DEPARTMENT OF PHYSICS
UNIVERSITY OF JYVÄSKYLÄ

# TRANSPORT PROPERTIES OF MESOSCOPIC SYSTEMS

**ANNA-MAIJA UIMONEN**

BACHELOR OF SCIENCE ESSAY

JYVÄSKYLlÄ, FINLAND
DECEMEMBER 2007

# Abstract

This text is a brief introduction to conductivity in mesoscopic systems. At first we study the diffusive transport, where the movement of charge carriers can be characterized by Boltzmann theory of conductivity. In the phase-coherent regime there is an interesting phenomenon called coherent or enhancement backscattering, which reduces the conductivity. A correction from this phenomenon to Drude conductivity is introduced. In the ballistic regime a totally different point of view has to be taken, in order to describe the conductance. With help of scattering approach the Landauer formula for conductance is obtained. The effect of noise in mesoscopic systems is considered. Finally, the relation between Kubo formula and Landauer formula is considered.

# Tiivistelmä

Teksti on lyhyt johdatus johtavuuteen mesoskooppisisa systeemeissä. Aluksi johtavuus ongelmaa tarkastellaan diffuusion avulla. Tällä rajalla johtavuutta voidaan kuvata Boltzmannin teorian avulla. "Vaihekoherentilla" rajalla on mielenkiintoinen ilmiö nimeltään "koherentti/vahvistettu" takaisinsironta, joka aiheuttaa johtavuuden pienenemistä. Tarkastelun kohteena on tämän ilmiön aiheuttama korjaus Druden johtavuuteen. Ballistisella rajalla johtavuuden ymmärtämiseksi täytyy ottaa aivan uusi näkökulma. Sirontateorian avulla voidaan johtaa Landaurein yhtälö johtavuudelle. Lopuksi käsitelään hieman kohinaa ja mitä se kertoo meille, sekä myös miten Kubo-yhtälö ja Landauerin yhtälö liittyvät toisiinsa.

# Contents

# 1 Introduction

Why is it interesting to study the conductance of small scale systems? One obvious reason is the decrease of the size of the components in modern technology, but how should this concern physicists at all? Should this not just be a problem of modern engineering?

When the sizes of the samples are reduced or single atoms or molecules are being investigated quantum phenomena step in to the arena. The conductance through a atomic scale contact is a truly quantum phenomenon as the observed steps in the conductance for example a function of the contact area indicate. By understanding the conductance at the small scale more understanding and information could be gained from the behavior of the nature and from the quantum phenomenon in general.

Classically the conductance of a macroscopic conductor is understood by motion of valence/ conduction electrons under a applied electric field. The number of atoms in the conductor is enormous, some integer multiple of Avogadro's number $N_A \approx 6,022 \cdot 10^{23}$ and thus the number of conduction electrons is of the same order of magnitude. In this case the conductivity can be expressed by the familiar Ohm's law, which states that the conductance $G$ is directly proportional to the cross-sectional area $A$ and inversely proportional to length L of a conductor,

$$G = \sigma A/L,$$

where $\sigma$ is a characteristic constant of a given material and it relates the current density to the electric field $j = \sigma E$.

But what happens when the size of the conductor is reduced to the scale of atoms and there are only few conduction electrons allowed to go though? The motion of the electrons cannot be described anymore just by diffusive motion along the conductor where the electrons every now and then scatter off by impurities and other electrons. That is because the typical distance between the scattering events is larger than the interparticle distance between atoms and the ohms' law does not apply anymore.

The electrons go through the atomic size conductor ballistically and to get a better description for the conduction it is preferable or even necessary to consider the electrons as a wave instead of particles. The Landauer formula describes the conductance at the ballistic regime and in its simplicity its states that the conductance is equivalent to the transmission probability of the incoming waves through the conductor. The

Landauer formula is obtained via second quantization, where the wave functions are replaced by creation and annihilation operators, which create and annihilate particles (electrons) in and from the many-particle states $|\sigma\rangle$.

This text is about the conductance in the mesoscopic regime. A rough definition for the mesoscopic conductor can be given by saying that the dimensions of mesoscopic conductor are larger than microscopic dimensions but still smaller than macroscopic, i.e., ohmic dimensions. A more specific definition can be given using length scales defined by different scattering mechanisms in the conductor.

# 2 Basic concepts

### 2.0.1 Different transport regimes

The identification of different transport regimes is done according to the relative sizes of various length scales of different scattering mechanisms. The two most fundamental length scales are the phase-coherence length $l_\phi$ and the elastic mean free path $l$. A short description of these length scales has been given in the article written by Agraït *et al.* [1] and in the notes written by Galperin [2].

The elastic mean free path $l$ is a measure of the distance between charge-carrier collisions with static impurities or phonons. It is defined as [2]

$$l = \tau v \quad \text{or} \quad l = \sqrt{D\tau}, \tag{2.1}$$

where $\tau$ is the relaxation time, $v$ is a typical velocity and $D$ is the diffusion constant. Typically $v$ can be taken to be Fermi velocity $v_F$.

The comparison of elastic mean-free-path with the sample dimensions gives the diffusive and ballistic transport regimes. When the length $L$ and the cross-sectional area $A$ of the conductor are larger than the mean free path $l$ i.e $l \ll L, A$ the motion of electrons can be considered to be diffusive. In the diffusive transport regime the charge carriers are moving randomly in different directions and this movement can be viewed as random walk with step size $l$.

In the cases when the elastic mean free path is larger than the length $L$ and the cross-sectional area of the conductor, i.e., $l > L, A$ the transport is said to be ballistic. In this regime the electron momentum is approximately constant and restricted by collisions with the boundaries of the sample. The conductivity $\sigma$ plays no role in the ballistic regime and the conductance $G$ is characterized by Landauer formula

$$G = \frac{2e^2}{h} \sum_{n=1}^{N} T_n, \tag{2.2}$$

which states that the conductance is proportional to the transmission probability $T_n$ through the "conductor".

The phase-coherence or phase-relaxation time or length are purely a quantum mechanical time- and length-scales and give the distance over which quantum coherence

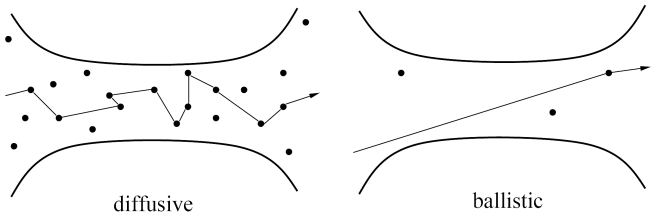



**Figure 2.1**: Diffusive and ballistic conductor [1].

is preserved, i.e. they describe the distance over which a particle or an electron must travel until its initial phase-coherence is lost. The processes that usually destroy the phase-coherence are electron-electron, electron-phonon interactions (scattering) and spin-flip processes. A common factor for all these processes according to Galperin [2] is that they break time-reversal symmetry. On the other hand the scattering of the charge carrier from the static time-independent potentials is time- reversible and does not lead to phase-relaxation [2].

In those processes which destroy the phase-coherence the charge carrier is suffering many collisions during a time $\tau_\phi$. Since the charge carriers can be assumed to move diffusively, an estimate to phase-coherence length $l_\phi$ is [2]

$$l_\phi = v_F \tau_\phi \quad \text{or} \quad l_\phi = \sqrt{D\tau_\phi}, \tag{2.3}$$

where $D$ is the diffusion constant.

In order to be able to state the criterion for an ohmic conductor there is still another length scale needed, namely the de Broglie wavelength of electrons

$$\lambda = \frac{h}{p}, \tag{2.4}$$

where $h$ is Planck's constant and $p$ is the momentum of the particle.

If the dimensions of the conductor are greater then all the introduced length scales, i.e., mean free path $l$, phase-coherence length $l_\phi$ and de Broglie wavelength $\lambda$, the conductor shows ohmic behavior. If the dimensions of the conductor are smaller than the phase-coherence length, i.e $L, A < l_\phi$ Ohm's law does not apply and the conductor is shows to the mesoscopic phenomena.

### 2.0.2 Drude conductivity

Drude was one of the first persons to develop the theory of the conductivity. His approach was based purely on classical theory of electromagnetism. This model is

discussed in the book written by Ashcroft & Mermin [3]. When a charged particle is placed in the electric field $\mathbf{E}$ it acquires a drift velocity $\mathbf{v}_{drift}$, which is parallel to the electric field.

But first, let us consider $n$ electrons in the unit volume, which all move with velocity $\mathbf{v}$. During a time $dt$ electrons will advance a distance $vdt$ in direction of $\mathbf{v}$. The total charge passing an area $A$ perpendicular to the direction of flow will be then $dQ = n(evdt)A$. The net current flow is [3]

$$\mathbf{j} = \frac{dQ}{dtA} - ne\mathbf{v}. \tag{2.5}$$

Now the $\mathbf{v}$ is the average electronic velocity. If there is no electric field present electrons are moving in random directions and $\mathbf{v}$ averages to zero. In the presence of the electric field the force acting on the electron is the Lorentz-force $\mathbf{F} = -e\mathbf{E}$. The solution to the equation of motion is now

$$m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} = \mathbf{F}. \tag{2.6}$$

The total velocity of electrons is

$$\mathbf{v} = \mathbf{v}_0 - \frac{e\mathbf{E}t}{m}, \tag{2.7}$$

where $\mathbf{v}_0$ is the velocity immediately after the collision and $-e\mathbf{E}t/m$ is the velocity caused by the electric field. Assuming that electrons will emerge from collision to random directions there will be no contribution from $\mathbf{v}_0$ to the average electron velocity and [3]

$$\langle \mathbf{v} \rangle = -\frac{t\mathbf{E}\langle t \rangle}{m}, \tag{2.8}$$

where $\langle t \rangle = \tau$ is the relaxation time. The current density is then [3]

$$\mathbf{j} = \left(\frac{ne^2\tau}{m}\right) \; E \tag{2.9}$$

where the Drude conductivity is obtained

$$\sigma = en\mu_e = \frac{ne^2\tau}{m}, \tag{2.10}$$

where the $\mu_e = e\tau/m$ is the electron mobility defined also as ratio of the drift velocity to the electric field $\mu_e = |v_{drift}/\mathbf{E}|$.

With help of this information it might also be possible to give a better explanations to that kind of phenomenon like adhesion, friction and wear. Drude conductivity is however a very restrictive and its fails to explain many observed phenomena. Examples of assumptions and failures of Drude model can be found in the literature for example from the book written by Ashcroft & Mermin [3].

# 3 Boltzmann conductivity

A little more detailed description for conductivity in the diffusive regime is given by the Boltzmann theory of conductivity. This theory has been considered for example in the books written by Stephen Elliot [4] and Michael Marder [5].

The collection of microscopic particles, in this case electrons, can be described by the distribution function $g(\mathbf{k})$. The electrical current density is then proportional to the integral over the first Brillouin zone of this function weighted by the velocity function $v(\mathbf{k})$ [4]

$$\mathbf{j} = -\frac{e}{4\pi^3}\int_{1stB.Z} v(\mathbf{k})g(\mathbf{k})d\mathbf{k}. \tag{3.1}$$

The distribution $g(\mathbf{k})$ is conserved as the system evolves in time according to Liouville's theorem. Therefore it must be [4]

$$g(t, \mathbf{r}, \mathbf{k}) = g(t + dt, \mathbf{r} + d\mathbf{r}, \mathbf{k} + d\mathbf{k}) \tag{3.2}$$

after a time interval $dt$ in the absence of collisions.

The real materials are never ideal and conduction electrons suffer collisions with impurities, phonons or other electrons. These collisions change the position and wavevector of individual electrons. A collision can transfer an electron from state $(\mathbf{r}, \mathbf{k})$, where $\mathbf{r}$ is the position of the electron right before the collision and $\mathbf{k}$ is the wavevectors of the electron, to a state $(\mathbf{r} + d\mathbf{r}\ , \mathbf{k} + d\mathbf{k})$. Taking this into account gives the more general expression for the equation (3.2) [4]

$$g(t + dt, \mathbf{r} + d\mathbf{r}, \mathbf{k} + d\mathbf{k}) - g(t, \mathbf{r}, \mathbf{k}) = \left(\frac{\partial g}{\partial t}\right)_s dt, \tag{3.3}$$

where $(\partial g/\partial t)_s$ refers to the temporal change in $g$ due to scattering.

This equation can be solved using $\dot{\mathbf{r}} = \mathbf{v}(\mathbf{k})$ and $\hbar\dot{\mathbf{k}} = -e(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \mathbf{F}(\mathbf{r}, \mathbf{k})$, where $\mathbf{F}$ is the force acting on the electron. So the equation (3.3) can be written in the first order approximation as [4]

$$\frac{\partial g}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}g + \mathbf{F}\cdot\frac{1}{\hbar}\nabla_{\mathbf{k}}g = \left(\frac{\partial g}{\partial t}\right)_s, \tag{3.4}$$

which reduces in the absence of magnetic fields to the following

$$\frac{\partial g}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}g - \frac{e}{\hbar}\mathbf{E}\cdot\nabla_{\mathbf{k}}g = \left(\frac{\partial g}{\partial t}\right)_s. \tag{3.5}$$

The probability for the scattering has to be proportional to the number of occupied electron $\mathbf{k}$ states $g(\mathbf{k})$, to the number of vacancies in the $\mathbf{k}'$ states $[1-g(\mathbf{k}')]$ and also proportional to the transition rate $W_{\mathbf{k}\mathbf{k}'}$, which is independent of $g(\mathbf{k})$ (independent electron approximation). Thus the probability can be written in the form [5]

$$\mathcal{P}(\mathbf{k}\to\mathbf{k}',t) = g(\mathbf{k})[1-g(\mathbf{k}')]\delta_{\sigma\sigma'}W_{\mathbf{k}\mathbf{k}'}, \tag{3.6}$$

where $\sigma$ refers to the spin. The potential was assumed to be nonmagnetic and thus the transition can only occur if the spin state does not change.

Taking into account both electrons that are jumping up into $\mathbf{k}$-states and those which are jumping down gives expression for the derivative $(\partial g/\partial t)_s$ [5]

$$\left(\frac{\partial g}{\partial t}\right)_s = \frac{V}{2}\int [d\mathbf{k}']\left(g(\mathbf{k}')[1-g(\mathbf{k})]W_{\mathbf{k}'\mathbf{k}} - g(\mathbf{k})[1-g(\mathbf{k}')]W_{\mathbf{k}\mathbf{k}'}\right), \tag{3.7}$$

where $[d\mathbf{k}] = \frac{2}{V}\sum_{\mathbf{k}} = \frac{2}{(2\pi)^3}\int d\mathbf{k}$.

This equation is equivalent to the relaxation time approximation if the scattering potentials are spherically symmetric and can be treated as weak perturbation. Secondly the energies of occupied electron states must be isotropic and depend only upon the magnitude of $\mathbf{k}$.

The assumption of weak a potential allows the use of Fermi's Golden Rule writing the transition rate between states $\mathbf{k}$ and $\mathbf{k}'$ [5].

$$W_{\mathbf{k}\mathbf{k}'} = \frac{2\pi}{\hbar}\delta(\mathcal{E}_{\mathbf{k}} - \mathcal{E}_{\mathbf{k}'})|\langle\mathbf{k}|\hat{U}|\mathbf{k}'\rangle|^2, \tag{3.8}$$

where the potential matrix element is

$$\langle\mathbf{k}|\hat{U}|\mathbf{k}'\rangle = \int d\mathbf{r}\psi^*_{n\mathbf{k}'}(\mathbf{r})U(\mathbf{r})\psi_{n\mathbf{k}}(\mathbf{r}). \tag{3.9}$$

The Bloch functions are taken to be normalized as

$$\int_{unitcell} d\mathbf{r}|\psi_{n\mathbf{k}}(\mathbf{r})|^2 = V_{unitcell}. \tag{3.10}$$

Because the potential is spherically symmetric $W_{\mathbf{k}\mathbf{k}'}$ is also symmetric and depends only the angle $\theta$ between the $\mathbf{k}$ and $\mathbf{k}'$ and so $W_{\mathbf{k},\mathbf{k}'} = W_{\mathbf{k}',\mathbf{k}}$. This allows to simplify equation (3.7) to the following

$$\Rightarrow \left(\frac{\partial g}{\partial t}\right)_s = \frac{V}{2}\int [d\mathbf{k}']W_{\mathbf{k}\mathbf{k}'}[g(\mathbf{k}') - g(\mathbf{k})] \tag{3.11}$$

At this point is an assumption is made that Eq. (3.4) has a solution which has the form [5]

$$g(\mathbf{k}) = f(\mathbf{k}) + \vec{\mathcal{C}}\cdot\mathbf{k}, \tag{3.12}$$

where $\vec{\mathcal{C}}$ is a vector either a constant or it is a function of $\mathcal{E}(\mathbf{k})$. The function $f(\mathbf{k})$ is the Fermi-Dirac distribution function

$$f(\mathbf{k}) = \frac{1}{\exp[\mathcal{E}(\mathbf{k}) - \mu]\beta + 1}, \tag{3.13}$$

where $\beta = 1/k_B T$.

