Geometric embeddings of metric spaces

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Preface

These notes form a slightly expanded version of the lectures that I gave in the Finnish Graduate School of Mathematics at the University of Jyväskylä in January 2003. The purpose of this mini-course was to introduce beginning graduate students to some easily accessible questions of current interest in metric geometry. Besides additional remarks and references, the only topics that were not discussed in the course, but are included here, are the proof of the existence of a Urysohn universal metric space and the proof of Semmes's theorem 4.5.

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1. Basic Concepts

Let

$$X = (X, d) = (X, d_X)$$

denote a metric space. Throughout these lectures, we will consider quite general metric spaces. However, the reader should not think of anything pathological here (like the discrete metric on some huge set). We are motivated by problems in analysis and geometry.

For background reading, see [8], [33], [23], [37], [65].

1.1. **Examples of metric spaces.** (a) Any subset of $\mathbb{R}^n$ with the inherited metric. Many, but by no means all, interesting metric spaces belong to this group. We can also equip $\mathbb{R}^n$ with the p-norms, where

$$|x|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty\,,$$

and

$$|x|_\infty = \max_{i=1,\dots,n} |x_i|\,,$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. The corresponding distance or metric is

$$|x - y|_p\,, \qquad 1 \le p \le \infty\,.$$

(b) Cones over metric spaces. Let Z be a metric space. The *metric cone $C(Z)$ over Z* is the completion of the product $Z \times (0, \infty)$ in the metric that is defined by

$$d((z, r), (z', r')) := \sqrt{r^2 + r'^2 - 2r \cdot r' \cos(d_Z(z, z'))}\,,$$

if $d_Z(z, z') \le \pi$, and by

$$d((z, r), (z', r')) := (r + r')\,,$$

if $d_Z(z, z') \ge \pi$. If Z is a closed (=compact, without boundary) smooth Riemannian manifold, then $C(Z)$ is a smooth Riemannian manifold, except perhaps at the tip (=the point added in the completion).

The cone C_ρ over the circle of length $\rho > 0$ is *flat* outside the tip in that it is locally isometric to a patch in $\mathbb{R}^2$. The tip is a singular (nonsmooth) point, except when $\rho = 2\pi$. The curvature on C_ρ can defined in a certain generalized sense; it is a Dirac measure at the tip of total mass $2\pi - \rho$.

Read more in [2], [14], [13], [9], [11].

(c) *Snowflake spaces*

$$(\mathbb{R}, |x - y|^\varepsilon)\,, \qquad 0 < \varepsilon < 1\,.$$

These are indeed metric spaces because the inequality

$$(a + b)^\varepsilon \le a^\varepsilon + b^\varepsilon$$

is valid whenever $a, b \geq 0$ and $0 < \varepsilon < 1$. The name stems from the fact that $(\mathbb{R}, |x-y|^\varepsilon)$, for $\frac{1}{2} < \varepsilon < 1$, admits a bi-Lipschitz embedding in $\mathbb{R}^2$, and the image resembles the boundary of a snowflake. (See 1.3 for the definition of a bi-Lipschitz embedding.)

More generally we say that a metric space (X, d^ε) is a *snowflake version* of (X, d) if $0 < \varepsilon < 1$.

Read more in [5], [72], [23], [37], [65].

(d) *Rickman's rug*

$$(\mathbb{R}^2, |x-x'| + |y-y'|^\varepsilon)\,, \qquad 0 < \varepsilon < 1\,.$$

This can be thought of as a product of a snowflake and a line. Here

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}\,.$$

Metrics like this exhibit an interesting asymmetry. They show up in problems related to the heat equation, in the theory of pinched curvature Riemannian manifolds, and the theory of quasiconformal and related mappings.

Read more in [72], [54], [3], [75].

(e) Various classical *fractals*, such as the Sierpinski carpet or gasket.

Read more in [26], [52], [42], [23], [41].

(f) *Sub-Riemannian manifolds* and, in particular, *Carnot groups.* These are intriguing metric spaces with complicated local geometry, but many nice global analytic and geometric properties. They arise in complex analysis, control theory, and partial differential equations.

Read more in [32], [68], [7], [36], [69].

(g) Spaces with generalized curvature bounds. These are metric spaces where curvature makes sense either as a measure, or as a metric quantity that can be said to be bounded, or bounded from above or below. (Compare (b).)

Read more in [2], [21], [56], [55], [14], [13].

1.2. **Curves in metric spaces.** A (compact) *curve* in X is a continuous mapping $\gamma : [a, b] \to X$, where $[a, b] \subset \mathbb{R}$ is an interval. A curve is *rectifiable*, if

$$\sup \sum_{i=0}^{n} d(\gamma(t_{i+1}), \gamma(t_i)) < \infty\,,$$

where the supremum is taken over all sequences

$$a = t_0 \leq t_1 \leq \cdots \leq t_{n+1} = b\,.$$

A metric space is *quasiconvex*, if there exists a constant $C \geq 1$ with the property that every pair of points x, y in the space can be joined by a curve whose length is no more than $C \cdot d(x, y)$.

The Sierpinski carpet and gasket, although fractal objects, are quasiconvex. The snowflake space $(\mathbb{R}, |x-y|^{\varepsilon})$, $0 < \varepsilon < 1$, contains no nonconstant rectifiable curves. In fact, no snowflake version of a metric space can contain nonconstant rectifiable curves. The only nonconstant rectifiable curves in Rickman's rug are those with constant y-coordinate.

By the aid of rectifiable curves, some differential analysis in metric spaces is possible via reduction to the one variable case.

Read more in [13], [38], [37], [60], [66].

1.3. **Embeddings.** A map $\varphi : X \to Y$ is an *embedding* if it is a homeomorphism onto its image. An embedding φ is *L-bi-Lipschitz*, $L \geq 1$, if

$$\frac{1}{L}\, d_X(a,b) \leq d_Y(\varphi(a),\varphi(b)) \leq L\, d_X(a,b)$$

whenever $a, b \in X$. If $L = 1$, then φ is an *isometric embedding* and X is *isometrically embedded* in Y.

1.4. **Open problem.** *Characterize metric spaces that are bi-Lipschitz embeddable in some* $\mathbb{R}^n$.

Very little progress has been made on this problem. See Section 3 for further discussion.

Read more in [48], [63], [37], [49].

1.5. **Motivation.** Metric spaces arise in nearly all areas of mathematics. The metric point of view has been useful even in group theory, where finitely generated groups can be equipped with a *word metric*. In analysis, the study of various function spaces has a long history, and *completeness* is an important concept there. The study of completeness of a class of geometric objects is more recent. For example, an important problem in Riemannian geometry concerns the degeneration of metrics on a fixed manifold; singular spaces occur as limits. Similarly, focusing on *graphs* of functions, rather than on their analytic expressions, can shed new light on classical problems as well. For example, the functions $x \mapsto x^n$, $n = 2, 4, 6, \ldots$, do not converge to a finite valued function on $\mathbb{R}$, but their graphs have a nice locally uniform limit as a subset of $\mathbb{R}^2$.

Read more in [31], [33], [16], [70], [71], [24], [21], [56], [55], [23].

1.6. **Kuratowski embedding.** *Every metric space X embeds isometrically in the Banach space $L^{\infty}(X)$ of bounded functions on X with sup-norm:*

$$||s||_{\infty} := \sup_{x \in X} |s(x)|\,, \qquad s : X \to \mathbb{R}.$$

Proof. Fix $x_0 \in X$ and define

$$x \mapsto s^x\,, \qquad s^x(a) = d(x,a) - d(a,x_0)\,.$$

Then by the triangle inequality

$$|s^x(a)| \le d(x,x_0)$$

and

$$|s^x(a) - s^y(a)| = |d(x,a) - d(y,a)| \le d(x,y)\,,$$

and equality occurs if $a = x$ or $a = y$. □

1.7. **Remark.** If X is bounded, we do not need to subtract the term $d(a,x_0)$ in the definition of s^x. In this case, the embedding is canonical. In the unbounded case, the embedding depends upon the choice of the base point x_0.

The letter s in s^x conveys the idea that the elements in $L^\infty(X)$ are bounded sequences $s = (s_a)$ indexed by $a \in X$.

Theorem 1.6 is due to to Kuratowski [43].

1.8. **Application.** If $f, g : X \to Y$, then the sum $f + g$ is meaningless in general. However, after embedding Y isometrically,

$$Y \hookrightarrow L^\infty(Y)\,,$$

we have

$$f + g : X \to L^\infty(Y).$$

For example, we can form the homotopy

$$h_t = (1-t)f + tg : X \to L^\infty(Y)\,, \quad 0 \le t \le 1.$$

Here is a simple application of the idea of enlarging the range from Y to $L^\infty(Y)$.

1.9. **Proposition.** *Let X be compact and $f : X \to Y$ continuous. Then there exists a sequence (f_i) of Lipschitz continuous maps $f_i : X \to L^\infty(Y)$ such that $f_i \to f$ uniformly as $i \to \infty$.*

Recall that a map $f : X \to Y$ is *Lipschitz*, or *L-Lipschitz*, $L \ge 1$, if

$$d_Y(f(x), f(y)) \le L\, d_X(x,y)$$

whenever $x, y \in X$.

1.10. **Remark.** In general, the conclusion of Proposition 1.9 cannot hold with $f_i(X) \subset Y$. For example, the identity map

$$[0,1] \to ([0,1], |x-y|^{\frac{1}{2}})$$

cannot be approximated by Lipschitz maps because the target space has no nonconstant rectifiable curves.

The proof for Proposition 1.9 requires some concepts that are useful in other contexts as well. These are discussed next.

1.11. **Maximal nets.** A *maximal ε-net*, $\varepsilon > 0$, in a metric space X is a collection of points

$$N = N(X, \varepsilon) \subset X$$

such that

$$d(x, x') \geq \varepsilon$$

whenever $x, x' \in N$ are distinct, and that

$$X = \bigcup_{x \in N} B(x, \varepsilon)\,.$$

Here and later,

$$B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

is an open ball centered at x with radius ε.

We often call maximal nets simply *nets*.

The following facts can be proved easily by using Zorn's lemma:

- Maximal ε-nets exist for every X and $\varepsilon > 0$.
- Given an ε-net $N(X, \varepsilon)$ and $\varepsilon' < \varepsilon$, there exists an ε'-net $N(X, \varepsilon')$ extending $N(X, \varepsilon)$, i.e.,

  $$N(X, \varepsilon) \subset N(X, \varepsilon')\,.$$

- Given $x_0 \in X$, $0 < R < R'$, $0 < \varepsilon' < \varepsilon$, and an ε-net

  $$N(X, R, \varepsilon) := N(B(x_0, R), \varepsilon)$$

  of the ball $B(x_0, R)$, then there exists an ε'-net

  $$N(X, R', \varepsilon') = N(B(x_0, R'), \varepsilon')$$

  extending $N(X, R, \varepsilon)$.

1.12. **Lipschitz partition of unity.** Let

$$N = N(X, \varepsilon) = (x_i)$$

be a countable ε-net in X. Write

$$B_i = B(x_i, \varepsilon)\,, \quad 2B_i = B(x_i, 2\varepsilon),$$

and assume that the collection $\{2B_i\}$ is locally finite in the following sense: only finitely many balls $2B_j$ meet a given ball $2B_i$. In particular, if X is compact, such nets exist for every $\varepsilon > 0$.

Define

$$\psi_i(x) = \min\{1, \frac{1}{\varepsilon}\,\mathrm{dist}(x, X \backslash 2B_i)\}.$$

Then

- $0 \leq \psi_i \leq 1$, $\psi_i | B_i = 1$, and $\psi_i | X \backslash 2B_i = 0$.
- ψ_i is $\frac{1}{\varepsilon}$-Lipschitz.
- $\psi := \sum_i \psi_i \geq 1$ and $\psi | 2B_i$ is $\frac{L_i}{\varepsilon}$-Lipschitz, where L_i is the number of balls $2B_j$ meeting $2B_i$.

