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ON FRACTIONAL SMOOTHNESS AND L_p-APPROXIMATION ON THE GAUSSIAN SPACE

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We consider Gaussian Besov spaces obtained by real interpolation and Riemann–Liouville operators of fractional integration on the Gaussian space and relate the fractional smoothness of a functional to the regularity of its heat extension. The results are applied to study an approximation problem in L_p for $2 \le p < \infty$ for stochastic integrals with respect to the d-dimensional (geometric) Brownian motion.

1. Introduction. This paper is devoted to Besov spaces defined on a Gaussian space, associated Riemann–Liouville operators of fractional integration, and approximation theory. As Gaussian space, we consider $L_p(\mathbb{R}^d, \gamma_d)$ with $2 \le p < \infty$ and $d\gamma_d = e^{-|x|^2/2}\,dx/(2\pi)^{d/2}$ being the d-dimensional standard Gaussian measure. The (Gaussian) Besov spaces are obtained by the real interpolation method and the approximation problem concerns an approximation of stochastic integrals in L_p. Some of the results are extensions of corresponding statements proved mainly in L_2; see [7, 8, 11, 12, 14, 17, 19, 20, 26, 31]. However, the L_2-theory and the L_p-theory for $2 < p < \infty$ on a Gaussian space may differ significantly. For example, the Meyer inequalities can be proved in L_2 using orthogonality by standard ideas, but they are considerably more involved in the L_p-case when $1 < p \neq 2 < \infty$ (see [23], Proposition 1.5.3, and [24]). Another example is the phenomenon that, for instance, for $2 < p < \infty$ and $f \in L_p(\mathbb{R}, \gamma_1)$, the orthogonal Hermite expansion does not necessarily converge in $L_p(\mathbb{R}, \gamma_1)$ (see [25]).

Regarding the multi-step L_p-approximation problem on the Gaussian space for $2 < p < \infty$ we study in this paper, we cannot exploit chaos expansion techniques like in [10] nor can we reduce the problem by orthogonality to a question about a one-step approximation as in the L_2-setting [7, 11] and BMO-setting [12]. The difference between the L_2- and the L_p-context for $2 < p < \infty$ is also visible by

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the fact that we have to describe the optimal L_p-approximation in Theorem 5.5 below by a Riemann–Liouville operator instead of the real interpolation spaces.

To explain the purpose of this paper in more detail, let us introduce some notation. We let $W = (W_t)_{t\in[0,1]}$ be a standard d-dimensional Brownian motion starting in zero defined on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,1]})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is complete and $(\mathcal{F}_t)_{t\in[0,1]}$ is the augmentation of the natural filtration and where we can assume that $\mathcal{F} = \mathcal{F}_1$. As processes driving the stochastic integrals, we use the Brownian motion and the coordinate-wise geometric Brownian motion, that is,

$$Y_t := \big(W_t^{(1)}, \dots, W_t^{(d)}\big)^\top \quad \text{and} \quad E := \mathbb{R}^d$$

or

$$Y_t := \big(e^{W_t^{(1)}-(t/2)}, \dots, e^{W_t^{(d)}-(t/2)}\big)^\top \quad \text{and} \quad E := (0,\infty)^d.$$

Then we have

$$dY_t = \sigma(Y_t)\,dW_t,$$

where Y is considered as a column vector and the $d \times d$-matrix $\sigma(y)$ is given by $\sigma(y) = I_d$ or $(\sigma_{ij}(y))_{i,j=1}^d = (\delta_{i,j} y_i)_{i,j=1}^d$, respectively, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. The parabolic differential operator associated to the diffusion Y is

$$\mathcal{A} := \frac{\partial}{\partial t} + \frac{1}{2}\sum_{k=1}^d \sigma_{kk}^2 \frac{\partial^2}{\partial y_k^2}.$$

Given a Borel-function $g : E \to \mathbb{R}$ with $g(Y_1) \in L_2$, we let

$$G(t,y) := \mathbb{E}\big(g(Y_1)|Y_t = y\big) \tag{1}$$

and notice that $G(1,y) = g(y)$. Integrability properties of G and its derivatives are given in Lemma A.2 below and are used implicitly in this paper. The function G solves the backward parabolic PDE

$$\mathcal{A}G = 0 \qquad \text{on } [0,1) \times E.$$

For $0 \le s < t < 1$, Itô's formula implies that

$$G(t,Y_t) - G(s,Y_s) = \int_s^t \nabla G(u,Y_u)\sigma(Y_u)\,dW_u \qquad \text{a.s.}, \tag{2}$$

where $\nabla G(t,x)$ is considered as a row vector. Furthermore,

$$g(Y_1) = \mathbb{E}g(Y_1) + \int_0^1 \nabla G(u,Y_u)\sigma(Y_u)\,dW_u \qquad \text{a.s.} \tag{3}$$

by $t \uparrow 1$, where the convergence takes place in L_2 [or later in L_p if $g(Y_1) \in L_p$ with $2 \le p < \infty$]. One purpose of this paper is to investigate Riemann approximations of the stochastic integral in (3) by the following quantities.

DEFINITION 1.1. (i) Let $\mathcal{T}^{\mathrm{rand}}$ be the set of all sequences of stopping times $\tau = (\tau_i)_{i=0}^n$ with $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_{n-1} < \tau_n = 1$ where $n = 1, 2, \ldots,$ such that τ_i is $\mathcal{F}_{\tau_{i-1}}$-measurable for $i = 1, \ldots, n-1$, that is,

$$\{\tau_i \in B\} \cap \{\tau_{i-1} \le t\} \in \mathcal{F}_t \qquad \text{for } t \in [0, 1] \text{ and } B \in \mathcal{B}([0, 1]).$$

(ii) Given a time-net $\tau = (\tau_i)_{i=0}^n \in \mathcal{T}^{\mathrm{rand}}$, $0 \le t \le 1$ and $g(Y_1) \in L_2$, we let

$$C_t(g(Y_1), \tau) := \int_0^t \nabla G(s, Y_s)\, dY_s - \sum_{i=1}^n \nabla G(\tau_{i-1}, Y_{\tau_{i-1}})(Y_{\tau_i \wedge t} - Y_{\tau_{i-1} \wedge t}),$$

$$C_t(g(Y_1), \tau, v) := \int_0^t \nabla G(s, Y_s)\, dY_s - \sum_{i=1}^n v_{\tau_{i-1}}(Y_{\tau_i \wedge t} - Y_{\tau_{i-1} \wedge t}),$$

where $v = (v_{\tau_{i-1}})_{i=1}^n$ is a sequence of random row vectors $v_{\tau_{i-1}} : \Omega \to \mathbb{R}^d$ measurable w.r.t. $\mathcal{F}_{\tau_{i-1}}$.

Let us briefly describe the contents of this paper, which continues and extends results from the preprint [28].

(1) In Theorem 3.1, we provide a characterization of functions $f : \mathbb{R}^d \to \mathbb{R}$ belonging to the Besov space $\mathbb{B}_{p,q}^{\theta}(\mathbb{R}^d, \gamma_d)$ by $F : [0, 1] \times \mathbb{R}^d \to \mathbb{R}$ with $F(t, x) := \mathbb{E}(f(W_1)|W_t = x)$. Roughly speaking, considering F as the heat extension of $f \in L_2(\mathbb{R}^d, \gamma_d)$, the regularity of this extension precisely describes the Besov regularity of f. Theorem 3.1 mainly relies on Proposition A.4, which might be of independent interest.

(2) Besides the real interpolation spaces $\mathbb{B}_{p,q}^{\theta}$, the Riemann–Liouville operator $D^{Y,\theta}$ from Section 4 provides an alternative way to describe the fractional regularity of a function $g : E \to \mathbb{R}$. It is defined as a functional of the Hessian matrices $(D^2 G(t, y))_{t \in [0,1)}$ by

$$D_t^{Y,\theta} g(Y_1) := \left(\int_0^t (1-u)^{1-\theta} H_G^2(u, Y_u)\, du \right)^{1/2}$$

with

$$H_G^2(u, y) := \sum_{k,l=1}^d \left| \left(\sigma_{kk}\sigma_{ll} \frac{\partial^2 G}{\partial y_k\, \partial y_l} \right)(u, y) \right|^2.$$

In Proposition 4.2, we relate $D^{Y,\theta}$ to the spaces $\mathbb{B}_{p,q}^{\theta}(\mathbb{R}^d, \gamma_d)$, which continues the analysis in [15], where this operator was used in a different form.

(3) In the literature [3–5, 7, 8, 11, 17, 22], the question of the behavior of the discretization error $C_t(g(Y_1), \tau, v)$ has been treated mostly using the L_2-norm, $\|C_t(g(Y_1), \tau, v)\|_{L_2}$, or by weak or stable limits of the re-scaled error processes $\lim_n \sqrt{n} C_t(g(Y_1), \tau_n, v_n)$, where τ_n is of cardinality $n + 1$. Many of

the L_2-results are of asymptotic nature as well, and, concerning random time-nets, only asymptotic statements were obtained. There is a general lower bound $\|C_1(g(Y_1);\tau)\|_{L_p} \ge \delta/\sqrt{n}$ for nets τ of cardinality $n+1$, see Remark 5.3. As one of the main results of this paper, we obtain in Theorem 5.5 a characterization when this lower bound is actually achieved by special time-nets. A particular case of this statement is:

THEOREM 1.2. *For $2 \le p < \infty$, $0 < \theta \le 1$, and $g(Y_1) \in L_p$, the following assertions are equivalent*:

(i) $\|D_1^{Y,\theta} g(Y_1)\|_{L_p} < \infty$,

(ii) $\sup_{n=1,2,\dots} \sqrt{n}\|C_1(g(Y_1), \tau_n^\theta)\|_{L_p} < \infty$, *where the time-nets* $\tau_n^\theta = (t_{i,n}^\theta)_{i=0}^n$ *are given by* $t_{i,n}^\theta := 1 - (1 - \frac{i}{n})^{1/\theta}$.

Using Proposition 4.2(iii), we can replace, in the case $p = 2$ and $0 < \theta < 1$, condition (i) in the theorem above by $f \in \mathbb{B}_{2,2}^\theta$ [with convention (9)], which is in accordance with the known one-dimensional L_2-case; see [14].

The point of Theorem 1.2 is the usage of adapted time-nets. In the literature, equidistant time-nets are often used in discretizations for simplicity. Therefore, we provide in Theorem 5.7 a description of the random variables that can be approximated in L_p with equidistant time-nets with a rate $n^{-\theta/2}$ for $0 < \theta < 1$ in terms of the Besov spaces $\mathbb{B}_{p,\infty}^\theta$. In particular, this theorem shows the loss of accuracy in the approximation when not using the optimal nets. A special case of this theorem is the following.

THEOREM 1.3. *For $2 \le p < \infty$, $0 < \theta < 1$ and $g(Y_1) \in L_p$ the following assertions are equivalent*:

(i) $f \in \mathbb{B}_{p,\infty}^\theta$ *with f given by* (9),

(ii) $\sup_{n=1,2,\dots} n^{\theta/2}\|C_1(g(Y_1);\tau_n)\|_{L_p} < \infty$, *where* $\tau_n = (i/n)_{i=0}^n$ *are the equidistant time-nets*.

(4) Theorems 1.2 and 1.3 (Theorems 5.5 and 5.7) are based on Theorem 5.1 which extends the curvature type description of the L_2-approximation error from [11] for deterministic nets to the L_p-error $\|C_1(g(Y_1),\tau)\|_{L_p}$ with $2 \le p < \infty$ and to random time-nets $\tau = (\tau_i)_{i=0}^n \in \mathcal{T}^{\mathrm{rand}}$. To illustrate Theorem 5.1, let us formulate a corollary that follows from Remark 5.2.

THEOREM 1.4. *For $2 \le p < \infty$ there is a constant $c_{(1.4)} \ge 1$ depending at most on p such that for all $g(Y_1) \in L_p$ and $\tau = (\tau_i)_{i=0}^n \in \mathcal{T}^{\mathrm{rand}}$ we have that*

$$\|C_1(g(Y_1),\tau)\|_{L_p} \sim_{c_{(1.4)}} \left\| \left(\sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t)\,dt \right)^{1/2} \right\|_{L_p}.$$

For example, to connect Theorem 1.4 to Theorem 1.2, we measure the size of a sequence $0 = t_0 \le \cdots \le t_{n-1} < t_n = 1$ by

$$\big|(t_i)_{i=0}^n\big|_\theta := \frac{|t_i - t_{i-1}|}{(1-t_{i-1})^{1-\theta}} \qquad \text{with } 0 < \theta \le 1.$$

We get $|\tau_n^\theta|_\theta \le 1/(\theta n)$, where the nets τ_n^θ are taken from Theorem 1.2, in contrast to $|(i/n)_{i=0}^n|_\theta = n^{-\theta}$ for the equidistant nets, and Theorem 1.4 yields

$$\big\|C_1\big(g(Y_1),\tau\big)\big\|_{L_p} \le c_{(1.4)}\big\|\sqrt{|\tau|_\theta} D_1^{Y,\theta} g(Y_1)\big\|_{L_p} \qquad \text{for } \tau \in \mathcal{T}^{\text{rand}}$$

so that the implication (i) ⇒ (ii) of Theorem 1.2 follows.

