \tag{5.12}$$

which is true for $\forall m \in J_{\frac{M}{2}}$ and $\forall l \in J_L$.

Proof. Firstly, let's introduce function

$$f_1^l[n] \triangleq g\left[(n - lM)_N\right] g\left[\left(n + \frac{M}{2}\right)_N\right].$$

It is symmetrical on the intervals $n \in \left[0; lM - \frac{M}{2}\right]$ for $\forall l \in J_L$, because

$$f_1^l\left[lM - \frac{M}{2} - n\right] = g\left[(-\frac{M}{2} - n)_N\right] g[(lM - n)_N] =$$
$$= g\left[(n + \frac{M}{2})_N\right] g[(n - lM)_N] = f_1^l[n]$$

and also on the intervals $n \in \left[lM - \frac{M}{2} + 1; N - 1\right]$, because for all $p \in \left[1; N - 1 - lM + \frac{M}{2}\right]$

$$f_1^l[N - p] = g\left[(N - p - lM)_N\right] g\left[\left(N - p + \frac{M}{2}\right)_N\right] =$$
$$= g\left[\left(p + lM - \frac{M}{2} + \frac{M}{2}\right)_N\right] g\left[\left(p + lM - \frac{M}{2} - lM\right)_N\right] = f_1^l\left[p + lM - \frac{M}{2}\right].$$

On the same intervals function $\sin\left(\frac{2\pi}{M}k\left(n-\frac{\pi}{2}\right)\right)$ is anti-symmetrical, i.e.

$$\begin{aligned}\sin\left(\frac{2\pi}{M}k\left(lM-\frac{M}{2}-n-\frac{M}{4}\right)\right)&=\sin\left(\frac{2\pi}{M}k\left(-n-\frac{3M}{4}\right)\right)=\\&=-\sin\left(\frac{2\pi}{M}k\left(n-\frac{M}{4}\right)\right),\\ \sin\left(\frac{2\pi}{M}k\left(lM-\frac{M}{2}+p-\frac{M}{4}\right)\right)&=\sin\left(\frac{2\pi}{M}k\left(p+\frac{M}{4}\right)\right)=\\&=-\sin\left(\frac{2\pi}{M}k\left(N-p-\frac{M}{4}\right)\right).\end{aligned}$$

Therefore orthogonality condition (5.5) of MWH basis $\mathcal{B}[J_N]$, which can be rewritten like

$$\begin{aligned}\Re\left\{\sum_{n=0}^{N-1}\jmath\, f_1^l[n]\exp\left(j\frac{2\pi}{M}k\left(n-\frac{\alpha}{2}\right)\right)\right\}&=-\sum_{n=0}^{N-1}f_1^l[n]\sin\left(\frac{2\pi}{M}k\left(n-\frac{M}{4}\right)\right)=\\&=-\sum_{n=0}^{lM-\frac{M}{2}}f_1^l[n]\sin\left(\frac{2\pi}{M}k\left(n-\frac{M}{4}\right)\right)-\\&-\sum_{n=lM-\frac{M}{2}+1}^{N-1}f_1^l[n]\sin\left(\frac{2\pi}{M}k\left(n-\frac{M}{4}\right)\right)=0,\end{aligned}$$

is true automatically for $\forall k\in J_M$ and for $\forall l\in J_M$, because in both sums there are multiplications of symmetrical and anti-symmetrical functions, which equals zero. It follows directly from here, that necessary and sufficient originality conditions of the basis $\mathcal{B}[J_N]$ is only condition (5.4).

Now it is necessary to introduce a new function

$$f_2^l[n]\triangleq g\left[(n-lM)_N\right]g\left[(n)_N\right].\tag{5.13}$$

It is N-periodical and symmetrical on the interval $n\in[0;\ lM]$, because

$$f_2^l[lM-n]=g\left[(-n+lM)_N\right]g\left[(-n)_N\right]=g\left[(n-lM)_N\right]g[n]=f_2^l[n],\tag{5.14}$$

and also on the interval $n\in[lM+1;N-1]$, because for all $p\in[1;N-lM-1]$

$$\begin{aligned}f_2^l[p+lM]&=g\left[(p+lM)_N\right]g\left[(p)_N\right]=\\&=g\left[(-p-lM)_N\right]g\left[(-p)_N\right]=f_2^l[N-p].\end{aligned}\tag{5.15}$$

Moreover, on the same intervals the symmetry of function $\cos\left(\frac{2\pi}{M}k(n-\frac{M}{4})\right)$ depends on the odevity of index k:

$$\cos\left(\frac{2\pi}{M}k(lM-n-\frac{M}{4})\right)=\cos\left(\frac{2\pi}{M}k(n-\frac{M}{4})+\pi k\right)=$$
$$=(-1)^k\cos\left(\frac{2\pi}{M}k(n-\frac{M}{4})\right),$$
$$\cos\left(\frac{2\pi}{M}k(lM+p-\frac{M}{4})\right)=\cos\left(\frac{2\pi}{M}k(N-p-\frac{M}{4})+\pi k\right)=$$
$$=(-1)^k\cos\left(\frac{2\pi}{M}k(N-p-\frac{M}{4})\right).$$

Condition (5.4), which can be rewritten in the following form:

$$\Re\left\{\sum_{n=0}^{N-1}f_2^l[n]\exp\left(j\frac{2\pi}{M}k\left(n-\frac{\alpha}{2}\right)\right)\right\}=\sum_{n=0}^{N-1}f_2^l[n]\cos\left(\frac{2\pi}{M}k\left(n-\frac{M}{4}\right)\right)=$$
$$=\sum_{n=0}^{Ml}f_2^l[n]\cos\left(\frac{2\pi}{M}k\left(n-\frac{M}{4}\right)\right)+\sum_{n=Ml+1}^{N-1}f_2^l[n]\cos\left(\frac{2\pi}{M}k\left(n-\frac{M}{4}\right)\right)=\delta_{l,0}\delta_{k,0}$$

becomes true automatically for odd k. In this case both sums in the left part of the last equality equals zero because of the multiplication of symmetrical and anti-symmetrical functions.

It rests to consider only the case, when k is even, i.e. when $k=2m,\ m\in J_{M/2}$. The DFT of function $f_2^l[n]$ is

$$F_2^l[v]=\sum_{n=0}^{N-1}f_2^l[n]\exp\left(-j\frac{2\pi}{N}vn\right),\ v\in J_N, \tag{5.16}$$

consequently

$$F_2^l[2Lm]=\sum_{n=0}^{N-1}f_2^l[n]\exp\left(-j\frac{2\pi}{N}2Lmn\right). \tag{5.17}$$

Taking into account the symmetry properties (5.14) and (5.15) of function $f_2^l[n]$ it is possible to show that

$$F_2^l[-2Lm]=F_2^l[2Lm]. \tag{5.18}$$

Indeed,

$$F_2^l[-2Lm] = \sum_{n=0}^{N-1} f_2^l[n] \exp\left(-j\frac{2\pi}{N}(-2Lm)\,n\right) =$$

$$= \sum_{n=0}^{lM} f_2^l[lM-n] \exp\left(j\frac{2\pi}{N}2Lmn\right) + \sum_{n=lM+1}^{N-1} f_2^l[N-n+lM] \exp\left(j\frac{2\pi}{N}2Lmn\right) =$$

$$= \sum_{n'=0}^{lM} f_2^l[n'] \exp\left(-j\frac{2\pi}{N}2Lmn'\right) \exp\left(j\frac{2\pi}{N}2LmlM\right) +$$

$$+ \sum_{n''=lM+1}^{N-1} f_2^l[n''] \exp\left(-j\frac{2\pi}{N}2Lmn''\right) \exp\left(j\frac{2\pi}{N}2Lm\,(lM+N)\right) =$$

$$= \sum_{n=0}^{N-1} f_2^l[n] \exp\left(-j\frac{2\pi}{N}2Lmn\right) = F_1^l[2Lm],$$

where two changes of variables $n' = lM - n$ and $n'' = N - n + lM$ have been used. For even $k = 2m$ the left side of condition (5.4) becomes

$$\Re\left\{\sum_{n=0}^{N-1} f_2^l[n] \exp\left(\jmath\frac{2\pi}{M}2m\left(n-\frac{M}{4}\right)\right)\right\} = (-1)^m \sum_{n=0}^{N-1} f_2^l[n] \cos\left(\frac{2\pi}{M}2mn\right) =$$

$$= \frac{(-1)^m}{2}\left(\sum_{n=0}^{N-1} f_2^l[n] \exp\left(\jmath\frac{2\pi}{M}2mn\right) + \sum_{n=0}^{N-1} f_2^l[n] \exp\left(-\jmath\frac{2\pi}{M}2mn\right)\right) =$$

$$= \frac{(-1)^m}{2}\left(F_2^l[2Lm] + F_2^l[-2Lm]\right).$$

The account of property (5.18) results in the following orthogonality condition

$$F_2^l[2Lm] = F_2^l[-2Lm] = \delta_{l,0}\delta_{m,0},\ \forall m \in J_{M/2};\ \forall l \in J_L, \tag{5.19}$$

which is the analogue of formula (5.4).

Remembering the definition (5.13) of function $f_2^l[n]$ and expression (5.17) for its DFT one can get from the equation (5.19) the required necessary and sufficient orthogonality condition (5.12) for the MWH basis. □

For the family of functions $\mathcal{E}[J_N]$ it is possible to receive orthogonality conditions by proving the theorem analogical to Theorem 5.1 for modified WH basis.

Theorem 5.3. *The necessary and sufficient conditions of orthogonality for the functions (5.11) in terms of regular scalar product have the form*

$$\langle g_{l,m}[n], g_{0,0}[n]\rangle = \delta_{m,0}\delta_{l,0} \tag{5.20}$$

for $\forall m \in J_{\frac{M}{2}}$ *and* $\forall l \in J_L$.

The proof of this theorem repeats almost completely the proof of Theorem 5.1 and won't be presented here.

It is necessary to note that from Theorem 5.3 and the structure of functions (5.10), (5.11) it follows directly that equation (5.12) is the necessary and sufficient condition of orthogonality for the family $\mathcal{E}[J_N]$. Thus, if the assumptions of the theorem 5.3 are true, then from the orthogonality of the basis $\mathcal{B}[J_N]$ follows the orthogonality of the family $\mathcal{E}[J_N]$ and vice versa.

5.1.3 Additional Orthogonality Conditions for the $\mathcal{E}[J_N]$ Family

Irrespective of the symmetry type of the basis' initializing function $g[n]$ it is possible to prove two additional orthogonality criteria of function $\mathcal{E}[J_N]$ in te time and frequency domains.

Theorem 5.4. *The necessary and sufficient orthogonality condition for the family of functions $\mathcal{E}[J_N]$ (5.10),(5.11) in the time domain is represented by the following expression*

$$\sum_{r=0}^{2L-1} g[(n-r\frac{M}{2})_N]g[(n-l\frac{M}{2})_N] = \frac{2}{M}\delta_{l,0} \tag{5.21}$$

for $\forall n \in J_N$ and $\forall l \in J_L$.