Set

$$\varphi_i = \frac{\psi_i}{\psi}\,.$$

Then

$$0 \le \varphi_i \le 1\,, \quad \varphi_i | X \backslash 2B_i = 0\,, \quad \text{and} \quad \sum_i \varphi_i = 1\,.$$

We further claim that φ_i is L_i'-Lipschitz, where

$$L_i' = \frac{2L_i}{\varepsilon}\,.$$

To this end, let first $x, y \in 2B_i$. Then

$$\begin{aligned}
|\varphi_i(x) - \varphi_i(y)| &= \frac{|\psi_i(x)\psi(y) - \psi_i(y)\psi(x)|}{\psi(y)\psi(x)} \\
&\le |\psi_i(x)\psi(y) - \psi_i(y)\psi(y)| + |\psi_i(y)\psi(y) - \psi_i(y)\psi(x)| \\
&\le \sup_{2B_i} |\psi| \frac{1}{\varepsilon} d(x,y) + \frac{L_i}{\varepsilon} d(x,y) \\
&\le L_i'\, d(x,y)\,.
\end{aligned}$$

Next, let $x \in 2B_i$ and $y \in X \backslash 2B_i$. Then

$$|\varphi_i(x) - \varphi_i(y)| = |\varphi_i(x)| \le |\psi_i(x)| \le \frac{1}{\varepsilon} d(x,y)\,.$$

This proves the claim.

We call a collection of functions $\{\varphi_i\}$ as above the *Lipschitz partition of unity associated with the net* N.

Proof of Proposition 1.9. The preceding notation and terminology understood, fix $\varepsilon > 0$ and an ε-net $N = (x_i)$. Define

$$f_\varepsilon(x) = \sum_i \varphi_i(x) f(x_i).$$

Then

$$f_\varepsilon : X \to L^\infty(Y)$$

is Lipschitz, because N is finite. Fix $x \in X$. Let I_x denote the set of those indices i such that $x \in 2B_i$. We find

$$\begin{aligned}
|f_\varepsilon(x) - f(x)| &= |\sum_{i \in I_x} \varphi_i(x)(f(x_i) - f(x))| \\
&\le \max_{i \in I_x} |f(x_i) - f(x)| \\
&\le \omega(2\varepsilon),
\end{aligned}$$

where ω is the modulus of continuity of the uniformly continuous map f. Because $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$, the proposition follows.

1.13. **Remark.** The *modulus of continuity* of a map $f : X \to Y$ is the function

$$\omega = \omega_f : [0, \infty) \to [0, \infty]$$

defined by

$$\omega(t) = \sup\{d_Y(f(x), f(y)) : d_X(x, y) \leq t\} .$$

If X is compact, then the modulus of continuity of every continuous map is bounded and satisfies $\omega(t) \to 0$ as $t \to 0$.

1.14. **Doubling spaces.** The functions φ_i in the Lipschitz partition of unity have Lipschitz constant C/ε, with C independent of ε, if the amount of overlap of the balls $2B_i$ is uniformly bounded.

We call X *doubling* if there exists $C \geq 1$ such that every ball in X can be covered by C balls with half the radius.

1.15. **Proposition.** *A metric space X is doubling if and only if there exist constants $C' > 0$ and $\alpha > 0$ such that*

$$\#\{\varepsilon R\text{-}net\ in\ B_R\} \leq \frac{C'}{\varepsilon^\alpha}, \quad 0 < \varepsilon \leq \frac{1}{2},$$

for every ball B_R of radius R in X, where $\#$ denotes the cardinality. The constants C', α, and the doubling constant C depend only on each other.

Proof. Exercise. □

1.16. **Assouad dimension.** The infimal $\alpha > 0$ such that the condition in Proposition 1.15 holds is called the *Assouad dimension* of X. It is always at least as large as the Hausdorff dimension of the space, and can be strictly larger.

1.17. **Examples.** (a) Every subspace of a doubling space is a doubling space (with the same constant).

(b) $\mathbb{R}^n$ and all its subsets are doubling with constant $C = C(n)$. This is a special case of (a) and (c).

(c) If X carries a nontrivial *doubling (Borel) measure* μ, i.e.,

$$\mu(B(x, 2R)) \leq C\mu(B(x, R))$$

for some $C \geq 1$ and for every ball $B(x, R)$ in X, then X is doubling.

Conversely, if X is a complete doubling space, then X carries a nontrivial doubling measure [77], [51].

(d) Infinite dimensional normed spaces are never doubling. In fact, a complete doubling space is always *proper*, i.e., closed balls in the space are compact.

1.18. **Open problem.** *Which subsets of* $\mathbb{R}^n$ *carry nontrivial doubling measures? Difficult and unsolved even for* $n = 1$.

Read more on doubling spaces in [5], [77], [23], [51], [57], [37], [65], [12], [50].

1.19. **Lipschitz extensions.** The Kuratowski embedding is useful in extending Lipschitz maps. Recall first the *McShane-Whitney extension lemma*:

If $A \subset X$ *and* $f : A \to \mathbb{R}$ *is* L*-Lipschitz, then there exists an* L*-Lipschitz function* $F : X \to \mathbb{R}$ *extending* f*, i.e.,* $F|A = f$.

Proof. The proof is simple. Define

$$f_a(x) = f(a) + L\, d(x, a)\,, \quad a \in A.$$

Then f_a is L-Lipschitz, $f_a \geq f(a)$, and $f_a(x) = f(x)$ if $x = a$. The function

$$F(x) = \inf_{a \in A} f_a(x)$$

is the required extension. □

1.20. **Remark.** For mappings $f : A \to Y$, $A \subset X$, Lipschitz extensions are not always possible simply for topological reasons. These extensions can also fail to exist for metric reasons. For example, consider the (Lipschitz) identity map

$$\{0, 1\} \to ([0, 1], |x - y|^{\frac{1}{2}})\,,$$

and observe that no Lipschitz extension to $[0, 1]$ is possible (cf. 1.10).

On the other hand, if $A \subset X$ and $f : A \to Y$ is L-Lipschitz, then there exists an L-Lipschitz map $F : X \to L^\infty(Y)$ extending f. (Here we understand that a Kuratowski embedding, with a base point, has been fixed for $Y \subset L^\infty(Y)$.) This assertion follows from the McShane-Whitney lemma applied to each *component* of f. Indeed, recall that the elements of $L^\infty(Y)$ can be thought of as bounded sequences indexed by Y. Thus,

$$f(a) = (f^a_y)_{y \in Y}\,, \quad |f^a_y - f^b_y| \leq ||f(a) - f(b)||_\infty \leq L\, d_X(a, b)\,,$$

and we can extend each of the real-valued functions

$$a \mapsto f^a_y\,, \quad y \in Y.$$

Get

$$F(x) = (F^x_y)_{y \in Y}\,, \quad |F^{x_1}_y - F^{x_2}_y| \leq L\, d_X(x_1, x_2)$$

for all $y \in Y$, $x_1, x_2 \in X$, so that

$$||F(x_1) - F(x_2)||_\infty \leq L\, d_X(x_1, x_2)\,.$$

1.21. **Example.** Let $f : X \to \mathbb{R}$ be a function such that

$$|f(x) - f(y)| \le (g(x) + g(y))\, d(x, y)\,, \quad x, y \in X,$$

for some $g : X \to [0, \infty)$. Then

$$X = \bigcup_{i=1}^{\infty} X_i$$

where $f|X_i$ is i-Lipschitz. Indeed, define

$$X_i := g^{-1}\{[0, i/2]\}.$$

If $X = \mathbb{R}^n$, we find by using Rademacher's theorem and the extension lemma that every function f as above is *approximately differentiable* almost everywhere. These observations lead, for example, to a.e. approximate differentiability of Sobolev functions.

Read more in [27], [25], [35].

1.22. **Fréchet embedding.** The Kuratowski embedding has the disadvantage that the receiving space $L^\infty(X)$ depends on X. Already in 1909, Fréchet [28] [1] observed the following fact:

Every separable metric space X isometrically embeds in the Banach space l^∞.

Recall that $l^\infty = L^\infty(\mathbb{N})$ is the space of bounded sequences equipped with the sup-norm.

Proof. Pick a dense countable subset $\{x_0, x_1, \ldots\}$ of X and consider

$$x \mapsto s^x\,, \quad s_i^x = d(x, x_i) - d(x_i, x_0).$$

Note that this is exactly the (unique) extension of the Kuratowski embedding, with base point x_0,

$$\{x_0, x_1, \ldots\} \hookrightarrow L^\infty(\{x_0, x_1, \ldots\}) \simeq l^\infty\,,$$

from a dense set to a complete space. The embedding $x \mapsto s^x$ is isometric because

$$|s_i^a - s_i^b| = |d(a, x_i) - d(b, x_i)| \le d(a, b)\,,$$

with equality obtained when x_i approximates a (or b). □

We have thus obtained a universal Banach space l^∞ which isometrically contains all separable metric spaces. The isometric embedding depends on the chosen dense subset, and on the base point x_0, but is otherwise concretely defined.

Note that the Lipschitz extension lemma 1.19 is valid for functions valued in l^∞, because the sup-norm is used and so component-wise extensions are valid.

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[1] The concept of an abstract metric space was first introduced in [28].

2. Gromov-Hausdorff convergence

Given $X, Y \subset Z$, we define the *Hausdorff distance between X and Y in Z* by

$$d_H(X,Y) := d_H^Z(X,Y) := \inf\{\varepsilon > 0 : X \subset Y(\varepsilon) \text{ and } Y \subset X(\varepsilon)\}\,,$$

where

$$A(\varepsilon) := \{x \in Z : \operatorname{dist}(z, A) < \varepsilon\} = \bigcup_{a \in A} B(a, \varepsilon)$$

is the ε-neighborhood of a set $A \subset Z$.

Obviously

$$d_H(X,Y) = d_H(Y,X)$$

and it is easy to see that

$$d_H(X,Y) \leq d_H(X,W) + d_H(W,Y)\,.$$

Thus, d_H behaves like a metric on all subsets of Z, except that it can take the value ∞, and that $d_H(X,Y) = 0$ does not necessarily imply that $X = Y$. Examples to this effect are easy to give.

2.1. **Proposition.** *We have $d_H(X,Y) > 0$ for closed subsets $X \neq Y$ of Z. In particular, d_H determines a metric on the collection of all compact subsets of a metric space Z.*

Proof. Exercise. □

Gromov-Hausdorff distance is a generalization of the Hausdorff distance; it measures how much two abstract metric spaces deviate from each other. This distance can be defined in several equivalent ways. We will only consider separable spaces, and employ the Fréchet embedding to this end.

The *Gromov-Hausdorff distance* between two separable metric spaces X, Y is

$$d_{GH}(X,Y) := \inf d_H^\infty(\varphi(X), \psi(Y))\,,$$

where

$$d_H^\infty := d_H^{l^\infty}$$

is the Hausdorff distance in l^∞ and where the infimum is taken over all isometric embeddings $\varphi : X \to l^\infty$ and $\psi : Y \to l^\infty$.

Obviously, if X and Y are isometric, then

$$d_{GH}(X,Y) = 0.$$

The converse is not true in general, and it is wise here to restrict to isometry classes of compact spaces. (Note that compact spaces are separable.)

2.2. **Proposition.** *The Gromov-Hausdorff distance determines a metric on the isometry classes of all compact metric spaces.*

The statement needs some explanation. Among compact metric spaces we define an equivalence relation, where $X \sim Y$ if and only if X is isometric to Y. An *isometry class of* X is the equivalence class determined by X under this relation.

