Visible and Nonexistent Trees of Mandelbrot Sets

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Virpi Kauko

List of included articles

This thesis consists of an introductory part and the following two publications:

[K1] **Trees of Visible Components in the Mandelbrot Set**
Virpi Kauko
Fundamenta Mathematicae **164** (2000), pp. 41–60

[K2] **Shadow Trees of Mandelbrot Sets**
Virpi Kauko
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University of Jyväskylä, 2002

Mandelbrotin joukon näkyvät ja olemattomat puut

Kaikki tietävät, ettei lohikäärmeitä ole olemassa. Mutta vaikka tämä yksinkertaistettu muotoilu saattaakin riittää maallikolle, se ei tyydytä tieteellisesti ajattelevaa mieltä.

– Stanislaw Lem, *Kyberias* [14]

Tämä tohtorinväitöstutkielma koostuu ohessa mainituista kahdesta artikkelista [K1] ja [K2] sekä tästä yhteenvedosta.

Mandelbrotin joukot ovat fraktaalisia tasokuvioita, joilla on keskeinen merkitys tutkittaessa kompleksitason kuvausten dynamiikkaa. Kun $d \geq 2$ on kokonaisluku, d-asteinen *Mandelbrotin joukko* $\mathcal{M}^d$ määritellään niiden parametrien c joukoksi, joita vastaavien polynomien $z \mapsto z^d + c$ iteroinnissa origon rata pysyy rajoitettuna. Joukko on kompakti, yhdesti yhtenäinen ja osittain järjestetty; sisuskomponentit liittyvät toisiinsa puumaisesti haarautuvina ketjuina. Kaikki ketjut yhtyvät suurimmassa komponentissa, keskusepisykloidissa, joka sisältää origon.

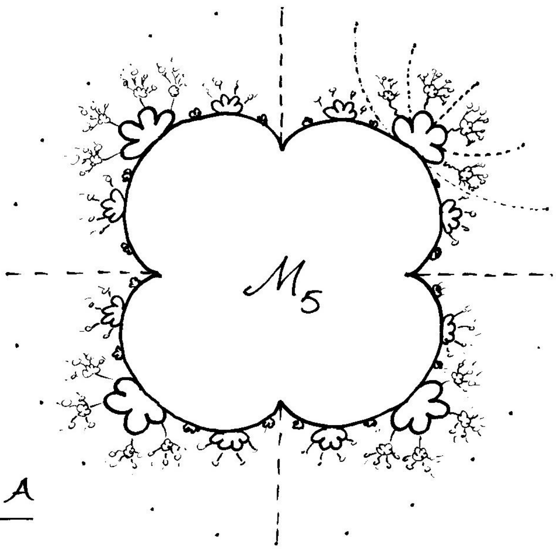


Fig. A

Tässä työssä tutkitaan joukon $\mathcal{M}^d$ puurakennetta kombinatoriselta kannalta, erityisesti Laun ja Schleicherin kehittämän teorian [12] pohjalta. Annetun parametrin origoon yhdistävästä ketjusta – eli sen *alapuolelta* – voidaan löytää mielivaltaisen monta komponenttia Laun–Schleicherin algoritmilla, joka perustuu siihen että kuhunkin parametriin liittyy kompleksidynaamisesti luonnollisella tavalla symboleista $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{d-1}, \mathbf{0}$ koostuva jono (joka ei ala symbolilla $\mathbf{0}$). Hyperbolisten sisuskomponenttien lohkot sekä näiden reunoilla olevat paraboliset pisteet ovat kombinatorisessa tarkastelussa erityisen tärkeitä, sillä niitä on numeroituva määrä ja niihin liittyvät symbolijonot ovat jaksollisia.

Annetusta hyperbolisesta lohkosta $\mathcal{K}$ *näkyvät* komponentit muodostavat numeroituvan määrän *näkyviä puita,* joissa kussakin on äärellinen määrä komponentteja. Samasta kantalohkosta kasvavat puut ovat usein keskenään kombinatorisesti (tai topologisestikin) ekvivalentteja, mutta tästä säännöstä, ns. *translaatioperiaatteesta,* on lukuisia poikkeuksia. Artikkelissa [K1] esitetään eräs vastaesimerkki neliöpolynomien tapauksessa ($d = 2$): kantakomponentin $\overline{\mathbf{1101110}}$ yhdessä puussa on yksi näkyvä komponentti vähemmän kuin toisessa (ks. 2.2 ja kuva E). Sen sijaan pätee heikompi tulos, jonka mukaan puiden latvat eli pienijaksoisimmat hyperboliset komponentit noudattavat tätä periaatetta. Todistuksen sivutuotteena saadaan tulos, joka rajoittaa komponenttien ”leveyttä”.

Näkyvien puiden kombinatorinen epäekvivalenssi liittyy siihen seikkaan, että jokaisella symbolijonolla ei ole vastaavaa Mandelbrotin joukon parametria; toisin sanoen jotkut jonot liittyvät *olemattomiin* parametreihin. LS-algoritmi toimii kuitenkin kaikilla symbolijonoilla, riippumatta siitä onko vastaava parametri olemassa vai ei.

> *Etevä tutkija Cerebron tarttuikin ongelmaan analyyttisesti ja löysi kolme erilaista lohikäärmetyyppiä: myyttisen, kimeerisen ja puhtaasti hypoteettisen. Ne olivat kaikki olemattomia, mutta kukin oli olematta aivan eri tavalla.*
>
> – S. Lem [14]

Itse asiassa symbolijonot muodostavat abstraktin metrisen avaruuden Σ^d, jossa LS-algoritmin avulla voidaan määritellä osittainen järjestysrelaatio. Samaa menetelmää laajentamalla saadaan myös välttämätön ehto jonoille, jotka voivat esiintyä annetun kantalohkon *yläpuolella* – eli origosta poispäin. Tällöin löydetään kaikki ”potentiaaliset” näkyvät puut; tämä *puunrakennusalgoritmi* kehitetään artikkelissa [K2]. Tämäkin algoritmi toimii riippumatta parametrien olemassaolosta, ja se antaakin myös olemattomia komponentteja. Edellä mainitussa [K1]:n esimerkissä se antaisi myös ”puuttuvan” komponentin symbolijonon $\overline{\mathbf{11011100}}$.

Symboliavaruudella Σ^d on täten hyvin määritelty puurakenne, ja se sisältää aidon osajoukon, joka on kombinatorisesti ekvivalentti tekijäavaruuden $\mathcal{M}^d/\simeq$ kanssa (missä ekvivalenssirelaatio $\simeq$ samaistaa saman symbolijonon omaavat parametrit). Siellä pätee translaatioperiaatteen vahva versio: annetusta jaksollisesta jonosta kasvavat puut ovat keskenään kombinatorisesti ekvivalentteja. Puissa voi silti olla ”varjo-oksia” (joille siis ei ole vastinetta oikeassa Mandelbrotin joukossa).

Eri symboliavaruudet ovat sisäkkäisiä, sillä merkeistä $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{d-1}$ koostuva jono on aina myös d:tä korkeampiasteisten symboliavaruuksien alkio. Olemattomuus ikäänkuin periytyy ylöspäin siinä mielessä, että mikäli jono vastaa (tietyllä tavalla) olematonta $\mathcal{M}^d$:n osaa, niin se ei toteudu myöskään $\mathcal{M}^{d+1}$:ssä. Toisaalta symbolijonot voidaan jakaa perheisiin siten, että jokaisesta perheestä jokin jono toteutuu riittävän korkea-asteisissa Mandelbrotin joukoissa.

> *Olettakaamme esimerkiksi, että on järjestetty lohikäärmeenmetsästys. Joukko metsästäjiä piirittää otuksen aseet tanassa, mutta löytää vain palaneen läikän maata sekä hajun josta ei voi erehtyä: lohikäärme on, kokiessaan tulleensa ahdistetuksi, livahtanut todellisesta kuvitteelliseen avaruuteen.*
>
> – S. Lem [14]

1 Introduction

Everyone knows that dragons don't exist. But while this simplistic formulation may satisfy the layman, it does not suffice for the scientific mind.

– Stanislaw Lem, *The Cyberiad* [14]

The Mandelbrot sets are fractal, compact, simply connected, and partially ordered sets in the complex plane; the interior components are linked to each other in chains starting from the origin and branching like trees. The Mandelbrot sets are central objects in the field of complex dynamics; they are defined by polynomials (of various degrees) but one comes across them when iterating many kinds of functions.

