DEPARTMENT OF PHYSICS
UNIVERSITY OF JYVÄSKYLÄ
RESEARCH REPORT No. 5/2007

STUDIES ON THE HAWKING RADIATION AND GRAVITATIONAL ENTROPY

BY
ARI PELTOLA

Academic Dissertation
for the Degree of
Doctor of Philosophy

To be presented, by permission of the Faculty of Mathematics and Science of the University of Jyväskylä, for public examination in Auditorium FYS-1 of the University of Jyväskylä on October 13, 2007 at 12 o'clock noon

Jyväskylä, Finland
August 2007

ISBN 978-951-39-3184-1
URN:ISBN:978-951-39-3184-1

Contents

Foreword

This dissertation is based on the research carried out at the Department of Physics in the University of Jyväskylä during the years 2002-2007. Personally, this period has been rather difficult for me due to some problems in my private life. For this reason, I am very delighted to see that the hard work has finally paid off. This dissertation, however, would not have been realized without the help of the following individuals and organizations. I am most indebted to all of them.

First, I want to express my gratitude to my supervisor and collaborator Dr. Jarmo Mäkelä for his most excellent and encouraging guidance throughout the work leading to this thesis, especially since it has been carried out along his official duties. I am also grateful to my supervisor Dr. Markku Lehto for his inspiring teaching and generous advice, as well as his constructive criticism. I thank Dr. Jorma Louko for the critique during the preparation of my papers and Dr. Matias Aunola for his invaluable help with the numerical analysis performed in the paper I. I am also grateful to Dr. Pasi Repo for introducing me to general relativity and the physics of black holes.

Warmest thanks also go to the whole staff at the Department of Physics for a pleasant working atmosphere. Especially, I want to thank Dr. Kari Rytkönen, Dr. Kimmo Kärkkäinen, Mr. Diego Meschini and Ms. Jonna Piilonen for their company and healthy sense of humour.

I am fortunate to have a most supporting family. I am grateful to my beloved wife Marjo for all the love and support during the ups and downs of my life. I also want to thank my brother, parents, and grandparents, as well as my in-laws, for all the unselfish help they have given.

Financial support provided by the Department of Physics, the Finnish Cultural Foundation, and Vilho, Yrjö and Kalle Väisälä Foundation is gratefully acknowledged.

Finally, and most importantly, I want to express my most sincere gratitude to my heavenly Father for all the blessings of my life.

Jyväskylä, August 2007

Ari Peltola

List of Publications

I J. Mäkelä and A. Peltola: "Spacetime Foam Model of the Schwarzschild Horizon", Phys. Rev. D **69**, 124008 (2004).

II A. Peltola and J. Mäkelä: "Radiation of the Inner Horizon of the Reissner-Nordström Black Hole", Int. J. Mod. Phys. D **15**, 817 (2006).

III J. Mäkelä and A. Peltola: "Gravitation and Thermodynamics: The Einstein Equation of State Revisited", gr-qc/0612078. Submitted for publication.

The author of this thesis has written the main parts of the papers **II** and **III** and participated in the writing of the paper **I**. The mathematical analysis in the paper **II**, and most of the mathematical analysis in the paper **III** has been performed by the author. The author has made a significant contribution to the calculations of the paper **I**.

Prologue

> *Physical reality is consistent with universal laws. Where the laws do not operate, there is no reality—we judge reality by the responses of our senses. Once we are convinced of the reality of a given situation, we abide by its rules.*
>
> Spock, stardate 4385.3 [1]

The story of modern science is rich and fascinating. The first big scientific breakthrough in the history of physics dates back to the later half of the 17th century when Sir Isaac Newton discovered his well-known theories of gravitation and classical mechanics. Newton's achievements in the field of science are hard to compete with. He discovered his theory of gravitation already in 1666, and over the centuries that theory has turned out to be amazingly accurate in describing the effects of gravitation. It predicts, for instance, the orbits of the major planets with a precision of a few seconds of arc over time intervals of several years. His theory of gravitation was finally published in his famous treatise entitled *Philosophiae Naturalis Principia Mathematica* in 1687. In this book he also introduced the three dynamical laws of classical mechanics. In fact, since "Principia" was the first successful attempt to formulate all known laws of physics in the 'language of mathematics' in a coherent manner, it may be regarded as the most important treatise in the history of modern science.

By the end of the 19th century, all known physical phenomena could be explained by *Newton's mechanics, thermodynamics* and *electrodynamics*. However, at the beginning of the 20th century physicists were confronted with severe problems in their scientific world view, and the solutions to these problems yielded two new theories: *general relativity* and *quantum theory*. These theories have turned out to be in beautiful harmony with the experiments and therefore, in our present stage of research, all the observed physical phenomena can be explained by the three foundational theories: quantum theory, thermodynamics and general relativity. Quantum theory tells us how particles behave when they interact with each other. Thermodynamics, in turn, describes how a large collection of particles behaves. The meaning of general relativity, however, is somewhat more diverse. First of all, general relativity is a theory of gravitation. On the other hand, it is also a theory of space and time. This is due to the idea that in some sense spacetime may be identified as the gravitational field itself. In the following pages we shall consider these three foundational theories in more detail and see how they are connected with the main subjects of this thesis.

General relativity, which was published by Albert Einstein in 1915 [2], was almost as epochal as Newton's theory. It is sometimes maintained that general relativity is difficult to understand. If so, the problem is not that the theory itself would be conflicting or complicated. On the contrary, it may be considered as one of the most beautiful theories ever developed. The problem is that general relativity forces us to change our classical conceptions of time and space in a very radical manner. Nevertheless, these changes are necessary if we want to achieve a deeper comprehension of Nature.

In general relativity space and time are no longer separated but together they constitute a four-dimensional continuum called *spacetime*. Einstein's ingenious idea was that matter interacts with spacetime in such a way that spacetime becomes *curved*. This interaction between matter and spacetime is described by *Einstein's field equation*. Furthermore, the paths of objects are determined by the geometry of spacetime: Objects in a free fall move along *geodesics*, i.e., routes of stationary length between spacetime points. Hence matter tells spacetime how to curve, whereas the geometry of spacetime tells matter how to move. In this sense, gravitation may be considered as a manifestation of the curvature of spacetime.

Einstein's field equation is a tensor equation, and therefore the mathematical structure behind the theory is based on tensor algebra. The use of tensors in general relativity is motivated by the principle of *general coordinate invariance*, which states that all laws of physics must be invariant under general coordinate transformations. In particular, this means that Einstein's field equation must be *covariant* with respect general coordinate transformations, i.e., the form of Einstein's equation cannot depend on the choice of coordinates. Since tensor equations are by definition covariant, it is natural that the physical quantities of general relativity are represented by tensors.

Today our understanding of gravitation is based on general relativity, and we have learnt that it describes gravitational effects with a very high accuracy. There are, however, implications of certain limits in its domain of applicability. The first curved solution to Einstein's field equation was found by Karl Schwarzschild in 1916 [3]. This solution describes spacetime outside a spherically symmetric mass distribution. Curiously, if the radius of the mass distribution is smaller than a certain value, known as the *Schwarzschild radius*

$$R_{\rm S} = \frac{2MG}{c^2}, \tag{1}$$

where $G \approx 6.67 \times 10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$ is Newton's gravitational constant, $c \approx 3.00 \times 10^8$ $\mathrm{ms^{-1}}$ is the speed of light and M is the mass of the body, spacetime contains a region where gravitational effects are so strong that not even light can escape out of it. This region is known as a *black hole*. The boundary of this region, a two-dimensional, spherical, spacelike hypersurface of spacetime with radius R_S, is called an *event horizon*. Inside the black hole all matter will be compressed into a single point. This point, in turn, is known as the *singularity*. At the singularity the mass density and the curvature of spacetime grow infinitely large, thus resulting in the breakdown of general relativity. The existence of spacetime singularities therefore implies that our knowledge of gravitation is still incomplete. This is one of the main reasons why physicists strive to find the *quantum theory of gravitation* or, in short, *quantum gravity*. By this theory one usually means some kind of a synthesis of quantum theory and general relativity. It is strongly believed that the quantum effects of gravitation are somehow able to prevent the formation of singularities.

Thermodynamics describes the properties of macroscopic systems by means of familiar macroscopic quantities such as pressure and temperature. In the first half of the 19th century the laws of thermodynamics were known only as phenomenological rules confirmed by experiments. However, through the visionary works of Boltzmann and Gibbs, the thermodynamical properties of macroscopic systems became viewed as statistical averages over their microscopic degrees of freedom. Investigation of statistical effects in systems consisting a large number of particles is called *statistical mechanics*.

The core of thermodynamics is given by the *four laws of thermodynamics*. The zeroth law of thermodynamics tells us that the temperature of an object in thermal equilibrium is constant. The first law of thermodynamics, in turn, is a manifestation of the principle of the conservation of energy. The second law of thermodynamics involves the concept of entropy. Entropy is a thermodynamical quantity which represents, in a sense, the disorder of a system: If the disorder of the system increases, so does the entropy. In more precise terms, the entropy S of a system is defined as

$$S = k_B \ln \Omega, \tag{2}$$

where $k_B \approx 1.38 \times 10^{-23}$ JK^{-1} is Boltzmann's constant and Ω is the number of microstates comprising the macrostate of the system. Macroscopic systems tend to end up in the state of maximum entropy. This property leads to the second law of thermodynamics, which states that the entropy of a system cannot decrease in any process. Putting differently, physical processes always proceed in the direction of increasing disorder. The third and the last law of thermodynamics is also of a fundamental nature. In effect, it tells us that the temperature $T = 0$ cannot be reached by means of a finite number of processes. Hence we can never halt the molecular motions of a system.

Quantum mechanics was developed in the early 20th century as a response to the need to understand the physical properties of atoms. The break impulse to quantum mechanics was the *quantum hypothesis* made by Max Planck in 1900. He proposed that electromagnetic radiation consists of certain discrete quanta whose energies are proportional to the frequency of the radiation. This hypothesis was motivated by the problems concerning the *blackbody radiation*. A black body is an ideal object which emits and absorbs all electromagnetic radiation. The purely wave-like behaviour of electromagnetic radiation could not explain the energy distribution of the blackbody radiation. The quantum hypothesis was the remedy for that problem, and the correct energy distribution was found by Planck in 1899. The properties of the blackbody radiation reveal that in some situations electromagnetic radiation acts like a flow of particles. The wave-like behaviour of light was, however, confirmed earlier. This is an example of *wave-particle dualism*, which was explicitly stated by Louis de Broglie in 1924: In some situations matter acts like particles and in some others like waves.

In quantum mechanics familiar measurable quantities, e.g., position and momentum, are replaced by *operators* which operate on the *state vector* $|\Psi\rangle$ of a system. An operator is an object which transforms state vectors to each other. The state vector, in turn, contains all available information of the system. A special example of an operator equation is an eigenvalue equation

$$\hat{A}|\Psi\rangle_a = a|\Psi\rangle_a, \tag{3}$$

where $\hat{A}$ is an operator, $|\Psi\rangle_a$ is an eigenstate of the operator $\hat{A}$ and a is an eigenvalue. One of the postulates of quantum mechanics states that when a measurable quantity, or observable, is replaced by its operator counterpart, then every eigenvalue of the operator is a possible result of measurement of that quantity. Hence, equation (3) represents a certain quantum-mechanical measurement process where a is identified as the result of the measurement.

The transition from classical to quantum mechanics may be performed, for instance, by means of a procedure called *canonical quantization*.[1] Assume that the classical theory to be quantized is first transformed into the *Hamiltonian form*. That is, the equations of motion are written in terms of the *phase space variables*, i.e., the *generalized coordinates* and the corresponding *generalized momenta*. Canonical quantization can be then performed in such a way that the phase space variables of the theory are replaced by their operator counterparts which obey the famous *canonical commutation relations*. In a simple one-particle problem, we can take the position x and the momentum p as our phase space variables and set the canonical commutation relation between their operator counterparts:

$$[\hat{x}, \hat{p}] = i\hbar, \tag{4}$$

$$[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = \hat{0}. \tag{5}$$

Here $\hbar \approx 1.05 \times 10^{-34}$ Js is the Planck constant and the *commutator* between operators $\hat{A}$ and $\hat{B}$ is defined as

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{6}$$

[1] An option for the canonical quantization is the path-integral quantization formulated by Feynman in 1948 [4].

It is remarkable that the standard methods of quantization are quite universal. In fact, similar procedure of quantization may be performed for various systems and, rather surprisingly, for almost any classical field.

How, then, are the methods of quantization applied in the vast world of modern physics? According to the best of our knowledge there exist *four fundamental interactions*: *strong, weak, electromagnetic and gravitational interaction*. The strong interaction, for instance, binds quarks together to form neutrons and protons. The weak interaction, in turn, is responsible for the beta-decay in atomic nucleus. The electromagnetic interaction expresses itself in various forms: It causes all chemical reactions, and it is responsible for almost all visible phenomena. Finally, the gravitational interaction makes apples fall from trees and determines the trajectories of celestial objects. In general, these interactions can be described by using the concept of *field*. Furthermore, fields—like particles—are governed by the laws of quantum physics. In the strong, the weak and the electromagnetic interaction the ordinary methods of quantization can be easily applied to the interacting fields, and as a result one may interpret these fields as a collection of particles. More precisely, interactions can be described as an exchange of certain mediating particles between the interacting particles. Unfortunately, an analogous description is not possible for the gravitational interaction: The resulting theory produces both mathematical and conceptional inconsistencies. For this reason quantum theory and general relativity are, in some sense, incompatible, and one is forced to seek for a different kind of way to apply the principles of quantum theory to gravity.

There is, however, one known phenomenon which contains an interplay between quantum theory and general relativity. In 1975 Stephen Hawking showed that black holes emit thermal radiation with the spectrum of that of a black body [5]. This radiation results from purely quantum-mechanical effects which take place in the immediate vicinity of the event horizon. An existence of such radiation implies that black holes, like any other macroscopic objects, have thermodynamical properties—including entropy. In fact, the existence of the black hole entropy was proposed even before Hawking by Jacob Bekenstein [6]. He based his proposal on certain similarities between the rules obeyed by the black hole horizons, and the second law of thermodynamics. Hawking was therefore able to confirm Bekenstein's conjecture, and it follows from his analysis that the entropy of a black hole is given by

$$S = \frac{1}{4}\frac{k_{\mathrm{B}}c^3}{\hbar G}A, \tag{7}$$

where A is the area of the event horizon of the hole. This result is known as the *Bekenstein-Hawking entropy law*.

