JYVÄSKYLÄ STUDIES IN COMPUTING 14

Kirsi Majava

Optimization-Based Techniques for Image Restoration

Esitetään Jyväskylän yliopiston informaatioteknologian tiedekunnan suostumuksella
julkisesti tarkastettavaksi yliopiston Agora rakennuksessa (Ag Aud. 2)
joulukuun 15. päivänä 2001 kello 17.

Academic dissertation to be publicly discussed, by permission of
the Faculty of Information Technology of the University of Jyväskylä,
in the Building Agora, Ag Aud. 2, on December 15, 2001 at 5 p.m.

UNIVERSITY OF JYVÄSKYLÄ

JYVÄSKYLÄ 2001

JYVÄSKYLÄ STUDIES IN COMPUTING 14

Kirsi Majava

Optimization-Based Techniques for Image Restoration

UNIVERSITY OF JYVÄSKYLÄ

JYVÄSKYLÄ 2001

Optimization-Based Techniques for Image Restoration

Editors
Tommi Kärkkäinen
Department of Mathematical Information Technology, University of Jyväskylä
Olli Ahonen and Pekka Olsbo
Publishing Unit, University Library of Jyväskylä

URN:ISBN:9513912213
ISBN 951-39-1221-3 (PDF)

ISBN 951-39-1113-6 (nid.)
ISSN 1456-5390

ABSTRACT

Majava, Kirsi
Optimization-Based Techniques for Image Restoration
Jyväskylä: University of Jyväskylä, 2001, 27 p. (+included articles)
(Jyväskylä Studies in Computing
ISSN 1456–5390; 14)
ISBN 951-39-1221-3
Finnish summary
Diss.

The intention of this work is to develop robust and efficient numerical methods for image restoration. This includes two goals: developing appropriate mathematical formulations for image restoration problems and efficient computational techniques for solving the resulting smooth and nonsmooth optimization problems.

This study is restricted to the noise reduction problem, that is, it is assumed that the observed image is degraded by a random noise and no blurring occurs. The noise reduction problem is formulated as a minimization problem consisting of a least squares fit and a regularization term. First, a bounded variational (BV) type regularization is studied, which makes it possible to find the discontinuities from the data. Due to the BV seminorm, the cost functional becomes nonsmooth, however, and an efficient numerical technique has to be employed. For this purpose, the so-called active-set methods based on augmented Lagrangian smoothing of the original optimization problem are developed. Convergence of the algorithms is established and efficient implementations are introduced.

The BV-regularized formulation recovers well the sharp edges of the image, but the result obtained using this technique consists of a staircase-like structure. Hence, some form of adaptivity is needed for an improved restoration capability. For this purpose, the SAC method based on a semi-adaptive, strictly convex formulation that better recovers smooth subsurfaces contained in the true image is proposed.

In the formulations considered, a regularization parameter controls the balance between the fitting term and the regularization term. A way to automatically determine the regularization parameter, without needing any a priori information on the amount of noise contained in the given image is also presented.

Efficiency and restoration capability of the methods are tested and illustrated through numerical experiments.

Keywords: image restoration, optimization techniques, BV regularization, noise reduction, active-set methods, image processing, semi-adaptivity

Author Kirsi Majava
Department of Mathematical Information Technology
University of Jyväskylä
Finland

Supervisor Professor Tommi Kärkkäinen
Department of Mathematical Information Technology
University of Jyväskylä
Finland

Reviewers Professor Jaroslav Haslinger
Department of Metal Physics
Charles University
Czech Republic

Professor Xue-Cheng Tai
Department of Mathematics
University of Bergen
Norway

Opponent Professor Anthony J. Kearsley
Mathematical and Computational Sciences Division
National Institute of Standards and Technology
USA

ACKNOWLEDGMENTS

I would like to take this opportunity to express my sincere gratitude to Prof. Tommi Kärkkäinen for introducing me into this field of research and for his excellent guidance and continuous support throughout my research. I am grateful to Prof. Jaroslav Haslinger and Prof. Xue-Cheng Tai for reviewing the manuscript and giving encouraging feedback. I would also like to thank my colleagues Dr. Kaisa Miettinen and Dr. Erkki Heikkola for their useful comments on the manuscript. Mrs. Tuula Blåfield deserves special thanks for her linguistic comments.

This work was financially supported by the Academy of Finland, The Finnish Cultural Foundation, and COMAS Graduate School of the University of Jyväskylä.

Finally, I want to express my appreciation to my family and friends for their support, especially to Petri for his patience and understanding.

Jyväskylä, 19 November, 2001

Kirsi Majava

CONTENTS

LIST OF INCLUDED ARTICLES

[A] T. KÄRKKÄINEN AND K. MAJAVA, *Nonmonotone and monotone active-set methods for image restoration, Part 1 : Convergence analysis*, Journal of Optimization Theory and Applications, 106 (2000), pp. 61–80.

[B] T. KÄRKKÄINEN AND K. MAJAVA, *Nonmonotone and monotone active-set methods for image restoration, Part 2 : Numerical results*, Journal of Optimization Theory and Applications, 106 (2000), pp. 81–105.

[C] T. KÄRKKÄINEN AND K. MAJAVA, *Determination of regularization parameter in monotone active set method for image restoration*, in Proceedings of the 3rd European Conference on Numerical Mathematics and Advanced Applications, P. Neittaanmäki, T. Tiihonen, P. Tarvainen, eds., World Scientific, Singapore, 2000, pp. 641–648.

[D] T. KÄRKKÄINEN AND K. MAJAVA, *SAC-methods for image restoration*, in Recent Advances in Applied and Theoretical Mathematics, N. Mastorakis, ed., World Scientific and Engineering Society, Greece, 2000, pp. 162–167.

[E] T. KÄRKKÄINEN, K. MAJAVA AND M. M. MÄKELÄ, *Comparison of formulations and solution methods for image restoration problems*, Inverse Problems, 17 (2001), pp. 1977–1995.

[F] T. KÄRKKÄINEN AND K. MAJAVA, *SAC - Semi-adaptive, convex optimization methodology for image restoration*, Report B 11/2001, Department of Mathematical Information Technology, University of Jyväskylä, 2001.

INTRODUCTION

This thesis is devoted to the study of different optimization formulations for image restoration problems and the development of efficient solution methods for these optimization problems. Let us first, however, give a general overview of the research field.

Image restoration is a fundamental task in image processing. In various applications of computer vision, image processing is usually started by removing or reducing noise and other distortions from the image taken by a digital camera or obtained using some other method, for example, ultrasound scan or computer tomography. After this, the aim is to identify different segments of the image for further treatment, such as recognition and classification. In such applications, it is essential that the restoration process preserves edges, since they define the location of different segments and objects in an image. Image restoration techniques are utilized, among many other fields, in telecommunications [11] and medical imaging [5, 21, 39, 40], where measured signals and images often contain measurement and quantization errors, i.e., noise.

Noise reduction is in many cases a necessity because all classical edge detection or segmentation methods rely on derivatives to some extent [31, 38]. However, the problem of numerical differentiation is ill-posed in the sense that small perturbations in the function (surface) to be differentiated may lead to large errors in the computed derivative [44, 54]. Hence, the derivatives of noisy images do not contain correct information and can thus be useless as such for edge detection or segmentation purposes.