In this work, however, we study their tree structures from a combinatorial point of view, mainly based on the theory developed in [19], [9], [7],[16], [1], [12], [2], [17]. The Mandelbrot set turns out to be equivalent to a proper subset of an abstract symbol space with a similar tree structure. However, this symbol space is more regular in the sense that certain trees are generally homeomorphic to each other, whereas in the actual Mandelbrot set there are exceptions: some trees are "missing". We develop an algorithm to construct local trees from a given starting point and find some conditions to determine whether a given symbolic sequence is realized by some element of a Mandelbrot set. In particular, we find classes of sequences some of which are never realized and some are realized by Mandelbrot sets of sufficiently high degrees.

Before discussing the content of the thesis in more detail (starting at §2) we provide a short survey of the backgroud theories, definitions and notation. For a more thorough introduction, see for example [6].

1.1 Dynamical and Parameter Planes

Iteration of a complex analytic function f yields a sequence $\{z, f(z), f^2(z), \dots\}$, the *orbit* of the initial point z. The point z is *stable* if it has an open neighborhood whose points stay close to each other in iteration (e.g., they all tend to a same periodic orbit, or the infinity). Otherwise z is *chaotic*, which means that orbits of nearby points escape far from z's orbit (e.g., one tends to a five-periodic orbit, another to a fixed point, and yet another to infinity). Stable points make up the *Fatou set* of f and chaotic points make up its *Julia set.* [5] [3]

Here the function to be iterated is a polynomial of the form $P_c : z \mapsto z^d + c$. The Julia sets of polynomials are either simply connected or have infinitely many components. The Julia set can also be defined as the boundary of the *filled-in Julia set*, which consists of the points z whose orbits stay bounded.

The *Mandelbrot set* of degree $d \geq 2$, denoted by $\mathcal{M}^d$, is defined as the set of parameters c for which any of the following equivalent conditions holds:

- Origin's orbit under iteration of the polynomial P_c (i.e., the sequence $\{0, c, c^d + c, (c^d + c)^d + c, \dots\}$) is bounded.
- Julia set of P_c is connected.
- Julia set of P_c contains the origin (the only critical point of P_c).

Julia sets live in the dynamical planes (of points z) of the polynomials, while the Mandelbrot set lives in the parameter plane (of points c). Figure A shows a sketch of the Mandelbrot set of degree five.

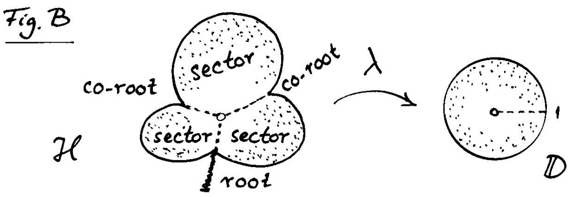
1.2 Hyperbolic Components and Sectors

The *multiplier* λ of a k-periodic orbit $\{P_c(z), \dots, P_c^k(z) = z\}$ is the derivative of the iterated map P_c^k at the periodic point z. The orbit is defined

- *attracting* if $|\lambda| < 1$,
- *neutral* if $|\lambda| = 1$, or $\lambda = e^{i2\pi\varphi}$; $\begin{cases} \textit{parabolic} \text{ if } \varphi \in \mathbb{Q} \\ \textit{irrationally neutral} \text{ otherwise} \end{cases}$
- *repelling* if $|\lambda| > 1$.

Any polynomial with just one critical point can have at most one non-repelling orbit [5, III.2] [15, §9,§10]. The set of parameters c whose polynomials have attracting orbits is an open subset of Mandelbrot set's interior, and its components are called *hyperbolic components*. The period of the attracting orbit is constant in each hyperbolic component, and this number is the *period* of the component.

The multiplier map $c \mapsto \lambda$ is an analytic $(d-1)$-to-one map from each component to the unit disk (hence conformal in the quadratic case). *Internal rays* are the preimages of the radial lines. In particular, the preimage rays of the positive real axis divides the component into $d-1$ *hyperbolic sectors* (see Figure B). These rays meet at the *center* of the component, the preimage of the origin. The polynomial at the center has a *superattracting* periodic orbit containing the origin. (In a neighbourhood of a superattracting fixed point, the map $z \mapsto \lambda$ is conformally conjugate to $z \mapsto z^{d-1}$). The map $c \mapsto \lambda$ extends to the boundary continuously but not smoothly: the preimage of the circle makes sharp cusps inwards at the preimages of 1. One of these points, the *root*, connects the component to the rest of $\mathcal{M}^d$; the other $d-2$ are the *co-roots*.



If the root of a component $\mathcal{H}$ is on the boundary of another hyperbolic sector $\mathcal{K}$, then $\mathcal{H}$ is a *satellite* of $\mathcal{K}$, otherwise $\mathcal{H}$ is *primitive*. The period of the satellite $\mathcal{H}$ is a multiple of the period of the base $\mathcal{K}$. Satellites are attached to the other parabolic points on $\partial\mathcal{K}$, i.e., ones with multiplier $\lambda = e^{i2\pi p/q}$ such that $0 < p/q < 1$. [9], [16] (Some satellites of the central component of $\mathcal{M}^5$ can be seen in Figure A.)

If the famous conjecture of local connectedness of the Mandelbrot set ("MLC") is true, then there are no other kinds of interior components [9].

1.3 External Rays

Information about the Julia sets and the Mandelbrot sets can be found studying the points *outside* them.

The exterior of the Mandelbrot set can be mapped conformally to the exterior of the unit disk by a map Φ. This proves that $\mathcal{M}^d$ is simply connected; Φ is like the Riemann Mapping, only its domain is a neighborhood of infinity as opposed to a finite point. The preimage of each radial line at direction $\theta \in \mathbb{R}/\mathbb{Z}$ is called *external parameter ray* at angle θ.

The conformal map is defined by $\Phi(c) = \phi_c(c)$, where the Böttcher coordinate ϕ_c is a mapping in the dynamical plane of each parameter c [9]. Preimages of radial lines under the map ϕ_c are the *dynamical rays* of c, or the external rays of its filled-in Julia set.

External rays accumulate to the boundary of the Mandelbrot set. If for some θ there is a unique limit point, then that ray *lands*. Two or more rays landing at one point then "pinch" off the set a part that is connected to the origin only via this common landing point [8]. The same is true for connected Julia sets.

All rays with rational angles land, some rays with irrational angles also land; for some parameters c all dynamical rays land, but for other parameters some irrational rays do not land. (That every *parameter* ray lands, the landing point depending continuously on the angle, would be an equivalent condition to MLC [9].)

1.4 Orbit Portraits

The combinatorial approach to $\mathcal{M}^d$ is based on the fact that every polynomial maps its dynamical rays to each other so that their angles get multiplied by d. Therefore we can study the iteration of the d-tupling (modulo 1) map σ_d on the circle as a "model" of iteration of polynomials in the complex plane.

If some dynamical rays with rational angles land at a periodic orbit $\mathcal{O} = \{z_1, \ldots, z_k\}$, and A_j consists of all external angles of z_j, then

- Each A_j has the same number, ν, of angles.
- The map $\sigma_d : A_j \to A_{j+1}$ is bijective and preserves the cyclic order of angles.
- The sets A_j are pairwise unlinked (i.e., any pair of them are contained in disjoint intervals of the circle).
- Every angle in $A_1 \cup \cdots \cup A_k$ has the same period, qk with some $q \in \mathbb{N}$.

The collection $\Theta = \{A_1, \ldots, A_k\}$ is the *orbit portrait* of $\mathcal{O}$. Every collection satisfying the conditions above occurs as the orbit portrait of some polynomials. An *essential* portrait is one with $\nu \geq 2$ or $\Theta = \{\{0\}\}$. A portrait is *primitive* if $q = 1$ and *satellite* otherwise; satellite portraits contain just one cycle of angles (so $\nu = q \geq 2$), whereas essential primitive portraits contain two cycles (so $\nu = 2$). Each A_j cuts the circle into ν intervals, and the unique shortest of the $k\nu$ intervals is the *characteristic interval* $[\theta_1, \theta_d]$ of Θ.

The theory of orbit portraits was developed by Milnor [16] in the quadratic case; generalizations to higher degrees are found in [10].