The form (3.12) is general enough when spatially uniform electric fields, uniform temperature gradients, and uniform magnetic fields are considered [5]. Substituting the assumption (3.12) to the Eq. (3.11) gives (the collisions are assumed to be elastic)

$$\Rightarrow \left(\frac{\partial g}{\partial t}\right)_s \approx \frac{V}{2}\int [d\mathbf{k}'] W_{\mathbf{k}\mathbf{k}'}[\mathcal{C}\cdot(\mathbf{k}' - \mathbf{k})]. \tag{3.14}$$

Because $W_{\mathbf{k}\mathbf{k}'}$ depends only on the angle between $\mathbf{k}$ and $\mathbf{k}'$ the component of $\mathbf{k}'$ parallel to $\mathbf{k}$ survives under the integration.

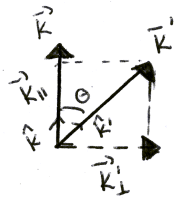


**Figure 3.1**: Vectors **k** and **k'**.

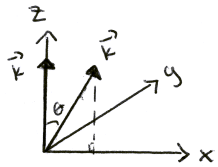


**Figure 3.2**: Choosing **k** along z-axis and **k'** in the direction of x-axis.

Writing the vector $\mathbf{k}'$ as $\mathbf{k}' = \mathbf{k}'_{||} + \mathbf{k}'_{\perp} = (\hat{k}\cdot\mathbf{k}')\hat{k} + \mathbf{k}'_{\perp}$. Choosing $\mathbf{k}$ along z-axis and **k'** in the direction of x-axis gives the easiest way to show that $\int dk' W_{\mathbf{k}\mathbf{k}'}\mathbf{k}'_{\perp} = 0$ and therefore

$$\Rightarrow \left(\frac{\partial g}{\partial t}\right)_s \approx \frac{V}{2}\int [d\mathbf{k}'] W_{\mathbf{k}\mathbf{k}'}[(g(\mathbf{k}) - f(\mathbf{k}))(1 - \hat{k}\cdot\hat{k}')] = -\frac{g(\mathbf{k}) - f(\mathbf{k})}{\tau(\mathbf{k})}, \tag{3.15}$$

where

$$\frac{1}{\tau(\mathbf{k})} = \frac{V}{2}\int [d\mathbf{k}'] W_{\mathbf{k}\mathbf{k}'}[1 - \hat{k}\cdot\hat{k}']. \tag{3.16}$$

In this way relaxation time approximation may be related to an integral over transition probabilities.

Equation (3.5) can be solved by using equation (3.15). In addition assuming that $g(\mathbf{k})$ is not a function of position and considering the case of a steady state, i.e, $\partial g/\partial t = 0$ we have [5]

$$-\frac{e}{\hbar}\mathbf{E}\cdot\nabla_{\mathbf{k}} g(\mathbf{k}) = -\frac{g(\mathbf{k}) - f(\mathbf{k})}{\tau(\mathbf{k})} \tag{3.17}$$

and solving this with respect to $g(\mathbf{k})$ gives[4]

$$g(\mathbf{k}) = f(\mathbf{k}) + \frac{e}{\hbar}\tau(\mathbf{k})[\mathbf{E}\cdot\nabla_{\mathbf{k}}g(\mathbf{k})]. \tag{3.18}$$

The linearised form of the equation (3.18) is obtained by approximating $g(\mathbf{k})$ by $f(\mathbf{k})$ in the right hand side

$$g(\mathbf{k}) \approx f(\mathbf{k}) + \frac{e}{\hbar}\tau(\mathbf{k})\mathbf{E}\cdot\nabla_{\mathbf{k}}f(\mathbf{k}). \tag{3.19}$$

This form of distribution function may be used to calculate the current density and thereby obtain the conductivity. Inserting equation (3.19) into (3.1) gives [4]

$$\mathbf{j} \simeq -\frac{e}{4\pi^3}\int v(\mathbf{k})[f(\mathbf{k}) + \frac{e}{\hbar}\tau(\mathbf{k})\mathbf{E}\cdot\nabla_{\mathbf{k}}f(\mathbf{k})]d\mathbf{k}. \tag{3.20}$$

For an electric field $\mathbf{E} = (E_x, 0, 0)$ applied in the x-direction $\mathbf{j} = (j_x, 0, 0)$ the current density is [4]

$$j_x \simeq -\frac{e}{4\pi^3}\int v_x(\mathbf{k})\left[f(\mathbf{k}) + \frac{e}{\hbar}\tau(\mathbf{k})E_x\frac{\partial f}{\partial k_x}\right]d\mathbf{k}. \tag{3.21}$$

The integral over $v_x(k)f(k)$ vanishes because of inversion symmetry about $\mathbf{k} = 0$ for the Brillouin zone. Moreover since [4]

$$\frac{\partial f}{\partial k_x} = \hbar v_x\frac{\partial f}{\partial \mathcal{E}}, \tag{3.22}$$

the current density is

$$\Rightarrow j_x = -\frac{e^2}{4\pi^3}E_x\int v_x^2(\mathbf{k})\tau(\mathbf{k})\frac{\partial f}{\partial \mathcal{E}}d\mathbf{k}. \tag{3.23}$$

And thus the conductivity $\sigma = j_x/E_x$ is [4]

$$\sigma = -\frac{e^2}{4\pi^3}\int v_x^2(\mathbf{k})\tau(\mathbf{k})\frac{\partial f}{\partial \mathcal{E}}d\mathbf{k}. \tag{3.24}$$

At the low temperature limit the energy derivative of the Fermi-Dirac function can be approximated by a delta function

$$\frac{\partial f}{\partial \mathcal{E}} \approx -\delta(\mathcal{E} - \mathcal{E}_F) \tag{3.25}$$

moreover since

$$d\mathbf{k} = dS_{\mathcal{E}}dk_{\perp} = dS_{\mathcal{E}}\frac{d\mathcal{E}}{|\nabla_{\mathbf{k}}\mathcal{E}|} = dS_{\mathcal{E}}\frac{d\mathcal{E}}{\hbar v(\mathbf{k})}. \tag{3.26}$$

The conductivity is obtained [4]

$$\sigma \simeq \frac{e^2}{4\pi^3\hbar}\int \frac{v_x^2(\mathbf{k})}{v(\mathbf{k})}\tau(\mathbf{k})\delta(\mathcal{E} - \mathcal{E}_F)dS_{\mathcal{E}}d\mathcal{E}, \tag{3.27}$$

$$\sigma \simeq \frac{e^2}{4\pi^3\hbar}\int_{\mathcal{E}=\mathcal{E}_F}\frac{v_x^2(\mathbf{k})}{v(\mathbf{k})}\tau(\mathbf{k})dS_{\mathcal{E}}. \tag{3.28}$$

The electrical conductivity of a metal can thus be expressed as a surface integral over the Fermi surface in **k**-space. The d.c. conductivity of a metal is proportional to the area of the Fermi surface available to conduction electrons.

From the expression (3.28) the Drude conductivity can also derived in the case of free electron gas and by taking average over the Fermi surface. The velocity in the x-direction is $v_x = \frac{1}{3}\langle\mathbf{v}\rangle$ and integrating over the Fermi sphere

$$\sigma = \frac{e^2}{4\pi^3\hbar}\frac{1}{3}v_F\tau(\mathcal{E}_F)4\pi k_F. \tag{3.29}$$

Because the Fermi velocity is $v_F = \hbar k_F/m$ and the density of electrons is $n = N/V = k_F/(3\pi^2)$, there is light in the end of the tunnel and the Drude conductivity is obtained

$$\sigma = \frac{e^2\tau n}{m}. \tag{3.30}$$

# 4 Localization

In real crystals it is not always possible to characterize electron wavefunctions by delocalized wavefunctions such as Bloch functions. Disorder and impurities in the real crystals cause electron wavefunctions to be localized. For example adding one impurity to a site in a perfect crystal gives rise to a single localized state. This kind of localization of electron wavefunction is called strong localization or Anderson localization.

In one and two dimensions the increase of impurities increase the amount of localized states and eventually all the states are localized. In three dimension only if the disordering wells have height of $9t$, where $t$ is the tight-binding hopping parameter, all the states are localized. The boundary between delocalized and localized states in three dimensions is called mobility edge.

The localized wavefunction has a exponentially decaying envelope around a particular site $\mathbf{r}_0$ [4]

$$\Psi = \exp\left[-\alpha_L(|\mathbf{r}-\mathbf{r}_0|)\right]\sum_n c_n \exp\left(i\psi_n\right)\phi(|\mathbf{r}-\mathbf{r}_0|), \tag{4.1}$$

where $\phi$ is an atomic wavefunction, $\alpha_L$ is the localization length and $\psi_n$ is the phase factor.

At the zero temperature the localized electrons do not diffuse and thus they do not give any contribution to the conductivity, which is different to the case when all the states are delocalized and the conductivity is finite as the equation (3.28) indicates. At nonzero temperatures the localized electron can only move from one localized site to another by phonon assisted hopping and thus a sample in which all states are localized is a bad conductor.

At low temperatures when the electron mobility is low, the phase-relaxation time can be much larger than the mean-free-path. This is the regime of quantum diffusion where the electron's wave nature reveals itself. In this regime there is a phenomenon called weak localization, which causes decrease of the conductivity. Weak localization is basically a consequence of interference between different scatterers.

When the resistance of a phase-coherent conductor is greater than $h/2e^2$ it is said to be in the strong-localization regime. Then the conductor's resistance no longer scales with length and shows large fluctuations if the scattering configuration or electron

wavelength is changed. The conductor is said to be weakly localized, if the length of the phase-coherent conductor is much less than the localization length.

The quantum interference in the phase-coherent regime increases the probability for electrons to be backscattered by potential fluctuations. This reduces the electron diffusion coefficient and hence the electrical conductivity. The weak localization correction to Drude conductivity is calculated in the article written by Beenakker & van Houten [6] and in the book written by Elliot [4]. Information about the localization can be also found in the notes written by Galperin [2] and in the seminar written by Kanduc [7].

## 4.1 Scaling theory of localization

Scaling theory of conductance gives a very powerful and easy way to consider the localization because there is no need to examine properties of a single wave function or Hamiltonian. The only thing needed to consider is how the solutions behave when the length scale is changed.

The physical idea behind this kind of scaling theory is to think a set of resistors or a set of conductors. It is assumed that any two of the resistors with a same resistance or conductors are indistinguishable. Because this text is about conductance let's think this via conductance and introduce a dimensionless conductance $g(L)$, which is a function of the scale $L$. The conductance of the whole conductor is assumed to depend only on the conductance of the smaller parts of length $L$ so that [8]

$$g(bL) = f(b, g(L)) \tag{4.2}$$

When $g$ is large the macroscopic transport theory and Ohm's law $G = \sigma A/L$ are valid. Considering only the ratio of the cross sectional are of the conductor to its length the conductance is [8]

$$G(L) = \sigma L^{d-2}, \tag{4.3}$$

where $d$ refers to the dimension. This is the large $g$ limit.

In the small $g$ limit all the states should be localized and so $g$ should fall off exponentially [8]

$$g = g_a E^{-\alpha_L L}, \tag{4.4}$$

where $g_a$ is a dimensionless constant.

The assumption (4.2) can also be expressed via following scaling hypothesis by Abrahams *et al.*[8]

$$\frac{d \ln g(L)}{d \ln L} = \beta(g(L)) \tag{4.5}$$

where $\beta$ is a continuous function. And thus in the large $g$ limit is

$$\lim_{g\to\infty} \beta_d(g) = d - 2 \tag{4.6}$$

And for the small $g$

$$\lim_{g\to 0} \beta_d(g) = \ln(g/g_a) \tag{4.7}$$

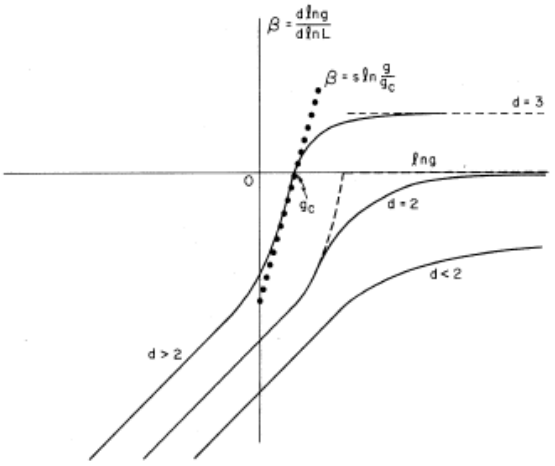



**Figure 4.1**: Plot of $\beta$. [8].

In one dimension the scaling function $\beta$ approaches value -1, which means that the conductance is decreased when the system size is increased.

For two dimensional systems the limit of $\beta$ is 0, consequently when the size of the conductor is large enough the conductance is independent of the size.

In three dimensions for large systems the $\beta$ reaches value 1 and for the small systems it approaches negative infinity. By continuity $\beta$ must go through zero which means at that point the conductance of the system is independent of the system size. For larger values of $g$ the conductance will increase and on the contrary when $g$ is smaller than zero point of $\beta$ the conductance will decrease with increasing sample size.

## 4.2 Weak localization

The strong localization caused the conductivity go to zero in the $T = 0K$ limit. So what is the effect of weak localization to the conductivity. First the probability

$P(\mathbf{r}, \mathbf{r}', t)$ for diffusion between $\mathbf{r}$ and $\mathbf{r}'$ of time t is needed and in Feynman path description this is [7].

$$\begin{aligned} P(\mathbf{r}, \mathbf{r}', t) &= |\langle \mathbf{r}|U(t_2, t_0)|\mathbf{r}'\rangle|^2 = |\sum_j \langle |\mathbf{r}|U(t_2, t_1)|\mathbf{r}''\rangle\langle \mathbf{r}''|U(t_1, t_0)|\mathbf{r}'\rangle|^2 \\ &\equiv \left|\sum_i A_i\right|^2 = \sum_i |A_i|^2 + \sum_{i \neq j} A_i A_k^*. \end{aligned} \tag{4.8}$$

where $U(t_2, t_0) = \exp\left[\frac{i\mathcal{H}}{\hbar}(t_2 - t_0)\right]$ is an unitary operator.

The first term in the summation in the last line of equation (4.8), corresponds the classical diffusion. The second term arises from the quantum interference. In the diffusive transport regime there is a plenty of different trajectories contributing to the sum and if the endpoints are different the interference term averages out.

In the case of backscattered trajectories the contributions from the sum (4.8) can be grouped in time reversed pairs. Time reversal invariance in isotropic space guarantees that the probability amplitudes $A^+$ and $A^-$ for clockwise and counter-clockwise propagation around the closed loop are identical $A^+ = A^- \equiv A$. Thus the coherent backscattering probability $|A^+ + A^-|^2 = 4|A|^2$ is twice the classical result [6].

The enhancement probability for return to the point of departure reduces the diffusion constant and hence the conductivity. This is the essence of weak localization.

Next the weak localization correction $\delta\sigma_{loc}$ to Drude conductivity $\sigma$ is calculated. Its magnitude is proportional to the probability that electron returns to the origin of a trajectory. Since $\delta\sigma_{loc}$ is assumed to be small one can estimate the probability from classical diffusion.

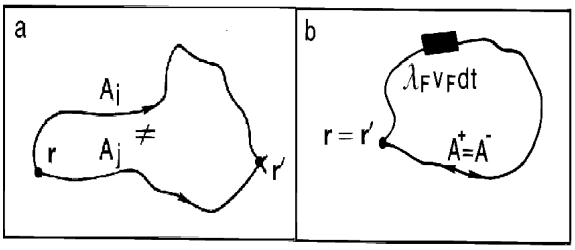



**Figure 4.2**: Mechanism of backscattering. a) Two paths with uncorrelated phases corresponding the probability amplitude. b) Time-reversed back-scattering paths [6].

Let $W(t)$ be the probability that an electron returns within a distance $d\mathbf{r}$ to the original starting point in time $t$. The correction to conductivity is given by [4]

$$\frac{\delta\sigma_{loc}}{\sigma} = -\frac{2\hbar}{m_e}\int_0^\infty W(t)e^{-t/\tau_\phi}dt. \tag{4.9}$$

The time-dependent loss of phase-coherence with characteristic rate $\tau_0$ due to inelastic scattering is accounted in the term $e^{-t/\tau_\phi}$. The ratio $\hbar/m_e \propto \lambda_F v_F$ comes form the flux tube of width $\lambda_F$ and length $v_F dt$ is swept out along a propagation path [4].

The return probability $W(t)$ in a 2D channel of width $L$ is given for times $t \gg \tau$ in the diffusive regime by [6]

$$W(t) = (4\pi Dt)^{-1}, \quad \text{if} \quad t \ll w^2/D \quad (2D), \tag{4.10}$$

$$W(t) = w^{-1}(4\pi Dt)^{-1/2}, \quad \text{if} \quad t \gg w^2/D \quad (1D). \tag{4.11}$$

where $w$ is the width of the channel.

