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Dynamic analysis for axially moving viscoelastic panels

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Abstract

In this study, stability and dynamic behaviour of axially moving viscoelastic panels are investigated with the help of the classical modal analysis. We use the flat panel theory combined with the Kelvin–Voigt viscoelastic constitutive model, and we include the material derivative in the viscoelastic relations. Complex eigenvalues for the moving viscoelastic panel are studied with respect to the panel velocity, and the corresponding eigenfunctions are found using central finite differences. **The governing equation for the transverse displacement of the panel is of fifth order in space, and thus five boundary conditions are set for the problem. The fifth condition is derived and set at the in-flow end for clamped-clamped and clamped-simply supported panels.** The numerical results suggest that the moving viscoelastic panel undergoes divergent instability for low

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values of viscosity. They also show that the critical panel velocity increases when viscosity is increased and that the viscoelastic panel does not experience instability with a sufficiently high viscosity coefficient. For the cases with low viscosity, the modes and velocities corresponding to divergent instability are found numerically. We also report that the value of bending rigidity (bending stiffness) affects the distance between the divergence velocity and the flutter velocity: the higher the bending rigidity, the larger the distance.
Keywords: Moving, Viscoelastic, Beam, Eigenvalues, Dynamic, Stability

1. Introduction

Stability of axially moving materials has been studied widely (e.g. Wickert and Mote, 1988) since such models have various applications in industry, e.g., paper webs, band saws, magnetic tapes and pipes containing flowing fluids. Recently, many scientists have taken an interest in viscoelastic moving materials, since industrial materials usually have viscoelastic characteristics (see e.g. Fung et al., 1997).

The first studies on vibrations of travelling elastic strings, beams, and bands include Sack (1954), Archibald and Emslie (1958), Miranker (1960), Swope and Ames (1963), Mote (1968), Mote (1972), Mote (1975), Simpson (1973), Ulsoy and Mote (1980), Chonan (1986), and Wickert and Mote (1990). These studies focused on free and forced vibrations including the nature of wave propagation in moving media and the effects of axial motion on the eigenfrequencies and eigenmodes. We also mention Ulsoy and Mote (1982), Lin and Mote (1995), Lin and Mote (1996), and Lin (1997), who have studied stability of travelling rectangular membranes and plates.

Archibald and Emslie (1958) and Simpson (1973) investigated the effects of axial motion on the frequency spectrum and eigenfunctions. It was shown that the natural frequency of each mode decreases as the transport speed increases, and that the travelling string and beam both experience divergence instability at a sufficiently high speed. Wickert and Mote (1990) studied stability of axially moving strings and beams using modal analysis and Green's function method. The expressions for the critical transport velocities were found analytically. However, Wang et al. (2005a) showed analytically that no static instability occurs for the transverse motion of a string at the critical velocity. Recently, Kong and Parker (2004) have found, by a perturbation analysis, closed-form expressions for the approximate frequency spectrum of axially moving beams with a small flexural stiffness.

Ulsoy and Mote (1982) and Lin (1997) studied the stability and vibration characteristics of axially moving plates. Ulsoy and Mote (1982) studied natural frequencies and stability of the moving plate using the Ritz method (and simplified boundary conditions at the free edges), and comparison with experimental data showed a good agreement with the analytical results. Lin (1997) showed by numerical analysis that the critical velocities predicted by the static analysis and the dynamic analysis coincide, and that the plate experiences divergence instability at the critical velocity. Kim et al. (2003) used modal spectral element formulation to analyse the eigenfrequencies with respect to the axial velocity of a moving plate. Luo and Hamidzadeh (2004) examined buckling of an axially moving plate using non-linear equations and the perturbation approach.

The first study on transverse vibration of travelling viscoelastic material was carried out by Fung et al. (1997) using a string model. Extending their work, they studied the material damping effect in their later research (Fung et al., 1998).

There are several studies on travelling viscoelastic materials concerning strings and beams. Chen and Zhao (2005) represented a modified finite difference method to simplify a non-linear model of an axially moving viscoelastic string. They studied the free transverse vibrations of elastic and viscoelastic strings numerically.

Chen et al. (2004), Yang and Chen (2005), Chen and Yang (2005), and Chen and Yang (2006), all investigated stability of axially moving viscoelastic beams in parametric transverse resonance. Yang and Chen (2005) studied the dynamic stability of axially moving viscoelastic beams in parametric resonance with time-pulsating speed. They found that the viscoelastic damping decreases the size of the instability region. Chen and Yang (2006) studied free vibrations of viscoelastic beams travelling between simple supports with torsion strings. They studied the viscoelastic effect by perturbing the similar elastic problem and using the method of multiple scales, and examined also the eigenfrequencies of the system illustrating the real parts of the eigenfrequencies.

Oh et al. (2004) and Lee and Oh (2005) have studied critical speeds, eigenvalues, and natural modes of axially moving viscoelastic beams using the spectral element model. They analysed dynamic behaviour of axially moving viscoelastic beams using modal analysis, performed a detailed eigenfrequency analysis, and reported that viscoelasticity did not affect the critical moving speed. They observed that, for an elastic beam above the divergence speed, the first and the second mode couple representing a coupled-mode flutter instability but for a viscoelastic beam no such coupled-mode flutter occurs: the first mode is unstable while the second mode remains stable.

Marynowski and Kapitaniak (2002) compared the Kelvin–Voigt and the Bürgers models in modelling of moving viscoelastic webs. The web was modelled as an axially moving (non-linear) beam with internal damping. For the linearised Kelvin–Voigt model, it was found that the beam exhibits divergent instability at some critical speed. In the case of non-linear Bürgers model, the critical speed decreased when the internal damping was increased, and the beam was found to experience the first instability in the form of flutter. Recently, they investigated non-linear vibrations of axially moving beams with time-dependent tension using the Zener model for internal damping (Marynowski and Kapitaniak, 2007). The critical transport speeds of the non-linear, parametrically excited viscoelastic beam with the Zener model were compared with the ones of the Bürgers and the Kelvin–Voigt models. The critical speeds predicted by the Zener and Bürgers models coincided. The Kelvin–Voigt model predicted a greater critical speed than the other two models.

Hou and Zu (2002) investigated non-linear free oscillations of moving viscoelastic strings at subcritical velocities. They studied numerically the frequencies and amplitudes with respect to the string velocity comparing three different models for viscoelasticity: the standard linear solid model, the Kelvin–Voigt model, and the Maxwell model. Zhang et al. (2007) investigated transverse non-linear vibrations of axially accelerating viscoelastic strings applying a complex-mode Galerkin approach. Zhang (2008) studied bifurcation for axially moving non-linear viscoelastic strings.

A few studies on transverse vibrations of axially moving viscoelastic plates have also been done. Hatami et al. (2008) studied free vibrations of axially moving viscoelastic plates using a finite strip method. They used the standard viscoelastic solid model. They found that, as for an elastic plate, also for a viscoelastic plate the vibration frequency decreases as axial velocity is increased and becomes zero at a critical velocity for the first frequency.

Zhou and Wang (2007) studied transverse vibration characteristics of axially moving viscoelastic rectangular plates. **They assumed the plate to be elastic in dilatation but viscoelastic in distortion, where the viscoelasticity was described by the Kelvin-Voigt law.** Zhou and Wang derived the dynamic equation for transverse displacements using the thin plate theory and the constitutive equations of the viscoelastic material in the Laplace domain. The curves of the moving speed and dimensionless complex frequencies of the axially moving viscoelastic plate were plotted. Two different combinations were used as boundary conditions: all four edges simply supported and two opposite edges simply supported but the remaining edges clamped. It was found that when the dimensionless delay time was very small (viscosity was small), the dynamic characteristics and stability of the axially moving viscoelastic plate were nearly the same as for an axially moving elastic plate. It was also reported that the increase of delay time did not alter the critical divergent moving speed in the first mode and that the plate did not exhibit a coupled-mode flutter. **(Similar behaviour for** viscoelastic beams was reported by Lee and Oh, 2005). Zhou and Wang (2008) continued their studies on transverse vibration characteristics of moving viscoelastic plates taking into account a parabolically varying plate thickness.