1.5 Kneading Sequences

The *kneading sequence of an angle* $\theta \in \mathbb{R}/\mathbb{Z}$,

$$K_d(\theta) = a_1 a_2 \ldots \in \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{d-1}, \, {}^{\mathbf{1}}_{\mathbf{0}}, \, {}^{\mathbf{2}}_{\mathbf{1}}, \ldots, \, {}^{\mathbf{0}}_{\mathbf{d-1}}\}^{\mathbb{N}},$$

is defined (according to [1], [19], [2], [12], [17] etc.) as follows: The d preimages $\frac{\theta+s}{d}$ ($s \in \mathbb{Z}_d$) of θ under the map σ_d divide the circle into d equal sectors. One of them contains $0 = 1$ and is labeled **0**, the others are labeled counter-clockwise **1** through $\mathbf{d-1}$. Now the iterates of θ travel around the circle hitting a sequence of sector labels on its way:

$$K_d(\theta)_n := \begin{cases} \mathbf{s} & \text{if } \sigma_d^{n-1}(\theta) \in \left(\frac{\theta+s-1}{d}, \frac{\theta+s}{d}\right) \\ \genfrac{}{}{0pt}{}{\mathbf{s+1}}{\mathbf{s}} & \text{if } \sigma_d^{n-1}(\theta) = \frac{\theta+s}{d} \end{cases}$$

(Kneading sequences of angles never start with **0**, because $\frac{\theta}{d} < \theta < \frac{\theta+d-1}{d}$ for all $0 < \theta < 1$.) An angle $\theta \in \mathbb{R}/\mathbb{Z}$ is k-periodic under the d-tupling (modulo 1) map on the circle if and only if it is of the form $\theta = t/(d^k - 1)$, where $t \in \mathbb{N}$.

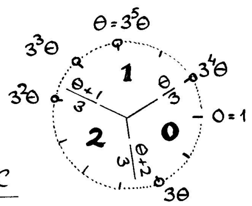


Fig. C

EXAMPLE 1 The angle $\theta = 3/11$ is five-periodic under tripling because $\sigma_3 : \frac{3}{11} \mapsto \frac{9}{11} \mapsto \frac{27}{11} = \frac{5}{11} \mapsto \frac{4}{11} \mapsto \frac{1}{11} \mapsto \frac{3}{11}$; note that $\frac{3}{11} = \frac{66}{3^5-1}$.
Its kneading sequence is $K_3(\frac{3}{11}) = \overline{\mathbf{1021}^{\mathbf{1}}_{\mathbf{0}}}$.

LEMMA 2 *When the angle θ moves counter-clockwise around the circle, the n^{th} entry in its kneading sequence changes from s to $s+1$ precisely when θ crosses a rational angle of the form $(rd+s)/(d^n-1)$.*

The proof (e.g., in [K1, 3.3, 3.4] for $d = 2$ and [K2, 2.6] for $d \geq 2$) is a simple application of definitions, but it has important consequences: for one, the two limit sequences

$$K_d^{\pm}(\theta) := \lim_{\varepsilon \to 0} K_d(\theta \pm \varepsilon) \in \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{d-1}\}^{\mathbb{N}}$$

exist for every $\theta \in \mathbb{R}/\mathbb{Z}$. For non-periodic angles they are equal; if θ is k-periodic, they differ exactly at indices nk for every $n \in \mathbb{N}$ so that $K_d^+(\theta)_{nk} = K_d^-(\theta)_{nk} + 1$.

1.6 Structure and Wakes

Identifying angles with equal kneading sequences in a certain fashion gives rise to the *Abstract Mandelbrot Set*, or "pinched-disk" model [8]. This set is homeomorphic to the boundary of the actual Mandelbrot set in case it is locally connected – otherwise it is homeomorphic to a factor space where certain parameters in $\mathcal{M}^d$ have been identified. A detailed discussion of this theory in the quadratic case is in [11].

The landing pattern of (rational) external rays in the parameter plane is proved (originally in the quadratic case in [9], using orbit portraits in [16], generalized for higher degrees in [10]) as the *Structure Theorems*, which include the following statements:

- External rays with periodic angles land at boundaries of hyperbolic components with the same period.
- Each hyperbolic component has exactly d rays of its own period landing at its boundary, two at its root and one at each co-root.

The pair of rays landing at the root of some component $\mathcal{C}$ divide the parameter plane into two domains, separating the *wake* of $\mathcal{C}$ from the origin. Denoting these rays' angles by θ_1, θ_d, every polynomial with parameter in the wake has a repelling orbit whose portrait's characteristic interval is $[\theta_1, \theta_d]$, and the root polynomial has a parabolic orbit with the same portrait.

The external rays landing at the co-roots, together with the internal rays separating the sectors of the component, then divide the wake into $d-1$ *sector wakes.*

1.7 Visible Trees

Each pair of wakes is obviously either disjoint or nested. Since each parameter c is contained in a well-defined family $U(c)$ of nested wakes, the Structure Theorems give rise to a *partial ordering* relation and an *equivalence* relation:

- $c \prec c' \iff U(c) \subset U(c')$ (c' is above c)
- $c \sim c' \iff U(c) = U(c')$

By identifying parameters that are not separated by rays with periodic angles we get rid of the problem with MLC; the word "parameter" may now refer to a "weird" component, if such exist, as well as a single point or a whole hyperbolic sector. The factor space $\partial\mathcal{M}^d/\sim$ is topologically equivalent to the Abstract Mandelbrot set [11].

Given two parameters $\mathcal{C} \prec \mathcal{A}$, the collection of all hyperbolic sectors $\mathcal{B}$ between them (i.e., $\mathcal{C} \prec \mathcal{B} \prec \mathcal{A}$) is called the *combinatorial arc*. If $\mathcal{A}$ is a hyperbolic sector or component, and none of the sectors between the two has period less than $\mathcal{A}$, then $\mathcal{A}$ is *visible* from $\mathcal{C}$ [12].

The set of hyperbolic components visible from a *base* sector are arranged in *visible trees*. Each tree $\mathcal{T}_{p/q}$ consists of a satellite component, *stem*, with rootpoint at internal angle p/q, and a finite number of primitive components above it. The tree $\mathcal{T}_{1/2}$ is called the *main tree*.

If a homeomorphism maps a visible tree $\mathcal{T}_{p_1/q_1}$ into another, $\mathcal{T}_{p_2/q_2}$, then the two trees are

- *topologically equivalent* if the homeomorphism preserves the embedding of the trees into the parameter plane (preserving or reversing orientation);
- *combinatorially equivalent* if the homeomorphism maps each hyperbolic sector with period n to a sector with period $(q_2 - q_1)k + n$, for some $k \in \mathbb{N}$.

Neither of these conditions imply the other. Combinatorial equivalence allows parts of the tree to be permuted around a branching point. In this work, we are only interested in combinatorial equivalence, which can be dealt with in terms of the symbolic dynamics rather than analytic or topological methods. Our central idea is to extend the tree structure for symbolic sequences that are not even realized by parameters in the complex plane, so it would not even make sense to talk about embedding the trees into the plane.

If the base sectors of two combinatorially equivalent trees have equal periods, then this common period k' must equal the factor k: any homeomorphism between two trees must map stems into stems, whose periods are $q_1 k'$, $q_2 k'$, so now (assuming $q_1 \neq q_2$) the definition implies $(q_2 - q_1)k + n = (q_2 - q_1)k + q_1 k' = q_2 k' \implies k = k'$.

CONJECTURE 3 (TRANSLATION PRINCIPLE) [12, 8.7]
Any two visible trees of a given base sector are combinatorially equivalent.

This statement is not true [K1] without further assumptions [12, 10.2] [K2]. The following weaker statement is true in general (in the quadratic case):

THEOREM 4 (PARTIAL TRANSLATION PRINCIPLE) [11, 1.24]
Let $\mathcal{C}$ be any hyperbolic sector in $\mathcal{M}^2$. Then all its visible trees $\mathcal{T}_{p/q}$, except perhaps $\mathcal{T}_{1/2}$, are combinatorially and topologically equivalent to $\mathcal{T}_{1/3}$.

1.8 Internal Address and the Lau–Schleicher Algorithm (LSA)

Every parameter in $\partial\mathcal{M}^d$ is the landing point (or at least accumulation point) of some external rays. If these rays have nonperiodic angles, they all have the same kneading sequence in $\{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{d-1}\}^{\mathbb{N}}$; we define this sequence as the kneading sequence of that parameter (class).

Parameters at the endpoints of periodic rays can also be given kneading sequences. It follows from the structure theorems (cf. 1.6) that $K_d^+(\theta) = K_d^-(\varphi)$ for any two angles $0 < \theta < \varphi < 1$ that are not separated by a wake boundary; in particular, for the two external angles of any hyperbolic sector.