Maybe the most intriguing aspect of black hole radiation is that it contains elements from quantum theory, thermodynamics and general relativity. Thus, one may say that in black hole radiation all the three foundational theories of physics meet for the first time. It is natural to expect that similar radiation processes would take place in the vicinity of other spacetime horizons as well. In this thesis, these issues are investigated when we consider the radiation effects of the *inner* horizon of a Reissner-Nordström black hole.

The concept of black hole entropy leaves us with a question about its microscopic origin. Classically, black holes have only three degrees of freedom: mass M, electric charge Q and angular momentum J. However, according to statistical mechanics, a black hole possessing the Bekenstein-Hawking entropy must have $\exp\left(\frac{1}{4}\frac{c^3}{\hbar G}A\right)$ microstates corresponding to the macrostate determined by M, Q, and J. It is one of the greatest challenges of modern physics to identify these microstates. In fact, it provides the main reason for the investigation of the properties of black holes: One expects that the quantum theory of gravitation should be able to reproduce the Bekenstein-Hawking entropy law in some low-energy limit. This is almost the only known clue in the search of quantum gravity.

There have been several attempts to explain the microscopic origin of the black hole entropy, of which the most well-known have been developed by the specialists in the fields of *string theory* and *canonical quantum gravity* [7, 8]. In string theory one is able to calculate the entropy of an *extreme black hole*[2] by counting the number of string states that have the same mass and charge at infinity as the black hole. It is most satisfying that this calculation leads to the correct Bekenstein-Hawking entropy law. However, there are no fully compelling derivations of black hole entropy outside the extreme case. In canonical quantum gravity, in turn, the microstates are thought to live in the surface of the event horizon. More precisely, if one performs a suitable canonical quantization to the classical spacetime outside the black hole and treats the horizon as a boundary, one obtains a theory which yields certain surface states. These states can be counted, and as a result one finds that the entropy is proportional to the horizon area. Unfortunately, the constant of proportionality depends on a free dimensionless parameter that cannot be determined by the theory. Nevertheless, this derivation is valid for more "realistic" black holes, such as Schwarzschild black holes.[3]

One may also try to explain the black hole entropy by investigating possible structures of spacetime at the length scales of the order of one *Planck length* $l_{\text{Pl}} = \sqrt{G\hbar/c^3} \sim 10^{-35}$ m. The constant l_{Pl} denotes the unit of length that can be built out of the fundamental constants G, c, and $\hbar$. It is generally believed that at those scales of distances the continuum structure of spacetime breaks down, and the effects of quantum gravity dominate. Consider, for instance, the possibility that spacetime is made of tiny Planck-length-sized black holes. This view may be supported by the fact that the energy needed to probe a spacetime region with a volume $\sim l_{\text{Pl}}^3$ is enough to shrink this region into a Planck-sized black hole. From the given spacetime structure it follows now that the event horizon of a macroscopic black hole consists of microscopic black holes as well, in a somewhat similar way as the surface of a macroscopic object consists of a layer of atoms. Assume then that each microscopic

[2]An extreme black hole is a special case of a Reissner-Nordström or a Kerr-Newman black hole, in which the inner and the outer horizons coincide.

[3]From the point of view of astrophysics, it is highly improbable that extreme black holes would exist in our universe.

hole contributes to the macroscopic horizon an area which is proportional to the area of its own horizon. If the microscopic holes have certain horizon area spectrum, one may calculate the entropy associated with the event horizon of the macroscopic black hole. One may identify this entropy as the entropy of the hole, and therefore this model provides one possible explanation for the origin of the black hole entropy. This idea will be discussed in more detail in this thesis.

There are also reasons to believe that some kind of a concept of entropy should be assigned to any spacelike two-surface, no matter whether the two-surface is part of a horizon or not.[4] Indeed, if one assumes that the entropy of any horizon stems from the microscopic structure of spacetime at the horizon, one may speculate on the idea that any spacelike two-surface should be associated with a certain concept of (statistical) entropy. This is because the microscopic structures of all spacelike two-surfaces should be (about) the same, no matter whether those two-surfaces are part of any horizon or not. Apart from these rather heuristic arguments, there are also more concrete reasons to extend the concept of entropy from horizons to arbitrary spacelike two-surfaces. Recently, it has been proposed that an accelerating two-plane may be associated with an entropy which is twice as large as the entropy associated with a horizon. This proposal was suggested by the results obtained in Ref. [9] by considering the flow entropy through an accelerating two-plane in spacetime filled with isotropic massless radiation in thermal equilibrium. In this thesis, we shall investigate the implications of this proposal. Our investigations are strongly motivated by an important discovery made by Ted Jacobson. In 1995 he found that Einstein's field equation can be derived from the proportionality of entropy and horizon area, together with the first law of thermodynamics [10]. This analysis suggests that Einstein's field equation may be nothing more than a thermodynamical equation of state of spacetime and matter fields. In this thesis, we shall do something similar: We shall consider the possibility that Einstein's field equation can be derived from the entropy associated with an accelerating two-plane and the first law of thermodynamics. This result, if true, would support the view that the equivalence between entropy and area is more general than one has previously expected.

This dissertation consists of three parts. Part I includes chapters 1 and 2, and it contains some results of classical general relativity. In chapter 1, we introduce some well-known results of black hole physics. Many of the issues discussed there are not directly connected with the main subjects of this thesis, but their purpose is to offer an extensive introduction to the properties of classical black holes—especially for the readers who are not experts of general relativity. Chapter 2 deals with the ADM formulation of general relativity. The ADM formulation gives the foundation for the quantum-mechanical models of black holes which will be discussed later in part III.

Part II deals with the Hawking radiation, and it consists of three chapters. First, in chapter 3, we shall briefly review some basic results of quantum field theory

[4]Here we use a somewhat unorthodox terminology where the entropy of a horizon is understood as an intrinsic property of such spacelike two-surface which determines the area of the given horizon. For instance, in the case of the Schwarzschild horizon, the entropy would be associated with the spacelike two-sphere $r = R_S$. This terminology, and reasons behind it, will be properly introduced in Sec. 7.4.

in curved spacetime. These results are of vital importance when investigating the radiation effects of black holes. Later on, in chapter 4, we shall give an introduction to the Hawking effect and its consequences. Finally, in chapter 5, we shall study the radiation effects of the Reissner-Nordström black hole—especially the radiation effects of its inner horizon. As a result, we shall find that the inner horizon emits radiation towards the singularity in a very similar way as the event horizon emits radiation out from the hole. When the backscattering effects are ignored, the energy distribution of the radiating particles obeys the normal blackbody spectrum.

Part III focuses on the gravitational entropy and it consists of three chapters. Our discussion begins in chapter 6 with the quantum-mechanical models of black holes. After that, in chapter 7, we shall introduce the spacetime foam model of the Schwarzschild horizon, where the horizon is build out of microscopic black holes. We shall show that it follows from the postulates of our model that the entropy associated with the horizon is proportional to its area. Moreover, we shall give certain geometrical arguments, which suggest that the constant of proportionality is, in natural units, one-quarter. Finally, in chapter 8, we perform an analysis where Einstein's field equation with a vanishing cosmological constant is derived by means of very simple thermodynamical arguments. Our derivation is based on a consideration of the properties of a very small, spacelike two-plane in a uniformly accelerating motion.

Part I

Classical Results

Chapter 1

Black Holes

Black holes are where God divided by zero.

Steven Wright [11]

Black holes are regions of space where the gravitational effects are so strong that even light cannot escape from those regions. The existence of such regions was proposed for the first time by Michell and Laplace already in the late 18th century (and probably independently of each other) [12,13]. Their arguments, however, were based on Newton's theory of gravitation. General relativity also predicts the existence of black holes, and the first black hole solution to Einstein's field equation was found by Schwarzschild in 1916 [3]. At first, black holes were thought to be only theoretical curiosities which would not exist in Nature. However, through the works of Chandrasekhar, Oppenheimer, Volkoff, and Snyder we have learnt that black holes are born, in some situations, as the final states of stars [14, 15]. Therefore, we may indeed expect that there exist black holes in our universe.

In this chapter, we shall give a short review of black hole physics. After discussing the gravitational collapse of stars, we shall define the concept of a black hole in more precise mathematical terms. Singularity theorems will also be presented. We shall also briefly discuss the uniqueness theorems concerning black holes, as well as the so-called black hole mechanics.

1.1 Formation of Black Holes

When a star has used all of its nuclear fuel, it begins to collapse due to its own mass. At the first stage of the collapse, the star turns into a *white dwarf*. At the matter densities characteristic of white dwarfs ($\rho \geq 10^5$g/cm^3), electrons are no longer strictly bounded by the atomic nuclei, and they can be modelled by the Fermi gas of electrons. Since electrons are fermions, they obey *Pauli's exclusion principle*. This creates a certain outward pressure which tries to halt the gravitational collapse. There is, however, an upper limit for the pressure originating from the electron gas. In fact, if the mass of the collapsing star exceeds a certain critical value of mass, namely 1.44 solar masses, the outward pressure cannot halt the collapse [16]. This critical mass is known as the *Chandrasekhar limit*.

A star with mass greater than the Chandrasekhar limit continues to collapse until it reaches the densities of neutron stars ($\rho \geq 10^{13}$g/cm^3). At these densities, another outward pressure mechanism is provided by the Fermi gas of neutrons. Again, it can be shown that this pressure has an upper limit. In 1939 Oppenheimer and Volkoff showed that if the mass of a neutron star exceeds a certain critical mass ($\sim$ 0.7 solar masses), the collapse cannot be halted and the star collapses completely [17]. More accurate calculations have been performed later, and the recent values of this critical mass are between 1.6 and 2 solar masses.

During the complete gravitational collapse, the star turns eventually into a black hole. More precisely, it can be shown that the radius of the completely collapsing star reaches the Schwarzschild radius

$$R_{\rm S} = 2M \tag{1.1}$$

within a finite *proper time*.[1] In this connection, the proper time means the time measured by an observer moving along with the surface of the star. After the Schwarzschild radius has been reached, the star continues its collapse but nothing can escape from the region bounded by the event horizon. All matter inside the black hole collapses finally into a singularity, that is, the star is ultimately compressed into a single point in space. It should be noted, however, that from the point of view of an outside observer, the process of a complete gravitational collapse appears somewhat different: According to an observer situated outside the collapsing star, the radius of the star approaches the Schwarzschild radius only asymptotically and never reaches it. This means that, in the astrophysical sense, complete gravitational collapse does not produce actual black holes. Nevertheless, there are good reasons to call such objects as black holes—even from the astrophysical point of view. It can be shown that all the measurable quantities of a completely collapsing star become indistinguishable from the quantities of real black holes within a finite (and reasonably short) time period. Therefore, even though a completely collapsing star never becomes an actual black hole from the point of view of an outside observer, it will collapse into an object which resembles perfectly a real black hole. In this thesis, as well as in astrophysics, these kinds of objects are always called 'black holes'. Usually, they are found in contexts totally different from the actual black holes, and therefore the chance for a confusion is minimal.

It is very difficult to understand the collapse of stars properly. In fact, the results introduced above rely heavily on the assumption of exact spherical symmetry, and at the present time there exist no models for a more realistic gravitational col-

---

[1]Unless otherwise stated, in this thesis we shall always use the natural units where $G = c = \hbar = k_{\rm B} = 1$.

lapse. However, as we shall see in the subsection 1.2.3, it is possible to predict the formation of the black hole singularities even in the context of non-spherical gravitational collapse without any explicit model. Furthermore, besides the collapse of stars, there are at least two other mechanisms for the formation of black holes. The first one is the gravitational collapse of the center of a cluster of stars [18]. This kind of collapse produces very massive black holes. The second one is a completely different kind of process. It is believed that in the early universe, when the matter density was extremely high, inhomogeneous regions of matter could have collapsed into black holes, instead of expanding with the universe. These objects are called *primordial black holes*. Even though their existence is still a controversial subject, it is often useful to study primordial black holes because they provide the only known mechanism for the birth of the actual black holes: For a primordial black hole all the matter can be trapped inside the event horizon—even from the point of view of an outside observer.

It is sometimes convenient to ignore the whole process of black hole formation, and simply assume that the black hole has existed eternally. These kinds of objects are called *eternal black holes*. The main advantage of the eternal black holes is their time symmetry. (The black holes formed by the gravitational collapse are not time symmetric due to the disappearance of mass behind the horizon.) Eternal black holes may be considered unphysical but in many situations they provide an excellent approximation for the real black holes. In this thesis, we shall study the properties of an eternal Reissner-Nordström black hole in chapter 5.

1.2 Black Holes and Singularities

It was mentioned in the previous section that, under certain conditions, a spherically symmetric mass distribution collapses into a singularity. One may still ask, what are the general conditions under which a spacetime singularity is formed. This kind of reasoning has led to the set of the so-called *singularity theorems*. Most of these theorems were found by Penrose and Hawking in the early 1970's.

General relativity predicts several kinds of physical singularities: cosmological, naked and black hole singularities.[2] The cosmological singularities, i.e., the Big Bang singularities, are predicted by the Friedmann models of cosmology where the universe is assumed to be homogeneous and isotropic (see, for instance, Ref. [19]). Surprisingly, it is possible to predict the existence of a cosmological singularity from much more general assumptions. These assumptions are discussed in detail in the subsection 1.2.3. Besides that, there exist several other statements concerning the formation of spacetime singularities. However, we wish to note that in this thesis we are not very interested in the naked singularities, that is, the singularities which are not causally separated from the rest of the universe by a horizon. It is widely believed that in any physically reasonable spacetime a naked singularity cannot form.

[2]By physical singularities we mean the singularities which cannot be removed by a coordinate transformation.

This proposal, known as the Cosmic Censorship Conjecture, was stated by Penrose already in 1969 but it still remains unproven [20].

In the following pages, we shall take a rather mathematical approach to the black hole physics. Our main goal will be the singularity theorems but we are also interested in other aspects of general relativity. For instance, up to this point the definition of a black hole has been somewhat vague. We have used such notions as 'gravitational field' and 'light' in order to specify what is meant by the objects called black holes. Interestingly, there is an alternative way to define a black hole. Before we are ready to formulate that definition, we should, however, consider some global properties of spacetime.

