Equations of State for White Dwarfs

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Thanks

Thanks to my supervisor Kimmo Tuominen for suggesting this interesting phenomenon as the subject of my Bachelor's Thesis.

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Abstract

This thesis is about deriving a few equations of state for white dwarfs below the regime of neutron drip. White dwarfs — also called degenerate dwarfs, composed mostly of electron-degenerate matter — are luminous and the color of the light they are emitting is white, hence their name. Because of the relatively enormous density, the gravitational potential of a white dwarf causes a collapse.

White dwarfs are classified as compact objects, meaning that their life begins when a star dies, and are therefore considered as one possibility of a final stage of stellar evolution since they are considered static over the lifetime of the Universe. Star death is a point where the most of its nuclear fuel has been consumed. After the birth, white dwarfs are slowly cooling, radiating away their residual thermal energy.

White dwarfs resist the gravitational collapse with electron degeneracy pressure. The temperature of white dwarfs is much higher than that of normal stars. These properties, together with exceedingly small size, are characteristic of white dwarfs. Cooling of white dwarfs offers information of solid state physics in a new setting — the circumstances of an original star can not be built up in a laboratory. Also, it would not be possible to realize the distance, which includes many advantages in sketching timescales and fundamental interactions by observation. More over, the evolution and the equation of state of white dwarfs provide us with more understanding of matter and physics describing the Universe.

In this study, the equation of state for white dwarf matter is derived first by treating the matter as ideal Fermi gas, then including also electrostatic forces and considering the effects of inverse β-decay. We conclude with an overview of the equation of gravitational potential energy arising from hydrostatic equilibrium.

The accuracy of the equation of state was concluded to depend on which interactions and phenomenon are included in the consideration. On the other hand, choosing the white dwarf model for an application depends significantly on the density of the matter, as well. The equations of state of ideal Fermi gas, with Coulomb correction and with the inverse β-decay correction were concluded to be accurate enough to provide a quantitatively adequate description of the phenomenon.

Contents

1 Introduction

1.1 The Luminous Living Dead

This study is about deriving a couple of equations of state for white dwarfs below the regime of neutron drip, which means densities less than 4×10^{11} g cm^{-3}. White dwarfs — also called degenerate dwarfs, composed mostly of electron-degenerate matter — are luminous and the color of the light they are emitting is white, hence their name. [1] Most observed white dwarfs have relatively high surface temperatures, between $8,000$ K and $40,000$ K [2]. The surface temperature of $\sim 10^4 K$ implies a white color [3].

White dwarfs are classified as compact objects, meaning that their life begins when a star dies. Star death is a point where the most of its nuclear fuel has been consumed. Like other compact objects such as black holes and neutron stars, white dwarfs have small radii, relatively large mass (in comparison to Sun, for instance) — and therefore have a large density, especially in the interior and at the surroundings of the core. [1], [4], [5], [6], [7] Despite the high surface temperature these stars are still considered cold, however, because temperature does not affect the equation of state [3].

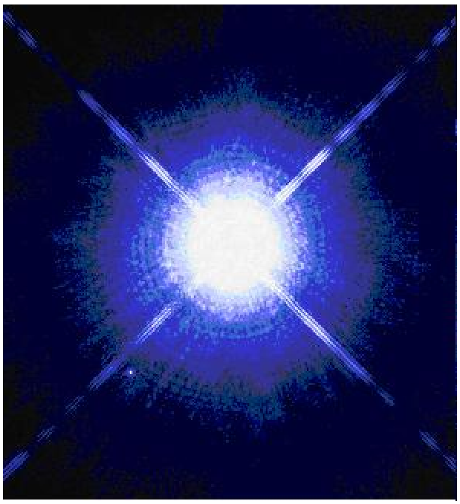

Figure 1: Hubble Space Telescope image of Sirius A along with its stellar companion Sirius B. (Figure taken from [8].)

White dwarfs are described as faint stars below the main sequence in the Hertzsprung–Russell diagram (Fig. 2), a plot of luminosity against spectral type (color), because of having high surface temperature and small diameter. [7], [9], [10] In other words, white dwarfs are less luminous than main-sequence stars of corresponding colors. [3], [9] It is also mentioned in the literature that the name given is deceptive because some white dwarfs are yellow and at least one is red. [9] While slowly cooling, the white dwarfs are changing in color from white to red and finally to black. [10] So, the visible radiation emitted by white dwarfs varies over a wide color range, from the blue-white color of a main sequence star to red of the red dwarf. [11] There ought to be a large number of invisible black dwarfs in the Milky Way. [10]

White dwarfs were established in the early 20^{th} century and have been studied and observed ever since. [1] They comprise an estimated 3% of all the stars of our galaxy. Because of their low luminosity, white dwarfs (except the very nearest ones) have been very difficult to detect at any reasonable distance and that is why there was very little observational data supporting the theory in the time of them being discovered. The companion of Sirius, discovered in 1915, was among the earliest to become known (Fig. 1). [6], [9], [10]

Because of the relatively enormous density (compared to regular still living stars), the gravitational field of a white dwarf causes a collapse. White dwarfs resist the gravitational collapse with electron degeneracy pressure. The temperature of white dwarfs is much higher than that of normal stars. These properties, together with exceedingly small size are characteristic of white dwarfs. After birth, the life and the life quality of these living dead objects depend on the equation of state — the white dwarfs spend their lives slowly cooling, radiating away their residual thermal energy. White dwarfs can be considered as one possibility of a final stage of stellar evolution since they are considered static over the lifetime of the Universe.