The novelty of Theorem 1.4 (Theorem 5.1) concerns the range $2 < p < \infty$ and the fact that certain fixed random nets (including all deterministic time-nets) are allowed, which distinguishes the result from previous asymptotic ones. As already pointed out, the techniques for the L_p-estimates differ significantly from the L_2-estimates because the problem cannot be translated into a one-step approximation problem nor can we use orthogonality. Moreover, the extension from deterministic nets to random nets does not seem to be straightforward as we still have to use the sub-class $\mathcal{T}^{\text{rand}}$ of random nets $(\tau_i)_{i=1}^n$ where τ_i is $\mathcal{F}_{\tau_{i-1}}$-measurable (see Remark 5.4). Our L_p-estimates can be seen as an interpolation between the L_2-estimates mentioned above and the weighted BMO-estimates from [12]. However, pure interpolation techniques do not seem to be sufficient yet to fully treat our problem.

2. Preliminaries.

Notation. We use $A \sim_c B$ for $A/c \le B \le cA$ whenever $A, B \ge 0$ and $c \ge 1$, $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$, and let $|\cdot|$ be the Euclidean norm for a vector or the Hilbert–Schmidt norm for a matrix. Given a random vector or a random matrix A, we write $\|A\|_{L_p} := \||A|\|_{L_p}$ and denote the transpose of A by $A^\top$.

Real interpolation. Let us recall the real interpolation method that we use to generate the (Gaussian) Besov spaces.

DEFINITION 2.1 ([1, 2]). Let (X_0, X_1) be a compatible couple of Banach spaces, that is, there exists a Hausdorff topological vector space in which both X_0 and X_1 are continuously embedded. Given $x \in X_0 + X_1$ and $\lambda > 0$, the K-functional is defined by

$$K(x,\lambda; X_0, X_1) := \inf\big\{\|x_0\|_{X_0} + \lambda\|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i\big\}.$$

Given $0 < \theta < 1$ and $1 \le q \le \infty$, we let $(X_0, X_1)_{\theta,q}$ be the space of all $x \in X_0 + X_1$ such that

$$\|x\|_{(X_0,X_1)_{\theta,q}} := \big\|\lambda^{-\theta} K(x,\lambda; X_0, X_1)\big\|_{L_q((0,\infty),(d\lambda)/\lambda)} < \infty.$$

The K-functional yields to one of the basic approaches to define intermediate spaces Y of a compatible couple of Banach spaces (X_0, X_1), that is, Banach spaces Y such that one has continues embeddings $X_0 \cap X_1 \hookrightarrow Y \hookrightarrow X_0 + X_1$. Assuming $X_1 \hookrightarrow X_0$ with norm one, this reduces to the embedding $X_1 \hookrightarrow Y \hookrightarrow X_0$. In this case, $K(x, \lambda; X_0, X_1) = \|x\|_{X_0}$ for $\lambda \in [1, \infty)$ which does not give any information. However, for $\lambda \in (0, 1)$ we have that

$$\lambda \|x\|_{X_0} \le K(x, \lambda; X_0, X_1) \le \|x\|_{X_0}.$$

The behavior of the function $\lambda \to K(x, \lambda; X_0, X_1)$ close to zero describes the distance of x to X_1: intuitively we can say that the closer the function is to a linear function in λ, the closer x is to X_1. In general, without the restriction $X_1 \hookrightarrow X_0$, the functionals

$$\left\|\lambda^{-\theta} K(x, \lambda; X_0, X_1)\right\|_{L_q((0,\infty),(d\lambda)/\lambda)}$$

examine the behavior of the K-functional (in particular at zero and at infinity) and lead to the spaces $(X_0, X_1)_{\theta,q}$. For $X_1 \hookrightarrow X_0$, we obtain the lexicographical ordering

$$(X_0, X_1)_{\theta_0,q_0} \subseteq (X_0, X_1)_{\theta_1,q_1} \quad \text{and} \quad (X_0, X_1)_{\eta,r_0} \subseteq (X_0, X_1)_{\eta,r_1}$$

if $0 < \theta_1 < \theta_0 < 1$, $1 \le q_0, q_1 \le \infty$, $0 < \eta < 1$, and $1 \le r_0 \le r_1 \le \infty$. The choice of the measure $d\lambda/\lambda$ ensures (also in the general case) the symmetry $(X_0, X_1)_{\theta,q} = (X_1, X_0)_{1-\theta,q}$.

Gaussian Sobolev and Besov spaces. We let $d \ge 1$ and γ_d be the standard Gaussian measure on $\mathbb{R}^d$. The space $L_2(\mathbb{R}^d, \gamma_d)$ is equipped with the orthonormal basis of generalized Hermite polynomials $(h_{k_1,\dots,k_d})_{k_1,\dots,k_d=0}^{\infty}$ given by

$$h_{k_1,\dots,k_d}(x_1, \dots, x_d) := h_{k_1}(x_1) \cdots h_{k_d}(x_d),$$

where $(h_k)_{k=0}^{\infty} \subset L_2(\mathbb{R}, \gamma_1)$ is the standard orthonormal basis of Hermite polynomials. The Sobolev space $\mathbb{D}_{1,2} = \mathbb{D}_{1,2}(\mathbb{R}^d, \gamma_d)$ consists of all $f \in L_2(\mathbb{R}^d, \gamma_d)$ such that

$$\sum_{k_1,\dots,k_d=0}^{\infty} \langle f, h_{k_1,\dots,k_d}\rangle_{L_2(\mathbb{R}^d,\gamma_d)}^2 \|\nabla h_{k_1,\dots,k_d}\|_{L_2(\mathbb{R}^d,\gamma_d)}^2 < \infty.$$

The space $\mathbb{D}_{1,2}$ is a Banach space under the norm

$$\|f\|_{\mathbb{D}_{1,2}} := \sqrt{\|f\|_{L_2(\mathbb{R}^d,\gamma_d)}^2 + \|Df\|_{L_2(\mathbb{R}^d,\gamma_d)}^2},$$

where, for $f \in \mathbb{D}_{1,2}$, the gradient Df is given by

$$Df := \sum_{k_1,\dots,k_d=0}^{\infty} \langle f, h_{k_1,\dots,k_d}\rangle_{L_2(\mathbb{R}^d,\gamma_d)} \nabla h_{k_1,\dots,k_d}.$$

Given $2 \le p < \infty$, the Banach space $\mathbb{D}_{1,p} \subseteq L_p$ is given by

$$\mathbb{D}_{1,p} := \{f \in \mathbb{D}_{1,2} : \|f\|_{\mathbb{D}_{1,p}} := (\|f\|_{L_p}^p + \|Df\|_{L_p}^p)^{1/p} < \infty\}.$$

Here and later, we use $\|f\|_{L_p} = \|f\|_{L_p(\mathbb{R}^d,\gamma_d)}$ and $\|Df\|_{L_p} = \|Df\|_{L_p(\mathbb{R}^d,\gamma_d)}$.

DEFINITION 2.2. For $0 < \theta < 1$ and $1 \le q \le \infty$, we let

$$\mathbb{B}_{p,q}^{\theta} := (L_p, \mathbb{D}_{1,p})_{\theta,q}$$

be the Gaussian Besov space on $\mathbb{R}^d$ of fractional smoothness θ and fine-index q.

Because $\mathbb{D}_{1,p}$ is not closed in L_p, we get a scale of spaces indexed by (θ, q), where the spaces are identical if and only if both indices coincide (see [21], Theorem 3.1). A typical function which has fractional smoothness is given by the following.

EXAMPLE 2.3. Let $d = 1$, $K \in \mathbb{R}$, $2 \le p < \infty$ and $0 \le \alpha < 1 - \frac{1}{p}$. Then one has that

$$f(x) := \begin{cases} ((x-K)^+)^{\alpha}, & \alpha > 0 \\ \chi_{[K,\infty)}(x), & \alpha = 0 \end{cases} \in \mathbb{B}_{p,\infty}^{(1/p)+\alpha},$$

which shows the trade-off between integrability and smoothness. This can be proved by verifying

$$K(f, \lambda; L_p, \mathbb{D}_{1,p}) \le c\lambda^{(1/p)+\alpha} \qquad \text{for } 0 < \lambda < 1.$$

Using canonical representations of functions of bounded variation, one can extend the case $\alpha = 0$ in Example 2.3 to certain functions of bounded variation by considering convex combinations $f(x) = \sum_{l=1}^{L} \beta_l \chi_{[K_l,\infty)}(x)$.

Burkholder–Davis–Gundy inequality. We use the Burkholder–Davis–Gundy inequality for Brownian martingales with values in a separable Hilbert space. An explicit formulation is as follows: assume for $i = 1, 2, \dots$ progressively measurable processes $(L_t^i)_{t\in[0,1]}$ with $L_t^i : \Omega \to \mathbb{R}^d$ considered as row vectors and such that

$$\sum_{i=1}^{\infty} \mathbb{E} \int_0^1 |L_t^i|^2 \, dt < \infty,$$

then, for all $1 < p < \infty$, there is a constant $c_{(4)} = c_{(4)}(p) \ge 1$ such that

$$\left\| \left(\sum_{i=1}^{\infty} \left| \int_0^1 L_u^i \, dW_u \right|^2 \right)^{1/2} \right\|_{L_p} \sim_{c_{(4)}} \left\| \left(\sum_{i=1}^{\infty} \int_0^1 |L_u^i|^2 \, du \right)^{1/2} \right\|_{L_p}. \tag{4}$$

3. Fractional smoothness on the Gaussian space. In this section, we characterize the Gaussian Besov spaces $\mathbb{B}^{\theta}_{p,q}$ by the behavior of G from (1) in the case $Y = W$. To make this more clear, we do a change of notation and replace g by f and G by F. This means that $f \in L_2(\mathbb{R}^d, \gamma_d)$ and

$$F(t,x) := \mathbb{E} f(x + W_{1-t}) \qquad \text{for } (t,x) \in [0,1] \times \mathbb{R}^d.$$

We also use the Hessian $d \times d$ matrix

$$D^2 F := \left(\frac{\partial^2 F}{\partial x_i \, \partial x_j} \right)_{i,j=1}^{d}.$$

One can check that

$$f \in \mathbb{D}_{1,2} \qquad \text{if and only if } \int_0^1 \| D^2 F(t, W_t) \|_{L_2}^2 \, dt < \infty. \tag{5}$$

Moreover, for all $f \in L_2(\mathbb{R}^d, \gamma_d)$ we have

$$\nabla F(t, W_t) = \nabla F(0,0) + \left(\int_0^t D^2 F(u, W_u) \, dW_u \right)^{\top} \qquad \text{a.s.} \tag{6}$$

for $0 \le t < 1$, where $\nabla F(t,x)$ is considered as a row vector. If $f \in \mathbb{D}_{1,2}$, then (6) can be extended to $t = 1$ with the convention $\nabla F(1, \cdot) := Df$. Now we generalize (5) to the scale of Besov spaces.

THEOREM 3.1. *Let $2 \le p < \infty$, $0 < \theta < 1$, $1 \le q \le \infty$ and $f \in L_p(\mathbb{R}^d, \gamma_d)$. Then*

$$\begin{aligned} \|f\|_{\mathbb{B}^{\theta}_{p,q}} &\sim_{c_{(3.1)}} \|f\|_{L_p} + \left\| (1-t)^{-\theta/2} \| F(1, W_1) - F(t, W_t) \|_{L_p} \right\|_{L_q([0,1),(dt)/(1-t))} \\ &\sim_{c_{(3.1)}} \|f\|_{L_p} + \left\| (1-t)^{(1-\theta)/2} \| \nabla F(t, W_t) \|_{L_p} \right\|_{L_q([0,1),(dt)/(1-t))} \\ &\sim_{c_{(3.1)}} \|f\|_{L_p} + \left\| (1-t)^{(2-\theta)/2} \| D^2 F(t, W_t) \|_{L_p} \right\|_{L_q([0,1),(dt)/(1-t))}, \end{aligned}$$

where $c_{(3.1)} \ge 1$ depends uniquely on (p, θ, q).

REMARK 3.2. Theorem 3.1 generalizes [14], Theorem 2.2, where $p = 2$ was considered, and [28], Lemma 4.7, which was proved for $2 < p < \infty$ and $q = \infty$.