1.2.1 Spacetime—the Final Frontier

Usually, it is reasonable to study such spacetimes only for which the direction of time is well-defined. We say that a metric spacetime $(\mathcal{M}, g_{\mu\nu})$ is *time orientable* if at every point $p \in \mathcal{M}$ it is possible to associate a light cone with the tangent space T_p in such a way that the light cone changes smoothly when one goes from one point to another.[3,4] In a time orientable spacetime, a differentiable mapping $\gamma : [0, 1] \to \mathcal{M}$ is called a *future directed timelike curve* if at any point $x^\mu(u)$ of $\gamma(u)$, the tangent vector

$$t^\mu(u) := \frac{d}{du} x^\mu(u) \tag{1.2}$$

is a future directed timelike vector.[5] If $t^\mu(u)$ is either a future directed timelike vector or null vector at any point of the curve, then the mapping γ is called a future directed *causal curve*. In a similar manner, one may define a *past directed* timelike curve and causal curve.

Consider again a time orientable spacetime. The *causal future* $J^+(p)$ of a point $p \in \mathcal{M}$ is a set of points q such that there exists a future directed causal curve γ^+ with properties $\gamma^+(0) = p$ and $\gamma^+(1) = q$. The *causal past* $J^-(p)$ of a point $p \in \mathcal{M}$ is defined in a similar way by using past directed causal curves. A closely related concept is the *chronological future (past)* $I^+(p)$ $(I^-(p))$ which is defined as a set of points q for which there exists a future (past) directed timelike curve γ in such a way that $\gamma(0) = p$ and $\gamma(1) = q$. These definitions allow us to define the causal future/past of a set $S \subset \mathcal{M}$,

$$J^\pm(S) = \bigcup_{p \in S} J^\pm(p), \tag{1.3}$$

and similarly the chronological future/past of a set $S \subset \mathcal{M}$,

$$I^\pm(S) = \bigcup_{p \in S} I^\pm(p). \tag{1.4}$$

[3] In this definition, each light cone is assumed to possess the local notions of future and past.

[4] From this point on, we shall sometimes denote a metric spacetime simply by $\mathcal{M}$.

[5] Note that the vectors at the point $p \in \mathcal{M}$ "live" in the tangent space T_p.

A subset S of a time orientable spacetime is called *achronal* if $I^+(S) \cap S = \emptyset$. For a closed achronal set S, the *future (past) domain of dependence* $D^+(S)$ ($D^-(S)$) consists of points $p \in \mathcal{M}$ for which every *past (future) inextendible* causal curve through p intersects S. Here the future (past) inextendible curve refers to such future (past) directed curve which has no *future (past) endpoint*. The point $p \in \mathcal{M}$, in turn, is said to be a future endpoint of a future directed curve $\gamma(u)$ if for every open neighbourhood $\mathcal{U}(p)$ there exists u_0 such that $\gamma(u) \in \mathcal{U}$ for all $u > u_0$. In a similar manner, one may define the past endpoint of a past directed curve.[6] By these definitions, the *domain of dependence* of S is then written as

$$D(S) = D^+(S) \cup D^-(S). \tag{1.5}$$

If a closed achronal set Σ has the property $D(\Sigma) = \mathcal{M}$, the set Σ is called the *Cauchy surface* of the spacetime $\mathcal{M}$. Correspondingly, if spacetime contains a Cauchy surface, that spacetime is called *globally hyperbolic*. Loosely speaking, global hyperbolicity means that spacetime is causally well-behaved, that is, one cannot change one's own past. In general relativity, the concept of time may be associated, in a certain sense, with a Cauchy surface of spacetime.

Another important property of spacetime is *asymptotical flatness*. In itself, this is a rather vast subject, and therefore the exact meaning of asymptotical flatness will not be discussed here. Let us simply define an asymptotically flat spacetime as a spacetime having the property

$$g_{\mu\nu} \overset{r\to\infty}{=} \eta_{\mu\nu} + \mathcal{O}(r^{-1}), \tag{1.6}$$

where $\eta_{\mu\nu}$ is the metric tensor of the flat spacetime and r is the (pseudo)distance measured from the origin of the chosen coordinate system. For a more detailed discussion about asymptotically flat spacetimes, the reader is advised to see Refs. [21, 22, 23]. When studying the asymptotic behaviour of spacetime, however, it is often convenient to adopt the notation introduced by Penrose to distinguish different kinds of spacetime infinities. At the heuristic level, one may use the following definitions.

(i) *Past null infinity* $\Im^-$ = boundary where all past inextendible null geodesics end.

(ii) *Future null infinity* $\Im^+$ = boundary where all future inextendible null geodesics end.

(iii) *Past timelike infinity* i^- = boundary where all past inextendible timelike geodesics end.

(iv) *Future timelike infinity* i^+ = boundary where all future inextendible timelike geodesics end.

[6] Note that an endpoint of a curve does not have to be contained in the curve.

(v) *Spatial infinity* i^0 = boundary where all the spacelike slices of spacetime end.

As an example, the infinities of Minkowski spacetime are illustrated in Fig.1.1. In a spacetime containing a black hole, these infinities have similar meanings.

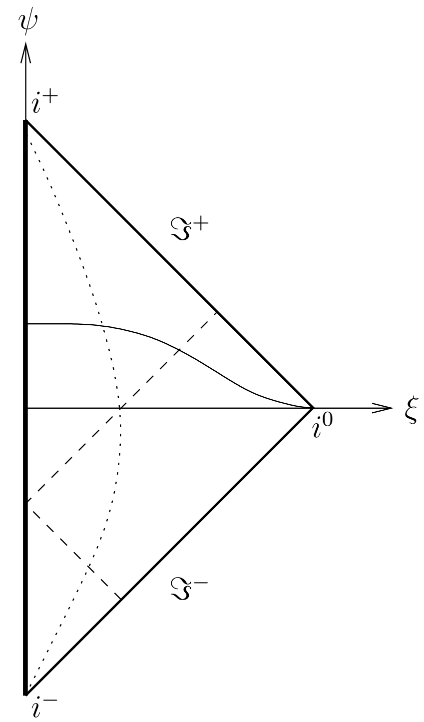

Figure 1.1. The conformal diagram of Minkowski spacetime. The dashed and the dotted line represent the worldlines of a null and a timelike observer, respectively. The continuous line, in turn, represents a spacelike surface. The asymptotically flat regions have been compressed to finite size by a conformal coordinate transformation from the spherical Minkowski coordinates t, r, θ, ϕ to the new spherical coordinates ψ, ξ, θ, ϕ [24]. In the diagram, the coordinates θ and ϕ have been suppressed.

Let $\mathcal{U} \subset \mathcal{M}$ be open. A *congruence* in $\mathcal{U}$ is a family of curves such that for every $p \in \mathcal{U}$ there is exactly one curve in this family going through p. We say that a congruence is *smooth* if the tangent vectors of the curves constitute a smooth vector field in $\mathcal{U}$. Now, let us consider a smooth congruence of timelike geodesics. We may normalize the tangent vector field ξ^μ of the geodesics to the unit length such that

$$\xi_\mu \xi^\mu = -1 \tag{1.7}$$

without losing any generality. The *expansion* θ, the *shear* $\sigma_{\mu\nu}$, and the *twist* $\omega_{\mu\nu}$ of the congruence are then defined as

$$\theta := q_{\mu\nu} B^{\mu\nu}, \tag{1.8}$$

$$\sigma_{\mu\nu} := \frac{1}{2}\big(B_{\mu\nu} + B_{\nu\mu}\big) - \frac{1}{3}\theta\, q_{\mu\nu}, \tag{1.9}$$

$$\omega_{\mu\nu} := \frac{1}{2}\big(B_{\mu\nu} - B_{\nu\mu}\big), \tag{1.10}$$

where

$$q_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu, \tag{1.11}$$

$$B_{\mu\nu} := \xi_{\mu;\nu}, \tag{1.12}$$

and '$;\nu$' stands for the covariant derivative with respect to x^ν. The tensor $q_{\mu\nu}$ can be interpreted as the "spatial metric" which determines the scalar product on a hypersurface orthogonal to the geodesics of the congruence. Moreover, it can be shown that the quantities θ, $\sigma_{\mu\nu}$, and $\omega_{\mu\nu}$ satisfy the equation

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\,\theta^2 - \sigma_{\mu\nu}\,\sigma^{\mu\nu} + \omega_{\mu\nu}\,\omega^{\mu\nu} + R_{\mu\nu}\,\xi^\mu \xi^\nu, \tag{1.13}$$

where τ is the proper time along the timelike geodesics and $R_{\mu\nu}$ is the *Ricci tensor* of spacetime. This equation is known as the *Raychaudhuri equation*.

Definition 1.2.1 *Let $(\mathcal{M}, g_{\mu\nu})$ be an asymptotically flat spacetime.*[7] *A black hole is a closed set $B \subset \mathcal{M}$ of the spacetime such that*

$$B := \mathcal{M} - J^-(\Im^+). \tag{1.14}$$

The boundary of B in $\mathcal{M}$,

$$H := \dot{J}^-(\Im^+) \cap \mathcal{M}, \tag{1.15}$$

where $\dot{J}^-(\Im^+)$ denotes the boundary of $J^-(\Im^+)$, is called the event horizon.

This is a natural definition for a black hole. The event horizon is an example of a *trapped surface* of spacetime. At the trapped surface all ingoing and outgoing light rays are trapped. More precisely, a compact, smooth, spacelike two-surface T of spacetime is a trapped surface if the expansion $\theta \leq 0$ for both the ingoing and the outgoing null geodesics orthogonal to T. Here the ingoing and the outgoing null geodesics denote the two families of null geodesics orthogonal to the two-surface T which intersect the surface from the opposite sides.

We are also interested in a more exact definition for a singular spacetime, i.e., for a spacetime containing a singularity.[8] Usually, the existence of a spacetime

[7] In fact, it is usually required that the spacetime containing a black hole is *strongly asymptotically predictable*. In essence, this means that the Cosmic Censorship Conjecture holds (see Ref. [23] for details). Either way, the definition of a black hole remains the same.

[8] Strictly speaking, singularities are not part of spacetime but in a singular spacetime the singular point is removed. In this sense, a spacetime never "contains" a singularity.

singularity is manifested by some kind of pathological behaviour in the spacetime metric, which makes the curvature of spacetime to "blow up". Unfortunately, there seems to be an endless variety of possible pathological behavior, and therefore the characterization of singularities by the behaviour of the curvature would be a hopeless task. Even worse, sometimes spacetimes may be regarded as singular even when their curvature does not really blow up. Nevertheless, for all spacetimes which may be considered singular in a physically meaningful sense, there appears to be one common property: There exist timelike or null inextendible geodesics which begin or end at the singularities, that is, there exist timelike or null *incomplete geodesics*. It is this feature which we adopt as the definition of a singularity.[9]

Definition 1.2.2 *A spacetime* $(\mathcal{M}, g_{\mu\nu})$ *is said to be singular if there exist timelike or null incomplete geodesics.*

There are indeed some objections to this definition. If there is a physical singularity in a spacetime, one would expect geodesic incompleteness for all the types of geodesics. It is, however, possible to construct a spacetime which is, for instance, spacelike and null geodesically complete but timelike geodesically incomplete. Even more examples against this definition may be given (see, for instance, Ref. [23]). Nevertheless, it is the most satisfactory proposal thus far and, in the end, it is clear that something physically pathological does occur in the spacetimes which are timelike or null geodesically incomplete.

In order to show that a spacetime contains incomplete geodesics, one must find an inextendible geodesic with a finite length. The length of a timelike geodesic is equal to the proper time of the observer moving along that geodesic. The situation is, however, completely different for the null geodesics since the separation between points in spacetime along any null geodesic is always zero. For this reason, a null geodesic is usually parametrized by the so-called *affine parameter* λ such that the tangent vector

$$k^\mu := \frac{\partial x^\mu}{\partial \lambda} \tag{1.16}$$

satisfies the equation

$$k^\mu k^\nu{}_{;\mu} = 0. \tag{1.17}$$

The *affine length* of a geodesic is then the difference between the values of the chosen affine parameter at its endpoints. The motivation for Eq. (1.17) can be found from the fact that when a timelike geodesic is parametrized by its length, the tangent vector of the geodesic obeys a similar equation. However, it is important to understand that affine parametrization is not unique: Every reparametrization $\lambda \to a\lambda + b$, where a and b are constants, is also affine.

[9] Since nothing moves along spacelike curves, the meaning of spacelike geodesic incompleteness is not clear. Therefore the definition of a singularity concerns only timelike or null geodesic incompleteness.

1.2.2 Energy Conditions and Generic Conditions

When we seek certain general conditions for the formation of spacetime singularities, we must also consider the properties of realistic matter. In general relativity, matter is described by the energy-momentum tensor $T^{\mu\nu}$, and therefore we are led to study the physical properties of this tensor. Although it is practically impossible to describe the exact form of the energy-momentum tensor for the realistic matter fields, it is possible to find certain inequalities which should be valid in any physically reasonable situation.

If the energy-momentum tensor satisfies the inequality

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq 0, \tag{1.18}$$

where ξ^{μ} is an arbitrary future directed timelike vector, we say that matter satisfies the *weak energy condition*. Since $T_{\mu\nu}\xi^{\mu}\xi^{\nu}$ may be interpreted as the energy density of matter measured by an observer whose 4-momentum is ξ^{μ}, the weak energy condition tells us that the energy density of matter can never be negative. This is a reasonable assumption, and it is true for all known classical fields. A bit more strict requirement is provided by the so-called *strong energy condition*, which is satisfied when

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq \frac{1}{2}\xi^{\mu}\xi_{\mu}T \tag{1.19}$$

for every timelike future directed vector ξ^{μ}. Again, this assumption seems physically justified, since it is generally believed that the stress components of $T^{\mu\nu}$ will not become so large and negative that Eq. (1.19) would be violated. Besides the above inequalities, there is at least one more condition that seems to hold for realistic matter: If ξ^{μ} is an arbitrary future directed timelike vector, then $-T^{\mu}_{\ \nu}\xi^{\nu}$ is either a future directed timelike vector or a null vector. This requirement is known as the *dominant energy condition*. The quantity $-T^{\mu}_{\ \nu}\xi^{\nu}$ represents the energy-momentum 4-current density of matter according to an observer whose 4-velocity is ξ^{μ}, and hence the dominant energy condition states that the speed of the energy flow of matter can never be greater than that of light.