1.2 Motivation for the Study of White Dwarfs

White dwarfs are believed to originate from light mass stars with masses $M \leq 4M_{\odot}$. The mass of a typical white dwarf is about one solar mass and the radii is about 5000 km. Mean densities of white dwarfs are around 10^6 cm^{-3}. [1], [3] According to some sources masses of white dwarfs are average half of the Sun's mass and their diameter is generally range between $1/4 - 4$ times the Earth's diameter, having densities of 2×10^5 times the density of the Sun and even greater values.

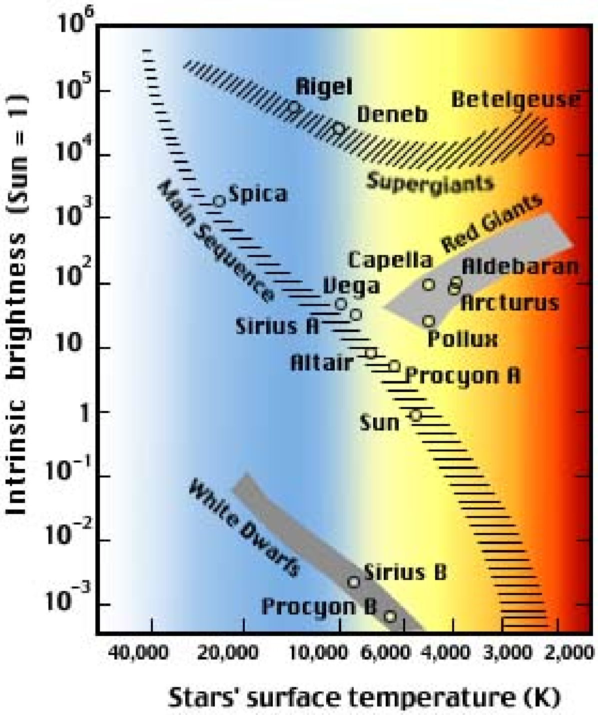

Figure 2: The Hertzsprung-Russell diagram. The diagonal is the main sequence, going from the hot and bright to the cooler and less bright. White dwarfs are in the lower–left. The Sun is found on the main sequence at luminosity 1 (absolute magnitude 4.8) and B–V color index 0.66 (temperature 5780K, spectral type G2). (Figure taken from [12].)

[9] According to [13] the maximum radius of equation of state is $R_{\max} = 0,021 R_{\odot}$, with mass of $M = 0,02 M_{\odot}$, where the solar radius is $R_{\odot} \approx 6,96 \times 10^{10}$ cm and the solar mass $M_{\odot} \approx 1,989 \times 10^{33}$ gm. The lowest possible value of the critical mass is $M_{\text{cre}} = 1,015 M_{\odot}$. [13] At high densities the degenerate electron gas becomes relativistic and when it is taken to be an ideal Fermi gas, dynamical instability will set in when the mass exceeds the critical mass. [3] This mass is given by [14] and [15] as $M_{\text{cre}} \approx 1,4587 M_{\odot}$, also known as the Chandrasekhar limit.

The mass of the best known white dwarf, Sirius B, was determined applying Kepler's Third Law to the binary star orbit (as the Sirius B is the binary companion of Sirius). In 1915 W.S. Adams discovered, that Sirius B had the spectrum of a white star. Adams reported measurements of the gravitational redshifts of several spectral lines emitted from the surface of Sirius B in his book in 1925 and Sir Arthur

Eddington applied the theory of general relativity to determine M/R ratio from the redshifts using these measurements in 1926. Eddington once wrote that "Adams killed two birds with one stone; he has carried out a new test of Einstein's general theory of relativity and confirmed our suspicion that matter 2000 times denser than platinum is not possible, but actually present in Universe". In 1926 P. Dirac formulated Fermi–Dirac statistics on the foundations established by E. Fermi, and R.H. Fowler applied Fermi–Dirac statistics to identify the pressure holding up the stars from gravitational collapse with electron degeneracy pressure. [1]

S. Chandrasekhar constructed the actual white dwarf models, taking into account the special relativistic effects in the degenerate equation of state in 1930 (Appendix A, Table 1 and Fig. 5). In 1932 L.D. Landau presented an explanation of the Chandrasekhar limit, the exact value depending on the composition of matter. In 1949 Kaplan discussed the mass-radius relation modifying effects of the general relativity. [1] In 1954 J. L. Greenstein published many reports about the study of white dwarfs. He described white dwarfs as "mainly a degenerate mass devoid of hydrogen, surrounded by a non-degenerate envelope 65 miles deep, and above this an atmosphere of a sort only a few hundred feet deep". Degenerate matter conforms to an equation of state different from the ordinary gas laws and according to the theory the radius of a completely degenerate star is inversely proportional to the mass, which cannot exceed 1,4 times the Sun's mass. [9] Greenstein reported on his studies of the spectra of 50 white dwarfs. Some spectra showed prominent helium absorption, others showed dark hydrogen lines and at least one spectra had only the hydrogen and the potassium lines of ionized calsium and a line of neutral calsium. [9], [16] In 1958 Harrison, Wakano and Wheeler incorporated inverse beta decay in the equation of state for white dwarf matter (Appendix A, Table 1 and Fig. 5). Chandrasekhar discovered the general relativistic instability of the white dwarfs in 1964 . [1]

White dwarfs are compact objects — they no longer burn nuclear fuel, but they are cooling by radiating thermal energy. An isolated white dwarf star cools to zero temperature, since it has no internal sources of energy. The pressure associated with matter at $T = 0$ supports the star against the gravitational collapse. [1], [10] The cooling of white dwarfs is not only a fascinating phenomenon but in addition offers information of solid state physics in a new setting — the circumstances of an original star can not be built up in a laboratory. Also, it would not be possible to realize the distance, which includes many advantages in sketching timescales and fundamental interactions by observation. More over, the evolution and the equation of state for white dwarfs together with the dependencies between the quantities affecting the equation of state can therefore be useful on Earth providing us more understanding of matter and physics describing the Universe.