Proof. Remembering the expression (5.16) for the DFT of function f_2^l defined by (5.13) it can be presented in the form of IDFT:

$$f_2^l[n] = \frac{1}{N}\sum_{\nu=0}^{N-1} F_2^l[\nu]\exp\left(\jmath\frac{2\pi}{N}n\nu\right).$$

Therefore for $\forall r \in J_{2L}$

$$f_2^l[n-r\frac{M}{2} = \frac{1}{N}\sum_{\nu=0}^{N-1} F_2^l[\nu]\exp\left(\jmath\frac{2\pi}{N}(n-r\frac{M}{2})\nu\right.$$

Let's sum up the last expression over r from 0 up to $2L-1$ and take into account the explicit from (5.13) of function $f_2^l[n]$:

$$\begin{aligned}\sum_{r=0}^{2L-1} f_2^l[n-r\frac{M}{2}] = \sum_{r=0}^{2L-1} g\left[(n-r\frac{M}{2})_N\right] g\left[(n-r\frac{M}{2}-lM)_N\right] = \\ = \frac{1}{N}\sum_{\nu=0}^{N-1}\exp\left(\jmath\frac{2\pi}{N}\nu n\right)F_2^l[\nu]\sum_{r=0}^{2L-1}\exp\left(\jmath\frac{2\pi}{2L}r(-\nu)\right).\end{aligned}$$

Notice, that

$$\frac{1}{2L}\sum_{r=0}^{2L-1}\exp\left(\jmath\frac{2\pi}{2L}r(-\nu)\right) = \begin{cases}1 & \text{for } \nu \text{ divisible by } 2L\\ 0 & \text{for all other } \nu\end{cases}$$

and taking into account that $\nu \in J_N$ it can be concluded that last expression is different from zero and equals $2L$ only when $\nu = 2Lm$ and $m \in J_{\frac{M}{2}}$. Thus

$$\sum_{r=0}^{2L-1} g\left[(n-r\frac{M}{2})_N\right] g\left[(n-r\frac{M}{2}-lM)_N\right] = \frac{2}{M}\sum_{m=0}^{\frac{M}{2}-1}\exp\left(\jmath\frac{4\pi}{M}mn\right)F_2^l[2Lm]. \tag{5.22}$$

It is obvious that when functions $\mathcal{E}[J_N]$ are orthogonal, i.e. when conditions (5.19) are fulfilled, then the equality (5.21) is also true. This proves that conditions (5.21) is necessary for the orthogonality of $\mathcal{E}[J_N]$.

To prove the sufficiency of this condition it is enough to notice, that when it is fulfilled, then from (5.22) it follows:

$$F_2^l[2Lm] = 0 \text{ for} \forall m \in J_{\frac{M}{2}}$$

and

$$\frac{2}{M} \sum_{m=0}^{\frac{M}{2}-1} \exp\left(\jmath \frac{4\pi}{M} mn\right) F_2^0[2Lm] = 1 \text{ for } \forall m \in J_{\frac{M}{2}}.$$

Moreover the last equality is true only when

$$F_2^0[2Lm] = \delta_{m,0} \text{ for } \forall m \in J_{\frac{M}{2}}.$$

Consequently, the necessary and sufficient orthogonality condition (5.19) is fulfilled. That proves the theorem. □

Analogical theorem on the necessary and sufficient orthogonality condition for the family of functions $\mathcal{E}[J_N]$ can be formulated in frequency domain:

Theorem 5.5. *The necessary and sufficient orthogonality condition for the family of functions $\mathcal{E}[J_N]$ (5.10),(5.11) in the frequency domain is represented by the following expression*

$$\sum_{k=0}^{M-1} G[(p+kL)_N] G^*[(p+kL-2Lm)_N] = \frac{1}{L}\delta_{m,0} \tag{5.23}$$

for $\forall p \in J_N$ and $\forall m \in J_{\frac{M}{2}}$.

Verification results for some of the criteria presented above are shown in paper [PII].

5.2 Nyquist Criterion and Theorem

Classical Nyquist's theorem and criterion of ISI absence is formulated for the per-symbol transmission model [16]. Thus it is necessary to overview it briefly to make the comparison with OFTDM model more obvious.

5.2.1 Per-Symbol Transmission Model

In practice, a transmitted signal $s(t)$ quite often can be represented as a sequence of the same impulses shifted in time [89, 16]:

$$s(t) = \sum_{k=-l}^{l} a_k p(t-kT), \tag{5.24}$$

where $t \in \mathbb{R}$ and

- $p(t)$ is the basic or initializing impulse;
- $\{a_k, k = 0, \pm 1, \ldots, \pm l\}$ is the sequence of modulating symbols from some finite alphabet, for example for binary transmission $a_k \in \{-1, 1\}$;
- $a_k p(t - kT)$ is the symbol corresponding to the kth time shift;
- $2l + 1$ is the length of the sequence, i.e. the number of modulating symbols in the sequence;
- T is time delay between neighboring symbols ans is called the symbol period.

$p(t)$ is a low-frequency impulse with band-limited limited spectrum. For that reason, it has finite energy $E_p = \int_{-\infty}^{\infty} p^2(t)dt$, but infinite duration. The model (5.24) itself describes the method for the transmission of information sequence $\{a_k\}$ over the channel of bandwidth W, where shifted in time impulses $p(t)$ are used as carriers.

An amplitude spectrum of impulse $p(t)$ is given by its FT $P(f) = \mathrm{FT}[p(t)]$, where $P(f) = 0$ for $|f| > W$ Its energy spectrum $G(f) \triangleq |P(f)|^2$ has the same property.

At the receiver the signal

$$r(t) = \tilde{s}(t) + w(t), \quad \tilde{s}(t) = \sum_{k=-l}^{l} a_k h(t - kT),$$

is observed, where $h(t) = p(t) * c(t) = \int_{-\infty}^{\infty} g(\tau)c(t-\tau)$, $c(t)$ describes channel amplitude response and $w(t) \sim \mathcal{N}(0, \sigma_w^2)$ is AWGN with zero mean and dispersion σ_w^2.

Optimal reception of this signal is realised with the help of the filter matched with the signal $\tilde{s}(t)$. An output of this filter is

$$y_s(t) = r(t) * \tilde{s}(-t) = \int_{-\infty}^{\infty} r(\tau)\tilde{s}(\tau - t)d\tau = \int_{-\infty}^{\infty} r(\tau) \sum_{k=-l}^{l} a_k h(\tau - kT - t)d\tau =$$

$$= \sum_{k=-l}^{l} a_k \int_{-\infty}^{\infty} r(\tau)h(\tau - kT - t)d\tau.$$

This filter can be realised as a Pulse Matched Filter (PMF) with impulse response $h_{\mathrm{PMF}}(t) = p(-t)$. Indeed,

$$y_s(0) = \sum_{k=-l}^{l} a_k \int_{-\infty}^{\infty} r(\tau)h(\tau - kT)d\tau = \sum_{k=-l}^{l} a_k y_k,$$

where $y_k \triangleq \int_{-\infty}^{\infty} r(\tau)h(\tau - kT)d\tau = (r(t) * h(-t))\,|_{t=kT}$ is the output of the filter matched with the impulse $p(t)$ taken in time moments $t = kT$. Furthermore,

$$y_k = \sum_{k'=-1}^{l} a'_k \int_{-\infty}^{\infty} h(\tau - k'T)h(\tau - kT)d\tau + w_k = \sum_{k'=-1}^{l} a'_k g(t - k'T),$$

where

$$g(t-k'T)=\int_{-\infty}^{\infty}h(\tau-k'T)h(\tau-t)=h(t)*h(-t)$$

is the pulse representing the response of the receiving filter to the input pulse $h(t-kT)$.

Shifted in time impulses $h(t-kT)$ are playing the role of orthonormal basis functions if the following condition is fulfilled:

$$\int_{-\infty}^{\infty}h(t-k'T)h(t-hT)dt=\delta_{k',k}. \tag{5.25}$$

It is called the *Nyquist criterion* (in the time domain). It guarantees, that transmitted symbols do not cause mutual interference, i.e. transmission takes place without ISI.

Upon the convolution theorem the spectrum representation of the impulse $g(t)$ is

$$G(f)=\mathrm{FT}[g(t)]=H(f)H^*(f)=|H(f)|^2,$$

where $h(f)=\int_{-W}^{W}H(f)e^{j2\pi ft}df$. Therefore, it is real-valued, positive and coincides with the power density spectrum of the impulse $h(t)$.

From the other hand, after the corresponding change of variables

$$g(t)=\int_{-\infty}^{\infty}h(t+\tau)h(\tau)d\tau,$$

i.e. it is, in fact, the autocorrelation of the impulse $h(t)$.

From the orthogonality criterion (5.25) it follows that

$$g((k'-k)T)=\begin{cases}1 & \text{, if } (k'-k)=0\\ 0 & \text{, otherwise}\end{cases}$$

or in an equivalent form

$$g(lT)=\delta_{0,l}=\begin{cases}1 & \text{, when } l=0\\ 0 & \text{, otherwise}\end{cases}$$

In other words, the Nyquist criterion to be true it is necessary and sufficient that the impulse $g(t)$ passes through zero in all points $t=kT$, except $t=0$.

An impulse $g(t)$ satisfying the condition $g(lT)=\delta_{0,l}$ is called Nyquist impulse. Corresponding per-symbol transmission method, satisfying the Nyquist condition for zero ISI is called the Nyquist transmission.

The Nyquist criterion can be formulated in the frequency domain with the help of FT:

$$g(lT)=\int_{-\infty}^{\infty}G(f)e^{j2\pi flT}df=\sum_{n=-\infty}^{\infty}\int_{(2n-1)}^{2n+1}G(f)e^{j2\pi flT}df=$$

$$=\sum_{n=-\infty}^{\infty}\int_{-\frac{1}{2T}}^{\frac{1}{2T}}G(f+\frac{n}{T})e^{j2\pi flT}df=\int_{-\frac{1}{2T}}^{\frac{1}{2T}}\sum_{n=-\infty}^{\infty}G(f+\frac{n}{T})e^{j2\pi flT}df. \tag{5.26}$$

It is easy to notice, that

$$G^D(f) \triangleq \sum_{n=-\infty}^{\infty} G(f+\frac{n}{T}), \quad -\frac{1}{2T} \le f \le \frac{1}{2T}$$

is by its sense a spectrum of the discrete signal $g(iT)$, $i = 0, \pm 1, \pm 2, \dots$. Indeed,

$$G^D(f) = T \sum_{i=-\infty}^{\infty} g(iT) e^{-2\pi f i T} = \mathrm{DFT}[g(iT)].$$

From the following condition

$$G^D(f) = T, \quad |f| \le \frac{1}{2T} \tag{5.27}$$

and expression (5.2.1) it follows that the Nyquist criterion in the time domain

$$g(lt) = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{j2\pi f l T} df = \frac{\sin(\pi l)}{\pi l} = \delta_{0,l}$$

is true. Therefore, condition (5.27) is the equivalent of the Nyquist condition of zero ISI in the frequency domain. Usually, this fact is formulated in the form of a theorem:

Theorem 5.6 (Nyquist)**.** *The necessary and sufficient condition of zero ISI, i.e. of the following condition for the impulse $g(t)$*

$$g(lT) = \delta_{0,l} = \begin{cases} 1 & \text{when } l = 0 \\ 0 & \text{for all other } l \ne 0 \end{cases}$$

is deduced to the condition on the FT $G(f)$

$$\sum_{n=-\infty}^{\infty} G(f+\frac{n}{T}) = T.$$

5.2.2 Analogue for OFTDM Signals

Let the OFTDM signal be transmitted over the Gaussian channel with ideal frequency response. The signal at the receiver is

$$r[n] = s[n] + w[n], \tag{5.28}$$

where $w[n] \sim \mathcal{N}(0, \sigma_w^2)$ is AWGN with zero mean and dispersion σ_w^2.

Optimal reception of such signal is performed by the scheme combined from a filter matched with the impulse response $h_{\mathrm{MF}}[n] \triangleq s^*[(-n)_N]$ and of the operator $\Re(\cdot)$ taking the real part of complex number. The output of such scheme is

$$y[n] = \Re(r[n] \otimes h_{\mathrm{cons}}[n]) = \Re\left(\sum_{m=0}^{N-1} r[m] s^*[(m-n)_N]\right), \tag{5.29}$$

where $\otimes$ means cyclic convolution of two functions.