Before we prove Theorem 3.1, we derive a corollary in the case $d = 1$ concerning the oscillation of a Borel function $f : \mathbb{R} \to \mathbb{R}$ given by

$$\mathrm{OSC}_p(f, x_0, s) := \left(\frac{1}{4s^2} \int_{Q(x_0,s)} |f(y) - f(z)|^p \, dy \, dz \right)^{1/p},$$

where $2 \le p < \infty$, $s > 0$, $x_0 \in \mathbb{R}$ and $Q(x_0, s) := \{(y,z) : |y - x_0| \le s, |z - x_0| \le s\}$.

COROLLARY 3.3. *For* $2 \le p < \infty$, $0 < \theta < 1$, $1 \le q \le \infty$ *and* $f \in \mathbb{B}^\theta_{p,q}$, *we have that*

$$\|s^{(1/p)-\theta}\mathrm{OSC}_p(f, x_0, s)\|_{L_q((0,1],(ds)/s)} \le c_{(3.3)}\|f\|_{\mathbb{B}^\theta_{p,q}},$$

where the constant $c_{(3.3)} > 0$ *depends at most on* (p, θ, q, x_0).

PROOF. From [15], Lemma 4.9, we know that

$$\mathrm{OSC}_p(f, x_0, \sqrt{1-t}) \le c(1-t)^{-1/(2p)}\|f(Y) - f(Z)\|_{L_p}$$

for $f \in L_p(\mathbb{R}, \gamma_1)$, $0 \le t < 1$ and a two-dimensional Gaussian vector (Y, Z) with $Y, Z \sim N(0, 1)$ and $\mathrm{cov}(Y, Z) = t$, where $c > 0$ depends at most on (x_0, p). Looking at [15], Proof of Proposition 4.5(iii), we see that

$$\|f(Y) - f(Z)\|_{L_p} \le 2\|f(W_1) - \mathbb{E}(f(W_1)|\mathcal{F}_t)\|_{L_p},$$

so that we can conclude by Theorem 3.1 of this paper. □

Now we turn to the proof of Theorem 3.1. We start with the following proposition.

PROPOSITION 3.4. *Let* $2 \le p < \infty$. *There exists a constant* $c_{(3.4)} \ge 1$ *depending at most on* p *such that for any* $0 < t < 1$,

$$K(f, \sqrt{1-t}; L_p, \mathbb{D}_{1,p}) \sim_{c_{(3.4)}} (\|f(W_1) - F(t, W_t)\|_{L_p} + \sqrt{1-t}\|f\|_{L_p}).$$

PROOF. (a) Fix $0 < t < 1$ and $\varepsilon > 0$. We find $f_0 \in L_p$ and $f_1 \in \mathbb{D}_{1,p}$ such that $f = f_0 + f_1$ and

$$\|f_0\|_{L_p} + \sqrt{1-t}\|f_1\|_{\mathbb{D}_{1,p}} \le K(f, \sqrt{1-t}; L_p, \mathbb{D}_{1,p}) + \epsilon.$$

For $F_i(t, x) := \mathbb{E}(f_i(W_1)|W_t = x)$ we obtain from (2) and (4) that

$$\begin{aligned}
&\|f(W_1) - F(t, W_t)\|_{L_p} \\
&\quad\le \|f_0(W_1) - F_0(t, W_t)\|_{L_p} + \left\|\int_t^1 \nabla F_1(u, W_u)\, dW_u\right\|_{L_p} \\
&\quad\le \|f_0(W_1) - F_0(t, W_t)\|_{L_p} + c_{(4)}\left(\int_t^1 \|\nabla F_1(u, W_u)\|^2_{L_p}\, du\right)^{1/2} \\
&\quad\le 2\|f_0\|_{L_p} + c_{(4)}\sqrt{1-t}\|f_1\|_{\mathbb{D}_{1,p}} \\
&\quad\le c[K(f, \sqrt{1-t}; L_p, \mathbb{D}_{1,p}) + \epsilon],
\end{aligned}$$

where $c := \max\{c_{(4)}, 2\}$ and we employed the facts that $2 \le p < \infty$ and that (6) yields

$$\|\nabla F_1(u, W_u)\|_{L_p} \le \|f_1\|_{\mathbb{D}_{1,p}} \qquad \text{for all } 0 \le u \le 1.$$

Letting $\epsilon \to 0$ and observing that $\sqrt{1-t}\|f\|_{L_p} \le K(f, \sqrt{1-t}; L_p, \mathbb{D}_{1,p})$ we achieve the first part of the desired inequality.

(b) For $0 < t < 1$, we set

$$g_t(x) := F(t, \sqrt{t}x) \quad \text{and} \quad h_t(x) := f(x) - F(t, \sqrt{t}x)$$

so that

$$\begin{aligned}\|g_t\|_{\mathbb{D}_{1,p}}^p &= \|F(t,\sqrt{t}x)\|_{L_p}^p + \|\nabla F(t,\sqrt{t}x)\sqrt{t}\|_{L_p}^p \\ &\le \|f\|_{L_p}^p + \|\nabla F(t, W_t)\|_{L_p}^p.\end{aligned}$$

Applying (2) for $Y = W$, (4), the fact that $\|\nabla F(t, W_t)\|_{L_p}$ is nondecreasing in t and that $2 \le p < \infty$, we estimate

$$\begin{aligned}\|F(t, W_t)\|_{L_p} &\le \left\|\int_0^t \nabla F(u, W_u)\, dW_u\right\|_{L_p} + \|F(0, W_0)\|_{L_p} \\ &\le c_{(4)}\|\nabla F(t, W_t)\|_{L_p} + |\mathbb{E}f(W_1)|.\end{aligned}$$

Thus,

$$\begin{aligned}\|g_t\|_{\mathbb{D}_{1,p}} &\le \|f(W_1) - F(t, W_t)\|_{L_p} + (1 + c_{(4)})\|\nabla F(t, W_t)\|_{L_p} + |\mathbb{E}f(W_1)| \\ &\le [1 + (1 + c_{(4)})c_{(A.3)}(1-t)^{-1/2}]\|f(W_1) - F(t, W_t)\|_{L_p} + |\mathbb{E}f(W_1)|,\end{aligned}$$

where we used Lemma A.3. Exploiting an independent Brownian motion $\widetilde{W}$ and the fact that the covariance structures of $(W_1, \sqrt{t}W_1 + \sqrt{1-t}\widetilde{W}_1)$ and $(W_1, W_{\sqrt{t}} + \widetilde{W}_{1-\sqrt{t}})$ are the same, we obtain for h_t that

$$\begin{aligned}\|h_t\|_{L_p} &= [\mathbb{E}|f(W_1) - \widetilde{\mathbb{E}}f(\sqrt{t}W_1 + \sqrt{1-t}\widetilde{W}_1)|^p]^{1/p} \\ &\le [\mathbb{E}\widetilde{\mathbb{E}}|f(W_1) - f(W_{\sqrt{t}} + \widetilde{W}_{1-\sqrt{t}})|^p]^{1/p} \\ &\le \|f(W_1) - F(\sqrt{t}, W_{\sqrt{t}})\|_{L_p} + \|F(\sqrt{t}, W_{\sqrt{t}}) - f(W_{\sqrt{t}} + \widetilde{W}_{1-\sqrt{t}})\|_{L_p} \\ &= 2\|f(W_1) - F(\sqrt{t}, W_{\sqrt{t}})\|_{L_p} \\ &\le 4\|f(W_1) - F(t, W_t)\|_{L_p},\end{aligned}$$

where in the last step $F(t, W_t)$ was inserted. Hence,

$$\begin{aligned}&K(f, \sqrt{1-t}; L_p, \mathbb{D}_{1,p}) \\ &\quad\le \|h_t\|_{L_p} + (1-t)^{1/2}\|g_t\|_{\mathbb{D}_{1,p}} \\ &\quad\le (1-t)^{1/2}|\mathbb{E}f(W_1)| + [5 + (1 + c_{(4)})c_{(A.3)}]\|f(W_1) - F(t, W_t)\|_{L_p}\end{aligned}$$

and the proof is complete. □

We are now ready to prove the main result of this section.

PROOF OF THEOREM 3.1. To verify the assumptions of Proposition A.4, we set

$$d^0(t) := \|f(W_1) - F(t, W_t)\|_{L_p},$$
$$d^1(t) := \|\nabla F(t, W_t)\|_{L_p},$$
$$d^2(t) := \|D^2 F(t, W_t)\|_{L_p},$$

$A := 2c_{(A.3)}\|f\|_{L_p}$ and $\alpha := c_{(4)} \vee c_{(A.3)}$. Then Lemma A.3 implies that

$$d^k(t) \le c_{(A.3)}(1-t)^{-k/2} d^0(t) \qquad \text{for } k = 1, 2.$$

By (2), (6), the Burkholder–Davis–Gundy inequalities (4) and $2 \le p < \infty$, we also see that

$$d^0(t) \le c_{(4)}\left(\int_0^t \|\nabla F(s, W_s)\|_{L_p}^2 \, ds\right)^{1/2} = c_{(4)}\left(\int_0^t [d^1(s)]^2 \, ds\right)^{1/2}$$

and

$$\begin{aligned} d^1(t) &\le \|\nabla F(0, W_0)\|_{L_p} + \left\|\int_0^t D^2 F(s, W_s)\, dW_s\right\|_{L_p} \\ &\le 2c_{(A.3)}\|f\|_{L_p} + c_{(4)}\left(\int_0^t \|D^2 F(s, W_s)\|_{L_p}^2 \, ds\right)^{1/2} \\ &= 2c_{(A.3)}\|f\|_{L_p} + c_{(4)}\left(\int_0^t [d^2(s)]^2 \, ds\right)^{1/2}, \end{aligned}$$

where we used Lemma A.3. Now, applying (21) on page 634 gives the equivalence between the last three expressions in Theorem 3.1. It remains to check that

$$\|f\|_{\mathbb{B}^\theta_{p,q}} \sim_c \|f\|_{L_p} + \left\|(1-t)^{-\theta/2}\|F(1, W_1) - F(t, W_t)\|_{L_p}\right\|_{L_q([0,1),(dt)/(1-t))}$$

for some $c = c(p, q, \theta) \ge 1$, which follows from Proposition 3.4. □

REMARK 3.5. In the literature, interpolation spaces on the Wiener (or Gaussian) space are considered, for example, in [14, 15, 18, 28, 30]. A classical approach is based on semi-groups. Instead of that, our approach uses the elementary Proposition A.4, which is not related to semi-groups and makes it therefore possible to apply Proposition A.4 in more general situations (see [13]). Regarding the present paper, Proposition A.4 opens the way to extend results from Sections 4 and 5 below to processes different from the (geometric) Brownian motion. Below we want to indicate a possible semi-group approach to Theorem 3.1:

(1) The first equivalence of Theorem 3.1 can be deduced in the case $q = p$ (q is the fine-tuning index in the interpolation, L_p the integrability of the underlying spaces) from [18], Remark on page 428. Using the simple observation [9], equation (6), one can transform Hirsch's condition into

$$\int_0^\infty s^{-(\theta p)/2} \| f(W_1) - F(e^{-s}, W_{e^{-s}}) \|_p^p \frac{ds}{s},$$

which is our condition, up to a different scaling.

(2) To consider the general case, that is, $q \neq p$, and also the other equivalences in Theorem 3.1, one can check general results about interpolation and semi-groups. There are two natural semi-groups one might use, the Ornstein–Uhlenbeck semi-group and the Poisson semi-group (see [27]). Roughly speaking, switching from the Ornstein–Uhlenbeck semi-group to the Poisson semi-group should result in a change of the main interpolation parameter η to our parameter $\theta = \eta/2$ in the corresponding formulas, cf. [29], Section 1.15.2. Now assume $2 \le p < \infty$ and $\xi = f(W_1) \in L_p$ and let $(T_s)_{s \ge 0}$ be the Ornstein–Uhlenbeck semi-group on L_p with generator Λ. Then [29], Section 1.13.2, gives

$$\| f \|_p + \big\| s^{-\eta} \| T_s \xi - \xi \|_p \big\|_{L_q((0,\infty),(ds)/s)} < \infty \tag{7}$$

for the interpolation space $(L_p, D(\Lambda))_{\eta,q}$ with $0 < \eta < 1$ and $1 \le q \le \infty$. By Mehler's formula, we have

$$T_s \xi = \widetilde{\mathbb{E}} f\big(f e^{-s} W_1 + \sqrt{1 - e^{-2s}} \widetilde{W}_1\big) = F\big(e^{-2s}, e^{-s} W_1\big)$$

for an independent Brownian motion $\widetilde{W}$. This would give a comparable statement to the first equivalence of Theorem 3.1 for the Ornstein–Uhlenbeck semi-group. To come closer to our statement, one can inspect the proof of Proposition 3.4, which gives $\| T_s \xi - \xi \|_p \le 4 \| f(W_1) - F(e^{-2s}, W_{e^{-2s}}) \|_p$. Inserting this upper bound into (7) would give an expression like in the first equivalence of Theorem 3.1. To get the full statement one would still need to try to upper bound $\| f(W_1) - F(t, W_t) \|_p$ by $\| T_s \xi - \xi \|_p$ in an appropriate way or to find an alternative way. Concerning the second and third equivalence of Theorem 3.1 one might try to exploit [29], Section 1.14.5.