Let us consider the reconstruction of an unknown image from given data which can represent, for example, a one-dimensional signal (Figure 1) or a two-dimensional image (Figures 2 and 3). In Figure 2, a grey-level image of a spine and the corresponding intensity plot are given. The image is distorted by uniformly distributed noise with a 30-per cent intensity compared to the maximum intensity of the original image. This noisy image is presented in Figure 3, together with the corresponding intensity plot. Looking at the noisy grey-level image in Figure 3, the image restoration problem does not seem difficult. This is because a human visual system automatically does some kind of image processing - we easily perceive edges even from a noisy image. However, it is the intensity plot that reveals the true noisy behaviour. Hence, it is clear that to estimate the quality of different images, it is not enough to look only at grey-level images.

It is often assumed that a noisy image, denoted by z, results from a degradation of the form [1, 6, 14, 16, 22, 45, 64]

$$z = Az^* + \eta \quad \text{in } \Omega, \tag{1}$$

where z^* is the true image, A is a linear operator (a blur modelled as a convolution, for instance), η represents a random noise, and $\Omega \subset R^d$ is the image domain. Blur can be caused, for example, by a defocused or shifted camera. In the

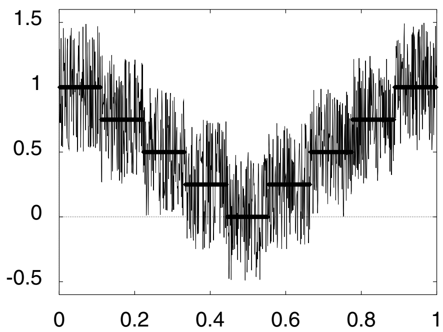

Figure 1: Example of original and noisy one-dimensional signals.

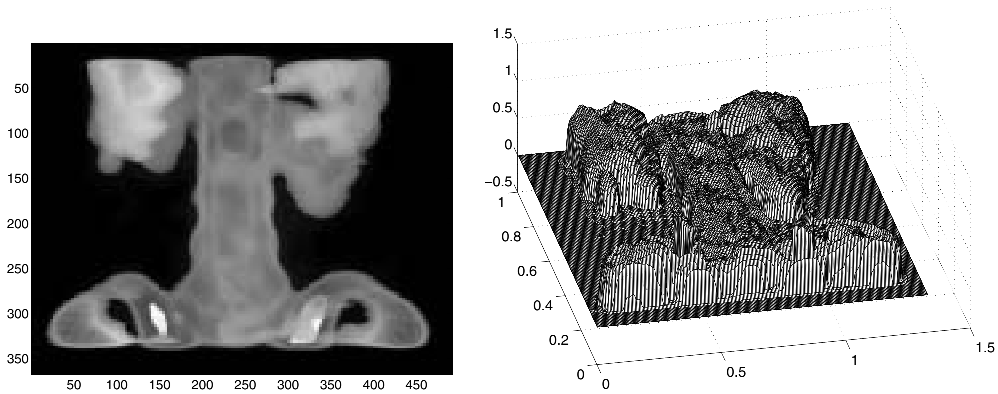

Figure 2: Original grey-level image and the corresponding intensity plot.

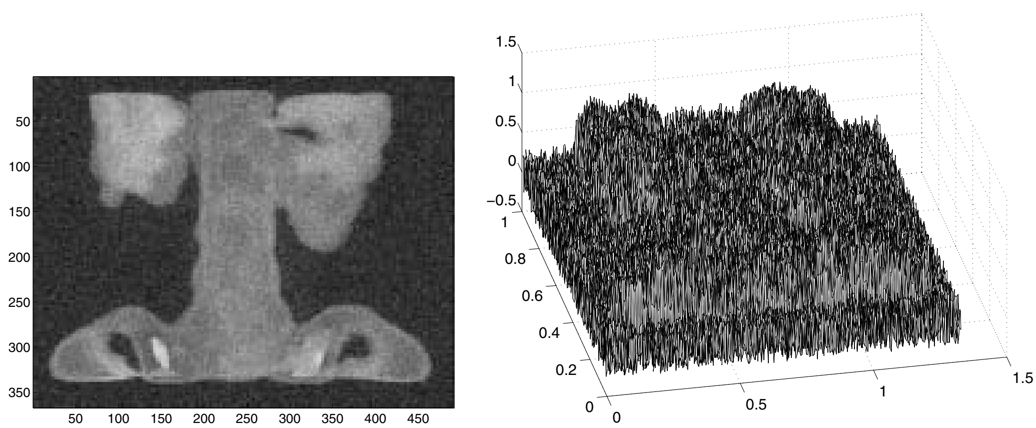

Figure 3: Noisy grey-level image and the corresponding intensity plot.

original form, the inverse problem (1) for determining z^* is generally ill-posed in the sense that the solution does not depend continuously on the data z. Stabilization techniques can be used to overcome this difficulty. In image restoration, these techniques can be roughly divided into three categories: statistical methods (e.g., Wiener filtering) [15, 31, 57], transform-based methods (e.g., filtering using Fourier or wavelet transforms) [31, 38], and optimization-based methods (e.g., constrained least squares or total variation -based methods) [27, 31, 45, 58]. Another class of methods gaining an increasing amount of attention in the image processing community are the methods based on partial differential equations (PDEs), more precisely, on nonlinear anisotropic diffusion [2, 3, 4, 13, 49, 51, 52]. Posed as evolution equations, these techniques are closely related to optimization-based methods, because their stationary solutions can often be interpreted to satisfy the optimality conditions of some optimization problem [49, 60]. In our work, however, we prefer to study the restoration problem using optimization formulations. We consider this form to be more illustrative, and, moreover, mathematical techniques on convex functionals [25] can be applied in the analysis. In addition, the actual numerical solution methods can be based on smooth and nonsmooth optimization techniques [46, 50].

Until recently, standard techniques for image restoration have been mainly linear. While these classical methods are computationally very fast, they tend to perform poorly if an image contains sharp edges. More precisely, linear methods tend to smooth edges out, and transform-based methods suffer from ringing effects (so-called Gibbs' phenomena) near edges. These difficulties are illustrated using one-dimensional examples, for example, in [18, 63]. As an example of the standard techniques, let us consider the well-known (nonlinear) median filter [31] which is commonly used for denoising images containing sharp edges. The method is fast and simple to implement, but the results are not satisfactory when the noise level is high or its distribution is not Gaussian. The result of a 3×3 median filter for the noisy image presented in Figure 3 is shown in Figure 4. The result is almost as noisy as the original noisy image. As the size of the mask gets larger, the image obtained becomes even more blurry and small details disappear (cf. Figures 5 and 6).