Zhou and Wang (2009) also studied dynamic stability of axially accelerating viscoelastic plates using the Floquet theory to find stability regions. They used harmonic vibrations in speed and studied their effect on stability. Yang et al. (2012) recently studied vibrations, bifurcation and chaos of axially moving viscoelastic plates using finite differences and a non-linear model for transverse displacements. They concentrated on bifurcations and chaos but also studied the dynamic characteristics of a linearised elastic model with the help of eigenfrequency analysis.

In all the discussed studies above, a partial time derivative has been used instead of a material derivative in the viscoelastic constitutive relations. Mockensturm and Guo (2005) suggested that the material derivative should be used. They studied non-linear vibrations and dynamic response of axially moving viscoelastic strings, and found significant discrepancy in the frequencies at which non-trivial limit cycles exist, comparing the models with the partial time derivative and the material time derivative. In Chen et al. (2008), Ding and Chen (2008), Chen and Wang (2009), and Chen and Ding (2010), the material derivative was also used in the viscoelastic constitutive relations. Ding and Chen (2008) studied stability of axially accelerating viscoelastic beams using the method of multiple scales and parametric resonance. Chen and Wang (2009) studied the stability of axially accelerating viscoelastic beams using the asymptotic perturbation analysis. In a recent research by Chen and Ding (2010), the steady-state response of transverse vibrations for axially moving viscoelastic beams was studied. Kurki and Lehtinen (2009) suggested, independently, that the material derivative in the constitutive relations should be used in their study concerning the in-plane displacement field of a travelling viscoelastic plate.

Recently, some studies using the material derivative in the viscoelastic constitutive relations for moving viscoelastic two-dimensional plates have been done. Marynowski (2010) studied free vibrations and stability of Levy-type viscoelastic plates. Marynowski compared a three-parameter Zener model and a two-parameter Kelvin–Voigt model for the viscoelasticity. It was found that the critical transport velocity predicted by the Zener model was higher than the one predicted by the Kelvin–Voigt model, which in turn was slightly higher than the critical velocity of an elastic plate. Tang and Chen (2012) studied stability in parametric resonance of moving viscoelastic plates with time-dependent travelling speed.

First studies on damping in the context of moving materials include Mahalingam (1957) and Ulsoy and Mote (1982). Mahalingam (1957) studied transverse vibrations of power transmission chains modelled as an axially moving string. Mahalingam considered periodic displacements and a damping force proportional to the transverse velocity of the form $c[(\cdot)_{,t} + V_0(\cdot)_{,x}]$, where c is the damping coefficient and V_0 is the constant velocity of the chain. As a numerical example, Mahalingam considered a simple roller chain and verified the results by experiments. Comparing the experiments and the numerical results, it was found that damping coefficient

of the above mentioned form gave a fairly good approximation. Mahalingam found that the resonant amplitudes of the fundamental frequency (eigenfrequency) decreased when the speed was increased. This was also verified by experiments. When damping of the form $c(\cdot)_{,t}$ was considered, it was found that the resonant amplitudes increased with an increased velocity. Ulsoy and Mote (1982) found similar results in their studies on vibrations of band saw blades. They also studied the damping of two-dimensional plate model and found that the real parts of the eigenvalues of the damped plate were negative when the plate velocity was below the critical value. This was the case for both forms $c(\cdot)_{,t}$ and $c[(\cdot)_{,t} + V_0(\cdot)_{,x}]$ of damping.

In Jeronen (2011), eigenvalues for a damped string were analysed. In the research, a second-order string model with damping terms of the form $a_1 w_{,t} + a_2 w_{,x}$ were investigated. The behaviour of the string was found to be stable when constants a_1 and a_2 were independent of the travelling speed. When the second constant was chosen to be $a_2 = V_0 a_1$ (V_0 is the string velocity), where the damping coefficients were assumed to be generated by the material derivative, it was found that the damping introduced an instability not existing in the non-damped system.

Models for pipes conveying fluid often share similarities with the models for axially moving materials. In Drozdov (1997), a pipe filled with a moving fluid is studied modelling the pipe as a viscoelastic beam driven by the forces caused by the fluid. Drozdov investigated stability of the system under a periodic flow. It was

found that for some parameter values, an increase in viscoelasticity resulted in a decrease in the critical fluid velocity while for other choices of parameters, an increase in viscoelasticity resulted in an increase in the critical velocity.

Recently, Wang et al. (2005b) derived a sixth order model for a curved viscoelastic pipe conveying fluid based on Hamilton's principle. Viscoelasticity of the pipe was modelled with the help of the Kelvin–Voigt model. As boundary conditions, they used clamped conditions for a curved pipe at both ends, which set the displacement, the first space derivative of the displacement and the second space derivative of the displacement to be zero. The stability of the viscoelastic curved pipe was studied by analysing the dimensionless complex frequency with respect to the fluid velocity. They found that when the dimensionless delay time was increased, the first, second and third mode do not couple. The viscoelastic pipe was found to undergo divergent instability in the first and second order modes and, for greater values of fluid velocity, single-mode flutter took place in the first order mode.

Using the material derivative in the viscoelastic constitutive relations for a beam model leads to a partial differential equation that is fifth-order with respect to the space coordinate. In Ding and Chen (2008), Chen and Wang (2009), and Chen and Ding (2010), the fifth-order dynamic equation was attained but four boundary conditions (in space) were used. We will use five boundary conditions (in space). For boundary value problems of higher order differential equations, see e.g. Agarwal (1986).

Although many studies on the viscoelastic moving beam model have been conducted, to our knowledge, the dynamic stability analysis with the eigenfrequencies has not been done before for this model. In this paper, we use the dynamic (modal) analysis to study the stability of an axially moving viscoelastic Kelvin-Voigt beam where the material derivative in the viscoelastic relations is used. **Marynowski and Kapitaniak (2002) and** Lee and Oh (2005) studied the same phenomenon but they did not use the material derivative in the viscoelastic constitutive relations. In this study, we use the material derivative but we also compare our results with the results obtained by the model with the partial time derivative.

The effects of the panel speed and the degree of viscosity on the stable regions are studied in numerical examples. The eigenfunctions corresponding to the critical speeds are found. Two different combinations of boundary conditions are used, and comparison with both elastic models and viscoelastic models with only a partial time derivative in viscoelastic relations are provided.

We use the term *panel* **(e.g. Bisplinghoff and Ashley, 1962; Banichuk et al., 2010, 2011)** here since we consider two-dimensional webs applying the Kirchhoff plate theory but reducing it to one dimension with the following assumption: for the moving web, we assume that the transverse displacement does not vary in the direction that is perpendicular to the moving direction of the web. Mathematically, the equations describing the out-of-plane displacement of the panel and the equations of the out-out-plane displacement of the beam coincide up to some terminological issues. This is to say that this research covers the case of axially moving beams after renaming some physical constants. We will use the terminology of the plate (panel) theory throughout the paper.

2. An axially moving viscoelastic panel

Consider an axially moving **thin** panel **or plate** made of viscoelastic material in a Cartesian coordinate system. See Fig. 1. **The panel is assumed**

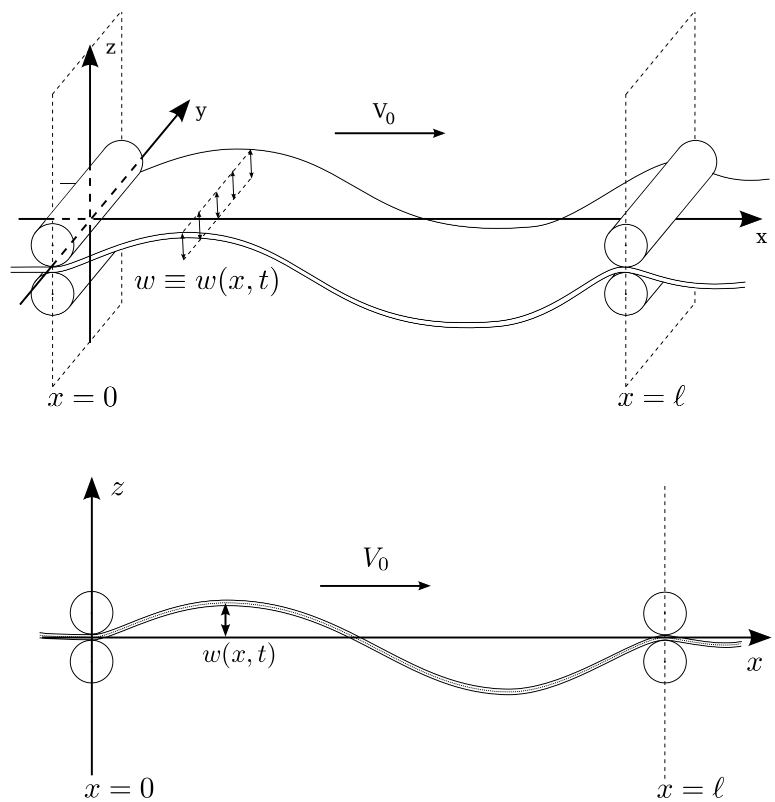


Figure 1: A travelling panel between two fixed supports. **The panel is assumed to undergo cylindrical deformation.**

to undergo cylindrical transverse deformation, that is, the transverse displacement does not vary in the y direction (Timoshenko

and Woinowsky-Krieger, 1959; Bisplinghoff and Ashley, 1962). The panel is supported at $x = 0$ and $x = \ell$, and the length of the unsupported open draw is ℓ. The travelling velocity of the panel is assumed to be constant and denoted by V_0. **The transverse displacement is denoted by the function $w = w(x, t)$.**