It can be seen, that once $n = 0$ the real part of correlation sum is formed at the output of the optimal receiver

$$y[0] = \Re(\langle r, s \rangle) = \Re\left(\sum_{n=0}^{N-1} r[n]s^*[n]\right) = \Re\left(r[n] \otimes s^*[(-n)_N]|_{n=0}\right).$$

Taking into account the presentation of the OFTDM signal (5.1), it can be shown that

$$y[0] = \sum_{k=0}^{M-1}\sum_{l=0}^{L-1}\left(c^R_{k,l}y^R_{k,l} + c^I_{k,l}y^I_{k,l}\right), \tag{5.30}$$

where

$$y^R_{k,l} \triangleq \Re\left(\sum_{n=0}^{N-1} r[n]\Psi^{*R}_{k,l}[n]\right) = \Re\left(r[n] \otimes \Psi^{*R}_{k,0}[(-n)_N]\Big|_{n=lM}\right),$$
$$y^I_{k,l} \triangleq \Re\left(\sum_{n=0}^{N-1} r[n]\Psi^{*I}_{k,l}[n]\right) = \Re\left(r[n] \otimes \Psi^{*I}_{k,0}[(-n)_N]\Big|_{n=lM}\right),$$

are the real outputs of filters matching with basis functions (5.2), (5.3).

Thus, an optimal reception of OFTDM signal can be performed with the help of a filter bank matched with basis functions of size $2M$, the output of which is measured in time moments divisible by M.

As shown in the previous section, the conditions which are applied on the signal structure and which ensure the absence of mutual interference are called the Nyquist criteria. It is obvious that the complexity of OFTDM signals does not allow the direct utilization of the Nyquist criteria formulated for the per-symbol transmittance model. However, the criteria can be generalized on a wider class of signals.

It can be seen, that in the case of Gaussian channel (5.28), zero ICI and ISI will be observed, when orthogonality conditions (5.6)-(5.8) of MWH basis are fulfilled. Indeed, if these conditions are true and noise is filtered out, then the signal at the output of optimal receiver (5.30) does not contain interference components and equals

$$y[0] = \sum_{k=0}^{M-1}\sum_{l=0}^{L-1}\left((c^R_{k,l})^2 + (c^I_{k,l})^2\right).$$

Therefore, conditions (5.6)-(5.8) describe the Nyquist criterion for the OFTDM signals. The criteria proved in this chapter can be used to formulate it in the form similar to (5.25):

Theorem 5.7 (Nyquist criteria for OFTDM signals)**.** *The necessary and sufficient condition of ICI and ISI absence in channel (5.29) with OFTDM signal are the orthogonality* *conditions of the MWH basis $\mathcal{B}[J_N]$, which are deduced to the following equalities:*

$$\Re\left(\sum_{n=0}^{N-1} g[(n-lM)_N]e^{\jmath\frac{2\pi}{M}k(n-\frac{\alpha}{2})}g^*[n]\right)=\delta_{k,0}\delta_{l,0}, \tag{5.31}$$

$$\Re\left(\sum_{n=0}^{N-1} g[(n-lM)_N]e^{\jmath\frac{2\pi}{M}k(n-\frac{\alpha}{2})}g^*[(n+\frac{M}{2})_N]\right)=0, \tag{5.32}$$

for $\forall k\in J_M$ and $\forall l\in J_L$.

Let's consider a particular case, when $g[n]$ is conjugated N-symmetrical function, i.e. $g[n]=g^*[n]$, and phase parameter is selected in the optimal way $\alpha=\frac{M}{2}$. Upon these conditions, the equality (5.32) becomes true automatically and (5.31) can be rewritten like

$$p(l,k)\triangleq\sum_{n=0}^{N-1} g[n]g[(n-lM)_N]e^{\jmath\frac{2\pi kn}{\frac{M}{2}}}=\delta_{l,0}\delta_{k,0} \tag{5.33}$$

for $\forall k\in J_{\frac{M}{2}}$ and $\forall l\in J_L$.

The Nyquist impulse for the OFTDM signal can be defined in the following form:

$$B(\tau,\nu)\triangleq\sum_{n=0}^{N-1} g[n]g^*[(n+\tau)_N]e^{-\jmath\frac{2\pi}{N}\nu n}, \tag{5.34}$$

where $\tau,\nu\in J_N$.

In spite of the Nyquist impulse defined for the per-symbol transmission model, impulse $B(\tau,\nu)$ depends on two variables (time and frequency) and has the sense of discrete ambiguity function

$$B(\tau,\nu)=\langle g_0[n,m],g_0[(n+\tau)_N,(m+\nu)_N]\rangle$$

of the base impulse $g_0[n,m]\triangleq g[n]e^{\jmath\frac{2\pi}{N}mn}$, $n,m\in J_N$.

From the expressions (5.33) and (5.34) it follows that

$$B(Ml,2Lk)=p(l,k)=\delta_{l,0}\delta_{k,0}. \tag{5.35}$$

Thus, the Nyquist impulse should be equal zero in all points (τ,ν) of the time-frequency plane, which are dividable by M and $2L$, consequently, except the point $(0,0)$, where it is different from zero.

Finally, from Theorems 5.4 and 5.5 it follows the generalised Nyquist theorem for the OFTDM signals:

Theorem 5.8 (Nyquist for OFTDM signals)**.** *Let the initializing function of MWH basis, which is used for the construction of the OFTDM signal, be conjugated N-symmetrical and the phase parameter be selected equal to $\frac{M}{2}$. Then the necessary and sufficient conditions of zero ICI and ISI in the Gaussian channel is the equality (5.35), which is equivalent in the time domain to (5.21) and in the frequency domain to (5.23).*

5.3 Conclusions of Orthogonality Criteria

Let's formulate the main conclusions which follow from the study in this chapter.

Firstly, it was proved that the orthogonality criteria of the MWH basis can be formulated in a quite compact form and also as conditions on the initializing functions or its DFT. Secondly, received orthogonality criteria have important theoretical considerations. They are the alternatives of Nyquist's criteria and theorem for OFDM signals. Their practical value is that they ensure the absence of ICI and ISI of the signal constructed from the basis function orthogonal in the time and frequency domains. Finally, the presented results open the way towards the formulation of an efficient basis synthesis algorithm [PVI]. The main idea is the replacement of a complicated matrix decomposition with simpler conditions on the initializing function.

6 CONCLUSION

Based on the study performed in the thesis, it is possible to formulate several important statements:

1. Based on the overview of the currently existing methods of time-frequency analysis and approaches for digital signal processing, it can be concluded that well-localized MWH bases are actual and promising tool for practical applications.
2. The main value of the received results is that the algorithms and programs developed present a complete mathematical and computational framework, necessary for the effective utilization of MWH bases from its synthesis to signal processing.
3. Sampling frequency is an important parameter: it defines time-frequency localization in the case when continuous function is used as an input in a basis generation algorithm. For the Gaussian function there is an optimal value of sampling parameter, which can be found from simulations.
4. N-periodical approximation of complex discrete symmetrical functions was constructed. Several types of symmetry have been studied.
5. The localization quality of the MWH basis can be additionally improved by the choice of phase parameter. Its optimal values were found theoretically for two types of symmetry of the initializing function.
6. The orthogonality conditions of the MWH basis can be formulated in a compact form, and also in the time and frequency domains. The generalized variant of the Nyquist theorem was formulated for OFTDM signals.
7. The orthogonality criteria of the MWH basis can be presented as the conditions on the corresponding Wiener basis. This result is used for the creation of the "fast" basis synthesis algorithm strongly based on FFT.
8. Poliphase decomposition was introduced in finite-dimensional N-periodical spaces $\mathbb{C}^N$. It turns out that its structure corresponds very well to the internal structure of OFTDM signals. This allows the development of the "fast" signal modulation algorithm with a number of operations of the same order with IFFT used in OFDM.

The generic scheme of the developed mathematical framework, including MWH basis construction and utilization, is shown on Figure 13. The numbers in the scheme correspond to the numbers in the list of statements presented above.

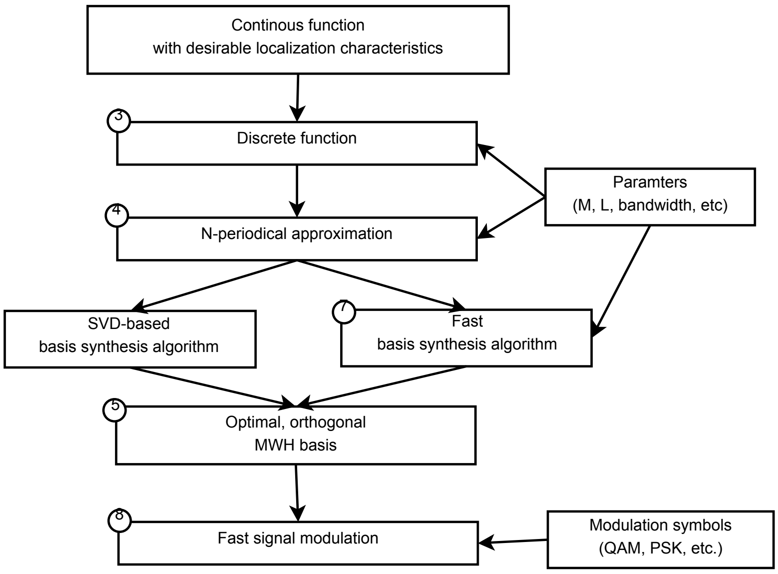


FIGURE 13 Scheme of the developed framework.

FT became a very popular and widely used instrument for signal processing in linear time-invariant systems. However, it is difficult to use in the study of short-time transient processes. For such more complicated phenomena, it is necessary to know the information about the spectrum localized in time. Synthesis of an universal basis, which could simplify the analysis of wide class of signals, is a very complicated problem. The most important examples of such bases are the WH (or Gabor) bases and wavelets. There are several reasons motivating the implementation and study of MWH bases in connection with wireless technologies:

- Shifted in time and frequency versions of prototype signal (initializing function or impulse) are used in many already existing applications, including digital communications, coding, voice recognition, spectral analysis, image compression etc. Thus, the main aim is the improvement of the characteristics of currently existing devices by optimizing their parameters without a complete change of the approach.
- Non-stationary media like radio channels can be characterized by two main effects: Doppler shifts and time delays. They belong to basic components of the WH group.
- It is well known that the requirements of symmetry and orthogonality cannot be fulfilled simultaneously for wavelets with compact support.
- As it is shown in the dissertation, signal processing algorithms using WH bases can be computationally efficient.

The practical value of the research is in the application of optimal orthogonal MWH bases as a core concept of OFTDM systems.

OFTDM technology can be used for the mitigation of ICI in currently existing standards such as WiMAX, WiFi, DVB, etc. However, the entire potential is realized, when the MWH basis is used in the synthesis filter bank at the transmitter and also in the analyzing filter bank of the receiver. This signal processing implicitly includes L-time oversampling and can be considered as a further development of Gabor's atoms method. In this more generic case, the family of initializing functions are shifted in time and frequency.

OFTDM technology is a perspective solution for next generation MC wireless mobile networks. The following advantages are realizable:

– Because of higher inter-carrier and time multiplexing limited frequency, resources are used in a more efficient way. Refuse from CP increases the spectral and power efficiency of the system.
– Due to better localization of the bases function, the level of out-of-band emission is reduced. It becomes possible to weaken the requirements to the output filter of the transmitter and to the GIs at the ends of the frequency band.
– Wireless telecommunication systems become more reliable against ICI and ISI. They can be adopted to the parameters of time-frequency dispersion.

Other benefits of OFTDM technology, including economic and other positive impacts, can be found in [PV]. For now, it is important to mention that this technology can be considered as one of the possible applications of the theoretical results considered in the thesis. In effect, the number of areas where it can be used is much wider. In particular, the developed methods and algorithms can be used for efficient time-frequency analysis of different processes. In this case, it is possible to perform flexible multi-level analysis of signals at the exit of registration devices. Examples of such devices include

– sensors in bio-medical devices;
– receivers of echo-signals in radio or hydro-radars;
– seismographic sensors, etc.

