Neutrino transport in coherent quasiparticle approximation

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Abstract

Coherence is a fundamental and peculiar property in all quantum theories. It allows for classically unexpected phenomena such as neutrino oscillation, where neutrinos of different kinds spontaneously transform into each other. Taking coherence properly into account in a complicated physical system is far from trivial.

Consider the following physical situation: Neutrinos travel in a temporally varying medium and nonzero temperature. To describe the behaviour of these neutrinos, special relativity, coherence, and interactions must be taken into account. A general formalism for the analysis of such situations is provided by coherent quasiparticle approximation (cQPA).

In addition to usual particle fields to describe neutrinos, cQPA also includes quasiparticle fields which describe the coherence between these particle fields. These excitations are as such undetectable, but their effect on the behaviour of particles leads to various coherence phenomena such as neutrino oscillations.

The main goal of this thesis is to write an equation of motion for Standard Model neutrinos in this situation. We briefly discuss the structure and meaning of cQPA, develop calculational tools, use these tools to calculate self energies, and finally use these self energies to write down the equation of motion under suitable assumptions.

A more detailed analysis of the obtained equation of motion would require heavy numerical calculations, which are beyond the scope of this work. Our main focus is not on the analysis of the equation, but on its derivation and understanding of the model that leads to it. We do, however, consider some immediate implications of the equation and find that it exhibits coherence as expected.

Finally, we discuss the possible applications of this equation and of cQPA in general. These include numerous phenomena regarding neutrino transport and the early Universe. Therefore cQPA can be expected to provide us with a better understanding of such phenomena and coherence in thermal quantum field theory in general.

Tiivistelmä

Koherenssi on perustavanlaatuinen ja erikoinen ominaisuus kaikissa kvanttiteorioissa. Se mahdollistaa klassisesti odottamattomia ilmiöitä kuten neutriino-oskillaation, jossa erilaiset neutriinot muuttuvat spontaanisti toisikseen. Koherenssin huomioiminen monimutkaisissa fysikaalisissa järjestelmissä ei kuitenkaan ole yksinkertaista.

Tarkastellaan seuraavaa fysikaalista tilannetta: Neutriinot kulkevat ajallisesti muuttuvassa väliaineessa nollasta poikkeavassa lämpötilassa. Näiden neutriinojen käytöksen kuvailu edellyttää suppean suhteellisuusteorian, koherenssin ja vuorovaikutusten huomiointia. Koherentti kvasihiukkasapproksimaatio (coherent quasiparticle approximation, cQPA) tarjoaa yleisen formalismin tällaisten tilanteiden tutkimiseen.

Tavallisten neutriinoja kuvaavien hiukkaskenttien lisäksi cQPA sisältää kvasihiukkaskenttiä, jotka kuvaavat näiden hiukkaskenttien välistä koherenssia. Näitä eksitaatioita ei voi suoraan havaita, mutta niiden vuorovaikutus hiukkasten kanssa johtaa monenlaisiin koherenssi-ilmiöihin, joista neutriino-oskillaatio on hyvä esimerkki.

Tämän opinnäytetyön päätavoite on kirjoittaa liikeyhtälö Standardimallin neutriinoille tässä tilanteessa. Esittelemme lyhyesti cQPA:n rakennetta ja merkitystä, kehitämme laskennallisia työkaluja, käytämme näitä työkaluja itseisenergioiden laskemiseen ja lopulta kirjoitamme liikeyhtälön sopivin oletuksin näiden itseisenergioiden avulla.

Saadun liikeyhtälön yksityiskohtaisempi tarkastelu edellyttäisi raskaita numeerisia laskuja, jotka jäävät tämän työn ulkopuolelle. Pääpaino ei ole yhtälön tutkimisessa, vaan sen johtamisessa ja sen pohjalla olevan mallin ymmärtämisessä. Esittelemme kuitenkin joitakin yhtälön välittömiä seurauksia ja toteamme, että koherenssi ilmenee odotetulla tavalla.

Lopuksi tarkastelemme tämän liikeyhtälön ja yleensä cQPA:n mahdollisia sovelluksia. Näihin sovelluksiin kuuluu useita neutriinokuljetukseen ja varhaiseen Maailmankaikkeuteen liittyviä ilmiöitä. Siksi cQPA:n voikin olettaa tarjoavan paremman ymmärryksen näihin ilmiöihin ja yleisemminkin koherenssiin termisessä kvanttikenttäteoriassa.

Contents

1 Introduction

Quantum mechanical coherence between distinct states is what allows non-classical behaviour in quantum mechanical systems; the probabilities emerging from a state written as a coherent superposition in some basis is in the very heart of the phenomenology of quantum mechanics (QM). Quantum field theory (QFT), as a relativistic formulation of many-body QM, is therefore also expected to take coherence phenomena into account. In the presence of nonlinearities due to interactions, however, the equations of motion in QFT defy analytic solutions, necessitating the use of various approximation schemes.

Neutrino oscillations present an excellent example of coherence[1]. In a weak charged current interaction process a neutrino is produced in pure flavour state. The mass of such a state is ill-defined and therefore the time evolution is rather complicated; it is best expressed as a superposition of neutrino mass eigenstates. The slightly different time evolution of different mass states is what leads to the observed oscillation in flavour basis. Having different kinematical properties, these different mass states tend to drift apart as the neutrino propagates. As the overlap between the wave packets of different mass states is gradually lost, the oscillations cease and the probability distribution in flavour basis no longer evolves in time. The classically unexpected yet significant phenomenon of neutrino oscillation thus vitally depends on coherence.

In this thesis we study coherent quasiparticle approximation (cQPA), an approximation scheme in QFT in nonzero temperature. In particular, we calculate the hermitean and absorptive (emissive) neutrino self energies to second order in the Fermi coupling constant in this scheme and find out that taking nonlocal coherence properly into account may lead to phenomenology substantially different from what would be expected when coherence is neglected.

The structure of this thesis is as follows: In Section 2 we briefly describe cQPA, and in Section 3 we present the momentum space Feynman rules and device a number of calculational tools for cQPA. Section 4 is devoted to the calculation of neutrino self energies in the framework of cQPA, and in Section 5 we present and analyse the Quantum Boltzmann equation arising from cQPA using the obtained self energies. Finally, a summary and outlook are given in Section 6.

[1] A more thorough discussion of coherence in neutrino oscillations can be found e.g. in Refs. [1] and [2].

2 Coherent quasiparticle approximation

The description of many physical situations require simultaneously taking into account finite temperature, special relativity, nonlocal quantum mechanical coherence and even thermodynamics out of equilibrium. Such situations include, for example, particle creation in the early universe and neutrino propagation in spatially or temporally varying background.

Coherent quasiparticle approximation (cQPA) is an approximation scheme capable of treating such physical systems. It was introduced by Herranen, Kainulainen, and Rahkila in Ref. [3] and reformulated in a more easily calculable form in Ref. [4]. The diagrammatic methods developed in Ref. [4] are used here to calculate leading order corrections to neutrino self energies due to weak interactions with the medium.

2.1 Thermal field theory

When doing QFT in vacuum in zero temperature, one is typically interested in scattering processes where long-lived particles interact by interchanging virtual particles. In nonzero temperature there is, however, a thermal distribution of various particles, and a propagating particle does not only interact with itself via spontaneous virtual excitations, but also with its surroundings. Moreover, particles in a thermal system are often short-lived, whence the asymptotic in- and out-states familiar from scattering theory are no longer meaningful. Similar phenomena take place also in zero temperature, when particles propagate and interact in a medium. Such phenomena can be investigated using thermal field theory (TFT).

There are two main formalisms for TFT: imaginary and real time. The imaginary time formalism is the one adopted in most introductory treatments of TFT (such as Refs. [5] and [6]). In this formulation one writes the time coordinate as $t = x^0 = -i\tau = -ix_4$ for some real $\tau = x_4$ (which is periodic with period β). Similarly one replaces p^0 with $-ip_4$ in the momentum space, and the Minkowskian structure of spacetime becomes an Euclidean one: $t^2 - \vec{x}^2 = -(\tau^2 + \vec{x}^2)$ and similarly for momentum. In this formulation the integration over energy appearing in the path integral representation of the propagators is replaced by a sum over discrete energies in a Euclidean space; this gives rise to the Matsubara (or imaginary time) propagator.

In the time integral appearing in the partition function it may be more convenient to choose a more complicated path in the complex plane than a (possibly slightly tilted) horizontal or vertical line. The Keldysh path $\mathcal{C}$, composed of three line segments joining $-T+i\varepsilon$, $+T$, $-T-i\varepsilon$, and $-T-i\beta$, where $-T$ is some large negative initial time (T is let tend to infinity), $\varepsilon > 0$

is small parameter which is let tend to zero, and β is the inverse temperature. Due to the boundary condition $\varphi(t,\vec{x}) = \varphi(t-i\beta,\vec{x})$ for bosonic fields φ this time path is periodic. A generic propagator $\Delta_{\mathcal{C}}(t,\vec{x};t',\vec{x}')$ splits to four parts: it is Δ^{++} (Δ^{--}) when both t and t' lie on the upper (lower) horizontal line segment and $\Delta^{<} = \Delta^{+-}$ when t is on the upper and t' on the lower line segment (and vice versa for $\Delta^{>} = \Delta^{-+}$). This is one formulation of the real time formalism.

Simple calculations tend to be easier to do in the imaginary time formalism, but more involved ones are often easier to handle in the real time formalism. The real time formalism also preserves the Minkowskian structure of the spacetime more explicitly.

All phenomena present in vacuum and zero temperature are also present when temperature is increased or a medium introduced. In the real time formalism vacuum and thermal phenomena can be separated (for example, the propagator can be written as a sum of a vacuum propagator and a thermal propagator) thus making it more straightforward to study changes in vacuum behaviour due to finite temperature or medium effects. In this thesis we follow this method.

For details on TFT beyond this relatively naive introduction, see for example the books by Kapusta [5] and Le Bellac [6].

2.2 A brief introduction to cQPA[2]

This is only a short introduction to cQPA. The practical Feynman rules needed here are given in Section 3.1 below. For more details on cQPA, see Refs. [3, 7, 8, 9, 10, 11, 4] and references therein. Here we follow the notational conventions of Ref. [4].

2.2.1 Propagators and self energies

In the study of non-equilibrium TFT, the fermionic Wightman functions $iS^{<}(u,v) = \langle\bar{\psi}(v)\psi(u)\rangle$ and $iS^{>}(u,v) = \langle\psi(u)\bar{\psi}(v)\rangle$ are of central interest[3]. These functions in a way describe the self-correlation of the fermionic field ψ between points u and v in a Minkowskian spacetime. The expectation values $\langle\cdot\rangle$ are calculated with respect to an unknown density operator.

We can also express the Wightman functions in terms of the relative and average coordinates $r = u - v$ and $x = (u+v)/2$; this is particularly

[2]This introduction follows mainly Ref. [4].

[3]It is a common convention to define $iS^{<}$ with an additional minus sign. See e.g. Ref [12].

convenient after a Fourier transformation in r (a Wigner transformation):

$$S^{<,>}(k,x) = \int \mathrm{d}^4 r \, e^{ik\cdot r} S^{<,>}(x+\frac{r}{2}, x-\frac{r}{2}). \tag{1}$$

In analogue to $iS^{<,>}$ we define the time ordered Green's function (Feynman propagator) iS^t and in turn the hermetian Green's function $S^h = S^t - (S^> - S^<)/2$. The self energies corresponding to iS (with any of the indices $<$, $>$, t, and h) are denoted by $i\Sigma$ (with the same indices).

Similarly we may define the retarded and advanced propagators as $S^{r,a} = S^t \pm S^{<,>}$ (so that $S^h = (S^r + S^a)/2$) and the anti-Feynman propagator $S^{\bar{t}}$ (with inverse time ordering). The antihermitean Green's function

$$\mathcal{A} = \frac{i}{2}(S^> + S^<) \tag{2}$$

is known as the spectral function[4].

In multiflavour formalism we include flavour indices so that in $iS_{ij}(u,v)$ the flavour index i corresponds to the coordinate u and similarly j to v. The flavour indices are suppressed where they can easily be inferred from the context.

For a more elaborate description of the various Green's functions, see Ref. [12].

2.2.2 Equations of motion and shell structure

We define the diamond operator (cf. Poisson brackets) as

$$\diamond = \frac{1}{2}(\partial_x^{(1)} \cdot \partial_k^{(2)} - \partial_k^{(1)} \cdot \partial_x^{(2)}). \tag{3}$$

It acts on a pair of functions (the bracketed indices refer to these functions) which depend on x and k. For two functions $f(k,x)$ and $g(k,x)$, for example,

$$\diamond\{f\}\{g\} = \frac{1}{2}(\partial_x f \cdot \partial_k g - \partial_k f \cdot \partial_x g). \tag{4}$$

Using Eq. (3) we may similarly define $\diamond^n\{f\}\{g\}$ for any $n \in \mathbb{N}$, and so also $e^{-i\diamond}$.

We denote by $m = m(x)$ the possibly space- and time-dependent and complex mass matrix, and write its hermitean and antihermitean parts as

[4] In the following we will only consider spectral functions for fermionic fields, whence it is written shortly as $\mathcal{A} = \mathcal{A}^\psi$.

$m_h = (m + m^\dagger)/2$ and $m_a = (m - m^\dagger)/(2i)$. Using these, we define the mass operators

$$\hat{m}_{0,5} f(k,x) = e^{-i\diamond}\{m_{h,a}(x)\}\{f(k,x)\}, \tag{5}$$

where we take $\partial_k m_{h,a} = 0$.

With these notations, the Wightman functions obey the equations

$$\begin{aligned}(\not{k} + \frac{i}{2}\not{\partial}_x - \hat{m}_0 - i\hat{m}_5\gamma^5)S^{<,>} - e^{-i\diamond}\{\Sigma^h\}\{S^{<,>}\}& \\ - e^{-i\diamond}\{\Sigma^{<,>}\}\{S^h\} = \pm\mathcal{C}_{\rm coll},&\end{aligned} \tag{6}$$

where the collision term is

$$\mathcal{C}_{\rm coll} = \frac{1}{2}e^{-i\diamond}(\{\Sigma^>\}\{S^<\} - \{\Sigma^<\}\{S^>\}). \tag{7}$$

Eq. (6) is the most fundamental equation of motion, but in practice impossible to solve in full generality.

It turns out [4] that in the mass eigenbasis and with suitable approximations the phase space structure of the homogeneous and isotropic Wightman functions is more complicated than naively expected. The phase space constraint equation for $iS^<_{ij}(k,x)$ in Eq. (6) is

$$\left(k^2 - \frac{m_i^2 + m_j^2}{2}\right)k_0^2 + \frac{1}{4}\left(\frac{m_i^2 - m_j^2}{2}\right)^2 = 0. \tag{8}$$

Defining $\omega_i = \omega_i(\vec{k}) = \sqrt{m_i^2 + \vec{k}^2}$, this gives rise to dispersion relations

$$k_0 = \pm\frac{1}{2}(\omega_i + \omega_j) \tag{9}$$

and

$$k_0 = \pm\frac{1}{2}(\omega_i - \omega_j). \tag{10}$$

In the case $m_i = m_j$ the dispersion relation of Eq. (9) gives the standard relation $k^2 = m_i^2$.

Corresponding to the four dispersion relations in Eqs. (9) and (10) there are four distribution functions describing the different shell occupations. These functions for Eq. (9) are $f^{m<}_{ijh\pm}$, which describe coherence between the mass eigenstates with on-shell energies $\pm\omega_i$ and $\pm\omega_j$ and helicity h. For Eq. (10) the corresponding functions are $f^{c<}_{ijh\pm}$, and they describe the coherence between the mass eigenstates with on-shell energies $\pm\omega_i$ and $\mp\omega_j$ and helicity h. No coherence between helicities h and $-h$ appears in this approximation.

Using the Feynman-Stückelberg interpretation we identify negative energy particles as antiparticles, and relate the elements of the distribution functions f on the flavour diagonal to the particle phase space densities by

$$n_{ih\vec{k}} = \frac{m_i}{\omega_i} f^{m<}_{iih+}, \qquad \bar{n}_{ih\vec{k}} = 1 + \frac{m_i}{\omega_i} f^{m<}_{iih-}. \tag{11}$$

The distribution functions for $iS^>$ are $f^{m>}_{ijh\pm} = \pm\frac{\omega_i}{m_i}\delta_{ij} - f^{m<}_{ijh\pm}$ and $f^{c>}_{ijh\pm} = -f^{c<}_{ijh\pm}$. One may also find the hermiticity relations $f^{m<}_{jih\pm} = (f^{m<}_{ijh\pm})^*$ and $f^{c<}_{jih\pm} = (f^{c<}_{ijh\mp})^*$.

The Feynman rules, especially Eq. (12), given below in Section 3.1 show how the shell structure appears in the Wightman functions in more detail.

3 Some calculational tools for cQPA

In this section we list the Feynman rules for cQPA and establish some auxiliary results which will be used in the self energy calculations in Section 4. The actual calculations will be more straightforward once these results are at hand and suitable notation for parts of the diagrams has been found.

3.1 Feynman rules

The Feynman rules of cQPA for calculating corrections to the fermion self energies $i\Sigma^{<,>}$ given in [4] are as follows (the Feynman rules relevant for the calculations done here are presented in Figs. 1 and 2):

1. Draw all perturbative two-particle irreducible diagrams and associate the usual symmetry factor and sign with them.

2. Associate with each vertex the normal vertex factor (not including a four-momentum conservation delta function). The vertex rules relevant here are listed in Fig. 2.

3. Associate a delta function $(2\pi)^4\delta^4(p_{\rm in} - p_{\rm out})$ with all vertices except the one next to the outgoing external fermion line.[5]

4. For fermion propagators substitute the propagator $iS^{<,>}_{ji,eff}(q,q')$ and integrate over both momenta: $\int \frac{d^4q}{(2\pi)^4}\frac{d^4q'}{(2\pi)^4}$. For the Z boson propagator,

[5] In the calculations of Section 4.2 it makes no difference to leading order in M_W^{-2} whether we drop the delta function from the end of the incoming or outgoing external fermion line.

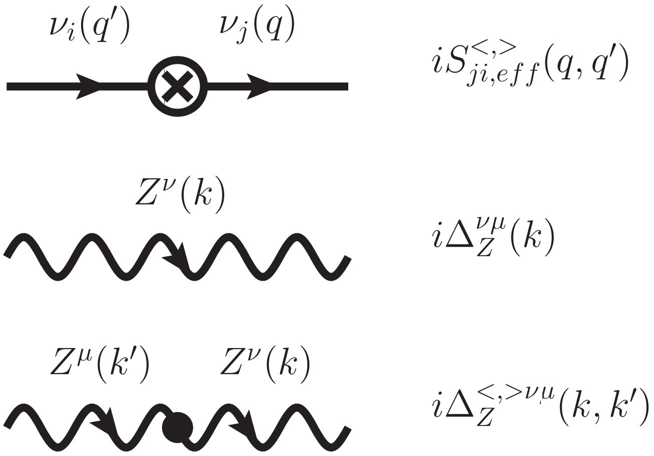


Figure 1: Feynman rules for the effective non-equilibrium neutrino propagator, the Z boson vacuum Feynman propagator, and the thermal Z boson propagator. See text for details.