Denoting the sectors of a given hyperbolic component by $\mathcal{H}^1, \ldots, \mathcal{H}^{d-1}$ and the angles by $\theta_1, \ldots, \theta_d$ (counter-clockwise from the root), we may thus define

$$K_d(\mathcal{H}^s) := K_d^+(\theta_s) = K_d^-(\theta_{s+1})$$

as the *kneading sequence of the hyperbolic sector*, and

$$\widehat{K}_d(\mathcal{H}) := K_d^-(\theta_1) = K_d^+(\theta_d)$$

as the kneading sequence of the rootpoint of the component, or the *root kneading sequence* of the component and its sectors.

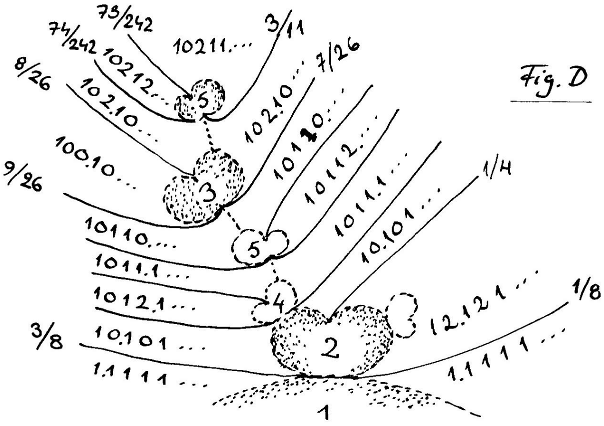
The *internal address* of any parameter $c \in \mathcal{M}^d$ is the sequence of integers

$$A(c) := n_1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \ldots$$

defined as follows: $n_1 = 1$, $c_1 = 0$. Among all pairs of periodic parameter rays separating c from c_j, exactly one pair has minimal period, n_{j+1}, and they land at a parameter c_{j+1}. The point c_j is the root of some hyperbolic component $\mathcal{H}_j$, and c sits in the wake of its sector number s_j. If $\mathcal{A}$ is a hyperbolic sector, the address ends with its period n_k and sector number s_k; otherwise it is infinite.

Given two parameters, $\mathcal{C} \prec \mathcal{A}$, compare $K_d(\mathcal{A}) = a_1 a_2 \ldots$ to $K_d(\mathcal{C}) = c_1 c_2 \ldots$; suppose they first differ at index n by $s := a_n - c_n \in \mathbb{Z}_d \setminus \{0\}$. Then n must be the minimal period of rational rays separating the two sectors; a lemma by Lavaurs [13] [12, 3.8] tells that this set of d rays is unique. It must land at the boundary of some hyperbolic component, $\mathcal{B}$, which hence must contain the unique sector in the combinatorial arc between $\mathcal{C}$ and $\mathcal{A}$ with the smallest period (thus, Lavaurs' lemma guarantees that internal address is well defined). The number of this sector is s because it tells how many rays of period n one must cross when walking around $\mathcal{B}$. Now the n-periodic kneading sequence of $\mathcal{B}$ is obtained by repeating the word $a_1 \dots a_n$. This is the basic step of Lau–Schleicher Algorithm (LSA) [12]. Iterating this step we find arbitrarily many hyperbolic sectors on a given combinatorial arc.

In case the lower parameter $\mathcal{C}$ is one of the sectors of the main epicycloid, then applying LSA iteratively yields the internal address of the upper parameter $\mathcal{A}$. In other words, LSA gives rise to a bijective mapping which takes kneading sequences into internal addresses.



EXAMPLE 5 The external ray with angle $\frac{3}{11}$ (compare to the previous example 1) lands at the boundary of a five-periodic hyperbolic component (Fig. D). Its two limit sequences are $\overline{\mathbf{10210}}$ and $\overline{\mathbf{10211}}$. The first translates into an infinite address

$$\widehat{A} := 1(1) \rightarrowtail 2(2) \rightarrowtail 3(1) \rightarrowtail 6(2) \rightarrowtail 7(2) \rightarrowtail 8(1) \rightarrowtail \dots$$

and the second into a finite one, $A := 1(1) \rightarrowtail 2(2) \rightarrowtail 3(1) \rightarrowtail 5(1)$. Hence the ray's landing point must be the root, and the other two rays with five-periodic angles landing at the same component must be the nearest ones that are greater than $\frac{3}{11}$ and whose kneading sequences differ only at the fifth character.

2 (Non)equivalence of Real Trees

The contents of the first article [K1] are a counterexample to the strong version of the Translation Principle (Conjecture 3), and a proof to a weaker version in the quadratic case:

THEOREM 6 (WEAK TRANSLATION PRINCIPLE) [12, 8.5]
Let $\mathcal{C}$ be any hyperbolic component in $\mathcal{M}^2$, k its period, and m the smallest period of hyperbolic components in its wake. Then the minimal period of components in any visible tree $\mathcal{T}_{p/q}$ is $m_q = (q-2)k + m$.

The proof in [12, 8.5] uses many properties of the dynamical plane, whereas in [K1] these are more "hidden" under the Structure Theorems (cf. 1.4 and 1.6). Moreover, our proof yields a couple of interesting corollaries.

2.1 Wake-widths

The *wake-width* of a hyperbolic sector is the difference of its two external angles. Since these are both periodic with the same period, say k, the wake-width is of the form $w/(d^k - 1)$, $w \in \mathbb{N}$. A hyperbolic sector is *narrow* if $w = 1$; or equivalently, if its wake contains no components with smaller periods. The two external rays landing at the root of each satellite component of a base sector at internal angle p/q bound the $\frac{p}{q}$*-subwake.*

LEMMA 7 *If the base has wake-width $w/(d^k - 1)$, then the $\frac{p}{q}$-subwake has width*

$$\Delta(\tfrac{p}{q}) = \frac{w(d^k - 1)}{d^{qk} - 1}.$$

The proof for the quadratic case in [18] is based on Douady's Tuning Algorithm [7]; another proof in [K2, §7] for general degree d uses the tree-growing algorithm [K2, §6].

The following result in [K1] is a kind of an extension to Structure Theorems: it gives a sufficient condition for two angles to be the endpoints of the characteristic interval of some orbit portrait. It serves as a link between the dynamics and the parameter plane.

LEMMA 8 [K1, 3.8]
Let θ and φ be two angles, periodic under doubling with periods equal to $k \in \mathbb{N}$. If $K^-(\theta) = K^+(\varphi)$, and the interval $]\theta, \varphi[$ contains no angles in the cycles of either θ or φ, then the parameter rays with these two angles land at the same point.

The proof relies on symbolic dynamics [1]. A consequence of this (related to the Lavaurs' Lemma [13] [12, 3.8]; cf. 1.8) is essential for proving Theorem 6:

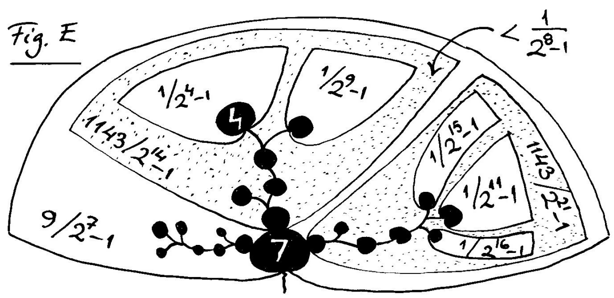
COROLLARY 9 [K1, 3.9] *Two parameter rays with angles $\theta = t/(2^n - 1)$ and $\varphi = (t+1)/(2^n - 1)$ either land at the same point, or else there is an angle with a period $i < n$ on $[\theta, \varphi]$.*

Now comparing the widths of subwakes, the Weak Translation Principle will follow from the next result, which also gives the period in terms of the width of the base:

LEMMA 10 [K1, 5.1] *If a hyperbolic component has wake-width* $t/(2^k - 1)$, *where* $t = 2^s + r$ *with maximal* s, *then the minimal period of components in the* $\frac{1}{2}$*-subwake is* $m = k + 1$ *if* $t = 1$, *and* $m = k - s$ *if* $t \geq 3$.