4. The Riemann–Liouville operator $D^{Y,\theta}$. Riemann–Liouville type operators are typically used to describe fractional regularity. We use these operators to replace the Besov regularity defined by real interpolation when we consider the approximation along adapted time-nets in Theorem 5.5 below. The operator, introduced in the following Definition 4.1, was also used in a slightly modified form in [15], where the weak convergence of the error processes was considered.

DEFINITION 4.1. For $g(Y_1) \in L_2$, $0 < \theta \le 1$ and $0 \le t \le 1$, we let

$$D_t^{Y,\theta} g(Y_1) := \left(\int_0^t (1-u)^{1-\theta} H_G^2(u, Y_u)\, du \right)^{1/2},$$

where, with G given by (1),

$$H_G^2(u,y) := \sum_{k,l=1}^{d} \left| \left(\sigma_{kk}\sigma_{ll} \frac{\partial^2 G}{\partial y_k \, \partial y_l} \right)(u,y) \right|^2.$$

From now on, we use the following convention: for $x = (x_1, \dots, x_d)^\top \in \mathbb{R}^d$ and $0 \le t \le 1$ we let

$$(8) \qquad y_k(t) = \begin{cases} x_k, & Y = W \\ e^{x_k - (t/2)}, & \text{else} \end{cases} \quad \text{and} \quad y(t) := \big(y_1(t), \dots, y_d(t)\big)^\top$$

and define the functions $f : \mathbb{R}^d \to \mathbb{R}$ and $F : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ as

$$(9) \qquad f(x) := g\big(y(1)\big) \quad \text{and} \quad F(t,x) := \mathbb{E} f(x + W_{1-t})$$

so that $f(W_1) = g(Y_1)$ and $F(t,x) = G(t, y(t))$. In the case that Y is the coordinate-wise geometric Brownian motion, this notation implies that

$$(10) \qquad y_k(t) y_l(t) \frac{\partial^2 G}{\partial y_k \, \partial y_l}\big(t, y(t)\big) = \frac{\partial^2 F}{\partial x_k \, \partial x_l}(t,x) - \delta_{k,l} \frac{\partial F}{\partial x_k}(t,x)$$

for $k, l = 1, \dots, d$. Let us summarize the connections between the Besov spaces and the operator $D^{Y,\theta}$ known to us.

PROPOSITION 4.2. *For $g(Y_1) \in L_p$ with $2 \le p < \infty$, the following assertions hold true*:

(i) *If $2 < p < \infty$ and $0 < \theta < 1$, then*

(a) $f \in \mathbb{B}_{p,2}^{\theta}$ *implies* $D_1^{Y,\theta} g(Y_1) \in L_p$,

(b) $D_1^{Y,\theta} g(Y_1) \in L_p$ *implies* $f \in \mathbb{B}_{p,\infty}^{\theta}$.

(ii) *If $2 \le p < \infty$, then $D_1^{Y,1} g(Y_1) \in L_p$ if and only if $f \in \mathbb{D}_{1,p}$.*

(iii) *If $0 < \theta < 1$, then $D_1^{Y,\theta} g(Y_1) \in L_2$ if and only if $f \in \mathbb{B}_{2,2}^{\theta}$.*

PROOF. (i)

(a) Because $2 \le p < \infty$, we see that

$$\begin{aligned} \| D_1^{Y,\theta} g(Y_1) \|_{L_p} &\le \left(\int_0^1 (1-t)^{1-\theta} \| H_G(t, Y_t) \|_{L_p}^2 \, dt \right)^{1/2} \\ &= \big\| (1-t)^{(2-\theta)/2} \| H_G(t, Y_t) \|_{L_p} \big\|_{L_2([0,1),(dt)/(1-t))}. \end{aligned}$$

Theorem 3.1 completes the proof, since in the case that Y is the Brownian motion, we have $H_G(t, Y_t) = |D^2 F(t, W_t)|$ and in the other case, we can use (10) and Theorem 3.1 again to see that

$$\big\| (1-t)^{(2-\theta)/2} \| H_G(t, Y_t) \|_{L_p} \big\|_{L_2([0,1),(dt)/(1-t))} < \infty.$$

(b) For all $0 < t \le 1$,

$$\|D_1^{Y,\theta} g(Y_1)\|_{L_p} \ge \left\| \left(\int_0^t (1-s)^{1-\theta} H_G^2(s, Y_s)\, ds \right)^{1/2} \right\|_{L_p}$$
$$\ge (1-t)^{(1-\theta)/2} \left\| \left(\int_0^t H_G^2(s, Y_s)\, ds \right)^{1/2} \right\|_{L_p}.$$

If Y is the Brownian motion, then we can bound this from below by

$$\frac{1}{c_{(4)}} (1-t)^{(1-\theta)/2} \|\nabla F(t, W_t) - \nabla F(0, W_0)\|_{L_p},$$

where we have used (4) and (6). This implies that

$$\|\nabla F(t, W_t)\|_{L_p} \le \|\nabla F(0, W_0)\|_{L_p} + c_{(4)} (1-t)^{(\theta-1)/2} \|D_1^{W,\theta} f(W_1)\|_{L_p}$$

and Theorem 3.1 can be used again. If Y is the coordinate-wise geometric Brownian motion, then we get from (10) that

$$\left\| \left(\int_0^t H_G^2(s, Y_s)\, ds \right)^{1/2} \right\|_{L_p}$$
$$\ge \left\| \left(\int_0^t H_F^2(s, W_s)\, ds \right)^{1/2} \right\|_{L_p} - \left\| \left(\int_0^1 |\nabla F(s, W_s)|^2\, ds \right)^{1/2} \right\|_{L_p}$$
$$\ge \frac{1}{c_{(4)}} \|\nabla F(t, W_t) - \nabla F(0, W_0)\|_{L_p} - \left\| \left(\int_0^1 |\nabla F(s, W_s)|^2\, ds \right)^{1/2} \right\|_{L_p}$$
$$\ge \frac{1}{c_{(4)}} \|\nabla F(t, W_t)\|_{L_p} - \frac{1}{c_{(4)}} \|\nabla F(0, W_0)\|_{L_p}$$
$$- \left\| \left(\int_0^1 |\nabla F(s, W_s)|^2\, ds \right)^{1/2} \right\|_{L_p},$$

where we again used the Burkholder–Davis–Gundy inequalities (4). Because the last two terms on the right-hand side are finite, we can conclude as in the case of the Brownian motion.

(ii) Because of (10) and $(\int_0^1 |\nabla F(t, W_t)|^2\, dt)^{1/2} \in L_p$, we get $D_1^{Y,1} g(Y_1) \in L_p$ if and only if $(\int_0^1 |D^2 F(t, W_t)|^2\, dt)^{1/2} \in L_p$. Using relations (5) and (6), one easily checks that this is equivalent to $f \in \mathbb{D}_{1,p}$.

(iii) Since (10) implies the equivalence of

$$\|D_1^{Y,\theta} g(Y_1)\|_{L_2}^2 = \int_0^1 (1-t)^{1-\theta} \|H_G(t, Y_t)\|_{L_2}^2\, dt < \infty$$

and $\int_0^1 (1-t)^{1-\theta} \|D^2 F(t, W_t)\|_{L_2}^2\, dt < \infty$, we can use Theorem 3.1. $\square$

5. An approximation problem in L_p. In the whole section, we use the convention (8) and (9).

Time-nets. Given a sequence $0 = t_0 \le \cdots \le t_{n-1} < t_n = 1$ and $0 < \theta \le 1$, we let

$$\begin{aligned}\left|(t_i)_{i=0}^n\right|_\theta &:= \sup_{i=1,\ldots,n} \sup_{t_{i-1}\le u<t_i} \frac{|t_i - u|}{(1-u)^{1-\theta}} = \sup_{i=1,\ldots,n} \frac{|t_i - t_{i-1}|}{(1-t_{i-1})^{1-\theta}},\\ \left|(t_i)_{i=0}^n\right| &:= \left|(t_i)_{i=0}^n\right|_1\end{aligned}$$

so that $|(t_i)_{i=0}^n|$ is the usual mesh-size. As special adapted deterministic time-nets we use $\tau_n^\theta = (t_{i,n}^\theta)_{i=0}^n$ defined by

$$t_{i,n}^\theta := 1 - \left(1 - \frac{i}{n}\right)^{1/\theta}.$$

For these time-nets,

$$|t_{i,n}^\theta - u| \le \frac{|t_{i,n}^\theta - u|}{(1-u)^{1-\theta}} \le \frac{|t_{i,n}^\theta - t_{i-1,n}^\theta|}{(1-t_{i-1,n}^\theta)^{1-\theta}} \le \frac{1}{\theta n} \qquad \text{for } u \in [t_{i-1,n}^\theta, t_{i,n}^\theta), \tag{11}$$

which implies that

$$|\tau_n^\theta| \le |\tau_n^\theta|_\theta \le \frac{1}{\theta n}. \tag{12}$$

Moreover, we have that

$$\frac{(1-t_{i-1,n}^\theta)^{1-\theta}}{|t_{i,n}^\theta - t_{i-1,n}^\theta|} \le \beta n \tag{13}$$

for some $\beta > 0$ independent from n.

The basic equivalence in L_p. The following result reduces the computation of the L_p-norm of the error processes defined in Definition 1.1 to an expression involving the curvature $H_G(t, Y_t)$ similar to a square function. This result generalizes [11], Theorem 4.4, proved for deterministic nets in the L_2-case.

THEOREM 5.1. *For $2 \le p < \infty$, there is a constant $c_{(5.1)} \ge 1$ depending at most on p such that for all $g(Y_1) \in L_p$ and $\tau = (\tau_i)_{i=0}^n \in \mathcal{T}^{\mathrm{rand}}$ we have that*

$$\|C_1(g(Y_1), \tau)\|_{L_p} \le c_{(5.1)} \left\| \left(\sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t)\, dt \right)^{1/2} \right\|_{L_p},$$

$$\inf_v \|C_1(g(Y_1), \tau, v)\|_{L_p} \ge \frac{1}{c_{(5.1)}} \left\| \left(\sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t)\, dt \right)^{1/2} \right\|_{L_p},$$

where the infimum is taken over all simple random vectors $v_{\tau_{i-1}}:\Omega\to\mathbb{R}^d$ *that are* $\mathcal{F}_{\tau_{i-1}}$*-measurable.*

REMARK 5.2. Both inequalities in Theorem 5.1 are proved by stopping at $0<T<1$ and letting $T\uparrow 1$. Therefore, it might be possible for one or both sides of an inequality to be infinite. However, this cannot be the case: step (b) of our proof for the trivial time-net $0=t_0<t_1=1$ gives by (15) that

$$\begin{aligned}\left\|\left(\int_0^T (T-t)H_G^2(t,Y_t)\,dt\right)^{1/2}\right\|_{L_p} &\le c\big\|\mathbb{E}\big(g(Y_1)|\mathcal{F}_T\big)-\mathbb{E}g(Y_1)-\nabla G(0,Y_0)(Y_T-Y_0)\big\|_{L_p}\\ &\le c\big\|g(Y_1)-\mathbb{E}g(Y_1)-\nabla G(0,Y_0)(Y_1-Y_0)\big\|_{L_p}<\infty\end{aligned}$$

so that

$$\begin{aligned}\left\|\left(\sum_{i=1}^n\int_{\tau_{i-1}}^{\tau_i}(\tau_i-t)H_G^2(t,Y_t)\,dt\right)^{1/2}\right\|_{L_p} \le \left\|\left(\int_0^1(1-t)H_G^2(t,Y_t)\,dt\right)^{1/2}\right\|_{L_p}<\infty.\end{aligned}$$

Following (16) from step (c), this implies that $\sup_{0\le T<1}\|C_T(g(Y_1),\tau)\|_{L_p}<\infty$, from which we can conclude that

$$\left(\int_0^1\sum_{i=1}^n\chi_{(\tau_{i-1},\tau_i]}(t)\big|\nabla G(\tau_{i-1},Y_{\tau_{i-1}})\sigma(Y_t)\big|^2\,dt\right)^{1/2}\in L_p$$

and $C_1(g(Y_1),\tau)\in L_p$. Finally, we have that

$$\inf_v\big\|C_1\big(g(Y_1),\tau,v\big)\big\|_{L_p}\le\big\|C_1\big(g(Y_1),\tau\big)\big\|_{L_p}, \tag{14}$$

where the infimum is taken over all simple random vectors $v_{\tau_{i-1}}:\Omega\to\mathbb{R}^d$ that are $\mathcal{F}_{\tau_{i-1}}$-measurable. The latter also implies that all three expressions—in particular the simple and optimal L_p-approximation—in Theorem 5.1 are equivalent up to a multiplicative constant.