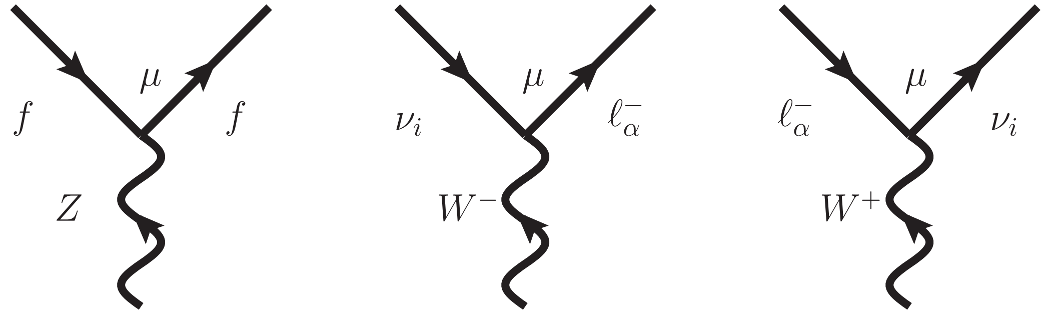
use $i\Delta_Z^{\nu\mu}(k)$ and integrate over k or use $i\Delta_Z^{<\nu\mu}(k,k')$ and integrate over k and k'.[6]

How and why to choose propagators for gauge bosons, will be discussed in Sections 3.6 and 4.1. The propagators $i\Delta_Z^{>\nu\mu}(k,k')$, $i\Delta_W^{<\nu\mu}(k,k')$, and $i\Delta_W^{>\nu\mu}(k,k')$ are formed similarly with $i\Delta_Z^{<\nu\mu}(k,k')$, and they need not be discussed separately.

For other fermions than neutrinos and the W boson we use similar propagators as those in Fig. 1 with obvious changes. The dot on the gauge boson propagator indicates the leading order expansion of the propagator as done in Section 3.6.

The vertex rules in Fig. 2 are exactly as they appear in the Standard Model. For neutrinos the coefficients $g^\nu_{V,A}$ both equal $\frac{1}{2}$. For charged leptons $g^\ell_V = -\frac{1}{2} + 2s_W^2$ and $g^\ell_A = -\frac{1}{2}$.[13] The coefficients c_W and $U_{\alpha i}$ in the vertex rules given in Fig. 2 are the cosine of the Weinberg angle and the elements of the leptonic mixing matrix (the Pontecorvo–Maki–Nakagawa–Sakata-matrix). No assumptions need to be made of the form of the PMNS-matrix or the number of lepton generations. The coefficient s_W in g^ℓ_V is the sine of the Weinberg angle.

[6]The propagator $i\Delta_Z^{<\nu\mu}(k,k')$ is given in Section 3.6, and for $i\Delta_Z^{\nu\mu}(k)$ we use the thermal propagator in unitary gauge: $i\Delta_Z^{\nu\mu}(k) = i\frac{-g_{\mu\nu}+k_\mu k_\nu/M_Z^2}{k^2-M_Z^2}$.



$\frac{-ig}{2c_W}\gamma^\mu(g_V^f - g_A^f\gamma^5)$ $\quad$ $\frac{-ig}{2\sqrt{2}}U_{\alpha i}\gamma^\mu(1-\gamma^5)$ $\quad$ $\frac{-ig}{2\sqrt{2}}U^*_{\alpha i}\gamma^\mu(1-\gamma^5)$

Figure 2: Feynman rules for weak interaction vertices.

The effective neutrino propagator (Wightman function) $iS^{<,>}_{ji,eff}$ in Fig. 1 is [4]

$$iS^{<,>}_{ji,eff} = \mathcal{A}_{jj}(q)F^{<,>}_{ji}(q,q')\mathcal{A}_{ii}(q'), \tag{12}$$

where the spectral function $\mathcal{A}$ is

$$\mathcal{A}_{ij}(k) = \pi\,\mathrm{sgn}(k_0)(\not{k} + m_i)\delta(k^2 - m_i^2)\delta_{ij} \tag{13}$$

and the effective two-point vertex F is defined as

$$F^{<,>}_{ij}(q,q') = 4(2\pi)^3\delta^3(\vec{q}-\vec{q}\,')\sum_{h,\pm} P_h(\hat{q})\theta^q_\pm(\theta^{q'}_\pm f^{m<}_{ijh\pm}(\vec{q}) + \theta^{q'}_\mp f^{c<}_{ijh\pm}(\vec{q})). \tag{14}$$

Here

$$P_h(\hat{q}) = \frac{1}{2}(1 + h\gamma^0\hat{q}\cdot\vec{\gamma}\gamma^5) \tag{15}$$

with $\hat{q} = \vec{q}/|\vec{q}|$ is the usual helicity projector and $\theta^q_\pm = \theta(\pm q^0)$.

When calculating corrections to the hermitean (dispersive) self energy Σ^h, we include an additional factor $-i$ to every graph and use the two-point function $F^<_{ji}(q,q')$, from which the vacuum contribution has been removed.

The distribution functions $f^{m,c}$ may depend on time, but this dependence is suppressed here for the sake of simplicity.

3.2 On-shell energies and momenta

We will write the on-shell energy corresponding to mass m_i and three-momentum $\vec{q}$ as $\omega_i(\vec{q}) = \sqrt{\vec{q}^2 + m_i^2}$. If the momentum $\vec{q}$ is implicitly clear from the context, it will be suppressed. To simplify the expressions further, we define the on-shell four-momentum $q^\mu_{i\pm} = (\pm\omega_i(\vec{q}), \vec{q})$, for which clearly $q^2_{i\pm} = m^2_i$.

For any three-momentum $\vec{q}$ we write its norm and the corresponding gamma matrix as $Q = |\vec{q}|$ and $\gamma_q = \vec{q} \cdot \vec{\gamma}$. Similarly we define the four-momentum $q_{0h} = (1, h\vec{q}/Q)$, which has the properties $\gamma^0 \not{q}_{0h} \gamma^0 = \not{q}_{0-h}$ and $q^2_{0h} = 0$. In the helicity projectors we will also use the normalized three-momentum $\hat{q} = \vec{q}/Q$.

If there are multiple momenta $\vec{q}_1$, $\vec{q}_2$ etc., we will write the on-shell four-momentum corresponding to $\vec{q}_1$ as $q^\mu_{1i\pm}$. Similarly we will use the notations Q_1, q_{10h}, and $\hat{q}_1$.

3.3 Fermion propagators

According to the Feynman rules presented in Section 3.1, a fermionic propagator is always of the form

$$\mathcal{A}_{jj}(q) F^{<,>}_{ji}(q, q') \mathcal{A}_{ii}(q') \tag{16}$$

and the four-momenta q and q' are integrated over. Using the definitions for the spectral function $\mathcal{A}$ and the two-point vertex function F, we find

$$\begin{aligned}
&\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{d}^4 q'}{(2\pi)^4} \mathcal{A}_{jj}(q) F^{<,>}_{ji}(q, q') \mathcal{A}_{ii}(q') G(q, q') \\
&= \sum_{h,\pm} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{2\omega_i(\vec{q}) 2\omega_j(\vec{q})} \\
&\quad (f^{m<,>}_{jih\pm}(\vec{q}) (\not{q}_{j\pm} + m_j) P_h(\hat{q}) (\not{q}_{i\pm} + m_i) G(q_{j\pm}, q_{i\pm}) \\
&\quad + f^{c<,>}_{jih\pm}(\vec{q}) (\not{q}_{j\pm} + m_j) P_h(\hat{q}) (\not{q}_{i\mp} + m_i) G(q_{j\pm}, q_{i\mp})).
\end{aligned} \tag{17}$$

Here the function $G(q, q')$ represents the rest of the diagram, which typically depends (through vertex delta functions) on both q and q'.

3.4 Dirac algebra

The following result of a trivial calculation will significantly help simplifying the Dirac algebra appearing in the calculations:

$$\begin{aligned}
&(1 - \gamma^5)(\not{q}_{j\pm} + m_j)(1 + h\gamma^0 \hat{q} \cdot \vec{\gamma}\gamma^5)(\not{q}_{i\pm'} + m_i)(1 + \gamma^5) \\
&= (1 - \gamma^5)(\pm\omega_j \gamma^0 - \gamma_q + m_j)(1 + \frac{h}{Q} \gamma^0 \gamma_q \gamma^5) \\
&\quad \times (\pm' \omega_i \gamma^0 - \gamma_q + m_i)(1 + \gamma^5) \\
&= 2(-hQ(m_i + m_j) \pm m_i \omega_j \pm' m_j \omega_i) \not{q}_{0h} (1 + \gamma^5).
\end{aligned} \tag{18}$$

Here $\pm$ and $\pm'$ are two independent signs. Similarly, we find

$$\begin{aligned}&(1+\gamma^5)(\not{q}_{j\pm}+m_j)(1+h\gamma^0\hat{q}\cdot\vec{\gamma}\gamma^5)(\not{q}_{i\pm'}+m_i)(1+\gamma^5)\\&=2(m_im_j-Q^2\pm\pm'\omega_i\omega_j+hQ(\pm'\omega_i\mp\omega_j))\gamma^0\not{q}_{0h}(1+\gamma^5).\end{aligned} \tag{19}$$

To be able to more conveniently use these results, we adopt the notations

$$A_{ji}^{\pm\pm'}(Q,h)=-hQ(m_i+m_j)\pm m_i\omega_j\pm' m_j\omega_i \tag{20}$$

and

$$B_{ji}^{\pm\pm'}(Q,h)=m_im_j-Q^2+(\pm\omega_j)(\pm'\omega_i)+hQ(\pm'\omega_i\mp\omega_j). \tag{21}$$

Using these, we get

$$\begin{aligned}&(1-\gamma^5)(\not{q}_{j\pm}+m_j)(1+h\gamma^0\hat{q}\cdot\vec{\gamma}\gamma^5)(\not{q}_{i\pm'}+m_i)(1+\gamma^5)\\&=2A_{ji}^{\pm\pm'}(Q,h)\not{q}_{0h}(1+\gamma^5)\end{aligned} \tag{22}$$

and

$$\begin{aligned}&(1+\gamma^5)(\not{q}_{j\pm}+m_j)(1+h\gamma^0\hat{q}\cdot\vec{\gamma}\gamma^5)(\not{q}_{i\pm'}+m_i)(1+\gamma^5)\\&=2B_{ji}^{\pm\pm'}(Q,h)\gamma^0\not{q}_{0h}(1+\gamma^5).\end{aligned} \tag{23}$$

Similarly, we find

$$\begin{aligned}&(1-\gamma^5)(\not{q}_{j\pm}+m_j)(1+h\gamma^0\hat{q}\cdot\vec{\gamma}\gamma^5)(\not{q}_{i\pm'}+m_i)(1-\gamma^5)\\&=2B_{ji}^{\pm\pm'}(Q,-h)\gamma^0\not{q}_{0-h}(1-\gamma^5)\end{aligned} \tag{24}$$

and

$$\begin{aligned}&(1+\gamma^5)(\not{q}_{j\pm}+m_j)(1+h\gamma^0\hat{q}\cdot\vec{\gamma}\gamma^5)(\not{q}_{i\pm'}+m_i)(1-\gamma^5)\\&=2A_{ji}^{\pm\pm'}(Q,-h)\not{q}_{0-h}(1-\gamma^5).\end{aligned} \tag{25}$$

Note how helicity changes sign in Eqs. (24) and (25). Also note that when we replace $(1+h\gamma^0\hat{q}\cdot\vec{\gamma}\gamma^5)$ by the helicity projector, we divide by two and thus lose the coefficient 2 in front of A and B.

Using Eq. (22), we find another result for three three-momenta $\vec{q}_n$ ($n=1,2,3$):

$$\begin{aligned}&\gamma^\sigma(1-\gamma^5)(\not{q}_{3j\pm_3}+m_j)(1+h_3\gamma^0\hat{q}_3\cdot\vec{\gamma}\gamma^5)(\not{q}_{3k\pm_3'}+m_k)\\&\quad\times\gamma^\lambda(1-\gamma^5)(\not{q}_{2k\pm_2}+m_k)(1+h_2\gamma^0\hat{q}_2\cdot\vec{\gamma}\gamma^5)(\not{q}_{2l\pm_1'}+m_l)\\&\quad\times\gamma^\nu(1-\gamma^5)(\not{q}_{1l\pm_1}+m_l)(1+h_1\gamma^0\hat{q}_1\cdot\vec{\gamma}\gamma^5)(\not{q}_{1i\pm_1'}+m_i)\\&\quad\times\gamma^\mu(1-\gamma^5)\\&=8A_{jk}^{\pm_3\pm_3'}(Q_3,h_3)A_{kl}^{\pm_2\pm_2'}(Q_2,h_2)A_{li}^{\pm_1\pm_1'}(Q_1,h_1)\\&\quad\times\gamma^\sigma\not{q}_{30h_3}\gamma^\lambda\not{q}_{20h_2}\gamma^\nu\not{q}_{10h_1}\gamma^\mu(1-\gamma^5).\end{aligned} \tag{26}$$

Furthermore, if we contract this result with $g_{\mu\lambda}g_{\nu\sigma}$, we obtain[7]

$$\begin{aligned}&-64A_{jk}^{\pm_3\pm_3'}(Q_3,h_3)A_{kl}^{\pm_2\pm_2'}(Q_2,h_2)A_{li}^{\pm_1\pm_1'}(Q_1,h_1)\\&\quad\times(q_{10h_1}\cdot q_{30h_3})\not{q}_{20h_2}(1-\gamma^5).\end{aligned}\tag{27}$$

The dot product above is simply $q_{10h_1}\cdot q_{30h_3}=1-\frac{h_1h_3}{Q_1Q_3}\vec{q}_1\cdot\vec{q}_3$.

Using Eqs. (22), (23), and (25) we find

$$\begin{aligned}&\gamma^\sigma(1-\gamma^5)(\not{q}_{3j\pm_3}+m_j)(1+h_3\gamma^0\hat{q}_3\cdot\vec{\gamma}\gamma^5)(\not{q}_{3k\pm_3'}+m_k)\\&\quad\times\gamma^\lambda(1-\gamma^5)(\not{q}_{2k\pm_2}+m_k)(1+h_2\gamma^0\hat{q}_2\cdot\vec{\gamma}\gamma^5)(\not{q}_{2l\pm_1'}+m_l)\\&\quad\times\gamma^\nu(1+\gamma^5)(\not{q}_{1l\pm_1}+m_l)(1+h_1\gamma^0\hat{q}_1\cdot\vec{\gamma}\gamma^5)(\not{q}_{1i\pm_1'}+m_i)\\&\quad\times\gamma^\mu(1-\gamma^5)\\&=8A_{jk}^{\pm_3\pm_3'}(Q_3,h_3)B_{kl}^{\pm_2\pm_2'}(Q_2,-h_2)B_{li}^{\pm_1\pm_1'}(Q_1,h_1)\\&\quad\times\gamma^\sigma\not{q}_{30h_3}\gamma^\lambda\gamma^0\not{q}_{20-h_2}\gamma^\nu\gamma^0\not{q}_{10h_1}\gamma^\mu(1-\gamma^5),\end{aligned}\tag{28}$$

and contracting this with $g_{\mu\lambda}g_{\nu\sigma}$ yields

$$\begin{aligned}&32A_{jk}^{\pm_3\pm_3'}(Q_3,h_3)B_{kl}^{\pm_2\pm_2'}(Q_2,-h_2)B_{li}^{\pm_1\pm_1'}(Q_1,h_1)\\&\quad\times\not{q}_{10-h_1}\not{q}_{30-h_3}\not{q}_{20h_2}(1-\gamma^5).\end{aligned}\tag{29}$$

In analogue to this we find

$$\begin{aligned}&\gamma^\sigma(1-\gamma^5)(\not{q}_{3j\pm_3}+m_j)(1+h_3\gamma^0\hat{q}_3\cdot\vec{\gamma}\gamma^5)(\not{q}_{3k\pm_3'}+m_k)\\&\quad\times\gamma^\lambda(1+\gamma^5)(\not{q}_{2k\pm_2}+m_k)(1+h_2\gamma^0\hat{q}_2\cdot\vec{\gamma}\gamma^5)(\not{q}_{2l\pm_1'}+m_l)\\&\quad\times\gamma^\nu(1-\gamma^5)(\not{q}_{1l\pm_1}+m_l)(1+h_1\gamma^0\hat{q}_1\cdot\vec{\gamma}\gamma^5)(\not{q}_{1i\pm_1'}+m_i)\\&\quad\times\gamma^\mu(1-\gamma^5)\\&=8B_{jk}^{\pm_3\pm_3'}(Q_3,-h_3)B_{kl}^{\pm_2\pm_2'}(Q_2,h_2)A_{li}^{\pm_1\pm_1'}(Q_1,h_1)\\&\quad\times\gamma^\sigma\gamma^0\not{q}_{30-h_3}\gamma^\lambda\gamma^0\not{q}_{20h_2}\gamma^\nu\not{q}_{10h_1}\gamma^\mu(1-\gamma^5),\end{aligned}\tag{30}$$

and the same contraction leads to

$$\begin{aligned}&32B_{jk}^{\pm_3\pm_3'}(Q_3,-h_3)B_{kl}^{\pm_2\pm_2'}(Q_2,h_2)A_{li}^{\pm_1\pm_1'}(Q_1,h_1)\\&\quad\times\not{q}_{20h_2}\not{q}_{10-h_1}\not{q}_{30-h_3}(1-\gamma^5).\end{aligned}\tag{31}$$

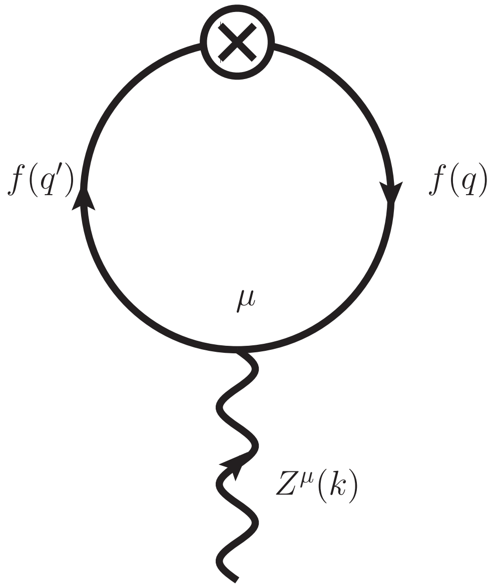


Figure 3: A fermion loop $C_f^\mu(k)$ with one external Z boson.

3.5 Fermion loops

In the diagrams we calculate in Section 4 various fermion loops arise. We calculate them here, with all the external vectors amputated. The three loops calculated here are those appearing in Figs. 3, 4, and 5.