It is simple to show that $m = k + 1$ when the wake is narrow and $k - s \leq m \leq k - s + 1$ otherwise; the hard part is to rule out the case "$m = k - s + 1$". This is done by noticing that in that case the base component would have wake-width in $]1/2^{k-s}, 1/(2^{k-s} - 1)[$, and then proving (using Corollary 9):

LEMMA 11 [K1, 5.6] *For any* $n \in \mathbb{N}$, *hyperbolic components (of any period) with wake-widths in* $]1/(2^{k-s} + 1), 1/(2^{k-s} - 1)[$ *exist nowhere in the parameter plane.*



2.2 Counterexample

An example of a hyperbolic component with two non-equivalent visible trees is the seven-periodic one at internal address $1 \rightarrowtail 3 \rightarrowtail 6 \rightarrowtail 7$ and kneading sequence $\overline{\mathbf{1101110}}$. The main tree $\mathcal{T}_{1/2}$ contains components with periods $4, 9, 10, 12, 13, 14$, and the tree $\mathcal{T}_{1/3}$ contains components with periods $11, 16, 17, 19, 20, 21$ (the same plus seven, like it should by Conjecture 3) – but, in addition, it has one with period 15 whose counterpart is missing from the main tree. This missing component would be 8-periodic; it is actually the "shadow satellite" of the four-periodic one (cf. 3.2).

The trees are constructed by calculating the widths of the wakes of the various components in the parameter plane, adding them up and seeing that there is no room for the wake of the eight-periodic component where it should have been if the Translation Principle were right. In [K2] we obtain a method to construct trees using just the kneading sequence or internal address of the base sector; this will be discussed in the following two sections.

3 Formal Symbol Space

The brilliant Cerebron, attacking the problem analytically, discovered three distinct kinds of dragon: the mythical, the chimerical, and the purely hypothetical. They were all, one might say, nonexistent, but each nonexisted in an entirely different way.

– S. Lem [14]

The example 2.2 not only shows that not all trees with common base are combinatorially equivalent; moreover, not all kneading sequences are realized by some angle. One may check all the 240 angles with exact period eight and find that none of them has $\overline{\mathbf{11011100}}$ as its kneading sequence. However, the LS-algorithm translates this sequence into $1 \rightarrowtail 3 \rightarrowtail 6 \rightarrowtail 7 \rightarrowtail 8$, which consequently does not correspond to any actual hyperbolic sector despite the fact that it looks like an internal address.

We call sectors (and other parameter classes) that would live in such "empty" addresses *nonexistent*, and study the combinatorial structure of the abstract space of addresses for its own right.

3.1 Kneading Sequences and Addresses

A *(formal) kneading sequence* is defined as an element $\mathbf{a} = a_1 a_2 a_3 \ldots$ of the symbol space $\{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{d-1}\}^{\mathbb{N}}$ with $a_1 \neq \mathbf{0}$. We denote by Σ^d the subspace of all such sequences equipped with the metric [1]

$$|\mathbf{a} - \mathbf{b}| := \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{d^i}.$$

By Lemma 2, the kneading mapping from the circle to Σ^d is continuous at nonperiodic angles and has an upper and a lower limit at periodic ones. However, the mapping is neither injective nor surjective; for example, $K(\frac{9}{56}) = K(\frac{11}{56}) = \mathbf{110}\overline{\mathbf{1}} \in \Sigma^2$ whereas $K(\theta) = \overline{\mathbf{11011100}}$ for no angle θ.

A *(formal) address* is a sequence $A := n_1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \ldots$ where

- $n_1, n_2, \ldots$ is a strictly increasing sequence of positive integers with $n_1 = 1$ (period numbers) and
- $s_j \in \mathbb{Z}_d \setminus \{0\}\ \forall j \in \mathbb{N}$ (sector numbers).

(Equivalently, $A := 1(r_1) \rightarrowtail 2(r_2) \rightarrowtail \ldots$ where $r_j \in \mathbb{Z}_d\ \forall j \in \mathbb{N}$; zero refers to the "forbidden sector".) The set of addresses is denoted by Λ^d. The LS-algorithm is obviously reversible and it defines a bijective mapping $\varkappa_d : \Lambda^d \to \Sigma^d$ that induces a topological structure to the address space from the metric above. Then $\varkappa_d(A) = \mathbf{a} = a_1 a_2 a_3 \ldots$ such that

$$\begin{aligned} a_1 \ldots a_{n_2-1} &= \mathbf{s}_1 \ldots \mathbf{s}_1, & a_{n_2} &= \mathbf{s}_1 + \mathbf{s}_2; \\ a_1 \ldots a_{n_j-1} &= (\overline{a_1 \ldots a_{n_{(j-1)}}})|_{n_j-1}, & a_{n_j} &= (\overline{a_1 \ldots a_{n_{(j-1)}}})_{n_j} + \mathbf{s}_j\ \forall j. \end{aligned}$$

Identifying all parameters in the Mandelbrot set that share a kneading sequence, we obtain a factor space $\mathcal{M}^d/\simeq$; this is contained in Λ^d (or equivalently, in Σ^d). It is a proper subset, because there are nonrealizable symbol sequences.

Finite addresses ending with n form a subspace $\Lambda_n^d \subset \Lambda^d$ with $(d-1)^2 d^{n-2}$ elements. The bijection $\varkappa_d$ takes them into n-periodic kneading sequences, but the set of all such, $\{\mathbf{a} : a_i = a_{n+i}\ \forall i\} =: \Sigma_n^d \subset \Sigma^d$, has $(d-1)d^{n-1}$ elements.

For each initial word $a_1 a_2 \ldots a_{n-1}$, exactly $d-1$ choices (of the d) for the next entry, a_n, corresponds to a finite address ending with n; their space is $\varkappa_d(\Lambda_n^d) = \dot{\Sigma}_n^d$. The remaining choice of a_n produces, under the mapping $\varkappa_d$, either an infinite address or a finite one ending with some proper divisor of n. Denoting by $\Sigma_{n,\infty}^d$ the set of sequences that have exact period n and correspond to infinite addresses, we thus have a disjoint union

$$\Sigma_n^d = \dot{\Sigma}_n^d \cup \Sigma_{n,\infty}^d \cup \bigcup_{k|n, k<n} \Sigma_k^d .$$

3.2 Primitive Addresses, Satellites and Shadow Satellites

Any formal address $B \in \Lambda_n^d$ has a kneading sequence $\varkappa_d(B) = \mathbf{b} = \overline{b_1 \ldots b_n} \in \dot{\Sigma}_n^d$; we define its *root sequence* $\widehat{\mathbf{b}} := \overline{b_1 \ldots b_n'}$ by replacing b_n by the one symbol b_n' such that $\widehat{\mathbf{b}} \notin \dot{\Sigma}_n^d$. There are now three possibilities as to where $\widehat{\mathbf{b}}$ is.

If B is realized by a hyperbolic sector $\mathcal{B}$, then $\widehat{\mathbf{b}}$ is its root kneading sequence as defined in 1.8 (see also Example 5). Then these three cases mean the following:

$\widehat{\mathbf{b}} \in \Sigma_{n,\infty}^d$: the infinite address $\varkappa^{-1}(\widehat{\mathbf{b}})$ lists a sequence of visible sectors approaching $\mathcal{B}$ from below, so $\mathcal{B}$ is part of a primitive component.

$\widehat{\mathbf{b}} \in \dot{\Sigma}_k^d$ for some $k \mid n$: the finite address $\varkappa^{-1}(\widehat{\mathbf{b}})$ is the internal address of another hyperbolic sector (with period k) and $\mathcal{B}$ is its satellite.

$\widehat{\mathbf{b}} \in \Sigma_{k,\infty}^d$ for some $k \mid n$: this is impossible.

Whether or not the hyperbolic sector $\mathcal{B}$ does exist, we now define:

- the *period* of $\mathcal{B}$ is the length (n) of its address B.
- $\mathcal{B}$ and its address B are $\begin{cases} \textit{primitive} \text{ if} & \widehat{\mathbf{b}} \in \Sigma_{n,\infty}^d, \\ \textit{satellite} \text{ if} & \widehat{\mathbf{b}} \in \dot{\Sigma}_k^d \text{ for some } k \mid n, \\ \textit{shadow satellite} \text{ if} & \widehat{\mathbf{b}} \in \Sigma_{k,\infty}^d \text{ for some } k \mid n. \end{cases}$

THEOREM 12 [K2, 5.10, 5.11]

- *Shadow satellites are nonexistent.*
- *For any numbers* k, q, *every primitive* k*-periodic hyperbolic component has a shadow satellite of period* qk.
- *The parent component of any shadow satellite is primitive.*

Thus we have found one class of nonexistent components.

3.3 Partial Ordering

We extend the definition of the "above"-relation for any addresses A, B, C in Λ^d (finite or infinite):

- If B is a subaddress of A (i.e., $A = B \rightarrowtail \ldots$), then $B \prec A$.
- If $C \prec A$ and the first step of LSA yields an address B, then $C \prec B \prec A$.
- If B is infinite and all its (finite) subaddresses are below A, then $B \prec A$.