REMARK 5.3. Theorem 5.1 provides an alternative way to prove the lower bound

$$\big\|C_1\big(g(Y_1),\tau^n\big)\big\|_{L_p}\ge\frac{\delta}{\sqrt{n}}$$

for some $\delta>0$, all $n=1,2,\dots$ and all nets $\tau^n=(\tau_i)_{i=1}^n\in\mathcal{T}^{\mathrm{rand}}$ whenever there is no row vector $v_0\in\mathbb{R}^d$ such that $g(Y_1)=\mathbb{E}g(Y_1)+v_0(Y_1-Y_0)$ a.s. This lower bound was obtained in [8] in the one-dimensional case using an asymptotic argument. To check the lower bound, observe for $0 \le a < b \le 1$ and $\rho_i := (a \vee \tau_i) \wedge b$ that

$$\begin{aligned}
&\left\| \left(\sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t)\, dt \right)^{1/2} \right\|_{L_p} \\
&\quad \ge \left\| \left(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) H_G^2(t, Y_t)\, dt \right)^{1/2} \right\|_{L_p} \\
&\quad \ge \left\| \inf_{a \le t < b} H_G(t, Y_t) \sum_{i=1}^{n} \frac{\rho_i - \rho_{i-1}}{\sqrt{2n}} \right\|_{L_p} \\
&\quad = \frac{b-a}{\sqrt{2n}} \left\| \inf_{a \le t < b} H_G(t, Y_t) \right\|_{L_p}.
\end{aligned}$$

Assume now $\sup_{0 \le a < b \le 1} \| \inf_{a \le t < b} H_G(t, Y_t) \|_{L_p} = 0$ and fix $0 < T < 1$ and $0 < a_n \uparrow T$. Then

$$0 = \lim_n \left\| \inf_{a_n \le t < T} H_G(t, Y_t) \right\|_{L_p} = \left\| \lim_n \inf_{a_n \le t < T} H_G(t, Y_t) \right\|_{L_p} = \| H_G(T, Y_T) \|_{L_p}$$

so that $H_G(T, Y_T) = 0$ a.s. for all $0 < T < 1$. Applying Theorem 5.1 for the trivial time-net $\{0, 1\}$ yields

$$\| g(Y_1) - \mathbb{E} g(Y_1) - \nabla G(0, Y_0)(Y_1 - Y_0) \|_{L_p} = 0.$$

PROOF OF THEOREM 5.1. (a) Assume a deterministic time $0 < T < 1$, two stopping times $0 \le a \le b \le T$ and that v_a is a simple $\mathcal{F}_a$-measurable random (row) vector. Exploiting relations (6) and (10) one quickly checks that

$$\left(\frac{\partial G}{\partial y_k}(b, Y_b) - v_a^k \right) \sigma_{kk}(Y_b) = m_a(k) + \sum_{l=1}^{d} \int_a^b \lambda_u^a(k, l)\, dW_u^l$$

with

$$m_a(k) := \left(\frac{\partial G}{\partial y_k}(a, Y_a) - v_a^k \right) \sigma_{kk}(Y_a)$$

and

$$\lambda_u^a(k, l) := \left(\sigma_{kk} \sigma_{ll} \frac{\partial^2 G}{\partial y_k \, \partial y_l} \right)(u, Y_u) + \left(\frac{\partial G}{\partial y_l}(u, Y_u) - v_a^l \right) \left(\sigma_{ll} \frac{\partial \sigma_{ll}}{\partial y_k} \right)(Y_u),$$

where $m_a := (m_a(1), \ldots, m_a(d))$ will be considered as a row vector.

(b) *Lower bound for* $\| C_1(g(Y_1), \tau, v) \|_{L_p}$: Let us fix $0 < T < 1$ and define $\rho_i := \tau_i \wedge T$ and $\alpha_{\rho_{i-1}} := v_{\tau_{i-1}} \chi_{\{\tau_{i-1} < T\}}$. Note that ρ_i and $\alpha_{\rho_{i-1}}$ are $\mathcal{F}_{\rho_{i-1}}$-measurable for

$i = 1, \dots, n$. Replacing v by α in the definitions of m and λ from step (a), it follows that

$$C_T\big(g(Y_1), \tau, v\big) = \sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} \bigg[m_{\rho_{i-1}} + \bigg[\int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} \, dW_u \bigg]^\top \bigg] dW_t.$$

Using (4) and the convexity inequality [6], pp. 104–105, p. 171, we achieve

$$\begin{aligned}
&\bigg\| \sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} \bigg[m_{\rho_{i-1}} + \bigg[\int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} \, dW_u \bigg]^\top \bigg] dW_t \bigg\|_{L_p}^p \\
&\quad \ge c_{(4)}^{-p} \mathbb{E} \Bigg(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} \bigg| m_{\rho_{i-1}} + \bigg[\int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} \, dW_u \bigg]^\top \bigg|^2 dt \Bigg)^{p/2} \\
&\quad \ge c_{(4)}^{-p} (p/2)^{-p/2} \mathbb{E} \Bigg(\sum_{i=1}^{n} \mathbb{E}_{\mathcal{F}_{\rho_{i-1}}} \int_{\rho_{i-1}}^{\rho_i} \bigg| m_{\rho_{i-1}} + \bigg[\int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} \, dW_u \bigg]^\top \bigg|^2 dt \Bigg)^{p/2} \\
&\quad \ge \big(c_{(4)} \sqrt{p/2}\big)^{-p} \mathbb{E} \Bigg(\sum_{i=1}^{n} \mathbb{E}_{\mathcal{F}_{\rho_{i-1}}} \int_{\rho_{i-1}}^{\rho_i} |m_{\rho_{i-1}}|^2 \, dt \Bigg)^{p/2} \\
&\quad = \big(c_{(4)} \sqrt{p/2}\big)^{-p} \mathbb{E} \Bigg(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} |m_{\rho_{i-1}}|^2 \, dt \Bigg)^{p/2},
\end{aligned}$$

where we used the assumption that ρ_i is $\mathcal{F}_{\rho_{i-1}}$-measurable. From this, we deduce that

$$\begin{aligned}
&\bigg\| \sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} \bigg[\int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} \, dW_u \bigg]^\top dW_t \bigg\|_{L_p} \\
&\qquad \le \bigg\| \sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} m_{\rho_{i-1}} \, dW_t \bigg\|_{L_p} \\
&\qquad\qquad + \bigg\| \sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} \bigg[m_{\rho_{i-1}} + \bigg[\int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} \, dW_u \bigg]^\top \bigg] dW_t \bigg\|_{L_p} \\
&\qquad \le c_{(4)} \Bigg\| \Bigg(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} |m_{\rho_{i-1}}|^2 \, dt \Bigg)^{1/2} \Bigg\|_{L_p} \\
&\qquad\qquad + \bigg\| \sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} \bigg[m_{\rho_{i-1}} + \bigg[\int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} \, dW_u \bigg]^\top \bigg] dW_t \bigg\|_{L_p} \\
&\qquad \le \big[c_{(4)}^2 \sqrt{p/2} + 1\big] \bigg\| \sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} \bigg[m_{\rho_{i-1}} + \bigg[\int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} \, dW_u \bigg]^\top \bigg] dW_t \bigg\|_{L_p}
\end{aligned}$$

so that

$$\begin{aligned}&\|C_T(g(Y_1),\tau,v)\|_{L_p}\\&\qquad\geq [c_{(4)}^2\sqrt{p/2}+1]^{-1}\left\|\sum_{i=1}^n\int_{\rho_{i-1}}^{\rho_i}\left[\int_{\rho_{i-1}}^{t\vee\rho_{i-1}}\lambda_u^{\rho_{i-1}}dW_u\right]^\top dW_t\right\|_{L_p}.\end{aligned}$$

We continue by writing

$$\begin{aligned}&\left\|\sum_{i=1}^n\int_{\rho_{i-1}}^{\rho_i}\left[\int_{\rho_{i-1}}^{t\vee\rho_{i-1}}\lambda_u^{\rho_{i-1}}dW_u\right]^\top dW_t\right\|_{L_p}\\&\qquad=\left\|\sum_{i=1}^n\int_0^1\left[\int_0^t[\chi_{(\rho_{i-1},t\vee\rho_{i-1}]}(u)\chi_{(\rho_{i-1},\rho_i]}(t)\lambda_u^{\rho_{i-1}}]\,dW_u\right]^\top dW_t\right\|_{L_p}\\&\qquad=\left\|\int_0^1\left[\int_0^t\mu^\rho(t,u)\,dW_u\right]^\top dW_t\right\|_{L_p}\end{aligned}$$

with the $d\times d$-matrix

$$\mu^\rho(t,u):=\sum_{i=1}^n\chi_{(\rho_{i-1},t]}(u)\chi_{(\rho_{i-1},\rho_i]}(t)\lambda_u^{\rho_{i-1}}=\sum_{i=1}^n\chi_{\{\rho_{i-1}<u\leq t\leq\rho_i\}}\lambda_u^{\rho_{i-1}}.$$

Here, we used again the condition that ρ_i is $\mathcal{F}_{\rho_{i-1}}$-measurable. By (4) and Lemma A.1 (note that $\rho_i\leq T<1$),

$$\begin{aligned}&\left\|\int_0^1\left[\int_0^t\mu^\rho(t,u)\,dW_u\right]^\top dW_t\right\|_{L_p}\\&\qquad\sim_{c_{(4)}}\left\|\left(\int_0^1\left|\int_0^1\mu^\rho(t,u)\,dW_u\right|^2dt\right)^{1/2}\right\|_{L_p}\\&\qquad\sim_{c_{(A.1)}}\left\|\left(\int_0^1\int_0^t|\mu^\rho(t,u)|^2\,du\,dt\right)^{1/2}\right\|_{L_p}\\&\qquad=\left\|\left(\sum_{i=1}^n\int_{\rho_{i-1}}^{\rho_i}(\rho_i-t)|\lambda_t^{\rho_{i-1}}|^2\,dt\right)^{1/2}\right\|_{L_p}.\end{aligned}$$

Letting $\delta=0$ if $Y=W$ and $\delta=1$ if Y is the geometric Brownian motion, this can be combined with

$$\begin{aligned}&\left\|\left(\sum_{i=1}^n\int_{\rho_{i-1}}^{\rho_i}(\rho_i-t)H_G^2(t,Y_t)\,dt\right)^{1/2}\right\|_{L_p}\\&\qquad-\delta\left\|\left(\sum_{i=1}^n\int_{\rho_{i-1}}^{\rho_i}(\rho_i-t)\left|(\nabla G(t,Y_t)-\alpha_{\rho_{i-1}})\sigma(Y_t)\right|^2dt\right)^{1/2}\right\|_{L_p}\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| \left(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) \bigl|\lambda_t^{\rho_{i-1}}\bigr|^2 \, dt \right)^{1/2} \right\|_{L_p} \\
&\leq \left\| \left(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) H_G^2(t, Y_t) \, dt \right)^{1/2} \right\|_{L_p} \\
&\quad + \delta \left\| \left(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) \bigl| (\nabla G(t, Y_t) - \alpha_{\rho_{i-1}}) \sigma(Y_t) \bigr|^2 \, dt \right)^{1/2} \right\|_{L_p}
\end{aligned}
$$

so that

$$
\begin{aligned}
&\| C_T(g(Y_1), \tau, v) \|_{L_p} \\
&\quad \geq [c_{(4)}^2 \sqrt{p/2} + 1]^{-1} c_{(4)}^{-1} c_{(\mathrm{A.1})}^{-1} \\
&\quad\quad \times \Biggl[\left\| \left(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) H_G^2(t, Y_t) \, dt \right)^{1/2} \right\|_{L_p} \\
&\quad\quad\quad - \delta \left\| \left(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} \bigl| (\nabla G(t, Y_t) - \alpha_{\rho_{i-1}}) \sigma(Y_t) \bigr|^2 \, dt \right)^{1/2} \right\|_{L_p} \Biggr].
\end{aligned}
$$

In the case of the Brownian motion, the last term disappears. In the case of the geometric Brownian motion, we apply again (4) to see that

$$
\left\| \left(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} \bigl| (\nabla G(t, Y_t) - \alpha_{\rho_{i-1}}) \sigma(Y_t) \bigr|^2 \, dt \right)^{1/2} \right\|_{L_p} \leq c_{(4)} \| C_T(g(Y_1), \tau, v) \|_{L_p}.
$$

Hence, in both cases, we have that

$$
\begin{aligned}
&\inf_v \| C_T(g(Y_1), \tau, v) \|_{L_p} \\
&\quad \geq \frac{1}{[c_{(4)}^2 \sqrt{p/2} + 1] c_{(4)} c_{(\mathrm{A.1})} + c_{(4)}} \\
&\quad\quad \times \left\| \left(\sum_{i=1}^{n} \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) H_G^2(t, Y_t) \, dt \right)^{1/2} \right\|_{L_p}.
\end{aligned} \tag{15}
$$

By $T \uparrow 1$, we obtain the lower bound of our theorem.