The viscoelasticity **of the material is described with** the **rheological** Kelvin–Voigt model **consisting of an elastic spring and a viscous damper connected in parallel. The spring element is described by the parameters E (the Young's modulus) and ν (the elastic Poisson ratio), and the damper by η (the viscous damping coefficient) and μ (the Poisson ratio for viscosity). See Fig. 2.**

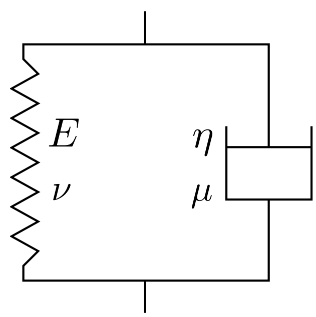


Figure 2: The rheological Kelvin–Voigt model.

The stress-strain relations under assumption of plane stress are (see e.g. Sobotka, 1984; Tang and Chen, 2012)

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) + \frac{\eta}{1-\mu^2}\left[(\varepsilon_{x,t} + V_0\varepsilon_{x,x}) + \mu(\varepsilon_{y,t} + V_0\varepsilon_{y,x})\right] , \\
\sigma_y &= \frac{E}{1-\nu^2}\left(\nu\varepsilon_x + \varepsilon_y\right) + \frac{\eta}{1-\mu^2}\left[\mu(\varepsilon_{x,t} + V_0\varepsilon_{x,x}) + (\varepsilon_{y,t} + V_0\varepsilon_{y,x})\right] , \\
\tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy} + \frac{\eta}{2(1+\mu)}(\gamma_{xy,t} + V_0\gamma_{xy,x}) ,
\end{aligned} \tag{1}$$

for the normal stresses σ_x and σ_y, the shear stress τ_{xy}, the normal strains ε_x and ε_y and the shear strain γ_{xy}. Denoting $\sigma = \sigma_x$ and $\varepsilon = \varepsilon_x$ and assuming small cylindrical deformations ($\varepsilon_y = -zw_{,yy}$), the relations (1) are reduced to

$$\sigma = \frac{E}{1-\nu^2}\varepsilon_x + \frac{\eta}{1-\mu^2}(\varepsilon_{x,t} + V_0\varepsilon_{x,x})\,. \tag{2}$$

For the bending moment $M = M_x$, we obtain

$$M = -\left[Dw_{,xxxx} + \Upsilon(w_{,xxt} + V_0 w_{,xxx})\right]\,, \tag{3}$$

where we have used the notations

$$D = \frac{Eh^3}{12(1-\nu^2)}\,, \qquad \Upsilon = \frac{\eta h^3}{12(1-\mu^2)}\,. \tag{4}$$

Note that we obtain the bending moment for a viscoelastic beam with the changes $D \leftrightarrow EI$ and $\Upsilon \leftrightarrow \eta I$ in Eq. (3).

We define the parameter λ as a creep time constant (Marynowski, 2008)

$$\lambda = \frac{\eta}{E}\,. \tag{5}$$

The unit of λ is the second. With the help of (5) and assuming that $\mu = \nu$, we may write

$$\Upsilon = \lambda D\,.$$

Proceeding in a similar way to Ding and Chen (2008) but using the stress-strain relation in (2), we finally obtain the dynamic

equilibrium

$$w_{,tt}+2V_0 w_{,xt}+\frac{\lambda D}{m}w_{,xxxxt}+(V_0^2-\frac{T_0}{m})w_{,xx}+\frac{D}{m}w_{,xxxx}+V_0\frac{\lambda D}{m}w_{,xxxxx}=0\,. \quad (6)$$

In Eq. (6), m is mass per unit area, and T_0 is a constant tension at the panel ends. **With the change $D \leftrightarrow EI$ (and $\lambda EI = \eta I$) in Eq. (6), one obtains the dynamic equation for an axially moving viscoelastic beam.**

Derivation of dynamic equation for moving viscoelastic beam is provided also by Ghayesh (2011). Derivation of dynamic equation for axially moving viscoelastic plates via Hamilton's principle is given in (Tang and Chen, 2012, Eq. (7)). Their equation is reduced into Eq. (6) when we assume that the displacement w does not vary in the y direction and that the axial velocity of the plate is constant.

Since Eq. (6) is of fifth order in space, we need five boundary conditions. We first assume that both the ends are clamped and, therefore, we have $w(0,t) = w(\ell,t) = 0$ and $w_{,x}(0,t) = w_{,x}(\ell,t) = 0$. These boundary conditions can be derived, e.g., by setting clamped boundary conditions for the panel in the reference frame moving with the panel and performing appropriate change of variable. For details, see e.g. Chen and Ding (2010). Since the panel is moving in the positive x direction, we seek the fifth condition at the in-flow end $x = 0$ indicating that we have more information there than at the out-flow end.

For the bending moment M in Eq. (3), we write the following

continuity condition (Flügge, 1975)

$$\lim_{\delta\to 0}\int_{-\delta}^{+\delta} M\,\mathrm{d}x = 0\,,$$

where M **is as described in Eq.** (3)**. Denoting the displacement of the panel in the domain** $x<0$ **by** w^-**, we obtain in the limit** $\delta\to 0$**:**

$$-Dw_{,x} + \Upsilon(w_{,xt} + V_0 w_{,xx}) + Dw^-_{,x} - \Upsilon(w^-_{,xt} + V_0 w^-_{,xx}) = 0 \tag{7}$$

at $x=0$**. Since** $w_{,x}(0,t)=0$ **and thus** $w_{,xt}(0,t)=0$**, by continuity of the panel (Flügge, 1975) also** $w^-_{,x}(0,t)=0$ **and thus** $w^-_{,xt}(0,t)=0$**. Substituting these into** (7)**, we obtain**

$$w_{,xx}(0,t) = w^-_{,xx}(0,t)\,.$$

That is, the second derivative of the panel deflections before and after the support must coincide. We set $w^-_{,xx}=0$ **and obtain the fifth condition**

$$w_{,xx}(0,t) = 0\,. \tag{8}$$

In the cases of an elastic panel ($\Upsilon=0$**) or a viscoelastic panel where partial time derivative is used instead of the material derivative in the constitutive relations, Eq.** (7) **does not produce additional conditions, which is a desired consequence.**

We study the problem using the following boundary conditions:

- A clamped boundary condition at the out-flow end and three conditions at the in-flow end:

$$w(0,t) = w_{,x}(0,t) = w_{,xx}(0,t) = 0\,, \qquad w(\ell,t) = w_{,x}(\ell,t) = 0\,. \tag{9}$$

We call (9) C$^+$-C conditions. If we remove the condition $w_{,xx}(0,t) = 0$, we obtain clamped–clamped (C-C) boundary conditions.

- A simply supported condition at the out-flow end and three conditions at the in-flow end:

$$w(0,t) = w_{,x}(0,t) = w_{,xx}(0,t) = 0\,, \qquad w(\ell,t) = w_{,xx}(\ell,t) = 0\,. \quad (10)$$

  We call (10) C$^+$-S conditions. If we remove the condition $w_{,xx}(0,t) = 0$, we obtain clamped–simply supported (C-S) conditions. **Note that the condition $w_{,xx}(\ell,t) = 0$ corresponds to the zero moment for an elastic panel but is considered as a mechanical condition or approximative for the viscoelastic panel.**

We use three boundary conditions at the in-flow end and two conditions at the out-flow end indicating that we have more information at the in-flow end. The three conditions at $x = 0$ (in (9) and (10)) are called C$^+$ conditions, since clamping is a stronger condition than being simply supported.