1 Optimization-based image restoration

To develop image restoration methods more suitable for images containing sharp edges, we need to modify the ill-posed inverse problem (1) in order to turn it into a well-posed problem. This can be achieved by using suitable regularization techniques. One approach is to obtain a restored image u as a solution of the constrained optimization problem

$$\min_u R(u) \quad \text{subject to} \quad \int_\Omega |Au - z|^2 \, dx = \sigma^2, \tag{2}$$

where R is a regularization functional measuring the irregularity of u in some sense, and σ^2 denotes the variance of the noise. Here, it is assumed that the noise

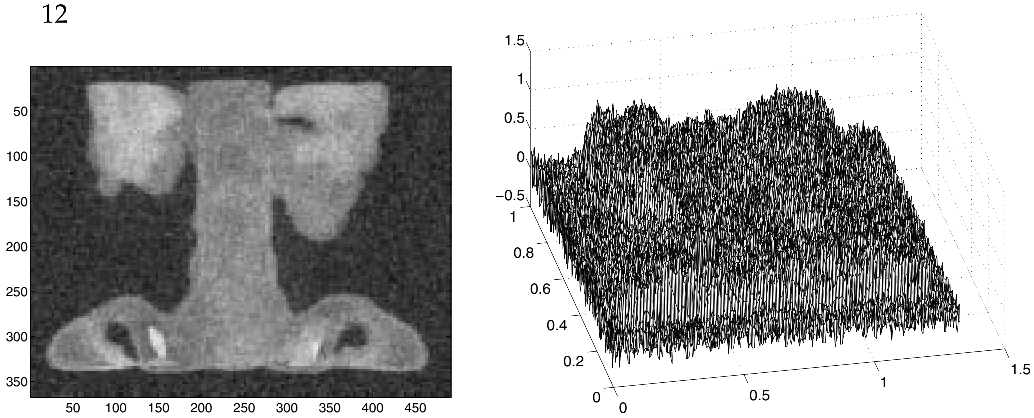

Figure 4: Reconstruction obtained by using a 3×3 median filter.

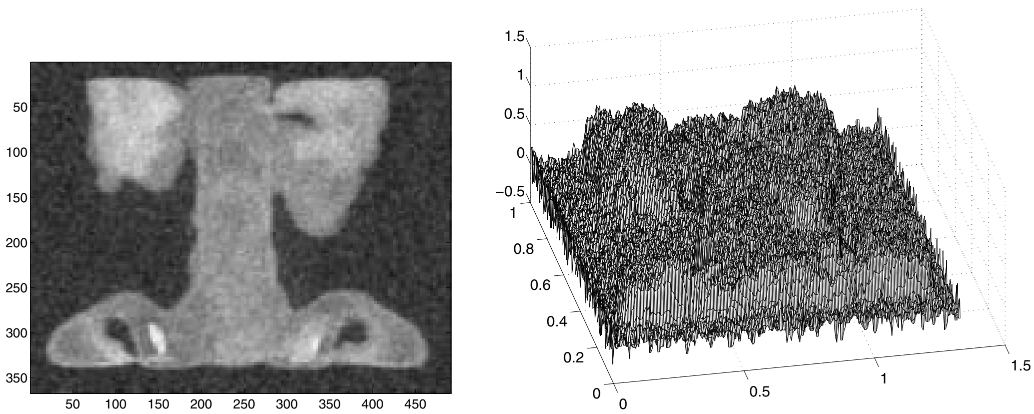

Figure 5: Reconstruction obtained by using a 5×5 median filter.

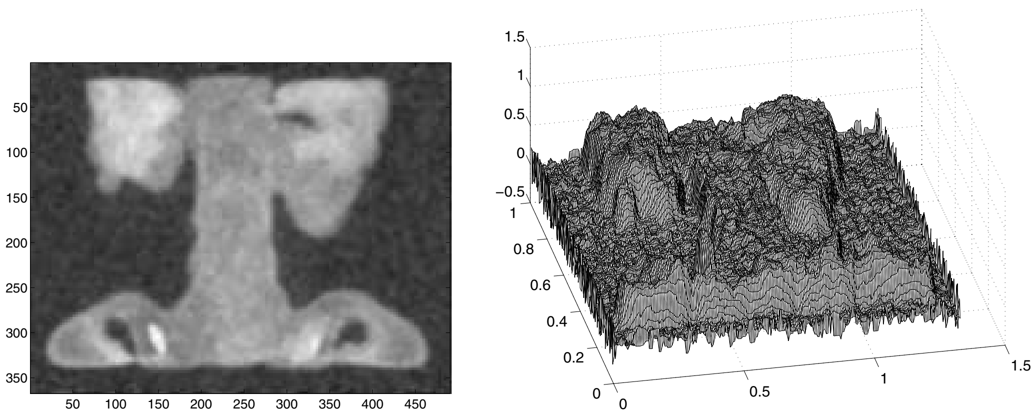

Figure 6: Reconstruction obtained by using a 9×9 median filter.

is Gaussian and has a zero mean ($|\eta|^2 \approx \sigma^2$). If there is no good estimate for the variance of the noise available, then the problem may be considered in the unconstrained, penalized form

$$\min_u \frac{1}{2}\int_\Omega |Au - z|^2\, dx + gR(u), \tag{3}$$

where the regularization parameter g controls the tradeoff between a good fit to the data and the regularity of the solution. Ideally, g should be chosen to be the reciprocal of the Lagrange multiplier for the equality constraint in problem (2). This penalized approach is known as the Tikhonov regularization [61]. Note that in the framework of Bayesian statistics, the use of a maximum a posteriori (MAP) estimation for image restoration leads typically to the very same optimization problem (3) (cf. [6, 40] and references therein).

A common approach in image restoration has been to use constrained least squares, that is, to use quadratic regularization functionals of the form $R(u) = |Qu|^2$, where Q is a linear operator [31]. Typical examples of this include $Q = I$ (the identity operator), $Q = \Delta$ (the Laplacian), and $Q = \nabla$ (the H^1 seminorm). However, the use of quadratic regularization functionals results with a linear least squares problem, which suffers from the property of linear methods mentioned earlier; it smears out sharp edges. Hence, it is not possible to properly recover discontinuities by using quadratic regularization functionals. For example, for $R(u) = |\nabla u|^2$, this is mathematically due to the fact that discontinuous functions do not have bounded H^1 seminorms.

In [58], the use of a bounded variational (BV) seminorm in the two-dimensional case (often called *total variation* of u),

$$R(u) = \int_\Omega |\nabla u|\, dx = \int_\Omega \sqrt{u_{x_1}^2 + u_{x_2}^2}\, dx_1 dx_2, \tag{4}$$

was proposed as a regularization functional. Here, the subscripts x_1 and x_2 denote the corresponding partial differentials. To be precise, let $\Omega \subset R^d$ be an open set. Then, a function $u \in L^1(\Omega)$ is said to be of bounded variation if

$$\sup\left\{\int_\Omega u(x)\,\mathrm{div}\, v(x)\, dx : v \in C_0^1(\Omega)^d, |v(x)| \le 1, x \in \Omega\right\} < \infty, \tag{5}$$

and the sup in (5) is denoted by $\int_\Omega |\nabla u|\, dx$ [29]. The space of all functions $u \in L^1(\Omega)$ with bounded variation is abbreviated by $BV(\Omega)$. Norm $|\cdot|$ in (5) denotes the Euclidean norm in R^d. Using a different norm in R^2 induces another equivalent seminorm on $BV(\Omega)$; for example, the norm $|\cdot|_\infty$ yields [12]

$$\int_\Omega |\nabla u|_1\, dx = \int_\Omega \Big(|u_{x_1}| + |u_{x_2}|\Big)\, dx_1 dx_2. \tag{6}$$

As (4) corresponds to the l_2-norm and (6) to the l_1-norm of the gradient, we use, in the sequel, this terminology to separate these two regularization methods. The basic practical difference between these two methods is that the l_2-norm is rotationally invariant, whereas the l_1-norm is not. Note that in the robust statistics, the use of the l_1-norm and the l_2-norm refers to the marginal median and spatial median, respectively [55].