In what follows, the pressure–density relation as an equation of state will be derived for white dwarf matter in three different sets of conditions. First by treating the matter as ideal Fermi gas, then taking electrostatic forces into account and considering the effects of inverse β–decay to the equation of state. We conclude with an overview of the equation of gravitational potential energy, arising from hydrostatic equilibrium.

2 Equation of State

2.1 ES of a Degenerate Ideal Fermi Gas

The pressure associated with matter at zero temperature supports the white dwarfs against the gravitational collapse. White dwarf matter has very high density and the dominant contribution to its pressure arises from the Pauli Exclusion Principle — the constituent fermions are forbidden to occupy identical quantum states. However, increasing density forces many of the particles very close to each other and apparently to higher–energy quantum states. Degeneracy pressure arises from this compression, resisting it.

The simplest case of equation of state can be derived with a single species of non–interacting fermions. Let us next consider electron gas at $T = 0$. Ignoring the electrostatic forces, the gas can be treated as ideal. Defining the Fermi momentum p_F by [17]

$$E_F \equiv \left(p_F^2 c^2 + m_e^2 c^4\right)^{1/2} \tag{1}$$

gives

$$n_e = \frac{2}{h^3} \int_0^{p_F} 4\pi p^2 \, dp = \frac{8\pi}{3h^3} p_F^3. \tag{2}$$

In terms of dimensionless relativity parameter,

$$x = \frac{p_F}{m_e c}, \tag{3}$$

we have

$$n_e = \frac{1}{3\pi^2 \lambda_e^3} x^3. \tag{4}$$

where $\lambda_e = \hbar/m_e c$ is the Compton wavelength of the electron. [1] The pressure of the electron gas is then given by [1]

$$\begin{aligned} P_e &= \frac{1}{3}\frac{2}{h^3}\int_0^{p_F} \frac{p^2c^2}{(p^2c^2 + m_e^2c^4)^{1/2}} 4\pi p^2\, dp \\ &= \frac{8\pi m_e^4 c^5}{3h^3}\int_0^x \frac{x^4 dx}{(1+x^2)^{1/2}}\, dx \\ &= \frac{m_e c^2}{\lambda_e^3}\phi(x) \\ &= 1.42180 \times 10^{25}\ \phi(x)\ \text{dyn cm}^{-2}, \end{aligned} \tag{5}$$

where[1] $\phi(x)$ is given parametrically in terms of x :

$$\phi(x) = \frac{1}{8\pi^2}\left\{x\left(1+x^2\right)^{1/2}\left(2x^2/3-1\right) + \ln\left[x + \left(1+x^2\right)^{1/2}\right]\right\}. \tag{6}$$

The energy density is given by

$$\varepsilon = \int E \frac{d\mathfrak{N}}{d^3x\ d^3p}\, d^3p, \tag{7}$$

where $E = \left(p^2c^2 + m^2c^4\right)^{1/2}$, m is the particle rest mass and $d\mathfrak{N}/d^3x\ d^3p$ is the number density in phase space. [1], [17] So, the energy density of electrons is given, similarly to equation (5), by [1]

$$\begin{aligned} \varepsilon_e &= \frac{2}{h^3}\int_0^{p_F} \left(p^2c^2 + m_e^2c^4\right)^{1/2} 4\pi p^2\, dp \\ &= \frac{m_e c^2}{\lambda_e^3}\chi(x), \end{aligned} \tag{8}$$

where

$$\chi(x) = \frac{1}{8\pi^2}\left\{x(1+x^2)^{1/2}(1+2x^2) - \ln\left[x + (1+x^2)^{1/2}\right]\right\}. \tag{9}$$

Considering $\phi(x)$ of equation (6) at the non–relativistic limit, $x \ll 1$, gives

$$\phi(x) \to \frac{1}{15\pi^2}\left(x^5 - \frac{5}{14}x^7 + \frac{5}{24}x^9 \ldots\right) \tag{10}$$

[1]Dyne is a unit of force specified in the CGS system of units, a predecessor of the modern SI. 1 dyn = 1 g cm s^{-2} = 10 μN

and for relativistic electrons, $x \gg 1$,

$$\phi(x) \to \frac{1}{12\pi^2}\left(x^4 - x^2 + \frac{3}{2}\ln 2x \ldots\right) . \tag{11}$$

[1] As we see, equations (10) and (11) differ — relativistic effects are significant.

The density of degenerate electrons is dominated by the rest mass of the ions:

$$\rho_0 = \sum_i n_i m_i, \tag{12}$$

where m_i is the mass of ion of species i. [1] Defining the mean baryon rest mass as

$$m_B \equiv \frac{1}{n}\sum_i n_i m_i = \frac{\sum_i n_i m_i}{\sum_i n_i A_i}, \tag{13}$$

where A_i is the baryon number of the ith species. Then the density is

$$\rho_0 = n m_B = \frac{n_e m_B}{Y_e}, \tag{14}$$

where $Y_e \equiv n_i/n$ is the mean number of electrons per baryon. If the quantity mean molecular weight per electron,

$$\mu_e = \frac{m_B}{m_u Y_e}, \tag{15}$$

is used, then

$$\rho_0 = \mu_e m_u n_e = 0.97395 \times 10^6 \mu_e x^3 \text{ g cm}^{-3}. \tag{16}$$