The fermion loop of Fig. 3 with one external Z boson (with momentum k) takes the following form according to the Feynman rules of Section 3.1:

$$\begin{aligned} C_f^\mu(k) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{d}^4 q'}{(2\pi)^4}(2\pi)^4\delta^4(k+q-q') \\ \times (-1)\,\mathrm{Tr}(\mathcal{A}_{ff}(q)F_{ff}^<(q,q')\mathcal{A}_{ff}(q')\frac{-ig}{2c_W}\gamma^\mu(g_V^f - g_A^f\gamma^5)), \end{aligned} \quad (32)$$

[7] Here we use the properties $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\mu = -2\gamma^\gamma\gamma^\alpha\gamma^\beta$ and $\gamma_\nu\gamma^\alpha\gamma^\beta\gamma^\nu = 4g^{\alpha\beta}$.

where f denotes the fermion in question. Using Eq. (17), we find

$$
\begin{aligned}
C_f^\mu = \frac{ig}{2c_W} \sum_{h,\pm} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{4\omega_f(\vec{q})^2} \\
&\times [f_{ffh\pm}^{m<}(\vec{q})(2\pi)^4 \delta^4(k) \\
&\quad \times \mathrm{Tr}((\not{q}_{f\pm} + m_f) P_h(\hat{q})(\not{q}_{f\pm} + m_f)\gamma^\mu (g_V^f - g_A^f \gamma^5)) \\
&+ f_{ffh\pm}^{c<}(\vec{q})(2\pi)^4 \delta(k^0 \mp 2\omega_f(Q))\delta^3(\vec{k}) \\
&\quad \times \mathrm{Tr}((\not{q}_{f\pm} + m_f) P_h(\hat{q})(\not{q}_{f\mp} + m_f)\gamma^\mu (g_V^f - g_A^f \gamma^5))].
\end{aligned}
\tag{33}
$$

By Eqs. (22) and (25) we have

$$
\begin{aligned}
&\mathrm{Tr}((\not{q}_{f\pm} + m_f) P_h(\hat{q})(\not{q}_{f\pm'} + m_f)\gamma^\mu (1 \pm'' \gamma^5)) \\
&= \frac{1}{2} A_{ff}^{\pm\pm'}(Q, \mp'' h)\, \mathrm{Tr}(\not{q}_{0\mp'' h} \gamma^\mu (1 \pm'' \gamma^5)) \\
&= 2 A_{ff}^{\pm\pm'}(Q, \mp'' h) q_{0\mp'' h}^\mu .
\end{aligned}
\tag{34}
$$

Writing $(g_V^f - g_A^f \gamma^5) = \frac{1}{2}(g_V^f - g_A^f)(1+\gamma^5) + \frac{1}{2}(g_V^f + g_A^f)(1-\gamma^5)$ and using the above result, we find

$$
\begin{aligned}
C_f^\mu(k) = \frac{ig}{2c_W} \sum_{h,\pm} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{4\omega_f(\vec{q})^2} \\
&\times [f_{ffh\pm}^{m<}(\vec{q})(2\pi)^4 \delta^4(k) \\
&\times ((g_V^f - g_A^f) A_{ff}^{\pm\pm}(Q, -h) q_{0-h}^\mu + (g_V^f + g_A^f) A_{ff}^{\pm\pm}(Q, h) q_{0h}^\mu) \\
&+ f_{ffh\pm}^{c<}(\vec{q})(2\pi)^4 \delta(k^0 \mp 2\omega_f(Q))\delta^3(\vec{k}) \\
&\times ((g_V^f - g_A^f) A_{ff}^{\pm\mp}(Q, -h) q_{0-h}^\mu + (g_V^f + g_A^f) A_{ff}^{\pm\mp}(Q, h) q_{0h}^\mu)].
\end{aligned}
\tag{35}
$$

If the distribution functions $f_{ffh\pm}^{m,c<}(\vec{q})$ are independent of either the direction of $\vec{q}$, then $C_f^i(k) = 0$ due to the symmetry in the integral. Even in the absence of this symmetry, the result for $C_f^\mu(k)$ can be further simplified by studying the components $C_f^0(k)$ and $C_f^i(k)$ separately, since $q_{0h}^0 = 1$ and $q_{0h}^i = hq^i/Q$.

A lepton loop with two external W bosons of momenta k and k' as shown in Fig. 4 takes the following form:

$$
\begin{aligned}
C_W^{<\nu\mu}(k, k') = \sum_{\alpha,\beta,i,j} \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \frac{\mathrm{d}^4 q_1'}{(2\pi)^4} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{\mathrm{d}^4 q_2'}{(2\pi)^4} \\
&\times (2\pi)^4 \delta^4(k' + q_2 - q_1')(2\pi)^4 \delta^4(k + q_2' - q_1) \\
&\times (-1)\, \mathrm{Tr}(\mathcal{A}_{\beta\beta}(q_1) F_{\beta\alpha}^<(q_1, q_1') \mathcal{A}_{\alpha\alpha}(q_1') \frac{-ig}{2\sqrt{2}} U_{\alpha j} \gamma^\mu (1 - \gamma^5) \\
&\quad \times \mathcal{A}_{jj}(q_2) F_{ji}^>(q_2, q_2') \mathcal{A}_{ii}(q_2') \frac{-ig}{2\sqrt{2}} U_{\beta i}^* \gamma^\nu (1 - \gamma^5)).
\end{aligned}
\tag{36}
$$

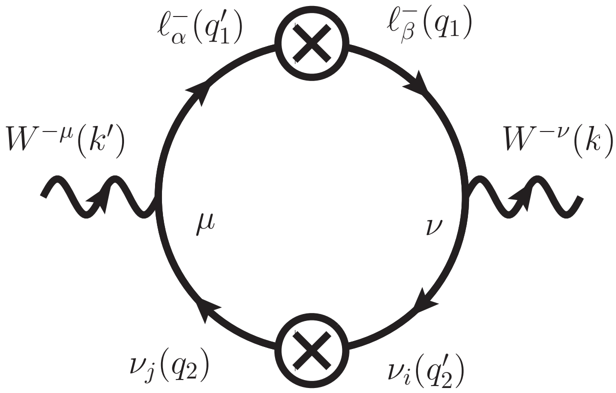


Figure 4: A fermion loop $C_W^{<\nu\mu}(k,k')$ with two external W bosons.

Using Eq. (17) gives

$$
\begin{aligned}
C_W^{<\nu\mu}(k,k') = \frac{g^2}{8} & \sum_{\alpha,\beta,i,j} \sum_{h_1,\pm_1,h_2,\pm_2} U_{\alpha j} U^*_{\beta i} \\
& \times \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{\mathrm{d}^3 q_2}{(2\pi)^3} \frac{1}{2\omega_\alpha(\vec{q}_1) 2\omega_\beta(\vec{q}_1)} \frac{1}{2\omega_i(\vec{q}_2) 2\omega_j(\vec{q}_2)} \\
& \times [f^{m<}_{\beta\alpha h_1 \pm_1}(\vec{q}_1) f^{m>}_{jih_2\pm_2}(\vec{q}_2) \\
& \quad \times (2\pi)^4 \delta^4(k' + q_{2j\pm_2} - q_{1\alpha\pm_1}) (2\pi)^4 \delta^4(k + q_{2i\pm_2} - q_{1\beta\pm_1}) \\
& \quad \times \mathrm{Tr}((\not{q}_{1\beta\pm_1} + m_\beta) P_{h_1}(\hat{q}_1)(\not{q}_{1\alpha\pm_1} + m_\alpha)\gamma^\mu(1-\gamma^5) \\
& \qquad \times (\not{q}_{2j\pm_2} + m_j) P_{h_2}(\hat{q}_2)(\not{q}_{2i\pm_2} + m_i)\gamma^\nu(1-\gamma^5)) \\
& + f^{m<}_{\beta\alpha h_1 \pm_1}(\vec{q}_1) f^{c>}_{jih_2\pm_2}(\vec{q}_2) \\
& \quad \times (2\pi)^4 \delta^4(k' + q_{2j\pm_2} - q_{1\alpha\pm_1}) (2\pi)^4 \delta^4(k + q_{2i\mp_2} - q_{1\beta\pm_1}) \\
& \quad \times \mathrm{Tr}((\not{q}_{1\beta\pm_1} + m_\beta) P_{h_1}(\hat{q}_1)(\not{q}_{1\alpha\pm_1} + m_\alpha)\gamma^\mu(1-\gamma^5) \\
& \qquad \times (\not{q}_{2j\pm_2} + m_j) P_{h_2}(\hat{q}_2)(\not{q}_{2i\mp_2} + m_i)\gamma^\nu(1-\gamma^5)) \\
& + f^{c<}_{\beta\alpha h_1 \pm_1}(\vec{q}_1) f^{m>}_{jih_2\pm_2}(\vec{q}_2) \\
& \quad \times (2\pi)^4 \delta^4(k' + q_{2j\pm_2} - q_{1\alpha\mp_1}) (2\pi)^4 \delta^4(k + q_{2i\pm_2} - q_{1\beta\pm_1}) \\
& \quad \times \mathrm{Tr}((\not{q}_{1\beta\pm_1} + m_\beta) P_{h_1}(\hat{q}_1)(\not{q}_{1\alpha\mp_1} + m_\alpha)\gamma^\mu(1-\gamma^5) \\
& \qquad \times (\not{q}_{2j\pm_2} + m_j) P_{h_2}(\hat{q}_2)(\not{q}_{2i\pm_2} + m_i)\gamma^\nu(1-\gamma^5)) \\
& + f^{c<}_{\beta\alpha h_1 \pm_1}(\vec{q}_1) f^{c>}_{jih_2\pm_2}(\vec{q}_2) \\
& \quad \times (2\pi)^4 \delta^4(k' + q_{2j\pm_2} - q_{1\alpha\mp_1}) (2\pi)^4 \delta^4(k + q_{2i\mp_2} - q_{1\beta\pm_1}) \\
& \quad \times \mathrm{Tr}((\not{q}_{1\beta\pm_1} + m_\beta) P_{h_1}(\hat{q}_1)(\not{q}_{1\alpha\mp_1} + m_\alpha)\gamma^\mu(1-\gamma^5) \\
& \qquad \times (\not{q}_{2j\pm_2} + m_j) P_{h_2}(\hat{q}_2)(\not{q}_{2i\mp_2} + m_i)\gamma^\nu(1-\gamma^5))].
\end{aligned}
\tag{37}
$$

Using Eq. (22)[8] we can evaluate the four traces above:

$$\begin{aligned}
&\mathrm{Tr}((\not{q}_{1\beta\pm_1} + m_\beta)P_{h_1}(\hat{q}_1)(\not{q}_{1\alpha\pm_1'} + m_\alpha)\gamma^\mu(1-\gamma^5) \\
&\quad \times (\not{q}_{2j\pm_2} + m_j)P_{h_2}(\hat{q}_2)(\not{q}_{2i\pm_2'} + m_i)\gamma^\nu(1-\gamma^5)) \\
&= 2A^{\pm_1\pm_1'}_{\beta\alpha}(Q_1,h_1)A^{\pm_2\pm_2'}_{ji}(Q_2,h_2)T^{\mu\nu}_{+}(q_{10h_1}, q_{20h_2}),
\end{aligned} \tag{38}$$

where we have denoted

$$\begin{aligned}
T^{\mu\nu}_{\pm}(a,b) &= \frac{1}{4}\,\mathrm{Tr}(\gamma^\nu \not{a}\gamma^\mu \not{b}(1\pm\gamma^5)) \\
&= a^\mu b^\nu + a^\nu b^\mu - a\cdot b g^{\mu\nu} \pm i\varepsilon^{\nu\gamma\mu\delta}a_\gamma b_\delta.
\end{aligned} \tag{39}$$

Thus

$$\begin{aligned}
&C^{<\nu\mu}_{W}(k,k') = \frac{g^2}{4}\sum_{\alpha,\beta,i,j}\sum_{h_1,\pm_1,h_2,\pm_2} U_{\alpha j}U^*_{\beta i} \\
&\times \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3}\frac{\mathrm{d}^3 q_2}{(2\pi)^3}\frac{1}{2\omega_\alpha(\vec{q}_1)2\omega_\beta(\vec{q}_1)}\frac{1}{2\omega_i(\vec{q}_2)2\omega_j(\vec{q}_2)} \\
&\times [f^{m<}_{\beta\alpha h_1\pm_1}(\vec{q}_1)f^{m>}_{jih_2\pm_2}(\vec{q}_2)A^{\pm_1\pm_1}_{\beta\alpha}(Q_1,h_1)A^{\pm_2\pm_2}_{ji}(Q_2,h_2) \\
&\quad \times (2\pi)^4\delta^4(k'+q_{2j\pm_2}-q_{1\alpha\pm_1})(2\pi)^4\delta^4(k+q_{2i\pm_2}-q_{1\beta\pm_1}) \\
&+ f^{m<}_{\beta\alpha h_1\pm_1}(\vec{q}_1)f^{c>}_{jih_2\pm_2}(\vec{q}_2)A^{\pm_1\pm_1}_{\beta\alpha}(Q_1,h_1)A^{\pm_2\mp_2}_{ji}(Q_2,h_2) \\
&\quad \times (2\pi)^4\delta^4(k'+q_{2j\pm_2}-q_{1\alpha\pm_1})(2\pi)^4\delta^4(k+q_{2i\mp_2}-q_{1\beta\pm_1}) \\
&+ f^{c<}_{\beta\alpha h_1\pm_1}(\vec{q}_1)f^{m>}_{jih_2\pm_2}(\vec{q}_2)A^{\pm_1\mp_1}_{\beta\alpha}(Q_1,h_1)A^{\pm_2\pm_2}_{ji}(Q_2,h_2) \\
&\quad \times (2\pi)^4\delta^4(k'+q_{2j\pm_2}-q_{1\alpha\mp_1})(2\pi)^4\delta^4(k+q_{2i\pm_2}-q_{1\beta\pm_1}) \\
&+ f^{c<}_{\beta\alpha h_1\pm_1}(\vec{q}_1)f^{c>}_{jih_2\pm_2}(\vec{q}_2)A^{\pm_1\mp_1}_{\beta\alpha}(Q_1,h_1)A^{\pm_2\mp_2}_{ji}(Q_2,h_2) \\
&\quad \times (2\pi)^4\delta^4(k'+q_{2j\pm_2}-q_{1\alpha\mp_1})(2\pi)^4\delta^4(k+q_{2i\mp_2}-q_{1\beta\pm_1})] \\
&\times T^{\mu\nu}_{+}(q_{10h_1}, q_{20h_2}).
\end{aligned} \tag{40}$$

A similar quark loop can be treated in exactly the same way. We only need to replace the PMNS-matrix U by the CKM-matrix V and use the masses of quarks instead of leptons.

For a fermion loop with two external Z's we may proceed in a similar manner. We take a fermion loop consisting of a fermion f[9] and sum over all

[8] We also need the trace formulas $\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} + g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta})$ and $\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^5) = 4i\varepsilon^{\alpha\beta\gamma\delta}$.

[9] Here i and j are used as generic flavour indices corresponding to the fermion f. This is natural when $f=\nu$, but for simplicity we use here the same indices when $f=\ell^-$, although α and β would be more natural.

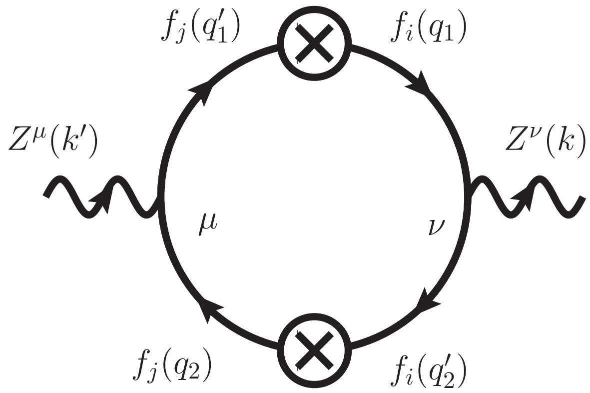


Figure 5: A fermion loop $C_Z^{<\nu\mu}(k,k')$ with two external Z bosons.

f. The loop is shown in Fig. 5 and takes the form

$$\begin{aligned}
C_Z^{<\nu\mu}(k,k') = \sum_f \sum_{i,j} \int & \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \frac{\mathrm{d}^4 q_1'}{(2\pi)^4} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{\mathrm{d}^4 q_2'}{(2\pi)^4} \\
& \times (2\pi)^4 \delta^4(k' + q_2 - q_1')(2\pi)^4 \delta^4(k + q_2' - q_1) \\
& \times (-1)\, \mathrm{Tr}(\mathcal{A}_{ii}(q_1) F_{ij}^<(q_1, q_1') \mathcal{A}_{jj}(q_1') \frac{-ig}{2c_W} \gamma^\mu (g_V^f - g_A^f \gamma^5) \\
& \quad \times \mathcal{A}_{jj}(q_2) F_{ji}^>(q_2, q_2') \mathcal{A}_{ii}(q_2') \frac{-ig}{2c_W} \gamma^\nu (g_V^f - g_A^f \gamma^5)).
\end{aligned} \tag{41}$$

We may generalize Eq. (38) to obtain the following two results:

$$\begin{aligned}
& \mathrm{Tr}((\not{q}_{1i\pm_1} + m_i) P_{h_1}(\hat{q}_1)(\not{q}_{1j\pm_1'} + m_j)\gamma^\mu(1 \pm \gamma^5) \\
& \quad \times (\not{q}_{2j\pm_2} + m_j) P_{h_2}(\hat{q}_2)(\not{q}_{2i\pm_2'} + m_i)\gamma^\nu(1 \pm \gamma^5)) \\
& = 2A_{ij}^{\pm_1\pm_1'}(Q_1, \mp h_1) A_{ji}^{\pm_2\pm_2'}(Q_2, \mp h_2) T_\mp^{\mu\nu}(q_{10\mp h_1}, q_{20\mp h_2})
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
& \mathrm{Tr}((\not{q}_{1i\pm_1} + m_i) P_{h_1}(\hat{q}_1)(\not{q}_{1j\pm_1'} + m_j)\gamma^\mu(1 \pm \gamma^5) \\
& \quad \times (\not{q}_{2j\pm_2} + m_j) P_{h_2}(\hat{q}_2)(\not{q}_{2i\pm_2'} + m_i)\gamma^\nu(1 \mp \gamma^5)) \\
& = 2B_{ij}^{\pm_1\pm_1'}(Q_1, \mp h_1) B_{ji}^{\pm_2\pm_2'}(Q_2, \pm h_2) \tilde{T}_\pm^{\mu\nu}(q_{10\pm h_1}, q_{20\pm h_2}),
\end{aligned} \tag{43}$$

where we have denoted

$$\tilde{T}_\pm^{\mu\nu}(a,b) = 2\delta_0^\mu T_\pm^{0\nu}(a,b) - T_\pm^{\mu\nu}(a,b). \tag{44}$$

Note that the traces in Eq. (43) are of the form $T_{\pm}^{\mu\nu}(q_{10\pm h_1}, q_{20\pm h_2})$ rather than $T_{\pm}^{\mu\nu}(q_{10\mp h_1}, q_{20\pm h_2})$. This is due to $\gamma^0 \not{q}_{10\mp h_1} \gamma^0 = \not{q}_{10\pm h_1}$.