The BV seminorm does not penalize discontinuities in u. Thus, it allows one to recover sharp edges of the original image. The superiority of the BV seminorm when images are "blocky", that is, piece-wise constant over a small number of subregions, has been demonstrated in many papers, using both theoretical [22, 56] and numerical [14, 17, 35] studies. We remark that the BV-regularized image restoration is a special case of anisotropic diffusion proposed in [52]; see details in [49].

Solution methods for BV-regularized problem

Due to the BV seminorm, the cost functional in (3) becomes nonsmooth, since the BV functional is nondifferentiable at locations where $|\nabla u| = 0$. Numerical methods for solving different variants of the (discretized) BV-regularized problem,

$$\min_u \int_\Omega |Au - z|^2 \, dx + g \int_\Omega |\nabla u| \, dx, \tag{7}$$

have been presented in numerous papers. Let us recall that often, such as in [23, 24, 58, 63] and in our work, the linear operator A is chosen to be identity, which leads us to the noise reduction problem. Note, however, that the restoration of z^* in (1) can be recovered in two steps: a noise reduction step, followed by a deblurring step. In deblurring, the inversion of A may also require the use of some stabilization technique. Numerical methods for solving (7) have usually been based on replacing the BV term $\int_\Omega |\nabla u| \, dx$ by a smooth term

$$\int_\Omega \sqrt{|\nabla u|^2 + \varepsilon} \, dx, \tag{8}$$

for a small positive ε. The main problem with this technique is that the related optimality condition contains a highly nonlinear and usually oscillating term $\nabla \cdot \left(\nabla u / \sqrt{|\nabla u|^2 + \varepsilon} \right)$ that one should linearize. Due to this term, the original Newton method does not work satisfactorily, in the sense that its domain of convergence is small. This is due to the regularity of the problem and especially true if the additional parameter ε is small. On the other hand, if ε is relatively large, then this term is well behaved, but the problem to be solved differs much from the original one [43].

The theoretical analysis of the minimization problem (7), also with the ε-smoothing, was conducted in [1]. In [58], explicit time marching was applied to obtain a gradient descent scheme for solving the ε-smoothed problem corresponding to (2). In [19, 24, 63, 64], a "lagged diffusivity" fixed point iteration was considered for solving the ε-smoothed unconstrained problem (7). The same approach was taken in [23], with the difference that the unbounded operator ∇u

was replaced by a stable approximation, and in [14], where a different smoothing technique was applied to the BV seminorm. This fixed point iteration can be viewed as a special case of the "half-quadratic regularization" scheme discussed in [27, 28]. In [16], a primal-dual method was proposed for solving problem (7), and in [6, 62], a more general regularization function $\phi(|\nabla u|)$ was used and the approach was based on a variational method. Different approaches for solving the BV-regularized problem have also been considered, for example, in [20, 47].

Since the use of the artificial smoothing parameter ε can be problematic, let us consider techniques for solving the image restoration problem in its original form. In [45], l_1-regularization (6) was used, and the method for minimizing the nonsmooth, constrained problem was based on an affine scaling strategy. The approach taken in [35] has, however, been our main topic of interest. There, the authors proposed to use active-set methods based on the augmented Lagrangian smoothing of the nonsmooth optimization problem

$$\min_{u\in H_0^1(\Omega)} \frac{1}{2}\int_\Omega |u-z|^2\,dx + \int_\Omega \Big(\frac{\mu}{2}|\nabla u|^2 + g|\nabla u|\Big)\,dx. \tag{9}$$

Here, μ is positive to ensure the coercivity of the cost functional in $H_0^1(\Omega)$ and it yields the unique solvability of the problem in this (Hilbert) space. The idea was to regularize the nonsmooth, unconstrained optimization problem (9) using Lagrange smoothing [7, 36]. More details of Lagrangian regularization techniques can be found in [26, 30, 34, 48]. Further, active-set methods based on regularized optimality conditions were described and analyzed. The characteristic feature of the active-set methods is that the original, nonsmooth optimization problem is transformed into a sequence of smoother, constrained problems for which Newton-like steps can be taken. The set of linear constraints, including the nonsmooth components of (9), is referred to as the active set. Note that similar methods can be developed also for other nonsmooth problems; see, for example, [7, 41]. In [35], l_2-regularization (4) and nonmonotone active-set algorithms were considered. The constrained optimization problem that appears in the inner iteration of active-set algorithms was treated by using a penalty method, which became expensive (in terms of storage requirement and CPU time) in two-dimensional problems.

In articles [A] and [B], which originate from the report [42], we studied and further developed the ideas of [35]. Article [A] was focused on the convergence of the active-set algorithms for solving problem (9). A one-dimensional image restoration problem and two formulations for the two-dimensional problem, corresponding to two definitions of the BV seminorm in (4) and (6), were considered. Unlike in [35], we noticed that, depending on the way the Lagrange multiplier is updated, the active-set algorithms are either nonmonotone or monotone. A rigorous convergence analysis of the one-dimensional algorithms and monotone two-dimensional algorithms was presented. Convergence proofs for nonmonotone algorithms were in principle the same as in [35], but we used different techniques in showing the details, some of which were missing in [35]. For instance,

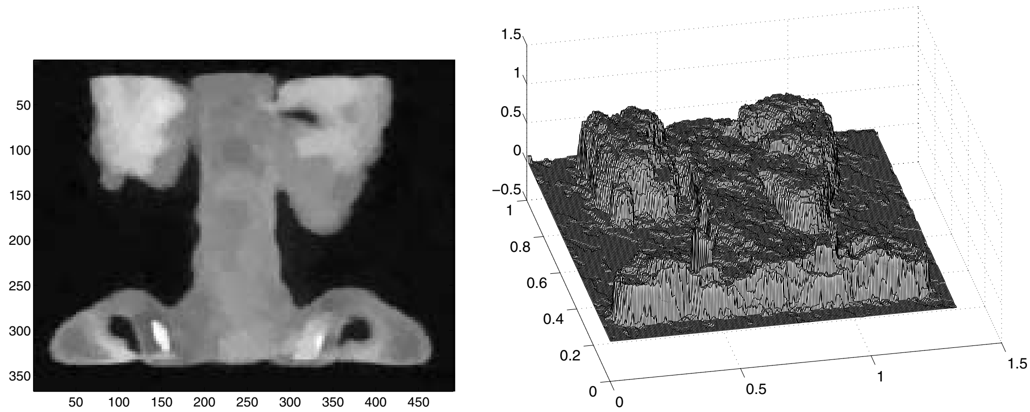

Figure 7: Result obtained by using the BV regularization.

our convergence analysis involved the explicit description of the change of the active (index) set between two iteration steps.