(c) *Upper bound for* $\|C_1(g(Y_1), \tau)\|_{L_p}$: For $0 < T < 1$, using the arguments and notation from step (b) and

$$
\nu_u^{\rho_{i-1}} := \bigl(\nabla G(u, Y_u) - \nabla G(\tau_{i-1}, Y_{\tau_{i-1}}) \chi_{\{\tau_{i-1} < T\}} \bigr) \sigma(Y_u),
$$

we obtain

$$
\begin{aligned}
&\|C_T(g(Y_1),\tau)\|_{L_p}\\
&\quad\le c_{(4)}c_{(\mathrm{A.1})}\left\|\left(\sum_{i=1}^n\int_{\rho_{i-1}}^{\rho_i}(\rho_i-t)H_G^2(t,Y_t)\,dt\right)^{1/2}\right\|_{L_p}\\
&\qquad+c_{(4)}c_{(\mathrm{A.1})}\delta\left\|\left(\int_0^T\int_0^t\left|\sum_{i=1}^n\chi_{\{\rho_{i-1}<u\le t\le\rho_i\}}\nu_u^{\rho_{i-1}}\right|^2du\,dt\right)^{1/2}\right\|_{L_p}\\
&\quad\le c_{(4)}c_{(\mathrm{A.1})}\left\|\left(\sum_{i=1}^n\int_{\tau_{i-1}}^{\tau_i}(\tau_i-t)H_G^2(t,Y_t)\,dt\right)^{1/2}\right\|_{L_p}\\
&\qquad+c_{(4)}c_{(\mathrm{A.1})}\delta\left\|\left(\int_0^T\int_0^t\sum_{i=1}^n\chi_{(\rho_{i-1},\rho_i]}(u)|\nu_u^{\rho_{i-1}}|^2\,du\,dt\right)^{1/2}\right\|_{L_p}.
\end{aligned}
$$

Because $2\le p<\infty$, we can continue by

$$
\begin{aligned}
&\left\|\left(\int_0^T\int_0^t\sum_{i=1}^n\chi_{(\rho_{i-1},\rho_i]}(u)|\nu_u^{\rho_{i-1}}|^2\,du\,dt\right)^{1/2}\right\|_{L_p}\\
&\quad\le\left(\int_0^T\left\|\left(\int_0^t\sum_{i=1}^n\chi_{(\rho_{i-1},\rho_i]}(u)|\nu_u^{\rho_{i-1}}|^2\,du\right)^{1/2}\right\|_{L_p}^2dt\right)^{1/2}\\
&\quad\le c_{(4)}\left(\int_0^T\left\|\int_0^t\sum_{i=1}^n\chi_{(\rho_{i-1},\rho_i]}(u)\nu_u^{\rho_{i-1}}\,dW_u\right\|_{L_p}^2dt\right)^{1/2}\\
&\quad=c_{(4)}\left(\int_0^T\|C_t(g(Y_1),\tau)\|_{L_p}^2\,dt\right)^{1/2}.
\end{aligned}
$$

Combining these estimates, we achieve

$$
\begin{aligned}
&\|C_T(g(Y_1),\tau)\|_{L_p}^2\\
&\quad\le 2c_{(4)}^2c_{(\mathrm{A.1})}^2\left\|\left(\sum_{i=1}^n\int_{\tau_{i-1}}^{\tau_i}(\tau_i-t)H_G^2(t,Y_t)\,dt\right)^{1/2}\right\|_{L_p}^2\\
&\qquad+2c_{(4)}^4c_{(\mathrm{A.1})}^2\int_0^T\|C_t(g(Y_1),\tau)\|_{L_p}^2\,dt.
\end{aligned}
$$

Gronwall's lemma thus implies that

$$
\begin{aligned}
&\|C_T(g(Y_1),\tau)\|_{L_p}\\
&\quad\le\sqrt{2}c_{(4)}c_{(\mathrm{A.1})}e^{c_{(4)}^4c_{(\mathrm{A.1})}^2}\left\|\left(\sum_{i=1}^n\int_{\tau_{i-1}}^{\tau_i}(\tau_i-t)H_G^2(t,Y_t)\,dt\right)^{1/2}\right\|_{L_p}.
\end{aligned}
\tag{16}
$$

Finally, by $T \uparrow 1$ we obtain the upper bound in Theorem 5.1. □

REMARK 5.4. Our proof requires the assumption that the stopping time τ_i is $\mathcal{F}_{\tau_{i-1}}$-measurable so that ρ_i is $\mathcal{F}_{\rho_{i-1}}$-measurable. For example, we need that the field $(\mu^\rho(t,u))_{t,u\in[0,1]}$ has the property that $\mu^\rho(t,u)$ is $\mathcal{F}_u$-measurable. Moreover, in step (b) we used $\mathbb{E}_{\mathcal{F}_{\rho_{i-1}}} \int_{\rho_{i-1}}^{\rho_i} |m_{\rho_{i-1}}|^2\,dt = \int_{\rho_{i-1}}^{\rho_i} |m_{\rho_{i-1}}|^2\,dt$.

Approximation with adapted time-nets in L_p. We recall that the nets τ_n^θ are given by

$$t_{i,n}^\theta = 1 - \left(1 - \frac{i}{n}\right)^{1/\theta}.$$

The following result extends [14], Theorem 3.2, from the one-dimensional L_2-setting, but see also [20], Theorem 1, for a related d-dimensional L_2-result.

THEOREM 5.5. *For $2 \le p < \infty$, $0 < \theta \le 1$ and $g(Y_1) \in L_p$, the following assertions are equivalent*:

(i) $\|D_1^{Y,\theta} g(Y_1)\|_{L_p} < \infty$.

(ii) $\sup_{\tau\in\mathcal{T}^{\mathrm{rand}}} \frac{\|C_1(g(Y_1),\tau)\|_{L_p}}{\|\sqrt{|\tau|_\theta}\|_{L_\infty}} < \infty$.

(iii) $\sup_{n\ge1} \sqrt{n}\|C_1(g(Y_1),\tau_n^\theta)\|_{L_p} < \infty$.

In particular, for all $\tau \in \mathcal{T}^{\mathrm{rand}}$,

$$\|C_1(g(Y_1),\tau)\|_{L_p} \le c_{(5.1)} \left\|\sqrt{|\tau|_\theta} D_1^{Y,\theta} g(Y_1)\right\|_{L_p}, \tag{17}$$

where $c_{(5.1)} \ge 1$ is the constant from Theorem 5.1.

For the proof, we need the following lemma that extends [14], Lemma 3.8.

LEMMA 5.6. *Let $0 < \theta \le 1$ and $0 < p < \infty$. Assume that $(\phi_t)_{t\in[0,1)}$ is a measurable process where all paths are continuous and nonnegative. Then the following assertions are equivalent*:

(i) *There exists a constant $c_1 > 0$ such that*

$$\left\|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - u)\phi_u\,du\right\|_{L_p} \le c_1 \sup_{1\le i\le n} \frac{t_i - t_{i-1}}{(1-t_{i-1})^{1-\theta}}$$

for all deterministic time-nets $0 = t_0 < t_1 < \cdots < t_n = 1$.

(ii) *There exists a constant $c_2 > 0$ such that, for all $n = 1, 2, \ldots$,*

$$\left\|\sum_{i=1}^n \int_{t_{i-1,n}^\theta}^{t_{i,n}^\theta} (t_{i,n}^\theta - u)\phi_u\,du\right\|_{L_p} \le \frac{c_2}{n}.$$

(iii) *There exists a constant* $c_3 > 0$ *such that*

$$\left\| \int_0^1 (1-u)^{1-\theta} \phi_u \, du \right\|_{L_p} \le c_3.$$

PROOF. The implications (iii) ⇒ (i) ⇒ (ii) are similar to [14], Lemma 3.8. For (ii) ⇒ (iii), take a sequence of deterministic nets $\tau^n = (t_i^n)_{i=0}^n$ with $0 = t_0^n < t_1^n < \cdots < t_n^n = 1$ such that

$$|\tau^n| \le \frac{\alpha}{n} \quad \text{and} \quad \sup_{1 \le i \le n} \frac{(1 - t_{i-1}^n)^{1-\theta}}{t_i^n - t_{i-1}^n} \le \beta n$$

for some $\alpha, \beta > 0$ independent from n [see, e.g., (11) and (13)]. For a fixed $0 < T < 1$, we define

$$N_T^n := \{ i \in \{1, \ldots, n\} : t_{i-1}^n < T \}$$

and observe that

$$\begin{aligned} &\int_0^T (1-u)^{1-\theta} \phi_u \, du \\ &\qquad \le \liminf_{n \to \infty} \sum_{i \in N_T^n} (1 - t_{i-1}^n)^{1-\theta} \phi_{t_{i-1}^n} (t_i^n - t_{i-1}^n) \end{aligned}$$

for all $\omega \in \Omega$ because ϕ is continuous on $[0, T]$. Hence,

$$\begin{aligned} &\left\| \int_0^T (1-u)^{1-\theta} \phi_u \, du \right\|_{L_p} \\ &\quad \le \left\| \liminf_{n \to \infty} \left[\sup_{1 \le i \le n} \frac{(1 - t_{i-1}^n)^{1-\theta}}{t_i^n - t_{i-1}^n} \right] \left[\sum_{i \in N_T^n} (t_i^n - t_{i-1}^n)^2 \phi_{t_{i-1}^n} \right] \right\|_{L_p} \\ &\quad \le \beta \left\| \liminf_{n \to \infty} n \left[\sum_{i \in N_T^n} (t_i^n - t_{i-1}^n)^2 \phi_{t_{i-1}^n} \right] \right\|_{L_p}. \end{aligned}$$

Noticing that $(t_i^n - t_{i-1}^n)^2 = 2 \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) \, du$ we continue with

$$\begin{aligned} &\beta \left\| \liminf_{n \to \infty} n \left[2 \sum_{i \in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) \, du \, \phi_{t_{i-1}^n} \right] \right\|_{L_p} \\ &\quad \le \beta \left\| \liminf_{n \to \infty} n \left[2 \sum_{i \in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) \phi_u \, du \right. \right. \\ &\qquad\qquad \left. \left. + \sum_{i \in N_T^n} \sup_{t_{i-1}^n \le u < t_i^n} |\phi_u - \phi_{t_{i-1}^n}| (t_i^n - t_{i-1}^n)^2 \right] \right\|_{L_p} \end{aligned}$$

$$
\begin{aligned}
&\le \beta \left\| \liminf_{n\to\infty} \left[2n \sum_{i\in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u)\phi_u \, du \right.\right. \\
&\qquad\qquad \left.\left. + \alpha \sup_{i\in N_T^n} \sup_{t_{i-1}^n \le u < t_i^n} |\phi_u - \phi_{t_{i-1}^n}| \right] \right\|_{L_p} \\
&\le \beta \left\| 2 \liminf_{n\to\infty} n \sum_{i\in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u)\phi_u \, du \right. \\
&\qquad \left. + \alpha \limsup_{n\to\infty} \sup_{i\in N_T^n} \sup_{t_{i-1}^n \le u < t_i^n} |\phi_u - \phi_{t_{i-1}^n}| \right\|_{L_p} \\
&= 2\beta \left\| \liminf_{n\to\infty} n \sum_{i\in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u)\phi_u \, du \right\|_{L_p} \\
&\le 2\beta \liminf_{n\to\infty} n \left\| \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u)\phi_u \, du \right\|_{L_p},
\end{aligned}
$$

where we used Fatou's lemma. Finally, by monotone convergence this implies that

$$
\left\| \int_0^1 (1-u)^{1-\theta} \phi_u \, du \right\|_{L_p} \le 2\beta \liminf_{n\to\infty} n \left\| \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u)\phi_u \, du \right\|_{L_p}. \qquad \square
$$

PROOF OF THEOREM 5.5. First, we employ Theorem 5.1 to confirm equation (17) by

$$
\begin{aligned}
&\|C_1(g(Y_1), \tau)\|_{L_p} \\
&\quad \le c_{(5.1)} \left\| \left(\sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t)\, dt \right)^{1/2} \right\|_{L_p} \\
&\quad \le c_{(5.1)} \left\| \sqrt{|\tau|_\theta} \left(\sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (1-t)^{1-\theta} H_G^2(t, Y_t)\, dt \right)^{1/2} \right\|_{L_p}.
\end{aligned}
$$

Part (i) $\Rightarrow$ (ii) follows from (17) and part (ii) $\Rightarrow$ (iii) from $|\tau_n^\theta|_\theta \le \frac{1}{\theta n}$ [see (12)]. To show that (iii) $\Rightarrow$ (i), we apply Theorem 5.1 and (14) to see that

$$
\frac{c}{\sqrt{n}} \ge \|C_1(g(Y_1), \tau_n^\theta)\|_{L_p} \ge \frac{1}{c_{(5.1)}} \left\| \left(\sum_{i=1}^n \int_{t_{i-1,n}^\theta}^{t_{i,n}^\theta} (t_{i,n}^\theta - t) H_G^2(t, Y_t)\, dt \right)^{1/2} \right\|_{L_p}.
$$

Lemma 5.6 completes the proof. $\square$

Approximation with equidistant time-nets in L_p. Here, we extend the L_2-results [7], Theorem 2.3, and [14], Theorem 3.5 for $q = \infty$, to the L_p-case, as well as [28], Theorem 1.2, which concerned deterministic time-nets and the one-dimensional Brownian motion, to random time-nets and the geometric Brownian motion.