Thus, in analogue to the W boson case, we obtain

$$\begin{aligned}
C_Z^{<\nu\mu}(k,k') &= \frac{g^2}{2c_W^2} \sum_f \sum_{i,j} \sum_{h_1,\pm_1,h_2,\pm_2} \\
&\times \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{\mathrm{d}^3 q_2}{(2\pi)^3} \frac{1}{2\omega_i(\vec{q}_1) 2\omega_j(\vec{q}_1)} \frac{1}{2\omega_i(\vec{q}_2) 2\omega_j(\vec{q}_2)} \\
&\times \{ f_{ijh_1\pm_1}^{m<}(\vec{q}_1) f_{jih_2\pm_2}^{m>}(\vec{q}_2) E_{\pm_1\pm_1\pm_2\pm_2}^{f\mu\nu}(\vec{q}_1, h_1, \vec{q}_2, h_2) \\
&\times (2\pi)^4 \delta^4(k' + q_{2j\pm_2} - q_{1j\pm_1}) (2\pi)^4 \delta^4(k + q_{2i\pm_2} - q_{1i\pm_1}) \\
&+ f_{ijh_1\pm_1}^{m<}(\vec{q}_1) f_{jih_2\pm_2}^{c>}(\vec{q}_2) E_{\pm_1\pm_1\pm_2\mp_2}^{f\mu\nu}(\vec{q}_1, h_1, \vec{q}_2, h_2) \\
&\times (2\pi)^4 \delta^4(k' + q_{2j\pm_2} - q_{1j\pm_1}) (2\pi)^4 \delta^4(k + q_{2i\mp_2} - q_{1i\pm_1}) \\
&+ f_{ijh_1\pm_1}^{c<}(\vec{q}_1) f_{jih_2\pm_2}^{m>}(\vec{q}_2) E_{\pm_1\mp_1\pm_2\pm_2}^{f\mu\nu}(\vec{q}_1, h_1, \vec{q}_2, h_2) \\
&\times (2\pi)^4 \delta^4(k' + q_{2j\pm_2} - q_{1j\mp_1}) (2\pi)^4 \delta^4(k + q_{2i\pm_2} - q_{1i\pm_1}) \\
&+ f_{ijh_1\pm_1}^{c<}(\vec{q}_1) f_{jih_2\pm_2}^{c>}(\vec{q}_2) E_{\pm_1\mp_1\pm_2\mp_2}^{f\mu\nu}(\vec{q}_1, h_1, \vec{q}_2, h_2) \\
&\times (2\pi)^4 \delta^4(k' + q_{2j\pm_2} - q_{1j\mp_1}) (2\pi)^4 \delta^4(k + q_{2i\mp_2} - q_{1i\pm_1}) \},
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
& E_{\pm_1\pm_1'\pm_2\pm_2'}^{f\mu\nu}(\vec{q}_1, h_1, \vec{q}_2, h_2) \\
&= \frac{1}{4}[(g_V^f + g_A^f)^2 A_{ij}^{\pm_1\pm_1'}(Q_1, h_1) A_{ji}^{\pm_2\pm_2'}(Q_2, h_2) T_+^{\mu\nu}(q_{10h_1}, q_{20h_2}) \\
&\quad + (g_V^f - g_A^f)^2 A_{ij}^{\pm_1\pm_1'}(Q_1, -h_1) A_{ji}^{\pm_2\pm_2'}(Q_2, -h_2) T_-^{\mu\nu}(q_{10-h_1}, q_{20-h_2}) \\
&\quad + (g_V^f + g_A^f)(g_V^f - g_A^f) B_{ij}^{\pm_1\pm_1'}(Q_1, -h_1) B_{ji}^{\pm_2\pm_2'}(Q_2, h_2) \\
&\qquad \times \tilde{T}_+^{\mu\nu}(q_{10h_1}, q_{20h_2}) \\
&\quad + (g_V^f + g_A^f)(g_V^f - g_A^f) B_{ij}^{\pm_1\pm_1'}(Q_1, h_1) B_{ji}^{\pm_2\pm_2'}(Q_2, -h_2) \\
&\qquad \times \tilde{T}_-^{\mu\nu}(q_{10h_1}, q_{20h_2})].
\end{aligned} \tag{46}$$

The tensors $T_{\pm}^{\mu\nu}(a,b)$ and $\tilde{T}_{\pm}^{\mu\nu}(a,b)$ defined in Eqs. (39) and (44) obey the following relations:

$$
\begin{aligned}
T_{\pm}^{\mu\nu}(a,b) &= T_{\mp}^{\nu\mu}(a,b) = T_{\mp}^{\mu\nu}(b,a) \\
T_{\pm}^{00}(a,b) &= a^0 b^0 + \vec{a}\cdot\vec{b} \\
T_{\pm}^{ii}(a,b) &= a^0 b^0 - \vec{a}\cdot\vec{b} + 2a_i b_i \\
T_{\pm}^{0i}(a,b) &= (-a^0\vec{b} - b^0\vec{a} \pm i\vec{a}\times\vec{b})_i \\
T_{\pm}^{ij}(a,b) &= a_i b_j + a_j b_i \pm i\varepsilon_{ijk}(b^0 a_k - a^0 b_k), \qquad i \neq j \\
\tilde{T}_{\pm}^{00}(a,b) &= T_{\pm}^{00}(a,b) \\
\tilde{T}_{\pm}^{0i}(a,b) &= T_{\pm}^{0i}(a,b) \\
\tilde{T}_{\pm}^{i0}(a,b) &= -T_{\pm}^{i0}(a,b) \\
\tilde{T}_{\pm}^{ij}(a,b) &= -T_{\pm}^{ij}(a,b).
\end{aligned} \tag{47}
$$

3.6 Gauge boson propagator expansion

The (anti-)Feynman propagator of a non-decaying boson of mass M is

$$i\Delta_{\nu\mu}(k) = i\frac{-g_{\mu\nu} + k_\mu k_\nu/M^2}{k^2 - M^2 + iM\Gamma}. \tag{48}$$

The leading decay width corrections is quadratic, and in the case of weak interactions $\Gamma \sim G_F$, so the arising corrections to the hermitean self energy of neutrinos calculated below in Section 4.2 is of order G_F^3. Such terms will, however, be neglected in the approximations described in more detail in Section 4.1, whence we drop decay width contributions entirely from $i\Delta_{\nu\mu}(k)$. We may expand this propagator to next-to-leading order as

$$\frac{-g_{\mu\nu} + k_\mu k_\nu/M^2}{k^2 - M^2} = M^{-2}g_{\mu\nu} - M^{-4}(k_\mu k_\nu - k^2 g_{\mu\nu}), \tag{49}$$

assuming that k^2 is much smaller than M^2. Where Fermi theory is sufficient, only the first term will be included. In the absence of decay width the full propagator $i\Delta_{\nu\mu}(k,k')$ factorizes in the form $i\Delta_{\nu\mu}(k,k') = (2\pi)^2\delta^4(k - k')i\Delta_{\nu\mu}(k)$, whence we will only use the propagator $i\Delta_{\nu\mu}(k)$. This is why the Feynman rules given in Section 3.1 were slightly asymmetric with respect to the description of momentum.

The case of the thermal propagator $i\Delta^{<}_{\nu\mu}(k,k')$ is different. Assuming that temperature is well below M, the gauge field is not significantly thermally excited. Thus the only nonzero term in $i\Delta^{<}_{\nu\mu}(k,k')$ to order M^{-4} arises from a lepton loop like the ones in Figs. 4 and 5. But as the fermion loops $C_W^{<\nu\mu}(k,k')$ and $C_Z^{<\nu\mu}(k,k')$ found above in Eqs. (40) and (45) in Section 3.5

do not conserve energy, we may not simply write $i\Delta^<_{\nu\mu}(k,k') = (2\pi)^2\delta^4(k-k')i\Delta^<_{\nu\mu}(k)$ with $i\Delta^<_{\nu\mu}(k) = M^{-3}g_{\mu\nu}\Gamma(k)$.[10]

Instead, we have the fermion loops $C_W^{<\nu\mu}(k,k')$ and $C_Z^{<\nu\mu}(k,k')$ between two leading order (anti-)Feynman propagators $i\Delta_{\nu\mu}(k) = iM^{-2}g_{\mu\nu}$, yielding

$$i\Delta_W^{<\nu\mu}(k,k') = M_W^{-4}C_W^{<\nu\mu}(k,k'). \tag{50}$$

and

$$i\Delta_Z^{<\nu\mu}(k,k') = M_Z^{-4}C_Z^{<\nu\mu}(k,k'). \tag{51}$$

Although energy is not conserved in these propagators, three-momentum is. This is manifestly visible in Eqs. 82 and 83.

4 Self energies

4.1 Outline

For the equation of motion (101) in Section 5 we need Hermitean and absorptive (emissive) self energies of the neutrinos. These are calculated in Sections 4.2 and 4.3, and further simplified in Section 4.4. The final forms of the self energies are listed concisely in Section 4.5 with some remarks.

The self energies are evaluated at four-momentum p and incoming and outgoing flavours i and j. To simplify the notations, we suppress p, i, and j and write $\Sigma^x = \Sigma^x_{ji}(p)$ for the various self energies Σ^x.

The calculations here will be done to order G_F^2 in the Fermi coupling constant

$$G_F = \frac{g^2}{4\sqrt{2}M_W^2}. \tag{52}$$

This order of expansion determines the diagrams we embark on to calculate and also the relevant expansion for the gauge boson propagators, which were discussed in Section 3.6.

As the expansion in the Fermi coupling constant is essentially an expansion in the inverse mass M_W^{-1}, we also make an expansion in the (not inverted) masses of neutrinos. The relevant approximations are given in Section 4.4.1. These approximations are not used in this thesis, but are presented because they are likely to be useful in further research.

We work with vacuum renormalized fields, whence we need not calculate any diagrams consisting of vacuum propagators only. We calculate two kinds

[10] In the vacuum theory we have the expansion $\frac{-g_{\mu\nu}+k_\mu k_\nu/M^2}{k^2-M^2} = M^{-2}g_{\mu\nu} - M^{-4}(k_\mu k_\nu - k^2 g_{\mu\nu}) - iM^{-3}\Gamma g_{\mu\nu}$, which suggests that $i\Delta^<_{\nu\mu}(k) = M^{-3}g_{\mu\nu}\Gamma(k)$.

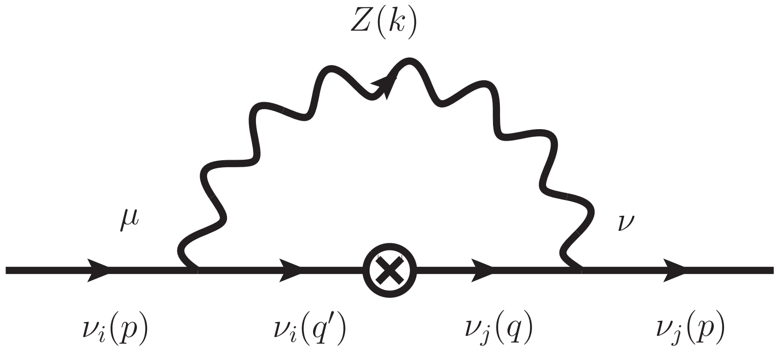


Figure 6: The Z loop diagram contributing to Σ^h, denoted by $\Sigma^{h,Z}$.

of self energies for neutrinos: the hermitean self energy describing the effect of the medium on the virtual cloud of particles surrounding a (bare) neutrino, and the absorptive self energy describing collisions with the medium.

4.2 Hermitean self energy

There are three diagrams contributing to the hermitean self energy Σ^h of neutrinos: the Z loop diagram of Fig. 6 (this contribution is denoted by $\Sigma^{h,Z}$), the W loop diagram of Fig. 7 ($\Sigma^{h,W}$), and the tadpole diagram of Fig. 6 ($\Sigma^{h,\mathrm{tad}}$).

Using the Feynman rules presented in Section 3.1, we obtain

$$\begin{aligned}\Sigma^{h,Z} = -i \int & \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{d}^4 q'}{(2\pi)^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-ig}{4c_W} \gamma_\nu (1-\gamma^5) \\ & \times \mathcal{A}_{jj}(q) F^<_{ji}(q,q') \mathcal{A}_{ii}(q') \frac{-ig}{4c_W} \gamma_\mu (1-\gamma^5) \\ & \times i\Delta_Z^{\nu\mu}(k) (2\pi)^4 \delta^4(k+q'-p).\end{aligned} \tag{53}$$

Using Eqs. (17) and (22), we find

$$\begin{aligned}\Sigma^{h,Z} = & \frac{ig^2}{16c_W^2} \sum_{h,\pm} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{2\omega_i(\vec{q})2\omega_j(\vec{q})} \\ & \times [f^{m<}_{jih\pm}(\vec{q}) A^{\pm\pm}_{ji}(Q,h) \gamma_\nu \not{q}_{0h} \gamma_\mu (1-\gamma^5)(2\pi)^4 \delta^4(k+q_{i\pm}-p) \\ & + f^{c<}_{jih\pm}(\vec{q}) A^{\pm\mp}_{ji}(Q,h) \gamma_\nu \not{q}_{0h} \gamma_\mu (1-\gamma^5)(2\pi)^4 \delta^4(k+q_{i\pm}-p)] \\ & \times i\Delta_Z^{\nu\mu}(k).\end{aligned} \tag{54}$$

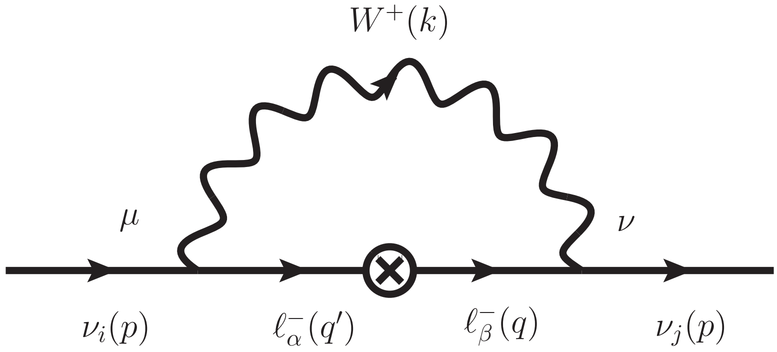


Figure 7: The W loop diagram contributing to Σ^h, denoted by $\Sigma^{h,W}$.

With the expansion (49) for $i\Delta_Z^{\nu\mu}(k,k')$ we obtain

$$\begin{aligned}\Sigma^{h,Z} = & \frac{-g^2}{16c_W^2}\sum_{h,\pm}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2\omega_i(\vec{q})2\omega_j(\vec{q})} \\ & \times[f^{m<}_{jih\pm}(\vec{q})A^{\pm\pm}_{ji}(Q,h)\gamma_\nu\not{q}_{0h}\gamma_\mu(1-\gamma^5)(2\pi)^4\delta^4(k+q_{i\pm}-p) \\ & + f^{c<}_{jih\pm}(\vec{q})A^{\pm\mp}_{ji}(Q,h)\gamma_\nu\not{q}_{0h}\gamma_\mu(1-\gamma^5)(2\pi)^4\delta^4(k+q_{i\pm}-p)] \\ & \times[(M_Z^{-2}+k^2M_Z^{-4})g^{\mu\nu}-M_Z^{-4}k^\mu k^\nu].\end{aligned} \tag{55}$$

The W loop diagram of Fig. 7 behaves similarly:

$$\begin{aligned}\Sigma^{h,W} = & \frac{-g^2}{8}\sum_{h,\pm}\sum_{\alpha,\beta}U_{\alpha i}U^*_{\beta j}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2\omega_\alpha(\vec{q})2\omega_\beta(\vec{q})} \\ & \times[f^{m<}_{\beta\alpha h\pm}(\vec{q})A^{\pm\pm}_{\beta\alpha}(Q,h)\gamma^\nu\not{q}_{0h}\gamma^\mu(1-\gamma^5)(2\pi)^4\delta^4(k+q_{\alpha\pm}-p) \\ & + f^{c<}_{\beta\alpha h\pm}(\vec{q})A^{\pm\mp}_{\beta\alpha}(Q,h)\gamma^\nu\not{q}_{0h}\gamma^\mu(1-\gamma^5)(2\pi)^4\delta^4(k+q_{\alpha\pm}-p)] \\ & \times[(M_W^{-2}+k^2M_W^{-4})g^{\mu\nu}-M_W^{-4}k^\mu k^\nu].\end{aligned} \tag{56}$$

For the tadpole diagram in Fig 8 we get

$$\Sigma^{h,\mathrm{tad}} = -i\sum_f\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-ig}{4c_W}\delta_{ij}\gamma_\mu(1-\gamma^5)i\Delta_Z^{\lambda\mu}(k)g_{\lambda\nu}C_f^\nu(k). \tag{57}$$

As for $\Sigma^{h,Z}$ and $\Sigma^{h,W}$, we use Eq. (49) for the propagator $i\Delta_Z^{\nu\mu}(k)$ to get

$$\begin{aligned}\Sigma^{h,\mathrm{tad}} = & \frac{-ig}{4c_W}\delta_{ij}\sum_f\int\frac{\mathrm{d}^4k}{(2\pi)^4} \\ & \times[(M_Z^{-2}+M_Z^{-4}k^2)C_f^\nu(k)\gamma_\nu-M_Z^{-4}k_\nu C_f^\nu(k)\not{k}](1-\gamma^5).\end{aligned} \tag{58}$$

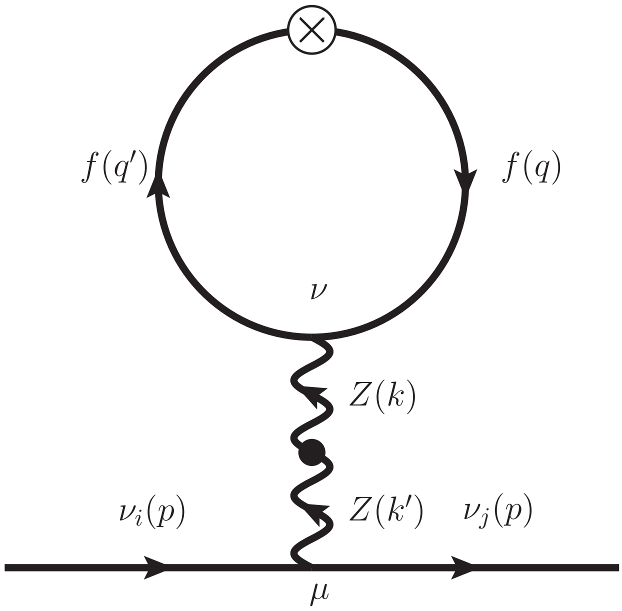


Figure 8: The tadpole diagram contributing to Σ^h, denoted by $\Sigma^{h,\text{tad}}$.