In article [B], the work of [A] was continued by including linearization in the algorithms considered. Convergence analysis in [A] was based on the assumption that the constrained minimization problem appearing in the inner iteration of the active-set algorithms is solved exactly. However, numerical tests have shown that it is enough to replace the exact solving by one Newton step and a line search (see, e.g., [35]). In the same way as in [35], we treated the one-dimensional non-monotone algorithm using a penalty method; for monotone algorithms, we presented an efficient, direct way for solving the inner iteration problem. Moreover, a modified regularization as well as a nested iteration -type technique [32] were proposed and tested in the two-dimensional algorithms. Compared to previously developed numerical methods, the main difference in our technique is the fact that no linear system has to be solved during the inner iterations. Thus, the storage requirement of our approach is $\mathcal{O}(N)$, that is, significantly smaller than in other methods. Furthermore, according to numerical experiments in [B], the CPU time for solving image restoration problems scales almost linearly and remains small, also for large problems, with hundreds of thousands of unknowns (when compared to other approaches, e.g., [53]). As an example of the restoration capability of the BV formulation, the restored image for the noisy image of Figure 3 is presented in Figure 7. The BV result yields a significant reduction of noise and an improvement of image quality. However, the result consists of a staircase-like structure which is not optimal for images with smooth subsurfaces, like in our example.

Choosing the value of the regularization parameter

In the unconstrained image restoration formulation (3), the regularization parameter g controls the balance between the fitting term and the regularization term.

If some extra information, like in (2), is available, then the value of g can be fixed precisely. General analysis on the determination of the regularization parameters is conducted in [59]. Unfortunately, this kind of precise information on noise statistics is usually not available in practical applications of image restoration.

In our experiments in [B], the value of g was chosen on the basis of the quality of the restored image - we chose g for which the result looked like the best. In [C], we assumed that the optimal choice of g minimizes the reconstruction error

$$e(u^*) := \sqrt{\frac{1}{n}\sum_{i=1}^{n}(u^* - z^*)_i^2}, \tag{10}$$

where u^* and z^* are the restored and true images, respectively, and n denotes the number of discretization points or pixels. A heuristic approach was presented in [C] to determine a near optimal value of g in the BV formulation (9) with respect to the reconstruction error $e(u^*)$ without using any extra information. The method was further tested and successfully applied in [E] and [F]. Notice that such an automatic determination of g improves possibilities to estimate the variance of noise in (2).

The result of the BV formulation in Figure 7 was computed with the value of g obtained by using the method of [C].

Towards a better formulation

The BV-regularized formulation for the image restoration problem is superior in recovering sharp edges of an image. However, as already pointed out, the result obtained using this technique consists of a staircase-like structure, which is not well-suited for images with smooth subsurfaces. In article [E], we studied whether, using generalizations of basic smoothing approaches for the BV seminorm, one can enhance the recovery of smooth subsurfaces contained in the true image. The regularization terms considered were the s-regularization

$$R_s(u) = \frac{1}{s}\int_\Omega |\nabla u|^s\, dx, \quad 1 < s \leq 2, \tag{11}$$

the ε-regularization (8), and the δ-regularization

$$R_\delta(u) = \int_{|\nabla u|>\delta}\left(|\nabla u| - \frac{\delta}{2}\right) dx + \frac{1}{2\delta}\int_{|\nabla u|\leq\delta}|\nabla u|^2\, dx, \tag{12}$$

where $\delta > 0$. Such formulations are studied also, for example, in [6, 10, 14, 33, 62], but in [E] (and in [43]), we analyzed and compared them more thoroughly. In this paper, we also compared different solution methods for the formulations considered and proposed a generalization of the active-set methods described in [35], [A], and [B], for the s-regularization (11). After the numerical study of these formulations, the first conclusion was that the formulations considered gave practically the same reconstructions when the parameters were chosen appropriately. It also became obvious that the computational results were not optimal in terms of the restoration properties. Hence, some form of adaptivity is needed if the aim is to recover both sharp edges and smooth subsurfaces of an image. Moreover, special techniques have to be applied for solving these problems, because general optimization methods tend to be inefficient when compared to the active-set method, for example.

Adaptive formulations for image restoration have been considered, for example, in [8, 14, 17, 37]. As soon as adaptivity comes along, formulations tend to become much more complicated. Adaptive formulations are often nonconvex or nonsmooth (or even both), and, in many cases, the number of unknowns and free parameters in these formulations is increased. The adaptive approach of [8] has been the most interesting to us. There, the idea was to use the BV regularization near edges, the smooth H_0^1 regularization in flat regions, and regularization (11), for $1 < s < 2$, between. The exponent s was chosen to be a gradient-driven function $s = s(|\nabla u|)$ and it made the formulation nonconvex.

In articles [D] and [F], we proposed a new method for image restoration, which is based on a semi-adaptive, strictly convex (SAC) formulation lying between no adaptivity and full adaptivity. The formulation is also smooth enough so that well-known solution methods, especially the conjugate gradient method, can be applied as a solver. In the conference paper [D], the basic idea was introduced and illustrated through a one-dimensional example. In article [F], the method was studied more thoroughly in the two-dimensional case.

The basic idea of the SAC method is similar to [8]: We use smooth regularization in the presumably smooth parts of an image and the BV regularization for edges. Unlike in [8], however, the division of the given image into differently regularized parts is made explicitly. The discrete regularization in the one-dimensional case to be considered is of the form

$$R_{SAC}(u) = \frac{g}{s_i} \sum_{i=1}^{n} |(Du)_i|^{s_i}, \quad 1 \leq s_i \leq 2, \tag{13}$$

where D denotes the backward difference approximation $(Du)_i = (u_i - u_{i-1})/h$ of the derivative for a mesh step size h. Here, the values of s_i are determined by using a reference solution. As a reference solution, we chose the result of the BV formulation because of its known characteristic features, the fast active-set algorithm for solving the BV problem, and the possibility to automatically fix g by using the method of [C]. The BV result was used in a very similar manner as a reference solution also in [60], where the semi-adaptivity was focused on making the regularization parameter g to vary within the BV framework in order to better preserve small details of an image.

The actual SAC method consists of the following three steps:

Step 1. Compute a reference solution $\bar{u}$ and fix the regularization parameter g.

Step 2. Determine $\{s_i\}, i = 1, \ldots, n$, using $\bar{u}$.

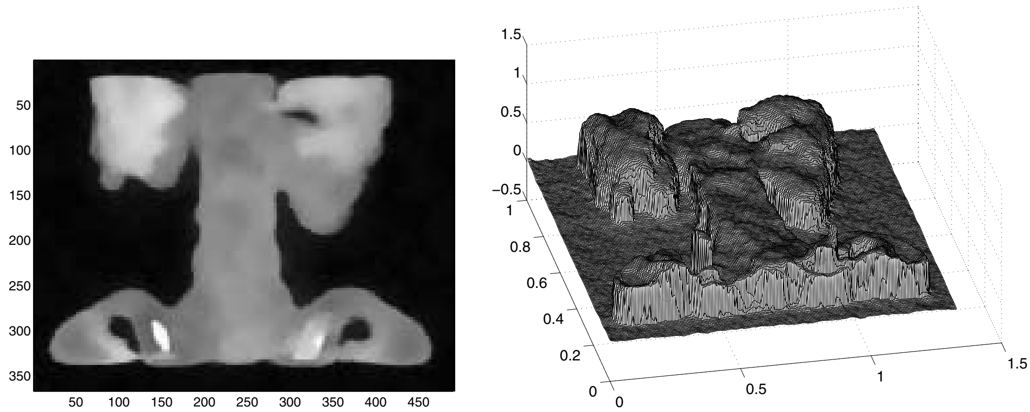

Figure 8: Result obtained by using the SAC method.