Further research topics include system level performance evaluation of OFTDM system using developed MWH bases under realistic channel conditions. Furthermore, the results of these simulations might be used in system level simulators, in particular, in WiFi, WiMAX or LTE modules of ns-3.

YHTEENVETO (FINNISH SUMMARY)

Tämä väitöskirja, jonka nimi on "Parannetun aika-taajuuslokalisaation omaavien ortogonaalisten signaali-kantojen synteesi ja soveltaminen liikkuvissa langattomissa tietoliikenneverkoissa"keskittyy kehittämään ja validioimaan viitekehystä, jota voidaan käyttää fyysisellä tasolla moderneissa langattomissa monikantoaaltojärjestelmissä, etenkin OFDM-järjestelmissä. Käytännön näkökulmasta liikkuva tietoliikenneverkko on erittäin haastava ympäristö palveluille, joiden tulisi toimia luotettavasti jopa huonoissa kanavatilanteissa. Tämä tutkimus ottaa kantaa moneen OFDM-teknologian ongelmaan, kuten huonoon signaali-kantojen lokalisointiin taajuustasossa ja syklisen etuliitteen aiheuttamaan pienentyneeseen hyötysuhteeseen. Näitä ongelmia ei voida ratkaista tehokkaasti klassisten WH-kantojen teoreettisten rajoitusten vuoksi.

Väitöskirjan tärkein menetelmä on paranneltujen aika-taajuus -lokalisoitujen ortogonaalisten signaalikantojen käyttö. Niiden käytännöllinen hyödyntäminen sisältää useita haasteita, jotka käydään läpi ja ratkaistaan tässä väitöskirjassa. Ensimmäinen ongelma on kantojen rakenne ja synteesi. MWH-kannat on valittu tutkimuksen pääkohteeksi. Niiden yleinen rakennusalgoritmi kehitettiin aloittaen annetuista jatkuvista funktioista, joilla on halutut lokaalit ominaisuudet. Todistettu ortogonaalisuuskriteeri johtaa algoritmin hyötysuhteen huomattavaan kasvuun. Toisessa ongelmassa optimoidaan kantojen taajuus- ja näytteistysparametrit. Viimeisenä esitellään laskennallisesti tehokas signaalin modulointialgoritmi. Nämä tulokset muodostavat OFTDM-tekniikan perusteet, jossa taajuusjakoista kanavointia täydennetään aikataajuusjaon kanavoinnilla.

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APPENDIX 1 MAIN ALGORITHMS

This appendix contains the listings of algorithms developed in the thesis. Algorithms have been realised and tested in Matlab.

APPENDIX 1.1 Construction of symmetrical N-periodic approximation

APPENDIX 1.1.1 Conjugated N-symmetry

```
function aprox = N_sym_aprox(Input)
%Input - sampled initial function on the interval (-N:N)

N = (length(Input)-1)/2; %the length of the period

%Symmetry check (g(n)=g*(-n))
for i=0:1:N
    if Input(N+1+i) ~= conj(Input(N+1-i))
        N+1+i
        N+1-i
        error('Symmetry condition is not fulfilled')
        break
        return
    end
end
if Input(2*N+1) ~= conj(Input(2*N+1))
    2*N+1
    error('Symmetry condition is not fulfilled')
    return
end

%Approximation construction
Input_r = Input(N+1:2*N);
A=0;
for tau=0:N-1 % Conjugated N-symetry
  if tau==0
      aprox(1)=Input(N+1)+Input(2*N+1);
      A = A + (abs(Input(N+1)+Input(2*N+1)))^2;
  else
      aprox(tau+1)=Input_r(1+tau)+ conj(Input_r(1+N-tau));
      A = A + (abs(Input_r(1+tau)+ conj(Input_r(1+N-tau))))^2;
  end
end
```

```
NInput=0; %Norm of the incoming function
for k=0:1:2*N-1
    NInput = NInput + (abs(Input(k+1)))^2;
end
A = A/NInput;

aprox = aprox./(sqrt(A));
```

APPENDIX 1.1.2 Conjugated $(N-1)$-symmetry

```
function aprox = N_1_sym_aprox(Input)
%Input - sampled initial function on the interval (-N:N-1)

N = length(Input)/2; %the length of the period
%Symmetry check  (g(n)=g*(-n-1))
for i=0:1:N-1
    if Input(N+1+i) == conj(Input(N-i))
        'Ok';
    else
        N+1+i
        N-i
        error('Symmetry condition is not fulfilled')
        break
        return
    end
end

%Approximation construction
Input_r = Input(N+1:2*N);
A=0;
for tau=0:N-1 % Conjugated (N-1) symmetry
    aprox(tau+1)=Input_r(tau+1)+conj(Input_r(N-tau));
    A = A + (abs(Input_r(1+tau)+ conj(Input_r(N-tau))))^2;
end
NInput=0; %Norm of the incoming function
for k=0:1:2*N-1
    NInput = NInput + (abs(Input(k+1)))^2;
end
A = A/NInput;

aprox = aprox./(sqrt(A));
```

APPENDIX 1.2 SVD-based basis construction

```
function H = MWHrSVD_SyMh (g, M, alfa)
%Input variables:
%g - initial non-orthogonal function
%M - number of sub-carriers
%alfa - phase parameter

N = length(g); %Length of the initialising function
L = N/M; %Number of symbols in OFTDM frame
j = sqrt(-1);

%Initialisation of matrices
Ur=zeros(N,M*L);
Ui=zeros(N,M*L);

%Construction of basis' matrix
for n=0:N-1
    gr=g(mod(n*ones(1,L)-M*[0:L-1],N)+1);
    gi=-j*g(mod(n*ones(1,L)+M/2-M*[0:L-1],N)+1);
    ee=exp(j*2*pi*(n-alfa/2)*[0:M-1].'/M);

    gr_=ee*gr.';
    Ur(n+1,:)=(gr_(:)).';
    gi_=ee*gi.';
    Ui(n+1,:)=(gi_(:)).';
end

GG=[Ur,Ui]; %Complex matrix of non-orthogonal MWH basis
GGr=[real(GG);imag(GG)]; %Real matrix of
%non-orthogonal MWH basis

%SVD decomposition
[V,S,W] = svd(GGr);
U = V*W';
Hr = U; %Real matrix of optimal orthogonal MWH basis

%Different ways to calculate difference norm
%between orthogonal and non-orthogonal bases
FSigmaNorm = sum((diag(S)-1).^2)
eenormn = norm(GGr- Hr,'fro')^2
FNorm = trace((GGr-U)*((GGr-U)'))
%Majorizing norm
B=eye(2*N);
```

```
F0Norm = trace(((GGr)*((GGr)')-B)*(((GGr)*((GGr)')-B)'))

%Complex matrix of optimal orthogonal MWH basis
H=Hr(1:N,:)+j*Hr(N+1:end,:);

%Initialising function of optimal orthogonal MWH basis
g_opt=[H(N/2+1:N,:);H(1:N/2,:)]
```

APPENDIX 1.3 Fast basis synthesis

```
function go = FastOrt (g, M)
%Input variables:
%g - initial non-orthogonal function
%M - number of sub-carriers

N = length(g); %Length of the initialising function
L = N/M; %Number of symbols in OFTDM frame

%Initialisation of matrices
G = zeros(2*L,N);
Z = zeros(2*L,N);
dZ = zeros(2*L,N);
Zo = zeros(2*L,N);
go = zeros(1,N);
NN = 0:1:N-1;

for i = 0:1:2*L-1
        G(i+1,:)=g(mod(NN+M/2*i,N)+1);
end
Z=fft(G);
dZ(1:L,:)=Z(L+1:2*L,:);
dZ(L+1:2*L,:)=Z(1:L,:);
Zo = Z.*2./sqrt(abs(Z).^2.*M + abs(dZ).^2.*M);

%Initialising function of orthgonal MWH basis
go = sum(Zo,1)./(2*L);
```

APPENDIX 1.4 Fast signal modulation

```
function signal = sigfastform(c, g, M, alfa)
%% Input parameters
```

```
% c - a colomn of information symbols
%(complex numbers) of length 2N.
% g - initialisation function of the basis
% M  - number of sub-carriers
% alfa - phase parameter of the basis
N = length(g);
L = N/M; % number of shifts in time domain
j = sqrt(-1); % imaginary 1

%% Initialisation of the matrices
D = zeros(M,2*L); D6 = zeros(M,2*L);
Cr = zeros(M,L); Ci = zeros(M,L);
Phi = zeros(2*L, 2*L); A = zeros (M, M);
E = zeros(M,L); E1 = zeros(M,L);
E2 = zeros(M,2*L); E3 = zeros(M,2*L);
W = zeros(M,N);

%% Preliminary steps
%1)Matrix of information symbols D
Cr = reshape(c(1:N),M,L);
Ci = reshape(c(N+1:2*N),M,L);
lr = 1:2:2*L-1;
li = 2:2:2*L-2;
D(:,lr) = Cr;
D(:,li) = Ci(:,2:L);
D(:,2*L) = Ci(:,1);
%2)Diagonal matrices Phi and A
Adiag = exp( (0:1:(M-1)).*(-j*pi/M*alfa) );
A = diag(Adiag,0);
Phidiag = ones(1,2*L);
Phidiag(2:2:2*L)= ones(1,L).*(-j);
Phi = diag(Phidiag,0);
%3)Poliphase components
for i = 1:1:M
    E(i,:) = g(i:M:N).';
end
E1 = fft(E,[],2);
E2 = [E1 E1];
E3(1:M/2,:)=E2(1:M/2,:);
for r = 0:1:2*L-1
    E3(M/2+1:M,r+1) = E2(M/2+1:M,r+1).*exp(ones(M/2,1)...
        *(-j*2*pi/(2*L)*r));
end
E3=E3*M;
```

```
%% Main algorithm
ticID = tic;
D(2:2:M,2:2:2*L) = -D(2:2:M,2:2:2*L);
D6 = ifft( fft((ifft(A*D*Phi)),[],2).*E3, [], 2);
signal = reshape (D6(1:M/2,:),N,1) +...
 reshape (D6(M/2+1:M,:),N,1);
calc_time = toc(ticID)
```

ORIGINAL PAPERS

PI

ORTHOGONAL WELL-LOCALIZED WEYL-HEISENBERG BASIS CONSTRUCTION AND OPTIMIZATION FOR MULTICARRIER DIGITAL COMMUNICATION SYSTEMS

by

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PII

ALGORITHMS FOR CONSTRUCTION OF ORTHOGONAL WELL-LOCALIZED BASES

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Algorithms for Construction of Orthogonal Well-Localized Bases

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Abstract—The paper considers an algorithm of the construction of orthogonal Weyl-Heisenberg bases which is based on the singular decomposition of the Gabor basis's matrix. The obtained basis has a good time-frequency localization, because its initializing function is close to the ideally localized Gaussian function. The sequence of the orthogonality conditions of the bases allows the formulation of a new computationally efficient algorithm of orthogonalization based on the fast Fourier transform. The modeling results confirm the identity of the initializing functions obtained as a result of these two algorithms and the good basis localization. The developed fast algorithm makes it possible to considerably expand the field of possible practical application of such bases, including the telecommunication devices based on the principle of orthogonal frequency-time division multiplexing (OFTDM).

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INTRODUCTION

Orthogonal well localized bases are applied to obtain time-frequency information on signals, in spectral analysis, in the processing of images, and in radiolocation. It should be noted that the expansion by orthogonal bases obtained by uniform shifts in the time and frequency of several initializing functions (the Weyl-Heisenberg generalized bases) yields a time-frequency description of signals similar to the wavelet transform [1]. From the Balian-Low theorem, it follows that the use of only one initializing function leads to the fact that the obtained Weyl-Heisenberg bases are poorly localized. Thus, the development of methods of synthesis of the generalized well localized Weyl-Heisenberg bases is highly important, regardless of the field of their further application.