The dominant energy condition implies the weak energy condition but otherwise these three statements are independent of each other. In that sense, the words ’weak’ and ’strong’ may be somewhat misleading.[10] Anyway, these conditions become very useful when they are combined with the results of general relativity. For instance, if we assume that the strong energy condition and Einstein’s field equation are satisfied, it can be shown that for all timelike and null vectors v^{μ} we have

$$R_{\mu\nu}v^{\mu}v^{\nu} \geq 0, \tag{1.20}$$

where $R_{\mu\nu}$ is again the Ricci tensor of the spacetime. It can also be shown that if we assume the weak instead of the strong energy condition, Eq. (1.20) holds for all

[10] Some references use the definition $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, where k^{μ} is arbitrary null vector, for the weak energy condition. In this case, the strong energy condition does in fact imply the weak energy condition.

null v^μ. As we soon shall see, these inequalities play major roles in the theory of singularities.

Besides the conditions mentioned above, there are also some other properties of spacetime which will become important later. A spacetime $(\mathcal{M}, g_{\mu\nu})$ is said to satisfy the *timelike generic condition* if for each timelike geodesic γ there exists at least one point $p \in \gamma$ at which

$$R_{\alpha\beta\mu\nu}\xi^\alpha\xi^\nu \neq 0, \tag{1.21}$$

where ξ^μ is again a tangent vector of the geodesic and $R_{\alpha\beta\mu\nu}$ is the *Riemann tensor* of the spacetime. Although this condition may not be valid in some idealized models, it is generally believed that in any realistic spacetime the timelike generic condition holds. Furthermore, we say that a spacetime satisfies the *null generic condition* if each null geodesic γ possesses at least one point $p \in \gamma$ where

$$k_{[\delta}R_{\alpha]\beta\mu[\nu}k_{\rho]}k^\beta k^\mu \neq 0. \tag{1.22}$$

Here k^μ is the tangent vector of the geodesic, and we have used a specific notation for the antisymmetric parts of tensors where, for instance,

$$C_{[\mu\nu]} = \frac{1}{2}\big(C_{\mu\nu} - C_{\nu\mu}\big), \tag{1.23}$$

where $C_{\mu\nu}$ is an arbitrary tensor.

1.2.3 Singularity Theorems

The idea behind the singularity theorems is to find certain general conditions which predict the existence of timelike or null incomplete geodesics. First, we shall state two theorems relevant to cosmology and trapped surfaces. These theorems were historically some of the first general statements concerning singularities. Later on, these theorems were also proved under significantly weaker hypotheses. After stating these theorems, we shall quote the Hawking-Penrose singularity theorem which contains a summary of conditions needed for the formation of a singularity.

Theorem 1.2.1 *Let $(\mathcal{M}, g_{\mu\nu})$ be a globally hyperbolic spacetime satisfying the condition $R_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ for all timelike ξ^μ. Furthermore, let Σ be a smooth Cauchy surface in $\mathcal{M}$ for which the expansion of the congruence of past directed timelike geodesics has the property $\theta \leq C < 0$, where C is a constant. In that case, there is no past directed timelike geodesic emanating from Σ with a geodesic length greater than $\frac{3}{|C|}$, i.e., all the past directed timelike geodesics are incomplete* [25].

The theorem given above shows that, under certain conditions, a globally hyperbolic universe has begun from a singular state a finite time ago. Consider now, for the sake of simplicity, the time orthogonal coordinates (see Sec. 2.2 for details). It is well known that then the expansion of the future directed timelike geodesics may be written as

$$\theta = \frac{1}{2}\dot{q}_{ab}q^{ab}, \tag{1.24}$$

where q_{ab} is the positive-definite metric on the Cauchy surface Σ, $a, b = 1, 2, 3$ and 'dot' stands for the time derivative. Now, if $\theta < 0$ for the past directed timelike geodesics then we must have $\theta > 0$ for the future directed timelike geodesics. Hence, the assumption that $\theta \leq C < 0$ at the Cauchy surface Σ means that at "some instant of time" the universe expands everywhere. As a result, the physical content of the theorem 1.2.1 may be expressed as follows: If in a globally hyperbolic spacetime Einstein's field equation is satisfied, the strong energy condition holds, and at some instant of time the universe expands everywhere, then there must have been the beginning of time.

As mentioned before, the assumptions used in theorem 1.2.1 can be weakened. It has been shown by Hawking that the hypothesis of global hyperbolicity can be replaced by an assumption that Σ is compact, i.e., the universe is closed [26]. There is, however, a price to pay: This assumption leads to the weakened conclusion where only one past directed geodesic must be incomplete. The unwanted hypotheses of this theorem have been eliminated in the Hawking-Penrose singularity theorem 1.2.3.

Theorem 1.2.2 *Let $(\mathcal{M}, g_{\mu\nu})$ be a connected*[11]*, globally hyperbolic spacetime possessing a non-compact Cauchy surface Σ. Moreover, suppose that $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ for all null k^{μ} and that the spacetime $\mathcal{M}$ contains a trapped surface T. If $\theta_0 < 0$ denotes the maximum value of the expansion θ at the surface T for both ingoing and outgoing null geodesics orthogonal to that surface, then there exists at least one inextendible future directed null geodesic from T with affine length $\lambda \leq \frac{2}{|\theta_0|}$.*

Historically, the theorem 1.2.2 was the first general statement concerning singularities, and it was proved by Penrose in 1965 [27]. In contrast to the theorem 1.2.1, which concerns singularities in the cosmological context, this theorem establishes geodesic incompleteness relevant to the gravitational collapse. In particular, the theorem 1.2.2 is extremely helpful when one considers non-spherical gravitational collapse: If the initial conditions of the gravitational collapse are sufficiently close to the initial conditions of the spherical collapse, it can be shown that a trapped surface must form. Therefore, non-spherical gravitational collapse may also develop a singularity. This means that even though we have only spherically symmetric models for the complete gravitational collapse, the formation of singularities requires no exact symmetry whatsoever. From the physical point of view, the content of the theorem 1.2.2 may be expressed as follows: If spacetime possesses a spacelike, compact two-surface such that future directed light rays cannot escape outside that surface, then there is at least one null geodesic emanating from the surface which ends somewhere. For example, one may consider the trapped surfaces $r = C < R_S$, where C

[11] A manifold $\mathcal{M}$ is said to be connected, if the only subsets which are both open and closed are the entire manifold $\mathcal{M}$ and the empty set $\emptyset$.

is a constant, in Schwarzschild spacetime. In that case, all null geodesics emanating from the sphere $r = C$ end at the black hole singularity.

The assumptions used in this theorem can also be weakened (mainly the assumption concerning global hyperbolicity). However, this leads to a slightly weaker conclusion in which we have no information which geodesic is incomplete. Most of the unwanted assumptions of the theorem 1.2.2 have been eliminated in the next theorem.

Theorem 1.2.3 *Let us assume that a spacetime $(\mathcal{M}, g_{\mu\nu})$ has the following four properties.*

1. $R_{\mu\nu}v^\mu v^\nu \geq 0$ *for all timelike or null vectors v^μ.*
2. *The timelike and null generic conditions hold.*
3. *There exist no closed timelike curves.*
4. *At least one of the following three conditions is satisfied:*
   (a) *$(\mathcal{M}, g_{\mu\nu})$ is a closed universe.*
   (b) *$(\mathcal{M}, g_{\mu\nu})$ has a trapped surface.*
   (c) *There exists a point $p \in \mathcal{M}$ such that the expansion θ of a congruence of either future or past directed null geodesics emanating from p becomes negative for each geodesic in this congruence.*

Then $(\mathcal{M}, g_{\mu\nu})$ contains at least one incomplete timelike or null geodesic [28].

It has been shown by Gannon that a fourth alternative for the condition 4 can be added, namely that the spacetime $(\mathcal{M}, g_{\mu\nu})$ contains a closed, achronal, edgeless set S which is *non-simply connected*[12] and "regular near infinity"[13] [29]. An asymptotically flat spacetime which initially has adequately non-trivial topology satisfies this assumption and therefore, assuming that conditions $1-3$ hold, it develops a singularity. Further results can be found in Refs. [29, 30].

Theorem 1.2.3 is a strong argument supporting the view that our universe is singular. In the present time, we have experimental evidence which suggests that our universe can be described by the Robertson-Walker models of cosmology with good accuracy—at least back until the decoupling epoch of matter and radiation [31]. In these models, however, the expansion of the past directed null geodesics emanating from the event representing us becomes negative much later than the time of decoupling. Therefore, it is likely that the assumption $4.(c)$ holds in our universe. Since it is reasonable to expect that the conditions $1-3$ are also satisfied, we have a strong reason to believe that our universe has begun from a singular state.

[12] A connected manifold $\mathcal{M}$ is said to be simply connected, if all closed curves in $\mathcal{M}$ can be continuously deformed into the trivial curve, for which $\gamma(u) = \gamma(0)$ for every $u \in [0, 1]$.

[13] See Refs. [23, 29] for the exact meaning of regularity.

1.3 Uniqueness Theorems and Black Hole Mechanics

Assuming that the Cosmic Censorship Conjecture holds, a black hole is expected to reach an equilibrium with its surroundings in a finite time regardless of the details of the gravitational collapse [32]. Therefore, the metric of the exterior spacetime region becomes eventually *stationary*, i.e., the metric is, at least in some frame of reference, independent of the time coordinate. A stationary black hole, moreover, can be *static*, axially symmetric or both [22]. In static spacetime, the metric is assumed to be stationary and symmetric under the time reversal $t \rightarrow -t$. Thus, all static spacetimes are stationary. Furthermore, it can be shown that the axial symmetry is a necessary property of stationary black holes.

There are only three different stationary black hole solutions in addition to the Schwarzschild black hole: *Reissner-Nordström* [33, 34], *Kerr* [35], and *Kerr-Newman* solutions [36]. A Reissner-Nordström black hole represents a static, spherically symmetric and electrically charged black hole. Both the Kerr and the Kerr-Newman solutions represent rotating black holes. The only difference between them is that the Kerr-Newman black hole is electrically charged. Clearly, the rotating black holes are not spherically symmetric, but it turns out that they are axially symmetric. There are many excellent books discussing the properties of these black hole solutions, so we shall not go into the details here. An introduction to the subjects like these can be found, for instance, in Ref. [19]. However, it should be noted that at this moment there is no precise mathematical description of any mechanism that would produce an electrically charged or a rotating black hole [37].

The most general black hole solution is the Kerr-Newman black hole. All other solutions can be regarded as its special cases. There are certain theorems, namely the so-called *uniqueness theorems*, which guarantee that the properties of a stationary black hole can be uniquely described by the three parameters of the Kerr-Newman solution: mass M, electric charge Q, and angular momentum J [38]. This statement was first proposed by Wheeler when he humorously stated that “black holes have no hair” [39]. The proof of the uniqueness of the Kerr-Newman solution has been put forward in a long chain of theorems by many authors. Logically, the first step is due to Hawking, when he showed that any stationary black hole is topologically spherical [22]. This theorem applies to both the electrically neutral and the charged black holes. Now, as we stated before, stationary black holes are either static or axisymmetric. We consider both of these cases separately.

The static solutions have been analyzed by Israel. In short, he showed that any static, topologically spherical vacuum solution must represent a Schwarzschild black hole [40]. This result has been generalized to the electrovacuum case as well [41]. Therefore, the only possible static electrovacuum black holes are Reissner-Nordström black holes.

According to the Carter-Robinson theorem, all stationary, axisymmetric topologically spherical vacuum solutions can be characterized by two parameters M and J [42, 43]. Therefore, they belong to the Kerr family of solutions. A similar result holds also for the Kerr-Newman black holes [44]. Therefore, since no other classical field can form around a black hole [45,46,47], all the properties of a stationary black hole are described by the mass M, the charge Q, and the angular momentum J.

It is an intriguing fact that in gravitational collapse, black holes "forget" all other properties of matter except mass, electric charge, and angular momentum. However, a somehow analogous phenomenon can be found in the thermodynamics of ordinary matter: When a thermodynamical system reaches an equilibrium with its surroundings, its properties can be described by few macroscopic quantities only. Since the thermodynamical properties of a system can be derived from the four laws of thermodynamics, one might suspect that there may be similar laws for stationary black holes as well. These theorems are known as the *four laws of black hole mechanics*, and they were formulated by Hawking, Bardeen and Carter in the beginning of the 70's [48,49].

The laws of black hole mechanics involve the quantity known as a *surface gravity*

$$\kappa := \frac{\sqrt{M^2 - a^2 - Q^2}}{2M(M + \sqrt{M^2 - a^2 - Q^2}\,) - Q^2}, \tag{1.25}$$

where M is the mass, $a = \frac{J}{M}$ is the angular momentum per mass and Q is the electric charge of the black hole.[14] For a static black hole, it describes the limiting value of the force exerted from the asymptotic infinity in order to hold a unit test mass in place at the horizon. However, for a rotating black hole such an interpretation cannot be given since in that case the horizon "spins" with respect to the infinity. By using this concept, the laws of black hole mechanics are now expressed as follows.

Theorem 1.3.1 (The Zeroth Law) *The surface gravity κ is constant over the horizon of a stationary black hole.*

Since this theorem resembles the zeroth law of thermodynamics, it gives us a reason to expect that the temperature of a stationary black hole is proportional to κ. Thus, the surface gravity seems to play, in a certain sense, the role of the temperature of the black hole.

Theorem 1.3.2 (The First Law) *Black holes satisfy the equation*

$$\delta M = \frac{1}{8\pi}\kappa\,\delta A + \Omega_{\mathrm{BH}}\,\delta J_{\mathrm{BH}} + \Phi_{\mathrm{BH}}\,\delta Q, \tag{1.26}$$

where M is the black hole mass, A is the area of the event horizon, Ω_{BH} is the angular velocity of the event horizon, J_{BH} is the angular momentum, and Φ_{BH} is the electric potential at the horizon.

In essence, the first law of black hole mechanics simply states that the mass-energy is conserved. When one compares this theorem with the first law of thermodynamics, one finds that the horizon area A has a similar role as entropy S in thermodynamics. This similarity can be seen even more clearly in the following theorem.

[14] For a general definition of the surface gravity, see Ref. [50].

Theorem 1.3.3 (The Second Law) *If the Cosmic Censorship Conjecture holds and* $R_{\mu\nu}k^\mu k^\nu \geq 0$ *for all null* k^μ*, then*

$$\delta A \geq 0 \tag{1.27}$$

in any (classical) process.