Because the diverging time integral of $W(t)$, some cut-off must be determined. Usually the weak localization is determined by the behavior of the return probability on the phase coherence time $\tau_\phi$, which provides a long time cut-off. [6]

Going down to shorter time scales $t \lesssim \tau$ the system approaches ballistic regime and equations (4.10) and (4.11) do not apply. In those cases the return probability is assumed to go smoothly down to zero. This can be accounted by factor $(1 - e^{t/\tau})$ to exclude those electrons that time $t$ have not been scattered. The short time cut-off becomes irrelevant for $\tau_\phi \gg \tau$. [6]

Summarizing all the previous the expressions for the 2D and 1D weak localization corrections are obtained [6]

$$\begin{aligned}
\delta\sigma_{loc} &= -\frac{2\hbar}{m_e}\sigma \int_0^\infty dt(4\pi Dt)^{-1}(1 - e^{t/\tau})e^{-t/\tau_\phi} \\
&= -\frac{\hbar e\rho(\mathcal{E})}{2m\pi} \int_0^\infty dt t^{-1}(1 - e^{t/\tau})e^{-t/\tau_\phi} \\
&= -g_s g_v \frac{e^2}{4\pi^2\hbar} \ln\left(1 + \frac{\tau_\phi}{\tau}\right), \quad \text{if} \quad l_\phi \ll w,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
\delta\sigma_{loc} &= -\frac{2\hbar}{m_e}\sigma \int_0^\infty dt w^{-1}(4\pi Dt)^{-1-2}(1 - e^{t/\tau})e^{-t/\tau_\phi} \\
&= -\frac{\hbar e\rho(\mathcal{E})}{2m\pi W} \int_0^\infty dt w^{-1}(t)^{-1-2}(1 - e^{t/\tau})e^{-t/\tau_\phi} \\
&= -g_s g_v \frac{e^2}{2\pi^2\hbar}\frac{l_\phi}{w}\left(1 - \left(1 + \frac{\tau_\phi}{\tau}\right)^{-1/2}\right), \qquad \text{if} \quad l_\phi \gg w,
\end{aligned} \tag{4.13}$$

where the expression $\sigma = e^2\rho(\mathcal{E})D$ ,with 2D density of states is used for Drude conductivity. The $g_s$ is the factor that accounts for spin degeneracy and $g_v$ accounts for the band degeneracy. The dimensionality is expressed in terms of phase coherence length $l_\phi = \sqrt{D\tau_\phi}$.

The ratio of weak localization correction to the Drude conductivity $\delta\sigma_{loc}/\sigma$ is of order $1/k_f l$ for 2D case and of order $(l_\phi/W)(1/k_f l)$ 1D case. The corrections to conductance are obtained in a similar fashion

$$\begin{aligned} \delta G_{loc} = (A/L)\delta\sigma_{loc} \propto \frac{e^2}{h}\frac{A}{L}, \quad 2D \\ \delta G_{loc} = (A/L)\delta\sigma_{loc} \propto \frac{e^2}{h}\frac{l_\phi}{L}, \quad 1D. \end{aligned} \tag{4.14}$$

# 5 Ballistic point contacts

Jan M. van Ruitenbeek et. al. are considering ballistic point contacts in reference [1]. First they give some introduction using Sharvin's approach and after that conductance is considered using the Landauer approach.

## 5.1 Sharvin's approach to ballistic point contacts

Sharvin was one of the first ones who considered small quantum contacts. He pointed out that the transport through small contacts resembles the problem of the flow of a dilute gas through a small hole. Semi-classically the current density can be written as

$$\mathbf{j}(\mathbf{r}) = \frac{2e}{L^3} \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}} g_{\mathbf{k}}(\mathbf{r}), \tag{5.1}$$

where $g_{\mathbf{k}}(\mathbf{r})$ is the semiclassical distribution function, which gives the occupation of state $\mathbf{k}$ at position $\mathbf{r}$ and $\mathbf{v}_{\mathbf{k}}$ is the group velocity of the electrons.

In absence of collisions of the right moving-states the occupation is fixed by the electrochemical potential within the left-hand-side electrode and conversely for the left-moving states. Thus for the applied voltage difference $V$ the right-moving states will be occupied to an energy $eV$ higher than the left-moving states. This results in a net current density

$$j = e\langle v_z \rangle \rho(\mathcal{E}_F) eV/2, \tag{5.2}$$

where $\rho(\mathcal{E}_F) = mk_f/\pi^2\hbar^2$ is the density of states at the Fermi level and $\langle v_z \rangle = \hbar k_F/2m$ is the average velocity in the positive z-direction.

The total current is obtained by integration over the contact

$$j = \int \frac{e^2 k_f^2 V}{4\pi^2 \hbar} dA = \frac{e^2 k_f^2 V}{2\pi h} \pi a^2 = \frac{e^2 k_f^2 a^2}{2h} V \tag{5.3}$$

where $a$ is the contact radius. Thus the conductance is given by

$$G_s = \frac{2e^2}{h} \left( \frac{k_f a}{2} \right)^2, \tag{5.4}$$

where $h$ is the Planck constant and $k_f$ is the Fermi wave vector.

## 5.2 Landauer formula

The motion of electrons in a one-dimensional wire, or though a quantum point contact, is characterized rather by wave motion than by a single electron just wandering around. The wave motion is considered to be energy conserving but on the other hand a wire with resistivity greater than zero dissipates energy. Is there a contradiction? Michael Marder examines this question in his book [5].

Landauer approached this problem by thinking, what it really means that there is a voltage drop in the circuit. He ended up to model the problem with two electron reservoirs, which are independently at thermal equilibrium. These reservoirs have different chemical potentials and they are connected with a channel whose conductance is to be measured.

When the electron is transmitted though the channel it must loose energy because the other reservoir is at lower potential and the arriving electron comes to equilibrium with the other electrons. So all dissipation occurs in the reservoirs, not in the channel.

What then makes the current to flow? When the potential $V > 0$ is applied the energy levels in the drain contact are lowered with respect to the source contact. And the applied potential maintains them at distinct electrochemical potentials separated by $qV$ i.e. [5]

$$\mu_L - \mu_R = qV, \tag{5.5}$$

giving rise to two different Fermi functions

$$f_L[\mathcal{E}(k)] = \frac{1}{\exp[\mathcal{E}(\mathbf{k}) - \mu_L]\beta + 1} \tag{5.6}$$

and

$$f_R[\mathcal{E}(k)] = \frac{1}{\exp[\mathcal{E}(\mathbf{k}) - \mu_R]\beta + 1}. \tag{5.7}$$

The net current arises because of imbalance between population of the modes moving from left to right and from right to left. The mode moving from left to right is fixed by the Fermi distribution on the left electrode, $f_L$ and mode form right to left is fixed by $f_R$. So the current is given by [1]

$$I = \frac{e}{L}\sum_{k\sigma} v_k(f_L(\mathcal{E}_k) - f_R(\mathcal{E}_k)) = \frac{e}{\pi}\int v(k)(f_L(\mathcal{E}_k) - f_R(\mathcal{E}_k))dk, \tag{5.8}$$

where $v(k)$ is electron group velocity ($\mathbf{v}_g = \partial\omega(\mathbf{k})/\partial\mathbf{k} = \frac{1}{\hbar}\partial\mathcal{E}(\mathbf{k})/\partial\mathbf{k}$), $L$ is the length of the conductor and $\sigma$ is the electron spin.

There are a few things to be noted out. First, no current will flow if $f_L(\mathcal{E}_k) = f_R(\mathcal{E}_k)$. At levels below both electrochemical potentials $\mu_L$ and $\mu_R$ is $f_L(\mathcal{E}_k) = f_R(\mathcal{E}_k) =$

1 and there is no contribution to the current. In the same way there are also no contribution to the current at levels that are above both potentials $\mu_L$ and $\mu_R$ since $f_L(\mathcal{E}_k) = f_R(\mathcal{E}_k) = 0$. It is only within a few $k_BT$ of potentials $\mu_L$ and $\mu_R$ that will have $f_L(\mathcal{E}_k) \neq f_R(\mathcal{E}_k)$ and current flows [1].

The current for one mode is [1]

$$\begin{aligned} I_n &= \frac{e}{\pi}\int v(k)(f_L(\mathcal{E}_k) - f_R(\mathcal{E}_k))dk = \frac{e}{\pi\hbar}\int \frac{\partial \mathcal{E}_k}{\partial k}(f_L(\mathcal{E}_k) - f_R(\mathcal{E}_k))\frac{dk}{d\mathcal{E}}d\mathcal{E} \\ &= \frac{e}{\pi\hbar}\int_{-\infty}^{\infty}(f_L(\mathcal{E}_k) - f_R(\mathcal{E}_k))d\mathcal{E}\overbrace{=}^{T=0K} = \frac{2e}{h}\int_{\mu}^{\mu+\delta\mu} d\mathcal{E} = \frac{2e}{h}\delta\mu = \frac{2e^2}{h}V. \end{aligned} \tag{5.9}$$

The notation is so that $\mu_R = \mu$ and $\mu_L = \mu + \delta\mu$, where $\delta\mu \ll \mu$.

In the zero temperature limit the conductance for one mode is given by

$$G_0 = \frac{2e^2}{h}. \tag{5.10}$$

The total current is given by the sum over all individual modes $I = \sum_n I_n$. Thus the total conductance is given by [2]

$$G = \frac{2e^2}{h}N. \tag{5.11}$$

If the wire is not ideal and there is some reflection only a part of the current is transmitted. Introducing the transmission probability $T_n$ of the mode $n$ allows to write the total current as [2]

$$I = \frac{2e}{h}\delta\mu\sum_{n=1}^{N}T_n = \frac{2e^2}{h}\left(\sum_{n=1}^{N}T_n\right)V \tag{5.12}$$

Defining the total transmission into $n$th channel as $T_n = \sum_{,} |t_{mn}|^2$, the total conductance is [2]

$$G = \frac{2e^2}{h}\sum_{n=1}^{N}T_n = \frac{2e^2}{h}\sum_{n,m=1}^{N}|t_{mn}|^2 = \frac{2e^2}{h}\mathrm{Tr}(\hat{t}\hat{t}^\dagger). \tag{5.13}$$

This is the Landauer formula or, being more specific two terminal Landauer formula and $\hat{t}$ is the matrix of scattering amplitudes. The Landauer equation then states that the conductance is proportional to the transmission probability or to the square of the coefficients $t_{mn}$, which give the ratio of amplitude for the incoming mode $m$ and the amplitude for the outgoing mode $n$. The scattering matrix relating the incoming and outgoing waves is [1]

$$\hat{S} = \begin{pmatrix} \hat{r} & \hat{t}' \\ \hat{t} & \hat{r}' \end{pmatrix} \tag{5.14}$$

The matrix $\hat{t}$ is not in general a square matrix but $\hat{t}^\dagger\hat{t}$ is a $N_L \times N_L$ matrix, where $L$ refers to the left lead. The current conservation requires that $T_{RL} = T_{LR} = \mathrm{Tr}[(\hat{t'}^\dagger t')]$, $(t_{nm} = (t_{mn})^*)$. [1]

Because $\hat{t}^\dagger\hat{t}$ is hermitian all its eigenvalues are real and the unitarity of scattering matrix ensures $\hat{t}^\dagger\hat{t} + \hat{r}^\dagger\hat{r} = \hat{I}$. From this follows that both $\hat{t}^\dagger\hat{t}$ and $\hat{r}^\dagger\hat{r}$ should become diagonal under same unitary transformation $\hat{U}$. [1]

The eigenchannels, which are the eigenvectors of matrices $\hat{t}^\dagger\hat{t}$ and $\hat{r}^\dagger\hat{r}$, correspond to invariant linear combination of the modes which are left invariant under the reflections of the sample. In the eigenchannel basis the conductance is [1]

$$G = \frac{2e^2}{h}\sum_n \tau_n, \tag{5.15}$$

where $\tau_n$ are the eigenvalues of the transmission matrix.

## 5.3 A more general expression

The generalization of Landauer formula to the nonzero temperatures and in the multi-terminal cases can also be derived. This more general expression for the Landauer- or Landauer-Büttiker formula given in the papers written by Büttiker [9] and Büttiker & Blanter [10].

Now it is assumed that the conductor is connected to electron baths via number of contacts $\alpha = 1, 2, 3, ..$ and each contact is characterized by an equilibrium Fermi function and chemical potential $\mu_\alpha$.

In task of calculating the current flowing from left reservoir to the right reservoir the first step is to introduce current density operator [10]

$$\hat{\mathbf{j}}(\mathbf{r}, t) = \frac{\hbar}{2mi}[\hat{\Psi}^\dagger\nabla\hat{\Psi} - (\nabla\hat{\Psi}^\dagger)\hat{\Psi}]. \tag{5.16}$$

It can be assumed that far from the sample the longitudinal and transverse motion of electrons is separable. In the longitudinal direction the system is open and characterized by continuous wave vector $k_l$. The energy in the longitudinal direction is $E_l = \hbar k_l/2m$. Because the motion in the transverse directions is quantized it can be described by a discrete index $n$. These states are referred to as transverse quantum channels. The transverse energy $E_{\alpha,n}$.depends also the probe index $\alpha$.

The total current entering the conductor at a contact is the total current in the probe

$\alpha$ is [10]

$$\begin{aligned}\hat{I}_\alpha(z,t) &= \int d(x,y) j_{z_\alpha}(\mathbf{r},t) \\ &= \frac{\hbar e}{2mi}\int d(x,y)\left[\hat{\Psi}^\dagger(\mathbf{r},t)\frac{\partial}{\partial z}\hat{\Psi}(\mathbf{r},t) - \frac{\partial}{\partial z}\left(\hat{\Psi}^\dagger(\mathbf{r},t)\right)\hat{\Psi}(\mathbf{r},t)\right],\end{aligned} \tag{5.17}$$

where $(x,y)$ are the transverse coordinates and $d(x,y)$ denotes the integral over the cross section of reservoir $\alpha$. The coordinate along the leads is denoted by $z$.

This problem can be solved most easily by introducing the creation $\hat{a}^\dagger(\mathcal{E})$ ,$\hat{b}^\dagger(\mathcal{E})$ and the annihilation $\hat{b}(\mathcal{E})$ , $\hat{a}(\mathcal{E})$ operators, which create and annihilate particles in the incoming channels and out coming channels with energy $\mathcal{E}$. These operators obey the familiar commutation relations and anticommutation relations for fermions and bosons respectively [10]

$$[\hat{a}^\dagger_{\alpha m}(\mathcal{E}), \hat{a}_{\beta n}(\mathcal{E}')]_\pm = \delta_{mn}\delta_{\alpha\beta}\delta(\mathcal{E}-\mathcal{E}'), \tag{5.18}$$

$$[\hat{a}_{\alpha m}(\mathcal{E}), \hat{a}_{\beta n}(\mathcal{E}')]_\pm = 0, \tag{5.19}$$

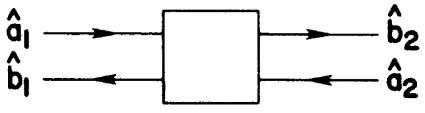
$$[\hat{a}^\dagger_{\alpha m}(\mathcal{E}), \hat{a}^\dagger_{\beta n}(\mathcal{E}')]_\pm = 0. \tag{5.20}$$

These operators are related via scattering matrix $\hat{S}$. [10]

$$\hat{b}_{\alpha m} = \sum_{\beta n}(\hat{s}_{\alpha\beta})_{mn}\hat{a}_{\beta n}. \tag{5.21}$$

The $(\hat{s}_{\alpha\beta})_{mn}$ is a ratio between the outgoing amplitude on mode $n$ in lead $\beta$ and the incoming amplitude on a mode $m$ in lead $\alpha$.