One of the most relevant applications of such bases is in the field of information transmission. In the systems with orthogonal frequency-time division multiplexing (OFTDM), it is these bases that make it possible to effectively withstand the difficult conditions in real channels, i.e., the interchannel interference caused by multibeam propagation, the Doppler shift, and impulse noise. In a disperse medium, in addition to the additive interference, noise as intersymbol interference (ISI) and interchannel interference (ICI) appear. Physically, the appearance of ISI and ICI in channels with frequency-time scattering is explained by a loss of orthogonality between the disturbed basis functions of the signal at the channel output. As a result, the procedure of signal demodulation is no longer optimal at the receiving side. Information from each subcarrier channel leaks to neighboring ones [3, 4].

In this way, in a medium with spatial-time scattering, well localized bases provide the best restoration of the signal. In particular, in systems of communication using OFTDM communications, these bases have the lowest sensitivity (robustness) to interchannel and intersymbol interference.

Up to now, one of the main problems in the application of well-localized bases has been the complexity of the algorithm of their formation. In particular, this is a substantial barrier in the use of such bases in subscriber telecommunication devices. The paper shows in detail how, from the algorithm of the formation of a well-localized Weyl-Heisenberg basis, through an ideally localized but nonorthogonal Gabor basis, by means of a number of orthogonality criteria, it is possible to come to a computationally effective algorithm by use of the fast Fourier transform. Thus, the procedure of basis construction is simplified, which leads to a considerable increase of its range of possible application.

GENERALIZED WEYL-HEISENBERG BASIS

An ensemble of discrete functions (or signals) $s[n]$, $n \in J_N \triangleq \{0, 1, \ldots, N-1\}$is defined as a linear envelope thrust on the Weyl-Heisenberg basis $\mathcal{B}[J_N]$:

$$s[n] = \sum_{k=0}^{M-1}\left(\sum_{l=0}^{L-1} c_{k,l}^R \psi_{k,l}^R[n] + \sum_{l=0}^{L-1} c_{k,l}^I \psi_{k,l}^I[n]\right), \quad n \in J_N, \tag{1}$$

$$\psi_{k,l}^R[n] = g[(n - lM)_{\mathrm{mod}N}]\exp\left(j\frac{2\pi}{M}k(n-\alpha/2)\right), \tag{2}$$

$$\psi_{k,l}^I[n] = -jg[(n + M/2 - lM)_{\mathrm{mod}N}]\exp\left(j\frac{2\pi}{M}k(n-\alpha/2)\right), \tag{3}$$

$$\mathcal{B}[J_N] = \{\psi_{k,l}^R[n], \psi_{k,l}^I[n]\}, \tag{4}$$

where $N = ML \geq M$ ($M \geq 2$ is even-numbered, L is any natural number), and α is a phase parameter.

The system of functions $\mathcal{B}[J_N]$ (generalized Weyl-Heisenberg basis) is normalized and orthogonal

$$\langle \psi_{k,l}^R(t), \psi_{k',l'}^R(t)\rangle_R = \delta_{k,k'}\delta_{l,l'}, \quad \langle \psi_{k,l}^I(t), \psi_{k',l'}^I(t)\rangle_R = \delta_{k,k'}\delta_{l,l'}, \quad \langle \psi_{k,l}^R(t), \psi_{k',l'}^I(t)\rangle_R = 0 \tag{5}$$

in the sense of a real scalar product, which is defined as the real part of a common scalar product:

$$\langle x[n], y[n]\rangle_R = \mathrm{Re}(\langle x[n], y[n]\rangle), \quad \langle x[n], y[n]\rangle = \sum_{n=0}^{N-1} x[n], y^*[n], \tag{6}$$

where (*) is the sign of complex conjugation.

The condition of orthogonality (5) can be presented in the matrix form

$$\mathrm{Re}(\mathbf{U}^*\mathbf{U}) = \mathbf{I}_{2N}, \tag{7}$$

where (*) is the symbol of the Hermitean conjugation; $\mathbf{I}_{2N}$ is the identity ($2N \times 2N$) matrix; and $\mathbf{U} = (\mathbf{U}_R, \mathbf{U}_I)$ is the block-structured rectangular matrix of dimensionality ($N \times 2N$), in which blocks $\mathbf{U}_R$, $\mathbf{U}_I$ are rectangular ($N \times N$) matrices composed of the columns of the appropriate basis functions $\psi_{k,l}^R = (\psi_{k,l}^R[0], \ldots, \psi_{k,l}^R[N-1])^T$, $\psi_{k,l}^I = (\psi_{k,l}^I[0], \ldots, \psi_{k,l}^I[N-1])^T$ for all values of indices $k = 0, \ldots, M-1$, $l = 0, \ldots, L-1$.

Note that formula (1) can be considered as the algorithm of the formation (modulation) of the OFTDM signal in discrete time. The appropriate algorithm of demodulation is

$$c_{k,l}^R = \langle s[n], \psi_{k,l}^R[n]\rangle_R, \quad c_{k,l}^I = \langle s[n], \psi_{k,l}^I[n]\rangle_R.$$

Let us consider in more detail the algebraic approach to the construction of the Weyl-Heisenberg bases proposed by V.P. Volchkov in [3].

We emphasize again that in practical applications, an important role is played just by the good localization of basis functions in both the frequency and time domain. However, the Heisenberg uncertainty principle restricts the possibility of improving such localization by the following inequality:

$$\left(\int_{-\infty}^{\infty}(t-\tau)^2|g(t)|^2 dt\right)\left(\int_{-\infty}^{\infty}(f-\nu)^2|G(f)|^2 df\right) \geq \frac{\|g(t)\|^4}{16\pi^2}.$$

The equality in this expression is achieved only when $g(t)$ is the Gauss function [5]. For all other functions, this expression turns into an inequality.

Unfortunately, the basis built up on the Gaussian as the initializing function (the Gabor basis) is not orthogonal.

Accordingly, the basic idea underlying the algorithm is to construct such a basis that, on the one hand, has the forming function close to the Gauss function, i.e., has good localization, and, on the other, is orthogonal.

We define the complex signal basis with the desirable characteristics of localization in time and frequency as a block matrix

$$\mathbf{G} = [\mathbf{G}_R, \mathbf{G}_I], \tag{8}$$

$$G_R(n, lM + k) = \psi^R_{k,l}[n]\big|_{g(t) = g_0(t)}, \quad G_I(n, lM + k) = \psi^I_{k,l}[n]\big|_{g(t) = g_0(t)}, \tag{9}$$

where $n = 0, ..., N - 1$, $l = 0, ..., L - 1$, $l = 0, ..., L - 1$.

Clearly, the forming function $g_0(t)$ determines the characteristics of the localization of the given basis. For the Gabor basis $\mathcal{B}_0[J_N]$ as $g(t)$, the Gaussian $g_0(t) = (2\sigma)^{1/4}\exp(-\pi\sigma t^2)$, $t \in \mathbb{R}$, having the best characteristics of localization by time and frequency, is chosen.

Note that since the Gabor basis is not always orthogonal, its matrix $\mathbf{G}$ does not fulfill the condition of orthogonality (5).

We denote by $M_{m,n}(\mathbb{F})$ the set of all matrices of dimension $m \times n$ over the field $\mathbb{F}$. If $m = n$, the abbreviated notation $M_n(\mathbb{F})$ is used. Here, $\mathbb{F}$ is either the field of real numbers $\mathbb{R}$, or the field of complex numbers $\mathbb{C}$; in addition, we denote the set of integers by $\mathbb{Z}$.

The problem of finding the matrix $\mathbf{U}$ of the generalized orthogonal Weyl-Heisenberg basis that is the closest to the matrix of the Gabor basis $\mathbf{G}$ by the matrix norm $\|\mathbf{A}\|^2_E = \mathrm{tr}(\mathbf{AA}^*)$ is considered.

Problem 1 (basic). *On the subset* $\mathfrak{U} = \{\mathbf{U} \in M_{N,2N}(\mathbb{C}) : \mathrm{Re}(\mathbf{U}^*\mathbf{U}) = \mathbf{I}_{2N}\}$ *of the complex orthogonal matrices for which the expression*

$$\mathrm{Re}(\mathbf{U}^*\mathbf{U}) = \mathbf{I}_{2N}, \tag{10}$$

is true, we find the optimal matrix $\mathbf{U}_{\mathrm{opt}}$ *that provides a minimum in the problem on the extremum*

$$\mathbf{U}_{\mathrm{opt}} : \min_{U \in \mathfrak{U}} \|\mathbf{G} - \mathbf{U}\|^2_E, \tag{11}$$

where $\mathbf{G} \in M_{N,2N}(\mathbb{C})$ *is the matrix of the Gabor basis.*

Note that the solution $\mathbf{U}_{\mathrm{opt}}$ will assign the orthogonal basis (4) with the best characteristics of localization.

Finding the extremum $\mathbf{U}_{\mathrm{opt}}$ is simplified if instead of problem 1 the following auxiliary problem is solved:

Problem 2 (auxiliary). *On the set* $\mathcal{V} = \{\mathbf{V} \in M_{2N}(\mathbb{R}) : \mathbf{V}^*\mathbf{V}) = \mathbf{I}_{2N}\}$ *of real orthogonal matrices, we find the optimal matrix* $\mathbf{V}_{\mathrm{opt}}$ *that provides a minimum in the problem*

$$\mathbf{V}_{\mathrm{opt}} : \min_{U \in \mathcal{V}} \|\mathbf{G}_B - \mathbf{V}\|^2_E, \tag{12}$$

where $\mathbf{G}_B = \begin{bmatrix} \mathrm{Re}\mathbf{G} \\ \mathrm{Im}\mathbf{G} \end{bmatrix} \in M_{2N}(\mathbb{R})$ *is assigned by the complex matrix of the Gabor basis* $\mathbf{G}$.

In [3], it is shown that the extremal problems (11) and (12) are equivalent, and their solutions are interrelated by the relation

$$\mathbf{U}_{\mathrm{opt}} = \mathbf{V}_{1\mathrm{opt}} + j\mathbf{V}_{2\mathrm{opt}}, \tag{13}$$

where matrices $\mathbf{V}_{1\mathrm{opt}}$ and $\mathbf{V}_{2\mathrm{opt}} \in M_{N \times 2N}(\mathbb{R})$are found from the block partition $\mathbf{V}_{\mathrm{opt}} = \begin{bmatrix} \mathbf{V}_{1\mathrm{opt}} \\ \mathbf{V}_{2\mathrm{opt}} \end{bmatrix}$.

Thus, matrix $\mathbf{U}_{\mathrm{opt}}$ of the sought for optimal Weyl-Heisenberg basis can be obtained from the solution of Problem 2 by formula (13). Then, matrix $\mathbf{V}_{\mathrm{opt}}$ is determined by the following theorem:

Theorem 1. *Optimal matrix* $\mathbf{V}_{\mathrm{opt}}$ *providing a maximum in extremum problem* (12) *is determined by the expression*

$$\mathbf{V}_{\mathrm{opt}} = \mathbf{SW}^T, \tag{14}$$

where $\mathbf{S}, \mathbf{W} \in \mathcal{V}$ *are a pair of real orthogonal matrices included in the singular transform of the matrix* $\mathbf{G}_B = \mathbf{S\Sigma W}^T$.

Matrices **S**, **W** are composed of eigenvectors of matrices $\mathbf{G}_B\mathbf{G}_B^T$ and $\mathbf{G}_B^T\mathbf{G}_B$, respectively; $\mathbf{\Sigma} = (\sigma_{i,j}) \in M_{2N}$ is the diagonal matrix in which on the main diagonal there are singular numbers $\sigma_i = \sigma_{ii}$ of matrix $\mathbf{G}_B$ in a nonincreasing order: $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{2N} \geq 0$.