4.3 Absorptive and emissive self energy

The absorptive and emissive self energies $\Sigma^<$ and $\Sigma^>$ can be obtained from the same calculations by reversing the signs $<$ and $>$ the distribution functions $f^{<,>}$. For simplicity, we will only calculate $\Sigma^<$ explicitly.

There are five diagrams contributing to the absorptive self energy $\Sigma^{<,>}$ of neutrinos:

1. the diagram of Fig. 9 with one Z boson (this contribution is denoted by $\Sigma^{<,>,Z}$),
2. the diagram of Fig. 10 with one W boson ($\Sigma^{<,>,W}$),
3. the diagram of Fig. 11 with two Z bosons ($\Sigma^{<,>,ZZ}$),
4. the diagram of Fig. 12 with one Z and one W ($\Sigma^{<,>,WZ}$), and
5. the similar diagram of Fig. 13 with the roles of Z and W inverted ($\Sigma^{<,>,ZW}$).

Note that there is no similar diagram for two W's[11].

[11] In such a diagram there will necessarily be a violation in fermion number or charge

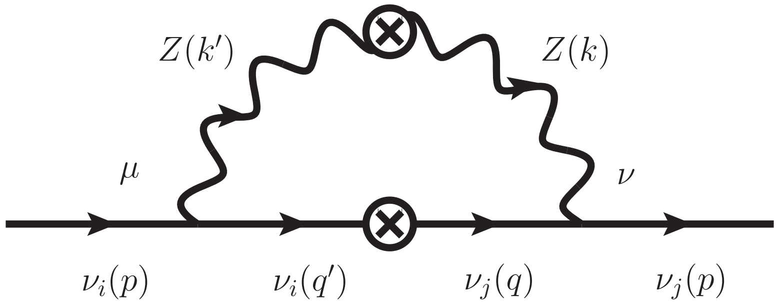


Figure 9: The one-Z diagram contributing to $\Sigma^{<,>}$, denoted by $\Sigma^{<,>,Z}$.

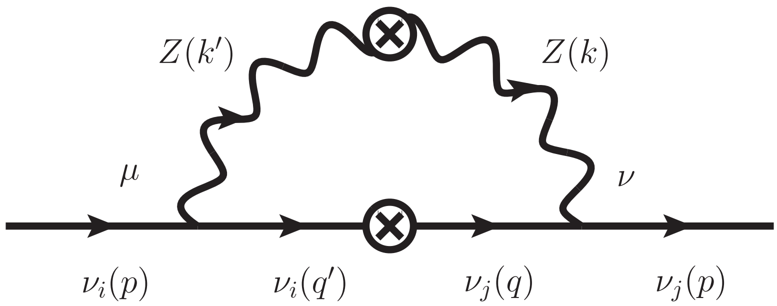


Figure 10: The one-W diagram contributing to $\Sigma^{<,>}$, denoted by $\Sigma^{<,>,W}$.

When calculating the self energy contribution $\Sigma^{<,Z}$ of Fig. 9, we may proceed as in the calculation of $\Sigma^{h,Z}$ in Section 4.2. We replace the propagator $i\Delta_Z^{\nu\mu}(k)$ of Eq. (49) with $i\Delta_Z^{\nu\mu}(k,k')$ of Eq. (51). We obtain

$$\begin{aligned}\Sigma^{<,Z} = \frac{ig^2}{16c_W^2}\sum_{h,\pm}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{1}{2\omega_i(\vec{q})2\omega_j(\vec{q})}\\ \times [f_{jih\pm}^{m<}(\vec{q})A_{ji}^{\pm\pm}(Q,h)\gamma_\nu \not{q}_{0h}\gamma_\mu(1-\gamma^5)(2\pi)^4\delta^4(k+q_{i\pm}-p)\\ + f_{jih\pm}^{c<}(\vec{q})A_{ji}^{\pm\mp}(Q,h)\gamma_\nu \not{q}_{0h}\gamma_\mu(1-\gamma^5)(2\pi)^4\delta^4(k+q_{i\pm}-p)]\\ \times M_Z^{-4}C_Z^{<\nu\mu}(k,k').\end{aligned} \tag{59}$$

For the contribution $\Sigma^{<,W}$ in Fig. 10 we may proceed in the same manner

conservation. For Majorana neutrinos such a diagram does exist, but is proportional to (Majorana) masses of neutrinos and would thus be omitted anyway in the approximations of Section 4.4.

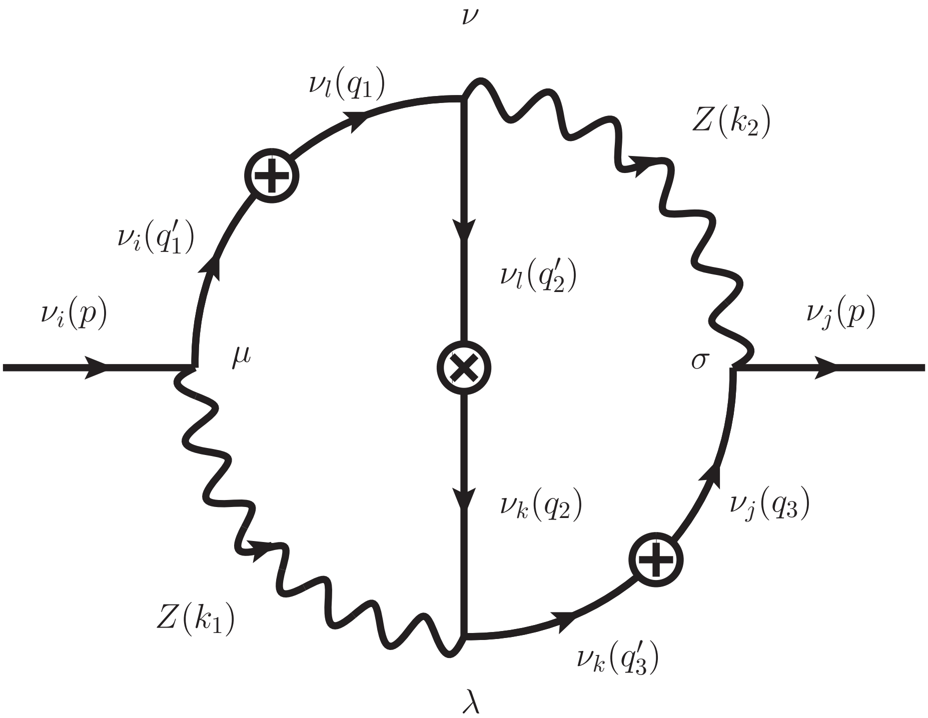


Figure 11: The two-Z diagram contributing to $\Sigma^{<,>}$, denoted by $\Sigma^{<,>,ZZ}$.

using Eq. (50) for the gauge boson propagator. We find

$$\begin{aligned}\Sigma^{<,W} = \frac{ig^2}{8} \sum_{h,\pm} \sum_{\alpha,\beta} U_{\alpha i} U^*_{\beta j} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{1}{2\omega_\alpha(\vec{q}) 2\omega_\beta(\vec{q})} \\ \times [f^{m<}_{\beta\alpha h\pm}(\vec{q}) A^{\pm\pm}_{\beta\alpha}(Q,h) \gamma_\nu \not{q}_{0h} \gamma_\mu (1-\gamma^5)(2\pi)^4 \delta^4(k + q_{\alpha\pm} - p) \\ + f^{c<}_{\beta\alpha h\pm}(\vec{q}) A^{\pm\mp}_{\beta\alpha}(Q,h) \gamma_\nu \not{q}_{0h} \gamma_\mu (1-\gamma^5)(2\pi)^4 \delta^4(k + q_{\alpha\pm} - p)] \\ \times M_W^{-4} C_W^{<\nu\mu}(k,k').\end{aligned} \tag{60}$$

By the Feynman rules presented in Section 3.1, we can easily construct the self-energy contribution $\Sigma^{<,ZZ}$ represented by Fig. 11:

$$
\begin{aligned}
\Sigma^{<,ZZ} = -i \sum_{k,l} \int & \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \frac{\mathrm{d}^4 q_1'}{(2\pi)^4} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{\mathrm{d}^4 q_2'}{(2\pi)^4} \frac{\mathrm{d}^4 q_3}{(2\pi)^4} \frac{\mathrm{d}^4 q_3'}{(2\pi)^4} \\
& \times \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \\
& \times \frac{-ig}{4c_W} \gamma_\sigma (1-\gamma^5) \mathcal{A}_{jj}(q_3) F^<_{jk}(q_3, q_3') \mathcal{A}_{kk}(q_3') \\
& \times \frac{-ig}{4c_W} \gamma_\lambda (1-\gamma^5) \mathcal{A}_{kk}(q_2) F^>_{kl}(q_2, q_2') \mathcal{A}_{ll}(q_2') \\
& \times \frac{-ig}{4c_W} \gamma_\nu (1-\gamma^5) \mathcal{A}_{ll}(q_1) F^<_{li}(q_1, q_1') \mathcal{A}_{ii}(q_1') \frac{-ig}{4c_W} \gamma_\mu (1-\gamma^5) \\
& \times i\Delta_Z^{\lambda\mu}(k_1) i\Delta_Z^{\sigma\nu}(k_2) (2\pi)^4 \delta^4(k_1' + q_1' - p) \\
& \times (2\pi)^4 \delta^4(k_2' + q_2' - q_1) (2\pi)^4 \delta^4(k_1 + q_2 - q_3').
\end{aligned}
\tag{61}
$$

Since we only wish to calculate the self energies to order M_Z^{-4} in Z boson mass, we can immediately drop next-to-leading terms in Eq. (49) and simply write

$$
i\Delta_Z^{\lambda\mu}(k_1) = i M_Z^{-2} g^{\lambda\mu} \tag{62}
$$

and similarly for $i\Delta_Z^{\sigma\nu}(k_2)$. With this and Eq. (17) we get

$$
\begin{aligned}
\Sigma^{<,ZZ} = +i\left(\frac{g}{4c_W M_Z}\right)^4 & \sum_{k,l}\sum_{h_1,\pm_1}\sum_{h_2,\pm_2}\sum_{h_3,\pm_3} \\
& \times \int \frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3}\frac{\mathrm{d}^3q_3}{(2\pi)^3}\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4} \\
& \times \frac{1}{2\omega_k(\vec{q}_3)2\omega_j(\vec{q}_3)}\frac{1}{2\omega_l(\vec{q}_2)2\omega_k(\vec{q}_2)}\frac{1}{2\omega_i(\vec{q}_1)2\omega_l(\vec{q}_1)} \\
& \times \gamma_\sigma(1-\gamma^5)(f^{m<}_{jkh_3\pm_3}(\vec{q}_3)(2\pi)^4\delta^4(k_1+q_{2k\pm_2}-q_{3k\pm_3}) \\
& \quad \times (\not{q}_{3j\pm_3}+m_j)P_{h_3}(\hat{q}_3)(\not{q}_{3k\pm_3}+m_k) \\
& \quad + f^{c<}_{jkh_3\pm_3}(\vec{q}_3)(2\pi)^4\delta^4(k_1+q_{2k\pm_2}-q_{3k\mp_3}) \\
& \quad \times (\not{q}_{3j\pm_3}+m_j)P_{h_3}(\hat{q}_3)(\not{q}_{3k\mp_3}+m_k)) \\
& \times \gamma_\lambda(1-\gamma^5)(f^{m>}_{klh_2\pm_2}(\vec{q}_2)(2\pi)^4\delta^4(k_2+q_{2l\pm_2}-q_{1l\pm_1}) \\
& \quad \times (\not{q}_{2k\pm_2}+m_k)P_{h_2}(\hat{q}_2)(\not{q}_{2l\pm_2}+m_l) \\
& \quad + f^{c>}_{klh_2\pm_2}(\vec{q}_2)(2\pi)^4\delta^4(k_2+q_{2l\mp_2}-q_{1l\pm_1}) \\
& \quad \times (\not{q}_{2k\pm_2}+m_k)P_{h_2}(\hat{q}_2)(\not{q}_{2l\mp_2}+m_l)) \\
& \times \gamma_\nu(1-\gamma^5)(f^{m<}_{lih_1\pm_1}(\vec{q}_1)(2\pi)^4\delta^4(k_1+q_{1i\pm_1}-p) \\
& \quad \times (\not{q}_{1l\pm_1}+m_l)P_{h_1}(\hat{q}_1)(\not{q}_{1i\pm_1}+m_i) \\
& \quad + f^{c<}_{lih_1\pm_1}(\vec{q}_1)(2\pi)^4\delta^4(k_1+q_{1i\mp_1}-p) \\
& \quad \times (\not{q}_{1l\pm_1}+m_l)P_{h_1}(\hat{q}_1)(\not{q}_{1i\mp_1}+m_i)) \\
& \times \gamma_\mu(1-\gamma^5)g^{\lambda\mu}g^{\sigma\nu}.
\end{aligned}
\tag{63}
$$

By Eqs. (26) and (27) and after carrying out the k_1 and k_2 integrals we obtain

$$
\begin{aligned}
\Sigma^{<,ZZ} = & -8i\left(\frac{g}{4c_W M_Z}\right)^4 \sum_{k,l}\sum_{h_1,\pm_1}\sum_{h_2,\pm_2}\sum_{h_3,\pm_3} \\
& \times \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3}\frac{\mathrm{d}^3 q_2}{(2\pi)^3}\frac{\mathrm{d}^3 q_3}{(2\pi)^3} \\
& \times \frac{1}{2\omega_k(\vec{q}_3)2\omega_j(\vec{q}_3)}\frac{1}{2\omega_l(\vec{q}_2)2\omega_k(\vec{q}_2)}\frac{1}{2\omega_i(\vec{q}_1)2\omega_l(\vec{q}_1)} \\
& \times [f^{m<}_{jkh_3\pm_3}(\vec{q}_3)f^{m>}_{klh_2\pm_2}(\vec{q}_2)f^{m<}_{lih_1\pm_1}(\vec{q}_1) \\
& \quad \times (2\pi)^4\delta^4(p+q_{2k\pm_2}-q_{1i\pm_1}-q_{3k\pm_3}) \\
& \quad \times A^{\pm_3\pm_3}_{jk}(Q_3,h_3)A^{\pm_2\pm_2}_{kl}(Q_2,h_2)A^{\pm_1\pm_1}_{li}(Q_1,h_1) \\
& + f^{m<}_{jkh_3\pm_3}(\vec{q}_3)f^{m>}_{klh_2\pm_2}(\vec{q}_2)f^{c<}_{lih_1\pm_1}(\vec{q}_1) \\
& \quad \times (2\pi)^4\delta^4(p+q_{2k\pm_2}-q_{1i\mp_1}-q_{3k\pm_3}) \\
& \quad \times A^{\pm_3\pm_3}_{jk}(Q_3,h_3)A^{\pm_2\pm_2}_{kl}(Q_2,h_2)A^{\pm_1\mp_1}_{li}(Q_1,h_1) \\
& + f^{m<}_{jkh_3\pm_3}(\vec{q}_3)f^{c>}_{klh_2\pm_2}(\vec{q}_2)f^{m<}_{lih_1\pm_1}(\vec{q}_1) \\
& \quad \times (2\pi)^4\delta^4(p+q_{2k\pm_2}-q_{1i\pm_1}-q_{3k\pm_3}) \\
& \quad \times A^{\pm_3\pm_3}_{jk}(Q_3,h_3)A^{\pm_2\mp_2}_{kl}(Q_2,h_2)A^{\pm_1\pm_1}_{li}(Q_1,h_1) \\
& + f^{m<}_{jkh_3\pm_3}(\vec{q}_3)f^{c>}_{klh_2\pm_2}(\vec{q}_2)f^{c<}_{lih_1\pm_1}(\vec{q}_1) \\
& \quad \times (2\pi)^4\delta^4(p+q_{2k\pm_2}-q_{1i\mp_1}-q_{3k\pm_3}) \\
& \quad \times A^{\pm_3\pm_3}_{jk}(Q_3,h_3)A^{\pm_2\mp_2}_{kl}(Q_2,h_2)A^{\pm_1\mp_1}_{li}(Q_1,h_1) \\
& + f^{c<}_{jkh_3\pm_3}(\vec{q}_3)f^{m>}_{klh_2\pm_2}(\vec{q}_2)f^{m<}_{lih_1\pm_1}(\vec{q}_1) \\
& \quad \times (2\pi)^4\delta^4(p+q_{2k\pm_2}-q_{1i\pm_1}-q_{3k\mp_3}) \\
& \quad \times A^{\pm_3\mp_3}_{jk}(Q_3,h_3)A^{\pm_2\pm_2}_{kl}(Q_2,h_2)A^{\pm_1\pm_1}_{li}(Q_1,h_1) \\
& + f^{c<}_{jkh_3\pm_3}(\vec{q}_3)f^{m>}_{klh_2\pm_2}(\vec{q}_2)f^{c<}_{lih_1\pm_1}(\vec{q}_1) \\
& \quad \times (2\pi)^4\delta^4(p+q_{2k\pm_2}-q_{1i\mp_1}-q_{3k\mp_3}) \\
& \quad \times A^{\pm_3\mp_3}_{jk}(Q_3,h_3)A^{\pm_2\pm_2}_{kl}(Q_2,h_2)A^{\pm_1\mp_1}_{li}(Q_1,h_1) \\
& + f^{c<}_{jkh_3\pm_3}(\vec{q}_3)f^{c>}_{klh_2\pm_2}(\vec{q}_2)f^{m<}_{lih_1\pm_1}(\vec{q}_1) \\
& \quad \times (2\pi)^4\delta^4(p+q_{2k\pm_2}-q_{1i\pm_1}-q_{3k\mp_3}) \\
& \quad \times A^{\pm_3\mp_3}_{jk}(Q_3,h_3)A^{\pm_2\mp_2}_{kl}(Q_2,h_2)A^{\pm_1\pm_1}_{li}(Q_1,h_1) \\
& + f^{c<}_{jkh_3\pm_3}(\vec{q}_3)f^{c>}_{klh_2\pm_2}(\vec{q}_2)f^{c<}_{lih_1\pm_1}(\vec{q}_1) \\
& \quad \times (2\pi)^4\delta^4(p+q_{2k\pm_2}-q_{1i\mp_1}-q_{3k\mp_3}) \\
& \quad \times A^{\pm_3\mp_3}_{jk}(Q_3,h_3)A^{\pm_2\mp_2}_{kl}(Q_2,h_2)A^{\pm_1\mp_1}_{li}(Q_1,h_1)] \\
& \times (q_{10h_1}\cdot q_{30h_3})\not{q}_{20h_2}(1-\gamma^5).
\end{aligned}
\tag{64}
$$

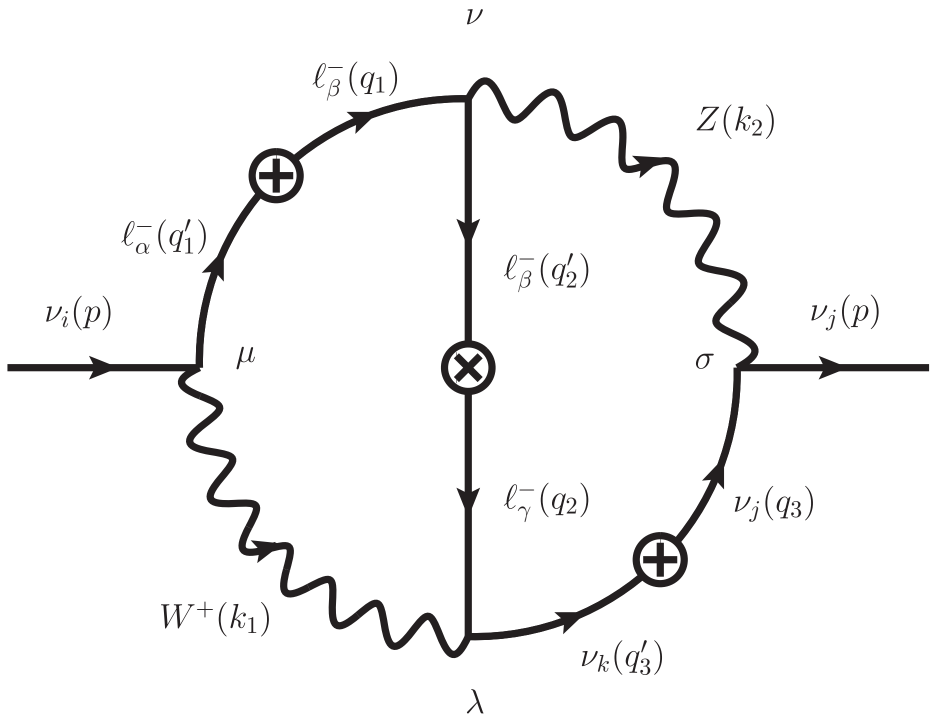


Figure 12: The W-Z diagram contributing to $\Sigma^{<,>}$, denoted by $\Sigma^{<,>,WZ}$.