The mean molecular weight is particularly useful in the non-degenerate limit, when the pressure is given by the perfect gas law [1], [17]

$$\begin{aligned} P &= \left(n_e + \sum_i n_i\right) kT \\ &= \frac{\rho_0}{\mu m_u} kT \end{aligned} . \tag{17}$$

Quantity m_u/m_B can be taken equal to unity, except when extreme accuracy is needed. The total density is $\rho = \rho_0 + \varepsilon_e/c^2$, but usually the term ε_e/c^2 is neglibly small. Combining equations (5), (6) and (16), the electron gas pressure, $P = P(\rho_0)$, can be written as a function of density as

$$
\begin{aligned}
P_e = \frac{m_e c^2}{8\pi^2 \lambda_e^3} & \Bigg\{ 1.0088 \times 10^{-2} \times \left(\frac{\rho_0}{\mu_e} \ \mathrm{g}^{-1} \ \mathrm{cm}^3 \right)^{1/3} \\
& \times \sqrt{1 + \left(1.0088 \times 10^{-2} \times \left(\frac{\rho_0}{\mu_e} \ \mathrm{g}^{-1} \ \mathrm{cm}^3 \right)^{1/3} \right)^2} \\
& \times \left(\left(\frac{\sqrt{2} \times 1.0088 \times 10^{-2}}{\sqrt{3}} \times \left(\frac{\rho_0}{\mu_e} \ \mathrm{g}^{-1} \ \mathrm{cm}^3 \right)^{1/3} \right)^2 - 1 \right) \qquad (18) \\
& + \ln \Bigg[\left(1.0088 \times 10^{-2} \times \left(\frac{\rho_0}{\mu_e} \ \mathrm{g}^{-1} \ \mathrm{cm}^3 \right)^{1/3} \right) \\
& + \sqrt{1 + \left(1.0088 \times 10^{-2} \times \left(\frac{\rho_0}{\mu_e} \ \mathrm{g}^{-1} \ \mathrm{cm}^3 \right)^{1/3} \right)^2} \Bigg] \Bigg\} .
\end{aligned}
$$

In this chapter we have derived the simplest equation of state for a single species of noninteracting fermions ignoring the electrostatic forces and treating the electron gas as ideal gas. Equations (5) and (16) or the combination of these two, as equation (18), give the pressure associated with white dwarf matter of this model at zero temperature. This ideal Fermi gas equation of state was first employed by Chandrasekhar in 1931.

2.2 Electrostatic Correction to ES

Let us now take the electrostatic interactions into account. This will lead to a correction to the equation of state due to electrostatic interactions among the ions and electrons in the white dwarf matter. The positive charges are not uniformly distributed in the gas. They are localized in the nuclei of the atoms, each of charge Z. This decreases the energy and pressure of the electrons. The mean distance between the electrons and the nucleus is smaller than the distance between the electrons — the attractive forces are greater than the repulsive ones.

The more the density increases, the more important the Coulomb effects become in a nondegenerate gas. The ratio of Coulomb energy and thermal energy is approximately

$$
\frac{E_c}{kT} = \frac{Ze^2/\langle r \rangle}{kT} \approx \frac{Ze^2 n_e^{1/3}}{kT}, \qquad (19)
$$

where Z is the charge of the nuclei, e is the charge of an electron, k is the Boltzmann constant, T is the temperature, $\langle r \rangle$ is the mean radius and n_e is the number density of electrons. [1] The ratio increases with increasing n_e. Approximation $\langle r \rangle \sim n_e^{-1/3}$ corresponds to the characteristic electron–ion separation. On the other hand, for a degenerate gas

$$\frac{E_c}{E'_F} = \frac{Ze^2/\langle r \rangle}{p_F^2/2m_e}, \tag{20}$$

in contrast to (19). [1]

Using equation (2), we have

$$\frac{E_c}{E'_F} = 2\left(\frac{1}{3\pi^2}\right)^{2/3} \frac{Z}{a_0}\frac{1}{n_e^{1/3}} = \left(\frac{n_e}{Z^3 \times 6 \times 10^{22}\ cm^{-3}}\right)^{-1/3}, \tag{21}$$

where $a_0 = \hbar/m_e e^2$ is the Bohr radius. In most astrophysically relevant degenerate gases $E_c \ll E'_F$. This implies n_e is to first approximation uniform — now we can derive an approximate expression for the electrostatic correction to the ideal degenerate Fermi gas equation of state. [1]

As the temperature approaches zero, $T \to 0$, the ions form a lattice, that maximizes the separation of ions. In the Wigner-Seitz approximation the gas is divided into neutral spheres of radius r_0 about each nucleus containing the Z number of electrons closest to the nucleus. A spherical shell lattice like this would have a volume of $4\pi r_0^3/3 = 1/n_N$, where n_N is the number density of the nuclei. The Wigner–Seitz approximation is very suitable for white dwarfs, because the number density n_e is more uniform than in typical non-uniform laboratory solids. [1]

Now, the let us calculate the energy of a uniform sphere of Z electrons (summing the potential energies of each sphere due to the electron–electron interactions)

$$\begin{aligned} E_{e-e} &= \int_0^{r_0} \frac{q}{r}\, dq \\ &= -\int_0^{r_0} \frac{Zer^2}{r_0^3}\, dq \\ &= \frac{3}{5}\frac{Z^2e^2}{r_0}, \end{aligned} \tag{22}$$

where the charge inside the radius r is $q = -Zer^3/r_0^3$. The total electrostatic energy of the electrons and the nucleus of charge

Ze is correspondingly (a sum of the potential energies of electron-ion interactions)

$$\begin{aligned} E_{e-i} &= Ze \int_0^{r_0} \frac{1}{r}\, dq \\ &= -\frac{3}{2}\frac{Z^2e^2}{r_0} \end{aligned} . \tag{23}$$