This theorem is often called *Hawking's area law of black holes*. The horizon area A can be expressed in terms of the so-called irreducible mass M_{ir}:

$$A = 16\pi M_{\text{ir}}^2. \tag{1.28}$$

The irreducible mass, in turn, has the form

$$M_{\text{ir}} = \frac{1}{2}\sqrt{\left(M + \sqrt{M^2 - a^2 - Q^2}\right)^2 + a^2}. \tag{1.29}$$

Therefore, theorem 1.3.3 may be written alternatively as

$$\delta M_{\text{ir}} \geq 0. \tag{1.30}$$

This equation sets, for instance, an upper limit for the energy extraction from a rotating black hole.[15] Besides that, theorem 1.3.3 has at least two other important consequences. Firstly, it restricts the amount of energy that can radiate away in black hole collisions: It can be shown that about 29% of the total mass can be radiated. Secondly, it forbids the bifurcation of a black hole. Bifurcation of a black hole would lead to a contradiction between the area law and the energy conservation.

Theorem 1.3.4 (The Third Law) *It is impossible to reach* $\kappa = 0$ *by physical processes.*

Again, one can see that there is an obvious connection to thermodynamics: The above theorem is strikingly similar to the third law of thermodynamics which tells us that $T = 0$ cannot be reached. Since the surface gravity is zero for the extreme black holes only, theorem 1.3.4 states that a non-extreme black hole cannot transform into an extreme black hole.

At first, the four laws of black hole mechanics were thought to be just analogous to the four laws of thermodynamics without any deeper physical meaning. However, when Hawking showed that a black hole has an entropy proportional to the area of its event horizon, he turned this analogy into an identity. Thus, the laws of black hole mechanics are not merely analogous to the laws of thermodynamics but they in fact describe the thermodynamics of black holes.

[15] Energy extraction from a rotating black hole is possible by means of the so-called Penrose process. For further information see, for example, Ref. [19].

Chapter 2

Hamiltonian Formulation of General Relativity

2.1 Introduction: Hamiltonian Formulation of Classical Mechanics

Beauty of a scientific theory is a concept which is not always easy to define. This property, however, is often regarded as a hallmark of a viable and profound theory. Although beauty manifests itself in various forms, the common feature seems to be an admirable, sometimes even astonishing simplicity of the theory. Even when the mathematical framework becomes complicated, the guiding principles always remain simple and unchanged.

Few scientific principles may be regarded as beautiful as *Hamilton's variational principle*. It dates back to the eighteenth and the nineteenth century when the concept of *action* was introduced into physics. The action is a certain real number S associated with the *history* of a physical system within some time interval $[t_\mathrm{i}, t_\mathrm{f}]$. The variational principle states that the only allowed histories of a (classical[1]) system are those for which the action is extremized, i.e., the action is stationary. As it is well known, the variational principle provides an elegant method for solving the problems of classical mechanics. Its beauty, however, goes beyond this feature. First of all, the Hamiltonian formulation of classical mechanics resulting from the variational principle can be deemed as the starting point of quantum mechanics. Secondly, the guiding principles of the variational methods can be utilized in many other branches of physics, including general relativity. In fact, even though the physical interpretations may vary greatly between different physical systems, the mathematical formalism stays the same in every case.

Before we proceed to discuss a certain Hamiltonian formulation of general relativity, namely the so-called Arnowitt-Deser-Misner (ADM) formulation, let us first recall the ideas of the action principle in classical mechanics. After all, the mathematical structure of the Hamiltonian mechanics is very similar to that of the

[1] Here the word 'classical' is used as a synonym for non-quantum.

ADM formulation.

A physical *system* in classical mechanics consists of mass points moving in three-dimensional Euclidean space. These mass points, in turn, interact with each other in a certain way. When a system is composed of n mass points with no constraints between them, one may introduce $3n$ *generalized coordinates* q_i, which are somehow related to the positions of the mass points ($i = 1, \ldots, 3n$). The complete knowledge about the history of the system requires complete knowledge about the values of every coordinate q_i at each instant of time $t \in [t_\mathrm{i}, t_\mathrm{f}]$. The action of the system is then defined as

$$S = \int_{t_\mathrm{i}}^{t_\mathrm{f}} L\big(q_i(t), \dot{q}_i(t); t\big)\, dt, \tag{2.1}$$

where the function L is known as the *Lagrangian* of the system and the time derivatives $\dot{q}_i(t) := \frac{d}{dt} q_i(t)$ are called *generalized velocities*. In classical mechanics, the Lagrangian is written as

$$L = T - U, \tag{2.2}$$

where T is the kinetic and U is the potential energy of the system. When one requires that the action (2.1) is stationary, one arrives at the *Lagrangian equations of motion*

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \tag{2.3}$$

which must be satisfied by every q_i. When U does not depend on the velocities, these equations are equivalent to the Newtonian equations of motion.

Equation (2.3) constitutes the *Lagrangian formulation* of classical mechanics. For a system with s degrees of freedom it gives a set of s second-order differential equations written for the generalized coordinates q_i. However, in *Hamiltonian formulation* of classical mechanics or, in short, in the *Hamiltonian mechanics*, one can express the equations of motion as first-order differential equations. Of course, there is a price to pay for this simplification: The number of independent differential equations increases from s to $2s$.

The starting point of Hamiltonian mechanics is to define the *canonical momentum*

$$p_i := \frac{\partial L}{\partial \dot{q}_i}. \tag{2.4}$$

Formally, the generalized coordinates and the canonical momenta are treated as independent variables, and together they span a $2s$-dimensional space known as the *phase space*. The quantities q_i and p_i, in turn, are usually referred as the *canonical variables*. In Hamiltonian mechanics, each state of the system corresponds to a certain point of the phase space. Spaces spanned individually by the generalized coordinates and the canonical momenta are called the *configuration space* and the *momentum space*, respectively.

If one is able to express the generalized velocities $\dot{q}_i$ in terms of the canonical momenta p_i, one may perform a change of variables from the set $(q_i, \dot{q}_i; t)$ to the set $(q_i, p_i; t)$.[2] Then the *Hamiltonian* of the system is

$$H(q_i, p_i; t) := \sum_{i=1}^{s} p_i \dot{q}_i - L(q_i, \dot{q}_i; t). \tag{2.5}$$

This definition stresses the fact that the Hamiltonian is always a function of q_i, p_i, and t, whereas the Lagrangian is considered as a function of q_i, $\dot{q}_i$, and t. This definition together with Eqs. (2.3) and (2.4), leads directly to the *Hamiltonian equations of motion*:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{2.6a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{2.6b}$$

where $i = 1, \ldots, s$. Under certain conditions, namely, when the kinetic energy can be written in the form

$$T = \sum_{i,j=1}^{s} f_{ij} \dot{q}_i \dot{q}_j, \tag{2.7}$$

where the function f_{ij} depends only on q_i, and the potential energy U is independent of both the generalized velocities $\dot{q}_i$ and the time t, the Hamiltonian H equals to the energy of the system.

There is no unique way to choose the phase space coordinates of a mechanical system, and therefore the choice of the coordinates is usually performed in such a way that the mechanical problem becomes as simple as possible. Since in the Hamiltonian mechanics the generalized coordinates and the canonical momenta are considered as independent variables, the coordinate transformations in the phase space must include simultaneous transformations of both the coordinates and the momenta. The transformation from the "old" phase space coordinates (q_i, p_i) to the "new" phase space coordinates (Q_i, P_i) may be written in the form

$$Q_i = Q_i(q_i, p_i; t), \tag{2.8a}$$

$$P_i = P_i(q_i, p_i; t), \tag{2.8b}$$

such that these equations are invertible. However, if we want to maintain the mathematical structure of the Hamiltonian mechanics, we are interested in those transformations only where the Hamiltonian equations of motion remain invariant. More precisely, we shall consider the transformations from the canonical variables (q_i, p_i)

[2] This kind of transformation is an example of the so-called *Legendre transformations*.

to the new variables (Q_i, P_i) such that the equations

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \tag{2.9a}$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i}, \tag{2.9b}$$

are valid for some function $K = K(Q_i, P_i; t)$. The function K takes the role of the Hamiltonian in the new set of coordinates.

It is easy to find a sufficient condition for the invariance of the Hamiltonian equations of motion. From Eq. (2.5) we see that the action may be written as

$$S = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \Big(\sum_{i=1}^{s} p_i \dot{q}_i - H \Big)\, dt. \tag{2.10}$$

If this action is varied with respect to the (independent) variables q_i and p_i, Eqs. (2.6) are reproduced. Similarly, if the equations of motion are expected to remain invariant, then the variation of the action

$$S' = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \Big(\sum_{i=1}^{s} P_i \dot{Q}_i - K \Big)\, dt \tag{2.11}$$

with respect to Q_i and P_i should lead to Eqs. (2.9). One therefore sees that the Hamiltonian equations remain invariant under the transformation (2.8) if

$$\sum_{i=1}^{s} p_i \dot{q}_i - H = \sum_{i=1}^{s} P_i \dot{Q}_i - K + \frac{dF}{dt}, \tag{2.12}$$

where F is any function of the phase space coordinates and of time t with continuous second derivatives. Coordinate transformations which follow Eq. (2.12) are called *canonical transformations* and the function F, in turn, is known as the *generating function*.

It turns out that the question whether the proposed coordinate transformation is canonical or not may be answered with the help of the so-called *Poisson brackets*. In general, the Poisson bracket between arbitrary functions $u(q_i, p_i)$ and $v(q_i, p_i)$ is written as

$$\{u, v\}_{(q,p)} := \sum_{i=1}^{s} \left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i} \right). \tag{2.13}$$

The Poisson brackets of the canonical variables q_i and p_i itself—the so-called *fundamental Poisson brackets*—are then

$$\{q_i, q_j\}_{(q,p)} = \{p_i, p_j\}_{(q,p)} = 0, \tag{2.14a}$$

$$\{q_i, p_j\}_{(q,p)} = \delta_{ij}. \tag{2.14b}$$

It can be shown that a coordinate transformation from the set (q_i, p_i) to the set (Q_i, P_i) is canonical if and only if the fundamental Poisson brackets remain invariant under that transformation. Putting differently, a coordinate transformation is canonical if and only if

$$\{q_i, q_j\}_{(q,p)} = \{q_i, q_j\}_{(Q,P)}, \tag{2.15a}$$
$$\{p_i, p_j\}_{(q,p)} = \{p_i, p_j\}_{(Q,P)}, \tag{2.15b}$$
$$\{q_i, p_j\}_{(q,p)} = \{q_i, p_j\}_{(Q,P)}, \tag{2.15c}$$

for every $i, j = 1, \ldots, s$. In fact, it can also be shown that *all* Poisson brackets remain invariant under canonical transformations. Therefore it is not necessary to include the references to the phase space coordinates.

2.2 ADM Formulation in Brief

We are now ready to transfer the ideas and concepts of the Hamiltonian mechanics to general relativity. First, it should be noted that there exist many different but equivalent Hamiltonian formulations of general relativity. This is a consequence of the fact that the generalized coordinates used in the gravitational action can be chosen freely. In this thesis, however, we shall concentrate on the ADM formulation of general relativity which was discovered by Arnowitt, Deser, and Misner in 1962 [51].

In general relativity, the concept of a system involves both the spacetime and the existing matter fields. In other words, it is a theory of fields, and the number of the degrees of freedom is therefore non-countable. In particular, the Lagrangian L is now replaced by the *Lagrangian density* $\mathcal{L}$ such that the gravitational action is written as

$$S = \int_{\mathcal{M}} \mathcal{L}\, d^4x, \tag{2.16}$$

where the integration is performed over the whole spacetime $\mathcal{M}$. In general, the Lagrangian density of general relativity is considered to be a function of the components of the metric tensor $g_{\mu\nu}$ and its derivatives $\partial_\alpha g_{\mu\nu}$ and $\partial_\alpha \partial_\beta g_{\mu\nu}$, and, of course, of the field variables of the matter. It is well known that Einstein's field equation in vacuum can be derived from the *Einstein-Hilbert action*

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} R\sqrt{-g}\, d^4x \tag{2.17}$$

by varying it with respect to $g_{\mu\nu}$.[3] Here R is the *Riemannian curvature scalar* and g denotes the determinant of the metric tensor $g_{\mu\nu}$.

Equation (2.17) constitutes the Lagrangian formulation of general relativity.

[3]Note that Einstein's field equation cannot be obtained directly from an equation similar to that of Eq. (2.3). This is because the Lagrangian density of general relativity depends also on the second-order derivatives of $g_{\mu\nu}$.

Hamiltonian formulation of a field theory, in turn, requires some sort of a slicing of spacetime into space and time. In general relativity there is an infinite number of ways to measure time, and none of them can be considered preferable to the others. However, when spacetime is globally hyperbolic, all the different notions of time are given by a certain function $t = t(x^a)$ of the spatial coordinates x^a $(a = 1, 2, 3)$ which, for given value of t, corresponds to exactly one Cauchy surface Σ_t. Therefore, the concept of time in general relativity can be associated with a certain hypersurface of spacetime. Furthermore, when the matter fields are ignored, the history of the system corresponds to the four-geometry of the subset of the spacetime which is bounded by two spacelike hypersurfaces $\Sigma_{t_{\rm i}}$ and $\Sigma_{t_{\rm f}}$ (see Fig. 2.1). For the sake of simplicity, we shall first assume that all matter fields are absent, and construct the ADM formulation for the vacuum solutions only. Then, at the end of this section, we shall briefly comment on some modifications caused by matter fields.

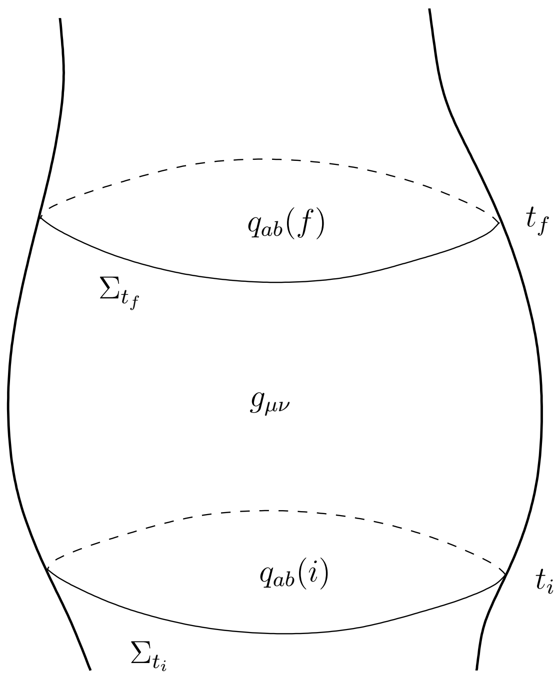

Figure 2.1. A three-dimensional analogue of spacetime. Within the time interval $[t_i, t_f]$ the history of the spacetime corresponds to the geometry of such subset of the spacetime which is bounded by the hypersurfaces Σ_{t_i} and Σ_{t_f}.