The matrix $\hat{S}$ has dimensions $(N_\alpha + N_\beta) \times (N_\alpha + N_\beta)$. Its size and matrix elements are depending the total energy $\mathcal{E}$.




$$\begin{pmatrix}\hat{b}_1\\ \hat{b}_2\end{pmatrix} = \begin{pmatrix}\hat{s}_{11} & \hat{s}_{12}\\ \hat{s}_{21} & \hat{s}_{22}\end{pmatrix}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix}$$

**Figure 5.1**: Second quantization representation for beam partitioning in the two terminal case [9].

The field operator in the each probe is defined as [10]

$$\hat{\Psi}(\mathbf{r},t) = \sum_m \int d\mathcal{E} e^{-i\mathcal{E}t/\hbar}\frac{\chi_{\alpha m}(x,y)}{[2\pi\hbar v_{\alpha m}(\mathcal{E})]^{1/2}}[\hat{a}_{\alpha m}e^{ik_{\alpha m}z} + \hat{b}_{\alpha m}e^{-ik_{\alpha m}z}], \tag{5.22}$$

the $\chi_{\alpha m}(x,y)$ is are the transverse wave functions in the probe $\alpha$ and channel $m$. The wave vector $k_{\alpha m}$ is $k_{\alpha m} = \hbar^{-1}[2m(\mathcal{E}-\mathcal{E}_{\alpha m})]^{1/2}$. Next substituting equation (5.22) into (5.17) and rearranging gives [10]

$$\begin{aligned}\hat{I}_\alpha(z,t) = \frac{e}{4\pi i m}\sum_{mn}\int d(x,y)d\mathcal{E}d\mathcal{E}' e^{i(\mathcal{E}'-\mathcal{E})t/\hbar}\frac{\chi^\dagger_{\beta n}\chi_{\alpha m}}{\sqrt{v_{\alpha m}(\mathcal{E})v_{\beta n}(\mathcal{E}')}}\\ \{i[k_{\beta n}(\mathcal{E}')+k_{\alpha m}(\mathcal{E})]\hat{a}^\dagger_{\beta n}(\mathcal{E}')\hat{a}_{\alpha m}(\mathcal{E})e^{i(k_{\alpha m}(\mathcal{E})-k_{\beta n}(\mathcal{E}'))z}\\ +i[k_{\beta n}(\mathcal{E}')-k_{\alpha m}(\mathcal{E})]\hat{a}^\dagger_{\beta n}(\mathcal{E}')\hat{b}_{\alpha m}(\mathcal{E})e^{-i(k_{\beta n}(\mathcal{E}')+k_{\alpha m}(\mathcal{E}))z}\\ +i[k_{\alpha m}(\mathcal{E})-k_{\beta n}(\mathcal{E}')]\hat{b}^\dagger_{\beta n}(\mathcal{E}')\hat{a}_{\alpha m}(\mathcal{E})e^{i(k_{\beta n}(\mathcal{E}')+k_{\alpha m}(\mathcal{E}))z}\\ -i[k_{\alpha m}(\mathcal{E})+k_{\beta n}(\mathcal{E}')]\hat{b}^\dagger_{\beta n}(\mathcal{E}')\hat{b}_{\alpha m}(\mathcal{E})e^{i(k_{\beta n}(\mathcal{E}')-k_{\alpha m}(\mathcal{E}))z}\}.\end{aligned} \tag{5.23}$$

Introducing the velocity of carriers $v_n(\mathcal{E}_{\alpha n}) = \hbar k_{\alpha n}/m$ in the $n$th transverse channel and assuming that the transverse wave functions are normalized to unity $\int \chi^\dagger_{\beta n}\chi_{\alpha m}d\mathbf{r} = \delta_{\alpha\beta}\delta_{mn}$ the previous equation can be simplified [10].

$$\begin{aligned}\hat{I}_\alpha(z,t) = \frac{e}{4\pi\hbar}\sum_{mn}\int \frac{d\mathcal{E}d\mathcal{E}' e^{i(\mathcal{E}'-\mathcal{E})t/\hbar}}{\sqrt{v_{\alpha m}(\mathcal{E})v_{\alpha m}(\mathcal{E}')}}\\ \{[v_{\alpha m}(\mathcal{E}')+v_{\alpha m}(\mathcal{E})]\hat{a}^\dagger_{\alpha m}(\mathcal{E}')\hat{a}_{\alpha m}(\mathcal{E})e^{i(v_{\alpha m}(\mathcal{E})-v_{\alpha m}(\mathcal{E}'))mz/\hbar}\\ +[v_{\alpha m}(\mathcal{E}')-v_{\alpha m}(\mathcal{E})]\hat{a}^\dagger_{\alpha m}(\mathcal{E}')\hat{b}_{\alpha m}(\mathcal{E})e^{-i(v_{\alpha m}(\mathcal{E}')+v_{\alpha m}(\mathcal{E}))mz/\hbar}\\ +[v_{\alpha m}(\mathcal{E})-v_{\alpha m}(\mathcal{E}')]\hat{b}^\dagger_{\alpha m}(\mathcal{E}')\hat{a}_{\alpha m}(\mathcal{E})e^{i(v_{\alpha m}(\mathcal{E}')+v_{\alpha m}(\mathcal{E}))mz/\hbar}\\ -[v_{\alpha m}(\mathcal{E})+v_{\alpha m}(\mathcal{E}')]\hat{b}^\dagger_{\alpha m}(\mathcal{E}')\hat{b}_{\alpha m}(\mathcal{E})e^{i(v_{\alpha m}(\mathcal{E}')-v_{\alpha m}(\mathcal{E}))mz/\hbar}\}.\end{aligned} \tag{5.24}$$

This equation can be simplified even further by noting that for all observable quantities, e.g., average current, noise or higher moments of the current distribution the energies $\mathcal{E}$ and $\mathcal{E}'$ are close to the each other. The velocities $v_m(\mathcal{E}_{\alpha m})$ vary with energy quite slowly typically on the scale of Fermi energy. Neglecting the energy dependence of the quantities mentioned earlier Eq. (5.24) can be reduced to much simpler form [10]

$$\hat{I}_\alpha(t) \approx \frac{e}{2\pi\hbar}\sum_m \int d\mathcal{E}d\mathcal{E}' e^{i(\mathcal{E}'-\mathcal{E})t/\hbar}[\hat{a}^\dagger_{\alpha m}(\mathcal{E}')\hat{a}_{\alpha m}(\mathcal{E}) - \hat{b}^\dagger_{\alpha m}(\mathcal{E}')\hat{b}_{\alpha m}(\mathcal{E})] \tag{5.25}$$

It is worth noting that $n^+_{\alpha m}(\mathcal{E}) = \hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\alpha m}(\mathcal{E})$ and $n^-_{\alpha m}(\mathcal{E}) = \hat{b}^\dagger_{\alpha m}(\mathcal{E})\hat{b}_{\alpha m}(\mathcal{E})$ are the operators for the occupation number of the incident carriers in probe $\alpha$ in channel $m$ and out-going carriers in probe $\alpha$ in channel $m$ respectively. So setting $\mathcal{E}' = \mathcal{E} - \hbar\omega$ and integrating over $\omega$ gives [10]

$$\hat{I}_\alpha(t) \approx \frac{e}{2\pi\hbar}\sum_m \int d\mathcal{E}[n^+_{\alpha m}(\mathcal{E}) - n^-_{\alpha m}(\mathcal{E})]. \tag{5.26}$$

Thus the current is a consequence of the imbalance between the population of incoming and outgoing states. Using Eq. (5.21), Eq. (5.25) can be expressed using only $\hat{a}$ operators alone [10]

$$\begin{aligned}\hat{I}_\alpha(t) = \frac{e}{2\pi\hbar}\sum_m \int d\mathcal{E}d\mathcal{E}' e^{i(\mathcal{E}'-\mathcal{E})t/\hbar}[\hat{a}^\dagger_{\alpha m}(\mathcal{E}')\hat{a}_{\alpha m}(\mathcal{E}) - \\ \sum_{\beta n}\sum_{\gamma k}[\hat{a}^\dagger_{\beta n}(\mathcal{E}')\hat{s}^\dagger_{\alpha\beta:nm}(\mathcal{E}')\hat{s}_{\alpha\gamma:mk}(\mathcal{E})\hat{a}_{\gamma k}(\mathcal{E})].\end{aligned} \tag{5.27}$$

Introducing the matrix

$$A^{nk}_{\beta\gamma}(\alpha,\mathcal{E}',\mathcal{E}) = \delta_{nm}\delta_{nk}\delta_{\beta\alpha}\delta_{\gamma\alpha} - \sum_m \hat{s}^\dagger_{\alpha\beta;km}(\mathcal{E}')\hat{s}_{\alpha\gamma:mk}(\mathcal{E}) \tag{5.28}$$

gives a nicer looking result [10]

$$\hat{I}_\alpha(t) = \frac{e}{2\pi\hbar}\sum_{\beta\gamma}\sum_{nk}\int d\mathcal{E}d\mathcal{E}' e^{i(\mathcal{E}-\mathcal{E}')t/\hbar}\hat{a}^\dagger_{\beta n}(\mathcal{E}')A^{nk}_{\beta\gamma}(\alpha,\mathcal{E},\mathcal{E}')\hat{a}_{\gamma k}(\mathcal{E}). \tag{5.29}$$

The quantum statistical average of the product of electron creation and annihilation operators for a Fermi gas in thermal equilibrium is [10]

$$\langle\hat{a}^\dagger_{\beta n}(\mathcal{E}')\hat{a}_{\gamma k}(\mathcal{E})\rangle = \delta_{\beta\gamma}\delta_{nk}\delta(\mathcal{E}'-\mathcal{E})f_\beta(\mathcal{E}). \tag{5.30}$$

Taking the average of (5.29) and using (5.30) the following is obtained [10]

$$\langle\hat{I}_\alpha\rangle = \frac{e}{h}\sum_{\beta n}\int d\mathcal{E}A^{nn}_{\beta\beta}(\alpha)f_\beta(\mathcal{E}) = \frac{e}{h}\sum_\beta\int d\mathcal{E}\mathrm{Tr}[\mathbf{A}_{\beta\beta}(\alpha)]f_\beta(\mathcal{E}). \tag{5.31}$$

For $\alpha = \beta$ is

$$\mathrm{Tr}[\mathbf{A}_{\alpha\alpha}(\alpha)] = \mathrm{Tr}(\mathbf{1}_\alpha - \hat{\mathbf{s}}^\dagger_{\alpha\alpha}\hat{\mathbf{s}}_{\alpha\alpha}) \equiv M_\alpha - R_{\alpha\alpha} = \mathrm{Tr}[1_\alpha] - \mathrm{Tr}[\hat{\mathbf{r}}^\dagger_{\alpha\alpha}\hat{\mathbf{r}}_{\alpha\alpha}], \tag{5.32}$$

where $M_\alpha = \mathrm{Tr}[\mathbf{1}_\alpha]$ is the total number of modes or quantum channels in reservoir $\alpha$ and $R_{\alpha\alpha} = \mathrm{Tr}[\hat{\mathbf{r}}^\dagger_{\alpha\alpha}\hat{\mathbf{r}}_{\alpha\alpha}]$ is the total probability for reflection for carriers in $\alpha$. For $\alpha \neq \beta$ one obtains [10]

$$\mathrm{Tr}[\mathbf{A}_{\beta\beta}(\alpha)] = -\mathrm{Tr}(\hat{\mathbf{s}}^\dagger_{\alpha\beta}\hat{\mathbf{s}}_{\alpha\beta}) \equiv -T_{\alpha\beta} = -\mathrm{Tr}[\hat{\mathbf{t}}^\dagger_{\alpha\beta}\hat{\mathbf{t}}_{\alpha\beta}], \tag{5.33}$$

where $T_{\alpha\beta} = \mathrm{Tr}[\hat{\mathbf{t}}^\dagger_{\alpha\beta}\hat{\mathbf{t}}_{\alpha\beta}]$ is the total probability for transmission. Thus the average incident current at lead $\alpha$ is obtained in terms of reflection and transmission currents

$$\langle\hat{I}_\alpha\rangle = \frac{e}{h}\int d\mathcal{E}\left[(M_\alpha - R_{\alpha\alpha})f_\alpha - \sum_\beta T_{\alpha\beta}f_\beta\right]. \tag{5.34}$$

If the chemical potentials are differing only by a small amount at the contacts the distribution functions can be expanded away from the equilibrium chemical potential $\mu$, $f_\alpha = -(\partial f/\partial \mathcal{E})(\mu_\alpha - \mu)$ [10]

$$\langle \hat{I}_\alpha \rangle = \frac{e}{h} \int d\mathcal{E} \left( -\frac{\partial f}{\partial \mathcal{E}} \right) \left[ (M_\alpha - R_{\alpha\alpha})\mu_\alpha - \sum_\beta T_{\alpha\beta}\mu_\beta \right]. \tag{5.35}$$

Using the fact that $M_\alpha = R_{\alpha\alpha} + \sum_\beta T_{\alpha\beta}$ then from the equation (5.35) one can get [10]

$$\langle \hat{I}_\alpha \rangle = \frac{e}{h} \int dE \left( -\frac{\partial f}{\partial E} \right) \sum_\beta T_{\alpha\beta}(\mu_\alpha - \mu_\beta) \tag{5.36}$$

In the linear regime [10]

$$\langle \hat{I}_\alpha \rangle = \sum_\beta G_{\alpha\beta} V_\beta. \tag{5.37}$$

Accounting also spin the conductance is then given by [10]

$$G_{\alpha\beta} = \frac{2e^2}{h} \int d\mathcal{E} \sum_\beta T_{\alpha\beta} \left( -\frac{\partial f}{\partial \mathcal{E}} \right), \tag{5.38}$$

which reduces to the equation (5.12) when the temperature is zero.

## 5.4 Conductivity and Green's functions

Transport problems are more traditionally approached by using linear response theory. In this case the conductivity is given by Kubo formula. The connection between Kubo formula and Landauer formula can be found with help of Green's functions. This concept have been considered by Agraït *et al.* in their article [1] and also in the book written by Economou [11].

When a finite electric field $\vec{E}$, oscillating with frequency $\omega$, is applied to the sample, the relation between the current density and the field is [1]

$$\vec{J}(\vec{r}) = \int d^3r' \sigma(\vec{r}, \vec{r'}, \omega) \vec{E}(\vec{r'}), \tag{5.39}$$

where $\sigma(\vec{r}, \vec{r'}, \omega)$ is the non-local conductivity tensor. In zero frequency limit the electric field is uniform in the mesoscopic sample and vanishes the leads.

First thing needed, when looking for expression for $\sigma(\omega)$ is the time-averaged power consumed by the system, i.e., the energy absorption rate due to electronic transitions

induced by the applied electric fields. The average over one cycle according to Agraït *et al.* [1] is

$$P = \langle \mathbf{E} \cdot \mathbf{J} \rangle = \mathrm{Re}\sigma(\omega) E^2, \tag{5.40}$$

where $\mathrm{Re}(\sigma)$ determines the dissipative part of the current and $\mathrm{Re}(\sigma) \to \sigma(0)$ for $\omega \to 0$.

The average power can also be computed by multiplying the energy absorbed by system during the transition $|\alpha\rangle \to |\beta\rangle$ induced by the field [11]

$$\mathcal{E}_{\alpha\beta} \equiv \hbar\omega_{\alpha\beta} = \mathcal{E}_\beta - \mathcal{E}_\alpha \tag{5.41}$$

by the transition rate $p_{\alpha\beta}$ and summing over all possibilities ($|\alpha\rangle \neq |\beta\rangle$) [11]

$$P = \sum_{\alpha\beta} \mathcal{E}_{\beta\alpha} p_{\alpha\beta} = \frac{1}{2} \sum_{\alpha\beta} \mathcal{E}_{\beta\alpha} (p_{\alpha\beta} - p_{\beta\alpha}). \tag{5.42}$$

The probability for the transition per unit time $p_{\alpha\beta}$ is [11]

$$p_{\alpha\beta} = f_\alpha (1 - f_\beta) W_{\alpha\beta}, \tag{5.43}$$

where $f_{\alpha,\beta}$, are the Fermi distributions and $W_{\alpha\beta}$ is the transition rate between states $|\alpha\rangle$ and $|\beta\rangle$ and obtained from Fermi's Golden rule [11]

$$W_{\alpha\beta} = \frac{2\pi}{\hbar} |\langle \alpha | \mathcal{H}_1 | \beta \rangle|^2 \delta(\mathcal{E}_\alpha - \mathcal{E}_\beta). \tag{5.44}$$

The perturbation $\mathcal{H}_1$ is simply the potential induced by electric field [11]

$$\mathcal{H}_1(t) = -exE(t) \tag{5.45}$$

and therefore [11]

$$W_{\alpha\beta} = \frac{2\pi e^2}{\hbar} |E(t)|^2 |\langle \alpha | \hat{x} | \beta \rangle|^2 \delta(\mathcal{E}_\alpha - \mathcal{E}_\beta). \tag{5.46}$$

Now substituting Eq. (5.46) and Eq. (5.43) into (5.42) and after that substituting the whole thing obtained into Eq. (5.40) one obtains $\mathrm{Re}\sigma(\omega)$ [11]

$$\mathrm{Re}\sigma(\omega) = \frac{\pi e^2}{\hbar} \sum_{\alpha\beta} |\langle \alpha | \hat{x} | \beta \rangle|^2 (f_\alpha - f_\beta)(\mathcal{E}_\alpha - \mathcal{E}_\beta) \delta(\mathcal{E}_\beta - \mathcal{E}_\alpha). \tag{5.47}$$

This expression can be recasted in terms of momentum matrix elements, since in the Heisenberg picture the time evolution of operator is given by [11]

$$i\hbar \frac{\partial \hat{A}}{\partial t} = [\hat{A}, \mathcal{H}]. \tag{5.48}$$

Noting that [11]

$$\hat{p}_x = m\frac{d\hat{x}}{dt} = im\frac{[\mathcal{H}, \hat{x}]}{\hbar} \tag{5.49}$$

he momentum matrix elements are

$$\langle\alpha|\hat{p}_x|\beta\rangle = \frac{im}{\hbar}\langle\alpha|\hat{x}|\beta\rangle(\mathcal{E}_\alpha - \mathcal{E}_\beta). \tag{5.50}$$

In the limit $\omega \to 0$ this gives

$$\sigma(0) = \frac{\pi e^2\hbar}{m^2}\int d\mathcal{E}\sum_{\alpha\beta}|\langle\alpha|\hat{p}_x|\beta\rangle|^2\left(-\frac{\partial f}{\partial\mathcal{E}}\right)\delta(\mathcal{E}-\mathcal{E}_\alpha)\delta(\mathcal{E}-\mathcal{E}_\beta). \tag{5.51}$$

This equation is called Kubo formula and it relates the conductivity to the equilibrium current fluctuations for a homogeneous conductor. The local conductivity is obtained by replacing $\langle\alpha|\hat{p}_x|\beta\rangle$ with $m\langle\alpha|\hat{j}(\mathbf{r})|\beta\rangle/e$, where $\hat{j}(\mathbf{r})$ is the current operator at position $\mathbf{r}$ [11].