The total Coulomb energy of the spherical cell of the lattice is then

$$E_c = E_{e-e} + E_{e-i} = -\frac{9}{10}\frac{Z^2e^2}{r_0}. \tag{24}$$

The cells are considered to be neutral in charge — the interactions between electrons and the nuclei of different cells can therefore be ignored. [1]

The electrostatic energy per electron is given by

$$\frac{E_c}{Z} = -\frac{9}{10}(\frac{4\pi}{3})^{1/3} Z^{2/3} e^2 n_e^{1/3}, \tag{25}$$

where

$$n_e = \frac{Z}{4\pi r_0^3/3}. \tag{26}$$

The corresponding pressure is

$$\begin{aligned} P_c &= n_e^2 \frac{d(E_c/Z)}{dn_e} \\ &= -\frac{3}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{4/3}, \end{aligned} \tag{27}$$

for which the ideal Chandrasekhar result is

$$P_0 \to \hbar c (3\pi^2)^{1/3} \frac{n_e^{4/3}}{4}. \tag{28}$$

In the extreme relativistic limit this leads to

$$\begin{aligned} \frac{P}{P_0} &= \frac{P_0 + P_c}{P_0} \\ &= 1 - \frac{2^{5/3}}{5}\left(\frac{3}{\pi}\right)^{1/3} \alpha Z^{2/3}, \end{aligned} \tag{29}$$

where $\alpha = e^2/\hbar c = 1/137$ is the fine structure constant. [1]

In the non-relativistic limit,

$$P_0 \to \hbar^2 (3\pi^2)^{2/3} \frac{n_e^{5/3}}{5m_e} \tag{30}$$

and

$$\frac{P}{P_0} = 1 - \frac{Z^{2/3}}{2^{1/3}\pi a_0 n_e^{1/3}}, \tag{31}$$

which predicts that $P = 0$ when

$$n_e = \frac{Z^2}{2\pi^3 a_0^3}, \tag{32}$$

corresponding to density

$$\rho_0 \approx 0.4Z^2 \text{ g cm}^{-3} \tag{33}$$

by equation (14) with $A \approx 2Z$. [1]

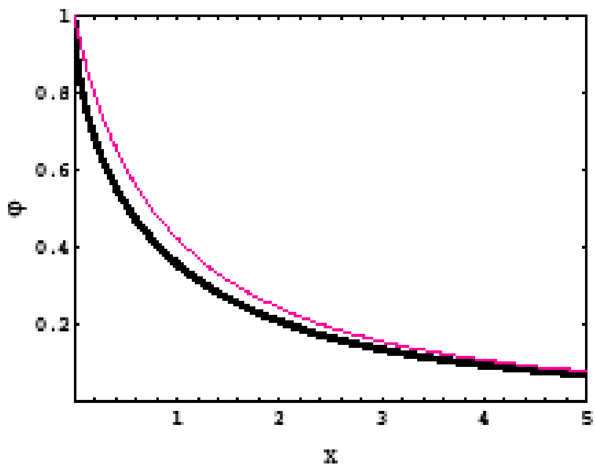

Figure 3: Solutions of the Thomas–Fermi function $\phi(x)$ (40) with BC's (41) and (42). The upper line refers to the exact numerical solution of equation (40), while the lower one corresponds to the parametric approximated solution reported in [18]. (Figure taken from [18].)

Let us now use the Thomas–Fermi method for the statistical treatment of the atomic structure. Assuming that electrons move in a slowly varying, approximately constant, spherically symmetric potential $V(r)$ and using free particle Fermi-Dirac statistics taking the interaction energy to be much less than the kinetic or potential energies of the individual electrons, all states up to $E = E_F$ are occupied and the energy is

$$E_F = -eV(r) + \frac{p_F^2(r)}{2m_e} = \text{constant}, \tag{34}$$

where p_F is the maximum momentum of electrons at r. Fermi energy, E_F, is independent of r because of the electron occupation. [1] Now, as in equation (2)

$$n_e = \frac{8\pi}{3h^3} p_F^3(r) = \frac{8\pi}{3h^3} \left\{2m_e \left[E_F + eV(r)\right]\right\}^{3/2}. \tag{35}$$

Combining this with the Poisson's equation for the potential,

$$\nabla^2 V = 4\pi e n_e \ + \ \text{nuclear contribution}, \tag{36}$$

we have

$$\frac{1}{r}\frac{d^2}{dr^2}(rV) = \frac{32\pi^2 e}{3h^3} \left[2m_e(E_F + eV)\right]^{3/2}. \tag{37}$$

Equation (37) is to be solved with two boundary conditions: since the nuclear contribution is a delta function about the origin,

$$\lim_{r\to 0} rV(r) = Ze, \tag{38}$$

for $r > 0$, and

$$\left.\frac{dV}{dr}\right|_{r_0} = 0, \tag{39}$$

since the electric field must vanish at the cell boundary r_0. [1]

Through simplifications, [1] equation (37) becomes the Thomas–Fermi equation: [1], [18]

$$\frac{d^2\phi}{dx^2} = \frac{\phi^{3/2}}{x^{1/2}} \tag{40}$$

with boundary conditions

$$\phi(0) = 1, \tag{41}$$

$$\phi'(x_0) = \frac{\phi(x_0)}{x_0}. \tag{42}$$

For the special case $\phi'(0) = -1.58897100$, when $x \to \infty$ then $\phi'(x_0) \to 0$ and $\phi(x_0) \to 0$. This zero pressure case corresponds to zero density and infinite radius — free atoms have infinite radius (Fig. 3). For $\phi'(0) = -1.58897100$, ϕ diverges as $x \to \infty$. [1], [18]