We transform the dynamic equation (6) into a dimensionless form. We introduce the transformations

$$x' = \frac{x}{\ell}\,, \qquad t' = \frac{t}{\tau}\,, \qquad w'(x',t') = \frac{w(x,t)}{h}\,. \quad (11)$$

Inserting (11) into (6), omitting the primes and multiplying by $m\ell^2/(T_0 h)$, we obtain

$$\frac{m\ell^2}{\tau^2 T_0} w_{,tt} + 2V_0 \frac{m\ell}{\tau T_0} w_{,xt} + \frac{\lambda D}{\tau \ell^2 T_0} w_{,xxxxt} + \left(\frac{V_0^2}{T_0/m} - 1\right) w_{,xx} + \frac{D}{\ell^2 T_0} w_{,xxxx} + V_0 \frac{\lambda D}{\ell^3 T_0} w_{,xxxxx} = 0\,. \quad (12)$$

We choose

$$\tau = \ell \sqrt{\frac{m}{T_0}}$$

as a characteristic time, and introduce the dimensionless problem parameters

$$c = \frac{V_0}{\sqrt{T_0/m}}\,, \qquad \alpha = \frac{D}{\ell^2 T_0}\,, \qquad \gamma\alpha = \frac{\lambda D}{\ell^3 \sqrt{m T_0}}\,, \tag{13}$$

where

$$\gamma = \frac{\lambda}{\tau} = \frac{\eta}{E} \frac{\sqrt{T_0}}{\ell \sqrt{m}} \tag{14}$$

is the dimensionless delay time. Zhou and Wang (2007) also defined the dimensionless delay time in a similar manner but considered a two-dimensional viscoelastic plate model with a different choice of the characteristic time.

Inserting (13) into (12), we finally have

$$w_{,tt} + 2cw_{,xt} + \gamma\alpha w_{,xxxxt} + (c^2 - 1)w_{,xx} + \alpha w_{,xxxx} + \gamma\alpha c w_{,xxxxx} = 0\,, \tag{15}$$

with the boundary conditions

$$\begin{aligned} & w(0,t) = w_{,x}(0,t) = w_{,xx}(0,t) = 0\,, \\ & w(1,t) = 0\,, \quad \text{and} \quad \begin{cases} w_{,x}(1,t) & = 0\,, \ \text{or} \\ w_{,xx}(1,t) & = 0\,. \end{cases} \end{aligned} \tag{16}$$

We represent the solution of the dynamic problem, Eqs. (15)–(16), in the form (the standard time harmonic trial function)

$$w(x,t) = W(x) e^{st}\,, \tag{17}$$

where

$$s = i\omega\,, \tag{18}$$

and ω is the characteristic **(dimensionless)** frequency of small transverse vibrations. Considering the system behaviour, s characterizes it in the following manner:

- If the imaginary part of s is non-zero
  - and the real part of s is zero, the panel vibrates harmonically with a small amplitude.
  - and the real part of s is positive, the amplitude of transverse vibrations grows exponentially (flutter).
  - and the real part of s is negative, the transverse vibrations are damped exponentially.
- If the imaginary part of s is zero
  - and the real part of s is zero, the panel has a critical point.
  - and the real part of s is positive, the panel displacement grows exponentially (divergence, buckling).
  - and the real part of s is negative, the panel displacement decreases exponentially.

The sign of the real part of s characterizes the stability of the panel: if $\mathrm{Re}\ s > 0$, the behaviour is unstable, and otherwise it is stable.

Insert (17) into (15), and obtain

$$s^2 W + s(2cW_{,x} + \gamma\alpha W_{,xxxx}) + (c^2 - 1)W_{,xx} + \alpha W_{,xxxx} + \gamma\alpha c W_{,xxxxx} = 0\,. \quad (19)$$

The boundary conditions for W are

$$
\begin{aligned}
& W(0) = W_{,x}(0) = W_{,xx}(0) = 0\,, \\
& W(1) = 0\,, \quad \text{and} \quad \begin{cases} W_{,x}(1) & = 0\,, \ \text{or} \\ W_{,xx}(1) & = 0\,. \end{cases}
\end{aligned} \tag{20}
$$

We study the stability of the travelling viscoelastic panel by solving Eqs. (19)–(20) with respect to the transport velocity.

Note that if we neglect the fifth-order derivative in (19) and formulate the buckling problem by setting $s = 0$ and neglecting the boundary condition $w_{,xx}(0) = 0$, we obtain the buckling problem for an elastic panel (beam), with the boundary conditions C-C or C-S depending on the boundary condition at the out-flow end.

For an axially moving elastic panel, the critical velocity corresponding to the divergent instability, can be found analytically. For a clamped-clamped elastic panel, the dimensionless critical velocity is expressed as

$$c_{\mathrm{cr}} = \sqrt{1 + 4\alpha\pi^2}\,. \tag{21}$$

Eq. (21) **was derived for elastic beams, e.g., in Wickert and Mote (1990). The corresponding critical mode is then**

$$W(x) = A\,\sin(\pi x)\,,$$

where A **is arbitrary constant.**

Similarly, one may obtain for a clamped-simply supported elastic panel:

$$c_{\mathrm{cr}} = \sqrt{1 + \alpha k_1^2}\,, \tag{22}$$

where k_1 is the smallest positive solution of

$$\tan k = k\,. \tag{23}$$

The critical mode for a clamped-simply supported elastic panel is

$$W(x) = A\left[k_1\cos(k_1 x) - \sin(k_1 x) + k_1 x - k_1\right],$$

where A is arbitrary constant.

3. Numerical solution

Let us outline the finite difference discretization approach. It is to find $\mathbf{w} = (w_1, \ldots, w_n)$ satisfying the discretised form of (19)–(20). We use central differences of second-order asymptotic accuracy

$$\begin{gathered}
w_{j,x} = \frac{w_{j+1} - w_{j-1}}{2\Delta x}\,, \quad w_{j,xx} = \frac{w_{j+1} - 2w_j + w_{j-1}}{(\Delta x)^2}\,, \\
w_{j,xxxx} = \frac{w_{j+2} - 4\,w_{j+1} + 6\,w_j - 4\,w_{j-1} + w_{j-2}}{(\Delta x)^4}\,, \\
w_{j,xxxxx} = \frac{w_{j+3} - 4\,w_{j+2} + 5\,w_{j+1} - 5\,w_{j-1} + 4\,w_{j-2} - w_{j-3}}{2(\Delta x)^5}\,.
\end{gathered} \tag{24}$$

The interval $[0, \ell]$ is divided to $n+1$ subintervals equal in length. The end points of the subintervals are labelled as $0 = x_0, x_1, x_2, \ldots, x_n, x_{n+1} = \ell$. We use two virtual points (w_{-2} and w_{-1}) at the in-flow end and one virtual (w_{n+2}) point at the out-flow end. From the boundary conditions (20), we get at the in-flow end:

$$\begin{aligned}
w_{-2} &= -w_2 \quad (\text{from } w_{,xx}(0) = 0), \\
w_{-1} &= w_1 \quad (\text{from } w_{,x}(0) = 0), \\
w_0 &= 0\,,
\end{aligned}$$

and at the out-flow end:

$$w_{n+1} = 0\,,$$
$$w_{n+2} = \begin{cases} w_n\,, & \text{(C)}\,, \\ -w_n\,, & \text{(S)}\,. \end{cases}$$

In Eqs. (24), $\Delta x = 1/(n+1)$. We use the following backward difference scheme (of second-order asymptotic accuracy) to calculate the fifth-order derivative at the out-flow end ($j = n$):

$$w_{j,xxxxx} = \frac{3\,w_{j+2} - 16\,w_{j+1} + 35\,w_j - 40\,w_{j-1} + 25\,w_{j-2} - 8\,w_{j-3} + w_{j-4}}{2(\Delta x)^5}\,.$$

We denote the derivative matrices by $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_4, \mathbf{K}_5$ built up with the help of (24) with the following correspondence:

$$\mathbf{K}_1 : W_{,x}\,, \qquad \mathbf{K}_2 : W_{,xx}\,, \qquad \mathbf{K}_4 : W_{,xxxx}\,, \qquad \mathbf{K}_5 : W_{,xxxxx}\,.$$