Proof. We need to show that $d_{GH}(X, Y) = 0$ implies the existence of a (surjective) isometry $\varphi : X \to Y$. Fix $\varepsilon > 0$ and choose a maximal ε-net

$$N_\varepsilon = \{x_1, x_2, \ldots, x_k\} \subset X.$$

We may assume that $X, Y \subset l^\infty$ and that

$$d_H^\infty(X, Y) < \varepsilon^2.$$

Write

$$|a - b| := ||a - b||_\infty$$

for $a, b \in l^\infty$. For each $i = 1, \ldots, k$ choose $y_i \in Y$ such that

$$|x_i - y_i| < \varepsilon^2,$$

to obtain a map $F_\varepsilon : N_\varepsilon \to Y$. We have

$$\begin{aligned} |F_\varepsilon(x_i) - F_\varepsilon(x_j)| &= |y_i - y_j| \\ &\le |y_i - x_i| + |x_i - x_j| + |x_j - y_j| \\ &\le 2\varepsilon^2 + |x_i - x_j| \\ &\le (1 + 2\varepsilon)|x_i - x_j|. \end{aligned}$$

Similarly,

$$\begin{aligned} |y_i - y_j| &\ge |x_i - x_j| - |x_i - y_i| - |x_j - y_j| \\ &\ge |x_i - x_j| - 2\varepsilon^2 \\ &\ge (1 - 2\varepsilon)|x_i - x_j|. \end{aligned}$$

Moreover, if $y \in Y$, then there are $x \in X$, $x_i \in N_\varepsilon$ such that

$$|y - y_i| \le |y - x| + |x - x_i| + |x_i - y_i| < \varepsilon^2 + \varepsilon + \varepsilon^2 < 2\varepsilon$$

for all small ε. It follows that, for ε small enough, the map $F_\varepsilon : N_\varepsilon \to Y$ is a $(1+10\varepsilon)$-bi-Lipschitz embedding whose image is 2ε-dense in Y, i.e., the 2ε-neighborhood of the image contains Y. The standard proof of the Arzela-Ascoli theorem then gives that $F_\varepsilon \to F_0$ as $\varepsilon \to 0$, where F_0 is an isometry of a dense subset of X onto a dense subset of Y. (One should use increasing nets $N_\varepsilon \subset N_{\varepsilon'}$, $\varepsilon' < \varepsilon$, here.) Such an isometry has an extension $X \to Y$. This completes the proof. $\square$

The following fundamental observation is due to Gromov [33].

2.3. **Gromov's compactness theorem (GCT).** *Let $\mathcal{M} = \mathcal{M}(C, N)$ be a family of metric spaces X such that*

- $\operatorname{diam}(X) \le C$,
- *#(every ε-net in X)$\le$ $N(\varepsilon)$,*

where C is a finite constant and $N : (0, \infty) \to (0, \infty)$ is a positive function. Then every sequence $(X_i) \subset \mathcal{M}$ contains a subsequence that converges in the Gromov-Hausdorff distance to a metric space X_∞. If, in addition, each X_i is compact, then X_∞ is compact.

We also have the following result (partially a corollary to 2.3):

2.4. **Theorem.** *Let $\mathcal{C}$ denote the collection of isometry classes of compact metric spaces. Then $(\mathcal{C}, d_{GH})$ is a contractible, complete, and separable metric space.*

Proof. $\mathcal{C}$ is contractible because, given $X = (X, d)$, the spaces

$$X_t := (X, t\, d)$$

converge in the Gromov-Hausdorff distance to a one point space as $t \to 0$.

Completeness follows from Gromov's compactness theorem, because it easily follows that for a Cauchy sequence $(X_i) \subset \mathcal{C}$,

- $\sup_i \operatorname{diam}(X_i) < \infty$
- $\sup_i \#(\text{every } \varepsilon\text{-net in } X_i) < \infty\,, \quad \varepsilon > 0.$

Finally, to prove separability, observe first that (the isometry classes of) finite spaces are dense in $\mathcal{C}$. On the other hand, given a finite set $X = \{x_1, x_2, \ldots, x_N\}$, the isometry classes of metrics on X are described by a subset of the set of symmetric $N \times N$-matrices, thus by a subset of $\mathbb{R}^{N^2}$. This is a separable space. ☐

Proof of the GCT. Let $(X_i) \subset \mathcal{M}$. We will pass to several subsequences in the course of the proof; these will all be denoted by (X_i).

For each i and each $j = 1, 2, \ldots$ choose increasing maximal 2^{-j}-nets

$$N_{i,j} := N\left(X_i, 2^{-j}\right) \subset N_{i,j+1}.$$

We assume, as we may, that there are infinitely many nets $N_{i,1} \ne \emptyset$. Moreover, we may assume that

$$n_1 := \#N_{i,1} \le N(2^{-1})$$

for all i.

Let $Y_1 = \{1, 2, \ldots, n_1\}$ and choose bijections

$$\psi_i : Y_1 \to N_{i,1} \subset X_i\,.$$

Pull back the metric d_{X_i} to Y_1, to obtain a metric

$$\delta_i(p, q) := d_{X_i}(\psi_i(p), \psi_i(q))\,, \quad p, q \in Y_1.$$

Write

$$Y_{i,1} := (Y_1, \delta_i).$$

We have

$$\delta_i : (Y_1 \times Y_1)\backslash\text{diagonal} \to [2^{-1}, C],$$

so we can find a subsequence of metrics (hence spaces) δ_i, such that

$$\delta_i \to e_1 : (Y_1 \times Y_1)\backslash\text{diagonal} \to [2^{-1}, C]\,,$$

where e_1 is a metric on Y_1.

Next, by passing to a further subsequence, we may assume that

$$n_2 := \# N_{i,2} \leq N(2^{-2})$$

for all i. Extend the bijections ψ_i to bijections

$$\psi_i : Y_2 \to N_{i,2} \subset X_i\,,$$

where $Y_2 = \{1, 2, \ldots, n_1, n_1 + 1, \ldots, n_2\}$. As above, get a convergent sequence of metrics δ_i on Y_2 that converges to a metric e_2 on Y_2, with

$$e_2 : (Y_2 \times Y_2)\backslash\text{diagonal} \to [2^{-2}, C].$$

Continuing in this manner, always passing to a subsequence when necessary, we get metrics

$$e_j : (Y_j \times Y_j)\backslash\text{diagonal} \to [2^{-j}, C],$$

where

$$Y_j = \{1, 2, \ldots, n_j\}\,, \quad n_j := \# N_{i,j},$$

for each i. In sum, the metric spaces

$$Z_j := (Y_j, e_j)$$

have the following property: for each j, there exists X_j from the original sequence, a 2^{-j}-net N_j in X_j, $\# N_j = n_j$, and a bijection

$$\psi_j : Z_j \to N_j$$

satisfying

$$|d_{X_j}(\psi_j(p), \psi_j(q)) - e_j(p, q)| < 10^{-j}$$

for $p, q \in Z_j$. (Instead of 10^{-j}, one can choose, in advance, any positive function of j here.) In particular,

$$\frac{1}{1 + \frac{1}{j}} \leq \frac{d_{X_j}(\psi_j(p), \psi_j(q))}{e_j(p, q)} \leq 1 + \frac{1}{j}.$$

The proof will be completed by two lemmas. It is easy to see that without loss of generality we can assume $n_j \to \infty$.

2.5. **Lemma.** *There exists a metric d on $\mathbb{N}$ such that $Z_j \to (\mathbb{N}, d)$ in d_{GH}.*

2.6. **Lemma.** *If X, Y are two bounded and separable metric spaces and if $\Psi : X \to Y$ is a $(1+\varepsilon)$-bi-Lipschitz bijection, then*

$$d_{GH}(X,Y) \le \varepsilon \max\{\mathrm{diam}(X)\,,\mathrm{diam}(Y)\}.$$

Assume the lemmas. Then

$$d_{GH}(Z_j,N_j) \le \frac{C}{j}.$$

Because also

$$d_{GH}(N_j,X_j) \le \, 2^{-j}$$

we have that

$$d_{GH}(X_j,(\mathbb{N},d)) \le d_{GH}(X_j,N_j) + d_{GH}(N_j,Z_j) + d_{GH}(Z_j,(\mathbb{N},d)) \to 0$$

as $j \to \infty$.

We also remark here that if, in the GCT, all spaces are compact, then the completion of the limit space $(\mathbb{N},d)$ is compact, too, because it is easily seen to be totally bounded. (In fact, every Gromov-Hausdorff limit of finite spaces is totally bounded.)

So it suffices to prove 2.5 and 2.6.

Proof of Lemma 2.5. Note first that

$$Z_j \subset Z_{j+1}$$

isometrically (e_{j+1} extends e_j), so the definition for d on $\mathbb{N}$ is obvious.

Because each Z_j is $(1+1/j)$-bi-Lipschitz to a maximal 2^{-j}-net N_j in X_j, with $N_j \subset N_{j+1}$, it is clear that for

$$p \in \mathbb{N}\backslash Z_j\,,\ p \in Z_k\,,\quad k > j,$$

we have

$$d(p,Z_j) \le d(p,p_{k-1}) + d(p_{k-1},p_{k-2}) + \cdots + d(p_{j+1},p_j)$$

for $p_l \in Z_l$, which is less than

$$2\sum_{k=j}^{\infty} 2^{-k}\,.$$

This expression $\to 0$ as $j \to \infty$, and the lemma follows. □

Proof of Lemma 2.6. We will define a metric d, an extension of both d_X and d_Y, in the disjoint union $Z = X \sqcup Y$ such that the Hausdorff distance between X and Y in Z is less than $C\,\varepsilon$, where

$$C = \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}.$$

This suffices.

To such end, define

$$d(x,y) := \inf_{a\in X}\{d_X(x,a) + d_Y(y,\Psi(a))\} + C\,\varepsilon$$

for $x \in X$ and $y \in Y$. To prove that this is a metric, we need to consider two cases.

Case 1. $x \in X$, and $y, y' \in Y$.

Then

$$\begin{aligned} d(x,y) &\le d_X(x,a) + d_Y(y,\Psi(a)) + C\,\varepsilon \\ &\le d_X(x,a) + d_Y(y,y') + d_Y(y',\Psi(a)) + C\,\varepsilon \end{aligned}$$

for all $a \in X$, so

$$d(x,y) \le d(x,y') + d(y',y).$$

Case 2. $x \in X$, and $y, y' \in Y$.

Then

$$\begin{aligned} d(y,y') &\le d_Y(y,\Psi(a)) + d_Y(\Psi(a),\Psi(a')) + d_Y(\Psi(a'),y') \\ &\le d_Y(y,\Psi(a)) + d_X(a,a') + C\,\varepsilon + d_Y(\Psi(a'),y') \\ &\le d_Y(y,\Psi(a)) + d_X(a,x) + d_X(x,a') \\ &+ C\,\varepsilon + d_Y(\Psi(a'),y') + C\,\varepsilon\,. \end{aligned}$$

By taking the infimum over all $a, a' \in X$, we get

$$d(y,y') \le d(y,x) + d(x,y')$$

as required.