$$\sigma(\mathbf{r},\mathbf{r}',0) = \pi\hbar\int d\mathcal{E}\sum_{\alpha\beta}\langle\alpha|\hat{j}(\mathbf{r})|\beta\rangle\langle\beta|\hat{j}(\mathbf{r}')|\alpha\rangle\left(-\frac{\partial f}{\partial\mathcal{E}}\right)\delta(\mathcal{E}-\mathcal{E}_\alpha)\delta(\mathcal{E}-\mathcal{E}_\beta). \tag{5.52}$$

By integration the current density along an arbitrary transversal cross-section $S$ oriented perpendicularly to the current direction $x$ the total current trough the sample s obtained [1]

$$I = \int_S dsJ_x(\mathbf{r}) = \int_S\int_V d^3r'\sigma(\mathbf{r},\mathbf{r}',0)E_x(\mathbf{r}'). \tag{5.53}$$

The integration over the volume $V$ can be divided into integration over cross-section $S'$ and along x-axis [1]

$$I = \int_S ds\int dx'\int_{S'} ds'\sigma(\mathbf{r},\mathbf{r}',0)E_x(\mathbf{r}'). \tag{5.54}$$

The integration over the transversal cross-section does not depend on its position at x-axis because of current conservation and therefore [1]

$$\int_S ds\langle\alpha|\hat{j}(\mathbf{r})|\beta\rangle = \int_{S'} ds\langle\alpha|\hat{j}(\mathbf{r}')|\beta\rangle. \tag{5.55}$$

This allows change the order of integration between $x'$ and the transversal cross-section $S'$. The total current is then [1]

$$I = \int_S ds\sigma(\mathbf{r},\mathbf{r}',0)\int_{S'} ds'E_x(\mathbf{r}'). \tag{5.56}$$

Integration of electric field along the x-axis gives the voltage drop along the sample $V$. This formula has the form $I = GV$ where the conductance $G$ is given by [1]

$$G = \int_S ds\sigma(\mathbf{r},\mathbf{r}',0). \tag{5.57}$$

Substituting to this the equation (5.52)

$$G = \pi\hbar \int dS \int d\mathcal{E} \sum_{\alpha\beta} \langle\alpha|\hat{j}(\mathbf{r})|\beta\rangle\langle\beta|\hat{j}(\mathbf{r}')|\alpha\rangle \left(-\frac{\partial f}{\partial \mathcal{E}}\right) \delta(\mathcal{E}-\mathcal{E}_\alpha)\delta(\mathcal{E}-\mathcal{E}_\beta) \tag{5.58}$$

and using the definition of current operator [1]

$$\langle\alpha|\hat{j}(\mathbf{r})|\beta\rangle = \frac{-i\hbar}{2}\left[\psi^*_\alpha(\mathbf{r})\nabla\psi_\beta(\mathbf{r}) - \psi_\beta(\mathbf{r})\nabla\psi^*_\alpha(\mathbf{r})\right] \tag{5.59}$$

gives [1]

$$\begin{aligned} G = \frac{\pi\hbar^3}{4}\int_S ds \int_S' ds' \int d\mathcal{E} \sum_{\alpha\beta} \frac{\partial}{\partial x}\left(\psi_\beta(\mathbf{r})\psi^*_\beta(\mathbf{r}')\right)\frac{\partial}{\partial x'}\left(\psi^*_\alpha(\mathbf{r})\psi_\alpha(\mathbf{r}')\right) \\ -\psi^*_\alpha(\mathbf{r})\psi_\alpha(\mathbf{r}')\frac{\partial}{\partial x}\frac{\partial}{\partial x'}\left(\psi_\beta(\mathbf{r})\psi^*_\beta(\mathbf{r}')\right) - \frac{\partial}{\partial x}\frac{\partial}{\partial x'}\psi_\beta(\mathbf{r})\psi^*_\beta(\mathbf{r}')\psi^*_\alpha(\mathbf{r})\psi_\alpha(\mathbf{r}') \\ +\frac{\partial}{\partial x}\left(\psi^*_\alpha(\mathbf{r})\psi_\alpha(\mathbf{r}')\right)\frac{\partial}{\partial x'}\left(\psi_\beta(\mathbf{r})\psi^*_\beta(\mathbf{r}')\right)\left(-\frac{\partial f}{\partial \mathcal{E}}\right)\delta(\mathcal{E}-\mathcal{E}_\alpha)\delta(\mathcal{E}-\mathcal{E}_\beta). \end{aligned} \tag{5.60}$$

The retarded and advanced Green functions $G^{r,a}(\mathbf{r},\mathbf{r}',E)$ are defined using eigenstates expansion as [1]

$$G^{r,a}(\mathbf{r},\mathbf{r}',\mathcal{E}) = \lim_{\eta\to 0}\sum_\alpha \frac{\psi^*_\alpha(\mathbf{r})\psi_\alpha(\mathbf{r}')}{\mathcal{E}-\mathcal{E}_\alpha \pm i\eta} \tag{5.61}$$

and the discontinuity can be expressed as [1]

$$G^a(\mathbf{r},\mathbf{r}',E) - G^r(\mathbf{r},\mathbf{r}',E) = 2\pi i\sum_\alpha \psi^*_\alpha(\mathbf{r})\psi_\alpha(\mathbf{r}')\delta(\mathcal{E}-\mathcal{E}_\alpha). \tag{5.62}$$

The conductance can be written in the form [1]

$$\begin{aligned} G = -\frac{e^2\hbar^3}{8\pi m^2}\int d\mathcal{E}\left(-\frac{\partial f}{\partial\mathcal{E}}\right)\int_S ds\int_{S'} ds'\frac{\partial}{\partial x}(\tilde{G}^a-\tilde{G}^r)\frac{\partial}{\partial x'}(G^a-G^r) \\ -(G^a-G^r)\frac{\partial}{\partial x}\frac{\partial}{\partial x'}(\tilde{G}^a-\tilde{G}^r) - \frac{\partial}{\partial x}\frac{\partial}{\partial x'}(G^a-G^r)(\tilde{G}^a-\tilde{G}^r) \\ +\frac{\partial}{\partial x}(G^a-G^r)\frac{\partial}{\partial x'}(\tilde{G}^a-\tilde{G}^r), \end{aligned} \tag{5.63}$$

where $G^a - G^r = G^a(\mathbf{r},\mathbf{r}',\mathcal{E}) - G^r(\mathbf{r},\mathbf{r}',\mathcal{E})$ and$\tilde{G}^a - \tilde{G}^r = \tilde{G}^a(\mathbf{r}',\mathbf{r},\mathcal{E}) - \tilde{G}^r(\mathbf{r}',\mathbf{r},\mathcal{E})$, it is also good to remember that $G^a(\mathbf{r},\mathbf{r}',\mathcal{E}) = [G^r(\mathbf{r}',\mathbf{r},\mathcal{E})]^*$. Taking the surfaces $S$ and $S'$ well inside the left and right lead then $\mathbf{r}$ and $\mathbf{r}'$ corresponds to points well inside the left and right leads the eigenstates are simple combinations of plane waves.

Each plane wave corresponds to a lead mode $n$ with eave number $k_n$ in the x-direction. Green function can be expressed in terms of modes wave functions $\chi_n$ [1]

$$G^{r,a}(\mathbf{r},\mathbf{r}',e) = \sum_{mn}\chi_m(\vec{\rho})\chi^*_n(\vec{\rho}')G^{r,a}_{mn}(x,x'), \tag{5.64}$$

the indexes $n$ and $m$ refer to modes on the left and right leads respectively and $\vec{\rho}$ and $\vec{\rho}'$ indicate the position on the transversal surfaces. For $x \to -\infty$ and $x' \to \infty$ the green functions behave as [1]

$$G^{r,a}_{mn}(x, x') \sim e^{\mp i k_n x} e^{\pm i k_m x'} \tag{5.65}$$

in this regime the derivatives can be computed easily if one does not get messed with indicies and finally the conductance is

$$G = \frac{e^2\hbar^3}{\pi m^2} \int d\mathcal{E} \left( -\frac{\partial f}{\partial \mathcal{E}} \right) \sum_{mn} k_m k_n |G^a_{mn}(x, x', \mathcal{E})|^2. \tag{5.66}$$

By comparing with equation (5.12) Landauer formula the transmission coefficient in term of Green functions is [1]

$$T_{12}(\mathcal{E}) = \hbar^2 \sum_{mn} v_n v_m |G^a_{mn}(x, x', \mathcal{E})|^2 \tag{5.67}$$

where $v_n = \hbar k_n / m$ is the velocity on the channel $n$.

# 6 Fluctuations

## 6.1 Universal conductance fluctuations

Normally, when $L \gg l$ and temperature $T$ is around classical regime the fluctuations in the electrical conductance $G$ are negligible. If the wire length $L$ divided into $L/l$ independently fluctuation segments connected in the series the standard deviation of the mean conductance is according to Elliot [4]

$$\delta G \cong (L/l)^{-1/2}\langle G\rangle, \tag{6.1}$$

where $\langle G\rangle$ is the average conductance. Therefore $\delta G$ can be considered negligible. In the phase-coherent regime on the other hand the fluctuation effects cannot simply be overruled. Let us assume that $l_\phi \sim L$ and that the transport is diffusive. In this case according Elliot, who is based his assumption to Al'tshuler-Lee-Stone theorem the magnitude of conductance fluctuations should have universal value,which is independent of the size of the sample [4]

$$\delta G \approx e^2/h. \tag{6.2}$$

Let's next take a look how one could end up to this kind of result. The conductance can be written using the classical Drude conductivity for 2D electron gas as [2]

$$G = \frac{w}{L}\frac{e^2}{h}\frac{k_F l}{2} = \frac{e^2}{h}\frac{\pi l}{2L}N, \tag{6.3}$$

where $N$ is the number of transverse wave guide modes (or 1D sub-bands) that are occupied at energy $\mathcal{E}_F = m_e v_F^2/2 = \hbar^2 k_F^2/2m_e$ in a wire width $w$, given by $N = k_F w/\pi$. The relaxation length is as usual $l = \tau v_F$.

When the current passes from source to a drain reservoir, the current is assumed to go trough the disordered region were the scattering can take place. It is also assumed that the reservoirs are in thermal equilibrium and that there is no phase-coherence between $N$ modes incident to disordered region. The different modes in this context can be viewed as quantum channels and when $L \gg l$ the transmission probability is from equation (6.3) $\pi l/2L$.

In order to get an expression for the fluctuations a multi channel Landauer formula $G = (e^2/\hbar)\sum_{\alpha,\beta=1}^{N} |t_{\alpha,\beta}|^2$ is the first building block. Making comparison with the

equation (6.3) gives the ensemble averaged transmission coefficient [2]

$$\langle |t_{\alpha\beta}|^2 \rangle = \pi l/2NL. \tag{6.4}$$

Current conservation means that the probabilities of transmission $|t_{\alpha\beta}|^2$ and reflection $|r_{\alpha\beta}|^2$ are related via [2]

$$\sum_{\alpha\beta=1}^{N} |t_{\alpha\beta}|^2 + \sum_{\alpha\beta=1}^{N} |r_{\alpha\beta}|^2 = N \tag{6.5}$$

and hence the average reflection probability is given by [2]

$$\langle |r_{\alpha\beta}|^2 \rangle = \frac{1}{N} - \langle |t_{\alpha\beta}|^2 \rangle = \frac{1}{N}\left(1 - \frac{\pi l}{2L}\right). \tag{6.6}$$

The reflection probabilities $|r_{\alpha\beta}|^2$ for different pairs $\alpha, \beta$ and $\alpha', \beta'$ of incident and reflected channels can be proven to be uncorrelated. This is basically consequence of the fact that electron reflection back into the reservoir is controlled only by a few scattering events. On the other hand the transmission probabilities $|t_{\alpha\beta}|^2$ might be correlated. Transmission between the source and the drain involves multiple scattering events and therefore leads to the correlation between different paths [2] .

In order to calculate conductance fluctuation $\delta G$ the averaging must be done using the reflection probability $|r_{\alpha\beta}|^2$.

The variance of the conductance is defined as usual [2]

$$\mathrm{var}(G) \equiv (\Delta G)^2 = \langle G^2 \rangle - \langle G \rangle^2. \tag{6.7}$$

Using the current conservation relation and Landauer formula for the conductance gives [2]

$$\mathrm{var}(G) = \left(\frac{e^2}{h}\right)^2 \mathrm{var}\left(\sum_{\alpha,\beta}^{N} |r_{\alpha,\beta}|^2.\right) = \left(\frac{e^2}{h}\right)^2 \mathrm{var}(|r_{\alpha\beta}|^2) \tag{6.8}$$

Because for reflection there are different Feynman paths and the variance can be written [2]

$$\mathrm{Var}(|r_{\alpha,\beta}|^2) = \langle |r_{\alpha,\beta}|^4 \rangle - \langle |r_{\alpha,\beta}|^2 \rangle^2. \tag{6.9}$$

A number $M$ of scattering events gives contribution with amplitude $A_i$ to the reflection probability $r_{\alpha\beta}$. The $A_i$'s can be taken to be uncorrelated. The different scattering sequences can be considered as different Feynman paths and the average scattering probability is then

$$\langle |r_{\alpha\beta}|^2 \rangle = \langle \sum_{mn} A_m A_n \rangle = \left| \sum_{m} A_m \right|^2. \tag{6.10}$$

The $\langle |r_{\alpha\beta}|^4 \rangle$ can be obtained by a similar fashion. Taking the summation over the paths and neglecting terms smaller by a factor $1/M \ll 1$ gives [2]

$$\begin{aligned} \langle |r_{\alpha\beta}|^4 \rangle &= \sum_{i,j,k,l=1}^{M} \langle A_i^\dagger A_j A_k^\dagger A_l \rangle \\ &= \sum_{i,j,k,l=1}^{M} [\langle |A_i|^2 \rangle \langle |A_k|^2 \rangle \delta_{ij}\delta_{kl} + \langle |A_i|^2 \rangle \langle |A_k|^2 \rangle \delta_{il}\delta_{jk}] \\ &= \sum_{i,j,k,l=1}^{M} \langle |A_i|^2 \rangle \langle |A_k|^2 \rangle [\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}] \\ &= 2\langle |r_{\alpha\beta}|^2 \rangle^2 \end{aligned} \tag{6.11}$$

and hence $\mathrm{Var}(|r_{\alpha\beta}|^2) = \langle |r_{\alpha\beta}|^2 \rangle^2$ and

$$\delta G \equiv \sqrt{\mathrm{var}(G)} = \frac{e^2}{h} \left( 1 - \frac{\pi l}{2L} \right). \tag{6.12}$$

In the diffusive limit $l \gg L$ the term $\pi l/2L$ can be approximated to zero and the value of fluctuations is then

$$\delta G = \frac{e^2}{h}, \tag{6.13}$$

which is independent of the sample size.

## 6.2 Noise in mesoscopic systems

Noise in different kind of systems gives information of the particle wave-duality. Albert Einstein was the first person to realize that electromagnetic fluctuations are different if the case of energy carrier is a wave or energy is carried by particles. Carlo Beenakker and Christian Schönenberger [12] are considering in their article the meaning of the noise and what we can learn about it.

The magnitude of energy fluctuations scales for classical particles as a square root of the mean energy and for classical waves it scales linearly with the mean energy. Since a photon is not a classical particle or classical wave the energy fluctuations have linear and square root dependence.

The particle (square root) contribution is typical for optical frequencies and the wave (linear) contribution is property of radio frequencies. So by measuring noise it is easy to define is the photon acting as a wave or as a particle. The noise fluctuations also define the difference between the radiation from laser and the radiation from a

black-body. For black-body the wave contribution is small but for a laser it is entirely absent.

Since electrons can also be considered as a particles or as a wave, there should be a similar diagnostic role of electrical current fluctuations as in the case of photons. The current fluctuations, which are a consequence of the discreteness of the electrical charge are known as shot noise.

In mesoscopic systems there are at least two major sources of noise, shot noise and thermal noise. These both are considered in this section, shot noise in a little more detail.

### 6.2.1 Thermal noise

Perhaps the one of the sources of are, is thermal fluctuations at non-zero temperatures. For the conductor at thermal equilibrium the voltage fluctuations give the value of the temperature $T$. In order to get more information about noise, one has to bring the electrons out of thermal equilibrium.

But first, something about thermal noise, which is also known as Johnsson-Nyquist noise. Ya. M. Blanter and M- Büttiker are discussing in their article [10] thermal noise by a few words. The non-zero temperature causes the occupation number $\langle n \rangle$ of the states of the system to fluctuate. The fluctuations in the occupation number give rise to equilibrium current fluctuations. In the external circuit these current fluctuations can be related to the conductance of the system via the fluctuation-dissipation theorem. Thus the investigation of equilibrium current fluctuations gives the same information than investigation of the conductance [10].

The Fermi distribution determines the thermodynamic average of occupation number $\langle n \rangle = f$. The probability of state being empty in equilibrium $1-f$ and the probability that the state is occupied is $f$. Fluctuations away from this average are given by [10]

$$(n - \langle n \rangle)^2 = n^2 - 2n\langle n \rangle + \langle n \rangle^2. \tag{6.14}$$

For a Fermi system $n^2 = n$ so fluctuations of the occupation number away from the average are [10]

$$\langle (n - \langle n \rangle)^2 \rangle = f(1 - f). \tag{6.15}$$

At the zero temperature limit these fluctuations vanish. The Fermi- distribution function is at high enough energies much smaller than one and in this case the factor $1-f$ can be replaced by one and the fluctuations are determined by the Maxwell-Boltzmann distribution.