Combinatorial arcs, visibility etc. can then be defined for nonexistent parameters as well. Now Λ^d is a partially ordered, simply connected topological space containing the factor space $\mathcal{M}^d/\simeq$.

If an address is realized, then so are all addresses below it (by LSA). On the other hand, if an address E is not realized, then nothing above it is realized either. Therefore there must be a unique *vanishing point* ν on the combinatorial arc from the origin to E that divides the arc into existent and nonexistent part.

4 Growing Trees

> *Suppose, for example, one organizes a hunt for such a dragon, surrounds it, closes in, beating the brush. The circle of sportsmen, their weapons cocked and ready, finds only a burned patch of earth and an unmistakable smell: the dragon, seeing itself cornered, has slipped from real to configurational space.*
>
> – S. Lem [14]

While the original LSA was a method to find new hyperbolic sectors *below* a given starting point (or towards the origin), the same method can be extended to find new sectors *above* the given base. We first define a useful concept to make the language simpler.

4.1 Visibility over a Sector, Narrow Addresses

Let $\mathcal{A}$ and $\mathcal{B}$ be hyperbolic sectors with periods m and n respectively, such that $\mathcal{B} \prec \mathcal{A}$, $n > m$, and no sector between $\mathcal{B}$ and $\mathcal{A}$ has period less than m. Then, by definition of visibility, $\mathcal{A}$ is visible from both $\mathcal{B}$ and any sector $\mathcal{C}$ below it from where $\mathcal{B}$ is, in turn, visible; we say that $\mathcal{A}$ is *visible over* $\mathcal{B}$. If no sector between them has period less than n, then $\mathcal{A}$ is *immediately visible over* $\mathcal{B}$.

Growing a visible tree, one must evidently find successive components immediately visible over the preceding one. Our tree-growing algorithm will be based on the following result, which gives a necessary condition for a hyperbolic sector to sit immediately visible over another: the periodic word of the lower one has the same subword at both ends.

LEMMA 13 [K2, 6.2] *If $\mathcal{A}$ is a hyperbolic sector immediately visible over another sector $\mathcal{B}$, then the periodic word $b_1 \dots b_n$ of $\mathcal{B}$ begins and ends with the same word $b_1 \dots b_{n-m} = b_{m+1} \dots b_n$. Moreover, m does not divide n.*

An address not satisfying this subword condition evidently cannot have addresses with less periods above it, so it must be realized by a *narrow* hyperbolic component, if at all. Hence we define a formal address to be *narrow* if the corresponding kneading sequence has no common subword at both ends of its periodic word. That all narrow components have narrow addresses follows from the fact that given a narrow, existent hyperbolic sector at address $1(s_1) \rightarrowtail \cdots \rightarrowtail k(s_k)$, any continuation $1(s_1) \rightarrowtail \cdots \rightarrowtail k(s_k) \rightarrowtail n(s_n)$ (with $n > k$) is realized in some visible tree [12, 10.2]. Therefore not even nonexistent components with periods less than k can be found above the narrow base sector.

4.2 Visible Tree Algorithm (VTA)

Lemma 13 gives the basic step of our algorithm; applying it iteratively one finds every address that *might* belong to a sector visible from the given base.

Let $\mathcal{C}$ with address $C = 1(s_1) \rightarrowtail \cdots \rightarrowtail k(s_k)$ and kneading sequence $\varkappa_d(C) = \overline{c_1 c_2 \dots c_k}$ be the base sector of the tree to be constructed.

1. Choose denominator $q \in \mathbb{N}$ of the internal angle to determine the tree's stem.
2. Choose sector number $s \in \mathbb{Z}_d \setminus \{0\}$. Replace the qk^{th} digit $c_{qk} = c_k$ in the base kneading sequence by $c_k + s$. This yields the sequence and address

$$\overline{(c_1 c_2 \dots c_k)^{q-1} c_1 c_2 \dots (c_k + s)} =: \overline{b_1 b_2 \dots b_n}$$
$$C_q^s = 1(s_1) \rightarrowtail \cdots \rightarrowtail k(s_k) \rightarrowtail qk(s) =: B = 1(s_1) \rightarrowtail \cdots \rightarrowtail n(s)$$

   for the s-sector of the q-satellite of C.
3. Find a number l such that $b_1 \dots b_l = b_{n-l+1} \dots b_n$, and set $m := n - l$. Now $\overline{b_1 \dots b_m}$ is the root sequence of a primitive address A immediately above B.
4. Choose sector number $r \in \mathbb{Z}_d \setminus \{0\}$. Replace b_m in the root sequence with $b_m + r \in \mathbb{Z}_d$ to obtain the kneading sequence and address for the r^{th} sector of $\mathcal{A}$.
5. Change notation ($A \to B$, $m \to n$, etc.) and repeat steps (3), (4), (5).

In other words:

- take the periodic word of a sector's kneading sequence;
- find a subword of length l in both ends of the periodic word, chop off the last l digits and change the last digit of the remaining word;
- continue upwards as long as possible, then take the next sector of the same component until you have found everything above that component, then step backwards and take the next sector of the lower component.

The resulting set is called the *(formal) visible tree* $\widetilde{\mathcal{T}}_{\cdot/q}$ of C (or $\mathcal{C}$). It contains a subtree that is combinatorially equivalent to the "real" tree $\mathcal{T}_{p/q}$.

Note that there is no "canonical" order in which to repeat from step (3). For example, the main tree for the address $1 \rightarrowtail 5 \in \mathcal{M}^2$ is obtained from the periodic word **11110 11111**. This starts and ends with four different subwords, namely **1111**, **111**, **11**, **1**, yielding four adjacent components with periods $6, 7, 8, 9$ like fingers in a hand. Now this address $1 \rightarrowtail 5$ is realized by four different satellite components of the main cardioid, and the branches of their main trees are in different order:

internal angle	counter-clockwise order
1/5	$6, 7, 8, 9$
2/5	$7, 9, 6, 8$
3/5	$8, 6, 9, 7$
4/5	$9, 8, 7, 6$

Thus the embedding into the complex plane is not determined by the symbolic sequences. However, for degrees higher than two, the branching of trees may also happen within a component. The sectors – and hence the branches above each sector – of any component do have their natural cyclic order in the combinatorial sense as well. The next example shows both kinds of branching.

$A = 1(1) \rightarrowtail 3(2) \rightarrowtail 6(2) \rightarrowtail 7(1)$, $\varkappa(A) = \overline{\mathbf{1101122}}$

1101122 1101120	$\rightarrowtail$				14(1)
1101122 1101121	$\rightarrowtail$				14(2)
1101122 110110	$\rightarrowtail$			13(1)	
1101122 111	$\rightarrowtail$		10(1)		
1101122 $*$	$\rightarrowtail$	8			
1101122 1$*$	$\rightarrowtail$	9			
1101122 112	$\rightarrowtail$		10(2)		
1101122 110111	$\rightarrowtail$			13(2)	
1101122 110$*$	$\rightarrowtail$		11		
1101122 1101$*$	$\rightarrowtail$		12		

$B = 1(1) \rightarrowtail 3(2) \rightarrowtail 6(2) \rightarrowtail 7(2)$, $\varkappa(B) = \overline{\mathbf{1101120}}$; similar to A

$C = 1(1) \rightarrowtail 3(2) \rightarrowtail 6(1) \rightarrowtail 7(1)$, $\varkappa(C) = \overline{\mathbf{1101112}}$

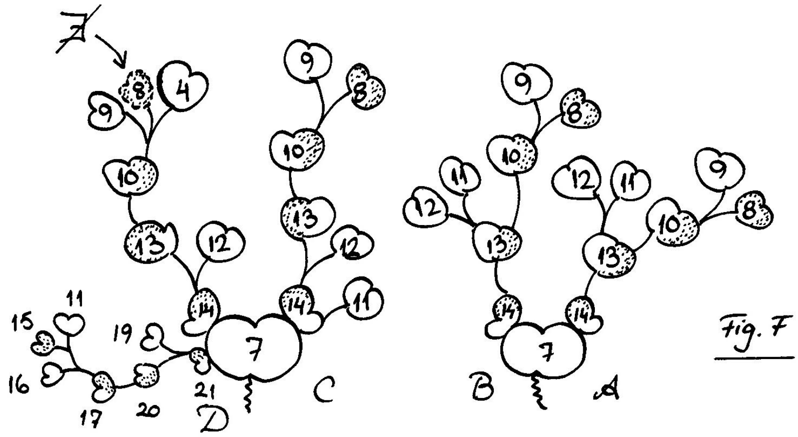
1101112 1101110	$\rightarrowtail$				14(1)
1101112 110$*$	$\rightarrowtail$			11	
1101112 1101111	$\rightarrowtail$				14(2)
1101112 1101$*$	$\rightarrowtail$			12	
1101112 110112	$\rightarrowtail$			13(1)	
1101112 110110	$\rightarrowtail$			13(2)	
1101112 111	$\rightarrowtail$		10(1)		
1101112 $*$	$\rightarrowtail$	8			
1101112 1$*$	$\rightarrowtail$	9			
1101112 112	$\rightarrowtail$		10(2)		