Inserting the matrices $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_4, \mathbf{K}_5$ into (19), we obtain the matrix equation

$$s^2\mathbf{w} + s\left[2c\mathbf{K}_1 + \gamma\alpha\mathbf{K}_4\right]\mathbf{w} + \left[(c^2-1)\mathbf{K}_2 + \alpha\mathbf{K}_4 + \gamma\alpha c\mathbf{K}_5\right]\mathbf{w} = 0\,. \tag{25}$$

Note that in the case $\alpha = 0$ or $c = 0$, we obtain a fourth-order equation needing only four boundary conditions. This has been taken into account: the virtual point w_{-2} is needed only by the matrix $\mathbf{K}_5$. When $\mathbf{K}_5$ is removed from the matrix equation (25), the boundary condition $w_{,xx}(0) = 0$ is simultaneously removed from the discretised problem. (It was numerically confirmed that when we decrease the value of α, the solution of (25) with the boundary conditions C$^+$-C approaches the solution of the corresponding elastic problem with the boundary conditions C-C (similarly the C$^+$-S solution approaches the elastic C-S solution). This was the case even if we selected $w_{-2} = w_2$ from $w_{,x}(0) = 0$, and $w_{-1} = -w_1$ from $w_{,xx}(0) = 0$.)

The matrix equation (25), which is a quadratic eigenvalue problem with respect to s, can be rewritten as

$$\begin{bmatrix} -\mathbf{M}_1 & -\mathbf{M}_0 \\ \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} s\mathbf{w} \\ \mathbf{w} \end{bmatrix} = s \begin{bmatrix} s\mathbf{w} \\ \mathbf{w} \end{bmatrix}, \tag{26}$$

where

$$\begin{aligned} \mathbf{M}_0 &= (c^2 - 1)\mathbf{K}_2 + \alpha \mathbf{K}_4 + \gamma\alpha c \mathbf{K}_5 \,, \\ \mathbf{M}_1 &= 2c\mathbf{K}_1 + \gamma\alpha \mathbf{K}_4 \,. \end{aligned} \tag{27}$$

The matrix equation (26) is now an eigenvalue problem of the standard form

$$\mathbf{A}\mathbf{y} = s\mathbf{y} \tag{28}$$

with

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}_1 & -\mathbf{M}_0 \\ \mathbf{I} & 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} s\mathbf{w} \\ \mathbf{w} \end{bmatrix}.$$

4. Numerical results

Problem (19)–(20) was solved using finite differences. In the discretizations, we fixed the problem parameters typical of a paper material shown in Table 1 (paper material constants have been measured for example by Yokoyama and Nakai, 2007). **The creep time constant λ was given several different values to examine the effect of viscosity.** In the finite difference method, we chose the number of computation points to be $n = 200$.

Using the physical parameters in Table 1, the dimensionless parameter α in Eq. (13) **gets the value** $\alpha = 1.8315 \cdot 10^{-7}$**. Creep time**

Table 1: Physical parameters used in the numerical examples.

T_0	m	ℓ	h	E	ν
500 N/m	0.08 kg/m^2	1 m	10^{-4} m	10^9 N/m^2	0.3

$\Rightarrow$	$D = Eh^3/(12 \cdot (1-\nu^2))$
	$9.1575 \cdot 10^{-5}$ Nm

constant λ **was given the values** $\lambda = 5{\cdot}10^{-5}$ **s**, $5{\cdot}10^{-4}$ **s**, **and** $5{\cdot}10^{-3}$ **s, the dimensionless delay time** γ **getting the values** $\gamma = 3.953 \cdot 10^{-3}$, $3.953 \cdot 10^{-2}$, **and** 0.3953**, respectively.**

In Figs. 3–6, three lowest eigenvalue pairs s, Eq. (18), are plotted with respect to the dimensionless panel velocity. In the numerical studies, it was found that for the parameter values in Table 1, the panel behaviour is stable with harmonic vibrations when the panel velocity is between 0 and 1 regardless of the value of **the dimensionless delay time** γ **(or the creep time constant** λ**)**, and the panel may experience divergence instability at a critical dimensionless velocity c_{cr} slightly above the value 1, depending on the value of γ. The eigenvalues between 0 and 1 behaved similarly in all of the studied cases and the behaviour is shown in Figs. 4–5 in the upper left corner of each sub-figure.

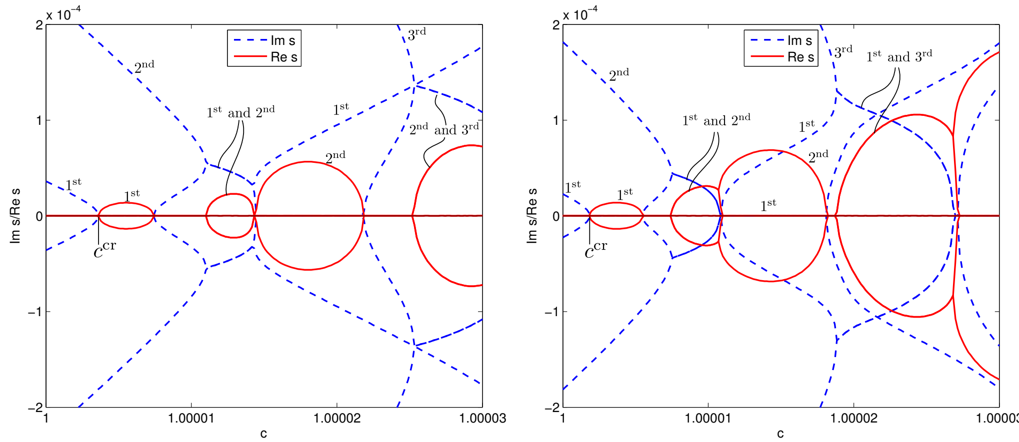
To analyse the behaviour close to the possible critical point more closely, the velocity range $1 \ldots 1.00003$ was studied (Figs. 3–6). As found previously for elastic beams (Wickert and Mote, 1990), at velocities greater than the divergence speed c_{cr}, a flutter instability region may appear. **In this study,** it was found that when the value of the parameter α was increased, the value of the critical velocity increased and the distance between the possible divergence speed and the possible flutter speed increased.

In Fig. 3, the eigenvalue spectra for moving elastic panels are shown with boundary conditions C-C and boundary conditions C-S. In Figs. 4–6, the eigenvalues spectra are shown for three different values of **the dimensionless delay time** $\gamma = 3.953 \cdot 10^{-3}$, $3.953 \cdot 10^{-2}$, **and** 0.3953**.**

Let us compare Figs. 3 and 4. It can be seen that for a **panel with small viscosity** ($\gamma = 3.953 \cdot 10^{-3}$), the results are close to that of elastic panels. The values of critical divergence velocities c_{cr} seem to coincide. The stable region after the divergence instability region seen in the behaviour of elastic panels in Fig. 3 seems to disappear when the viscoelasticity is introduced to the model (Fig. 4). In the case of elastic panels for both types of boundary conditions, the first and second mode couple, representing a coupled-mode flutter. At greater values of velocity and in the case with C-C boundary conditions, the second and the third mode couple, and with C-S boundary conditions, the first and the third mode couple. All these couplings are removed when the viscoelasticity is introduced.

In Fig. 4, in the upper right corner of both sub-figures, the eigenmodes corresponding to the critical velocities c_{cr} (solid line) and the eigenmodes of corresponding elastic problems (dashed line) are shown. The solutions are very close to each other for elastic problems and viscoelastic problems with a small viscosity. The eigenmodes were found by solving Problem (19)–(20) with $s = 0$, and the critical velocities given by this static analysis and the dynamic analysis were found to be the same.