In short, the basic idea of the ADM formulation is to decompose spacetime into space and time, and take the coordinates of the configuration space to be the components of the metric tensor q_{ab} of the Cauchy surface $t =$ constant. For a globally hyperbolic spacetime, this kind of decomposition is always possible. Moreover, this choice is practical because the complete knowledge of the spatial metric q_{ab}, together with the information about the choice of the time coordinate, provides all the information about the geometry of spacetime. The coordinates of spacetime, in turn, are chosen such that the metric of spacetime takes the form of the so-called *ADM metric*:

$$\begin{aligned} ds^2 &= -N^2 dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \\ &= -(N^2 - N^a N_a)dt^2 + 2q_{ab}N^a dx^b dt + q_{ab}\, dx^a dx^b. \end{aligned} \tag{2.18}$$

Here the (smooth) functions $N = N(t, x^1, x^2, x^3)$ and $N^a = N^a(t, x^1, x^2, x^3)$ ($a = 1, 2, 3$) are called the *lapse function* and the *shift vector*, respectively. The lapse function measures the rate of flow of the proper time with respect to the time coordinate t when one moves orthogonally to Σ_t, whereas the shift vector measures the tangential "shift" of the spacelike coordinates x^a during the proper time interval Ndt (see Fig. 2.2). It should be noted that one may, at first glance, conclude that the Hamiltonian formalism loses its generality due to the certain choice of coordinates. After all, the *diffeomorphism invariance* is one of the cornerstones of general relativity. However, since N and N^a are arbitrary functions of the spacetime coordinates, their role can be interpreted as providing the general coordinate transformations in the spacetime. Thus, in the ADM formulation, the diffeomorphism invariance is embedded within the functions N and N^a.

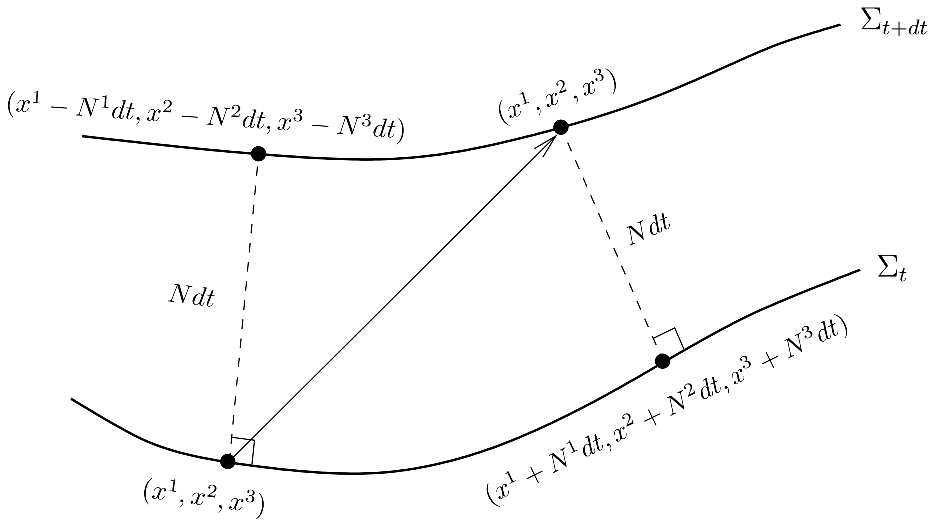

Figure 2.2. The geometrical interpretation of the ADM metric. When an observer moves orthogonally to Σ_t, the infinitesimal change of his proper time, $d\tau$, equals to Ndt. Moreover, on the hypersurface Σ_{t+dt} the point $(x^1 - N^1dt, x^2 - N^2dt, x^3 - N^3dt)$ instead of the point (x^1, x^2, x^3) lies on the line which is orthogonal to the hypersurface Σ_t at the point (x^1, x^2, x^3). For the sake of clarity, the orthogonality is depicted as in positive-definite spacetime.

It can be shown that when all the boundary terms are neglected, the gravitational action (2.17) can be written in terms of q_{ab}, N, and N^a such that

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} N\sqrt{q}\,(K_{ab}K^{ab} - K^2 + \mathcal{R})\,dt\,d^3x, \tag{2.19}$$

where q is the determinant of the metric tensor q_{ab}, $\mathcal{R}$ is the Riemannian curvature scalar on the hypersurface Σ_t, K_{ab} is the so-called *exterior curvature tensor* and $K = q^{ab}K_{ab}$ (see, for instance, Ref. [19]).[4] We denote the Lagrangian in Eq. (2.19) by

$$\mathcal{L}_{\text{eff}} := \frac{1}{16\pi G} N\sqrt{q}\,(K_{ab}K^{ab} - K^2 + \mathcal{R}) \tag{2.20}$$

in order to make a distinction to the Lagrangian in Eq. (2.17) which differs from $\mathcal{L}_{\text{eff}}$ by some boundary terms. The exterior curvature tensor K_{ab} measures how the hypersurfaces Σ_t are embedded in spacetime. This concept is useful since the components q_{ab} describe the *intrinsic* geometry of the hypersurface only, and they tell nothing about the exterior curvature of the surface. In general, the exterior curvature tensor of a spacelike hypersurface may be obtained from the equation

$$K_{ab} = -n_{\mu;b}\, b^{\mu}{}_{(a)}, \tag{2.21}$$

where $b^{\mu}{}_{(a)}$ is the tangent vector of the coordinate curve related to x^a and n_μ is the unit normal of the hypersurface satisfying the relations

$$g_{\mu\nu}n^{\mu}b^{\nu}{}_{(a)} = 0, \tag{2.22a}$$

$$g_{\mu\nu}n^{\mu}n^{\nu} = -1, \tag{2.22b}$$

for all $a = 1, 2, 3$.[5] In the case of the hypersurfaces Σ_t, for which the time coordinate t is constant, the exterior curvature tensor takes a very simple form:

$$K_{ab} = -|g^{00}|^{-1/2}\Gamma^{0}_{ab}. \tag{2.23}$$

Furthermore, in the so-called *time orthogonal coordinates*, that is, when $N = 1$ and $N^a = 0$, one finds that

$$K_{ab} = -\frac{1}{2}\dot{q}_{ab}. \tag{2.24}$$

If one now performs a coordinate transformation such that $dt \rightarrow N dt$, one finds that $\frac{\partial}{\partial t} \rightarrow \frac{1}{N}\frac{\partial}{\partial t}$. Moreover, it may be easily shown that in the infinitesimal coordinate transformation where $x^a \rightarrow x^a + N^a dt$, the metric tensor transforms as $q_{ab} \rightarrow q_{ab} - N_{a|b}dt - N_{b|a}dt$, where the symbol '$|$' means the covariant derivative on the

[4] In this section, we shall adopt an approach to the ADM formulation, where we first assume that the variations of the field variables and their derivatives always vanish on the boundaries of a given spacetime. Since this is not always true, we shall later, if needed, complement the action by the required boundary terms.

[5] Some references use the different sign convention in the definition of the exterior curvature tensor. Here we have adopted the convention which agrees, for example, with Arnowitt, Deser, and Misner [51], and with Misner, Thorne, and Wheeler [19].

hypersurface Σ_t. It is therefore easy to convince oneself that in the presence of a non-zero shift and when the lapse function differs from unity, we must have

$$K_{ab} = \frac{1}{2N}(-\dot{q}_{ab} + N_{a|b} + N_{b|a}). \tag{2.25}$$

This result, of course, can be obtained in a more precise fashion from Eq. (2.21) also.

In the ADM formalism, one chooses the spatial metric q_{ab}, the lapse function N, and the shift vector N^a as the field variables. However, as it can be seen from Eqs. (2.19) and (2.25), there are no time derivatives corresponding to the variables N or N^a in the action. Therefore, the lapse function and the shift vector are not dynamical variables of the theory but their role is somewhat similar to *Lagrange's undetermined multipliers* in classical mechanics. The dynamical variables of the ADM formulation are the components of the spatial metric q_{ab}, and one easily sees that the corresponding canonical momenta may be written in the form

$$p^{ab} := \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{q}_{ab}} = \frac{1}{16\pi G}\sqrt{q}\,(K^{ab} - q^{ab}K). \tag{2.26}$$

The *Hamiltonian density* is then defined as

$$\mathcal{H}_{\text{eff}} := p^{ab}\dot{q}_{ab} - \mathcal{L}_{\text{eff}}, \tag{2.27}$$

and when the boundary terms are again neglected, one finds that

$$\mathcal{H}_{\text{eff}} = N\mathcal{H} + N_a\mathcal{H}^a, \tag{2.28}$$

where we have defined

$$\mathcal{H} := \frac{1}{2}(16\pi G)\, G_{abcd}\, p^{ab}p^{cd} - \frac{\sqrt{q}}{16\pi G}\mathcal{R}, \tag{2.29}$$

$$\mathcal{H}^a := -2p^{ab}{}_{|b}. \tag{2.30}$$

The functions $\mathcal{H}$ and $\mathcal{H}^a$ are often called the *super-Hamiltonian* and the *supermomentum*, respectively. The function G_{abcd} in Eq. (2.29), in turn, is given by the definition

$$G_{abcd} := \frac{1}{\sqrt{q}}(q_{ab}q_{cd} - q_{ac}q_{bd} - q_{ad}q_{bc}), \tag{2.31}$$

and it is known as the *Wheeler-DeWitt metric*. These definitions lead to the action

$$S = \int_{\mathcal{M}} \left(p^{ab}\dot{q}_{ab} - N\mathcal{H} - N_a\mathcal{H}^a\right) dt\, d^3x, \tag{2.32}$$

which is called the Hamiltonian, or the canonical, form of the ADM action.

One is now able to write down the Hamiltonian equations of motion of general relativity. It can be shown that the dynamical equations for the phase space variables

q_{ab} and p^{ab} are of the form

$$\dot{q}_{ab} = \frac{\partial \mathcal{H}_{\text{eff}}}{\partial p^{ab}}, \tag{2.33a}$$

$$\dot{p}^{ab} = -\frac{\partial \mathcal{H}_{\text{eff}}}{\partial q_{ab}} + \partial_c\Big[\frac{\partial \mathcal{H}}{\partial(\partial_c q_{ab})}\Big] - \partial_c\partial_d\Big[\frac{\partial \mathcal{H}}{\partial(\partial_c\partial_d q_{ab})}\Big]. \tag{2.33b}$$

These equations resemble greatly Eqs. (2.6) found in the previous section. The only difference is that there are two extra terms on the right hand side of Eq. (2.33b). These terms are due to the fact that the Lagrangian density of general relativity depends, in addition to q_{ab} and $\partial_\mu q_{ab}$, also on the second-order derivatives of q_{ab}. The most important thing, however, is that Eqs. (2.33) are equivalent to those Einstein's field equations which are obtained by varying the action (2.17) with respect to the spacelike components g_{ab} of the metric tensor. Furthermore, the variations of the ADM action with respect to N and N^a are also required to vanish, and, as one easily sees from Eq. (2.28), these variations yield

$$\mathcal{H} = 0, \tag{2.34a}$$

$$\mathcal{H}^a = 0. \tag{2.34b}$$

These equations are known as the *Hamiltonian constraint* and the *diffeomorphism constraints*, respectively. As expected, Eq. (2.34a) is equivalent to Einstein's field equation which is obtained by varying the Einstein-Hilbert action with respect to g_{00}, whereas Eq. (2.34b) may be obtained by varying the Einstein-Hilbert action with respect to g_{a0}. The importance of Eqs. (2.34) lies in the fact that they ensure that the ADM action is independent of N and N^a. In other words, they imply that the ADM formulation does not depend on the decomposition of spacetime. Since q_{ab} has 6 independent components, the four constraints in Eqs. (2.34) also imply that the real number of the degrees of freedom per spacetime point is not 6 but $6 - 4 = 2$.

What, then, does the ADM formalism tell us? In the Hamiltonian mechanics we defined the Hamiltonian function which, under certain conditions, equals to the energy of a given system. Can we use familiar concepts, such as energy, in the ADM formulation as well? Although the concept of energy is pretty complicated in general relativity, there are indeed some special cases, where the answer to this question is affirmative. The *Hamiltonian* of pure gravity can be written as

$$H := \int_{\Sigma_t} \mathcal{H}_{\text{eff}}\, d^3x, \tag{2.35}$$

In the presence of matter, in turn, the Hamiltonian density $\mathcal{H}_{\text{eff}}$ must be supplemented by two new terms $\mathcal{H}_{\text{matter}}$ and $\mathcal{H}^a_{\text{matter}}$. These are known as the Hamiltonian and the momentum density of matter, respectively. Therefore, if matter fields are present, we take the Hamiltonian of spacetime to be

$$H := \int_{\Sigma_t} [N(\mathcal{H} + \mathcal{H}_{\text{matter}}) - N_a(\mathcal{H}^a + \mathcal{H}^a_{\text{matter}})] \, d^3x, \tag{2.36}$$

and, as a consequence, the constraint equations take the form

$$\mathcal{H} + \mathcal{H}_{\text{matter}} = 0, \tag{2.37a}$$

$$\mathcal{H}^a + \mathcal{H}^a_{\text{matter}} = 0. \tag{2.37b}$$

Up to this point, we have paid no attention to the possible non-vanishing boundary terms of the action. Indeed, when spacetime is spatially compact, no such terms emerge, and therefore, according to the constraint equations, the Hamiltonian vanishes:

$$H = 0. \tag{2.38}$$

This result may be understood such that the total energy of a compact universe is zero. However, if spacetime is asymptotically flat, it can be shown that the ADM action must be supplemented by certain boundary terms. One of these terms reads

$$S^{\text{ADM}}_{\partial\Sigma} = -\int_{t_i}^{t_f} N_+(t) \, E^{\text{ADM}} \, dt, \tag{2.39}$$

where $N_+(t)$ is the lapse function as r approaches infinity and

$$E^{\text{ADM}} := \lim_{r\to\infty} \frac{1}{16\pi G} \oint_{\partial\Sigma_t} (\partial_m q_{mn} - \partial_n q_{mm}) \, dS^n \tag{2.40}$$

is the so-called *ADM energy* (see, for instance, Ref. [23]). The symbol '$\lim_{r\to\infty}$' is used here to denote a process where the two-dimensional spatial boundary of Σ_t is transferred to the spatial infinity. If the coordinates x^a on the hypersurfaces Σ_t are chosen such that they coincide with the Minkowski system of coordinates at the spatial infinity, the boundary term (2.39) is the only non-vanishing boundary contribution of an asymptotically flat spacetime. In that case, the Hamiltonian is written as

$$H = \int_{\Sigma_t} \mathcal{H}_{\text{eff}} \, d^3x + N_+(t) \, E^{\text{ADM}}, \tag{2.41}$$

and when the constraint equations are satisfied,

$$H = N_+(t) \, E^{\text{ADM}}. \tag{2.42}$$

Furthermore, if the foliation Σ_t of spacetime is chosen such that the time coordinate t becomes the Minkowski time coordinate at the spatial infinity, the Hamiltonian equals the ADM energy. For example, in Schwarzschild spacetime, one simply has

$$E^{\text{ADM}} = M, \tag{2.43}$$

implying, as expected, that the energy of Schwarzschild spacetime equals to the Schwarzschild mass M.