The pressure at the boundary of the cell can now be computed using the free particle expression as:

$$\begin{aligned} P &= \frac{2}{\hbar^3}\frac{1}{3}\int_0^{p_F} \frac{p^2}{m_e} 4\pi p^2 \, dp \\ &= \frac{8\pi}{15h^3 m_e} p_F^5(r_0). \end{aligned} \tag{43}$$

Writing second equality in (35) as

$$\begin{aligned} p_F &= \{2m_e\left[E_F + eV(r_0)\right]\}^{1/2} \\ &= \left\{2m_e\left[\frac{Ze^2\phi(x_0)}{x_0\mu}\right]\right\}^{1/2}, \end{aligned} \tag{44}$$

where $r = x\mu$ and

$$\mu = \left(\frac{9\pi^2}{128Z}\right)^{1/3} a_0, \tag{45}$$

equation (43) becomes

$$\begin{aligned} P &= \frac{8\pi}{15h^3 m_e}\left\{2m_e\left[\frac{Ze^2\phi(x_0)}{x_0\mu}\right]\right\}^{5/2} \\ &= \frac{1}{10\pi}\frac{Z^2e^2}{\mu^4}\left[\frac{\phi(x_0)}{x_0}\right]^{5/2} \end{aligned} \tag{46}$$

and the density is thus given by the total rest mass of the cell,

$$\rho_0 = \frac{Am_B}{4_\pi \mu^3 x_0^3/3}. \tag{47}$$

Now the electrostatic interactions have been taken into account. Equations (46) and (47) give the equation of state with the Coulomb correction.

2.3 Correction to the ES due to Inverse β–decay

Next we will provide an overview of the effects of inverse β–decay to the equation of state. Let us consider the ideal, cold gas consisting of free neutrons, protons and electrons. At high densities, examining such $n-p-e$ gas requires a correction to the equation of state due to inverse β–decay: [1]

$$e^- + p \rightarrow n + \nu, \tag{48}$$

also known as electron capture [19]. Now, let us assume that the neutrinos generated in the reaction escape from the system and ignore them. Reaction (48) can proceed when the electrons acquire enough energy to balance the mass difference between protons and neutrons,

$$(m_n - m_p)c^2 = 1.29 \text{ MeV}, \tag{49}$$

and is effective process for transforming protons into neutrons if β–decay (also known as β^-- and neutron β-decay [19], [20]),

$$n \to p + e^- + \bar{\nu}, \tag{50}$$

does not significantly occur. Reaction (50) is blocked if the density is high enough — if all the electron energy levels in the Fermi sea are occupied, there is no available position for the emitted electron to fill. [1], [21] This implies that there is a critical density for the reaction (48), as well.

Let us now examine the properties of $n-p-e$ mixture by assuming that they all are in equilibrium. Let us express reaction (48) by means of chemical potentials:

$$\mu_e + \mu_p = \mu_n + \mu_\nu = \mu_n, \tag{51}$$

where we have set the chemical potential and thus, the number density of neutrinos to zero. Now, defining

$$x_i = \frac{p_{iF}}{m_i c}, \quad i = e, p, n \tag{52}$$

as in (3), and since

$$\mu_e = \left[(p_{eF} c)^2 + m_e^2 c^4 \right]^{1/2}, \tag{53}$$

equation (51) becomes [1]

$$m_e \left(1 + x_e^2\right)^{1/2} + m_p \left(1 + x_p^2\right)^{1/2} = m_n \left(1 + x_n^2\right)^{1/2}. \tag{54}$$

The equal amount of protons and electrons implies charge neutrality, which leads to [1]

$$\begin{gathered} \frac{1}{3\pi^2 \lambda_e^3} x_e^3 = \frac{1}{3\pi^2 \lambda_p^3} x_p^3 \text{ and thereby to} \\ m_e x_e = m_p x_p. \end{gathered} \tag{55}$$

The equation of state including inverse β–decay correction is then given by three equations — by the electron gas pressure

$$P = \frac{m_e c^2}{\lambda_e^3} \phi(x_e) + \frac{m_p c^2}{\lambda_p^3} \phi(x_p) + \frac{m_n c^2}{\lambda_n^3} \phi(x_n), \tag{56}$$

the energy density

$$\varepsilon = \frac{m_e c^2}{\lambda_e^3} \chi(x_e) + \frac{m_p c^2}{\lambda_p^3} \chi(x_p) + \frac{m_n c^2}{\lambda_n^3} \chi(x_n), \tag{57}$$

and the number density

$$n = \frac{1}{3\pi^2\lambda_p^3}x_p^3 + \frac{1}{3\pi^2\lambda_n^3}x_n^3, \tag{58}$$

where χ is defined in equation (9). [1]

The escaping neutrinos cause an energy loss and affect the equation of state as well. Nevertheless, the neutrinos have been ignored in deriving these equations. So, actually these results describe an equilibrium system of zero charge, fixed baryon and lepton numbers — the lepton number is chosen to have its minimum possible value in the limit $n_\nu \to 0$. Thermodynamical equilibrium is not reached in an open system and the composition of neutrons, protons and electrons should be determined by rate equations for the reactions.

2.4 Hydrostatic Equilibrium and Gravitational Potential Energy

In this section, let us take the gravitational forces of white dwarfs under consideration. If the star is in a steady state, the gravitational force balances the pressure force of the white dwarf matter at every point and vice versa (Fig. 4). Let us consider an infinitesimal fluid element of white dwarf matter between radius r and $r + dr$ and an infinitesimal area dA perpendicular to the radial direction. To maintain the steady state, the gravitational force exerted on the mass dm of the fluid must be equal in magnitude and opposite in direction to the force that the pressure of the fluid exerts on the area dA.