Thermal noise can be found at all frequencies up to the quantum limit at $kT/h$. In experiments the fluctuations around a some frequency $\nu$ are considered. If the frequency interval under a consideration is $\Delta\nu$ then thermal noise has an electrical power of $4kT\Delta\nu$ [12].

### 6.2.2 Shot noise

When there was not much information available in the thermal noise, shot noise for example detects the open transmission channels in a disordered wire.

Shot noise refers to the time-independent current fluctuations due to the discreteness of the electron charge and it is an important quantity for characterizing transport properties of mesoscopic systems. To investigate shot noise one has to observe the non-equilibrium (transport) state of the system. The origin of shot noise is considered in the papers written by Büttiker [9] and Büttiker & Blaneter [10]

In order to understand the origin of shot noise it is helpful first to consider just one particle incident upon a barrier. At the barrier particle is either transmitted with probability $T$ or reflected with probability $R = 1 - T$. The incident state is characterized by an occupation number $n_{in}$, transmitted state by $n_T$ and reflected state $n_R$ [10].

When this experiment is repeated many times one gets the average of the occupation numbers and fluctuations away from these values. The averages are simply $\langle n_{in}\rangle = 1$, $\langle n_T\rangle = T$ and $\langle n_R\rangle = R$. Because there is only one particle incident upon a barrier the averages of the squares of the occupation numbers are $\langle n_T^2\rangle = T$ and $\langle n_R^2\rangle = R$. [10].

The mean squared fluctuations in the incident beam vanish $(n_{in} - \langle n_{in}\rangle)^2 = 0$. The mean squared fluctuations for the transmitted and reflected state can be found with help of the product $\langle n_T n_R\rangle$. The product $n_T n_R$ vanishes for each experiment since in each event the particle is either transmitted or reflected. Thus the mean squares of the transmitted and reflected beam and their correlations are given by [10]

$$\langle(\Delta n_T)^2\rangle = \langle(\Delta n_R)^2\rangle = -\langle\Delta n_T \Delta n_R\rangle = TR, \tag{6.16}$$

where $\Delta n_{T,R} = n_{T,R} - \langle n_{T,R}\rangle$. The transmitted and reflected beams are thus correlated.

In order to consider a more realistic example, it is assumed that the incident beam is occupied with the probability $f$, which equals to Fermi distribution function. Now the averages of the occupation numbers of incident, transmitted and reflected beams are $\langle n_{in}\rangle = f$, $\langle n_T\rangle = fT$ and $\langle n_R\rangle = fR$ respectively. Since still only one particle is

considered for each event the product $n_T n_R$ vanishes and so [10]

$$\langle(\Delta n_T)^2\rangle = Tf(1-Tf), \tag{6.17}$$
$$\langle(\Delta n_R)^2\rangle = Rf(1-Rf), \tag{6.18}$$
$$\langle\Delta n_T \Delta n_R\rangle = -TRf^2. \tag{6.19}$$

In the zero temperature limit $f = 1$ and the earlier results are obtained. In the limit $T = 1$ there are still fluctuations in the transmitted state.

To relate the above results to the fluctuations of the current in a conductor, the perfect conductor is assumed. The conductor is also assumed to guide the reflected and transmitted carriers away from the conductor so that the incident, reflected and transmitted current can be discussed separately [10].

In a narrow energy interval $d\mathcal{E}$ the incident current is $dI_{in} = ev(\mathcal{E})d\rho(\mathcal{E})$, where $d\rho(\mathcal{E})$ is the density of carriers per unit length in this energy range. The density in the energy interval $d\mathcal{E}$ is determined by the density of states per unit length $\nu(\mathcal{E}) = n_{in}(\mathcal{E})d\rho/d\mathcal{E}$. The density of states for perfect conductors is $\nu(\mathcal{E}) = 1/(2\pi\hbar v(E))$ and so $dI_{in} = \frac{e}{2\pi\hbar} n_{in}(\mathcal{E})d\mathcal{E}$. It's good to point out again the link between occupation numbers and current. The total current is $I_{in} = \frac{e}{2\pi\hbar}\int n_{in}(\mathcal{E})d\mathcal{E}$ and the average current [10]

$$\langle I_{in}\rangle = \frac{e}{2\pi\hbar}\int f(\mathcal{E})d\mathcal{E}. \tag{6.20}$$

In similar way

$$\langle I_T\rangle = \frac{e}{2\pi\hbar}\int f(\mathcal{E})Td\mathcal{E} \tag{6.21}$$
$$\langle I_R\rangle = \frac{e}{2\pi\hbar}\int f(\mathcal{E})Rd\mathcal{E} \tag{6.22}$$

The above results can be applied to investigate time-dependent current fluctuations because equation so $dI_{in} = \frac{e}{2\pi\hbar} n_{in}(\mathcal{E})d\mathcal{E}$ holds also for time dependent occupation numbers. In long time intervals the current fluctuation in a narrow energy interval is $dI_{in} = (e/2\pi\hbar)n_{in}(\mathcal{E}, t)d\mathcal{E}$. Integrating and invoking a Fourier transform gives in the low frequency limit [10]

$$I(\omega) = (e/2\pi\hbar)\int d\mathcal{E}n(\mathcal{E}, \mathcal{E}+\hbar\omega). \tag{6.23}$$

As a consequence the fluctuations in the current and the fluctuations in the occupation numbers are directly related. The current noise power in the zero frequency limit is $S = e^2\int dES_{nn}(E)$. At each small energy interval particles arrive at a rate $d\mathcal{E}/(2\pi\hbar)$ and contribute to the means square fluctuation to the noise power. Because $S_{nn}(\mathcal{E}) =$

$(1/\pi\hbar)\langle\Delta n\Delta n\rangle$ and therefore [10]

$$S_{I_{in}I_{in}} = 2\frac{e^2}{2\pi\hbar}\int d\mathcal{E} f(1-f), \tag{6.24}$$

$$S_{I_TI_T} = 2\frac{e^2}{2\pi\hbar}\int d\mathcal{E} Tf(1-Tf), \tag{6.25}$$

$$S_{I_RI_R} = 2\frac{e^2}{2\pi\hbar}\int d\mathcal{E} Rf(1-Rf). \tag{6.26}$$

And because of correlation between transmitted and reflected beam [10]

$$S_{I_TI_R} = -2\frac{e^2}{2\pi\hbar}\int d\mathcal{E} TfRf. \tag{6.27}$$

If the factor $(1-Tf)$ in Eq.(6.25) can be replaced by one i.e. either $T$ or $f$ is very small gives the Schottky's/Poisson's result for shot noise [10]

$$S_{I_TI_T} = 2e\langle I\rangle \tag{6.28}$$

Schottky's/Poisson's result corresponds to the uncorrelated arrival of particles with a distribution function of time intervals between arrivals which is Poison $P(t) = \tau^{-1}\exp(-\Delta t/\tau)$, $\tau$ is the mean square time interval for carriers. [10]

Correlations can reduce the noise below the value (6.28). Interactions which lead to correlation are for example coulomb repulsion and Pauli principle. Though, at metals the coulomb repulsion is strongly screened and thus ineffective. [12]

### A more general derivation

In order to evaluate the general current fluctuation spectra belonging to equation (5.35) one need to consider the Fourier amplitude of the current operator [9]

$$\hat{I}_\alpha(\omega) = \int dt e^{i\omega t}\hat{I}_\alpha(t) \tag{6.29}$$

substituting (5.29) and integrating over $t$ and $\mathcal{E}'$ gives [9]

$$\begin{aligned}\hat{I}_\alpha(\omega) &= \frac{e^2}{2\pi\hbar}\sum_{\beta\gamma}\sum_{mn}\int dt d\mathcal{E} d\mathcal{E}' e^{i(\mathcal{E}-\mathcal{E}'+\hbar\omega)t/\hbar}\hat{a}^\dagger_{\beta m}(\mathcal{E})A^{mn}_{\beta\gamma}(\alpha,\mathcal{E},\mathcal{E}')\hat{a}_{\gamma n}(\mathcal{E}') \\ &= \frac{e^2}{2\pi\hbar}\sum_{\beta\gamma}\sum_{mn}\int d\mathcal{E} d\mathcal{E}'\delta(\mathcal{E}-\mathcal{E}'+\hbar\omega)\hat{a}^\dagger_{\beta m}(\mathcal{E})A^{mn}_{\beta\gamma}(\alpha,\mathcal{E},\mathcal{E}')\hat{a}_{\gamma n}(\mathcal{E}') \\ &= \frac{e}{\hbar}\sum_{\beta\gamma}\sum_{mn}\int d\mathcal{E}\hat{a}^\dagger_{\beta m}(\mathcal{E})A^{mn}_{\beta\gamma}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega)\hat{a}_{\gamma n}(\mathcal{E}+\hbar\omega)\end{aligned} \tag{6.30}$$

where the matrix $A^{mn}_{\beta\gamma}$ is [9]

$$A^{mn}_{\beta\gamma} = \delta_{\alpha\beta}\delta_{\alpha\gamma}\delta_{mn} - \sum_k \hat{s}^{\dagger}_{\alpha\beta;mk}(\mathcal{E})\hat{s}_{\alpha\gamma:kn}(\mathcal{E}') \tag{6.31}$$

The calculation of current-current fluctuation spectra $S_{\alpha\beta}(\omega)$ is done by using Fourier-transformed correlation function for the current in contact $\alpha$ to the contact $\beta$ as [9]

$$\hat{S}_{\alpha\beta}(t-t') \equiv \frac{1}{2}\langle \Delta\hat{I}_\alpha(t)\Delta\hat{I}_\beta(t') + \Delta\hat{I}_\beta(t')\Delta\hat{I}_\alpha(t)\rangle. \tag{6.32}$$

Applying the Fourier transform gives [9]

$$2\pi\hat{S}_{\alpha\beta}(\omega)\delta(\omega-\omega') = \langle \Delta\hat{I}_\alpha(\omega)\Delta\hat{I}_\beta(\omega') + \Delta\hat{I}_\beta(\omega')\Delta\hat{I}_\alpha(\omega)\rangle \tag{6.33}$$

where the operator $\Delta\hat{I}_\alpha(t) \equiv \hat{I}_\alpha(t) - \langle\hat{I}_\alpha(t)\rangle$ , which measures the fluctuations away from the average.

First let's consider the expectation value [9]

$$\langle \Delta\hat{I}_\alpha(\omega)\Delta\hat{I}_\beta(\omega')\rangle = \langle \hat{I}_\alpha(\omega)\hat{I}_\beta(\omega')\rangle - \langle\hat{I}_\alpha(\omega)\rangle\langle\hat{I}_\beta(\omega'\rangle \tag{6.34}$$

the first term is (substituting equation (6.30) twice) [9]

$$\begin{aligned}\langle \hat{I}_\alpha(\omega)\hat{I}_\beta(\omega')\rangle = \frac{e^2}{\hbar}\int d\mathcal{E}d\mathcal{E}' \sum_{\gamma\delta\epsilon\zeta}\sum_{mnkl}\langle \hat{a}^{\dagger}_{\gamma m}(\mathcal{E}')A^{mn}_{\gamma\delta}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega)\hat{a}_{\delta n}(\mathcal{E}+\hbar\omega)\\ \times\hat{a}^{\dagger}_{\epsilon k}(\mathcal{E}')A^{kl}_{\epsilon\zeta}(\beta,\mathcal{E}',\mathcal{E}'+\hbar\omega)\hat{a}_{\zeta l}(\mathcal{E}'+\hbar\omega)\rangle\end{aligned} \tag{6.35}$$

The equation (6.35) reduces to [9]

$$\begin{aligned}\langle \hat{I}_\alpha(\omega)\hat{I}_\beta(\omega')\rangle = \frac{e^2}{\hbar}\int d\mathcal{E}d\mathcal{E}' \sum_{\gamma\delta\epsilon\zeta}\sum_{mnkl}A^{mn}_{\gamma\delta}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega)A^{kl}_{\epsilon\zeta}(\beta,\mathcal{E}',\mathcal{E}'+\hbar\omega')\\ \times\langle \hat{a}^{\dagger}_{\gamma m}(\mathcal{E}')\hat{a}_{\delta n}(\mathcal{E}+\hbar\omega)\hat{a}^{\dagger}_{\epsilon k}(\mathcal{E}')\hat{a}_{\zeta l}(\mathcal{E}'+\hbar\omega)\rangle\end{aligned} \tag{6.36}$$

Similarly the second term is [9]

$$\begin{aligned}\langle \hat{I}_\alpha(\omega)\rangle\langle\hat{I}_\beta(\omega'\rangle = \frac{e^2}{\hbar}\int d\mathcal{E}d\mathcal{E}' \sum_{\gamma\delta\epsilon\zeta}\sum_{mnkl}A^{mn}_{\gamma\delta}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega)A^{kl}_{\epsilon\zeta}(\beta,\mathcal{E}',\mathcal{E}'+\hbar\omega')\\ \times\langle \hat{a}^{\dagger}_{\gamma m}(\mathcal{E}')\hat{a}_{\delta n}(\mathcal{E}+\hbar\omega)\rangle\langle\hat{a}^{\dagger}_{\epsilon k}(\mathcal{E}')\hat{a}_{\zeta l}(\mathcal{E}'+\hbar\omega)\rangle\end{aligned} \tag{6.37}$$

combining equations (6.35) (6.37) gives [9]

$$\begin{aligned}\langle \Delta\hat{I}_\alpha(\omega)\Delta\hat{I}_\beta(\omega')\rangle = \frac{e^2}{\hbar}\int d\mathcal{E}d\mathcal{E}' \sum_{\gamma\delta\epsilon\zeta}\sum_{mnkl}A^{mn}_{\gamma\delta}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega)A^{kl}_{\epsilon\zeta}(\beta,\mathcal{E}',\mathcal{E}'+\hbar\omega')\\ \times\Delta_{\hat{a}^{\dagger}_{\gamma m}(\mathcal{E}')\hat{a}_{\delta n}(\mathcal{E}+\hbar\omega)\hat{a}^{\dagger}_{\epsilon k}(\mathcal{E}')\hat{a}_{\zeta l}(\mathcal{E}'+\hbar\omega)}\end{aligned} \tag{6.38}$$

where [9]

$$\begin{aligned}\Delta_{\hat{a}^\dagger_{\gamma m}(\mathcal{E}')\hat{a}_{\delta n}(\mathcal{E}+\hbar\omega)\hat{a}^\dagger_{\epsilon k}(\mathcal{E}')\hat{a}_{\zeta l}(\mathcal{E}'+\hbar\omega)} &= \langle\langle \hat{a}_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')\hat{a}_{\gamma k}(\mathcal{E}'')\hat{a}_{\delta l}(\mathcal{E}''')\rangle\rangle \\ &- \langle\langle \hat{a}_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')\rangle\rangle \langle\langle \hat{a}_{\gamma k}(\mathcal{E}'')\hat{a}_{\delta l}(\mathcal{E}''')\rangle\rangle .\end{aligned} \tag{6.39}$$

Now the problem is to calculate the expectation values of the products of the $\hat{a}$ operators. It is instructive first consider the expectation value for the product $\hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')$ between states $|\sigma\rangle$, which is a many-particle state, specified by occupation numbers $\sigma_{\alpha m}(\mathcal{E})$ for each incident channel $m$. The product $\hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')$ is the occupation number operator in the case when $\alpha = \beta$, $m = n$ and $\mathcal{E} = \mathcal{E}'$. For that operator is $\hat{N}|\sigma\rangle = \sigma_{\alpha m}(\mathcal{E})|\sigma\rangle$ and therefore [9]

$$\langle\sigma|\hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')|\sigma\rangle = \delta(\mathcal{E}-\mathcal{E}')\delta_{\alpha\beta}\delta_{mn}\sigma_{\alpha m}(\mathcal{E}) \tag{6.40}$$

The expectation value for a product of four $a$ operators $\langle\sigma|\hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')\hat{a}^\dagger_{\gamma k}(\mathcal{E}'')\hat{a}_{\delta l}(\mathcal{E}''')|\sigma\rangle$ is little bit more complicated. the product can be nonzero only if there is two pairs of $a$ operators with same indices and arguments i.e. $a^\dagger_{\alpha m}(\mathcal{E})$ and $a_{\alpha m}(\mathcal{E})$ [9].

So clearly there are contributions from both normal and exchange paring. For normal pairing the indices and arguments are ($\alpha = \beta$, $\gamma = \delta$, $m = n, k = l$, $\mathcal{E} = \mathcal{E}$, $\mathcal{E}'' = \mathcal{E}'''$) the operators are automatically in right order. But for the exchange pairing ($\alpha = \delta$, $\beta = \gamma$, $m = l$, $n = k$, $\mathcal{E} = \mathcal{E}'''$, $\mathcal{E}' = \mathcal{E}''$) a little algebra has to be done [9].