The value of the achieved extreme in problem (12) is

$$\min_{V \in \tilde{V}} \|\mathbf{G}_B - \mathbf{V}\|_E^2 = \|\mathbf{G}_B - \mathbf{V}_{\text{opt}}\|_E^2 = \sum_{i=1}^{2N} (\sigma_i - 1)^2, \tag{15}$$

and the optimal forming function $g[n]$ $n \in J_N$ for basis (4) is set by the first column of this matrix, i.e.,

$$g[n] = U_{\text{opt}}(n, 1). \tag{16}$$

The deficiency of this algorithm is that at its core lies a singular transform (Procrustean transform), which requires a greater number of mathematical operations and is inefficient, in particular, at high values of N (several thousand and over).

ORTHOGONALITY CRITERIA AND THE WIENER BASIS

Since the only forming function $\mathcal{B}[J_N]$ actually lies at the core of the basis $g[n]$, though shifted by phase, it is possible to simplify the algorithm for obtaining it by using the orthogonality criteria of the basis as the condition for this function.

We consider special criteria of the orthogonality of the Weyl-Heisenberg basis allowing a computationally more effective algorithm.

We dwell on the case where $g[n]$ is a real function and has the property of the conjugated N symmetry: $g[n] = g^*[(-n)_N]$.

This type of symmetry is in correspondence with the optimal value of the phase parameter $\alpha = (M/2)_{\operatorname{mod} M}$ [6]. Note that when α is not optimal, the forming function for the Weyl-Heisenberg basis, resulting from the orthogonolization algorithm, is no longer symmetrical and its localization is worse.

Under the above-mentioned conditions of symmetry, it is possible to show that the basis

$$\begin{gathered} \mathrm{E}(J_N) = \{g_{l,m}[n]\}, \quad \langle g_{l,m}[n], g_{l',m'}[n]\rangle = \delta_{l,l'}\delta_{m,m'}, \\ g_{l,m}[n] = g[(n - lM)_N]\exp\left(j\frac{4\pi}{M}mn\right), \quad m \in J_{M/2}, \quad l \in J_L \end{gathered} \tag{17}$$

is orthonormal in the sense of a common scalar product only in the case where the generalized Weyl-Heisenberg basis constructed on the same initializing function $g[n]$ is orthonormal in the sense of a real scalar product (6), and the following theorem [7] holds:

Theorem 2. *A necessary and sufficient condition for the orthonormality of basis* $\mathrm{E}[J_N]$ (17) *(and of the Weyl-Heisenberg basis* $\mathcal{B}[J_N]$*) in the time domain is the following equality*:

$$\sum_{r=0}^{2L-1} g[(n - rM/2)_N]g[(n - rM/2 - lM)_N] = (2/M)\delta_{l,0}, \quad \forall n \in J_N, \quad \forall l \in J_L. \tag{18}$$

We take the next step and transfer from the orthogonality condition of the reduced basis $\mathrm{E}[J_N]$ to the condition of the orthogonality of the appropriate Wiener bases.

Definition 1 [8]. *The body of functions $f_0, f_1, \ldots, f_{N-1}$ of the linear subspace $V \subset \tilde{\mathbb{C}}^N$ ($\tilde{\mathbb{C}}^N$ is the space of the N periodic complex functions of the integer argument) is called the Weiner basis of subspace V, if the discrete-periodic analogue of the Wiener theorem is good for it; functions $\{g[n - kN/J]\}_{k=0}^{J-1}$ form the basis of the space V if and only if $g \in \tilde{\mathbb{C}}^N$ and is presentable as $g = \sum_{k=0}^{J-1} a_k f_k$, where all a_k differ from zero.*

Therefore, the existence of the Wiener basis is necessary and sufficient for the existence of function g, whose vector of shifts

$$\mathbf{g[n]} \triangleq (g[n], g[(n - N/J)_N], \ldots, g[(n - N/J)_N])^T \tag{19}$$

is the basis of the subspace V.

Definition 2. *The transformation allowing the transfer from the vector of shifts to the orthogonal Wiener basis is called the Wiener transform and can be written as*

$$\eta_k^{M/2}[n] = \sum_{r=0}^{2L-1} g[(n - rN/2L)_N]\exp\left(\frac{2\pi j}{2L}rk\right). \tag{20}$$

The inverse Wiener transformation will be determined in the following way:

$$g[n] = \frac{1}{2L}\sum_{k=0}^{2L-1}\eta_k[n]. \tag{21}$$

We formulate the basic properties of the Wiener transformation:

(1) $\eta_k[(n \pm pN/2L)_N] = \eta_k[n]\exp\left(\pm j\frac{2\pi}{2L}kp\right)$.

(2) $\eta_{k\pm 2L}[n] = \eta_k[n]$.

(3) If $g[n]$ is a real function, then $\eta_k{}^*[n] = \eta_{-k}[n]$.

Using (7) and the above-mentioned properties, we prove the following theorem:

Theorem 3. *The necessary and sufficient condition for orthogonality of basis* $\mathrm{E}[J_N]$ (17) *in the time domain is the following equation*:

$$\left|\eta_k^{M/2}[n]\right|^2 + \left|\eta_{k+L}^{M/2}[n]\right|^2 = 4/M. \tag{22}$$

Proof. *First, we will prove the necessity of this condition. The components of orthogonality condition* (18) *are presented as the inverse Wiener transformation*

$$g[(n - pM/2)_N] = \frac{1}{2L}\sum_{k=0}^{2L-1}\eta_k^{M/2}[(n-pM/2)_N] = \frac{1}{2L}\sum_{k=0}^{2L-1}\eta_k^{M/2}[n]\exp\left(-j\frac{2\pi}{2L}kp\right),$$

$$g[(n - pM/2 - lM)_N] = \frac{1}{2L}\sum_{k=0}^{2L-1}\eta_k^{M/2}[n]\exp\left(-j\frac{2\pi}{2L}kp\right)\exp\left(-j\frac{2\pi}{2L}k2l\right).$$

Then, the left part of orthogonality condition (18) is

$$\frac{1}{4L^2}\sum_{p=0}^{2L-1}\sum_{k_1=0}^{2L-1}\eta_{k_1}^{M/2}[n]\sum_{k_2=0}^{2L-1}\eta_{k_2}^{M/2}[n]\exp\left(-j\frac{2\pi}{2L}(k_1+k_2)p\right)\exp\left(-j\frac{2\pi}{2L}k_2 2l\right)$$

$$= \frac{1}{4L^2}\sum_{k_1=0}^{2L-1}\sum_{k_2=0}^{2L-1}\eta_{k_1}^{M/2}[n]\eta_{k_2}^{M/2}\exp\left(-j\frac{2\pi}{2L}k_2 2l\right)[n]\sum_{p=0}^{2L-1}\exp\left(-j\frac{2\pi}{2L}(k_1+k_2)p\right).$$

And,

$$\sum_{p=0}^{2L-1}\exp\left(-j\frac{2\pi}{2L}(k_1+k_2)p\right) = \begin{cases} 2L, k_1 + k_2 = 2Lq, & q \in \mathbb{Z} \\ 0, & \forall k_1, k_2 : k_1 + k_2 \neq 2Lq. \end{cases}$$

Then, k_1 and k_2 change from 0 to $2L$, so this sum differs from zero only if $k_2 = -k_1 + 2L$. This fact is taken into orthogonality condition

$$1/2L\sum_{k_1=0}^{2L-1}\eta_{k_1}^{M/2}[n]\eta_{-k_1+2L}^{M/2}[n]\exp\left(j\frac{2\pi}{2L}k_1 2l\right)\exp\left(-j\frac{2\pi}{2L}2L2l\right)$$

$$= 1/2L\sum_{k_1=0}^{2L-1}\eta_{k_1}^{M/2}[n]\eta_{k_1}^{*M/2}[n]\exp\left(j\frac{2\pi}{L}k_1 l\right) = 1/2L\sum_{k_1=0}^{2L-1}\left|\eta_{k_1}^{M/2}[n]\right|^2\exp\left(j\frac{2\pi}{L}k_1 l\right)$$

$$= 1/2L\sum_{k_1=0}^{L-1}\left|\eta_{k_1}^{M/2}[n]\right|^2\exp\left(j\frac{2\pi}{L}k_1 l\right) + 1/2L\sum_{k_1=L}^{2L-1}\left|\eta_{k_1}^{M/2}[n]\right|^2\exp\left(j\frac{2\pi}{L}k_1 l\right)$$

$$= 1/2L\sum_{k_1=0}^{L-1}\left|\eta_{k_1}^{M/2}[n]\right|^2\exp\left(j\frac{2\pi}{L}k_1 l\right) + 1/2L\sum_{k'=0}^{L-1}\left|\eta_{k'+L}^{M/2}[n]\right|^2\exp\left(j\frac{2\pi}{L}k' l\right)$$

$$= 1/2L\sum_{k=0}^{L-1}\left(\left|\eta_{k_1}^{M/2}[n]\right|^2 + \left|\eta_{k+L}^{M/2}\right|[n]^2\right)\exp\left(j\frac{2\pi}{L}kl\right) = (2/M)\delta_{l,0},$$

where the properties of the Wiener transformation are utilized, $k' = k_1 - L$.

Taking into account that $\delta_{l,0}$ can be presented as the inverse discrete Fourier transform from unities

$$\delta_{l,0} = \frac{1}{L}\sum_{k=0}^{L-1} 1\exp\left(j\frac{2\pi}{L}kl\right),$$

we obtain the desired type of orthogonality criterion.

The sufficiency of condition (22) can be directly seen, having performed all actions applied in the proof of the necessity of this condition in the reverse order.

FAST ALGORITHM FOR CONSTRUCTION OF THE ORTHOGONAL BASIS

The criterion proved in Theorem 3 allows us to form an effective procedure of orthogonalization.

We take a certain function $g_0[n]$, $n \in J_N$ that is conjugated N symmetrical. To obtain good localization of the orthogonal basis as $g_0[n]$, we should choose the Gauss function.

We build the Wiener basis in the following form:

$$\eta_k^{M/2}[n] = \frac{2\tilde{\eta}_k^{M/2}[n]}{\sqrt{M\left|\tilde{\eta}_k^{M/2}[n]\right|^2 + M\left|\tilde{\eta}_{k-L}^{M/2}[n]\right|^2}}, \tag{23}$$

where $\tilde{\eta}_k^{M/2}[n] = \sum_{r=0}^{2L-1} g_0[(n - rM/2)_N]\exp\left(\frac{2\pi j}{2L}rk\right)$.

It is easy to demonstrate that with the direct substitution of expression (23) into the orthogonality criterion (22), we obtain an equality (in doing this, Property 2 of the Wiener transformation for $\tilde{\eta}_k^{M/2}[n]$ is used). It follows from this that the basis $E[J_N]$ and, respectively, also the Weyl-Heisenberg basis $\mathcal{B}[J_N]$ constructed on function $g[n]$ obtainable by the inverse Wiener transformation from basis $\eta_k^{M/2}[n]$ are orthogonal. The obtained forming function $g[n]$ has the same property of conjugated N symmetry as the initial function $g_0[n]$.