For $\Sigma^{<,WZ}$ (see Fig. 12) the calculation is similar. For charged leptons $g_V^\ell = -\frac{1}{2}+2s_W^2$ and $g_A^\ell = -\frac{1}{2}$, so the chiral part of the coupling of the charged lepton to Z is

$$\begin{aligned}(g_V^\ell - g_A^\ell\gamma^5) &= \frac{1}{2}(g_V^\ell - g_A^\ell)(1+\gamma^5) + \frac{1}{2}(g_V^\ell + g_A^\ell)(1-\gamma^5) \\ &= s_W^2(1+\gamma^5) + \frac{1}{2}(2s_W^2 - 1)(1-\gamma^5).\end{aligned} \tag{65}$$

Thus we may use Eqs. (30) and (31) to see that we may replace the structure

$$
\begin{aligned}
&-8i\left(\frac{g}{4c_W M_Z}\right)^4 \sum_{k,l}\sum_{h_1,\pm_1}\sum_{h_2,\pm_2}\sum_{h_3,\pm_3}\int[\cdots] \\
&\times \sum_{\pm_1',\pm_2',\pm_3'} f^{m,c<}_{jkh_3\pm_3}(\vec{q}_3) f^{m,c>}_{klh_2\pm_2}(\vec{q}_2) f^{m,c<}_{lih_1\pm_1}(\vec{q}_1) \\
&\times A^{\pm_3\pm_3'}_{jk}(Q_3,h_3) A^{\pm_2\pm_2'}_{kl}(Q_2,h_2) A^{\pm_1\pm_1'}_{li}(Q_1,h_1) \\
&\times (2\pi)^4\delta^4(p+q_{2k\pm_2}-q_{1i\pm_1'}-q_{3k\pm_3'}) \\
&\times (q_{10h_1}\cdot q_{30h_3})\not{q}_{20h_2}(1-\gamma^5)
\end{aligned}
\tag{66}
$$

in Eq. (64) by

$$
\begin{aligned}
&-8i\frac{g^4}{8^2 c_W^2 M_Z^2 M_W^2}\sum_{\alpha,\beta,\gamma,k} U_{\alpha i}U^*_{\gamma k}\sum_{h_1,\pm_1}\sum_{h_2,\pm_2}\sum_{h_3,\pm_3}\int[\cdots] \\
&\times \sum_{\pm_1',\pm_2',\pm_3'} f^{m,c<}_{jkh_3\pm_3}(\vec{q}_3) f^{m,c>}_{\gamma\beta h_2\pm_2}(\vec{q}_2) f^{m,c<}_{\beta\alpha h_1\pm_1}(\vec{q}_1) \\
&\times (2\pi)^4\delta^4(p+q_{2\gamma\pm_2}-q_{1\alpha\pm_1'}-q_{3k\pm_3'}) \\
&\times [(s_W^2-\frac{1}{2})A^{\pm_3\pm_3'}_{jk}(Q_3,h_3)A^{\pm_2\pm_2'}_{\gamma\beta}(Q_2,h_2)A^{\pm_1\pm_1'}_{\beta\alpha}(Q_1,h_1) \\
&\quad\times (q_{10h_1}\cdot q_{30h_3})\not{q}_{20h_2} \\
&-\frac{1}{2}s_W^2 B^{\pm_3\pm_3'}_{jk}(Q_3,h_3)B^{\pm_2\pm_2'}_{\gamma\beta}(Q_2,-h_2)A^{\pm_1\pm_1'}_{\beta\alpha}(Q_1,h_1) \\
&\quad\times \not{q}_{20h_2}\not{q}_{10-h_1}\not{q}_{30-h_3}](1-\gamma^5)
\end{aligned}
\tag{67}
$$

to obtain $\Sigma^{<,WZ}$ instead of $\Sigma^{<,ZZ}$.

To find $\Sigma^{<,ZW}$, we use Eqs. (28) and (29). The structure of $\Sigma^{<,ZZ}$ pre-

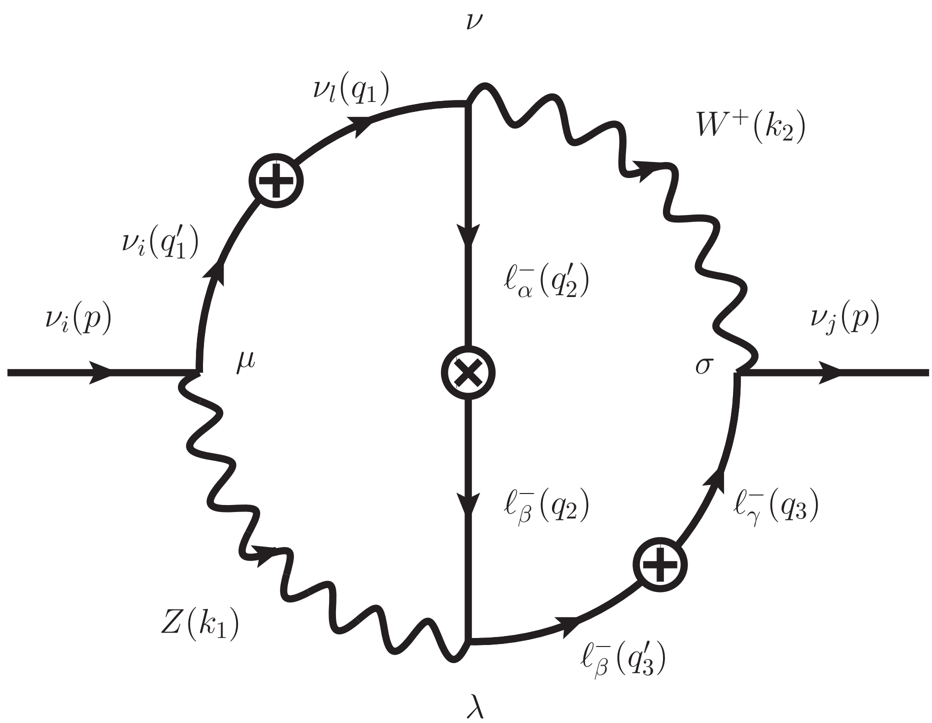


Figure 13: The Z-W diagram contributing to $\Sigma^{<,>}$, denoted by $\Sigma^{<,>,ZW}$.

sented in Eq. (66) can be replaced by

$$\begin{aligned}
&-8i\frac{g^4}{8^2 c_W^2 M_Z^2 M_W^2}\sum_{l,\alpha,\beta,\gamma} U_{\alpha l}U^*_{\gamma j}\sum_{h_1,\pm_1}\sum_{h_2,\pm_2}\sum_{h_3,\pm_3}\int[\cdots] \\
&\times \sum_{\pm_1',\pm_2',\pm_3'} f^{m,c<}_{\gamma\beta h_3\pm_3}(\vec{q}_3) f^{m,c>}_{\beta\alpha h_2\pm_2}(\vec{q}_2) f^{m,c<}_{lih_1\pm_1}(\vec{q}_1) \\
&\times (2\pi)^4\delta^4(p+q_{2\beta\pm_2}-q_{1i\pm_1'}-q_{3\beta\pm_3'}) \\
&\times [(s_W^2-\frac{1}{2})A^{\pm_3\pm_3'}_{\gamma\beta}(Q_3,h_3)A^{\pm_2\pm_2'}_{\beta\alpha}(Q_2,h_2)A^{\pm_1\pm_1'}_{li}(Q_1,h_1) \\
&\quad\times (q_{10h_1}\cdot q_{30h_3})\not{q}_{20h_2} \\
&-\frac{1}{2}s_W^2 A^{\pm_3\pm_3'}_{\gamma\beta}(Q_3,h_3)B^{\pm_2\pm_2'}_{\beta\alpha}(Q_2,h_2)B^{\pm_1\pm_1'}_{li}(Q_1,h_1) \\
&\quad\times \not{q}_{10-h_1}\not{q}_{30-h_3}\not{q}_{20h_2}](1-\gamma^5).
\end{aligned} \tag{68}$$

The additional factor $-\frac{1}{2}$ in front of the second term in Eqs. (67) and (68) is due to the fact that in Eqs. (29) and (31) we have a factor 32 instead of the factor -64 in Eq. (27).

4.4 Approximations and simplifications

Before simplifying further the expressions for the self energies obtained in Sections 4.2 and 4.3 we make the assumptions that there are no significant particle–antiparticle correlations. In practice, this implies that the distribution functions f^c are all identically zero.

We also assume that the distributions f^m are isotropic, that is $f^{m<,>}_{jih\pm}(\vec{q}) = f^{m<,>}_{jih\pm}(Q)$. We remark that this can only hold in one specific frame, because Q is not Lorentz-invariant.

Furthermore, we rescale the functions A and B defined in Eqs. (20) and (21) as follows:

$$\tilde{A}^{\pm\pm'}_{ji}(Q,h) = \frac{1}{2\omega_i 2\omega_j} A^{\pm\pm'}_{ji}(Q,h) \tag{69}$$

and

$$\tilde{B}^{\pm\pm'}_{ji}(Q,h) = \frac{1}{2\omega_i 2\omega_j} B^{\pm\pm'}_{ji}(Q,h). \tag{70}$$

Because we have taken $f^c = 0$, we only need the functions $\tilde{A}^{\pm\pm}_{ji}(Q,h)$ and $\tilde{B}^{\pm\pm}_{ji}(Q,h)$. For these

$$\tilde{A}^{\pm\pm}_{ji}(Q,h) = \frac{1}{4}\left[-hQ\frac{m_i+m_j}{\omega_i\omega_j} \pm \frac{m_i}{\omega_i} \pm \frac{m_j}{\omega_j}\right] \tag{71}$$

and

$$\tilde{B}^{\pm\pm}_{ji}(Q,h) = \frac{1}{4}\left[\frac{m_i m_j}{\omega_i\omega_j} - \frac{Q^2}{\omega_i\omega_j} + 1 \pm hQ\left(\frac{1}{\omega_j} - \frac{1}{\omega_i}\right)\right]. \tag{72}$$

For the diagonal elements we have

$$\tilde{A}^{\pm\pm}_{ii}(Q,h) = \frac{m_i}{2\omega_i}\left[-h\frac{Q}{\omega_i} \pm 1\right] \tag{73}$$

and

$$\tilde{B}^{\pm\pm}_{ii}(Q,h) = \frac{m_i^2}{2\omega_i^2}. \tag{74}$$

4.4.1 Approximations for $\tilde{A}_{ji}^{\pm\pm}(Q,h)$ and $\tilde{B}_{ji}^{\pm\pm}(Q,h)$

In the massless case $m_i = m_j = 0$ we have $\tilde{A}_{ji}^{\pm\pm}(Q,h) = \tilde{B}_{ji}^{\pm\pm}(Q,h) = 0$. For small masses, we approximate $\omega_i^{-1} \approx Q^{-1} - \frac{1}{2}Q^{-3}m_i^2$ to find

$$\begin{aligned}\tilde{A}_{ji}^{\pm\pm}(Q,h) \approx \frac{-h \pm 1}{8}\left[2\frac{m_i}{Q} + 2\frac{m_i}{Q} - \left(\frac{m_i^3}{Q^3} + \frac{m_j^3}{Q^3}\right)\right] \\ + \frac{h}{8}\left(\frac{m_i m_j^2}{Q^3} + \frac{m_j m_i^2}{Q^3}\right)\end{aligned} \tag{75}$$

and

$$\tilde{B}_{ji}^{\pm\pm}(Q,h) \approx \frac{1 \pm h}{8}\frac{m_i^2}{Q^2} + \frac{1 \mp h}{8}\frac{m_j^2}{Q^2} + \frac{1}{4}\frac{m_i m_j}{Q^2}, \tag{76}$$

where terms of order m_i^4 and higher in neutrino mass have been dropped.

For charged leptons the mass scale may not be negligible. But in this case flavour coherences is presumably small in comparison with neutrinos, and the distributions $f_{\alpha\beta h\pm}^{m,c<,>}(\vec{q})$ may be assumed to be diagonal.

4.4.2 Fermion loops

With the approximations at hand, the fermion loop $C_f^\mu(k)$ in Eq. (35) becomes

$$\begin{aligned}C_f^\mu(k) = \frac{ig}{2c_W}(2\pi)^4\delta^4(k)\delta_0^\mu \sum_{h,\pm}\int_0^\infty \frac{Q^2\mathrm{d}Q}{2\pi^2} f_{ffh\pm}^{m<}(Q) \\ \times [g_V^f(\tilde{A}_{ff}^{\pm\pm}(Q,h) + \tilde{A}_{ff}^{\pm\pm}(Q,-h)) \\ + g_A^f(\tilde{A}_{ff}^{\pm\pm}(Q,h) - \tilde{A}_{ff}^{\pm\pm}(Q,-h))].\end{aligned} \tag{77}$$

In the absence of helicity dependence in $f_{ffh\pm}^{m<}(Q)$, this simplifies to

$$\begin{aligned}C_f^\mu(k) = \frac{ig}{2c_W}(2\pi)^4\delta^4(k)\delta_0^\mu \sum_{h,\pm}\int_0^\infty \frac{Q^2\mathrm{d}Q}{2\pi^2} \\ \times f_{ffh\pm}^{m<}(Q)\left(\pm g_V^f\frac{m_f}{\omega_f} - g_A^f\frac{m_f Q}{\omega_f^2}\right),\end{aligned} \tag{78}$$

where Eq. (73) was used.

It is convenient to separate the delta functions and the coupling constant g from the loop; to this end, we write

$$C_f^\mu(k) = ig(2\pi)^4\delta^4(k)\delta_0^\mu\tilde{C}_f \tag{79}$$

whether helicity dependence is included or not.

The loop of Eq. (40) becomes

$$\begin{aligned} C_W^{<\nu\mu}(k,k') = {} & \frac{g^2}{4}(2\pi)^3\delta^3(\vec{k}-\vec{k}') \sum_{\alpha,\beta,i,j} \sum_{h_1,\pm_1,h_2,\pm_2} U_{\alpha j}U^*_{\beta i} \\ & \times \int \frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3} f^{m<}_{\beta\alpha h_1\pm_1}(Q_1) f^{m>}_{jih_2\pm_2}(Q_2) \\ & \times \tilde{A}^{\pm_1\pm_1}_{\beta\alpha}(Q_1,h_1)\tilde{A}^{\pm_2\pm_2}_{ji}(Q_2,h_2) \\ & \times (2\pi)^5\delta(k'^0 \pm_2 \omega_j(Q_2) \mp_1 \omega_\alpha(Q_1)) \\ & \times \delta(k^0 \pm_2 \omega_i(Q_2) \mp_1 \omega_\beta(Q_1))\delta^3(\vec{k}+\vec{q}_2-\vec{q}_1) \\ & \times T_+^{\mu\nu}(q_{10h_1},q_{20h_2}). \end{aligned} \tag{80}$$

Similarly for the loop with two Z bosons in Eq. (45) we obtain

$$\begin{aligned} & C_Z^{<\nu\mu}(k,k') = \frac{g^2}{2c_W^2}(2\pi)^3\delta^3(\vec{k}-\vec{k}') \sum_f \sum_{i,j} \sum_{h_1,\pm_1,h_2,\pm_2} \\ & \times \int \frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3} f^{m<}_{ijh_1\pm_1}(\vec{q}_1) f^{m>}_{jih_2\pm_2}(\vec{q}_2) \\ & \times (2\pi)^5\delta(k'^0 \pm_2 \omega_j(Q_2) \mp_1 \omega_j(Q_1)) \\ & \quad \times \delta(k^0 \pm_2 \omega_i(Q_2) \mp_1 \omega_i(Q_1))\delta^3(\vec{k}+\vec{q}_2-\vec{q}_1) \\ & \times \frac{1}{4}\{(g_V^f+g_A^f)^2\tilde{A}^{\pm_1\pm_1}_{ij}(Q_1,h_1)\tilde{A}^{\pm_2\pm_2}_{ji}(Q_2,h_2)T_+^{\mu\nu}(q_{10h_1},q_{20h_2}) \\ & \quad + (g_V^f-g_A^f)^2\tilde{A}^{\pm_1\pm_1}_{ij}(Q_1,-h_1)\tilde{A}^{\pm_2\pm_2}_{ji}(Q_2,-h_2)T_-^{\mu\nu}(q_{10-h_1},q_{20-h_2}) \\ & \quad + (g_V^f+g_A^f)(g_V^f-g_A^f) \\ & \qquad \times [\tilde{B}^{\pm_1\pm_1}_{ij}(Q_1,-h_1)\tilde{B}^{\pm_2\pm_2}_{ji}(Q_2,h_2)\tilde{T}_+^{\mu\nu}(q_{10h_1},q_{20h_2}) \\ & \qquad + \tilde{B}^{\pm_1\pm_1'}_{ij}(Q_1,h_1)\tilde{B}^{\pm_2\pm_2'}_{ji}(Q_2,-h_2)\tilde{T}_-^{\mu\nu}(q_{10h_1},q_{20h_2})]\}. \end{aligned} \tag{81}$$

We write the Eqs. (80) and (81) as

$$C_W^{<\nu\mu}(k,k') = g^2(2\pi)^3\delta^3(\vec{k}-\vec{k}')\tilde{C}_W^{<\nu\mu}(k,k') \tag{82}$$

and

$$C_Z^{<\nu\mu}(k,k') = \frac{g^2}{c_W^2}(2\pi)^3\delta^3(\vec{k}-\vec{k}')\tilde{C}_Z^{<\nu\mu}(k,k'), \tag{83}$$

thus defining the functions $\tilde{C}_W^{<\nu\mu}(k,k')$ and $\tilde{C}_Z^{<\nu\mu}(k,k')$.