Next, fix $y \in Y$ and take $x = \Psi^{-1}(y)$. Then

$$d(x,y) \le d_X(\Psi^{-1}(y),a) + d_Y(y,\Psi(a)) + C\,\varepsilon$$

for each $a \in X$. In particular, we can take $a = \Psi^{-1}(y)$ so that $d(x,y) \le C\,\varepsilon$. By symmetry,

$$d_{GH}(X,Y) \le C\,\varepsilon.$$

The lemma follows. □

2.7. **Concrete description of Gromov-Hausdorff convergence.** Let X, Y be separable and bounded spaces, and suppose that there exist maximal ε-nets $N(X,\varepsilon)$ and $N(Y,\varepsilon)$ that are bi-Lipschitz equivalent with constant $(1+\varepsilon)$. Then

$$d_{GH}(X,Y) \le \varepsilon\ \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\} + 2\varepsilon\,.$$

In fact, the proof of Lemma 2.6 shows that there is a metric in the disjoint union $Z = X \sqcup Y$ so that the Hausdorff distance between X, Y in Z satisfies the preceding inequality. More precisely, the first part of the proof of Lemma 2.6 goes unchanged, just take the infimum over all $a \in N(X,\varepsilon)$. To compute the Hausdorff distance, we argue

$$\begin{aligned} d_{GH}(X,Y) &\le d_{GH}(X, N(X,\varepsilon)) \\ &\quad + \varepsilon\ \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\} + d_{GH}(N(Y,\varepsilon), Y)\,. \end{aligned}$$

The proof of the following result (stated for simplicity for compact spaces) is a straightforward exercise.

2.8. **Proposition.** *A sequence of compact spaces* (X_i) *converges in* d_{GH} *if and only if there are isometries* $\Psi_i : X_i \to l^\infty$ *such that*

$$d_H^\infty(\Psi_i(X_i), Z) \to 0\,, \quad i \to \infty,$$

for some compact subset $Z \subset l^\infty$.

2.9. **GCT for noncompact spaces.** Recall that a metric space is said to be proper if its closed balls are compact.

Let $((X_i, x_i))$ be a sequence of pointed proper spaces. We say that $((X_i, x_i))$ Gromov-Hausdorff converges to a (pointed proper) space (Z, z) if

$$\overline{B}(x_i, R) \to \overline{B}(z, R)$$

in the Gromov-Hausdorff distance for each $R > 0$.

Here and later

$$\overline{B}(x, R) = \{y \in X : d(x, y) \leq R\}$$

denotes a closed ball.

The GCT together with a standard diagonalization argument gives the following important result.

2.10. **Theorem.** *A sequence of proper, uniformly doubling metric spaces has a Gromov-Hausdorff convergent subsequent.*

By a *uniformly doubling sequence* we mean a sequence of spaces that are all doubling with the same constant (cf. 1.14).

In particular, we have the following corollary.

2.11. **Theorem.** *Let* (X, d) *be a locally compact doubling space, and let* $x_0 \in X$. *Then every sequence* (ε_i), $\varepsilon_i \to 0$, *contains a subsequence* (ε_{i_j}) *such that the sequence* $\left(X, \varepsilon_{i_j}^{-1}\, d, x_0\right)$ *Gromov-Hausdorff converges to some proper, doubling metric space* (W, w).

2.12. **Tangent spaces.** A pointed space $W = (W, w)$ as in 2.11 is called *a tangent space of* X *at* x_0.

Tangent spaces reflect the local (infinitesimal) structure of X. The concept of a tangent space is used in powerful "blow-up" or rescaling methods. For example, the infinitesimal behavior of functions often translates into the global behavior of rescaled limit functions on tangent spaces. In general, a space X as in 2.11 has many tangent spaces at a given point $x_0 \in X$.

Read more in [23], [15], [40], [41], [17], [18], [19].

3. Some fundamental embeddings

We have seen that every separable metric space isometrically embeds in l^∞. This Fréchet embedding has two shortcomings. First, if we want to embed a given metric space "a point at a time", it is not clear that the Fréchet embedding helps here. For example, a given finite set

$$\{x_1, x_2, \ldots, x_n\} \subset X$$

embeds via

$$x_j \mapsto (d(x_j, x_1) - d(x_1, x_0), \ldots, d(x_j, x_n) - d(x_n, x_0), 0, \ldots) \subset l^\infty,$$

while

$$\{x_1, x_2, \ldots, x_n, x_{n+1}\} \subset X$$

embeds via

$$x_j \mapsto (d(x_j, x_1) - d(x_1, x_0), \ldots, d(x_j, x_{n+1}) - d(x_{n+1}, x_0), 0, \ldots) \subset l^\infty,$$

which is in general different.

Second, l^∞ is not separable itself; it would be desirable to have a separable Banach space as a universal target.

This section addresses the two shortcomings.

3.1. **Urysohn universal space.** A metric space (U, e) is called *Urysohn universal* if it is separable and complete, and has the following property: given any finite subspace $X \subset U$ and any one point metric extension of X,

$$(X^*, d^*) = (X \cup \{x^*\}, d^*), \quad d^*|X \times X = e,$$

then there is a point $\Psi(x^*) \in U$ such that

$$e\,(\Psi(x^*), x) = d^*(x^*, x)$$

for all $x \in X$.

The following property of Urysohn universal spaces is clear from the definitions.

3.2. **Proposition.** *Let U be Urysohn universal, and let*

$$X = \{x_1, x_2, \ldots, x_n, \ldots\}$$

be a countable metric space. Then every isometric embedding

$$\Psi_n : \{x_1, x_2, \ldots, x_n\} \to U$$

can be extended to an isometric embedding

$$\Psi_{n+1} : \{x_1, x_2, \ldots, x_n, x_{n+1}\} \to U.$$

In particular, every separable metric space embeds isometrically in U.

The extension property for finite subspaces, stipulated in the definition for Urysohn universal spaces, is stronger than merely receiving isometrically every separable metric space. In fact, we have the following description of Urysohn universal spaces.

3.3. **Proposition.** *Let U be a separable and complete metric space that contains an isometric image of every separable metric space. Then U is Urysohn universal if and only if U has the following transitivity property: every isometry between finite subsets of U extends to an isometry of U onto itself.*

The proof of Proposition 3.3 is left to the reader.

The next theorem, due to Urysohn[2] [73], guarantees that Urysohn universal spaces exist.

3.4. **Theorem.** *A Urysohn universal space exists. Moreover, up to isometry there is only one such space.*

Proof. We prove the existence of a Urysohn universal space. The second assertion is left to the reader.

Given an arbitrary metric space $X = (X, d)$, denote by $E(X)$ the collection of functions $f : X \to \mathbb{R}$ that satisfy both

$$|f(x) - f(y)| \le d(x, y) \le f(x) + f(y)$$

whenever $x, y \in X$, and

$$f(x) = \inf\{d(x, y) + f(y) : y \in Y\}$$

for all $x \in X$, where $Y \subset X$ is a finite set (allowed to depend on f).

A set Y as above is called a *support of* f. For example, if $x_0 \in X$, then the function

$$f_{x_0}(x) := d(x_0, x)$$

belongs to $E(X)$ with support $Y = \{x_0\}$.

Alternatively, $E(X)$ can be described as the collection of all those 1-Lipschitz functions $f : X \to \mathbb{R}$ that are McShane-Whitney extensions of 1-Lipschitz functions (see 1.19) defined on finite subsets (supports) of X and satisfying the condition

$$d(x, y) \le f(x) + f(y)$$

for x and y in the support.

We equip $E(X)$ with the sup-metric,

$$d_E(f, g) := \sup\{|f(x) - g(x)| : x \in X\}\,.$$

It follows from the definitions that d_E is indeed a metric. Moreover, we have that X embeds isometrically in $(E(X), d_E)$ via

$$x_0 \mapsto f_{x_0}\,, \qquad f_{x_0}(x) = d(x_0, x)\,.$$

Note that this embedding is canonical, cf. 1.7.

It is easy to check that

$$d_E(f, x_0) = d_E(f, f_{x_0}) = f(x_0)$$

[2]Pavel Samuilovich Urysohn (1898-1924) drowned in Brittany at the age of 26 while swimming. The cited article [73] was published posthumously.

for each $x_0 \in X$ and $f \in E(X)$, where we have identified $x_0 \in X$ and the function f_{x_0}.

If we denote by $E_n(X)$ the subspace of $E(X)$ consisting of those functions in $E(X)$ that have a support of cardinality at most n, then

$$X \subset E_1(X) \subset E_2(X) \subset \dots ,$$

and

$$E(X) = \bigcup_{n=1}^{\infty} E_n(X) .$$

Finally, starting with an arbitrary separable metric space X, we define by induction

$$X_0 := X , \qquad X_{n+1} := E(X_n) , \quad n = 0, 1, 2, \dots .$$

We then claim that the metric completion $\overline{X}_\infty$ of the space

$$X_\infty := \bigcup_{n=0}^{\infty} X_n$$

is Urysohn universal. Note that the metric d in X_∞ is unambiguously defined, as each X_n sits inside X_{n+1} canonically and isometrically.

To prove the claim, we first observe that $\overline{X}_\infty$ is complete by definition; it is also separable because functions that take rational values in their supports are dense in every $E(X)$. Next, let $\{x_1, \dots, x_n\}$ be a finite subset of $\overline{X}_\infty$, and let d^* be a metric in a set $\{x_1, \dots, x_n, x^*\}$ satisfying $d^*(x_i, x_j) = d(x_i, x_j)$. Let $f : \overline{X}_\infty \to \mathbb{R}$ be the McShane-Whitney extension of the function

$$x_i \mapsto d^*(x_i, x^*) .$$

Next, fix $\epsilon > 0$ and choose points

$$x_1^\epsilon, \dots, x_n^\epsilon \in X_{m_\epsilon}$$

such that

$$d(x_i^\epsilon, x_i) < \epsilon .$$

Then the McShane-Whitney extension $f_\epsilon : X_{m_\epsilon} \to \mathbb{R}$ of the function

$$x_i^\epsilon \mapsto f(x_i^\epsilon)$$

belongs to

$$E(X_{m_\epsilon}) = X_{m_\epsilon + 1} \subset \overline{X}_\infty .$$

It is easy to see from the definitions that $y := (f_\epsilon)$, $\epsilon > 0$, forms a Cauchy sequence in X_∞, so that $y \in \overline{X}_\infty$. Moreover,

$$d(y, x_i) = \lim_{\epsilon \to 0} d(f_\epsilon, x_i) = f(x_i) = d^*(x_i, x^*) ,$$

as required. This completes the proof of the theorem. $\square$

3.5. **Remark.** It is hard to find a proof of Urysohn's theorem in modern sources. There is a proof in [33, pp. 78–79], but crucial details are missing. (See however [29].)

Read more in [33], [29], [76].

The following theorem due to Banach [6] in 1932 provides a concrete separable target for all separable spaces.

3.6. **Theorem.** *Every separable metric space embeds isometrically in* $\mathcal{C}[0,1]$.

Here $\mathcal{C}[0,1]$ is the (separable) Banach space of continuous real-valued functions on the closed unit interval equipped with the sup-norm.

3.7. **Remark.** The Banach space $\mathcal{C}[0,1]$ cannot be Urysohn universal, for it cannot satisfy the transitivity requirement given in Proposition 3.3. This is easily ascertained, by using the fact that every isometric bijection between Banach spaces is an affine map [8, p. 341].

The proof of Theorem 3.6 requires the following auxiliary result, interesting in its own right.

3.8. **Theorem.** *Given a compact metric space* X*, there is a continuous surjection from the Cantor set* C *to* X.

In the statement, the *Cantor set* C is the standard $\frac{1}{3}$-Cantor set. Alternatively, one can think of C as an arbitrary compact, perfect, and totally disconnected space; then the Cantor set refers to the unique such space, up to homeomorphism.

Read more in [23], [47, Chapter 2, Section 6], [44, Chapter XV, Section 8].

Let us assume Theorem 3.8 for the moment.

Proof of Theorem 3.6. Let X be a separable metric space. By the Fréchet embedding, we can assume that $X \subset l^\infty$. Moreover, by considering the completion of the span of a countable dense set in X, we can assume that X itself is a separable Banach space. (Indeed, the finite sums of the form

$$\sum q\,x\,, \quad q \in \mathbb{Q}\,, \quad x \in D,$$

where $D \subset X$ is countable and dense, are dense in the span.)