Maybe the most surprising feature of the ADM energy is that it does not depend on the internal structure of spacetime but on the properties of the spacetime boundaries only. In fact, this is a manifestation of a general principle: In asymptotically flat spacetimes, the total energy, momentum and angular momentum of spacetime can be read off from the boundaries of spacelike hypersurfaces. In essence, the boundary terms corresponding to the momentum and the angular momentum of spacetime result from a non-vanishing shift vector N^a at the asymptotic infinity. Therefore, in many cases these boundary terms may vanish if one chooses the coordinates x^a in a convenient way. (Note that the boundary term (2.39) cannot be removed by a choice of coordinates. This is because this boundary term originates from the behaviour of the lapse function $N(t)$ at the spatial infinity. The lapse function, in turn, cannot vanish at the infinity since that would completely freeze the time evolution of the hypersurfaces Σ_t at the infinity. For the more detailed investigation on the boundary terms, see Ref. [52].) In general, one can show that the total flat-space boundary contribution of pure gravity may be written as

$$S^{\text{tot}}_{\partial\Sigma} = -\int_{t_i}^{t_f} \Big(N_+(t) E^{\text{ADM}} + N^a_+(t) P^{\text{ADM}}_a + \omega^b L^{\text{ADM}}_b \Big) dt, \tag{2.44}$$

where $N^a_+(t)$ is the shift vector as r goes to infinity,

$$P^{\text{ADM}}_a := -2 \lim_{r\to\infty} \oint_{\partial\Sigma_t} {p_a}^b \, dS_b \tag{2.45}$$

is known as the *ADM momentum* of spacetime, the quantity ω^b is the angular velocity of the spacetime coordinates x^a around the Cartesian coordinate system at the asymptotic infinity, and

$$L^{\text{ADM}}_b := 2\epsilon_{abc} \lim_{r\to\infty} \oint_{\partial\Sigma_t} x^c p^{ai} \, dS_i, \tag{2.46}$$

where ϵ_{abc} is the antisymmetric Levi-Civita symbol, is called the *ADM angular momentum* of spacetime.[6] Hence the gravitational action of an asymptotically flat space-

[6]Here the coordinates x^a are chosen to rotate (with extremely small angular velocity) around the z-axis of the Cartesian coordinate system at the asymptotic infinity.

time with appropriate boundary terms finally takes the form

$$S = \int_{t_i}^{t_f} \int_{\Sigma_t} \left(p^{ab}\dot{q_{ab}} - \mathcal{H}_{\text{eff}}\right) d^3x\, dt - \int_{t_i}^{t_f} \Big(N_+(t)E^{\text{ADM}} + N_+^a(t)P_a^{\text{ADM}} + \omega^b L_b^{\text{ADM}}\Big) dt. \qquad (2.47)$$

Part II

Hawking Radiation

Chapter 3

Quantum Fields in Curved Spacetime

In this part of the thesis, we shall consider the radiation effects of spacetime horizons. The basic ingredient of our analysis will be quantum field theory in curved spacetime. It is a rather vast and complicated subject to discuss exhaustively. Fortunately, there is no need to dwell on the details of the formalism of the theory: It turns out that only few basic results are necessary in our analysis. We shall therefore give only a brief review of these results. Readers interested in a more detailed discussion should peruse, for instance, Refs. [50, 53] and references therein.

In principle, the ideas of ordinary (flat spacetime) quantum field theory may be generalized to curved spacetime in a straightforward manner. In curved spacetime, the ordinary derivatives are simply replaced by the covariant ones, and the concept of time is replaced by a Cauchy surface of spacetime. The physical interpretation of the theory, however, is not so readily accomplished. For instance, it turns out that some familiar concepts, such as the concepts of particle and vacuum, are not always well defined. In fact, there may be many different vacuum states. The reason for this is the absence of global symmetries in curved spacetime. In flat spacetime, the inertial observers are related to each other by the Poincaré transformations. This makes possible to define a unique vacuum state for the inertial observers which is invariant under the action of the Poincaré group. In general, there are no global symmetries in curved spacetime, and therefore the vacuum depends on the observer and the geometry of spacetime. In some special case—for instance, in static spacetimes—it is possible to define a natural vacuum state, whereas in more complicated (and realistic) spacetimes the physical meaning of a particle vacuum becomes more or less obscure. In this thesis, however, we shall not discuss these conceptual problems of the theory any further. Rather, we shall concentrate on its mathematical formulation to the extent that is necessary in this work.

For the sake of simplicity, consider a *Klein-Gordon field*, or a *scalar field* $\phi(x^\mu)$ in curved spacetime. The Lagrangian density of the field ϕ may be written as

$$\mathcal{L}_{\text{KG}} = -\frac{1}{2}\sqrt{-g}\,\left[g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + m^2\phi^2\right], \tag{3.1}$$

where g denotes the determinant of the metric tensor $g_{\mu\nu}$ and m is the mass of the field quanta. When the action

$$S = \int \mathcal{L}_{\text{KG}}\, d^4x \tag{3.2}$$

is varied with respect to the field variable ϕ, one arrives at the Klein-Gordon equation

$$(g^{\mu\nu} D_\mu D_\nu - m^2)\phi = 0, \tag{3.3}$$

where D_μ denotes the covariant derivative. When spacetime is globally hyperbolic, we define the inner product between two solutions ϕ_1 and ϕ_2 of Eq. (3.3) as

$$\langle \phi_1 | \phi_2 \rangle = -i \int_\Sigma \left[\phi_1 (\partial_\mu \phi_2^*) - (\partial_\mu \phi_1) \phi_2^* \right] d\Sigma^\mu, \tag{3.4}$$

where * stands for the complex conjugate, Σ is a Cauchy surface of spacetime and $d\Sigma^\mu := n^\mu d\Sigma$ such that $d\Sigma$ is the volume element on the Cauchy surface and n^μ is the future directed timelike unit vector orthogonal to Σ. The inner product (3.4) remains invariant under diffeomorphic coordinate transformations and its value is independent of the choice of the Cauchy surface.

Consider then a complete set of solutions u_i to Eq. (3.3). We take these solutions to be orthogonal, i.e., satisfying the conditions

$$\langle u_i | u_j \rangle = \delta_{ij}, \tag{3.5a}$$
$$\langle u_i^* | u_j^* \rangle = -\delta_{ij}, \tag{3.5b}$$
$$\langle u_i | u_j^* \rangle = 0. \tag{3.5c}$$

The general solution ϕ to Eq. (3.3) may then be expanded as

$$\phi = \sum_i (a_i u_i + a_i^\dagger u_i^*), \tag{3.6}$$

where the coefficients $a_i, a_i^\dagger \in \mathbb{C}$.

The canonical quantization of the theory may be performed as follows. Define the canonical momentum conjugate p to ϕ as

$$p := \frac{\partial \mathcal{L}}{\partial (n^\mu \partial_\mu \phi)}. \tag{3.7}$$

The classical variables ϕ and p are then replaced by the operators $\hat{\phi}$ and $\hat{p}$ which satisfy the canonical commutation relations

$$[\hat{\phi}(\Sigma, P), \hat{p}(\Sigma, P')] = i\, \delta^3(P, P'), \tag{3.8a}$$
$$[\hat{\phi}(\Sigma, P), \hat{\phi}(\Sigma, P')] = 0, \tag{3.8b}$$
$$[\hat{p}(\Sigma, P), \hat{p}(\Sigma, P')] = 0, \tag{3.8c}$$

for all $P, P' \in \Sigma$. From the operator counterpart of Eq. (3.6),

$$\hat{\phi} = \sum_i (\hat{a}_i u_i + \hat{a}_i^\dagger u_i^*), \tag{3.9}$$

it follows that the operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ obey the commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \tag{3.10a}$$

$$[\hat{a}_i, \hat{a}_j] = 0, \tag{3.10b}$$

$$[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0. \tag{3.10c}$$

In these equations, † stands for the Hermitian conjugate of an operator.

The operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ can be interpreted as operators which annihilate and create quantum states. However, because of the obscure notions of particle and vacuum, these states do not necessarily represent particles. To further specify the meaning of particle vacuum, consider two complete sets of orthogonal solutions to the Klein-Gordon equation: $\{u_i : i \in \mathcal{I}\}$ and $\{u_i' : i \in \mathcal{I}\}$, where $\mathcal{I}$ is an index set. Because both of the sets are complete, the modes u_i' can be written in terms of the modes u_i such that

$$u_i' = \sum_j \big(A_{ij} u_j + B_{ij} u_j^*\big), \tag{3.11}$$

where $A_{ij}, B_{ij} \in \mathbb{C}$. Conversely, it turns out that

$$u_i = \sum_j \big(A_{ji}^* u_j' - B_{ji} u_j'^*\big). \tag{3.12}$$

These relations are known as the *Bogolubov transformations* and the numbers A_{ij} and B_{ij} are called the *Bogolubov coefficients* [54]. One can easily show that

$$\sum_k \big(A_{ik} A_{jk}^* - B_{ik} B_{jk}^*\big) = \delta_{ij} \tag{3.13a}$$

$$\sum_k \big(A_{ik} B_{jk} - B_{ik} A_{jk}^*\big) = 0. \tag{3.13b}$$

The Bogolubov transformations impose the following relations between the annihilation and the creation operators:

$$\hat{a}_i = \sum_j \big(A_{ji} \hat{a}_j' + B_{ji}^* \hat{a}_j'^\dagger\big), \tag{3.14a}$$

$$\hat{a}_i' = \sum_j \big(A_{ij}^* \hat{a}_j - B_{ij}^* \hat{a}_j^\dagger\big). \tag{3.14b}$$

Clearly, the vacuum associated with the modes u_i is different from the vacuum associated with the modes u_i' if $B_{ij} \neq 0$. To see that this is the case, we denote the vacuum annihilated by the operator $\hat{a}_i$ by $|0\rangle$ and, correspondingly, the vacuum annihilated by the operator $\hat{a}'_i$ by $|0'\rangle$. It follows now from Eq. (3.14a) that

$$\hat{a}_i|0'\rangle = \sum_j B^*_{ji}|1'_j\rangle \neq 0. \tag{3.15}$$

Moreover, the expectation value for the number of u_i-mode particles in the vacuum $|0'\rangle$ is

$$n_i = \langle 0'|\hat{a}^\dagger_i \hat{a}_i|0'\rangle = \sum_j |B_{ji}|^2. \tag{3.16}$$

This means that when $B_{ij} \neq 0$, the modes u_i and u'_i do not share a common vacuum state but the vacuum $|0'\rangle$ contains $\sum_j |B_{ji}|^2$ particles in the mode u_i.

Chapter 4

An Introduction to the Hawking Effect

4.1 Black Hole Radiation

After Hawking had published his area theorem, Bekenstein proposed that the area of the event horizon of a black hole is in fact a measure of its entropy. More precisely, Bekenstein suggested that the entropy of a black hole is, in SI units,

$$S = \gamma \, \frac{k_{\mathrm{B}} c^3}{\hbar G} A, \tag{4.1}$$

where γ is a constant of the order of unity [6]. By information theoretic arguments, he was even able to conjecture a certain value for the constant γ, namely $\frac{\ln 2}{8\pi}$, which later turned out to be wrong. However, inspired by Bekenstein's ideas, Hawking was able to show that the correct value of this constant is one-quarter [5]. Therefore, in SI units, the black hole entropy may be written as

$$S = \frac{1}{4} \frac{k_{\mathrm{B}} c^3}{\hbar G} A. \tag{4.2}$$

This result is known as the *Bekenstein-Hawking entropy law* for black holes.

The entropy of a black hole is manifested by the radiation process which takes place at the event horizon of the hole. Hawking's original analysis was based on the properties of quantum field theory in curved spacetime. In short, he considered the behaviour of the vacuum states of the massless Klein-Gordon field when the field was transported from $\Im^-$ to $\Im^+$ near the event horizon of a (collapsing) Schwarzschild black hole (see Fig. 4.1). Quite surprisingly, it turns out that the vacuum states at $\Im^-$ are different from the vacuum states at $\Im^+$. As a result, an observer at the future null infinity observes a flux of particles coming from the immediate vicinity of the event horizon. (Note that since Schwarzschild spacetime is asymptotically flat and stationary, one has well-defined concepts of energy and particles at the regions very far away from the black hole.) The expectation value for the number of particles coming out of the hole with angular frequency ω agrees with the Planck distribution for blackbody radiation at the *Hawking temperature*

$$T_{\mathrm{H}} = \frac{1}{8\pi M} \overset{\mathrm{SI}}{=} \frac{\hbar c^3}{8\pi G k_{\mathrm{B}} M}, \tag{4.3}$$

which, in terms of the surface gravity κ, may be written as

$$T_{\mathrm{H}} \overset{\mathrm{SI}}{=} \frac{\hbar\kappa}{2\pi c k_{\mathrm{B}}}. \tag{4.4}$$

This temperature leads directly to the Bekenstein-Hawking entropy introduced in Eq. (4.2). It turns out that these results are independent of the details of the gravitational collapse and, as one might therefore expect, they hold for eternal black holes as well.