For the exchange pairing of the indicies (fermions) [9]

$$\begin{aligned}\hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')\hat{a}^\dagger_{\beta n}(\mathcal{E}'')\hat{a}_{\alpha m}(\mathcal{E}''') &= \hat{a}^\dagger_{\alpha m}(\mathcal{E})[\delta(\mathcal{E}'-\mathcal{E}'') - \hat{a}^\dagger_{\beta n}(\mathcal{E}'')\hat{a}_{\beta n}(\mathcal{E}')]\hat{a}_{\alpha m}(\mathcal{E}''') \\ &= \hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\alpha m}(\mathcal{E}''')[\delta(\mathcal{E}'-\mathcal{E}'') - \hat{a}^\dagger_{\beta n}(\mathcal{E}'')\hat{a}_{\beta n}(\mathcal{E}')],\end{aligned} \tag{6.41}$$

where the last equivalence follows from the fact that $[\hat{a}_{\alpha m}, \hat{a}_{\beta n}]_+ = 0$ and $[\hat{a}^\dagger_{\alpha m}, \hat{a}_{\beta n}]_+ = 0$ because $\alpha \neq \beta$ [9].

Thus for four $\hat{a}$ operators the quantum mechanical expectation value is (taking also bosons into account) [9]

$$\begin{aligned}&\langle\sigma|\hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')\hat{a}^\dagger_{\gamma k}(\mathcal{E}')\hat{a}_{\delta l}(\mathcal{E}''')|\sigma\rangle = \\ &\delta(\mathcal{E}'-\mathcal{E}''')\delta(\mathcal{E}'-\mathcal{E}'')\delta_{\alpha\delta ml}\delta_{\beta\gamma nk}\sigma_{\alpha m}(\mathcal{E})[1\mp\sigma_{\gamma k}(\mathcal{E}'')] \\ &+\delta(\mathcal{E}-\mathcal{E}')\delta(\mathcal{E}''-\mathcal{E}''')\delta_{\alpha\beta mn}\delta_{\gamma\delta kl}\sigma_{\alpha n}(\mathcal{E})\sigma_{\gamma m}(\mathcal{E}'')\end{aligned} \tag{6.42}$$

The last term in equation (6.42) is according to equation (6.40) [9]

$$\delta(\mathcal{E}-\mathcal{E}')\delta(\mathcal{E}''-\mathcal{E}''')\delta_{\alpha\beta mn}\delta_{\gamma\delta kl}\sigma_{\alpha n}(\mathcal{E})\sigma_{\gamma m}(\mathcal{E}'') = \langle\sigma|\hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')|\sigma\rangle\langle\sigma|\hat{a}^\dagger_{\gamma k}(\mathcal{E}'')\hat{a}_{\delta l}(\mathcal{E}''')|\sigma\rangle \tag{6.43}$$

and that is why instead of the equation (6.42) one can also write [9]

$$\begin{aligned}&\langle\sigma|\hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')\hat{a}^\dagger_{\gamma k}(\mathcal{E}'')\hat{a}_{\delta l}(\mathcal{E}''')|\sigma\rangle - \langle\sigma|\hat{a}_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')|\sigma\rangle\langle\sigma|\hat{a}_{\gamma k}(\mathcal{E}'')\hat{a}_{\delta l}(\mathcal{E}''')|\sigma\rangle \\ &= \delta(\mathcal{E}-\mathcal{E}''')\delta(\mathcal{E}'-\mathcal{E}'')\delta_{\alpha\delta ml}\delta_{\beta\gamma nk}\sigma_{\alpha m}(\mathcal{E})[1\mp\sigma_{\gamma k}(\mathcal{E}')]\end{aligned} \tag{6.44}$$

The statistical average of the occupation probability is $\langle\sigma_{\alpha m}(\mathcal{E})\rangle = f_\alpha(\mathcal{E})$ i.e. the distribution function of the reservoir $\alpha$ independent of the channel index. The statistical average of the equation (6.40) is then [9]

$$\langle\langle\sigma|\hat{a}^\dagger_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')|\sigma\rangle\rangle = \delta(\mathcal{E}-\mathcal{E}')\delta_{\alpha\beta mn}f_\alpha(\mathcal{E}) \tag{6.45}$$

And because $\langle\sigma_{\alpha m}(\mathcal{E})\sigma_{\beta n}(\mathcal{E}')\rangle = f_\alpha(\mathcal{E})f_\beta(\mathcal{E}')$ the statistical average of equation can be shown to be [9]

$$\begin{aligned}\langle\langle\sigma|\hat{a}_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')\hat{a}_{\gamma k}(\mathcal{E}'')\hat{a}_{\delta l}(\mathcal{E}''')|\sigma\rangle\rangle - \langle\langle\sigma|\hat{a}_{\alpha m}(\mathcal{E})\hat{a}_{\beta n}(\mathcal{E}')|\sigma\rangle\rangle\langle\langle\sigma|\hat{a}_{\gamma k}(\mathcal{E}'')\hat{a}_{\delta l}(\mathcal{E}''')|\sigma\rangle\rangle \\ = \delta(\mathcal{E}-\mathcal{E}''')\delta(\mathcal{E}'-\mathcal{E}'')\delta_{\alpha\delta ml}\delta_{\beta\gamma nk}f_\alpha(\mathcal{E})[1\mp f_\gamma(\mathcal{E}')]\end{aligned} \tag{6.46}$$

Now using equation (6.41) the equation (6.38) has the form [9]

$$\begin{aligned}\langle\Delta\hat{I}_\alpha(\omega)\Delta\hat{I}_\beta(\omega')\rangle = \frac{e^2}{\hbar}\int d\mathcal{E}d\mathcal{E}'\sum_{\gamma\delta\epsilon\zeta}\sum_{mnkl}A^{mn}_{\gamma\delta}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega)A^{kl}_{\epsilon\zeta}(\beta,\mathcal{E}',\mathcal{E}'+\hbar\omega') \\ \times\delta(\mathcal{E}-\mathcal{E}'-\hbar\omega')\delta(\mathcal{E}'-\mathcal{E}-\hbar\omega)\delta_{\gamma\zeta ml}\delta_{\delta\epsilon nk}f_\gamma(1\mp f_\zeta(\mathcal{E}+\hbar\omega))\end{aligned} \tag{6.47}$$

Integrating with respect to $\mathcal{E}'$ gives [9]

$$\begin{aligned}\langle\Delta\hat{I}_\alpha(\omega)\Delta\hat{I}_\beta(\omega')\rangle = \frac{e^2}{\hbar}\int d\mathcal{E}\sum_{\delta\gamma}\mathrm{Tr}[A_{\delta\gamma}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega)A_{\gamma\delta}(\beta,\mathcal{E},\mathcal{E}+\hbar\omega)] \\ \times f_\gamma(\mathcal{E})[1\mp f_\delta(\mathcal{E}+\hbar\omega)]\delta(\omega+\omega')\end{aligned} \tag{6.48}$$

Next consider the expectation value of two current operators with their order interchanged. Then [9]

$$\begin{aligned}\langle\Delta\hat{I}_\beta(\omega')\Delta\hat{I}_\alpha(\omega)\rangle = \frac{e^2}{\hbar}\int d\mathcal{E}'d\mathcal{E}\sum_{\gamma\delta\epsilon\zeta}\sum_{mnkl}A^{kl}_{\epsilon\zeta}(\beta,\mathcal{E}',\mathcal{E}'+\hbar\omega')A^{mn}_{\gamma\delta}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega) \\ \times\Delta_{\hat{a}^\dagger_{\epsilon k}(\mathcal{E}')\hat{a}_{\zeta l}(\mathcal{E}'+\hbar\omega')\hat{a}^\dagger_{\gamma m}(\mathcal{E}')\hat{a}_{\delta n}(\mathcal{E}+\hbar\omega)}\end{aligned} \tag{6.49}$$

And similar calculation gives [9]

$$\begin{aligned}\langle\Delta\hat{I}_\beta(\omega')\Delta\hat{I}_\alpha(\omega)\rangle = \frac{e^2}{\hbar}\int d\mathcal{E}\sum_{\delta\gamma}\mathrm{Tr}[A_{\gamma\delta}(\beta,\mathcal{E}+\hbar\omega,\mathcal{E})A_{\delta\gamma}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega)] \\ \times f_\delta(\mathcal{E}+\hbar\omega)[1\mp f_\gamma(\mathcal{E})]\delta(\omega+\omega')\end{aligned} \tag{6.50}$$

Because the order of the matrices can be interchanged under the trace the contribution to spectral density from second term is obtained [9]

$$\begin{aligned}\langle\Delta\hat{I}_\beta(\omega')\Delta\hat{I}_\alpha(\omega)\rangle = \frac{e^2}{\hbar}\int d\mathcal{E}\sum_{\delta\gamma}\mathrm{Tr}[A_{\gamma\delta}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega) \\ \times A_{\delta\gamma}(\beta,\mathcal{E}+\hbar\omega,\mathcal{E})]f_\delta(\mathcal{E}+\hbar\omega)[1\mp+f_\gamma(\mathcal{E})]\delta(\omega+\omega')\end{aligned} \tag{6.51}$$

Combining both contributions gives [9]

$$S_{\alpha\beta}(\omega) = \frac{e^2}{2\pi\hbar}\int d\mathcal{E}\sum_{\delta\gamma}\mathrm{Tr}[A_{\gamma\delta}(\alpha,\mathcal{E},\mathcal{E}+\hbar\omega) \times A_{\delta\gamma}(\beta,\mathcal{E},\mathcal{E}+\hbar\omega)]\{f_\gamma(\mathcal{E})[1\mp+f_\delta(\mathcal{E}+\hbar\omega)]+f_\delta(\mathcal{E}+\hbar\omega)[1\mp+f_\gamma(\mathcal{E})]\} \tag{6.52}$$

This equation can be used to investigate the frequency dependence of the spectral densities [9].

**Low-frequency fluctuations** In the zero frequency limit one gets [10]

$$S_{\alpha\beta}(0) = \frac{e^2}{h}\int d\mathcal{E}\sum_{\delta\gamma}\mathrm{Tr}[A_{\delta\gamma}(\alpha,\mathcal{E},\mathcal{E}) \times A_{\gamma\delta}(\beta,\mathcal{E},\mathcal{E})]f_\gamma(\mathcal{E})[1\mp f_\delta(\mathcal{E})]f_\delta(\mathcal{E})[1\mp f_\gamma(\mathcal{E})] \tag{6.53}$$

For $\alpha = \beta$ the terms proportional to $f_\gamma(\mathcal{E})[1\mp f_\delta(\mathcal{E})]$ and terms proportional to $f_\delta(\mathcal{E})[1\mp f_\gamma(\mathcal{E})]$ give same contribution to fluctuation spectrum. In the case of $\alpha\neq\beta$ the terms $f_\gamma f_\delta$ and $f_\delta f_\gamma$ give same contribution. And because [10]

$$\sum_\delta f_\delta\left(\mathrm{Tr}[A_{\gamma\delta}(\alpha)A_{\delta\gamma}(\beta)]\right) = \sum_\gamma f_\gamma\left(\mathrm{Tr}[A_{\gamma\delta}(\alpha)A_{\delta\gamma}(\beta)]\right) \tag{6.54}$$

the terms linear in $f$ are also identical. In the zero-frequency limit the current-fluctuation spectra can therefore be written as [10]

$$S_{\alpha\beta}(0) = 2\frac{e^2}{h}\int d\mathcal{E}\sum_{\delta\gamma}\mathrm{Tr}[A_{\delta\gamma}(\alpha)A_{\gamma\delta}(\beta)] \times\{f_\gamma(\mathcal{E})[1\mp f_\delta(\mathcal{E})] \tag{6.55}$$

The thermal equilibrium or low temperature limits are also evaluated at low frequency limit. So the following equations are based on this equation.

**Equilibrium noise** At thermal equilibrium all the distribution functions $f_\alpha$ have the same chemical potential and hence they are identical [10].

$$S_{\alpha\beta}(\omega) = 2\frac{e^2}{\hbar}\int d\mathcal{E}\sum_{\delta\gamma}\mathrm{Tr}[A_{\delta\gamma}(\alpha,\mathcal{E},\mathcal{E})A_{\gamma\delta}(\beta,\mathcal{E},\mathcal{E})]f(\mathcal{E})[1\mp f(\mathcal{E})] \tag{6.56}$$

No using the following equations $f(1\mp f) = -k_BT\partial f/\partial\mathcal{E}$ and [10]

$$\sum_{\gamma\delta}\mathrm{Tr}(s^\dagger_{\alpha\gamma}s_{\alpha\delta}s^\dagger_{\beta\delta}s_{\beta\gamma}) = \delta_{\alpha\beta}N_\alpha, \tag{6.57}$$

which follows from the unitarity of the scattering matrix. The trace is taken over the transverse channel indices and $N_\alpha$ is the number of channels open in the lead $\alpha$ [10]. Then the noise at the thermal equilibrium is according to Blanter & Büttiker [10]

$$S_{\alpha\beta}(\omega) = \frac{e^2 k_B T}{\pi\hbar} \int d\mathcal{E} \left(-\frac{\partial f}{\partial \mathcal{E}}\right) \left[2N_\alpha \delta_{\alpha\beta} - \mathrm{Tr}(\hat{s}^\dagger_{\alpha\beta}\hat{s}_{\alpha\beta} + \hat{s}^\dagger_{\beta\alpha}\hat{s}_{\beta\alpha})\right] \tag{6.58}$$

Which is the Nyqvist-Johnson noise. It is a consequence of the thermal fluctuations of occupation numbers in the reservoirs.

The equilibrium noise does not provide any new information in addition to conductance measurements. But it can be used to test theoretical assumptions and in a calibration of experiments.

**Zero-temperature limit** At this limit the noise is considered only for Fermi system because for Bose systems the Bose condensation has to be taken into account. Now $f(1-f) = -k_B T \partial f/\partial \mathcal{E}$ so all terms containing Fermi functions in this kind of fashion vanish. This allows to do some restrictions in the summations over the probe indicies. The summation can be restricted to case when $\delta \neq \gamma$. In this case the $\hat{A}$ matrices are $\hat{A} = \hat{s}^\dagger_{\alpha\gamma}\hat{a}_{\gamma\delta}$ and thus [9]

$$S_{\alpha\beta}(\omega) = 2\frac{e^2}{h} \sum_{\gamma\neq\delta} \int d\mathcal{E} \mathrm{Tr}[\hat{s}^\dagger_{\alpha\gamma}\hat{s}_{\alpha\delta}\hat{s}^\dagger_{\beta\delta}\hat{s}_{\beta\gamma}] f_\gamma(\mathcal{E})[1 - f_\delta(\mathcal{E})], \tag{6.59}$$

where the Fermi functions are step functions.

Correlations of the current in the same lead $S_{\alpha\alpha}$ are positive,

$$S_{\alpha\alpha} = 2\frac{e^2}{h} \sum_{\gamma\neq\delta} \int d\mathcal{E} \mathrm{Tr}[\hat{s}^\dagger_{\alpha\gamma}\hat{s}_{\alpha\delta}\hat{s}^\dagger_{\alpha\delta}\hat{s}_{\alpha\gamma}] f_\gamma(\mathcal{E})[1 - f_\delta(\mathcal{E})], \tag{6.60}$$

since their signs are determined by positively defined quantities $\mathrm{Tr}[\hat{s}^\dagger_{\alpha\gamma}\hat{s}_{\alpha\delta}\hat{s}^\dagger_{\beta\delta}\hat{s}_{\beta\gamma}]$. The correlations from different leads $S_{\alpha\beta}$ are negative. This can be seen using property $\sum_\delta \hat{s}_{\alpha\delta}\hat{s}^\dagger_{\beta\delta} = 0$ then $\sum_{\delta\neq\gamma} \hat{s}_{\alpha\delta}\hat{s}^\dagger_{\beta\delta} = -\hat{s}_{\alpha\gamma}\hat{s}^\dagger_{\beta\gamma}$. The contribution from linear terms from equation (6.59) is proportional to

$$-\sum_\gamma \mathrm{Tr}[\hat{s}^\dagger_{\alpha\gamma}\hat{s}_{\alpha\gamma}\hat{s}^\dagger_{\beta\gamma}\hat{s}_{\beta\gamma}] f_\gamma(\mathcal{E}). \tag{6.61}$$

At the zero temperature limit the Fermi functions are either zero or one and thus the following is valid:

$$-\sum_\gamma \mathrm{Tr}[\hat{s}^\dagger_{\alpha\gamma}\hat{s}_{\alpha\gamma}\hat{s}^\dagger_{\beta\gamma}\hat{s}_{\beta\gamma}] f_\gamma(\mathcal{E}) f_\gamma(\mathcal{E}). \tag{6.62}$$

These are the diagonal terms quadratic in $f$ omitted in Eq. (6.59). Finally the correlation between different probes $\alpha$ and $\beta$ is according to Büttiker [9]

$$S_{\alpha\beta}(\omega) = -2\frac{e^2}{h}\sum_{\gamma\neq\delta}\int d\mathcal{E}\mathrm{Tr}[\hat{s}^\dagger_{\alpha\gamma}\hat{s}_{\alpha\gamma}\hat{s}^\dagger_{\beta\gamma}\hat{s}_{\beta\gamma}]f_\gamma(\mathcal{E})f_\delta(\mathcal{E}), \tag{6.63}$$

The negativity of $s_{\alpha\beta}$ comes also from the conservation of electric charge. If the spectral function $S_{\alpha\alpha}$ is positive then all cross-correlations must be negative [10].