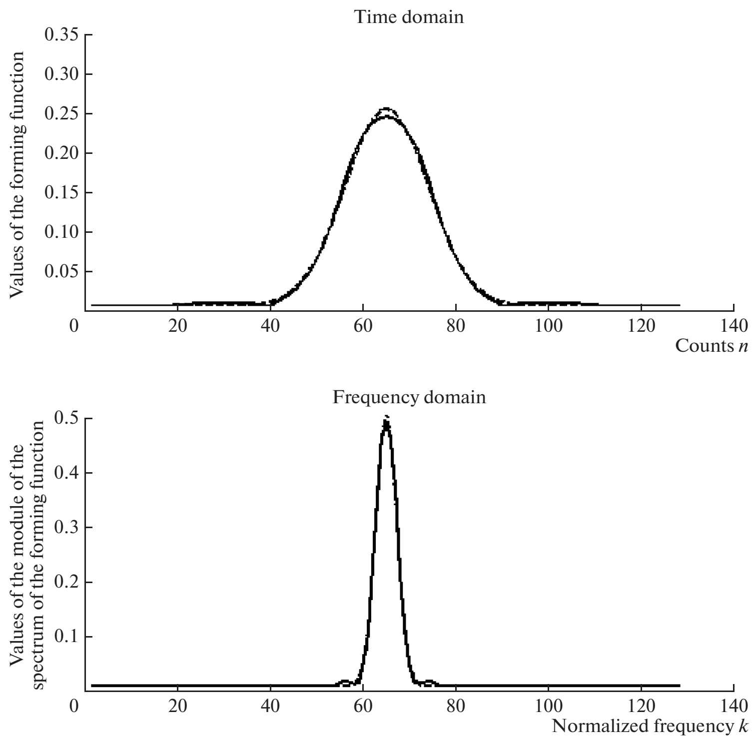
Note that obtaining the elements of the Wiener basis (20) based on function $g_0[n]$ is actually the discrete Fourier transform (DFT), for the implementation of which special fast algorithms have been developed. Their utilization allows a considerable increase in the computational effectiveness of the orthogonalization algorithm.

Let us formulate the set of sequential steps of the algorithm:

1. We build matrix $\mathbf{Z}$ of dimensionality $N \times 2L$, the lines of which are vectors of shifts $\mathbf{g}_0[\mathbf{n}]$ of type (19) of the function $g_0[n]$, $n \in J_N$.

2. To perform the Wiener transformation of function $g_0[n]$ (i.e., to obtain functions $\tilde{\eta}_k^{M/2}[n]$), we apply the fast discrete Fourier transform over the lines of matrix $\mathbf{Z}$. As a result, we obtain matrix $\mathbf{Z}_\mathbf{f}$ of dimensionality $N \times 2L$, the lines of which are functions $\tilde{\eta}_k^{M/2}[n]$, $n \in J_N$ answering all $k = 0, 1, ..., 2L - 1$.

3. Since functions $\tilde{\eta}_k^{M/2}[n]$ are $2L$ periodic by index k (Property 2 of the Wiener transformation), to obtain functions $\tilde{\eta}_{k-L}^{M/2}[n]$, it is sufficient to interchange the positions of the first and the last L lines of



Graphs of the Gauss function and obtained basis-forming function $g[n]$ in the frequency and time domains.

matrix $\mathbf{Z_f}$. As a result, we obtain matrix $\mathbf{dZ_f}$, the lines of which are functions $\tilde{\eta}_{k-L}^{M/2}[n]$, $n \in J_N$, for each $k = 0, 1, ..., 2L-1$.

4. We obtain the Wiener basis $\eta_k^{M/2}[n]$ in accordance with formula (23). To do this, we build matrix $\mathbf{V} = \dfrac{2\mathbf{Z_f}}{\sqrt{M|\mathbf{Z_f}|^2 + M|\mathbf{dZ_f}|^2}}$, where all operations (multiplication by the number, division, squaring, etc.) are performed over the elements of the matrix. The lines of matrix $\mathbf{V}$ are the elements of the Wiener basis $\eta_k^{M/2}[n]$.

5. In order to obtain the desired forming function $g[n]$, it remains only to apply the inverse Wiener transformation of the columns of matrix $\mathbf{V}$.

MODELING RESULTS

As a result of the performed mathematical modeling of the operation of the algorithms of the Weyl-Heisenberg generalized basis built on the orthogonal transform and new fast transformations, it has been found that using the same initializing functions $g_0[n]$, the same forming functions are obtained. The figure shows the graphs of the initializing Gauss function chosen as $g_0[n]$ (dashed curves) and obtained as a result of both algorithms of the forming function $g[n]$ of the orthogonal well-localized basis in the time and frequency domain (solid curves).

From the performed study, it can be concluded that the developed algorithm achieves two purposes: first, it is computationally effective due to the application of the Fourier fast transform and, second, it allows the construction of bases with good localization due to their closeness to the ideally localized Gauss function.

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PIII

EFFICIENT WH-OFTDM SIGNAL PROCESSING

by

D. A. Petrov, T. Hämäläinen 2010

The International Conference on Ultra-Modern Telecommunications (ICUMT)

PIV

ULTRAWIDEBAND SIGNALS CONSTRUCTED FROM GENERALIZED WEYL-HEISENBERG BASES

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PV

BETTER PERFORMANCE OF MOBILE DEVICES IN TIME-FREQUENCY DISPERSIVE CHANNELS USING WELL-LOCALIZED BASES

by

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Better Performance of Mobile Devices in Time-Frequency Dispersive Channels Using Well-Localized Bases

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Abstract. The main aim of this paper is to show on the conceptual level the practical and commercial benefits of signal construction and processing technology based on orthogonal frequency and time domains multiplexing (OFTDM). This technology utilizes mathematical framework of special orthogonal bases with the best time-frequency localization. Higher spectral and energy efficiency of telecommunication systems together with robustness against complex realistic channel conditions are achieved. In particular, the level of interference between subcarrier channels in time and in frequency domain is minimized. This approach can have wide implementation in wideband mobile networks (WiMAX, LTE), digital television (DVB-T/H) and other telecommunication systems.

Keywords: WiMAX, LTE, OFDM, OFTDM, Well-localized bases, time-frequency dispersion.

1 Introduction

Constriction of high-speed wireless digital telecommunication systems often faces the problem that real radio channel (propagation media) is time-frequency dispersive [1]. In particular this follows from the fact that radio signal comes to the receiver through multiple paths after many reflections from the nonstationary media inhomogeneities like city buildings, moving objects, hydrometeors, ionospheric layers, etc. Among the examples of such dispersive channels are wideband multiple access radio lines (mobile WiMAX, LTE), digital television (DVB-T / H), short-wave and ultra-short-wave radio lines.

Currently one of the most popular physical layer technologies for data transmission is multiplexing with orthogonal frequency division (OFDM). There is no doubt that this technology will be also used in future telecommunication standards.

As a result of time-frequency dispersion of the OFDM signal such effects as multipath propagation, amplitude-phase fading, Doppler shift and spreading are observed at the receiving side. Those effects are even stronger if receiver is moving in a car, situated in the building or near the strong source of electromagnetic emission

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like airport. As a consequence intercarrier (ICI) and intersimbol interferences (ISI) considerably worsen the quality of the received signal. Moreover ICI cannot be compensated or filtrated by regular digital processing methods.

It is necessary to mention that the structure and properties of OFDM signals are determined by the basis, which was used for its construction (signal basis). In classical OFDM systems the role of such basis plays the family of rectangular initializing functions shifted in time and frequency. Thus OFDM signals are constructed as the linear combination of such basis functions with real or complex data symbols (determined by the signal constellation: QAM, PSK, ets.) as the coefficients. Channel equalization is simplified because OFDM may be viewed as many slowly modulated narrowband signals rather than one rapidly modulated wideband signal.

In channels with time-frequency dispersion complex multiplicative interference affects the signal in addition to additive noise. Such interference has the factor of time dispersion characterized by multiplicative action in frequency domain, which is equivalent to the convolution with the signal in time domain. For the factor of frequency dispersion it is vice versa: multiplicative action in time domain and convolution in frequency domain. As a result signal basis is distorted, it becomes nonorthogonal and Nyquist theorem [2] is not fulfilled any more.

In other words the appearance of ICI and ISI in time-frequency dispersive channels is caused by the loss of orthogonality between disturbed basic functions. The demodulation procedure in the receiver becomes nonoptimal. The leakage of information from every channel subcarrier to the neighboring channels takes place. Notably the value of this mutual interference depends on the time-frequency localization of signal basis functions and is determined by the support of their uncertainty functions. The faster decay the tails of the uncertainty function the better is time-frequency localization of the signal basis and thus the less is the level of ISI and ICI.

The low symbol rate of OFDM systems allows to use a guard interval or cyclic prefix (CP) between consecutive OFDM symbols. The length of the CP - T_s is longer than the time dispersion of the channel. Because of that it is possible to handle time spreading and eliminate ISI. Thus the effect of time dispersion can be effectively compensated but with the loss of spectral and energy efficiency. In particular, the cyclic prefix coasts a loss of spectral efficiency by $T_s/(T_s+T_0)$, where T_0 is the initial OFDM symbol duration [3]. It also implies the same order of power loss. In spite of several efforts [4], [5] towards the reduction of these overheads CP rests very simple and effective approach.

The rectangular form of forming functions used in classical OFDM systems in time domain correspond to sinc(x) or sin(x)/x functions in frequency domain. It is not optimal from the point of ICI. The level of out-of-band emission is overrated.

This is one of the reasons of connection breaks when a subscriber enlarges its velocity or when the signal/noise limit is exceeded. It causes synchronization upsets or inaccurate assessments of channel parameters when frequency dispersion and ICI are strong.

Now we will briefly consider several known methods of ICI reduction:

- In telecommunication systems based on OFDM principals signal basis functions represent the segments of harmonics. In frequency domain they have slowly decaying tails. It is possible to improve the robustness of such signals against Doppler effects by spreading the spectrum of basic functions, e.g. by shortening the duration of harmonics. This results in spreading of the signal's spectrum and in extension of distance between subcarriers. However such changes are not always allowable in terms of existing standard. In addition they do not rescue against the part of intercarrier interference caused by overlapping of side lobes of sinc functions.
- Sometimes ICI could be additionally mitigated by adding guard intervals in the form of dummy subcarriers at the borders of frequency range and between informational subcarriers.
- Other approach to the decision of ICI problem is based on the usage of window Fourier transform. In this case nothing is changed at the transmitting side but at the receiver the initial orthogonal FFT basis is replaced with nonorthogonal basis of weighted FFT with better localization in frequency domain. This method allows to reduce the component of ICI caused by overlapped side lobes of basis' functions. However window function spreads the spectrum of each subcarrier at the receiver. This results in overlapping of main peaks of basic functions and again in the leak of information from one subcarrier to another but in different from. In addition refusal from orthogonality increases the noise level.
- Next approach is based on the generalization of Nyquist-Kotel'nikov-Shannon sampling theorem. The main idea of this approach is based on oversampling and usage of series with well-localized core functions for signal interpolation. In other words received signal is discretized with frequency much higher than critical Nyquist frequency. After that for signal reconstruction so called atomic functions [5] are used. Family of such functions shifted in frequency domain can also be considered as a signal basis [6]. Because of oversampling the main lobe of atomic function can be selected in correspondence with frequency range of subcarrier channel of OFDM signal. Moreover such functions have fast decaying side lobes. This approach seems to be one of the most promising. Nevertheless basis constricted from atomic functions cannot be always made orthogonal. This complicates signal processing and reduces robustness against noise interference.

From the foresighted analysis it follows that the problem of ICI reduction in mobile OFDM systems is still actual and does not always have satisfactory decision.

2 Orthogonal Frequency and Time Domains Multiplexing

The main idea our research is to use well-localized basis function instead of rectangular ones used in classical OFDM. The optimal localization and "tuning" of basis parameters reduce the out-of-band emission and mutual interference of subcarriers in frequency domain.

Utilization of well-localized bases requires more complex synthesis and processing procedures. That is why the important part of the research is devoted to the development of computationally efficient methods which are comparable to discrete

Fourier transform (DFT) used in classical OFDM. Thus the whole scope of works includes the following stages:

1. Determine the type and structure of signal basis that fulfills several important requirements:
   - From the digital nature of signal processing it follows that the basis should be discrete and defined in finite number of points (finite support of basic functions). This makes clearer it's practical utilization and future technical realization.
   - Of cause, basis should be orthogonal.
   - Basis should have good time-frequency localization under some criteria.
   - The symmetry of basis' functions is not an imperative requirement but there is no reason to make overlapping of functions stronger from one of the sides in the signal.
2. After the structure of the basis was determined it was necessary to develop efficient methods of its synthesis.
3. In addition to that it was necessary to explain theoretically the choice of basis parameters which allow to adjust localization characteristics.
4. The matrix form of the basis gives rather straightforward approach to signal modulation and demodulation: these operations can be performed by vector matrix multiplication. However it is possible to achieve much better computational efficiency taking into account the structural particularities of basis' functions.