Step 3. Solve the image restoration problem with regularization R_{SAC}.

In [F], we explained in detail how these steps have been realized and illustrated their behaviour using a set of example images. Moreover, we proposed an improved algorithm to automatically determine the value of g by using the method of [C]. The example images considered in [F] are very different by nature. However, after once fixing the parameters of Step 2, the method gave satisfactory results in all examples. The result of the SAC method for the noisy image of Figure 3 is presented in Figure 8. The sharp edges are as in the BV result, but the smooth parts of the original image do not contain a staircase-like structure any more.

In [F], we also discussed three characteristics of images (cf. [37]) - flat, smoothly varying and edge subregions - and introduced the use of a suitable histogram of a compound gradient information to distinguish and illustrate them. Studying the properties of images using gradient information captured into a histogram turned out to be a useful tool in comparing the quality of the reconstructed images as well.

2 Conclusions and future research topics

In this work, image restoration formulations and their solution methods were discussed. Active-set methods for solving the BV-regularized image restoration problem were described. Convergence of the algorithms was established and efficient implementations were introduced. The SAC method based on a semi-adaptive, strictly convex formulation that better recovers smooth subsurfaces contained in the true image was proposed. A way to automatically determine the regularization parameter without needing any a priori information on the amount of noise contained in the given image was presented. Efficiency and restoration capability of the methods were illustrated through numerical experiments.

One difficulty related to the research field is to quantify the quality of the reconstructed images, for example, for comparing different restoration methods. In the references that we have found, the quality of the reconstructed images was usually estimated by presenting only the grey-level images and intensity plots. Contour plots are also often used for comparing the quality of the reconstructed images [47]. However, it is clear that these techniques do not fully capture the overall quality of the reconstructed image. In [F], we noticed that, especially, if an image contains a lot of small details, a plot of a compound gradient information contains useful and additional information when compared to a grey-level image or an intensity plot. As a measure of quality, we in our work used the reconstruction error $e(u^*)$ in (10). However, $e(u^*)$ mainly measures the jumps and does not mind small variations. Hence, if a formulation tends to smear edges, the optimal result with respect to $e(u^*)$ is visually not the best one (cf, for example, δ-formulation in [E]). To conclude, $e(u^*)$ contains quantitative information, but how an image actually looks like is also a qualitative matter. Thus, we should be able to combine these two characteristics to obtain a robust error measure for comparing different reconstructions.

The scope of image restoration starting from theoretical analysis and ending up with real digital images in various application fields is extremely broad. There probably exists no single method having an optimal performance (in terms of *both* restoration properties and computational efficiency) for all kinds of restoration problems. The SAC approach includes basic steps for realizing an image restoration algorithm with proper restoration and decent computational efficiency properties with automatic determination of free parameters. The proposed substeps can and should be modified for different application areas when doing practical restoration. Hence, instead of a single method, we consider the proposed technique a methodology for image restoration and analysis. For a specific application area, the SAC method can be tuned as follows:

1) Collect a representative set of samples from the application domain.

2) For the set of samples, study the histograms of the compound gradient information to reveal the amount of noise and basic characteristics of the images.

3) Compute the value of the regularization parameter g over the set of samples. If g is about the same size for all the samples, it can be fixed for the unknown images to be restored.

4) Compute the histograms for the BV results and realize Step 2 in the SAC method. In various application domains, images with different characteristics are produced, and, hence, Step 2 must be tuned individually for each application domain.

5) Carry out tests concerning stopping criteria and the number of SAC iterations (cf. [F]).

To this end, there are many possibilities to continue the work in this research field. One should develop a more efficient solution method for the SAC problem, for example, by realizing the generalized active-set method of [E] to the two-dimensional case. Because the active-set algorithm is not (easily) parallelizable, also other solution methods for the optimization problems should be considered, such as operator-splitting methods. Our future research will also involve generalizing our methods to colour image processing [9], including segmentation techniques [49] to our denoising methods, and applying our methods to real applications.

3 Author's contribution

Finally, I report my contribution in the presented papers [A]-[F] written together with other authors. The whole course of research has been inseparable teamwork with Professor Kärkkäinen. The main ideas may have come from him, but I have processed them from ideas to practical methods. I have made almost all the implementations and conducted numerical testing. The results were interpreted together.

REFERENCES

[1] R. ACAR AND C. VOGEL, *Analysis of bounded variation penalty methods for ill-posed problems*, Inverse Problems, 10 (1994), pp. 1217–1229.

[2] L. ALVAREZ, P. L. LIONS, AND J. M. MOREL, *Image selective smoothing and edge detection by nonlinear diffusion II*, SIAM Journal on Numerical Analysis, 29 (1992), pp. 845–866.

[3] L. ALVAREZ AND L. MAZORRA, *Signal and image restoration using shock filters and anisotropic diffusion*, SIAM Journal on Numerical Analysis, 31 (1994), pp. 590–605.

[4] L. ALVAREZ AND J. MOREL, *Formalization and computational aspects of image analysis*, Acta Numerica, (1994), pp. 1–59.

[5] S. ARRIDGE, *Optical tomography in medical imaging*, Inverse Problems, 15 (1999), pp. 41–93.

[6] G. AUBERT AND L. VESE, *A variational method in image recovery*, SIAM Journal on Numerical Analysis, 34 (1997), pp. 1948–1979.

[7] D. P. BERTSEKAS, *Constrained Optimization and Lagrange Multiplier Methods*, Academic Press, New York, New York, 1982.

[8] P. BLOMGREN, *Total Variation Methods for Restoration of Vector Valued Images*, PhD thesis, University of California, 1998.

[9] P. BLOMGREN AND T. CHAN, *Color TV: total variation methods for restoration of vector valued images*, IEEE Transactions on Image Processing, 7 (1998), pp. 304–309.

[10] P. BLOMGREN, T. CHAN, AND P. MULET, *Extensions to total variation denoising*, in Proceedings of SPIE. Advanced Signal Processing Algorithms, Architectures, and Implementations vol 7, F. Luk, ed., vol. 3162, Bellingham (WA), 1997, SPIE.

[11] A. CARLSON, *Communication systems. An introduction to signals and noise in electrical communication*, McGraw-Hill Book Company, Singapore, 1986.

[12] E. CASAS, K. KUNISCH, AND C. POLA, *Regularization by functions of bounded variation and applications to image enhancement*, Applied Mathematics and Optimization, 40 (1999), pp. 229–257.

[13] F. CATTÉ, P. L. LIONS, J. M. MOREL, AND T. COLL, *Image selective smoothing and edge detection by nonlinear diffusion*, SIAM Journal on Numerical Analysis, 29 (1992), pp. 182–193.