In Fig. 5, the eigenvalue spectra and critical eigenmodes are shown for **a**



(a) (b)

Figure 3: The first three eigenvalue pairs for moving elastic panels ($\gamma = 0$) plotted against the dimensionless velocity c. Solid lines present real parts of eigenvalues (Re s) and dashed lines present imaginary parts (Im s). The points representing critical velocities c_{cr} are labelled. (a) Boundary conditions C-C. (b) Boundary conditions C-S.

dimensionless delay time γ ten times greater than in the case analysed above. The changes in the spectra are radical. The values of critical velocities are greater than for the corresponding elastic panels. Also the shapes of the corresponding critical eigenmodes are changed. More changes in the spectra can be reported: in the case of C^{+}-C boundary conditions, the divergence instability region is slightly wider $\gamma = 3.953 \cdot 10^{-2}$ than for $\gamma = 3.953 \cdot 10^{-3}$. However, the unstable region after the divergence instability region has now become stable, or more precisely, the panel vibrates with a damped amplitude. The second mode is stable for all values of velocity. In the case of C^{+}-S boundary conditions (Fig 5b), the unstable region still exists after the divergence instability and the second mode is unstable (with divergence-type instability) for some range of velocities greater than c_{cr}.

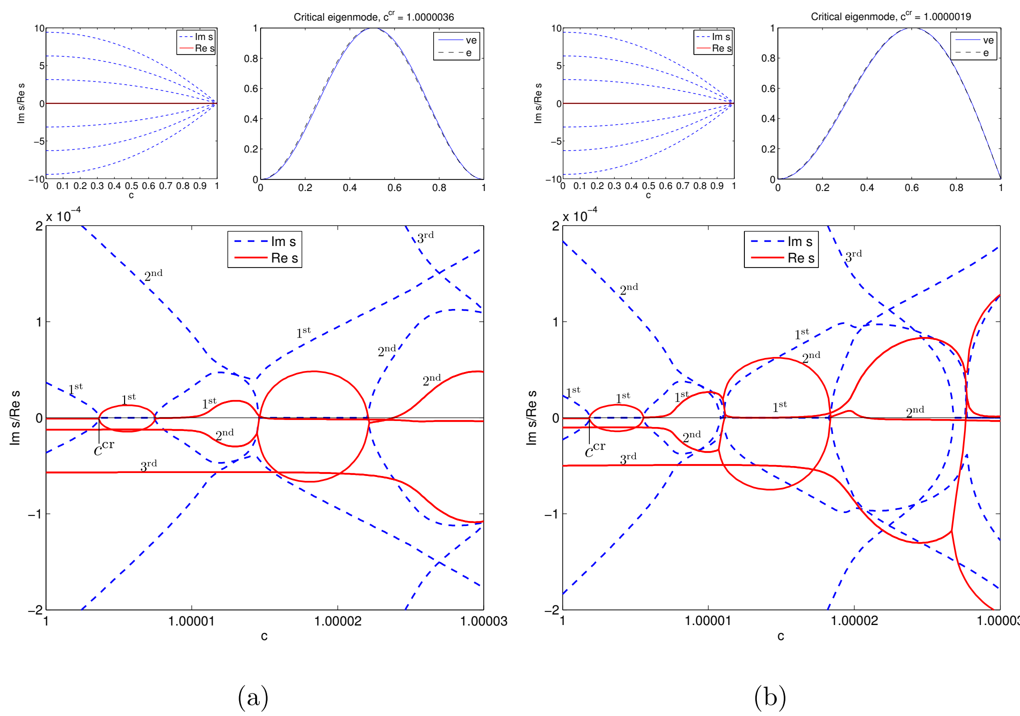
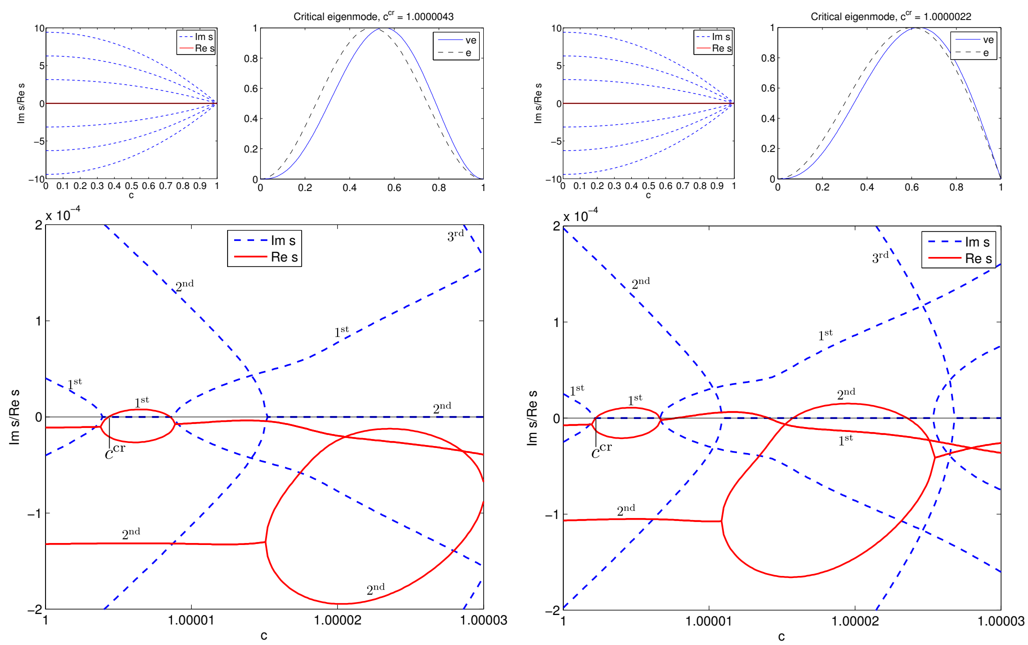
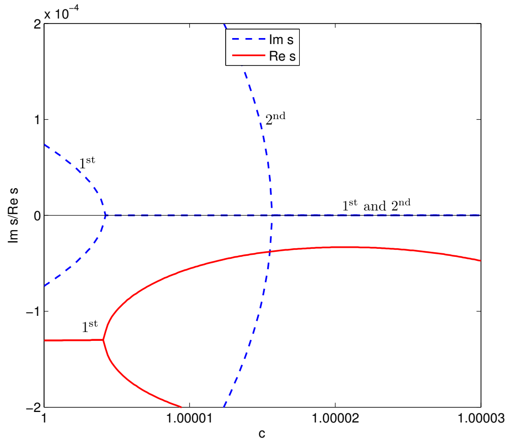


Figure 4: The first three eigenvalue pairs for moving viscoelastic panels with respect to the dimensionless velocity c. **Dimensionless delay time** $\gamma = 3.953 \cdot 10^{-3}$ ($\lambda = 5 \cdot 10^{-5}$ s, almost elastic). Solid lines represent real parts of eigenvalues (Re s) and dashed lines represent imaginary parts (Im s). The points representing critical velocities c_{cr} are labelled. In each sub-figure, the behaviour of the eigenvalues s between $c = 0$ and $c = 1$ is shown up left. The eigenmode corresponding to the (dimensionless) critical velocity c_{cr} is shown up right. (a) Boundary conditions C$^+$-C. (b) Boundary conditions C$^+$-S.

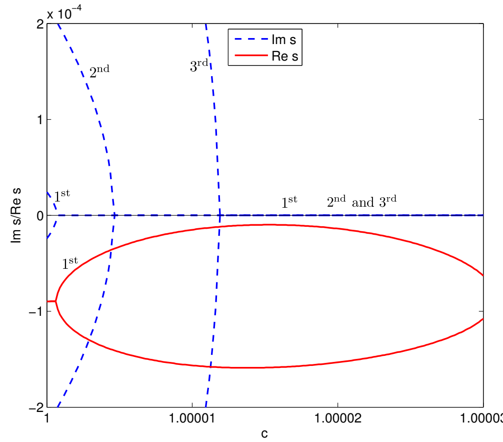


(a) (b)

Figure 5: The first three eigenvalue pairs for moving viscoelastic panels with respect to the dimensionless velocity c. **Dimensionless delay time** $\gamma = 3.953 \cdot 10^{-2}$ ($\lambda = 5 \cdot 10^{-4}$ s). Solid lines represent real parts of eigenvalues (Re s) and dashed lines represent imaginary parts (Im s). The points representing critical velocities c_{cr} are labelled. In each subfigure, the behaviour of the eigenvalues s between $c = 0$ and $c = 1$ is shown up left. The eigenmode corresponding to the (dimensionless) critical velocity c_{cr} is shown up right. (a) Boundary conditions C$^+$-C. (b) Boundary conditions C$^+$-S.