By the *Banach-Alaoglu theorem*, the closed unit ball U of the dual space X^*, equipped with the weak*-topology, is a compact metrizable space [79, p. 32]. The separability of X is used to infer the last assertion about metrizability.

Recall that

$$U = \{\Lambda : X \to \mathbb{R} \text{ linear with } |\Lambda(x)| \le ||x||_X \text{ for all } x \in X\},$$

and that $\Lambda_i \to \Lambda$ in the weak*-topology if and only if $\Lambda_i(x) \to \Lambda(x)$ for all $x \in X$.

Next, let

$$h : C \to U$$

be a surjective continuous map guaranteed by Theorem 3.8. We define the map

$$X \to \mathcal{C}(C)\,, \quad x(t) = \langle h(t), x\rangle \quad \text{for} \quad t \in C\,,$$

where x is identified with its image, and a bracket notation $\langle \Lambda, x\rangle = \Lambda(x)$ is used for the dual action. Indeed, because $X \subset X^{**}$, we have

$$C \xrightarrow{h} U \xrightarrow{x} \mathbb{R}$$

so that the map $X \to \mathcal{C}(C)$ is simply $x \mapsto x \circ h$.

We observe

$$|x(t) - x'(t)| = |\langle h(t), x - x'\rangle| \le ||x - x'||_X$$

for each $t \in C$. On the other hand, given $x, x' \in X$, $x \ne x'$, there exists, by the Hahn-Banach theorem, an element $\Lambda \in U$ such that

$$\langle \Lambda, x - x'\rangle = ||x - x'||_X.$$

By surjectivity, $\Lambda = h(t)$ for some $t \in C$, and we conclude that

$$||x - x'||_{\mathcal{C}[0,1]} = ||x - x'||_X.$$

It follows that the embedding $X \to \mathcal{C}(C)$ is isometric. By using the linear extension in the complementary intervals of C, we further have an isometric embedding

$$\mathcal{C}(C) \to \mathcal{C}[0,1]\,.$$

The proof is complete. □

Proof of Theorem 3.8. Let X be a compact, and let

$$N_1 = N(X, 2^{-1}) = \{x_1, x_2, \ldots, x_{n_1}\}$$

be a maximal 2^{-1}-net in X. By repeating points, if necessary, we may assume that

$$n_1 = 2^{m_1}$$

for some integer m_1. Here and below we consider finite point sets on X that are like maximal nets, but a point in the set may be counted more than once; we call these sets nets as well.

Consider the m_1th stage in the Cantor construction. Thus, we have $n_1 = 2^{m_1}$ intervals of length 3^{-m_1},

$$I^1_1, I^1_2, \ldots, I^1_{n_1}.$$

Define

$$h_1 : \{I^1_1, I^1_2, \ldots, I^1_{n_1}\} \to N_1\,, \quad I^1_i \mapsto x_i.$$

Next, extend N_1 to a 2^{-2}-net N_2. We group the points in N_2 into n_1 disjoint groups,

$$N_2^1, N_2^2, \dots, N_2^{n_1}, \quad N_2^i \subset B\left(x_i, 2^{-1}\right),$$

each containing

$$n_2 = 2^{m_2}$$

elements (counted with multiplicity). Then we subdivide each interval I_i^1 from the previous construction into $n_2 = 2^{m_2}$ intervals of length $3^{-m_1-m_2}$,

$$I_{i_1}^2, I_{i_2}^2, \dots, I_{i_{n_2}}^2,$$

and define

$$h_2 : \{I_{i_1}^2, I_{i_2}^2, \dots, I_{i_{n_2}}^2\} \to N_2,$$

where I_{ij}^2 maps to the corresponding point (depending on j) in N_2^i.

Continuing in this manner, we get (locally constant) maps

$$h_k : C_k \to N_k,$$

where C_k is the $(m_1 + \cdots + m_k)$th stage in the Cantor construction, N_k is a maximal 2^{-k}-net in X (with points counted with multiplicity). Moreover h_k has the property that

$$|h_k(x) - h_{k+l}(x)| \leq 2 \cdot 2^{-k}.$$

In particular,

$$h(x) = \lim_{k\to\infty} h_k(x)$$

exists for each $x \in C = \bigcap C_k$. This map is clearly continuous (by construction, it is a uniform limit of continuous functions) and surjective. This proves the theorem. □

3.9. **Embeddings in finite dimensional spaces.** Thus far we have only studied embeddings of metric spaces in infinite dimensional (Banach) spaces. The case of finite dimensional targets is much harder, even when we consider bi-Lipschitz embeddings in place of isometries. It is a wide open problem to decide which metric spaces can be embedded bi-Lipschitzly in some Euclidean space. We next discuss some examples along these lines.

3.10. **Theorem.** *Every closed (=compact without boundary) Riemannian manifold embeds smoothly and bi-Lipschitzly in some Euclidean space.*

Proof. Cover X^n by finitely many balls $B_1, B_2, \dots, B_N$ of radius $\varepsilon > 0$ such that for each i there exist 2-bi-Lipschitz diffeomorphisms

$$\varphi_i : 5B_i \to \varphi_i(5B_i) \subset \mathbb{R}^n.$$

(See [39, Chapter 1, Section 6].) Without loss of generality we assume that

$$|\varphi_i(x)| \geq 1$$

for all i and $x \in 5B_i$.

Let $u_i \in C_0^\infty(2B_i)$ be such that $0 \le u_i \le 1$ and $u_i|B_i = 1$, and let $v_i \in C_0^\infty(5B_i)$ be such that $0 \le v_i \le 1$ and $v_i|4B_i = 1$. Then set

$$\varphi(x) := \sum_{i=1}^{N} \varphi_i(x)\, u_i(x) + \sum_{i=1}^{N} \varphi_i(x)\, v_i(x)\,,$$

where we think

$$\varphi : X \to \mathbb{R}^{nN} \times \mathbb{R}^{nN} = \mathbb{R}^{2nN}.$$

Obviously φ is smooth, and hence Lipschitz. We will show that φ is bi-Lipschitz. To this end, an equivalent norm

$$|a| = \sum_{j=1}^{2nN} |a_j|\,, \quad a = (a_1, a_2, \ldots, a_{2nN})\,,$$

can be used in $\mathbb{R}^{2nN}$.

Let us assume first that $d(x, y) > 3\,\varepsilon$. Then there exists i such that $u_i(x) = 1$ and $u_i(y) = 0$. Thus,

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\ge |\varphi_i(x)\, u_i(x) - \varphi_i(y)\, u_i(y)| \\ &= |\varphi_i(x)| \\ &\ge 1 \ge \frac{d(x, y)}{\operatorname{diam} X}. \end{aligned}$$

On the other hand, if $d(x, y) \le 3\,\varepsilon$, then there exists i such that $v_i(x) = 1 = v_i(y)$. Thus,

$$|\varphi(x) - \varphi(y)| \ge |\varphi_i(x) - \varphi_i(y)| \ge \tfrac{1}{2}\, d(x, y).$$

The theorem follows. □

3.11. **Remark.** According to a celebrated and deep theorem of Nash [53], every smooth Riemannian manifold, whether compact or not, embeds isometrically in some finite dimensional Euclidean space. Here, however, the isometry has to be interpreted intrinsically; that is, the image has to be considered as a submanifold of the surrounding Euclidean space.

Read more in [34].

Partition of unity type arguments can be used more generally. The following theorem is due to Aharoni [1]; the proof is due to Assouad [4].

3.12. **Theorem.** *There exists an absolute constant $K \le 12$ such that every separable metric space embeds K-bi-Lipschitzly in the (separable) Banach space c_0.*

Recall that c_0 is the Banach subspace of l^∞ consisting of all bounded sequences (x_i) with $x_i \to 0$ as $i \to \infty$.

Proof. Fix $x_0 \in X$, and write $B(r) = B(x_0, r)$. We claim that there exists a doubly infinite sequence of subsets

$$Q_{k,j} \subset X\,, \quad k \in \mathbb{Z}\,, \quad j \geq 0\,,$$

satisfying the following:

(i) $\bigcup_{j\geq 0} Q_{k,j} = X \backslash B(\frac{3}{2} \cdot 2^{-k})$,
(ii) $\operatorname{diam} Q_{k,j} \leq 2 \cdot 2^{-k}$,
(iii) Given $k \in \mathbb{Z}$ and $x \in X$, $B(x, 2^{-k-1})$ meets only finitely many $Q_{k,j}$.

Thus, the $Q_{k,j}$'s form a sort of *locally finite tesselation* of $X \backslash B(\frac{3}{2} \cdot 2^{-k})$ by sets with diameter not exeeding $2 \cdot 2^{-k}$.

Assume, for a moment, the existence of the $Q_{k,j}$'s and define

$$\varphi_{k,j}(x) = \max\{12 \cdot (2^{-k-1} - \operatorname{dist}(x, Q_{k,j}))\,, 0)\,\}\,.$$

Then

- $\varphi_{k,j}$ is 12-Lipschitz.
- $0 \leq \varphi_{k,j} \leq 12 \cdot 2^{-k-1}$ and $\varphi_{k,j} | B(2^{-k-1}) = 0$.
- Given $x \in X$ and $\varepsilon > 0$, then $\varphi_{k,j}(x) > \varepsilon$ only for finitely many k, j.

Now we index the standard basis of c_0 by $\mathbb{Z} \times \{0, 1, 2, \ldots\}$, to have $(e_{k,j})$, and define

$$\varphi(x) := \sum_{k,j} \varphi_{k,j}(x) e_{k,j}\,, \quad \varphi : X \to c_0\,.$$

Clearly φ is 12-Lipschitz. We will show that φ is 12-bi-Lipschitz. Fix $x \neq y$ and choose $k \in \mathbb{Z}$ such that

$$3 \cdot 2^{-k} < d(x, y) \leq 3 \cdot 2 \cdot 2^{-k}.$$

We may assume that $x \notin B(\frac{3}{2} \cdot 2^{-k})$, so that $x \in Q_{k,j}$ for some j. Thus

$$\varphi_{k,j}(x) = 12 \cdot 2^{-k-1}.$$

On the other hand,

$$d(y, Q_{k,j}) \geq d(y, x) - \operatorname{diam} Q_{k,j} \geq 3 \cdot 2^{-k} - 2 \cdot 2^{-k} = 2^{-k}$$

so that

$$\varphi_{k,j}(y) = 0\,.$$

Thus

$$|\varphi(x) - \varphi(y)| \geq 12 \cdot 2^{-k-1} \geq \frac{12 \cdot 2^{-k-1}}{3 \cdot 2 \cdot 2^{-k}} d(x, y) = d(x, y)\,,$$

which is more than sufficient.

It suffices to construct the sets $Q_{k,j}$. Fix a countable dense set $\{y_0, y_1, \ldots\}$ in X. Set

$$Q_{k,0} = B(y_0, 2^{-k}) \backslash B\left(\tfrac{3}{2} \cdot 2^{-k}\right)\,,$$
$$Q_{k,j} = B(y_j, 2^{-k}) \backslash (B(\tfrac{3}{2} \cdot 2^{-k}) \cup Q_{k,0} \cup \cdots \cup Q_{k,j-1})\,.$$

Only property (iii) requires an argument: Fix $k \in \mathbb{Z}$, $x \in X$, and consider $B(x, 2^{-k-1})$. Arbitrarily close to x there is y_i, so that $B(y_i, 2^{-k}) \supset B(x, 2^{-k-1})$ and so that for no index j after i can $Q_{k,j}$ meet $B(x, 2^{-k-1})$.

The proof is now complete. □

3.13. **Remark.** An easy sharpening of the above argument shows that one can take any $K > 4$ in Theorem 3.12. It is not known what the best constant is. Aharoni [1] shows that for every K-bi-Lipschitz embedding

$$\varphi : l^1 \to c_0$$

one must have $K \geq 2$, where l^1 is the (separable) Banach space of absolutely summable sequences.