4.4.3 Hermitean self energy

With these notations and approximations with $c_W M_Z = M_W$, we obtain from Eq. (55)[12]

$$\begin{aligned}\Sigma^{h,Z} &= \frac{g^2}{16}\sum_{h,\pm}\int_0^\infty \frac{\mathrm{d}Q}{2\pi^2}Q^2 f^{m<}_{jih\pm}(Q)\tilde{A}^{\pm\pm}_{ji}(Q,h)\\ &\times [2M_W^{-2}\gamma^0\\ &\quad + c_W^2 M_W^{-4}(p^2+2Q^2+3m_i^2-2hQ(p^0\pm\omega_i)-2p^0(p^0\mp\omega_i))\gamma^0\\ &\quad + 2c_W^2 M_W^{-4}(p^0+hQ\mp\omega_i)\gamma_p](1-\gamma^5).\end{aligned} \tag{84}$$

All terms antisymmetric in $\vec{q}$ have dropped, since they vanish because of the assumed spherical symmetry of $f^{m<}_{jih\pm}(Q)$.

The same simplifications for Eq. (56) yield

$$\begin{aligned}\Sigma^{h,W} &= \frac{g^2}{8}\sum_{\alpha\beta}U_{\alpha i}U^*_{\beta j}\sum_{h,\pm}\int_0^\infty \frac{\mathrm{d}Q}{2\pi^2}Q^2 f^{m<}_{\beta\alpha h\pm}(Q)\tilde{A}^{\pm\pm}_{\beta\alpha}(Q,h)\\ &\times [2M_W^{-2}\gamma^0\\ &\quad + M_W^{-4}(p^2+2Q^2+3m_\alpha^2-2hQ(p^0\pm\omega_\alpha)-2p^0(p^0\mp\omega_\alpha))\gamma^0\\ &\quad + 2M_W^{-4}(p^0+hQ\mp\omega_\alpha)\gamma_p](1-\gamma^5).\end{aligned} \tag{85}$$

Let us then turn to the tadpole diagram of Fig. 8, which is presented in Eq. (58). Eq. (79) demonstrates that $C_f^\nu(k)$ is proportional to $\delta^4(k)$, whence we obtain

$$\Sigma^{h,\mathrm{tad}} = \frac{g^2}{4M_W^2}c_W\delta_{ij}\left(\sum_f \tilde{C}_f\right)\gamma^0(1-\gamma^5), \tag{86}$$

where the coefficient $\tilde{C}_f$ is defined in Eq. (79).

4.4.4 Absorptive self energy

For the term $\Sigma^{<,Z}$ and $\Sigma^{<,W}$ of Eqs. (59) and (60), we use Eqs. (83) and (82) to obtain

$$\begin{aligned}\Sigma^{<,Z} = \frac{ig^4}{16M_W^4}\sum_{h,\pm}\int &\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}k'^0}{2\pi}\\ &\times f^{m<}_{jih\pm}(Q)\tilde{A}^{\pm\pm}_{ji}(Q,h)\gamma_\nu \not{q}_{0h}\gamma_\mu(1-\gamma^5)\\ &\times \tilde{C}_Z^{<\nu\mu}(p-q_{i\pm},(k'^0,\vec{p}-\vec{q}))\end{aligned} \tag{87}$$

[12] In the term of order M_Z^{-4} we have $k^2\gamma_\mu\not{q}_{0h}\gamma^\mu - \not{k}\not{q}_{0h}\not{k} = -k^2\not{q}_{0h} - 2k\cdot q_{0h}\not{k}$ and in the leading term $k^2\gamma_\mu\not{q}_{0h}\gamma^\mu = -2k^2\not{q}_{0h}$.

and

$$
\begin{aligned}
\Sigma^{<,W} = & \frac{ig^4}{8M_W^4} \sum_{h,\pm} \sum_{\alpha,\beta} U_{\alpha i} U_{\beta j}^* \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathrm{d}k'^0}{2\pi} \\
& \times f^{m<}_{\beta\alpha h\pm}(Q) \tilde{A}^{\pm\pm}_{\beta\alpha}(Q,h) \gamma_\nu \not{q}_{0h} \gamma_\mu (1-\gamma^5) \\
& \times \tilde{C}_W^{<\nu\mu}(p - q_{\alpha\pm}, (k'^0, \vec{p} - \vec{q})).
\end{aligned}
\tag{88}
$$

We note that in Eqs. (80) and (81) k'^0 only appears in a delta function $(2\pi)\delta(k'^0 - \dots)$. Thus we get easily for any four-vector a

$$
\begin{aligned}
\tilde{C}_W^{<\nu\mu}(a) = & \int \frac{\mathrm{d}k'^0}{2\pi} \tilde{C}_W^{<\nu\mu}(a, (k'^0, \vec{a})) \\
= & \frac{1}{4} \sum_{\alpha,\beta,i,j} \sum_{h_1,\pm_1,h_2,\pm_2} U_{\alpha j} U_{\beta i}^* \\
& \times \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{\mathrm{d}^3 q_2}{(2\pi)^3} f^{m<}_{\beta\alpha h_1 \pm_1}(Q_1) f^{m>}_{jih_2\pm_2}(Q_2) \\
& \times \tilde{A}^{\pm_1\pm_1}_{\beta\alpha}(Q_1, h_1) \tilde{A}^{\pm_2\pm_2}_{ji}(Q_2, h_2) \\
& \times (2\pi)^4 \delta^4(a + q_{2i\pm_2} - q_{1\beta\pm_1}) \\
& \times T_+^{\mu\nu}(q_{10h_1}, q_{20h_2})
\end{aligned}
\tag{89}
$$

and

$$
\begin{aligned}
\tilde{C}_Z^{<\nu\mu}(a) = & \int \frac{\mathrm{d}k'^0}{2\pi} \tilde{C}_Z^{<\nu\mu}(a, (k'^0, \vec{a})) \\
= & \frac{1}{2} \sum_f \sum_{i,j} \sum_{h_1,\pm_1,h_2,\pm_2} \\
& \times \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{\mathrm{d}^3 q_2}{(2\pi)^3} f^{m<}_{ijh_1\pm_1}(Q_1) f^{m>}_{jih_2\pm_2}(Q_2) \\
& \times (2\pi)^4 \delta^4(a + q_{2i\pm_2} - q_{1j\pm_1}) \\
& \times \frac{1}{4} \{ (g_V^f + g_A^f)^2 \tilde{A}^{\pm_1\pm_1}_{ij}(Q_1, h_1) \tilde{A}^{\pm_2\pm_2}_{ji}(Q_2, h_2) \\
& \quad \times T_+^{\mu\nu}(q_{10h_1}, q_{20h_2}) \\
& \quad + (g_V^f - g_A^f)^2 \tilde{A}^{\pm_1\pm_1}_{ij}(Q_1, -h_1) \tilde{A}^{\pm_2\pm_2}_{ji}(Q_2, -h_2) \\
& \quad \times T_-^{\mu\nu}(q_{10-h_1}, q_{20-h_2}) \\
& \quad + (g_V^f + g_A^f)(g_V^f - g_A^f) \\
& \quad\quad \times [\tilde{B}^{\pm_1\pm_1}_{ij}(Q_1, -h_1) \tilde{B}^{\pm_2\pm_2}_{ji}(Q_2, h_2) \tilde{T}_+^{\mu\nu}(q_{10h_1}, q_{20h_2}) \\
& \quad\quad + \tilde{B}^{\pm_1\pm_1'}_{ij}(Q_1, h_1) \tilde{B}^{\pm_2\pm_2'}_{ji}(Q_2, -h_2) \tilde{T}_-^{\mu\nu}(q_{10h_1}, q_{20h_2})] \}.
\end{aligned}
\tag{90}
$$

With these notations,

$$\begin{aligned}\Sigma^{<,Z} = &\frac{ig^4}{16M_W^4}\sum_{h,\pm}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\\ &\times f^{m<}_{jih\pm}(Q)\tilde{A}^{\pm\pm}_{ji}(Q,h)\gamma_\nu \not{q}_{0h}\gamma_\mu(1-\gamma^5)\\ &\times \tilde{C}^{<\nu\mu}_Z(p-q_{i\pm})\end{aligned} \tag{91}$$

and

$$\begin{aligned}\Sigma^{<,W} = &\frac{ig^4}{8M_W^4}\sum_{h,\pm}\sum_{\alpha,\beta}U_{\alpha i}U^*_{\beta j}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\\ &\times f^{m<}_{\beta\alpha h\pm}(Q)\tilde{A}^{\pm\pm}_{\beta\alpha}(Q,h)\gamma_\nu \not{q}_{0h}\gamma_\mu(1-\gamma^5)\\ &\times \tilde{C}^{<\nu\mu}_W(p-q_{\alpha\pm}).\end{aligned} \tag{92}$$

With the approximations at hand, the absorptive self energy $\Sigma^{<,ZZ}$ in Eq. (64) becomes

$$\begin{aligned}\Sigma^{<,ZZ} = &\frac{-ig^4}{32M_W^4}\sum_{k,l}\sum_{h_1,\pm_1}\sum_{h_2,\pm_2}\sum_{h_3,\pm_3}\\ &\times\int\frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3}\frac{\mathrm{d}^3q_3}{(2\pi)^3}\\ &\times f^{m<}_{jkh_3\pm_3}(\vec{q}_3)f^{m>}_{klh_2\pm_2}(\vec{q}_2)f^{m<}_{lih_1\pm_1}(\vec{q}_1)\\ &\times(2\pi)^4\delta^4(p+q_{2k\pm_2}-q_{1i\pm_1}-q_{3k\pm_3})\\ &\times\tilde{A}^{\pm_3\pm_3}_{jk}(Q_3,h_3)\tilde{A}^{\pm_2\pm_2}_{kl}(Q_2,h_2)\tilde{A}^{\pm_1\pm_1}_{li}(Q_1,h_1)\\ &\times(q_{10h_1}\cdot q_{30h_3})\not{q}_{20h_2}(1-\gamma^5).\end{aligned} \tag{93}$$

Using Eqs. (66), (67), and (68), we also find

$$
\begin{aligned}
\Sigma^{<,WZ} = & \frac{-ig^4}{8M_W^4} \sum_{\alpha,\beta,\gamma,k} U_{\alpha i} U^*_{\gamma k} \sum_{h_1,\pm_1} \sum_{h_2,\pm_2} \sum_{h_3,\pm_3} \\
& \times \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{\mathrm{d}^3 q_2}{(2\pi)^3} \frac{\mathrm{d}^3 q_3}{(2\pi)^3} \\
& \times f^{m<}_{jkh_3\pm_3}(\vec{q}_3) f^{m>}_{\gamma\beta h_2\pm_2}(\vec{q}_2) f^{m<}_{\beta\alpha h_1\pm_1}(\vec{q}_1) \\
& \times (2\pi)^4 \delta^4(p + q_{2\gamma\pm_2} - q_{1\alpha\pm_1} - q_{3k\pm_3}) \\
& \times [(s_W^2 - \frac{1}{2}) \tilde{A}^{\pm_3\pm_3}_{jk}(Q_3,h_3) \tilde{A}^{\pm_2\pm_2}_{\gamma\beta}(Q_2,h_2) \tilde{A}^{\pm_1\pm_1}_{\beta\alpha}(Q_1,h_1) \\
& \quad \times (q_{10h_1} \cdot q_{30h_3}) \not{q}_{20h_2} \\
& - \frac{1}{2} s_W^2 \tilde{B}^{\pm_3\pm_3}_{jk}(Q_3,h_3) \tilde{B}^{\pm_2\pm_2}_{\gamma\beta}(Q_2,-h_2) \tilde{A}^{\pm_1\pm_1}_{\beta\alpha}(Q_1,h_1) \\
& \quad \times \not{q}_{20h_2} \not{q}_{10-h_1} \not{q}_{30-h_3}](1-\gamma^5)
\end{aligned}
\tag{94}
$$

and

$$
\begin{aligned}
\Sigma^{<,ZW} = & \frac{-ig^4}{8M_W^4} \sum_{l,\alpha,\beta,\gamma} U_{\alpha l} U^*_{\gamma j} \sum_{h_1,\pm_1} \sum_{h_2,\pm_2} \sum_{h_3,\pm_3} \\
& \times \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{\mathrm{d}^3 q_2}{(2\pi)^3} \frac{\mathrm{d}^3 q_3}{(2\pi)^3} \\
& \times f^{m<}_{\gamma\beta h_3\pm_3}(\vec{q}_3) f^{m>}_{\beta\alpha h_2\pm_2}(\vec{q}_2) f^{m<}_{lih_1\pm_1}(\vec{q}_1) \\
& \times (2\pi)^4 \delta^4(p + q_{2\beta\pm_2} - q_{1i\pm_1} - q_{3\beta\pm_3}) \\
& \times [(s_W^2 - \frac{1}{2}) A^{\pm_3\pm_3'}_{\gamma\beta}(Q_3,h_3) A^{\pm_2\pm_2'}_{\beta\alpha}(Q_2,h_2) A^{\pm_1\pm_1'}_{li}(Q_1,h_1) \\
& \quad \times (q_{10h_1} \cdot q_{30h_3}) \not{q}_{20h_2} \\
& - \frac{1}{2} s_W^2 A^{\pm_3\pm_3'}_{\gamma\beta}(Q_3,h_3) B^{\pm_2\pm_2'}_{\beta\alpha}(Q_2,h_2) B^{\pm_1\pm_1'}_{li}(Q_1,h_1) \\
& \quad \times \not{q}_{10-h_1} \not{q}_{30-h_3} \not{q}_{20h_2}](1-\gamma^5).
\end{aligned}
\tag{95}
$$

4.5 Summary of self energies

The hermitean self energy consists of three parts:

$$
\Sigma^h = \Sigma^{h,Z} + \Sigma^{h,W} + \Sigma^{h,\mathrm{tad}},
\tag{96}
$$

and the parts were described above:

- $\Sigma^{h,Z}$ (Fig. 6): Eq. (84).

- $\Sigma^{h,W}$ (Fig. 7): Eq. (85).
- $\Sigma^{h,\mathrm{tad}}$ (Fig. 8): Eqs. (86), (79), and (77).

The absorptive self energy is a similar sum:

$$\Sigma^{<} = \Sigma^{<,Z} + \Sigma^{<,W} + \Sigma^{<,ZZ} + \Sigma^{<,WZ} + \Sigma^{<,ZW}, \tag{97}$$

whose terms were also calculated in the preceeding sections:

- $\Sigma^{<,Z}$ (Fig. 9): Eqs. (91), (83), and (81).
- $\Sigma^{<,W}$ (Fig. 10): Eqs. (92), (82), and (80).
- $\Sigma^{<,ZZ}$ (Fig. 11): Eq. (93).
- $\Sigma^{<,WZ}$ (Fig. 12): Eq. (94).
- $\Sigma^{<,ZW}$ (Fig. 13): Eq. (95).

We remark again that in the fermion loops $\tilde{C}_Z^{<\nu\mu}(k,k')$ and $\tilde{C}_W^{<\nu\mu}(k,k')$ defined by Eqs. (83) and (82) should also include summation over quark loops. In the case of $\tilde{C}_Z^{<\nu\mu}(k,k')$ this is already implicitly included in the sum over all fermions. The quark loops should be added to $\tilde{C}_W^{<\nu\mu}(k,k')$; we replace the PMNS-matrix U by the CKM-matrix V and use the masses of quarks instead of leptons, as already mentioned in Section 3.5.

The mass of t quark is even greater than that of Z [14, 15], so there is no reason to include $t\bar{t}$-loops in $\tilde{C}_Z^{<\nu\mu}(k,k')$ or any loops containing t in $\tilde{C}_W^{<\nu\mu}(k,k')$. If the temperature is so high that the t field is thermally excited to a significant extent, then also the gauge boson should be, rendering the present approximation scheme inadequate.

The reader should also bear in mind that despite only $\Sigma^{<}$ was explicitly calculated, we will also need $\Sigma^{>}$ in Section 5. We may obtain $\Sigma^{>}$ from $\Sigma^{<}$ by interchanging all distribution functions $f_{ijh\pm}^{m<}$ and $f_{ijh\pm}^{m>}$. The particle–antiparticle correlation terms $f_{ijh\pm}^{c<,>}$ were assumed to be negligible in Section 4.4.

5 Equation of motion

The equation of motion in cQPA is Eq. (6). It is an equation of motion for the Wightman functions $S^{<,>}$ and contains the self energies Σ^{h} and $\Sigma^{<,>}$ as parameters.

In Ref. [16] the equation of motion (6) was reduced to an equation of motion for the hermitean local correlator $\bar{S}^{<,>}_{\vec{k}}(t,t) = iS^{<,>}_{\vec{k}}(t,t)\gamma^0$:

$$
\begin{aligned}
\partial_t \bar{S}^{<}_{\vec{k}}(t,t) = -i\big[\tilde{H}_{\vec{k},eff}(t), \bar{S}^{<}_{\vec{k}}(t,t)\big]^m \\
- \frac{1}{2}\Big(\big\{\tilde{\Sigma}^{>}_{\vec{k}}(t), \bar{S}^{<}_{\vec{k}}(t,t)\big\}^m - \big\{\tilde{\Sigma}^{<}_{\vec{k}}(t), \bar{S}^{>}_{\vec{k}}(t,t)\big\}^m\Big)
\end{aligned}
\tag{98}
$$

with the effective Hamiltonian

$$
\begin{aligned}
\tilde{H}_{\vec{k},eff}(t) &= H_{\vec{k}0}(t) + \tilde{\Sigma}^h_{\vec{k}}(t) \\
\tilde{\Sigma}_{\vec{k}}(t) &= \int \frac{\mathrm{d}k^0}{2\pi}\, \bar{\Sigma}_{eff}(k,t)\mathcal{A}(k)\mathcal{A}^{-1}_{\mathbf{k}0} \\
\bar{\Sigma}^h &= \gamma^0 \Sigma^h \\
\bar{\Sigma}^{<,>} &= \gamma^0 i \Sigma^{<,>}
\end{aligned}
\tag{99}
$$

and the generalized (anti)commutators[13]

$$
\big[A,B\big]^m = AB - B^\dagger A^\dagger, \quad \{A,B\}^m = AB + B^\dagger A^\dagger. \tag{100}
$$

This matrix approach is complicated to handle, but it can be reduced to a set of scalar equations.