(a)



(b)

Figure 6: The first three eigenvalue pairs for moving viscoelastic panels with respect to the dimensionless velocity c. **Dimensionless delay time** $\gamma = 0.3953$ ($\lambda = 5 \cdot 10^{-3}$ s). Solid lines represent real parts of eigenvalues (Re s) and dashed lines represent imaginary parts (Im s). (a) Boundary conditions C$^+$-C. (b) Boundary conditions C$^+$-S.

When the **dimensionless delay time** γ is further increased, the real part of also the lowest eigenvalue stays negative, and no critical point or loss of instability can be detected. This can be seen in Fig. 6. Since no real part of the eigenvalues crosses the x axis, Problem (19)–(20) with $s = 0$ has no solution in such cases. This suggests that high viscosity makes the model stable at any velocity.

The limit values for the **dimensionless delay time** γ were found numerically using the bisection method. For C$^+$-C boundary conditions, **after** $\gamma \approx 0.1022$ **(**$\lambda \approx 1.29 \cdot 10^{-3}$ **s),** the real part of the first eigenvalue does not become positive, and also the other (higher) eigenvalues are behaving similarly. For C$^+$-S boundary conditions, the value after which the real part of s remains non-positive is $\gamma \approx 0.2183$ **(**$\lambda \approx 2.76 \cdot 10^{-3}$ **s).**

The results of critical velocities for different **values of dimensionless delay time and the corresponding the creep time constant** are collected in Table 2 including the limit cases. One may notice that the critical velocities of viscoelastic panels approach the critical velocities of elastic panels as **the dimensionless delay time** γ **approaches** zero. **The critical velocities for the elastic panels were calculated analytically with the help of Eq.** (21) **(see also Wickert and Mote, 1990) and Eqs.** (22)–(23)**.** The limit value of the **dimensionless delay time** γ needed for stabilization is higher for the C^+-S boundary conditions than for the C^+-C conditions suggesting that the latter case is more stable than the previous one. **The analytically calculated critical velocities for elastic panels coincided with the numerically calculated critical velocities from the dynamic analysis.**

Table 2: The results for critical velocities.

		c_{cr}		$(V_0)_{\mathrm{cr}}$ (m/s)	
γ	λ (s)	C($^+$)-C	C($^+$)-S	C($^+$)-C	C($^+$)-S
0	0	1.0000036	1.0000018	79.0572	79.0571
$3.953 \cdot 10^{-3}$	$5 \cdot 10^{-5}$	1.0000036	1.0000019	79.0572	79.0571
$3.953 \cdot 10^{-2}$	$5 \cdot 10^{-4}$	1.0000043	1.0000022	79.0573	79.0571
0.1022	$1.29 \cdot 10^{-3}$	1.0000087	–	79.0576	–
0.2183	$2.76 \cdot 10^{-3}$	–	1.0000099	–	79.0577

As mentioned above, different values for the parameter α were examined. Qualitatively, the behaviour of eigenvalue spectra was the same independent of the choice of parameter values in Table 1: for viscosities high enough, all the modes were stable for all the values of velocities. **We also studied the behaviour of the eigenvalue spectra with the parameter choices correponding to a beam made of steel. The parameters were chosen similarly to Lee and Oh (2005). Also with this choice of parameters, it was found that when the dimensionless delay time was high enough, the viscoelastic beam was stable for all values of velocities.**

Since several studies exist of the model where the *partial time derivative* was used instead of the material derivative in the viscoelastic relations, we compared the eigenvalue spectra of these two different models using boundary conditions C($^+$)-C and **dimensionless delay time** $\gamma = 3.953 \cdot 10^{-2}$. See Fig. 7. Also greater values for **the dimensionless delay time** were examined but it was seen that the magnitude of the **dimensionless delay time** did not affect (qualitatively) the eigenvalue spectrum in the case of the model with the partial time derivative. The eigenvalue spectrum of this model is very close to the one of the elastic C-C panel (see Fig. 3(a)) but the real parts of s are slightly negative before the critical velocity.

The behaviours predicted by the two different models are totally different. **The model where the partial time derivative was used predicts smaller value for the critical velocity than the model with the material derivative. The behaviours at supercritical speeds between the two models differ from each other in many ways. Firstly after the divergent instability, there is a second stable region by the model with $\eta(\cdot)_{,t}$ but the panel is damped by the model with $\eta[(\cdot)_{,t} + V_0(\cdot)_{,x}]$. Secondly after this, the panel undergoes couple-mode flutter by the model with $\eta(\cdot)_{,t}$ but is stable with damping vibrations by the model**

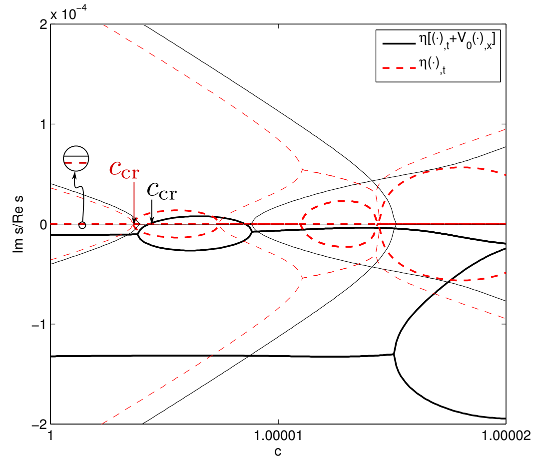


Figure 7: Comparison of two different models. Solid lines show the spectra of the model with the material derivative in viscoelastic relations. Dashed lines represent the model with the partial time derivative in the viscoelastic relations. Real parts of eigenvalues are plotted in the bold line, imaginary parts in the light line. Boundary conditions C($^+$)-C. **Dimensionless delay time** $\gamma = 3.953 \cdot 10^{-2}$ ($\lambda = 5 \cdot 10^{-4}$ s).

with $\eta[(\cdot)_{,t} + V_0(\cdot)_{,x}]$**.**

In Fig. 8, the dynamic behaviour of the viscoelastic panel at sub-critical and super-critical velocities is illustrated. The space discretization was done via finite differences as reported in Sec. 3 and the time discretization was done via the fourth-order Runge–Kutta method. The initial displacement ($w(x, 0)$) was the critical eigenmode with $\gamma = 3.953 \cdot 10^{-2}$ (shown in Fig. 5(a)), and $w_{,t}(x, 0) = 0$ initially. Two different values for **the dimensionless delay time** studied, $\gamma = 3.953 \cdot 10^{-2}$ ($\lambda = 5 \cdot 10^{-4}$ s) and $\gamma = 0.3953$ ($\lambda = 5 \cdot 10^{-3}$ s), and boundary conditions C$^+$-C were used. With the help of Figs. 5 and 6, appropriate sub-critical and super-critical velocities were chosen, $c = 0.99$ (sub-critical) and $c = 1.000005$ (super-critical). Fig. 8

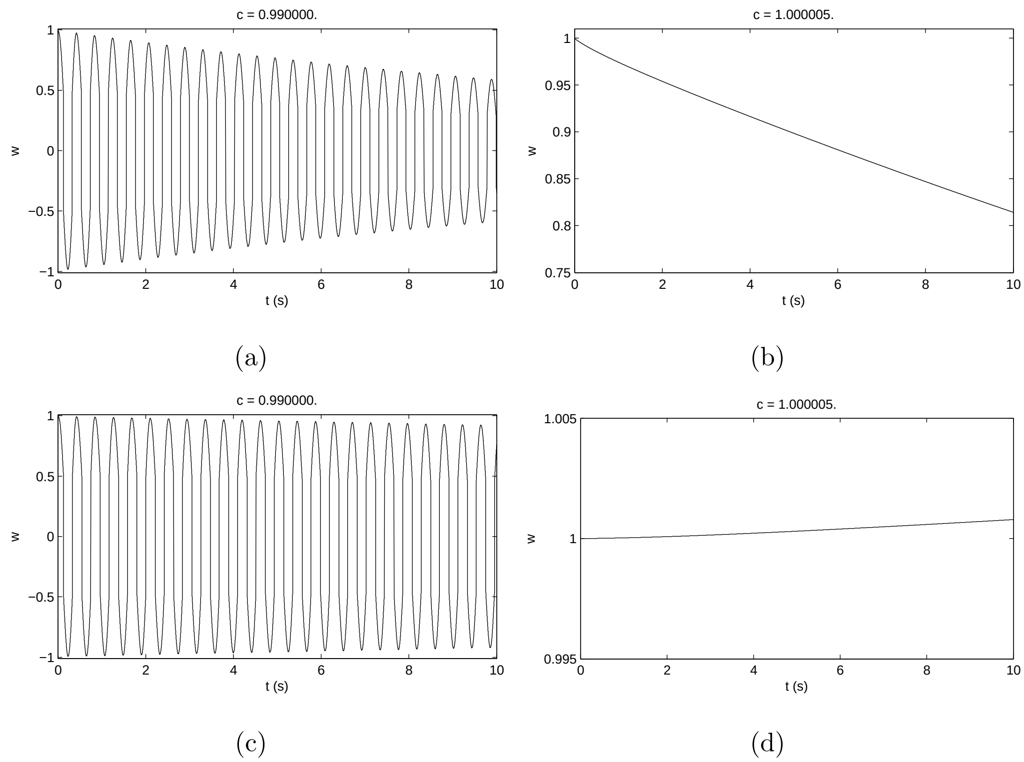


Figure 8: The dynamic behaviour of the displacement maxima for the first 10 seconds. Boundary conditions C$^+$-C.

presents the time behaviour of the displacement maxima for the first 10 seconds.