$D = 1(1) \rightarrowtail 3(2) \rightarrowtail 6(1) \rightarrowtail 7(2)$, $\varkappa(D) = \overline{\mathbf{1101110}}$

1101110 1101111	$\rightarrowtail$				14(1)
1101110 1101$*$	$\rightarrowtail$			12	
1101110 110112	$\rightarrowtail$			13(1)	
1101110 110110	$\rightarrowtail$			13(2)	
1101110 111	$\rightarrowtail$		10(1)		
110$*$	$\rightarrowtail$	4			
1101110 $*$	$\rightarrowtail$	8			
1101110 1$*$	$\rightarrowtail$	9			
1101110 112	$\rightarrowtail$		10(2)		
1101110 1101112	$\rightarrowtail$				14(2)

1101110 1101110 1101111	$\rightarrowtail$				21(1)
1101110 1101110 1101$*$	$\rightarrowtail$			19	
1101110 1101110 110112	$\rightarrowtail$			20(1)	
1101110 1101110 110110	$\rightarrowtail$			20(2)	
1101110 1101110 111	$\rightarrowtail$		17(1)		
1101110 110$*$	$\rightarrowtail$	11			
1101110 1101110 $*$	$\rightarrowtail$	15			
1101110 1101110 1$*$	$\rightarrowtail$	16			
1101110 1101110 112	$\rightarrowtail$		17(2)		
1101110 1101110 1101112	$\rightarrowtail$				21(2)

EXAMPLE 14 We grow the visible tree $\widetilde{\mathcal{T}}_{\cdot/2}$ for four sectors in Λ_7^3 sharing the period sequence $1 \rightarrowtail 3 \rightarrowtail 6 \rightarrowtail 7$ in their addresses, and also the tree $\widetilde{\mathcal{T}}_{\cdot/3}$ for one of them (Figure F). Both sectors of each narrow component are marked with the symbol $*$.



The eight-periodic component is nonexistent because it is a shadow satellite of the four-periodic one; compare to 2.2 and 3.2.

4.3 Shadow Trees

VTA obviously works independently of the existence of the base sector, like LSA. In particular, a tree can be grown on any shadow satellite of a primitive hyperbolic component $\mathcal{C}$ just like on a regular satellite: take q copies of the periodic word of $\mathcal{C}$'s root sequence, change the last digit, find matching subwords at both ends and proceed as in 4.2. Such a tree is called the *shadow tree* $\widetilde{\mathcal{V}}_{\cdot/2}$ of $\mathcal{C}$.

Components in a shadow tree of $\mathcal{C}$ are not visible *from* $\mathcal{C}$, but they are visible *over* the shadow satellite, so they may appear in visible trees of some sectors below $\mathcal{C}$. Components in a shadow tree are obviously nonexistent.

EXAMPLE 15 Each shadow-q-tree of the (realizable) address
$A := 1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 6$ with periodic word **12101**$*$ consists of the shadow q-satellite and one primitive address of length $(q-1)\cdot 6 + 5$ behind the second sector of the stem; see Figure G.

121012 121010	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 12(1)$
121012 121011	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 12(2)$
121012 1210$*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 11$
121012 121012 121010	$1(1) \rightarrowtail \cdots \rightarrowtail 10(1) \rightarrowtail 14(2) \rightarrowtail 16(1) \rightarrowtail 18(1)$
121012 121012 121011	$1(1) \rightarrowtail \cdots \rightarrowtail 10(1) \rightarrowtail 14(2) \rightarrowtail 16(1) \rightarrowtail 18(2)$
121012 121012 1210$*$	$1(1) \rightarrowtail \cdots \rightarrowtail 10(1) \rightarrowtail 14(2) \rightarrowtail 16(1) \rightarrowtail 17$

The root kneading sequence $\overline{\mathbf{121012}}$ corresponds to an infinite address

$$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 14(2) \rightarrowtail 16(1) \rightarrowtail 20(2) \rightarrowtail 22(1) \rightarrowtail \ldots$$

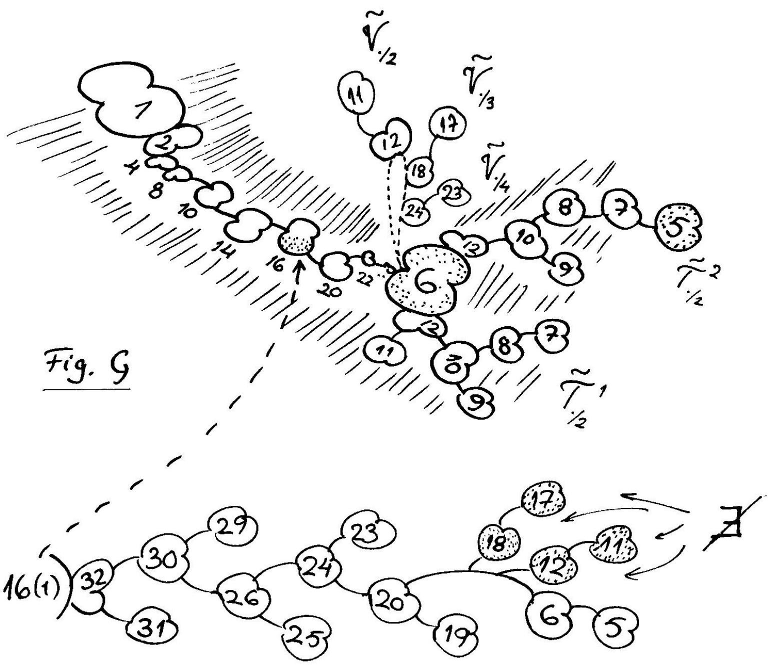


Fig. 9

The main tree $\widetilde{\mathcal{T}}_{\cdot/2}$ of the ten-periodic sector on the way contains the nonexistent components with periods 11 and 12; the main tree of the 16-periodic component contains in addition the nonexistent components with periods 18 and 17, and so on.

4.4 Structure of Trees

As it will turn out, certain trees are combinatorially equivalent. The goal in [K2, §6] is to prove the following theorems for formal visible trees and shadow trees:

THEOREM 16 [K2, 6.17] *For every* q*, the tree* $\widetilde{\mathcal{T}}_{\cdot/q}$ *of* $C := 1(s_1) \rightarrowtail \cdots \rightarrowtail k(s_k) \in \Lambda_k^d$ *consists of* k *components, with periods in* $\{(q-2)k+3, \ldots, qk\}$.

THEOREM 17 (STRONG TRANSLATION PRINCIPLE) [K2, 6.14, 6.21]
All formal visible trees $\widetilde{\mathcal{T}}_{\cdot/q}$ *of any base sector* $\mathcal{K}$ *are combinatorially equivalent. All shadow trees* $\widetilde{\mathcal{V}}_{\cdot/q}$ *of* $\mathcal{K}$ *are also combinatorially equivalent to each other.*

The proof is divided into a set of lemmas, which also show other interesting combinatorial properties of the visible trees.

LEMMA 18 [K2, 6.6] *The visible tree* $\widetilde{\mathcal{T}}_{\cdot/2}$ *of the address*
$C = 1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \cdots \rightarrowtail n_{k-1}(s_{k-1}) \rightarrowtail n_k(s_k)$ *contains a chain of addresses* A_j *such that* $A_j \in \Lambda_{n_k+n_j}^d$ *for all* $j \in \{1, \ldots, k\}$ *and* $A_1 \succ A_2 \succ \cdots \succ A_{k-1} \succ A_k$. *Each* A_j *is immediately visible over* A_{j+1}*, and the sector number of* A_j *is* $d - s_j$.

(This lemma was inspired by an observation by Dierk Schleicher, p.c.) It is proved by comparing the tree-growing algorithm VTA to the LSA, and noticing that one chain of steps in VTA happens exactly as LSA reversed. We call such a chain the *secondary trunk* of the tree. It is shaded in Figure F.