The most convenient scalar to use an equation of motion for the distribution functions $f^{m<,>}_{ijh\pm}$. This can be done by taking projections and tracing over Dirac indices as in Ref. [4]. The main equation of interest is Eq. (3.14) of Ref. [4]:

$$
\partial_t f^m_{ijh\pm} = \mp i2\Delta\omega_{ij} f^m_{ijh\pm} - \frac{\mathbf{k}^2 \bar{m}_{ij}}{2\Omega^2_{mij}} \{\tfrac{m'}{\omega^2}, f^-_{h\pm}\}^m_{ij} + iX^m_{h\pm}[f]_{ij} + \mathcal{C}^m_{h\pm}[f]_{ij}. \tag{101}
$$

The first term on the right hand side generates neutrino oscillations in the interactionless limit, the second one depends on the time derivative of mass, the third one is due to mixing gradients and the fourth is due to collisions with the thermal background. A more detailed description of the notations used can be found in Section 3.1 of Ref. [4]. We will return to this equation in Section 5.3 once we are ready to formulate it in the present situation.

The present situation differs slightly from what was done in Ref. [4] in the following ways:

[13] These are (up to a constant) the hermitean and antihermitean parts of the matrix AB, and thus $[AB,C]^m = [A,BC]^m = [ABC,1]^m$ and similarly for $\{\cdot,\cdot\}^m$, so the notation is slightly redundant. The purpose of this notation becomes clear when the particle–antiparticle correlations are included and one has to introduce the corresponding (anti)commutators $[\cdot,\cdot]^c$ and $\{\cdot,\cdot\}^c$. See Ref. [4] for details.

- We take the neutrino masses and mixing to be constant, thus dropping all terms proportional to m' or Ξ'.[14]
- We have neglected the particle–antiparticle correlations $f^{c<,>}_{ijh\pm}$.
- We have taken Σ^h into account, although it was neglected in the treatment in Ref. [4]. We manifest this by replacing Ξ' by $-\Sigma^h$. This is natural, because in the absence of Σ^h the effective Hamiltonian is $\hat{H}_0 - \Xi'$ but with mixing gradients neglected, we have the effective Hamiltonian $\hat{H}_0 + \Sigma^h$.

Before proceeding, we adopt the following notations:

$$\begin{aligned}
\omega_i &= \omega_i(P) \\
\bar{m}_{ij} &= \frac{1}{2}(m_i + m_j) \\
\Delta m_{ij} &= \frac{1}{2}(m_i - m_j) \\
\bar{\omega}_{ij} &= \frac{1}{2}(\omega_i + \omega_j) \\
\Delta\omega_{ij} &= \frac{1}{2}(\omega_i - \omega_j) \\
\Omega^2_{mij} &= \bar{m}^2_{ij} - \Delta\omega^2_{ij} = \frac{1}{2}(\omega_i\omega_j - P^2 + m_i m_j) \\
\left(\frac{m}{\omega}\right)_{ij} &= \frac{m_i}{\omega_i}\delta_{ij} \\
(v_{\vec{p}})_{ij} &= \frac{P}{\omega_i}\delta_{ij} \\
N_{ijh\pm} &= \frac{\bar{\omega}_{ij}}{\bar{m}_{ij}} \pm h\frac{P\bar{m}_{ij}}{\Omega^2_{mij}} = \frac{\frac{1}{2}(m_i\omega_j + m_j\omega_i) \pm hP\bar{m}_{ij}}{\Omega^2_{mij}}
\end{aligned} \tag{102}$$

Note that here we use the p as the four-momentum instead of k used in Ref. [4].

5.1 Hermitean self energy

In Eq. (3.27) and the related discussion in Ref. [4] it was shown that the mixing gradient term is $\Xi'_{ij} = \Xi'^{+}_{ij} + \Xi'^{-}_{ij}\gamma^5$. In the calculations of Section 4 we found that $\Sigma^h = (a^h\gamma^0 + b^h\gamma_p)P_L$ for some flavour matrices a^h and b^h[15].

[14] The matrix Ξ' is defined by $\Xi'^{\pm}_{ij} \equiv \frac{i}{2}\big(V\partial_t V^\dagger \pm U\partial_t U^\dagger\big)_{ij}$, where U and V are the unitary matrices used to (locally) diagonalize the mass matrix. For more details, see Ref. [4].

[15] We remark that the upper index h here stands for 'hermitean' and is not related to helicity. The lower index appearing below in ξ_h is a helicity index.

The hermitean self energy appears as $P_h\gamma^0\Sigma^h$ when one solves for $f^{m<}_{ijh\pm}$ from the equation of motion similar to Eq. (6) to find Eq. (101). By calculation, we find

$$P_h\gamma^0(a^h\gamma^0 + b^h\gamma_p)P_L = P_h(a^h + hPb^h\gamma^5)P_L = P_h(a^h - hPb^h)P_L. \quad (103)$$

Since $P_h\Xi' = P_h(\Xi'^+ + \Xi'^-\gamma^5)$ and $P_h\gamma^0\Sigma^h$ is to replace $P_h\Xi'$, we can take $\xi_{hij} = \Xi'^+_{ij} = -\Xi'^-_{ij}$, where

$$\xi_h = \frac{1}{2}(a^h - hPb^h). \quad (104)$$

With $\xi = \Xi'^+ = -\Xi'^-$, the mixing gradient term in Eq. (101) can be written as (cf. Eq. (3.16) of Ref. [4])

$$\begin{aligned} X^m_{h\pm}[f]_{ij} = \frac{1}{2}&\left[\xi_h\left(1 \mp hv_{\vec{p}} + N_{ijh\mp}\frac{m}{\omega}\right), f^{m<}_{h\pm}\right]^m_{ij} \\ &- \frac{P\Delta\omega_{ij}}{2\Omega^2_{mij}}\left\{\xi_h\left(v_{\vec{p}} \mp h\right), f^{m<}_{h\pm}\right\}^m_{ij}. \end{aligned} \quad (105)$$

5.2 Collision term

In Ref. [4] the absorptive self energy is parametrized as

$$P_h i\Sigma^{<,>}_{eff}(p,t) = P_h\left(\gamma^0 A^{<,>}_h + \hat{\vec{p}}\cdot\gamma\, B^{<,>}_h + C^{<,>}_h + ih\gamma^5 D^{<,>}_h\right). \quad (106)$$

In the calculations of Section 4 we found that $i\Sigma^{<,>} = (a^{<,>}\gamma^0 + b^{<,>}\gamma_p)P_L$ for some flavour matrices $a^{<,>}$ and $b^{<,>}$. A straightforward (cf. Eqs. (103) and (104)) calculation yields

$$P_h(a^{<,>}\gamma^0 + b^{<,>}\gamma_p)P_L = \alpha^{<,>}_h \not{p}_{0h} P_L = \alpha^{<,>}_h P_h \not{p}_{0h}, \quad (107)$$

where we have defined the flavour matrix

$$\alpha^{<,>}_h = \frac{1}{2}(a^{<,>} - hPb^{<,>}). \quad (108)$$

Using the parametrization (106), we find $A^{<,>}_h = \alpha^{<,>}_h$, $B^{<,>}_h = -h\alpha^{<,>}_h$, and $C^{<,>}_h = D^{<,>}_h = 0$.

In this case the collision term appearing in Eq. (101) is (cf. Eq. (4.21) of Ref. [4])

$$\begin{aligned} \mathcal{C}^m_{h\pm}[f]_{ij} = -\frac{1}{4}\Bigg(&\left\{\alpha^>_h(1 \mp hv_{\vec{p}} + N_{ijh\mp}\frac{m}{\omega}), f^{m<}_{h\pm}\right\}^m_{ij} \\ &- \frac{P\Delta\omega_{ij}}{\Omega^2_{mij}}\left[\alpha^>_h(v_{\vec{p}} \mp h), f^{m<}_{h\pm}\right]^m_{ij} \\ &- [> \leftrightarrow <]\Bigg). \end{aligned} \quad (109)$$

5.3 Formulation of the equation of motion

Inserting Eqs. (105) and (109) to Eq. (101), we obtain the equation of motion in the present situation:

$$\partial_t f^m_{ijh\pm} = \mp i2\Delta\omega_{ij} f^m_{ijh\pm} + X^m_{h\pm}[f]_{ij} + \mathcal{C}^m_{h\pm}[f]_{ij}, \tag{110}$$

where $X^m_{h\pm}[f]_{ij}$ and $\mathcal{C}^m_{h\pm}[f]_{ij}$ are as they were defined in Eqs. (105) and (109). This is the fundamental equation of motion, which we shall consider in more detail in the subsequent sections.

In Eq. (110) the distribution $f^m_{ijh\pm}$ only depends on three-momentum $\vec{p}$, so the following question arises: On what energy should one evaluate ξ_h and $\alpha_h^{<,>}$? The proper shell is that of the middle index in the sum, that is [4]

$$\xi_{hij}(p^0, \vec{p}) A_{jk}(\vec{p}) = \xi_{hij}(\omega_j(P), \vec{p}) A_{jk}(\vec{p}). \tag{111}$$

For practical numerical calculations the choice of shell may not make much of a difference if the self-energies do not depend too strongly on energy.

We also remark that the flavour matrices ξ_h and $\alpha_h^{<,>}$ can be obtained by

$$\xi_h = \frac{1}{4}\,\mathrm{Tr}(\not{p}_{0-h}\Sigma^h) \quad \text{and} \quad \alpha_h^{<,>} = \frac{1}{4}\,\mathrm{Tr}(\not{p}_{0-h} i\Sigma^{<,>}), \tag{112}$$

although this may not be practically any more convenient than calculating Σ^h and $i\Sigma^{<,>}$ and identifying the components a and b as in Eqs. (104) and (108).

5.4 Preliminary analysis

To gain a further insight to Eq. (110), our equation of motion, let us consider it in a couple of simple special cases. A proper analysis of the implications of the equation calls for a numerical approach, which is omitted here; instead, we do simplifying approximations and recover the behaviour predicted by simpler models.

The approximations are coarse and scenarios overly simplified in the following discussion; this is so to illustrate the basic phenomenology without flooding the analysis with technical details.

5.4.1 No interactions

Let us first consider the case of no interactions. If densities of the neutrinos and medium are low, this provides a reasonable approximation. The hermitean self energy in Eq. (96) is proportional to g^2 and the absorptive self energy to g^4. Letting $g = 0$ thus makes these self energies vanish. Setting $M_W = M_Z = \infty$ would have the same effect.

In this situation the equation of motion (110) simplifies dramatically:

$$\partial_t f^m_{ijh\pm} = \mp i 2\Delta\omega_{ij} f^m_{ijh\pm}. \tag{113}$$

The equations for different sets of indices i, j, h, and $\pm$ fully decouple. Given an initial condition $f^m_{ijh\pm}(\vec{p}, t=0)$, the above equation has the solution

$$f^m_{ijh\pm}(\vec{p}, t) = \exp(\mp i 2\Delta\omega_{ij}(\vec{p})t) f^m_{ijh\pm}(\vec{p}, 0). \tag{114}$$

Suppose now that the initial data is such that the distributions $f^m_{ijh\pm}(\vec{p}, t = 0)$ are peaked near some momentum $\vec{k}$ such that $\left|\vec{k}\right|$ is much larger than neutrino masses. Then

$$2\Delta\omega_{ij}(\vec{p}) \approx \frac{m_i^2 - m_j^2}{2\left|\vec{k}\right|} = \frac{\Delta m^2_{ij}}{2E}. \tag{115}$$

We use the notation E for $\left|\vec{k}\right|$, since it gives the (approximate) energy for all neutrinos.

Suppose furthermore that the initial state contains no antineutrinos. Since neutrinos and antineutrinos are completely decoupled in the absence of interactions, no antineutrinos will appear at any time. Different helicities are not coupled either, so we drop the helicity index completely.

With these approximations, we obtain the time evolution

$$f^m_{ij}(\vec{p}, t) = \exp\left(-i\frac{\Delta m^2_{ij}}{2E} t\right) f^m_{ij}(\vec{p}, 0). \tag{116}$$

The diagonal elements $f^m_{ii}(\vec{p}, t)$ describing particle densities (cf. Eq. (11)) remain constant, whereas the off-diagonals describing quantum coherence between different flavours oscillate with angular frequencies $\Delta m^2_{ij}/2E$. When one transforms this time evolution to flavour basis, $f^m_{\alpha\beta}(\vec{p}, t) = U_{\alpha i} U^*_{\beta j} f^m_{ij}(\vec{p}, t)$, the oscillation appears also in the diagonal elements, making it measurable (via weak charged current interactions).

In other words, we recover the usual neutrino oscillations. One should note that neutrino oscillations are a kinematical phenomenon and are therefore present even in the absence of any interactions. But if coherence is omitted, no oscillations occur, as expected. For a more elaborate discussion of neutrino oscillations in QFT (and, in particular, comparison between QM and QFT), see Ref. [17].

5.4.2 Diagonal distribution

Let us consider the case when the distribution functions $f^m_{ijh\pm}$ are diagonal in flavour indices i and j. The equation of motion (110) does not guarantee that an initially diagonal distribution remains diagonal, but under some simplifying assumptions the off-diagonal entries do remain at zero.

When the distribution functions are diagonal, so are the hermitean and absorptive self energies of Section 4.4.3 and 4.4.4 that do not contain W-bosons ($\Sigma^{h,Z}$, $\Sigma^{h,\mathrm{tad}}$, $\Sigma^{<,Z}$, and $\Sigma^{<,ZZ}$). Similarly the diagonality of the charged lepton distribution functions implies that $\Sigma^{h,W}$ is diagonal in the flavour basis. The structures of $\Sigma^{<,WZ}$ and $\Sigma^{<,ZW}$ are more complicated in this respect, but are nevertheless greatly simplified from the general case.

Let us neglect neutrino mixing[16]. Since the distribution functions are diagonal and there is no mixing, all self energies are diagonal in flavour. Using Eq. (104) we find that

$$X^m_{h\pm}[f]_{ij} = 0. \tag{117}$$

That is, hermitean self energies do not contribute to time evolution in this case, since the corresponding term in the equation of motion (110) vanishes.

If collisions were neglected, we would have $\partial_t f^m_{iih\pm} = 0$ by Eq. (110). Including collisions leads to nontrivial time evolution. The matrix $\alpha_h^{<,>}$ as defined in Eq. (108) is diagonal, so Eq. (109) yields

$$\mathcal{C}^m_{h\pm}[f]_{ii} = -\,(1 \mp hv_i)\,(\alpha^>_{hii} f^{m<}_{iih\pm} - \alpha^<_{hii} f^{m>}_{iih\pm}), \tag{118}$$

where $v_i = P/\omega_i$ is the velocity corresponding to the momentum P and flavour i.

With this and Eq. (11) the equation of motion becomes

$$\partial_t n_{ih} = (1 - hv_i)[\alpha^<_{hii} - (\alpha^>_{hii} + \alpha^<_{hii}) n_{ih}] \tag{119}$$

for particles and

$$\partial_t \bar{n}_{ih} = (1 - hv_i)[\alpha^>_{hii} - (\alpha^>_{hii} + \alpha^<_{hii}) \bar{n}_{ih}] \tag{120}$$

for antiparticles. These equations have an immediate interpretation: The term $\alpha^>_{hii} + \alpha^<_{hii}$ corresponds to decay and $\alpha^<_{hii}$ ($\alpha^>_{hii}$) is a source term for (anti)particles.

The correspondence between $\alpha^>_{hii} + \alpha^<_{hii}$ and decay is very natural. The decay width is [4] $\Gamma = \frac{i}{2}(\Sigma^> + \Sigma^<)$, whence Eq. (112) gives

$$\alpha^>_h + \alpha^<_h = \frac{1}{2}\,\mathrm{Tr}(\not{p}_{0-h}\Gamma). \tag{121}$$

[16] The same result could be obtained by neglecting all diagrams which include W-bosons.

Eqs. (119) and (120) give decoupled equations of motion for particles and antiparticles of different flavours. All coherence is absent, but this is only due to the assumptions that there is initially no coherence and that neutrino mixing is trivial.

In a more realistic case of nontrivial mixing, an initially diagonal distribution functions do not remain diagonal. The self energy terms containing W-bosons are in a generic situation nondiagonal, whence also $\partial_t f^m_{ijh\pm}$ is nondiagonal by Eq. (110). That is, mixing in weak charged current interactions generates coherence between neutrinos of different flavour.

As an example of the relation between mixing and coherence, consider neutrino production. If a neutrino is created in a process of weak charged current from a charged lepton of well-defined flavour (say, an electron), not only is a superposition between massive neutrino states created, but also coherence between them. This is by no means a surprise; rather, this observation explains how cQPA can naturally and simply explain essential phenomena in neutrino oscillations in the QFT framework.

6 Conclusions and outlook

The main result of this work is Eq. (110). This is the equation of motion for neutrinos in the situation described in the beginning of Section 5. This equation of motion with its terms as summarised in Sections 4.5, 5.1, and 5.2 gives a tool to analyse time evolution of a neutrino field.

Phenomena that could be studied with this equation of motion (or a suitable variant of it) include, for example, electroweak phase transition in the early Universe and neutrino oscillations.[10] In addition, models of coherent baryogenesis in the early Universe or neutrino transport in a supernova explosion could be conveniently described within the cQPA formalism. There is an additional difficulty to the case of supernovae: special relativity gives even locally a bad approximation of the structure of spacetime under such extreme conditions, and therefore the full time evolution of a supernovae is beyond the reach of the current formalism.

To fully grasp the implications of Eq. (110), one needs to solve the equation numerically. The preliminary analysis given in Section 5.4 demonstrates that at a simple level the phenomenology of this equation of motion is as expected. In particular, the role of coherence in neutrino oscillations is stressed and explained in the framwork of cQPA.

Applicability of the calculational tools developed in Section 3 is by no means limited to neutrinos in the Standard Model. With these tools, the self energy calculations of Section 4 could be carried out for any mixed massive spin-$\frac{1}{2}$ particles. These self energies lead to an equation of motion similar to Eq. (110). With numerous simplifying assumptions, such as those used in Section 5, the equation of motion turns out to be rather simple. This does not, however, mean that the resulting phenomenology is trivial; the coherence effects remain significant even in the interactionless limit described in Section 5.4.1.

All these applications have to do with weak interactions. It would be interesting to see how these tools could be generalized to, for example, quantum chromodynamics (QCD). Weak interactions come with a nontrivial mixing also in the quark sector and coherence may play an important role even in the absence of weak interactions, but due to otherwise strong (nonperturbative) interactions of the quarks the resulting phenomenology is difficult to analyse. Without perturbative diagrammatic expansions for the self energies the calculations of Section 4 and the subsequent analysis would be dramatically changed.

In the light of the discussion of Section 5.4, it appears that cQPA is a natural tool to study and explain phenomena where quantum mechanical coherence, interacting field theory, effects due to nonzero temperature and temporally varying medium, and special relativity have to be taken into account simultaneously. A calculationally feasible tool to analyse such phenomena can greatly help us understand, for example, the physics of the early Universe.

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