For a spherically symmetric distribution of matter

$$m(r) = \int_0^r \rho 4\pi r^2\, dr, \tag{59}$$

in other words,

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho, \tag{60}$$

where $m(r)$ is the mass interior to a radius r and $\rho \approx \rho_0$ is the rest-mass density in consideration of non-relativistic matter. [1] Now, the net outward pressure force on mass $dm = \rho\ dA\ dr$ is

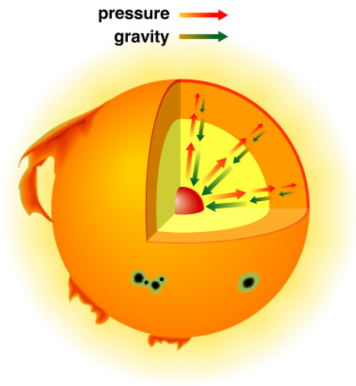

Figure 4: Hydrostatic equilibrium. The internal pressure outward balances the force of gravity inward. (Figure taken from [22].)

$F_p = -dP\, dA = -[P(r+dr) - P(r)]dA$ and therefore in equilibrium, $\nabla P = -\rho\nabla\Phi$, as

$$\begin{aligned} -\frac{dP}{dr} dr dA &= \frac{Gm(r)}{r^2} dm \\ \frac{dP}{dr} &= -\frac{Gm(r)\rho}{r^2}, \end{aligned} \tag{61}$$

where G is the gravitational constant and Φ is the gravitational potential. [1] Consequently, this leads to the virial theorem. Integrating by parts, the gravitational potential energy of white dwarf is

$$\begin{aligned} W &= -\int_0^R \frac{Gm(r)}{r} \rho 4\pi r^2 \, dr \\ &= \int_0^R \frac{dP}{dr} 4\pi r^3 \, dr \\ &= -3 \int_0^R P 4\pi r^2 \, dr. \end{aligned} \tag{62}$$

As a simple example, consider characterizing the fluid by an adiabatic equation of state

$$P = K\rho_0^\Gamma, \tag{63}$$

where K and Γ are constants. Note, that ideal Fermi gas equation of state (18) has this form in extreme relativistic, $\Gamma = 4/3$, and non-relativistic, $\Gamma = 5/3$, limits. With this form, the energy density of the fluid becomes

$$\varepsilon' = \frac{P}{\Gamma - 1} \tag{64}$$

and with the first law of thermodynamics this results in

$$d\frac{\varepsilon}{\rho_0} = -Pd\frac{1}{\rho_0}, \tag{65}$$

since the changes are adiabatic. [1] Integration gives

$$\varepsilon = \rho_0 c^2 + \frac{P}{\Gamma - 1}, \tag{66}$$

which leads to $\varepsilon' \equiv \varepsilon - \rho_0 c^2$. Now equation (62) can be rewritten as

$$W = -3(\Gamma - 1)U, \tag{67}$$

where

$$U = \int_0^r \varepsilon' 4\pi r^2 \, dr, \tag{68}$$

is the total internal energy of the white dwarf. [1]

3 Conclusion

In previous chapters, we derived the equation of state for white dwarfs below the regime of neutron drip — first by treating the white dwarf matter as ideal Fermi gas, then taking electrostatic forces into account and finally including also the effects of inverse β–decay into the consideration. We concluded with an overview of the gravitational potential energy of the star.

Equation (5) and (16) give the equation of state for white dwarfs ignoring electrostatic forces and treating the electron gas as ideal. Equation (18) combines equations (5) and (16). This cold, ideal Fermi gas pressure–density relation is the simplest case of equation of state for a single species of noninteracting fermions.

The equation of state including the electrostatic correction is given by equations (46) and (47). The principal electrostatic correction arises from the non–uniformly distributed, nuclei–centered positive charges and decreases the pressure and energy of electrons in the gas. The Coulomb correction is relatively small, but nevertheless important for high-density white dwarfs. At low densities it is no longer a good approximation to consider the electron gas uniform — for instance, the laboratory value for iron is $\rho_0 = 7.86$ g cm^{-3} and equation (33) gives $\rho_0 = 250$ g cm^{-3}. Electron shell effects mask the statistical effects at laboratory densities, but as the densities increase — say, to a few times laboratory densities and higher — statistical approach works fine for the equation of state.

The equation of state of the ideal, cold $n - p - e$ gas including the effects of inverse β–decay is given by equations (56), (57) and (58). It was concluded that high densities require a correction to the equation of state due to inverse β–decay. The equilibrium nuclide become more neutron rich with increasing densities. There is a critical density limit for the inverse β–decay, reaction (48)— the nuclei of the white dwarf matter are stabilized against the β–decay, reaction (50), by the filled Fermi sea of electrons.

In the end, the gravitational forces of white dwarfs were taken under consideration. If the star is in a steady state, the gravitational force balances the pressure force of the white dwarf matter at every point and vice versa — this lead us to the hydrostatic equilibrium equation (61). The gravitational potential energy of the star is given by equations (62) and (67) and the internal energy by (68).

The escaping neutrinos cause an energy loss and affect the equation of state, as well. Nevertheless, the neutrinos have been ignored in deriving these equations. So, actually these results describe an equilibrium system of zero charge, fixed baryon and lepton numbers — the lepton number is chosen to have its minimum possible value in the limit $n_\nu \rightarrow 0$. Thermodynamical equilibrium is not reached in an open system and the composition of neutrons, protons and electrons should be determined by rate equations for the reactions.