Steps from 1 to 4 are described in more details in works [7], [8], [9] and will be briefly considered in section 3. They form the mathematical framework of the OFTDM technology.

The next part of the research which is on the go now is more practical and includes following steps:

5. Analyze on the link level how channel conditions influence the characteristics of OFTDM system. In particular, it is necessary to receive bloc error rates (BLER) for different values of interference and block size. It will be the input for the next step.
6. Analyze the performance of the OFDTM based system on the system level and compare it to classical OFDM systems.

Future benefits of proposed approach are given in section 4.

3 Well-Localized Weyl-Heisenberg Bases and Signal Structure

Fourier transform is a powerful instrument of signal analysis in linear time invariant systems. Nevertheless it is complicated to use it for short-term or transitional processes when we need information about spectrum localized in time. Development of some universal basis (analogical to Fourier basis) which could simplify the processing of most types of signals is a very difficult problem [10]. Several known examples of such bases exit including wavelet bases, bases constructed from splines and atomic functions, etc. Weyl-Heisenberg bases were initially derived from Gabor bases and can be constructed by discrete shifts in time and frequency of initializing function (or family of initializing functions in more general case).

The quality of time-frequency localization of such bases is limited by two constrains:

- Fundamental Heisenberg uncertainty principle which states that with improvement of localization of function in time domain we lose in frequency localization and vice versa. Mathematically it is described by the following relation: $\sigma_t^2 \sigma_\omega^2 \geq 1/4$, where σ_t^2 and σ_ω^2 is variance in time and frequency domain correspondingly. The equality is attained only for Gaussian function.
- From the Balian-Low theorem, it follows that the use of only one initializing function leads to the fact that the obtained Wey-Heisenberg bases are poorly localized in the case of maximum density of discrete time-frequency lattice.

However the last constraint can be overcame by the use of two types of initializing functions in the basis and special orthogonality condition. Thus the number of basis' functions doubles in compare to classical Weyl-Heizenberg basis used in OFDM. At the same time instead of complex QAM modulation coefficients their real and imaginary parts are used so that changes should be made manly in the physical layer. This idea was firstly proposed in the paper [11]. The main difference and advantage at the same time of our approach is that the basis is initially considered in the finite-dimensional space of N-periodical functions. $N = L \cdot M$, where $M \geq 2$ is the number of subcarriers, L - any natural number unequal zero which corresponds to the number of shifts in time domain.

Generalized Weyl-Heisenberg basis $\mathcal{B}[J_N]$ and transmitted OFTDM signal $s(t)$ in discrete time can be presented in the following form

$$s[n] = \sum_{k=0}^{M-1} \left(\sum_{l=0}^{L-1} c_{k,l}^R \, \psi_{k,l}^R[n] - \sum_{l=0}^{L-1} c_{k,l}^I \, \psi_{k,l}^I[n] \right), \; n \in J_N . \tag{1}$$

$$\psi_{k,l}^R[n] = g\left[(n - lM)_{\mathrm{mod}\,N} \right] \exp\left(j \frac{2\pi}{M} k (n - \alpha/2) \right), \tag{2}$$

$$\psi_{k,l}^I[n] = -jg\left[(n + M/2 - lM)_{\mathrm{mod}\,N} \right] \exp\left(j \frac{2\pi}{M} k (n - \alpha/2) \right), \tag{3}$$

$$\mathcal{B}[J_N] \overset{def}{=} \left\{ \psi_{k,l}^R[n], \psi_{k,l}^I[n] \right\}, \tag{4}$$

where $c_{k,l}^R = \mathrm{Re}(a_{k,l})$ and $c_{k,l}^I = \mathrm{Im}(a_{k,l})$ are real and imaginary parts of complex information QAM symbols $a_{k,l}$ used in OFDM; $s[n] = s(nT/M)$; $g[n] = g[nT/M]$ and $g[n + M/2]$ - initializing functions; $J_N = \{0, 1, ..., N-1\}$; α - phase parameter.

The system of basic functions $\mathcal{B}[J_N]$ is orthogonal in terms of real scalar product defined on the Hilbert space of discrete functions on J_N

$$\left\langle x[n], y[n] \right\rangle_R = \mathrm{Re}\sum_{n=0}^{N-1} x[n]\cdot y^*[n], \tag{5}$$

where $*$ is the sign of complex conjugation.

Matrix representation of basis (4) $\mathbf{U} = (\mathbf{U}_R, \mathbf{U}_I)$ is a $N\times 2N$ block matrix with blocks $\mathbf{U}_R$, $\mathbf{U}_I$ - square $N\times N$ matrixes constructed from columns of basis' functions $\vec{\psi}^R_{k,l} = \left(\psi^R_{k,l}[0],...,\psi^R_{k,l}[N-1]\right)^T$ and $\vec{\psi}^I_{k,l} = \left(\psi^I_{k,l}[0],...,\psi^I_{k,l}[N-1]\right)^T$ for all indexes $k = 0,...,M-1$, $l = 0,...,L-1$. This presentation makes easier theoretical investigation of the basis. In particular, orthogonality condition in matrix form is

$$\mathrm{Re}\left(\mathbf{U}^*\mathbf{U}\right) = \mathbf{I}_{2N}, \tag{6}$$

where $\mathbf{I}_{2N}$ is $2N\times 2N$ identity matrix; signal modulation and demodulation will look like

$$\vec{S}^T = \mathbf{U}\vec{C}^T; \quad \vec{C}^T = \mathrm{Re}\left\{\mathbf{U}^*\vec{S}^T\right\}. \tag{7}$$

It is necessary to mention that Weyl-Heisenberg basis constructed from rectangular functions is orthogonal in time domain only because these functions do not overlap. Thus this orthogonality is artificial in some sense. In real dispersive channel consecutive OFDM symbols will overlap and that is why it is impossible to refuse from cyclic prefix. In OFTDM this problem does not exits ever more. Basis functions can be overlapped not only in frequency but also in time domain if orthogonality conditions (5) or (7) are fulfilled. Time-frequency structure of OFTDM signal is described on Fig-1. Thus CP can be used less often mainly to divide OFTDM symbols and for synchronization purpose.

In addition to that generalized Weyl-Heisenberg bases can have much better localization in frequency domain without the loss in spectral efficiency. Initializing functions $g[n]$ in (2), (3) can be selected in such a way that the matrix $\mathbf{U}_{opt}$ of the basis will minimize the following functional on the space of orthogonal matrixes: A

$$\mathbf{U}_{opt} : \min_{\mathbf{U}\in\mathfrak{A}} \left\|\mathbf{G}-\mathbf{U}\right\|_E^2, \tag{8}$$

where $\mathbf{G}$ is the matrix of some nonorthogonal basis with desired localization characteristics; $\|\mathbf{A}\|_E^2 = \mathrm{tr}(\mathbf{AA}^*)$ is a Frobenius norm. The quality of localization can be estimated, for example, with the help of ambiguity function:

$$A(\tau,\upsilon) \triangleq \sum_{n=0}^{N-1} g[n]\, g^*\left[(n+\tau)_N\right]\exp\left(-j\frac{2\pi}{N}\upsilon n\right). \tag{9}$$

In general case any function with necessary localization properties can be used as an impute to problem (8). In particular, the ambiguity function of orthogonal generalized Weil-Heisenberg basis constructed from Gaussian function is presented on Fig. 2. The size of side lobes is very small in compare to the main lobe.

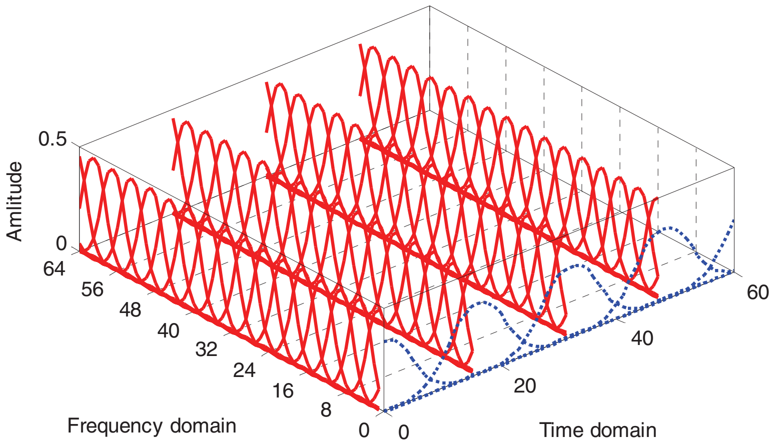

Fig. 1. Time-frequency structure of generalized Weyl-Heisenberg basis. Basis functions in time domain (dotted lines) and in frequency domain (solid lines) are shown.

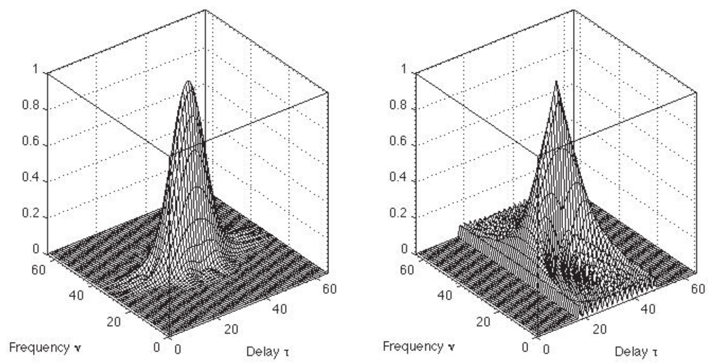

Fig 2. Module of ambiguity function of initializing function of generalized Weyl-Heisenberg function constructed from Gaussian function (left) and rectangular impulse (write)

Presented figures show two main advantages of proposed technique: firstly it is denser packing of the signal not only in frequency but also in time domain (Fig. 1) and, secondly, better localization in frequency domain (Fig. 2).

4 Conclusions

Of cause more simulation results are still required to justify the advantage of OFTDM scheme over classical OFDM. In particular in work [9] it has been already shown that

OFTDM signals have higher robustness against Doppler shift. Nevertheless the model used in that paper was rather simple and didn't include such important blocks as coding, interleaving, etc. Thus our future work includes such necessary and logical steps as:

- Extension of WiMAX link level model with OFTDM modulator.
- Utilization of link level model output in system level simulator [12].

In the conclusion it is important to formulate expected advantages of OFTDM technology:

- Application of OFTDM instead of OFDM on physical layer in channels with time-frequency dispersion improves spectral and energy efficiency. This effect is achieved because of additional intersymbol multiplexing used in OFTDM.
- In OFTDM signals the level of out of band emission is lower. Thus the requirements on the quality of transmitter's filter and on the guard intervals on the edges of the frequency range can be weaken.
- It is possible to improve the robustness of the system against ICI an ISI and to adopt better to the parameters of time-frequency dispersion.

In addition to factors mentioned above several economic benefits can be mentioned:

- One of the main effects of proposed approach is better interference resistance of the system and thus better reception quality of mobile users. It means that in equal conditions guaranteed quality will be achieved for the lower value of signal to interference and noise ratio (SINR). As a result less number of base stations is required in a given service area.
- Bad reception quality results in the necessity to use lower modulation indexes. Throughput and the number serviced subscribers go down. Thus in the same conditions OFTDM technology makes it possible to transmit data to larger number of users in compare to OFDM realisation. Increase in the maximum number of users with the constant number of base stations gives direct increase of profits.
- Improvement of the quality of services is the important factor in competitive struggle. It stimulates the demand for the new devices with better performance based in particular on proposed OFDTM technology.

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by

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