The results resemble the behaviour predicted by the dynamic analysis. For $\gamma = 0.3953$ at a sub-critical velocity, the panel vibrates with a damped amplitude, and at a super-critical velocity, the panel displacement decreases exponentially. For $\gamma = 3.953 \cdot 10^{-2}$ at a super-critical velocity, the displacement grows exponentially and thus the panel is unstable. At the sub-critical velocity, the panel vibrations are damped but slower than for the panel with $\gamma = 0.3953$.

The dynamic solver was verified with the solutions of the static divergence problems (for viscosities low enough). The critical eigenmode was taken as an initial displacement which stayed constant in time as expected.

5. Discussion

Ulsoy and Mote found already in 1982 that, for a damping force proportional to $\beta_1 w_{,t} + \beta_2 c w_{,x}$ ($\beta_1 \neq 0$), an axially moving plate undergoes damping vibrations, i.e. the real parts of the eigenvalues are negative, when the plate velocity is smaller than the critical velocity. This is characteristics also for viscoelastic beams and plates and has been reported by several researchers, including Oh et al. (2004), Lee and Oh (2005), Hatami et al. (2008) and Zhou and Wang (2007, 2008).

As known for both elastic and viscoelastic moving beams and plates, the natural frequencies (or imaginary parts of the eigenvalues) tend to decrease with the growth of the transport speed. This characteristics was found in this research, and has been reported in the studies mentioned above but also in Marynowski and Kapitaniak (2002), Chen and Yang (2006), and Tang and Chen (2012) for viscoelastic materials.

Hatami et al. (2008) studied the effect of the relaxation time on the complex natural frequencies of the axially moving viscoelastic plates. They found out that, for velocities lower than the critical velocity, the real part of the complex frequency increases and the imaginary part of it increases linearly when the relaxation time

is increased. In our results, similar characteristics can be seen. For velocities lower than the critical velocity when the creep time constant is increased, the absolute value of the real parts of the eigenvalues (corresponding to the imaginary part of the complex frequency) are clearly increasing and the imaginary parts of the eigenvalues (corresponding to the real part of the complex frequency) seem to increase slightly.

In the studies by Oh et al. (2004) and Lee and Oh (2005) for axially moving viscoelastic beams, it was found that the critical speed of the elastic beam and the viscoelastic beam coincide. The similar conclusion was reported for viscoelastic plates by Hatami et al. (2008) and Zhou and Wang (2007). Recently, Marynowski (2010) reported that for a Levy-type viscoelastic plate, the critical transport speed for Kelvin–Voigt was slightly greater than for an elastic plate. Marynowski also reported that an increase in the relaxation time caused the critical transport speed to increase. Resembling the results obtained by Marynowski for Levy-type plates, in our study, it was found that the critical velocity for a viscoelastic one-dimensional thin panel or beam, is greater than the critical velocity for the corresponding elastic model. We also found that when the creep time constant (or dimensionless delay time) was increased, the critical panel velocity increased.

In the eigenvalue spectra of moving viscoelastic beams or plates, the effect that the system does not experience couple-mode flutter typical of elastic materials, is well-known, and has been reported, e.g., by Lee and Oh (2005) and Zhou and Wang (2007). Similar behaviour has been found in the studies on stability of viscoelastic pipes conveying fluid (Wang et al., 2005b). Our results coincide with the previously reported findings.

For a moving Kelvin-Voigt beam, Marynowski and Kapitaniak (2002) found the system to undergo divergent instability at some critical velocity. At supercritical transport speeds and with small internal damping, the beam was found to experience divergent and flutter instabilities, and between these two areas, there was a second stability area. The second stable region was found to disappear when the internal damping was increased.

Lee and Oh (2005) and Zhou and Wang (2007) also reported that viscoelasticity may remove the second stability region, which may appear in the case of an elastic beam or plate, and instead of stable behaviour the first order mode undergoes single-mode flutter. For small values of viscosity, the similar phenomenon was found in the results obtained in this study. However, for greater values of viscosity, this unstable region became stable with damping vibrations.

When the creep time constant (dimensionless delay time) is increased, the results in Sec. 4 show that the third mode becomes stable, and with further increase of the creep time constant, the second mode becomes stable, and finally also the first mode becomes stable for all values of velocities. The phenomenon that the higher modes become stable when the viscosity is increased can

be seen also in the results by Lee and Oh (2005), Zhou and Wang (2007, 2008), and Wang et al. (2005b).

As a new result, it was found that when the creep time constant (and the dimensionless form of it) was big enough, the first three modes remained stable for all values of velocities. The stability at all values of velocity cannot be guaranteed by numerical tests but the stability of the first three (or as many as wanted) natural modes at the expected *critical velocity* can be reported. **To our knowledge, this effect introduced by viscoelasticity has not been reported previously for axially moving viscoelastic materials.**

In this case of a moving viscoelastic panel (beam), it was found that the viscosity (damping) removed the instability existing in the elastic (non-damped) system contrary to the case of a moving damped string (Jeronen, 2011).

6. Conclusions

In this paper, the eigenvalues (related to eigenfrequencies) of viscoelastic axially moving panels (beams) were studied. **The studied dynamic equation for the viscoelastic panel was fifth order in space, and thus five boundary conditions were needed. Clamped conditions at the panel ends produced four boundary conditions, and the fifth condition at the in-flow end was derived with the help of continuity of the panel. The problem was examined also using the (elastic) simply supported condition at the out-flow end.** A dynamic equation describing the transverse vibrations was examined with the help of classical modal analysis. The material derivative was used instead of the partial time derivative in the viscoelastic relations of the Kelvin–Voigt model. The eigenvalues were calculated with respect to the axial panel velocity, and the effect of the viscosity on the spectrum was studied.

The results were compared to other studies, in which dynamic stability of axially moving viscoelastic materials was studied. In our case, the material derivative was used in the viscoelastic relation instead of the partial time derivative, which was used in the other studies. Our results partly coincided with the previous results, but also many new aspects were found.

- In the cases where the viscosity was small, the results of models where the partial time derivative or the material derivative in the viscoelastic relations coincided qualitatively. Firstly, the critical divergence velocities were the same as for corresponding elastic models. Secondly, the coupled-mode flutters at higher velocities than the divergence speed typical of elastic models were removed when viscoelasticity was introduced to the models.

- For greater values of viscosity, several new results, not reported before for viscoelastic moving materials, were found. As viscosity was increased, the value of the critical (divergence) velocity increased. While increasing viscosity, the third mode became stable for all values of velocities and, with further increase, also the second mode became stable for all values of velocities.

- For high enough values of viscosity, all the natural modes remained stable for all values of velocities. The limiting value of viscosity high enough for this behaviour was found numerically. It was noticed that, consequently, the static problem (buckling problem) has no solution if viscosity is high enough.

- The eigenvalue spectra and the critical eigenmodes were seen to approach the ones of the corresponding elastic problems as viscosity approached zero.

Viscosity is known for its damping characteristics and therefore these results for the investigated model are seen as realistic.

This research can be considered important for the discussion about which model should be used for moving viscoelastic materials. The obtained results highlight the differences between the predictions given by the model where the partial time derivative was used and the model where the material derivative was used in the viscoelastic relations.

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