[14] A. CHAMBOLLE AND P. L. LIONS, *Image recovery via total variation minimization and related problems*, Numerische Mathematik, 76 (1997), pp. 167–188.

[15] A. CHAN AND J. MELOCHE, *Image restoration*, Journal of Statistical Planning and Inference, 65 (1997), pp. 233–254.

[16] T. CHAN, G. GOLUB, AND P. MULET, *A nonlinear primal-dual method for total variation-based image restoration*, SIAM Journal on Scientific Computing, 20 (1999), pp. 1964–1977.

[17] T. CHAN, A. MARQUINA, AND P. MULET, *High-order total variation-based image restoration*, SIAM Journal on Scientific Computing, 22 (2000), pp. 503–516.

[18] T. CHAN AND P. MULET, *Iterative methods for total variation image restoration*, in Iterative methods in scientific computing, R. Chan, T. Chan, and G. Golub, eds., Springer-Verlag, Singapore, 1997, pp. 359–381.

[19] ——, *On the convergence of the lagged diffusivity fixed point method in total variation image restoration*, SIAM Journal on Numerical Analysis, 36 (1999), pp. 354–367.

[20] T. CHAN, S. OSHER, AND J. SHEN, *The digital TV filter and nonlinear denoising*, Tech. Rep. CAM report 99-34, Department of Mathematics, University of California, Los Angeles, 1999.

[21] D. DOBSON AND F. SANTOSA, *An image enhancement technique for electrical impedance tomography*, Inverse Problems, 10 (1994), pp. 317–334.

[22] D. C. DOBSON AND F. SANTOSA, *Recovery of blocky images from noisy and blurred data*, SIAM Journal on Applied Mathematics, 56 (1996), pp. 1181–1198.

[23] D. C. DOBSON AND O. SCHERZER, *Analysis of regularized total variation penalty methods for denoising*, Inverse Problems, 12 (1996), pp. 601–617.

[24] D. C. DOBSON AND C. R. VOGEL, *Convergence of an iterative method for total variation denoising*, SIAM Journal on Numerical Analysis, 34 (1997), pp. 1779–1791.

[25] I. EKELAND AND R. TEMAM, *Convex Analysis and Variational Problems*, vol. 1 of Studies in Mathematics and Its Applications, North-Holland, Amsterdam, Holland, 1976.

[26] M. FORTIN AND R. GLOWINSKI, *Augmented Lagrangian Methods: Applications to the Numerical Solution of Boundary Value Problems*, vol. 15 of Studies in Mathematics and Its Applications, North-Holland, Amsterdam, Holland, 1983.

[27] D. GEMAN AND G. REYNOLDS, *Constrained restoration and the recovery of discontinuities*, IEEE Transactions on Pattern Analysis and Machine Intelligence, 14 (1992), pp. 367–382.

[28] D. GEMAN AND C. YANG, *Nonlinear image recovery with half-quadratic regularization*, IEEE Transactions on Image Processing, 4 (1995), pp. 932–946.

[29] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, vol. 80 of Monographs in Mathematics, Birkhäuser, Boston, Massachusetts, 1984.

[30] R. GLOWINSKI AND P. L. TALLEC, *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, vol. 9 of SIAM Studies in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1989.

[31] R. C. GONZALEZ AND R. E. WOODS, *Digital Image Processing*, Addison-Wesley, 1993.

[32] W. HACKBUSCH, *Multi-Grid Methods and Applications*, Springer Series in Computational Mathematics, Springer-Verlag, Berlin, Germany, 1985.

[33] U. HERMANN AND D. NOLL, *Adaptive image reconstruction using information measures*, SIAM Journal on Control and Optimization, 38 (2000), pp. 1223–1240.

[34] K. ITO AND K. KUNISCH, *Augmented Lagrangian formulation of nonsmooth, convex optimization in Hilbert spaces*, in Lecture Notes in Pure and Applied Mathematics. Control of Partial Differential Equations and Applications, E. Casas, ed., vol. 174, Marcel Dekker, New York, New York, 1995, pp. 107–117.

[35] ——, *An active set strategy based on the augmented Lagrangian formulation for image restoration*, M2AN Mathematical Modelling and Numerical Analysis, 33 (1999), pp. 1–21.

[36] ——, *Augmented Lagrangian methods for nonsmooth, convex optimization in Hilbert spaces*, Nonlinear Analysis. Theory, Methods & Applications, 41 (2000), pp. 591–616.

[37] ——, *BV-type regularization methods for convoluted objects with edge, flat and grey scales*, Inverse Problems, 16 (2000), pp. 909–928.

[38] A. JAIN, *Fundamentals of digital image processing*, Prentice Hall, New Jersey, 1989.

[39] C. JOHNSON AND A. SOFER, *A primal-dual method for large-scale image reconstruction in emission tomography*, SIAM Journal on Optimization, 11 (2000), pp. 691–715.

[40] J. KAIPIO, V. KOLEHMAINEN, E. SOMERSALO, AND M. VAUHKONEN, *Statistical inversion and Monte Carlo sampling methods in electrical impedance tomography*, Inverse Problems, 16 (2000), pp. 1487–1522.

[41] T. KÄRKKÄINEN, K. KUNISCH, AND P. TARVAINEN, *Primal-dual active set methods for obstacle problems*, Tech. Rep. B 2/2000, Department of Mathematical Information Technology, University of Jyväskylä, 2000.

[42] T. KÄRKKÄINEN AND K. MAJAVA, *Nonmonotone and monotone active set methods for image restoration*, Tech. Rep. B 9/1999, Department of Mathematical Information Technology, University of Jyväskylä, 1999.

[43] T. KÄRKKÄINEN, K. MAJAVA, AND M. M. MÄKELÄ, *Comparison of formulations and solution methods for image restoration problems*, Tech. Rep. B 14/2000, Department of Mathematical Information Technology, University of Jyväskylä, 2000.

[44] I. KNOWLES AND R. WALLACE, *A variational method for numerical differentiation*, Numerische Mathematik, 70 (1995), pp. 91–110.

[45] Y. LI AND F. SANTOSA, *A computational algorithm for minimizing total variation in image restoration*, IEEE Transactions on Image Processing, 5 (1996), pp. 987–995.

[46] M. M. MÄKELÄ AND P. NEITTAANMÄKI, *Nonsmooth Optimization. Analysis and Algorithms with Applications to Optimal Control*, World Scientific, Singapore, 1992.

[47] A. MARQUINA AND S. OSHER, *Explicit algorithms for a new time dependent model based on level set motion for nonlinear deblurring and noise removal*, SIAM Journal on Scientific Computing, 22 (2000), pp. 387–405.

[48] A. MIELE, P. E. MOSELEY, AND E. E. CRAGG, *On the method of multipliers for mathematical programming problems*, Journal of Optimization Theory and Applications, 10 (1972), pp. 1–33.

[49] J. M. MOREL AND S. SOLIMINI, *Variational Methods in Image Segmentation*, vol. 14 of Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, Boston, 1995.

[50] J. NOCEDAL AND S. WRIGHT, *Numerical Optimization*, Springer series in operations research, Springer-Verlag, New York, 1999.

[51] S. OSHER AND L. RUDIN, *Feature-oriented image enhancement using shock filters*, SIAM Journal on Numerical Analysis, 27 (1990), pp. 919–940.