3.14. **Assouad embedding.** As mentioned earlier, there is no known characterization of metric spaces that admit bi-Lipschitz embedding in some Euclidean space. The following theorem of Assouad [5], in contrast, gives a beautiful characterization for doubling spaces in terms of embeddings.

3.15. **Theorem.** *A metric space* (X, d) *is doubling if and only if the snowflake space* (X, d^ε) *admits a bi-Lipschitz embedding in some Euclidean space for every* $0 < \varepsilon < 1$.

3.16. **Remarks.** (a) There are examples (none of which are trivial) of doubling spaces that do not embed bi-Lipschitzly in any Euclidean space. Thus Theorem 3.15 is, in a sense, sharp.

Read more in [61], [62], [65], [48], [45], [46].

(b) Theorem 3.15 is sharp also in the sense that isometric embeddings of doubling snowflake spaces in some $\mathbb{R}^n$ do not always exist. Consider, for example, the space

$$X = ([0, 1], |t - s|^{\frac{1}{2}}).$$

Then there is an isometric embedding in a Hilbert space,

$$X \to L^2[0, 1]\,, \quad t \mapsto \chi_{[0,t]}\,,$$

where $\chi.$ denotes the characteristic function of a set, but no such embedding can be found in some $\mathbb{R}^n$.

To see this, fix $N \geq 1$. Consider N orthogonal vectors in $X \subset L^2$,

$$f_1 = \chi_{[0, \frac{1}{N}]}, \dots, f_i = \chi_{[\frac{i}{N}, \frac{i+1}{N}]}, \dots, f_N = \chi_{[\frac{N-1}{N}, 1]}\,.$$

We observe that

$$||f_i||_{L^2} = \frac{1}{\sqrt{N}}\,, \quad ||f_i - f_j||_{L^2} = \frac{\sqrt{2}}{\sqrt{N}}\,,$$

for $i \neq j$. If we now assume that the linear span of X inside L^2 (after an appropriate isometric embedding) has finite dimension n, then upon rescaling the preceding norm expressions, we find points $x_1, \dots, x_N$ in the unit sphere of $\mathbb{R}^n$ with pairwise distance at least $\sqrt{2}$. This is an obvious contradiction for $N \gg n$.

(c) Assouad [5] has shown that $([0,1], |t-s|^\varepsilon)$, $0 < \varepsilon < 1$, embeds bi-Lipschitzly in

$$\mathbb{R}^{[\frac{1}{\varepsilon}]+1}.$$

This is sharp by a result of Väisälä [74]: the image of $[0,1]$ in $\mathbb{R}^n$ under such a map must have Hausdorff dimension $< n$.

(d) One can embed every snowflaked $\mathbb{R}^n$, $(\mathbb{R}^n, |x-y|^\epsilon)$ with $0 < \epsilon < 1$, in the Hilbert space l^2 of square summable sequences. Indeed, by setting

$$x \mapsto f_x\,, \quad f_x(t) := e^{\sqrt{-1}\langle x \cdot t\rangle} - 1\,,$$

we obtain an element f_x of the Hilbert space $L^2(\mathbb{R}^n; \mu)$, where

$$d\mu(t) := c(\epsilon, n)\, |t|^{-2\epsilon - n}\, dt\,, \qquad t \in \mathbb{R}^n\,,$$

and it is straightforward to check that

$$||f_x - f_y||^2_{L^2(\mathbb{R}^n;\mu)} = |x-y|^{2\epsilon}$$

under an appropriate choice for the constant $c(\epsilon, n)$.

This fact was pointed out to me by Eero Saksman. It also follows from more general, similar and rather elementary facts about positive definite kernel functions on metric spaces given in [8, Chapter 8]. Definite functions on metric spaces were introduced long ago by Schoenberg and von Neumann [3][58], [78], who studied isometric embeddability of metric spaces in Hilbert spaces via Fourier analysis.

(e) The study of definite functions as in [8, Chapter 8] also leads to the proof that every snowflaked (separable) Hilbert space $(l^2, |x-y|_2^\epsilon)$ embeds isometrically in l^2. One is led to wonder which Banach spaces have the property that they isometrically or bi-Lipschitzly contain snowflaked versions of themselves. Besides l^2, the space $\mathcal{C}[0,1]$ has this property (isometrically) by Theorem 3.6, and c_0 (bi-Lipschitzly) by Theorem 3.12. I do not know if this question has been studied in the literature.

Proof of Theorem 3.15. Let $N_0 \subset X$ be a maximal 1-net. Then by the doubling condition

$$\# N_0 \cap B(x, 12) \le M$$

for all $x \in X$ and for some M independent of x. We claim that there exists a "coloring map"

$$\kappa : N_0 \to \{e_1, \ldots, e_M\} = \text{standard basis of } \mathbb{R}^M$$

[3] I thank Piotr Hajłasz for these references.

with
$$\kappa(a) \neq \kappa(b) \quad \text{if} \quad d(a,b) < 12\,.$$
To see this, enumerate $N_0 = \{a_0, a_1, \ldots\}$ and suppose that a map
$$\kappa_i : \{a_1, \ldots, a_i\} \to \{e_1, \ldots e_M\}$$
has been defined so that the claim holds for κ_i. Because
$$\#\{a_1, \ldots, a_i\} \cap B(a_{i+1}, 12) \;\leq\; M - 1\,,$$
we can assign a value $\kappa_{i+1}(a_{i+1})$ to extend κ_i so that the claim holds for κ_{i+1}.

We now define a map $\varphi_0 : X \to \mathbb{R}^M$,
$$\varphi_0(x) = \sum_{a_i \in N_0} \max\{2 - d(x, a_i), 0\}\,\kappa(a_i).$$
(Compare the proof of Theorem 3.12.)

Then
$$\frac{1}{2} \cdot 8 < d(x,y) \leq 8$$
implies
$$|\varphi_0(x) - \varphi_0(y)| \geq 1\,,$$
while
$$|\varphi_0(x) - \varphi_0(y)| \leq C \min\{d(x,y), 1\}$$
for every $x, y \in X$, with $C \geq 1$ independent of x and y.

Now we also have that
$$\#N_j \cap B(x, 2^{-j} \cdot 12) \leq M$$
for all $x \in X$, and $j \in \mathbb{Z}$, where N_j is a maximal 2^{-j}-net and we similarly obtain maps
$$\varphi_j : X \to \mathbb{R}^M$$
with the property
$$|\varphi_j(x) - \varphi_j(y)| \geq 1$$
if
$$2^{-j-1} \cdot 8 < d(x,y) \leq 2^{-j} \cdot 8\,.$$

(This follows directly by applying the construction of φ_0 to the metric space $(X, 2^j \cdot d)$.) Moreover,
$$|\varphi_j(x) - \varphi_j(y)| \leq C \min\{2^j\, d(x,y), 1\}\,.$$

Next, consider $\mathbb{R}^{2N}$ with the standard basis $\{\widehat{e}_j\}$ cyclically extended to all $j \in \mathbb{Z}$. We claim that the map
$$\varphi(x) := \sum_{j \in \mathbb{Z}} 2^{-\varepsilon j} \varphi_j(x) \otimes \widehat{e}_j$$
is bi-Lipschitz $(X, d^\varepsilon) \to \mathbb{R}^M \otimes \mathbb{R}^{2N}$, provided N is large enough (depending on the given data only).

Here we have normalized (as we may) the maps φ_j so that $\varphi_j(x_0) = 0$ for a fixed base point $x_0 \in X$.

First note that the above series converges, because

$$|\varphi(x)| = |\varphi(x) - \varphi(x_0)| \leq C \sum_{j\in\mathbb{Z}} 2^{-\varepsilon j} \min\{2^j\, d(x,x_0), 1\} < \infty\,.$$

Next, fix $x, y \in X$ and choose $k \in \mathbb{Z}$ such that

$$2^{-k-1} \cdot 8 < d(x,y) \leq 2^{-k} \cdot 8\,.$$

Then

$$\begin{aligned}
|\varphi(x) - \varphi(y)| &\leq \sum_{j\geq k+1} 2^{-\varepsilon j} |\varphi_j(x) - \varphi_j(y)| + \sum_{j\leq k} 2^{-\varepsilon j} |\varphi_j(x) - \varphi_j(y)| \\
&\leq C \cdot (2^{-\varepsilon k} + 2^{k(1-\varepsilon)} d(x,y)) \\
&\leq C \cdot d(x,y)^{\varepsilon}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\varphi(x) - \varphi(y)| &\geq \Big| \sum_{-N+k<j\leq N+k} 2^{-\varepsilon j} (\varphi_j(x) - \varphi_j(y)) \otimes \widehat{e}_j \Big| \\
&\quad - \sum_{j>N+k} 2^{-\varepsilon j} |\varphi_j(x) - \varphi_j(y)| \\
&\quad - \sum_{j\leq -N+k} 2^{-\varepsilon j} |\varphi_j(x) - \varphi_j(y)| \\
&\geq 2^{-\varepsilon k} |\varphi_k(x) - \varphi_k(y)| - c \cdot 2^{-\varepsilon(N+k)} \\
&\quad - c \cdot 2^{-\varepsilon(-N+k)} 2^{-N+k} d(x,y) \\
&\geq 2^{-\varepsilon k} |\varphi_k(x) - \varphi_k(y)| - \frac{1}{2} \cdot 2^{-\varepsilon k} \\
&\geq c\, 2^{-\varepsilon k} \geq c \cdot d(x,y)^{\varepsilon}\,,
\end{aligned}$$

provided N is large enough. This completes the proof.

4. Strong A_∞-deformations

In this section, we consider metric spaces that result when the standard Euclidean metric on $\mathbb{R}^n$ is deformed by a weight. For smooth positive weights this procedure is called a *conformal deformation*; in our case, the weight is allowed to have singularities in a controlled manner. We follow David and Semmes [22], [59].

4.1. **Metric doubling measures.** Let μ be a nontrivial doubling measure in $\mathbb{R}^n$ (see 1.17 (c)). For $x, y \in \mathbb{R}^n$ set

$$d_\mu(x,y) := \mu(B_{x,y})^{1/n}\,,$$

where

$$B_{x,y} := B(x, |x-y|) \cup B(y, |x-y|)\,.$$

Then d_μ satisfies all the axioms for a metric on $\mathbb{R}^n$ except perhaps the triangle inequality. We call μ a *metric doubling measure* if there exists a constant $C \geq 1$ such that

$$d_\mu(x,y) \leq C \inf \sum_{i=0}^{N} d_\mu(x_i, x_{i+1}),$$

where the infimum is taken over all sequences of points

$$x = x_0, x_1, \ldots, x_{N+1} = y$$

in $\mathbb{R}^n$. If μ is a metric doubling measure, then the above infimum determines a metric δ_μ on $\mathbb{R}^n$ that satisfies

$$\delta_\mu(x,y) \leq d_\mu(x,y) \leq C\,\delta_\mu(x,y)$$

whenever $x, y \in \mathbb{R}^n$.

It is easy to see that every doubling measure on $\mathbb{R}$ is a metric doubling measure. For $n \geq 2$, metric doubling measures possess subtle geometric and analytic features; in particular, not every doubling measure is a metric doubling measure.

Read more in [63], [61], [62].

4.2. **Strong A_∞ weights.** The following theorem was proved by David and Semmes in [22].

4.3. **Theorem.** *Every metric doubling measure μ on $\mathbb{R}^n$, $n \geq 2$, is of the form $d\mu = w\,dx$, where w is an A_∞ weight.*

We omit the proof. The idea behind this result goes back to Gehring's fundamental work [30].