THEOREM 5.7. *For $2 \le p < \infty$, $0 < \theta < 1$ and $g(Y_1) \in L_p$, the following assertions are equivalent*:

(i) $f \in \mathbb{B}^{\theta}_{p,\infty}$.

(ii) $\sup_{\tau \in \mathcal{T}^{\mathrm{rand}}} \frac{\|C_1(g(Y_1);\tau)\|_{L_p}}{\||\tau|^{\theta/2}\|_{L_\infty}} < \infty$.

(iii) $\sup_{n=1,2,\dots} n^{\theta/2} \|C_1(g(Y_1);\tau_n)\|_{L_p} < \infty$, *where $\tau_n = (i/n)_{i=0}^n$ are the equidistant time-nets*.

In particular, for $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ with $p \le q, r \le \infty$ and for all $\tau \in \mathcal{T}^{\mathrm{rand}}$,

$$\begin{aligned}\|C_1(g(Y_1),\tau)\|_{L_p} \le c_{(5.1)} &\left(\int_0^1 \|\sqrt{\psi(t)}\|^2_{L_q} (1-t)^{\theta-2}\,dt\right)^{1/2} \\ &\times \sup_{t\in[0,1)} (1-t)^{1-(\theta/2)} \|H_G(t,Y_t)\|_{L_r},\end{aligned} \tag{18}$$

where $c_{(5.1)} \ge 1$ is the constant from Theorem 5.1 and

$$\psi(t,\omega) := \Big(\max_{i=1,\dots,n} |\tau_i(\omega) - \tau_{i-1}(\omega)|\Big) \wedge (1-t).$$

REMARK 5.8. The order for the equidistant nets can also be obtained from Theorem 5.5 under the condition $\|D_1^{Y,\theta} g(Y_1)\|_{L_p} < \infty$ because $|(i/n)_{i=0}^n|_\theta = n^{-\theta}$.

PROOF OF THEOREM 5.7. To verify (18), we use Theorem 5.1 and derive that

$$\begin{aligned}\|C_1(g(Y_1),\tau)\|_{L_p} &\le c_{(5.1)} \left\| \left(\sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t,Y_t)\,dt\right)^{1/2} \right\|_{L_p} \\ &\le c_{(5.1)} \left\| \left(\int_0^1 \psi(t) H_G^2(t,Y_t)\,dt\right)^{1/2}\right\|_{L_p} \\ &\le c_{(5.1)} \left(\int_0^1 \|\sqrt{\psi(t)} H_G(t,Y_t)\|^2_{L_p}\,dt\right)^{1/2} \\ &\le c_{(5.1)} \left(\int_0^1 \|\sqrt{\psi(t)}\|^2_{L_q} \|H_G(t,Y_t)\|^2_{L_r}\,dt\right)^{1/2} \\ &\le c_{(5.1)} \left(\int_0^1 \|\sqrt{\psi(t)}\|^2_{L_q} (1-t)^{\theta-2}\,dt\right)^{1/2} \\ &\quad \times \sup_{t\in[0,1)} (1-t)^{1-(\theta/2)} \|H_G(t,Y_t)\|_{L_r}.\end{aligned}$$

Part (i) ⇒ (ii): we first observe that for

$$|\tau|(\omega) = \max_{i=1,\dots,n} \big|\tau_i(\omega) - \tau_{i-1}(\omega)\big|$$

we can compute (for $q = \infty$)

$$\begin{aligned}
\int_0^1 & \|\sqrt{\psi(t)}\|_{L_\infty}^2 (1-t)^{\theta-2}\,dt \\
&= \||\tau|\|_{L_\infty} \int_0^{1-\||\tau|\|_{L_\infty}} (1-t)^{\theta-2}\,dt + \int_{1-\||\tau|\|_{L_\infty}}^1 (1-t)^{\theta-1}\,dt \\
&= \||\tau|\|_{L_\infty} \frac{1}{1-\theta}\big(\||\tau|\|_{L_\infty}^{\theta-1} - 1\big) + \frac{1}{\theta}\||\tau|\|_{L_\infty}^{\theta} \\
&\le \frac{1}{\theta(1-\theta)} \||\tau|\|_{L_\infty}^{\theta},
\end{aligned}$$

so that letting $q = \infty$ and $r = p$ in (18) we obtain

$$\big\|C_1\big(g(Y_1),\tau\big)\big\|_{L_p} \le \frac{c_{(5.1)}}{\sqrt{\theta(1-\theta)}} \||\tau|\|_{L_\infty}^{\theta/2} \sup_{t\in[0,1)} (1-t)^{1-(\theta/2)} \big\|H_G(t,Y_t)\big\|_{L_p}.$$

It remains to check that

$$\sup_{t\in[0,1)} (1-t)^{1-(\theta/2)} \big\|H_G(t,Y_t)\big\|_{L_p} < \infty,$$

whenever $f \in \mathbb{B}_{p,\infty}^{\theta}$. This follows from Theorem 3.1, where we additionally use (10) and the a priori estimate

$$\sup_{t\in[0,1)} (1-t)^{1/2} \big\|\nabla F(t,W_t)\big\|_{L_p} < \infty \tag{19}$$

from Lemma A.3 if Y is the geometric Brownian motion.

The implication (ii) ⇒ (iii) is trivial.

Part (iii) ⇒ (i): employing Theorem 5.1 and (14), we achieve

$$\begin{aligned}
cn^{-\theta/2} &\ge \big\|C_1\big(g(Y_1),\tau_n\big)\big\|_{L_p} \\
&\ge \frac{1}{c_{(5.1)}} \left\| \left(\sum_{i=1}^n \int_{(i-1)/n}^{i/n} \left(\frac{i}{n} - t\right) H_G^2(t,Y_t)\,dt \right)^{1/2} \right\|_{L_p} \\
&\ge \frac{1}{c_{(5.1)}} \left\| \left(\int_{(n-1)/n}^1 (1-t) H_G^2(t,Y_t)\,dt \right)^{1/2} \right\|_{L_p} \\
&\ge \frac{1}{c_{(5.1)}} \left\| \left(\int_{(n-1)/n}^1 (1-t) H_G^2\left(1-\frac{1}{n}, Y_{1-(1/n)}\right) dt \right)^{1/2} \right\|_{L_p} \\
&= \frac{1}{c_{(5.1)}} \sqrt{\frac{1}{2}\frac{1}{n}} \left\| H_G\left(1-\frac{1}{n}, Y_{1-(1/n)}\right) \right\|_{L_p},
\end{aligned}$$

where we use in the last inequality the martingale property of the processes

$$\left(\left(\sigma_{kk}\sigma_{ll}\frac{\partial^2 G}{\partial y_k\,\partial y_l}\right)(t, Y_t)\right)_{t\in[0,1)}.$$

The estimate above means that

$$\left\|H_G\left(1-\frac{1}{n}, Y_{1-(1/n)}\right)\right\|_{L_p} \le \sqrt{2}cc_{(5.1)}n^{1-(\theta/2)}$$

for all $n = 2, 3, \ldots$. Consequently,

$$\|H_G(t, Y_t)\|_{L_p} \le 2^{1-(\theta/2)}\sqrt{2}cc_{(5.1)}(1-t)^{(\theta/2)-1},$$

which follows from the monotonicity of $\|H_G(t, Y_t)\|_{L_p}$. Theorem 3.1 completes the proof, where we use (19) again. □

6. Further extensions. We see different open questions and possible extensions, and briefly indicate some of them here: first, one should clarify whether Theorem 5.1 holds true without the additional assumption on the stopping times that τ_i is $\mathcal{F}_{\tau_{i-1}}$-measurable. Second, the investigation to what extend the results of this paper can be extended to path dependent terminal conditions $g(Y_{r_1}, \ldots, Y_{r_L})$ and their limits would possibly require new techniques and yield to a deeper insight into the approximation problem (cf. [9]). Finally, an extension to more general diffusions would be of interest, but might require a modification of the Besov spaces (see [7]) and a comparison of these modified spaces to the spaces we have used in this paper. As described in Remark 3.5, Proposition A.4 below, which does not relay on semi-groups, might be useful in this respect.

APPENDIX

A key step in the proof of Theorem 5.1 is the following-known formulation of the Burkholder–Davis–Gundy inequalities.

LEMMA A.1. *Assume that $\mu : [0,1] \times [0,1] \times \Omega \to \mathbb{R}^{d\times d}$ satisfies the following assumptions*:

(i) $\mu : [0,1] \times [0,u] \times \Omega \to \mathbb{R}^{d\times d}$ *is* $\mathcal{B}([0,1]) \times \mathcal{B}([0,u]) \times \mathcal{F}_u$*-measurable for all* $u \in [0,1]$.

(ii) $\int_0^1\int_0^1 \mathbb{E}|\mu(t,u)|^2\,du\,dt < \infty$, *where* $\int_0^1 \mathbb{E}|\mu(t,u)|^2\,du < \infty$ *for all* $t \in [0,1]$.

(iii) $(\int_0^1 \mu(t,u)\,dW_u)_{t\in[0,1]}$ *is a measurable modification.*

Then, for $1 < p < \infty$, *there exists a constant* $c_{(A.1)} \ge 1$ *depending only on* p *such that*

$$\left\|\left(\int_0^1\left|\int_0^1 \mu(t,u)\,dW_u\right|^2 dt\right)^{1/2}\right\|_{L_p} \sim_{c_{(A.1)}} \left\|\left(\int_0^1\int_0^1 |\mu(t,u)|^2\,du\,dt\right)^{1/2}\right\|_{L_p}.$$

PROOF. For the convenience of the reader, we sketch the proof. By a further modification, we can assume that $((\int_0^1 \mu(t,u)\,dW_u)(\omega))_{t\in[0,1]} \in L_2[0,1]$ for all $\omega \in \Omega$ because of assumption (ii). Assume that $(h_n)_{n=0}^\infty$ is the orthonormal basis of Haar-functions in $L_2[0,1]$ and that $\mu_k(t,u)$ is the kth row of $\mu(t,u)$. Letting

$$L_u^{n,k} := \int_0^1 h_n(t)\mu_k(t,u)\,dt$$

and using a stochastic Fubini argument we see that

$$\left\| \left(\int_0^1 \left| \int_0^1 \mu(t,u)\,dW_u \right|^2 dt \right)^{1/2} \right\|_{L_p} = \left\| \left(\sum_{n=0}^\infty \sum_{k=1}^d \left| \int_0^1 L_u^{n,k}\,dW_u \right|^2 \right)^{1/2} \right\|_{L_p}.$$

Using the Burkholder–Davis–Gundy inequalities (4), we obtain that

$$\begin{aligned} \left\| \left(\int_0^1 \left| \int_0^1 \mu(t,u)\,dW_u \right|^2 dt \right)^{1/2} \right\|_{L_p} &\sim_{c_{(4)}} \left\| \left(\sum_{n=0}^\infty \sum_{k=1}^d \int_0^1 |L_u^{n,k}|^2\,du \right)^{1/2} \right\|_{L_p} \\ &= \left\| \left(\sum_{k=1}^d \int_0^1 \int_0^1 |\mu_k(t,u)|^2\,dt\,du \right)^{1/2} \right\|_{L_p} \\ &= \left\| \left(\int_0^1 \int_0^1 |\mu(t,u)|^2\,dt\,du \right)^{1/2} \right\|_{L_p}. \qquad \square \end{aligned}$$

LEMMA A.2. *Let* $1 < p < \infty$, $g(Y_1) \in L_p$ *and* $0 < t < 1$, *and let* $a = (a_1, \dots, a_d)$ *be a multi-index of differentiation. Assume that* G *is given by* (1). *Then*

$$\left\| \sup_{0\le s\le t} \left| D_y^a G(s, Y_s) \right| \right\|_{L_q} < \infty \qquad \textit{for } 0 < q < q(p,t) := \frac{p-1+t}{t}.$$

SKETCH OF THE PROOF. We use the notation (8) and (9) and consider first the case that Y is the Brownian motion. A simple direct computation gives the hyper-contraction property

$$\left| D_x^a F(t,x) \right| \le C(q,t,a) \|f\|_{L_p(\mathbb{R}^d,\gamma_d)} e^{|x|^2/(2tq)}$$

for $0 < t < 1$ and $0 < q < q(p,t)$. Moreover, the identity

$$D_x^a F(s,x) = \mathbb{E} D_x^a F(t, x + W_{t-s})$$

for $0 \le s \le t < 1$ directly implies that $(D_s^a F(s, W_s))_{s\in[0,t]}$ is an L_q-martingale. Therefore, we can exploit Doob's maximal inequality for $1 < q < q(p,t)$ to conclude

$$\mathbb{E} \sup_{0\le s\le t} \left| D_x^a F(s, W_s) \right|^q < \infty \qquad \text{for all } 0 < q < q(p,t). \tag{20}$$

The case of the geometric Brownian motion can be deduced from the case of the Brownian motion. Using the notation (8) and (9) to switch between the Brownian motion and the geometric Brownian motion, we get for $0 \le t < 1$ that

$$D_y^a G(t, Y_t) = \left[\prod_{k=1}^{d} (Y_t^k)^{-a_k}\right] \sum_{0 \le b \le a} \kappa_a^b D_x^b F(t, W_t),$$

where $0 \le b \le a$ is the coordinate-wise ordering and κ_a^b are fixed coefficients. Using (20), the integrability properties of the geometric Brownian motion and Hölder's inequality, we conclude that

$$\mathbb{E} \sup_{0 \le s \le t} \left|D_y^a G(s, Y_s)\right|^q < \infty \qquad \text{for all } 0 < q < q(p, t). \qquad \square$$

The following estimates are known for more general processes than the Brownian motion (see [16] and [9], Remark 3). In our case, they can be easily verified by using the martingale property of the processes $(\nabla F(t, W_t))_{t \in [0,1)}$ and $(D^2 F(t, W_t))_{t \in [0,1)}$.