More complete equation of state would take the effects of escaping neutrinos into account, determine the proportions of the mixture of the nuclei, free electrons and neutrons, include effects arising from the nuclear interactions and reactions and so on. More accurate white dwarf models including such and even more corrections do exist, developed for instance by Salpeter, Salpeter and Zapolsky, Feynmann and Metropolis and Teller, Harrison and Wheeler, Baym and Pethick and Sutherland, just to mention the most famous ones (Appendix A, Table 1 and Fig. 5). [1], [23]

The accuracy of the equation of state was concluded to depend on the assumptions and approximations made during the derivation — the precision of the equation of state describing white dwarfs highly depends on the chooses of which interactions and phenomena are included into the considerations. On the other hand, choosing the white dwarf model for an application depends significantly on the density of the matter, as well. At the low–density regime, up to 10^4 g cm^{-3} [1], the Feynmann–Metropolis–Teller and the Thomas–Fermi–Dirac models are considered relevant. In the high–density limit, the Thomas–Fermi result (46) reduces to (31). However, for higher densities, Chandrasekhar ideal gas result with the Coulomb correction in equations (27) and (29) is adequate. The equations of state of ideal Fermi gas, with Coulomb correction and the inverse β–decay correction were concluded to be accurate enough to describe phenomenon in general, over the whole density range considered here.

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A Representative Equations of State

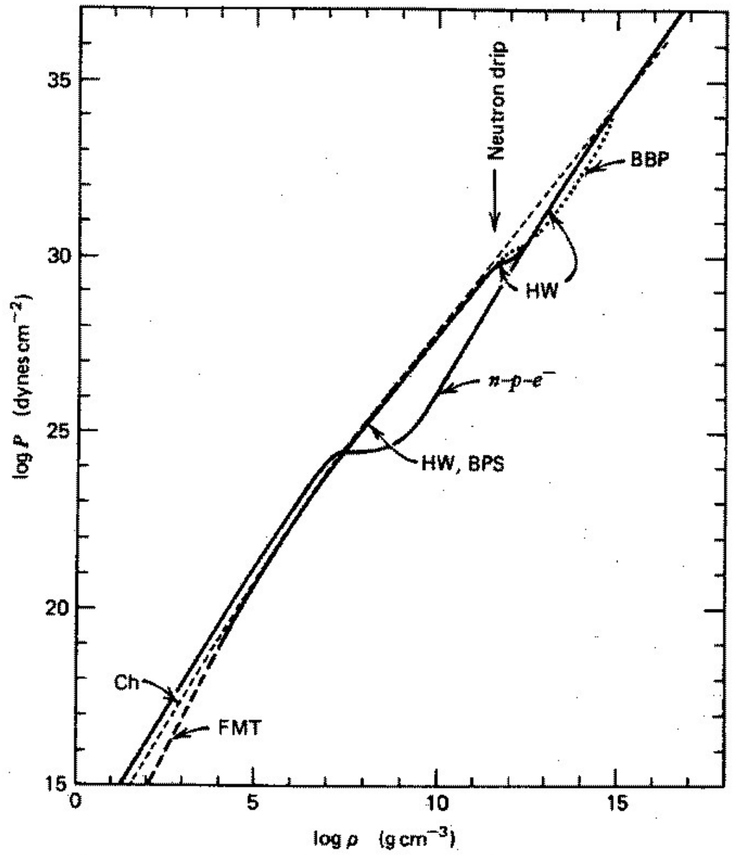

Figure 5: Equations of states for white dwarfs. Chandrasekhar (Ch) equation of state is based on the ideal degenerate electron gas model. Harrison–Wheeler (HW) equation of state is based on the liquid drop model of the nucleus. The Baym–Pethick–Sutherland (BPS) uses Coulomb interaction energy of all the nuclei and the electrons in addition to HW. (Figure taken from [1].)

Table 1: Representative Equations of State Below Neutron Drip (Table adapted from [1].)

Equation of State	Density Regime	(g cm^{-1})	Composition	Theory
Chandrasekhar (1931) ideal electron gas	$0 \leq$	$\rho \leq \infty$	e^- (nuclei specified by μ_e)	Noninteracting electrons
Ideal $n - p - e^-$ gas	$0 \leq$	$\rho \leq 1.2 \times 10^7$	e^-, p	Equilibrium matter
	$1.2 \times 10^7 <$	$\rho \leq \infty$	e^-, p, n	
Feynman–Metropolis–Teller (1949; FMT)	$7.9 \leq$	$\rho \leq 10^4$	e^- and $^{56}_{26}Fe$	Thomas–Fermi–Dirac atomic model
Harrison–Wheeler (1958)	$7.9 \leq$	$\rho \leq 10^4$		Same as FMT
	$10^4 <$	$\rho \leq 10^7$	e^- and $^{56}_{26}Fe$	Noninteracting electrons
	$10^7 <$	$\rho \leq 3 \times 10^{11}$	e^- and equilibrium nuclide	Semi-empirical mass formula; equilibrium matter
	Above $3 \times 10^{11} <$	$\rho \leq 4 \times 10^{12}$	e^-, n and equilibrium nuclide	
	"Neutron Drip" $4.5 \times 10^{12} <$	$\rho \leq \infty$		Same as ideal $n - p - e^-$
Baym–Pethcik–Sutherland (1971)	$7.9 \leq$	$\rho \leq 10^4$		Same as FMT
	$10^4 <$	$\rho \leq 8 \times 10^6$	e^- and $^{56}_{26}Fe$	Ideal electrons with Coulomb lattice corrections
	$8 \times 10^6 <$	$\rho \leq 4.3 \times 10^11$	e^- and equilibrium nuclide	Laboratory nuclear energies (with extrapolations); Coulomb lattice energy; equilibrium matter