One can also define the noise conductances in both of the cases, which are according to Büttiker [9]

$$G_{\gamma\delta}(\alpha\alpha) \equiv \frac{e^2}{h}\mathrm{Tr}[\hat{s}^\dagger_{\alpha\gamma}\hat{s}_{\alpha\delta}\hat{s}^\dagger_{\alpha\delta}\hat{s}_{\alpha\gamma}] \tag{6.64}$$

and

$$G_{\gamma\delta}(\alpha\beta) \equiv \frac{e^2}{h}\mathrm{Tr}[\hat{s}^\dagger_{\alpha\gamma}\hat{s}_{\alpha\gamma}\hat{s}^\dagger_{\beta\gamma}\hat{s}_{\beta\gamma}] \tag{6.65}$$

### The information obtained from noise

As mentioned already earlier the shot noise was a consequence of the discreteness of the electron charge and it appears only in the presence of transport. From the noise measurements it is possible to obtain information which is not available in the conductance measurements.

The noise measurements can provide information for example on the time-dependent correlations between electrons. It detects also open transmission channels in a disordered wire, which is discussed briefly in the next section.

Shot noise measures the unit transferred charge in a tunnel junction and it can be used to detect the fractional $e/3$ charge carriers in the fractional quantum Hall effect. So the noise measurements can provide information on the charge statistic of the charge carrier particles or quasiparticles. This in other hand can provide information about the internal energy scales and the potential profiles of the mesoscopic system.

Beenakker and Schönenberger have been considering also a one more diagnostic of noise [12]. In quantum dots or electron billiards, if the shape of the confining potential is irregular enough, the classical dynamics is chaotic. Traces of this chaos can be searched in the quantum mechanical properties [12].

In electron billiards with help of shot noise the deterministic scattering can be distinguished from stochastic scattering. The deterministic scattering is characteristic for particles when on the other hand the stochastic scattering is deterministic for waves [12].

The initial momentum and position of the particle fix its trajectory. These initial conditions will determine is the particle is reflected or transmitted. The scattering is thus noiseless all the time. When the particle dynamics is deterministic the wave dynamics is stochastic, because of quantum uncertainty. The wave dynamics is thus noisy [12].

Because the origin of the shot noise is also in the motion of the electron charge the measurements of noise can be sensitive to the symmetry of the orbital part of the wavefunction. Therefore the measurement of nose power could give information whether the state is singlet or triplet. The noise power is enhanced for a singlet state and suppressed for a triplet state. This piace of information can be used the investigate wehther the state is entangled or not according to Blanter and Büttiker.[10]

## 6.3 Quantum point contact

Quantum point contact is usually described as a constriction between two metallic reservoirs. Usually the constriction is formed by depleting two dimensional electron gas with a help of a number of gates. Changes in the gate voltage lead to a variation of the width of the constriction and electron concentration. The transport through the constriction is assumed to be ballistic i.e. all the sizes in the contact are assumed to be smaller than any scattering length. In quantum point contacts the width of the constriction is comparable to the Fermi wavelength [5].

One way to model a quantum point contact is to consider it as a ballistic channel between infinitely high potential walls. If the distance between the walls $d(x)$ is taken to be varying slowly the transverse and longitudinal motions can be separated in an adiabatic way. This reduces the problem to one-dimensional motion in an adiabatic potential $U(x) = \pi^2 n^2 \hbar^2 / 2md^2(x)$ [13].

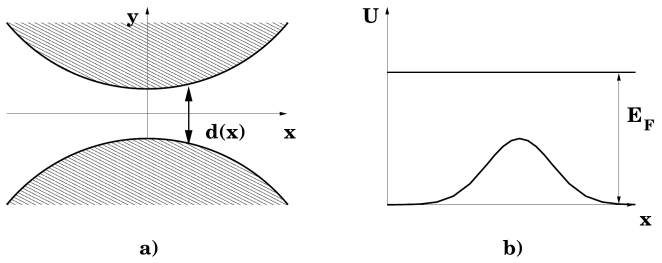



**Figure 6.1**: Geometry of the quantum point contact in the hard-wall model (a) and the effective potential for one-dimensional motion (b) [13].

A smooth external potential can be treated semiclassically. Then the channels with

$n < k_F d_{min}/\pi$ are open and transparent ($T_n = 1$) and the other channels are closed ($T_n = 0$). Because the conductance $G = (e^2/2\pi\hbar)\sum_n T_n$ is proportional to the number of open channels there are plateaus as a function of the gate voltage. At the plateaus shot noise is zero since all the channels are either open or closed [13].

The semiclassical approximation failes when the Fermi energy lies close to the top of the potential in one of the transverse channels. The transmission coefficient for this channel increases from zero to one due to quantum tunnelling through the barrier and quantum reflection at the barrier. The transition from one plateau to next is associated with a peak at noise spectrum [13].

A more realistic description takes into account that the potential is smooth in the transverse direction. The constriction can be thought as a bottleneck of the form of saddle [13].

$$V(x,y) = V_0 - \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2 \tag{6.66}$$

The constant $V_0$ denotes the potential at the saddle point and the curvatures of the potential are expressed in terms of $\omega_x$ and $\omega_y$.

The transmission probability for this constriction is according to Büttiker [13]

$$\begin{aligned} T_n(E) &= \delta_{mn}[1 + \exp(-\pi\epsilon_n)]^{-1} \\ \epsilon_n &\equiv \left[E - \hbar\omega_y\left(n + \frac{1}{2}\right) - V_0\right]/\hbar\omega_x \end{aligned} \tag{6.67}$$

Only non-zero transmission probabilities are for cases when the incident channel and outgoing channel are different. In the case of $\epsilon_n \ll 0$ the transmission probability is small $T_n \approx \exp(\pi\epsilon_n)$ and when $\epsilon_n \gg 0$ transmission probability is close to one $T_n \approx 1 - \exp(-\pi\epsilon_n)$. The transition from zero to one in the transmission probability occurs when $\epsilon_n \approx 0$ i.e. as the energy passes the value $V_0 + \hbar\omega_y(n + 1/2)$ [13].

The size of energy interval needed for the transmission is determined by $\hbar\omega_x$ Total transmission probability in the case of one equilibrium reservoir to other is the sum $T = \sum_{mn} T_{mn}$ and so the conductance is [13]

$$G = \frac{e^2}{h}T \tag{6.68}$$

Total transmission probability and the two terminal conductance thus show series of steps if the opening of quantum channel is small compared to the channel separation.

The zero temperature shot noise can be determined by using equation (6.71) [13]

$$S = \frac{e^3|V|}{\pi\hbar}\sum_n T_n(1 - T_n) \tag{6.69}$$

which is non-zero only if $T_n \neq 0, 1$ in the case of transition.

### 6.3.1 Shot noise in two terminal conductors

Next the shot noise for two terminal conductor at the zero-temperature is considered. Denoting leads as $R$ right and $L$ left the current conservation assures that $S \equiv S_{LL} = S_{RR} = -S_{RL} = -S_{LR}$. Unitarity of $\hat{S}$ matrix $\hat{S}^\dagger \hat{S} = \hat{1}$ gives $\hat{r}^\dagger \hat{r} + \hat{t}^\dagger \hat{t} = \hat{1}$. Then the formula for shot noise can be simplified to[10]

$$S = \frac{e^2}{\pi\hbar} \mathrm{Tr}(\hat{r}^\dagger \hat{r} \hat{t}^\dagger \hat{t}) eV \tag{6.70}$$

where the scattering matrix elements are evaluated at Fermi level. Using transmission $T_n$ and reflection probabilities $R_n = 1 - T_n$ noise can be determined as a product of these two variables [10]

$$S = \frac{e^3 V}{\pi\hbar} \sum_n T_n (1 - T_n). \tag{6.71}$$

In particular neither open nor closed channels contribute to the shot noise. Maximal contribution is attained with channels $T_n = 1/2$. The reduction of noise due to the Pauli principle is described by a factor $1 - T_n$. Without Pauli principle the noise would be just Poisson noise (6.28).

In the one-dimensional conductor electrons above the Fermi level about $eV$ will enter at the conductor at a rate $eV/h$. The number of attempted transmissions in a time $\tau$ will be $\tau eV/h$. In the zero-temperature there will be no fluctuation in this number since each state contains only one electron. Because the transmission occurs at the probability $T_1$, which is different from 0 or 1, there is fluctuation in the transmitted charge $Q$. The transmission probability obeys binomial distribution and the mean squared fluctuation $\langle \delta Q^2 \rangle$ of the charge is [12]

$$\langle \delta Q^2 \rangle = e^2 (\tau eV/h) T_1 (1 - T_1). \tag{6.72}$$

Since the noise is given by $S = (2/\tau)\langle \delta Q^2 \rangle$ the equation (6.72) is just the equation (6.71) for one channel. The multichannel expression is obtained from (6.72) just by summing over all channels since the fluctuations in different channels are independent [12].

# 7 Quantum dot and coulomb blockade

The quantum dot consists of two quantum point contacts in series. Thus quantum dot can be to be considered a structure which is more complex than a single quantum point contact. The quantum point contact is considered in the Elliot's book [4] and in the Galperin's notes [2].

Quantum dots can be formed using metallic films or semiconductor materials or the confinement can be due to electrostatic potentials. The conductance in the quantum dots exhibits a series of sharp peaks. These peaks are almost periodic at very low temperatures. Each of the peaks corresponds to an addition of electron to the dot [4].

If the dot is initially electrically neutral it requires an energy $e^2/2C$ when an electron is added to it ($C$ is the total capacitance between dot and its surroundings). Thus an energy barrier $e^2/2C$ must be surmounted in order to make current flow [4] .

For an electron to tunnel into the dot its energy must be greater than the Fermi energy of the contact by $e^2/2C$ and for a hole the energy must be lower by $e^2/2C$. So, there is a total gap of width $e^2/C$ in the tunneling density of states. If the temperature is low enough $k_BT < e^2/2C$ neither electron or holes can flow. (Typically $C \simeq 10^{-16}F$ and thus $T \lesssim 1K$) [4]

Altering the gate voltage tunneling current may be able to made to flow, since the energy needed to add the charge is changed. The energy of dot with charge $Q$ under a gate voltage $V_g$ is [4]

$$\mathcal{E} = QV_g + \frac{Q^2}{2C} \tag{7.1}$$

The graph effective capacitance $C$ which takes into account polarization of the electrodes is a parabola with the minimum at [2]

$$Q = Q_0 = -CV_g \tag{7.2}$$

The energy as a function of the number $N$ electrons at the dot is [2]

$$E(N) = -NeV_g + \frac{N^2e^2}{2C}. \tag{7.3}$$

The difference [2]

$$E(N+1) - E(N) = -eV_g + N\frac{e^2}{C} \tag{7.4}$$

at certain values of $V_g$ [2]

$$V_{GN} = N\frac{e}{C} \tag{7.5}$$

the difference vanishes. So only at those values of the gate voltage resonant transfer is possible. Otherwise one has to pay for the transfer i.e. only inelastic process can contribute. As a result, at [2]

$$k_B T \leq \frac{e^2}{2C} \tag{7.6}$$

the linear conductance is exponentially small if the condition (7.5) is met. This phenomenon is called the Coulomb blockade of conductance [2].

As a result of the Coulomb blockade, electron tunnel one-by-one, and the conductance vs. gate voltage dependence is a set of sharp peaks. The fact allows one to create a so-called single electron transistor [2].

# 8 Limitations of the scattering approach

Although the scattering approach provides a quite powerfull method for analyzing the conduction properties in the mesoscopic systems it has some restrictions, which have been considered by Agraït *et al.* in their article [1]. The scattering approach is mainly a phenomenological theory. Inputs of this theory are the scattering properties of the sample, contained in the $\hat{S}$ matrix. Scattering approach is also a one particle theory and valid as long as inelastic scattering process can be neglected.

Considering electron propagation through the sample as a fully quantum coherent process is a strong assumption of scattering theory. This description is basically valid at zero temperature and only for electrons at Fermi-energy. Under a finite bias voltage and finite temperatures this simple description may not be valid [1].

The transmission coefficients in the scattering matrix can be both energy and voltage dependent. This dependence can be determined through the precise shape of the electrostatic potential profile developing in the sample, which should in principle be calculated self-consistently [1].

Inelastic scattering due to electron-phonon and electron-electron collisions may limit the coherent propagation of electrons through the sample. Also the presence of strong coulomb interactions may alter completely the description of transport given by the scattering approach [1].

The electron-electron and electron-phonon interactions can be accounted for if one uses the non-equilibrium Green's function formalism or the density functional methods. With help of NEGF one could also handle the problems arising from the finite bias voltage.

# 9 Final words

Now the concepts of conductance and conductivity have been considered in the diffusive and ballistic regimes. The effect of phase-coherence to the conductivity was also under a consideration. The main idea of this text, before I started to write, was to be a brief introduction to the conductance in the mesoscopic systems and an estimate how well this is accomplished is left to the reader.

In diffusive regime an estimate to the conductance was found semiclassically by using relaxation-time approximation. During the derivation it was assumed that the interaction potential is weak, which made the use of Fermi's golden rule appropriate. The scattering process was also assumed to be elastic, which follows from the delta function in the Fermi's golden rule. The spherically symmetric potential simplified the life so as did the assumption of the isotropically distributed of electron states. This is a huge list of assumptions and when the temperature was taken to be near to zero it is clear that the obtained result for conductivity cannot be valid. Nevertheless, it is good that this theory gives also the Drude conductivity as a limit. But if the interaction cannot be taken to be weak or the inelastic scattering processes are involved, then one has to move to use a more powerfull methods like NEGF.

What if the electrons cannot be considered as particles anymore but as waves? An example of quantum diffusion was the weak localization which caused the reduction of diffusion coefficient and hence the reduction of the conductivity. The weak localization was basically a consequence of the interference between electron waves, which caused the wavefunction to depend on the position.

The criterion for strong localization was the quantum conductance $e^2/h$. Phase-coherent conductor which conductance is smaller belonged to the strong localization regime. The effects of strong localization was considered phenomenologically in terms of scaling theory of localization. But the story does not tell anything about full quantum mechanical treatment of the effects to the conductance in the regime of strong localization, which in principle could be interesting.

The reduction of the sample size causes the transport to move into the ballistic regime. When the electron moves through a ballistic conductor its wavevector and momentum do not change and the transmission can be considered to happen with certain probability. Landauer formula related these transmission probabilities to the conductance.

The transmission coefficient can be expressed in terms of Green's functions. This provides a bridge between the scattering approach and linear response theory from which the conductance is obtained more traditionally. The Green's functions have a lot of importance in further theories of conductance like in the non-equilibrium Green's function methods, in the tight binding models of conductance or when the theory of conductance is build purely on the basis of Green's functions.

The conductance fluctuations showed to have a general value at least in the diffusive regime. The value of fluctuations is therefore independent of the sample size. This result was obtained in the zero temperature and when one wants to investigate fluctuation in the nonzero temperatures a different approach is needed.

Another type of fluctuations is a noise and especially the shot noise was under consideration. The shot noise was a consequence of the discreteness of charge carrier electric charge. By measuring this noise one can obtain more information of systems which is not available in the conductance measurements. Those things which can be found in the measurements were listed in the their own chapter so I am not going to sum up twice.

This was basically the summary of the text. There are plenty of phenomenon which could have been included to this text, like for example quantum Hall effect and especially the fractional quantum Hall effect and the effects of magnetic fields. But that would have caused this text just to be too broad, long and messy.

# References


[1] Nicolas Agraït, Alfrendo Levy Yeyati, and Jan M. van Ruitenbeek. Quantum properties of atomic-sized conductors. *Phys Rep.*, 377:81–279, 2003.

[2] Yuri M. Galperin. Quantum Transport, Lecture Notes, Lund University, (1998).

[3] Neil W. Ashcroft and N. David Mermin. *Solid State Physics.* Books/Cole, USA, 1976.

[4] Stephen Elliot. *The Physics and Chemistry of Solids.* Wiley, West Sussex, England, 2006.

[5] Michael P. Marder. *Condensed Matter Physics.* Wiley, 2000.

[6] C.W.J Beenakker and H. van Houten. Quantum transport in semiconductor nanostructures. *Solid State Phys. (cond-mat/0412664)*, 44:1–228, 1991.

[7] Matej Kanduc. Weak Localization (seminar), University of Ljubljani, (2005).

[8] E. Abrahams, P. W. Anderson, D.C Licciardello, and T.V. Ramakrishnan. Scaling theory of localization: absence of quantum diffusion in two dimensions. *Phys. Rev. Lett.*, 42(10):673, 1991.

[9] M. Büttiker. Scattering theory of current intensity noise correleations in conductors and wave guides. *Phys. Rev. B*, 46(19):12485, 1992.

[10] Ya. M. Blanter and M. Büttiker. Shot noise in mesoscopic conductors. *Phys.Rep. (cond-mat/9910158)*, 336(1), 2000.

[11] E.N. Economou. *Green's Functions in Quantum Physics.* Springer, Berlin, Germany, 2006.

[12] Carlo Beenakker and Christian Schönenberger. Quantum shot noise. *Phys. Tod.*, page 37, 2003.

[13] M. Büttiker. Quantized transmission of a saddle-point constriction. *Phys. Rev. B*, 41(11), 1989.