A nonnegative locally integrable function w on $\mathbb{R}^n$ is called a *weight.* A weight w is an A_∞ *weight* if there exist a constant $C \geq 1$ and a positive number $\epsilon > 0$ such that

$$\Big(\frac{1}{|B|}\int_B w^{1+\epsilon}\,dx\Big)^{1/(1+\epsilon)} \leq C\,\frac{1}{|B|}\int_B w\,dx$$

for every ball $B \subset \mathbb{R}^n$. Here and hereafter,

$$|E| := \int_E dx$$

denotes the Lebesgue measure of a measurable set.

The A_∞ weights that arise as densities of metric doubling measures are called *strong A_∞ weights* in the literature. There are many interesting open questions about strong A_∞-weights.

Read more in [64], [37], [10].

4.4. A_1**-weights.** A weight w on $\mathbb{R}^n$ is called an A_1 *weight* if there exists a constant $C \geq 1$ such that

$$\frac{1}{|B|}\int_B w\,dx \;\leq\; C\,\mathrm{essinf}_B w$$

for every ball $B \subset \mathbb{R}^n$. It is not difficult to see that every A_1 weight is a strong A_∞ weight. Let δ_w denote the corresponding metric. Thus,

$$\delta_w(x,y) \leq\; w(B_{x,y})^{1/n} \;\leq\; C\,\delta_w(x,y)\,,$$

for some constant $C \geq 1$ independent of x and y. Here and hereafter, we write

$$w(E) := \int_E w\,dx\,,$$

if w is a weight on $\mathbb{R}^n$.

We will prove the following theorem of Semmes [59].

4.5. **Theorem.** *Let w be an A_1 weight on $\mathbb{R}^n$, $n \geq 2$. Then the metric space $(\mathbb{R}^n, \delta_w)$ admits a bi-Lipschitz embedding in some $\mathbb{R}^N$.*

It is not known how low the receiving dimension N in Theorem 4.5 can be.

4.6. **Open problem.** *Let w be an A_1 weight on $\mathbb{R}^n$, $n \geq 2$. Is $(\mathbb{R}^n, \delta_w)$ bi-Lipschitz homeomorphic to $\mathbb{R}^n$?*

It even seems to be an open problem if one can choose N such that it depends only on n and on the constant in the A_1 condition. See [59, p. 228].

4.7. **Remark.** (a) As explained in [22], [59], [10], an affirmative answer to the question in 4.6 would imply that with every A_1 weight w on $\mathbb{R}^n$ there is associated a quasiconformal homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$C^{-1}J_f(x) \;\leq\; w(x) \;\leq\; C\,J_f(x)$$

for almost every $x \in \mathbb{R}^n$, for some constant $C \geq 1$. Here J_f denotes the Jacobian determinant of the derivative matrix of f.

A theorem of Coifman and Rochberg [20], [68, p. 214] asserts that for every Radon measure ν on $\mathbb{R}^n$, and for every $0 < \delta < 1$, the function

$$w := (M\nu)^\delta$$

is an A_1 weight provided it is finite almost everywhere, where $M\nu$ denotes the Hardy-Littlewood maximal function of the measure ν. In fact, all A_1 weights essentially arise in this manner.

It follows from the Coifman-Rochberg theorem that given any set of Lebesgue measure zero in $\mathbb{R}^n$, there is an A_1 weight that takes the value ∞ (in an appropriate sense) on that set. Thus, an affirmative answer to the above question would imply that given any set of measure zero in $\mathbb{R}^n$, there exists a quasiconformal homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with Jacobian determinant infinite (in an appropriate sense) on that set. Theorem 4.5 implies that a mapping with such Jacobian always can be found, if we allow for a larger class of mappings with range in some higher dimensional Euclidean space.

(b) The metric space $(\mathbb{R}^n, \delta_w)$, and all its bi-Lipschitz images, possesses many good geometric properties, if w is a strong A_∞ weight. For example, it is easy to see that w determines an *Ahlfors n-regular measure* on the space, which means by definition that

$$C^{-1} r^n \leq w(B_w(r)) \leq C r^n$$

for every metric ball $B_w(r)$ of radius $r > 0$ in $(\mathbb{R}^n, \delta_w)$, for some constant $C \geq 1$. In particular, $(\mathbb{R}^n, \delta_w)$ has Hausdorff dimension n and the n-dimensional Hausdorff measure satisfies an Ahlfors n-regularity condition as well.

(c) There exist strong A_∞ weights w on $\mathbb{R}^n$ for each $n \geq 2$ such that the corresponding space $(\mathbb{R}^n, \delta_w)$ is not bi-Lipschitz embeddable in any Hilbert space, or even in any uniformly convex Banach space. See [46], [61].

Proof of Theorem 4.5. We give a reasonably detailed proof, but leave some straightforward checking for the reader. I would like to point out that Semmes proves in [59] a more general theorem, where the conclusion of Theorem 4.5 is established under the hypothesis that w satisfies a stability property in terms of taking powers. A_1 weights have such a stability property. We forgo the general case here.

We now fix an A_1 weight w in $\mathbb{R}^n$, $n \geq 2$, and let $C \geq 1$ denote a generic positive constant that only depends on n and on w.

The A_1 condition readily implies that w satisfies

$$w(x) \leq w^*(x) := \sup_{B, x \in B} \frac{1}{|B|} \int_B w \, dx \leq C\, w(x)$$

for almost every x. Here we use the uncentered Hardy-Littlewood maximal function in the middle (see [67, Ch. I]). This minor smoothing of w is required only to have the sets $\{w^* > \lambda\}$ open for each $\lambda \in \mathbb{R}$. We could also work with convolutions w_ϵ of w, and obtain estimates independent of ϵ. It is somewhat more elementary to use the maximal function, especially because we only use the definition and some of its direct consequences.

Next, we fix a large real number $L > 0$, to be determined later so as to depend only on n and w. Then set

$$\Omega_j := \{x \in \mathbb{R}^n : w^*(x) > L^{jn}\}, \quad j \in \mathbb{Z},$$

and define

$$j_0 := \inf\{j : \Omega_{j+1} \neq \mathbb{R}^n\}\,.$$

It may happen that $j_0 = -\infty$.

As in the proof of Assouad's Theorem 3.15, the desired embedding will be defined rather explicitely by using locally defined building blocks that are maps to some Euclidean space. The A_1 condition is used to control the resulting infinite sum. For the building blocks, we employ a Whitney decomposition in the open sets Ω_j.

We divide each Ω_j, $j > j_0$, into countably many pairwise essentially disjoint cubes $\{Q_{j,k}\}$ such that

$$C^{-1}\mathrm{dist}(Q_{j,k}, \mathbb{R}^n \setminus \Omega_j) \;\leq\; \mathrm{diam}(Q_{j,k}) \;\leq\; \epsilon\,\mathrm{dist}(Q_{j,k}, \mathbb{R}^n \setminus \Omega_j)\,,$$

where $0 < \epsilon < \frac{1}{100}$ is a small constant to be chosen at a later stage. Such a decomposition can be found by appropriately subdividing the cubes in the standard Whitney decomposition (as constructed in [67, p. 16], for example).

If $j_0 > -\infty$, then define

$$M_0 := \inf\,\Big(\frac{w(B)}{|B|}\Big)^{1/n}\,,$$

where the infimum is taken over all balls B such that

$$B \cap (\mathbb{R}^n \setminus \Omega_{j_0+1}) \neq \emptyset\,.$$

Next, for $j > j_0$, set

$$M_{j,k} := \Big(\frac{w(Q_{j,k})}{|Q_{j,k}|}\Big)^{1/n}\,.$$

Note that $M_{j,k}$ is roughly the distance distortion between the metric δ_w and the Euclidean metric at the scale of the cube $Q_{j,k}$.

Fix Lipschitz bump functions $\theta_{j,k}$, with Lipschitz constant not exceeding $2(\mathrm{diam}(Q_{j,k}))^{-1}$, such that

$$0 \leq \theta_{j,k} \leq 1\,, \qquad \theta_{j,k}|2Q_{j,k} = 1\,, \qquad \theta_{j,k}|\mathbb{R}^n \setminus 3Q_{j,k} = 0\,.$$

By setting

$$\varphi_{j,k} := \theta_{j,k}\,(\sum_l \theta_{j,l})^{-1}\,,$$

we obtain a Lipschitz partition of unity for Ω_j,

$$\sum_k \varphi_{j,k} \equiv 1$$

with the Lipschitz condition (cf. 1.12)

$$|\varphi_{j,k}(x) - \varphi_{j,k}(y)| \;\leq\; C\,(\mathrm{diam}(Q_{j,k}))^{-1}|x-y|\,, \quad x, y \in \mathbb{R}^n\,.$$

Further auxiliary mappings are defined as follows. Denote by $q_{j,k}$ the center point of the cube $Q_{j,k}$. If $j_0 > -\infty$, then define

$$f_0 := M_0\,x\,, \qquad \text{if } x \in \mathbb{R}^n \setminus \Omega_{j_0+1}\,,$$

and

$$f_0(x) := M_0 \sum_k \varphi_{j_0+1,k}(x)\, q_{j_0+1,k}\,, \qquad \text{if } x \in \Omega_{j_0+1}\,.$$

If $j_0 = -\infty$, then define $f_0 \equiv 0$. For $j > j_0$, set

$$h_{j,k}(x) := M_{j,k}\,(x - q_{j,k})\,, \qquad \text{if} \quad x \in 3Q_{j,k} \setminus \Omega_{j+1}\,,$$

and

$$h_{j,k}(x) := M_{j,k} \sum_l \varphi_{j+1,l}(x)\,(q_{j+1,l} - q_{j,k})\,, \qquad \text{if} \quad x \in 3Q_{j,k} \cap \Omega_{j+1}\,,$$

and then

$$f_{j,k} := \theta_{j,k}\, h_{j,k}\,.$$

We understand that $f_{j,k}$ is everywhere defined; it vanishes outside $3Q_{j,k}$.

Next we record some estimates for the given functions. Define

$$d_j(x) := \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega_j)\,.$$

In the proof of the ensuing proposition, we use the estimate

$$M_0 \;\le\; L^{j_0+1}\,, \quad \text{if } j_0 > -\infty\,,$$

which easily follows from the definitions.

4.8. **Proposition.**

(1) $\qquad |f_0(x) - M_0\, x| \le\; C\, M_0\, \epsilon\, d_{j_0+1}(x)\,, \quad \text{if } j_0 > -\infty\,,$

(2) $\qquad |f_0(x) - f_0(y)| \;\le\; C\, L^{j_0+1}|x-y|\,,$

(3) $\qquad |f_{j,k}(x)| \;\le\; C\, w(Q_{j,k})^{1/n}\,,$

(4) $\qquad |f_{j,k}(x) - f_{j,k}(y)| \;\le\; C\, L^{j+2}|x-y|$

for every $x, y \in \mathbb{R}^n$ *and* j, k.

Proof. To prove (1) and (2), we may assume that $x \in \Omega_{j_0+1}$. Then

$$\begin{aligned}|f_0(x) - M_0\, x| &= M_0 \Big|\sum_k \varphi_{j_0+1,k}(x)\,(q_{j_0+1,k} - x)\Big| \\ &\le\; C\, M_0\, \epsilon\, d_{j_0+1}(x)\,,\end{aligned}$$

which gives (1).

To prove (2), we consider first the case when

$$|x-y| > \frac{1}{10}\, d_{j_0+1}(x)\,.$$

Then, by using (1),

$$\begin{aligned}|f_0(x) - f_0(y)| &\le\; |f_0(x) - M_0\, x| + M_0\, |x-y| + |f_0(y) - M_0\, y| \\ &\le\; C\, M_0\, |x-y|\,.\end{aligned}$$

Thus (2) follows in this case. We also have that

$$|f_0(x) - f_0(y)| \;\le\; M_0\,|\sum_k (\varphi_{j_0+1,k}(x) - \varphi_{j_0+1,k}(y))(q_{j_0+1,k} - x)|\,;$$

and because only a fixed number of cubes $3Q_{j_0+1,k}$ meet x, we easily see that the last quantity is less than

$$C\,M_0\,|x-y|\,,$$

provided

$$|x-y| \le \frac{1}{10}\,d_{j_0+1}(x)\,.$$

Thus (2) follows in all cases.