[52] P. PERONA AND J. MALIK, *Scale-space and edge detection using anisotropic diffusion*, IEEE Transactions on pattern analysis and machine intelligence, 12 (1990), pp. 629–639.

[53] C. POLA AND C. SAGASTIZABAL, *Infeasible predictor-corrector interior-point method applied to image restoration in the presence of noise*, Journal of Optimization Theory and Applications, 104 (2000), pp. 517–538.

[54] A. QUARTERONI AND A. VALLI, *Numerical Approximation of Partial Differential Equations*, vol. 23 of Springer series in computational mathematics, Springer-Verlag, Berlin Heidelberg, 1994.

[55] C. R. RAO, *Methodology based on the L_1-norm in statistical inference*, The Indian Journal of Statistics, 50 (1988), pp. 289–313.

[56] W. RING, *Structural properties of solutions to total variation regularization problems*, M2AN Mathematical Modelling and Numerical Analysis, 34 (2000), pp. 799–810.

[57] B. RIPLEY, *Statistics, images, and pattern recognition*, The Canadian Journal of Statistics, 14 (1986), pp. 83–111.

[58] L. I. RUDIN, S. OSHER, AND E. FATEMI, *Nonlinear total variation based noise removal algorithms*, Physica D, 60 (1992), pp. 259–268.

[59] O. SCHERZER, H. W. ENGL, AND K. KUNISCH, *Optimal a posteriori parameter choice for Tikhonov regularization for solving nonlinear ill-posed problems*, SIAM Journal on Numerical Analysis, 30 (1993), pp. 1796–1838.

[60] D. STRONG, *Adaptive total variation minimizing image restoration*, PhD thesis, University of California, Los Angeles, 1997.

[61] A. N. TIKHONOV AND V. Y. ARSENIN, *Solutions of Ill-Posed Problems*, Wiley, New York, 1977.

[62] L. VESE, *A study in the BV space of a denoising-deblurring variational problem*, Applied Mathematics and Optimization, 44 (2001), pp. 131–161.

[63] C. R. VOGEL AND M. E. OMAN, *Iterative methods for total variation denoising*, SIAM Journal on Scientific Computing, 17 (1996), pp. 227–238.

[64] ——, *Fast, robust total variation-based reconstruction of noisy, blurred images*, IEEE Transactions on Image Processing, 7 (1998), pp. 813–824.

YHTEENVETO (FINNISH SUMMARY)

Tutkimuksen tarkoituksena on kehittää luotettavia ja tehokkaita numeerisia menetelmiä kuvan laadun parantamiseksi. Tämä sisältää kaksi tavoitetta: sopivien matemaattisten mallien kehittäminen kuvanparannustehtäville ja tehokkaiden laskennallisten menetelmien kehittäminen malleissa esiintyvien sileiden ja epäsileiden optimointitehtävien ratkaisemiseksi.

Tässä työssä rajoitutaan kohinanpoisto-ongelmaan, jolloin oletetaan, että havaitussa kuvassa ei esiinny muita vääristymiä kuin satunnaista kohinaa. Kohinanpoistotehtävä muotoillaan minimointitehtävänä, joka koostuu ns. pienimmän neliösumman sovitustermistä ja siloitustermistä. Aluksi tarkastellaan ns. BV-tyyppistä (eng. bounded variation) siloitusta, jonka ansiosta kuvasta löydetään epäjatkuvuudet. BV-tyyppinen siloitus tekee kustannusfunktiosta kuitenkin epäsileän. Tämän epäsileän optimointitehtävän ratkaisemiseksi työssä kehitetään ns. aktiivijoukkomenetelmiä, jotka perustuvat alkuperäisen optimointitehtävän siloittamiseen täydennetyllä Lagrangen menetelmällä. Aktiivijoukkoalgoritmien konvergenssit todistetaan ja algoritmeille esitetään tehokkaita numeerisia toteutuksia.

BV-siloituksen ansiosta kuvasta löydetään jyrkät rajapinnat, mutta saadulla ratkaisulla on porrasmainen rakenne. Jotta myös kuvan sileät osat voitaisiin palauttaa paremmin, tarvitaan mukautuvia malleja. Työssä esitetään ns. SAC-menetelmä, joka perustuu osittain mukautuvaan, aidosti konveksiin optimointitehtävään. Sen lisäksi että SAC-menetelmä löytää jyrkät rajapinnat, se palauttaa paremmin myös kuvan sileät osat.

Tarkasteltavissa malleissa ns. siloitusparametri määrää sovitus- ja siloitustermien suhteen. Työssä esitetään myös tapa, jolla siloitusparametri voidaan määrätä automaattisesti tarvitsematta tietää kuvassa olevan kohinan määrää.

Kehitettyjen menetelmien tehokkuutta testataan ja havainnollistetaan numeeristen esimerkkien avulla.

1 ROPPONEN, JANNE, Software risk management - foundations, principles and empirical findings. 273 p. Yhteenveto 1 p. 1999.

2 KUZMIN, DMITRI, Numerical simulation of reactive bubbly flows. 110 p. Yhteenveto 1 p. 1999.

3 KARSTEN, HELENA, Weaving tapestry: collaborative information technology and organisational change. 266 p. Yhteenveto 3 p. 2000.

4 KOSKINEN, JUSSI, Automated transient hypertext support for software maintenance. 98 p. (250 p.) Yhteenveto 1 p. 2000.

5 RISTANIEMI, TAPANI, Synchronization and blind signal processing in CDMA systems. - Synkronointi ja sokea signaalinkäsittely CDMA järjestelmässä. 112 p. Yhteenveto 1 p. 2000.

6 LAITINEN, MIKA, Mathematical modelling of conductive-radiative heat transfer. 20 p. (108 p.) Yhteenveto 1 p. 2000.

7 KOSKINEN, MINNA, Process metamodelling. Conceptual foundations and application. 213 p. Yhteenveto 1 p. 2000.

8 SMOLIANSKI, ANTON, Numerical modeling of two-fluid interfacial flows. 109 p. Yhteenveto 1 p. 2001.

9 NAHAR, NAZMUN, Information technology supported technology transfer process. A multi-site case study of high-tech enterprises. 377 p. Yhteenveto 3 p. 2001.

10 FOMIN, VLADISLAV V., The process of standard making. The case of cellular mobile telephony. - Standardin kehittämisen prosessi. Tapaustutkimus solukkoverkkoon perustuvasta matkapuhelintekniikasta. 107 p. (208 p.) Yhteenveto 1 p. 2001.

11 PÄIVÄRINTA, TERO, A genre-based approach to developing electronic document management in the organization. 190 p. Yhteenveto 1 p. 2001.

12 HÄKKINEN, ERKKI, Design, implementation and evaluation of neural data analysis environment. 229 p. Yhteenveto 1 p. 2001.

13 HIRVONEN, KULLERVO, Towards Better Employment Using Adaptive Control of Labour Costs of an Enterprise. 118 p. Yhteenveto 4 p. 2001.

14 MAJAVA, KIRSI, Optimization-Based Techniques for Image Restoration. 27 p. (142 p.) Yhteenveto 1 p. 2001.