The branch of the tree leading to the component with shortest period is called the *primary trunk*. The secondary trunks of all sectors of the same base component are combinatorially equivalent [K2, 6.8] but their primary trunks may be different.

As shown in [12, 10.2], the (real) tree is well-behaved if the base sector is narrow, so we are particularly interested in the non-narrow case. For the next couple of lemmas we assume the tree to contain a sector with period less than the base's.

LEMMA 19 [K2, 6.15] *If there is an address* A *of length* $m < k$ *above an address* $C \in \Lambda^d_k$, *then* A *is above the* 2*-satellite of* C.

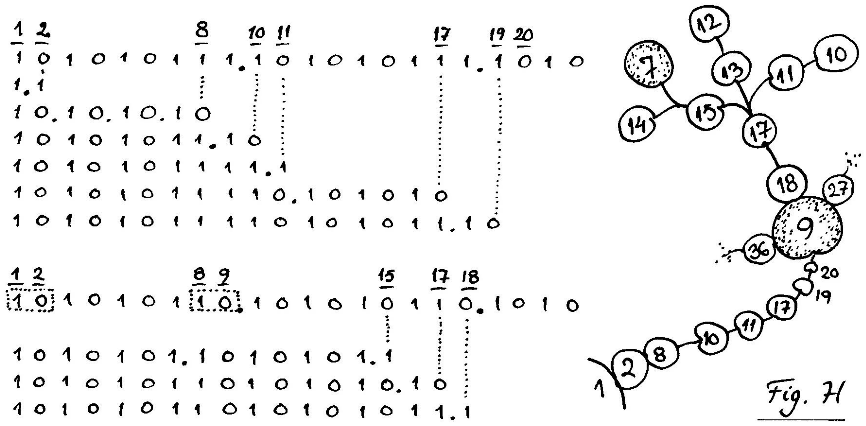


Fig. H

The proof of this result follows a similar argument as Lemma 18, showing that the Lau–Schleicher Algorithm from A down to C works almost identically as from the main cardioid up to the root of C – except that it ends at period 2k, the stem of the main tree. Figure H shows an example with $C := 1 \rightarrowtail 2 \rightarrowtail 8 \rightarrowtail 9 \in \Lambda^2_9$ and $A := 1 \rightarrowtail 2 \rightarrowtail 7$. Notice the two trunks of the tree branching off each other.

The proofs of the two theorems above are completed by an induction step [K2, 6.16], which states that if the tree $\widetilde{\mathcal{T}}_{\cdot/q}$ contains addresses A and B, then the tree $\widetilde{\mathcal{T}}_{\cdot/q+1}$ contains addresses A' and B' in an equivalent arrangement and with periods $k + |A|$, $k + |B|$. Again, the proof relies on VTA.

Another consequence of these same results is a certain kind of symmetry between formal visible trees and shadow trees of the same base component.

COROLLARY 20 [K2, 6.19]
Let $K \in \Lambda^d_k$ *be the base component. The following conditions are equivalent:*

- *The tree* $\widetilde{\mathcal{T}}^s_{./2}$ *based at sector* K^s *contains a formal address* B *with length* $|B| =: n < k$ *immediately visible over* K^s.
- *The formal tree* $\widetilde{\mathcal{T}}^r_{./2}$ *based at any other sector* K^r $(r \neq s)$ *contains a* B' *with* $|B'| = k + n$*;* B' *is immediately visible over the* $(s - r)^{th}$ *sector of the stem satellite.*
- *If* K *is primitive, then the shadow tree* $\widetilde{\mathcal{V}}_{./2}$ *of* K *contains an address* B' *with* $|B'| = k + n$*, immediately visible over the* s^{th} *sector of the stem satellite.*
  If K *is a* q*-satellite of another address* D*, then the formal* $2q$*-tree of* D *contains a* B' *with* $|B'| = k + n$ *immediately visible over the* s^{th} *sector of the stem satellite.*

Assuming there are formal addresses B *and* B' *as above, the tree containing* B *has an address* $A \succ B$ *with length* $m < n$ *if and only if the tree (formal or shadow) containing* B' *has an address* $A' \succ B'$ *with* $|A'| = k + m$.

It follows that for each number n less than the period of the base component $\mathcal{K}$, there can be at most one n-periodic component above the whole component $\mathcal{K}$ [K2, 6.20]. Denoting by $\widetilde{\mathcal{S}}(\mathcal{K})$ the tree consisting of $\mathcal{K}$ and all components visible over it, we get another corollary [K2, 6.21]: *All shadow trees of* $\mathcal{K}$ *are combinatorially equivalent to* $\widetilde{\mathcal{S}}(\mathcal{K})$. (See Example 15 and Figure G.)

These results are useful in figuring out which parts of a formal visible tree are nonexistent.

5 To Be or Not to Be?

We now know different levels of nonexisting hyperbolic components:

1. ones in shadow trees of existing sectors
2. ones visible from sectors of kind (1)
3. ones in shadow trees of sectors of kinds (1) and (2)
4. ones visible from sectors of kinds (2) and (3)
5. . . .

Components of the first kind are visible from some existing sectors, whereas the other kinds are only visible from other nonexistent sectors.

All the kinds of nonexistent components listed above have the property that they are either shadow satellites of existing sectors, or sit above such. In other words, their vanishing points are the roots of existing, primitive hyperbolic components. We call them *shadow components*. It remains open whether being shadow component is the *only* reason to nonexist. This question is being studied in [4].

All formal addresses come in families sharing the period numbers; addresses differing only at sector numbers are called here *siblings*. Any component address with j steps has $(d-1)^j$ siblings (including itself). The main results of [K2, §8] are the following:

THEOREM 21 [K2, 8.8]
Every component $\mathcal{A}$ of period n or less has an existing sibling in $\mathcal{M}^d$, if $d \geq n/2$.

THEOREM 22 [K2, 8.12] *If a shadow component $\mathcal{F}$ is visible from an existent sector $\mathcal{G}$, then $\mathcal{F}$ sits in the main tree of $\mathcal{G}$.*

This is complementary to the Partial Translation Principle [11, 3.78] (Theorem 4).

The proofs are largely based on the results in [K2, §6]. In addition, some reasons to *exist* are needed; being visible from a *narrow* sector was one [12, 10.2]. This argument together with iterative construction of trees and calculation of wake-widths (cf. 2.1, [K2, §7]) can be used to prove the following lemmas, which are useful for detecting existing components in a given tree.

LEMMA 23 [K2, 8.5] *For all d, every address of the form $1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail n_3(s_3) \rightarrowtail n_4$ is realized by a hyperbolic component in $\mathcal{M}^d$.*

LEMMA 24 [K2, 8.7] *If B is a non-narrow address, then at least one of the addresses above it (given by VTA) with length less than B, is realized.*

Finally we show that nonexistence is "inherited upwards", but existing siblings are found nearby:

- If a kneading sequence $\mathbf{b} = \overline{b_1 \ldots b_n} \in \dot{\Sigma}_n^d$ refers to a shadow sector $\mathcal{B}$ (and thus is not realized in $\mathcal{M}^d$), then $\mathbf{b}$ is not realized in $\mathcal{M}^{d+1}$ either.
- If the last subaddress of $\mathcal{B}$ is realized by a sector $\mathcal{C} \in \mathcal{M}^{d+1}$ with kneading sequence $\overline{b_1 \ldots b_k}$ ($\frac{n}{2} < k < n$), then $\overline{b_1 \ldots b_{k-1} \mathbf{d}}$ refers to a narrow sector $\mathcal{C}'$ of the same component. Hence $\overline{(b_1 \ldots b_{k-1} \mathbf{d}) b_{n-k} \ldots b_n}$ refers to an existing sibling $\mathcal{B}' \in \mathcal{M}^{d+1}$ of $\mathcal{B}$ ($\mathcal{B}' \succ \mathcal{C}'$).

See Example 14 and Figure F: the nonexistent eight-periodic component above the non-narrow sector $\mathcal{D}$ has an existing sibling above the narrow sector $\mathcal{C}$. The kneading sequence of $\mathcal{D}$ only contains symbols $\mathbf{0}, \mathbf{1}$, whereas that of $\mathcal{C}$ also contains symbols $\mathbf{2}$.

In some cases this result improves Theorem 21: existing siblings can be found with degrees d a lot less than $n/2$.

If all nonexistent components were shadow components, then there would be stronger corollaries:

- Nonexistent components can only be found in the main tree of the last subaddress.
- A kneading sequence consisting of d different symbols is realized either in all Mandelbrot sets with degree at least d, or in none.
- Every address has an existing sibling in $\mathcal{M}^3$.

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