Inequality (3) follows from the definitions and from the fact that

$$M_{j,k} \;\le\; C\,w(Q_{j,k})^{1/n}(\mathrm{diam}(Q_{j,k}))^{-1}\,.$$

Estimate (4) requires a little more work. We need the following lemma, which is a rather straightforward consequence of the A_1 condition and Jensen's inequality. We leave the proof to the reader (who may also consult [59, Lemma 7.14]).

4.9. **Lemma.** *Let Q, Q' be two cubes such that $Q \subset \Omega_j$, that*

$$Q' \cap (Q \setminus \Omega_{j+1}) \ne \emptyset\,,$$

and that $Q' \subset 100Q$. Then

$$|\frac{1}{|Q'|}\int_{Q'} \log w\,dx - \frac{1}{|Q|}\int_Q \log w\,dx| \;\le\; n\,\log L + C\,.$$

Now to prove (4), we observe that analogously to (1) and (2), we have the following estimates:

$$|h_{j,k}(x) - M_{j,k}\,(x - q_{j,k})| \le C\,M_{j,k}\,\epsilon\,d_{j+1}(x)\,,$$

and

$$|h_{j,k}(x) - h_{j,k}(y)| \;\le\; C\,L^{j+2}|x-y|\,.$$

The proof of the first inequality if analogous to the proof of (1), and, armed with Lemma 4.9, the proof of the second inequality is analogous to the proof of (2).

Therefore, by using

$$|f_{j,k}(x) - f_{j,k}(y)| \;\le\; |\theta_{j,k}(x)||h_{j,k}(x) - h_{j,k}(y)| + |h_{j,k}(y)||\theta_{j,k}(x) - \theta_{j,k}(y)|\,,$$

we only need the estimate

$$|h_{j,k}(y)| \;\le\; C\,w(Q_{j,k})^{1/n} \;\le\; C\,L^j\,(\mathrm{diam}(Q_{j,k}))\,,$$

which follows from the definitions. The proposition is proved. □

We consider an integer $M = m_1 \cdot m_2$, where the integers m_1 and m_2 will be selected momentarily (and separately).

Arguing as in the proof of Theorem 3.15, we can find a map

$$\kappa : \{Q_{j,k}\} \to \{1, 2, \dots, M\}$$

such that

$$\kappa(Q_{j,k}) = \kappa(Q_{j',k'})$$

implies both that $j \equiv j' \mathrm{mod} m_1$, and that $k = k'$ if also $j = j'$ and

$$\mathrm{dist}(3Q_{j,k}, 3Q_{j',k'}) < \frac{1}{2}\, \mathrm{dist}(Q_{j,k} \cup Q_{j',k'}, \mathbb{R}^n \setminus \Omega_j)\,.$$

The integer m_2 is chosen here large enough so that the second of these implications is possible. Then define, for $i = 1, 2, \dots, M$,

$$f_i := \sum_{\{(j,k):\kappa(Q_{j,k})=i\}} f_{j,k}$$

and

$$g_i := \sum_{\{(j,k):\kappa(Q_{j,k})=i\}} g_{j,k}\,,$$

where

$$g_{j,k} := L^j\, \mathrm{dist}(x, \mathbb{R}^n \setminus 2Q_{j,k})\,.$$

We claim that

$$F := (f_0, f_1, \dots, f_M, g_1, \dots, g_M)\,, \quad F : (\mathbb{R}^n, \delta_w) \to R^{2M+1}\,,$$

is a bi-Lipschitz embedding.

We first show that F is Lipschitz. By Proposition 4.8, estimate (2), f_0 is Lipschitz. To handle the remaining components, it is convenient to consider a fixed pair (f_i, g_i). Given an integer $j_1 > j_0$, we use estimate (4) in Proposition 4.8 and observe that each term in the sum

$$\sum_{j_0<j\le j_1} \sum_k (|f_{j,k}(x) - f_{j,k}(y)| + |g_{j,k}(x) - g_{j,k}(y)|)$$

is bounded by

$$C\, L^{j+2}|x-y|\,.$$

Because both $f_{j,k}$ and $g_{j,k}$ are supported in the cube $3Q_{j,k}$, we also find that only a bounded number of terms are nonzero, for each given j. This implies that the sum has a bound

$$C\, L^{j_1+2}\, |x-y|$$

for large enough L.

Next, we infer from estimate (3) in Proposition 4.8, and from the definitions, that the sum

$$\sum_{j_1<j} \sum_k (|f_{j,k}(x)| + |g_{j,k}(x)|)$$

has a bound

$$C \sum_{j_1<j} w(Q_{j,k})^{1/n} ,$$

where $Q_{j,k} \subset \Omega_j$ is a cube such that $x \in 3Q_{j,k}$. Now the definition for Ω_j implies that

$$\operatorname{dist}(x, \mathbb{R}^n \setminus \Omega_{j+1}) \;\leq\; C\, L^{-1} \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega_j) ,$$

which in turn gives that

$$w(Q_{j+1,k'}) \;\leq\; C\, L^{-\alpha} w(Q_{j,k})$$

if $x \in Q_{j+1,k'} \subset Q_{j,k}$, where $\alpha > 0$ depends only on n and w. (This latter claim follows from the doubling property of w.) Therefore,

$$\sum_{j_1<j} \sum_k (|f_{j,k}(x)| + |g_{j,k}(x)|) \;\leq\; C\, w(Q_{j_1+1,k})^{1/n} ,$$

where either $x \in Q_{j_1+1,k}$ or $x \notin \Omega_{j_1+1}$.

To prove the Lipschitz condition for the pair (f_i, g_i), we use the preceding two bounds as follows. Fix $x, y \in \mathbb{R}^n$ and let j_1 be the largest integer j for which $B(x, 10|x-y|) \subset \Omega_j$. By splitting the sums involving $f_{j,k}$ and $g_{j,k}$ as above, with this choice of j_1, we obtain that

$$|f_i(x) - f_i(y)| + |g_i(x) - g_i(y)| \;\leq\; C\, L^{j_1+2} |x-y| + C\, \delta_w(x,y) .$$

On the other hand,

$$L^j \;\leq\; C\, (w(B_{x,y}))^{1/n} |x-y|^{-1}$$

in this case, whence the Lipschitz condition follows.

Notice that for the Lipschitz condition, we did not need the coloring map κ which was used to split the $f_{j,k}$'s into different generations. This splitting is used next in the proof of the lower estimate

$$|F(x) - F(y)| \;\geq\; C^{-1} \delta_w(x,y) , \qquad x, y \in \mathbb{R}^n .$$

To that end, fix x, y, and define j_1 as above. In particular, we have that

$$|x-y| > \frac{1}{10} d_{j_1+1}(x) .$$

If $j_1 = j_0 > -\infty$, then

$$\begin{aligned} |F(x) - F(y)| \;&\geq\; |f_0(x) - f_0(y)| \\ &\geq\; M_0\, |x-y| - |f_0(x) - M_0\, x| - |f_0(y) - M_0\, y| \\ &\geq M_0 \Big(|x-y| - C\, \epsilon\, (d_{j_0+1}(x) + d_{j_0+1}(y)) \Big) \\ &\geq\; \frac{1}{2}\, M_0\, |x-y| , \end{aligned}$$

by estimate (1) in Proposition 4.8, provided ϵ is chosen small enough. It is also easy to see, by employing Lemma 4.9, that

$$M_0\,|x-y| \;\geq\; C\,\delta_w(x,y)\,,$$

which gives the claim if $j_1 = j_0$.

Assume next that $j_1 > j_0$. Fix a cube Q_{j_1,k_1} that contains x, and let $i = \kappa(Q_{j_1,k_1})$ be the color of the cube. We need to distinguish two cases depending upon whether y is in $2Q_{j_1,k_1}$ or not. Suppose first that $y \in 2Q_{j_1,k_1}$. In this case,

$$\begin{aligned} |F(x)-F(y)| \;&\geq\; |f_i(x)-f_i(y)| \\ &\geq\; |f_{j_1,k_1}(x)-f_{j_1,k_1}(y)| - \sum_{(j,k)\in A} |f_{j,k}(x)-f_{j,k}(y)|\,, \end{aligned}$$

where

$$A = \{(j,k) : (j,k) \neq (j_1,k_1)\,,\ \kappa(Q_{j,k}) = i\}\,.$$

To estimate the first term on the right hand side, we observe that $\theta_{j_1,k_1}(x) = \theta_{j_1,k_1}(y) = 1$, which gives

$$\begin{aligned} |f_{j_1,k_1}(x)-f_{j_1,k_1}(y)| =\; &|h_{j_1,k_1}(x)-h_{j_1,k_1}(y)| \\ \geq\; &M_{j_1,k_1}\,|x-y| - |h_{j_1,k_1}(x) - M_{j_1,k_1}\,(x-q_{j_1,k_1})| \\ &- |h_{j_1,k_1}(y) - M_{j_1,k_1}\,(y-q_{j_1,k_1})| \\ \geq\; &M_{j_1,k_1}\Big(|x-y| - C\,\epsilon\,(d_{j_1+1}(x)+d_{j_1+1}(y))\Big) \\ \geq\; &\frac{1}{2}M_{j_1,k_1}|x-y|\,. \end{aligned}$$

It is again easy to see from the basic (doubling) properties of w, by employing Lemma 4.9, that

$$M_{j_1,k_1}|x-y| \;\geq\; C^{-1}L^{-1}\delta_w(x,y)\,,$$

and it suffices to show that the sum

$$\sum_{(j,k)\in A} |f_{j,k}(x)-f_{j,k}(y)|$$

is small compared to $C^{-1}L^{-1}\delta_w(x,y)$. To do so, observe first that the indices of the type (j_1,k) do not contribute to the sum, which therefore splits into two sums

$$\sum_1 := \sum_{m_1+j_1\leq j}\ \sum_k |f_{j,k}(x)-f_{j,k}(y)|$$

and

$$\sum_2 := \sum_{j_0<j\leq j_1-m_1}\ \sum_k |f_{j,k}(x)-f_{j,k}(y)|\,.$$

By now it is straightforward to verify that

$$\sum_1 + \sum_2 \le C\Big((C\,L^{-\alpha})^{m_1-1} + L^{-m_1}\Big)\,\delta_w(x,y)\,,$$

which means that a large enough choice for m_1 gives

$$|F(x)-F(y)| \ge |f_{j_1,k_1}(x) - f_{j_1,k_1}(y)| \ge C^{-1}L^{-1}\,\delta_w(x,y)\,.$$

Finally, we assume that $y \notin 2Q_{j_1,k_1}$. This time we estimate

$$\begin{aligned}|F(x)-F(y)| &\ge |g_i(x)-g_i(y)| \\ &\ge |g_{j_1,k_1}(x) - g_{j_1,k_1}(y)| - \sum_{(j,k)\in A} |g_{j,k}(x)-g_{j,k}(y)|\,,\end{aligned}$$

where A is as before. Arguing as above, and using the definition for the functions $g_{j,k}$, it is easily ascertained that

$$|g_{j_1,k_1}(x) - g_{j_1,k_1}(y)| \ge C^{-1}\delta_w(x,y)\,,$$

and that the sum

$$\sum_{(j,k)\in A} |g_{j,k}(x)-g_{j,k}(y)|$$

is small in comparison.

This completes the proof of Theorem 4.5.

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