LEMMA A.3. *Let $2 \le p < \infty$. Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is measurable with $f \in L_p(\mathbb{R}^d, \gamma_d)$ and that $F : [0, 1] \times \mathbb{R}^d \to \mathbb{R}$ is given by $F(t, x) := \mathbb{E} f(x + W_{1-t})$. Then there exists a constant $c_{(A.3)} > 0$ depending only on p such that, for all $0 \le t < 1$,*

(i) $\|\nabla F(t, W_t)\|_{L_p} \le c_{(A.3)} (1 - t)^{-1/2} \| f(W_1) - F(t, W_t)\|_{L_p}$,
(ii) $\|D^2 F(t, W_t)\|_{L_p} \le c_{(A.3)} (1 - t)^{-1} \| f(W_1) - F(t, W_t)\|_{L_p}$.

Next, we state some Hardy type inequalities we have used in the paper.

PROPOSITION A.4. *Let $0 < \theta < 1$, $2 \le q \le \infty$ and let $d^k : [0, 1) \to [0, \infty)$, $k = 0, 1, 2$, be measurable functions. Assume that*

$$\frac{1}{\alpha}(1 - t)^{k/2} d^k(t) \le d^0(t) \le \alpha \left(\int_t^1 \left[d^1(s)\right]^2 ds \right)^{1/2} \qquad \textit{for } t \in [0, 1)$$

and for $k = 1, 2$, and that

$$d^1(t) \le A + \alpha \left(\int_0^t \left[d^2(u)\right]^2 du \right)^{1/2} \qquad \textit{for } t \in [0, 1)$$

for some $A \ge 0$ and $\alpha > 0$. Then

$$\left\|(1 - t)^{-\theta/2} d^0(t)\right\|_{L_q([0,1),(dt)/(1-t))} \sim_{c_{(A.4)}} \left\|(1 - t)^{(1-\theta)/2} d^1(t)\right\|_{L_q([0,1),(dt)/(1-t))}$$

and

$$\begin{aligned}&\|(1-t)^{(2-\theta)/2} d^2(t)\|_{L_q([0,1),(dt)/(1-t))}\\&\quad\le c_{(A.4)}\|(1-t)^{-\theta/2} d^0(t)\|_{L_q([0,1),(dt)/(1-t))}\\&\quad\le c_{(A.4)}^2\big[A+\|(1-t)^{(2-\theta)/2} d^2(t)\|_{L_q([0,1),(dt)/(1-t))}\big],\end{aligned}$$

where $c_{(A.4)} \ge 1$ depends at most on (α, θ, q). If the functions d^1 and d^2 are nondecreasing, then the inequalities are true for $1 \le q < 2$ as well.

From Proposition A.4, it follows that

$$A + \|(1-t)^{(k-\theta)/2} d^k(t)\|_{L_q} \sim_{c_{(21)}} A + \|(1-t)^{(l-\theta)/2} d^l(t)\|_{L_q} \tag{21}$$

for $L_q = L_q([0,1), \frac{dt}{1-t})$, $k, l = 0, 1, 2$ and $c_{(21)} := [1 + c_{(A.4)}]^2$. To prove Proposition A.4, we need:

LEMMA A.5. *Let $0 < \theta < 1$, $2 \le q \le \infty$ and let $\phi : [0, 1) \to [0, \infty)$ be a measurable function. Then there is a constant $c_{(A.5)} > 0$, depending at most on θ, such that*

$$\begin{aligned}&\left\|(1-t)^{(1-\theta)/2}\left(\int_0^t \phi(u)^2\,du\right)^{1/2}\right\|_{L_q([0,1),(dt)/(1-t))}\\&\qquad\le c_{(A.5)}\|(1-t)^{1-(\theta/2)}\phi(t)\|_{L_q([0,1),(dt)/(1-t))}.\end{aligned}\tag{22}$$

Moreover, if ϕ is nondecreasing, the inequality is true for $1 \le q < 2$ as well.

PROOF. (a) For $2 \le q \le \infty$, we can use Hardy's inequality (see, e.g., [1], Theorem 3.3.9): for $-\infty < \lambda < 1$ and $1 \le r < \infty$, and a measurable $\psi : (0, \infty) \to [0, \infty)$,

$$\left(\int_0^\infty \left[t^{1-\lambda}\int_t^\infty \psi(s)\frac{ds}{s}\right]^r \frac{dt}{t}\right)^{1/r} \le \frac{1}{1-\lambda}\left(\int_0^\infty [t^{1-\lambda}\psi(t)]^r \frac{dt}{t}\right)^{1/r}$$

and the same with the supremum norm if $r = \infty$. With the notation $r := \frac{q}{2}$, $g(t) = [\phi(t)]^2$, and $\lambda = \theta$, we compute, in the case $2 \le q < \infty$,

$$\begin{aligned}&\left\|(1-t)^{(1-\theta)/2}\left(\int_0^t \phi(u)^2\,du\right)^{1/2}\right\|^2_{L_q([0,1),(dt)/(1-t))}\\&\qquad= \left(\int_0^1 \left[(1-t)^{1-\theta}\int_0^t g(u)\,du\right]^r \frac{dt}{1-t}\right)^{1/r}\\&\qquad= \left(\int_0^\infty \left[s^{1-\theta}\int_s^\infty h(v)\,dv\right]^r \frac{ds}{s}\right)^{1/r},\end{aligned}$$

where $h(v) = g(1-v)\chi_{(0,1]}(v)$. Now we use Hardy's inequality for $\psi(v) = vh(v)$ and continue with

$$\begin{aligned}
&\left(\int_0^\infty \left[s^{1-\theta}\int_s^\infty \psi(v)\frac{dv}{v}\right]^r \frac{ds}{s}\right)^{1/r}\\
&\qquad \le \frac{1}{1-\theta}\left(\int_0^\infty [s^{1-\theta}\psi(s)]^r \frac{ds}{s}\right)^{1/r}\\
&\qquad = \frac{1}{1-\theta}\left(\int_0^\infty [s^{2-\theta}h(s)]^r \frac{ds}{s}\right)^{1/r}\\
&\qquad = \frac{1}{1-\theta}\|(1-t)^{1-(\theta/2)}\phi(t)\|^2_{L_q([0,1),(dt)/(1-t))}
\end{aligned}$$

and the proof is complete for $2 \le q < \infty$. The case $q = \infty$ is analogous.

(b) For $1 \le q < 2$, we use a different argument. First, we define $r := \frac{2}{q}$ so that $1 < r \le 2$. For $0 < T < 1$, we compute

$$\begin{aligned}
&\int_0^1 (1-t)^{(1-\theta)/r}\left(\int_0^t \chi_{[T,1)}(u)\,du\right)^{1/r}\frac{dt}{1-t}\\
&\qquad = \int_0^1 (1-t)^{(1-\theta)/r}(t-T)_+^{1/r}\frac{dt}{1-t}\\
&\qquad \le (1-T)^{1/r}\int_T^1 (1-t)^{((1-\theta)/r)-1}\,dt\\
&\qquad = c\int_T^1 (1-t)^{(2-\theta)/r}\chi_{[T,1)}(t)\frac{dt}{1-t}
\end{aligned}$$

with $c := \frac{2-\theta}{1-\theta}$. This proves the desired inequality for $\psi^{(T)}(t) := \chi_{[T,1)}(t)$. Next, we define $\psi := \phi^q$ so that $\psi^r = \phi^2$. By assumption, ϕ is nondecreasing, and so is ψ, too. Now, we can approximate ψ from below by a sum of functions like $\psi^{(T)}$: for each integer $n \ge 1$, we find $\alpha_k^n \ge 0$, $k = 0, \dots, 2^n - 1$ and $0 = t_0^n < t_1^n < \cdots < t_{2^n-1}^n < t_{2^n}^n = 1$ such that

$$\psi_n(t) := \sum_{k=0}^{2^n-1}\alpha_k^n \psi^{(t_k^n)}(t) \to \psi(t)$$

for almost all $t \in [0,1)$ and $\psi_{n-1} \le \psi_n$ for all $n \ge 2$. Then, since $r \ge 1$,

$$\begin{aligned}
&\int_0^1 (1-t)^{(1-\theta)/r}\left(\int_0^t \psi_n(u)^r\,du\right)^{1/r}\frac{dt}{1-t}\\
&\qquad \le \int_0^1 (1-t)^{(1-\theta)/r}\sum_{k=0}^{2^n-1}\alpha_k^n\left(\int_0^t \psi^{(t_k^n)}(u)\,du\right)^{1/r}\frac{dt}{1-t}
\end{aligned}$$

$$\leq \sum_{k=0}^{2^n-1} \alpha_k^n c \int_0^1 (1-t)^{(2-\theta)/r} \psi^{(t_k^n)}(t) \frac{dt}{1-t}$$
$$= c \int_0^1 (1-t)^{(2-\theta)/r} \psi_n(t) \frac{dt}{1-t}$$

and the claim follows by monotone convergence. □

PROOF OF PROPOSITION A.4. (a) Our assumptions imply for all $1 \leq q \leq \infty$ that

$$\|(1-t)^{(1-\theta)/2} d^1(t)\|_{L_q([0,1),(dt)/(1-t))} \leq \alpha \|(1-t)^{-\theta/2} d^0(t)\|_{L_q([0,1),(dt)/(1-t))}$$

and

$$\|(1-t)^{(2-\theta)/2} d^2(t)\|_{L_q([0,1),(dt)/(1-t))} \leq \alpha \|(1-t)^{-\theta/2} d^0(t)\|_{L_q([0,1),(dt)/(1-t))}.$$

(b) Next, we observe that

$$\begin{aligned} &\|(1-t)^{-\theta/2} d^0(t)\|_{L_q([0,1),(dt)/(1-t))} \\ &\quad \leq \alpha \left\| (1-t)^{(1-\theta)/2} \left(\frac{1}{1-t} \int_t^1 [d^1(s)]^2 \, ds \right)^{1/2} \right\|_{L_q([0,1),(dt)/(1-t))} \\ &\quad \leq \alpha \theta^{-\max\{1/2,1/q\}} \|(1-t)^{(1-\theta)/2} d^1(t)\|_{L_q([0,1),(dt)/(1-t))}, \end{aligned}$$

where we used [14], formula (14) (the condition that ψ in [14] is continuous in the case $1 \leq q < 2$ is not necessary).

(c) To prove the remaining inequality, we continue from (b) with Lemma A.5 to

$$\begin{aligned} &\|(1-t)^{(1-\theta)/2} d^1(t)\|_{L_q([0,1),(dt)/(1-t))} \\ &\quad \leq \left\| (1-t)^{(1-\theta)/2} \left[A + \alpha \left(\int_0^t d^2(u) \, du \right)^{1/2} \right] \right\|_{L_q([0,1),(dt)/(1-t))} \\ &\quad \leq A \|(1-t)^{(1-\theta)/2}\|_{L_q([0,1),(dt)/(1-t))} \\ &\qquad + \alpha c_{(A.5)} \|(1-t)^{1-(\theta/2)} d^2(t)\|_{L_q([0,1),(dt)/(1-t))}. \end{aligned}$$
□

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