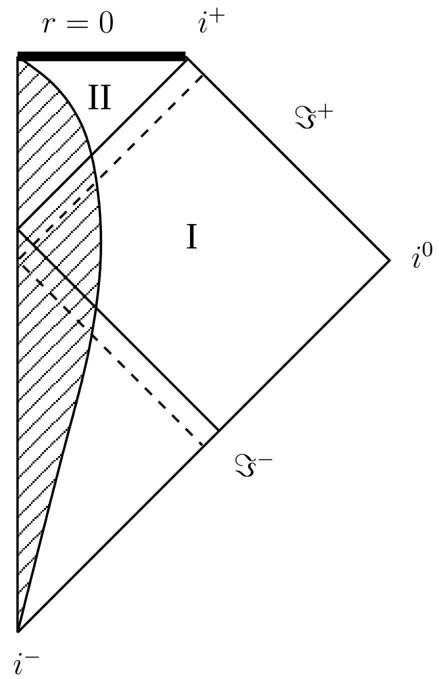

Figure 4.1. The conformal diagram of an infinitely collapsing spherical mass distribution. The massless Klein-Gordon particles are transported from $\Im^-$ to $\Im^+$ along the dashed line. The exterior region I is causally separated from the black hole region II by the event horizon. The shaded region represents the collapsing mass distribution.

The black hole radiation is a somewhat confusing subject since black holes are by definition “regions of no escape”. For this reason, some heuristic explanations for the origin of the radiation have been put forward. Maybe the most famous one is due to Hawking. He proposed that the spontaneous pair production process near the event horizon provides a mechanism for the radiation. In normal conditions, a virtual particle-antiparticle pair annihilates itself very rapidly after its emergence. In the vicinity of the event horizon, however, it is possible that the member of the pair with negative energy is swallowed by the black hole before the annihilation, and so the the other member with positive energy is free to escape away from the hole. This process is discussed in more detail in chapter 5, where similar ideas are used as a motivation to study the radiation effects associated with the inner horizon of a Reissner-Nordström black hole.

Another possible explanation for the Hawking radiation is provided by the tunneling effect through the event horizon. This effect follows from the assumption that the correct outgoing wave function should be a specific superposition of the outgoing waves both inside and outside the black hole. It can be shown that the outgoing positive energy waves inside the hole correspond to the ingoing negative energy waves, and therefore the black hole radiation may be viewed as a tunneling effect of particles out of the hole [55]. Nevertheless, as useful as these explanations may be, one should not forget that they are only heuristic pictures of the Hawking effect.

4.2 Physical Aspects of Black Hole Radiation

The Hawking radiation has various physical consequences. First of all, as one can see from Eq. (4.3), the temperature of the hole increases when its mass decreases, i.e., small holes are hotter than large ones. Thus, if a hole is so large that its Hawking temperature becomes smaller than the cosmic background temperature, then, in effect, the black hole only absorbs radiation. For this reason, it is impossible, even in principle, to detect large holes by the radiation they emit. Leaving that aside, when a black hole radiates, it loses its mass and becomes hotter and hotter. In other words, the specific heat of a black hole is negative. Since the hole loses mass during the radiation, the hole eventually evaporates. The final stages of this evaporation process are still unknown, because the current theories probably break down when the mass of the hole is of the order of the *Planck mass* $m_{\text{Pl}} = \sqrt{\hbar c/G} \sim 10^{-8}$ kg. However, a rough estimate for the lifetime of a macroscopic Schwarzschild black hole can be calculated from the Stefan-Boltzmann law

$$\frac{dE}{dt} = -\sigma T_{\text{H}}^4 A, \tag{4.5}$$

where $\sigma = \pi^2 k_{\text{B}}^4/(60\hbar^3 c^2)$ is the Stefan-Boltzmann constant and $A = 4\pi R_{\text{S}}^2$ is the event horizon area. By using $E = Mc^2$ we find that

$$\frac{dM}{dt} = -\frac{\hbar c^4}{15360\,\pi\, G^2 M^2}, \tag{4.6}$$

which leads to the evaporation time

$$t_{\text{ev}} = \frac{5120\,\pi\, G^2}{\hbar c^4} M^3 \approx 8 \cdot 10^{-17}\ \text{s/kg}^3 \times M^3. \tag{4.7}$$

For a black hole with one solar mass $M_\odot \approx 2 \times 10^{30}$ kg, the lifetime is of the order 10^{67} a, which is much greater than the age of our universe.

The temperature in Eq. (4.3) is not completely realistic. In a more precise analysis, one should take into account the backscattering of the radiation from the "gravitational potential" of the black hole. These effects have been studied carefully, and it has turned out that the radiation spectrum is not purely Planckian (see, for instance, Ref. [53]). However, if the hole is in a heat bath, then the fraction of backscattered ingoing particles will be the same as is the fraction of outgoing particles removed by the backscattering. Therefore, the ratio of emission and absorption does not depend on the details of the backscattering but it remains identical to that of a black body. In this sense, the radiation distribution can be considered as "thermal".

Since Hawking's original analysis was based on the quantization of the Klein-Gordon field, it neglected the possible effects resulting from the spins of the radiating particles. This might seem a severe problem since the radiation consists mostly of neutrinos and photons.[1,2] Nevertheless, the results remain essentially the same even if we take into account the spins of the particles. In fact, for photon, neutrino and linearized graviton fields the temperature of the radiation is exactly the Hawking temperature. The greatest difference lies in the backscattering which depends on the spins of the particles of the field. More precisely, backscattering is more efficient for particles with greater mass and angular momentum.

The Hawking effect has been studied also for the event horizons appearing in other black hole solutions, as well as for some other horizons of spacetime. A short introduction to these matters can be found in Ref. [53]. For a charged black hole, Eq. (4.4) still holds, but the presence of a background electric field gives rise to the spontaneous creation of charged particle-antiparticle pairs. The hole prefers to emit particles with the charge of the same sign as the hole has, and therefore the hole has a tendency to discharge itself. As it comes to the rotating black holes, the situation is far more complicated. In particular, due to the axisymmetric metric of spacetime, the emission of particles will be asymmetric around the hole. However, the fact that makes the Reissner-Nordström and Kerr-Newman holes totally different from the Schwarzschild black hole is an existence of *inner* horizons. The possible radiation effects of the inner horizons are the main subjects of the chapter 5.

The Hawking radiation is a *semiclassical* result. This means that matter fields are assumed to follow the laws of quantum physics while spacetime is considered as a non-dynamical background. However, in a more realistic situation we expect the gravitational field to have quantum effects as well. It is not known very well what is the region of validity of the semiclassical approximation. Generally, it is believed that the quantum effects of gravity become significant at the Planck length scale where distances are of the order $l_{\rm Pl} = \sqrt{G\hbar/c^3} \sim 10^{-35}$ m. One also expects that

[1] In fact, the only spinless particle predicted by the standard model of particle physics is the Higgs particle. However, at the present time there is no indisputable experimental evidence of the existence of such particle.

[2] It has been estimated that for the Schwarzschild black hole the radiation consists about 81% neutrinos, 17% photons and 2% gravitons [56].

the effects of quantum gravity take place when the gravitational field goes through a rapid time-dependent change within about one Planck time. Therefore, the semi-classical gravity should be valid unless we consider microscopic black holes, the gravitational effects near the spacetime singularities, or the very early epoch of the universe.

In addition to Hawking's original work, there are other approaches to the problem of the black hole radiation. Maybe the most famous one is the path-integral approach initiated by Gibbons and Hawking [57, 58]. The path-integral quantization in non-relativistic quantum mechanics was developed by Feynman as an alternative to the canonical quantization [4]. The generalization of this method may be used as a tool to quantize (or perhaps *try* to quantize) gravitational field. Unfortunately, the path integrals concerning gravity are extremely hard to evaluate. However, if one confines himself to the semiclassical approximation, where spacetime is considered as a rigid background, one finds that the first-order approximation reproduces the Bekenstein-Hawking entropy law. An obvious advantage of the path-integral approach is that the thermal properties of black holes are no more due to the behaviour of quantized matter at the event horizon but they are of purely geometrical origin. This suggests that entropy and temperature may be seen as intrinsic properties of gravitation. Besides the path-integral approach, there are numerous other derivations of the Hawking effect based on substantially different physical assumptions. A review of different approaches to the black hole radiation can be found in Ref. [59]. The variety of different viewpoints on the Hawking radiation manifests the fact that the origin of the radiation cannot be wholly understood by our current theories.

4.3 Black Hole Entropy

The Bekenstein-Hawking entropy law has several important consequences. First of all, since in natural units

$$\delta S = \frac{1}{4}\,\delta A, \tag{4.8}$$

the first law of black hole mechanics, theorem 1.3.2, can be written as

$$\delta M = T_{\mathrm{H}}\,\delta S_{\mathrm{BH}} + \Omega_{\mathrm{BH}}\,\delta J_{\mathrm{BH}} + \Phi_{\mathrm{BH}}\,\delta Q. \tag{4.9}$$

The dramatic change caused by this substitution is that this equation describes now *thermodynamics* rather than mechanics of the black hole. For this reason, this equation is usually called the first law of black hole thermodynamics.

Secondly, as the black hole radiates, its mass, and therefore the area of its event horizon, decreases. This contradicts the second law of black hole mechanics and, more importantly, the second law of thermodynamics. A remedy for this problem was proposed by Bekenstein. He suggested that although the entropy of the black hole decreases, the total entropy $S_{\text{tot}} = S_{\text{ext}} + S_{\text{BH}}$, where S_{ext} is the entropy of the exterior spacetime region, is a non-decreasing function of time in any process [60].

Putting differently,

$$\delta S_{\text{tot}} \geq 0 \text{ in any process.} \tag{4.10}$$

This statement is known as the *generalized second law of thermodynamics*.

However, the most important consequence of the black hole entropy is, at least from the point of view of quantum gravity, its statistical interpretation: Since a black hole has an entropy $S = \frac{1}{4}\frac{k_{\mathrm{B}}c^3}{\hbar G}A$, we expect that the hole has $\exp\left(\frac{1}{4}\frac{c^3}{\hbar G}A\right)$ microstates corresponding to its macrostate. Thus, a macroscopic hole has an enormous amount of quantum-mechanical degrees of freedom compared to the three classical ones predicted by the no-hair theorems. The existence of these microstates raises many intriguing questions. Do these microstates correspond to the quantum states of the collapsing matter inside the black hole, or are these degrees of freedom connected with the quantized matter fields on a background geometry. Or could it be possible that the notion of black hole entropy stems from the microscopic structure of spacetime itself? In the spirit of the last question, we shall introduce a spacetime foam model of the Schwarzschild horizon in chapter 7.

Chapter 5

Radiation of the Inner Horizon of the Reissner-Nordström Black Hole

5.1 Introduction

Hawking's celebrated paper on black hole radiation came as a great surprise to almost everyone working in the field of general relativity. Till then it was strongly believed that black holes are totally black, i.e., neither matter nor radiation can be emitted by black holes. Indeed, this is what one would expect on purely classical grounds. However, now we know that when one takes into account quantum-mechanical effects near the event horizon of a hole, one finds that the black hole does radiate thermal radiation with a spectrum similar to that of a black body.

As mentioned in the previous chapter, one way to understand the origin of this radiation is to consider spontaneous particle-antiparticle pair production near the event horizon of a black hole. Normally, such a pair annihilates itself very rapidly. In the vicinity of the event horizon, however, it is possible that the members of a virtual pair become separated by the horizon such that the annihilation is prevented. One member of the pair escapes away from the hole with positive energy to contribute to the Hawking radiation, while the other one with negative energy is swallowed by the black hole. Therefore, an observer outside the hole, that is, at the region I of the Fig. 5.1.(a), observes a flux of quanta with positive energy which seems to come out of the black hole. This event is illustrated in Fig. 5.1.(a) in the case of a Schwarzschild black hole.

It is well known that the outer horizon of a Reissner-Nordström black hole radiates in a similar way as the event horizon of a Schwarzschild black hole. However, it is interesting to see what kind of phenomenon is predicted by the virtual pair production mechanism if one looks at the *inner horizon* of the Reissner-Nordström black hole. Consider a maximally extended Reissner-Nordström spacetime (see Fig. 5.1.(b)). It is easy to see that the causal relationship between the regions V' and IV' is similar to that between the regions I and II, respectively. Therefore, as shown in the Fig. 5.1.(b), a virtual particle-antiparticle pair which emerges very close to the inner horizon $r = r_-$ in the region V' can avoid annihilation if either the particle or the antiparticle falls into the region IV' and the other one remains in V'. Therefore the pair production mechanism implies that the inner horizon does radiate and, moreover, that the radiation is directed *inwards*, towards the singularity. This line of reasoning, however, provides no information about the radiation itself. Especially, it remains unclear what is the energy distribution of the radiating quanta.

After Hawking's original work, there have been various derivations of the Hawking effect with different physical assumptions. Curiously, very little is known about the radiation of the inner horizons of the Reissner-Nordström and the Kerr-Newman black holes. This feature can, at least to some extent, be regarded as a consequence of the fact that inside the inner horizon there are no spacetime regions analogous to the regions $\Im^+$ or $\Im^-$. After all, Hawking's original work was based on the analysis of the properties of the Klein-Gordon field at $\Im^+$ and $\Im^-$. To the best of our knowledge, the only explicit calculation considering the radiation of the inner horizons was performed by Wu and Cai by means of the analytic continuation of the Klein-Gordon field [61]. However, as a result of their analysis they found that the temperature of the inner horizon is *negative* and this seems to contradict the general attitude towards the black hole thermodynamics [62], as well as the very foundations of thermodynamics themselves. Thus, the true nature of the radiation of the inner horizon is still somewhat unclear.

The aim of this chapter is to perform a detailed analysis of the radiation of the inner horizon of the Reissner-Nordström black hole. We would like to point out, however, that our analysis predicts very few astrophysical consequences because no mechanism for the formation of Reissner-Nordström black holes is known. One of the main reasons why the full Reissner-Nordström spacetime is not considered astrophysically real is a phenomenon called *mass inflation* near the inner horizon of the Reissner-Nordström black hole. It is known from the works of Poisson and Israel that when one considers the spherical collapse of a charged star then, at least in a somewhat idealized situation, the flux of particles emitted by the collapsing star and its backscattered counterpart near the inner horizon of the Reissner-Nordström spacetime provoke an enormous inflation of the internal mass parameter of the black hole [63]. Eventually the mass parameter becomes large enough to form a singularity at the inner horizon and in effect to freeze the evolution of spacetime. The inflation of the mass parameter at the inner horizon does not have any implications in the region outside the black hole since these regions are causally separated.

In this chapter, we take the maximally extended Reissner-Nordström spacetime as the starting point of our analysis. We would like to emphasize that here the Reissner-Nordström spacetime is only considered as a mathematical solution to the combined Maxwell-Einstein field equations, and the whole problem concerning the formation of such spacetime is completely ignored. One could ask, of course, why are we interested in the properties of the full Reissner-Nordström spacetime if it

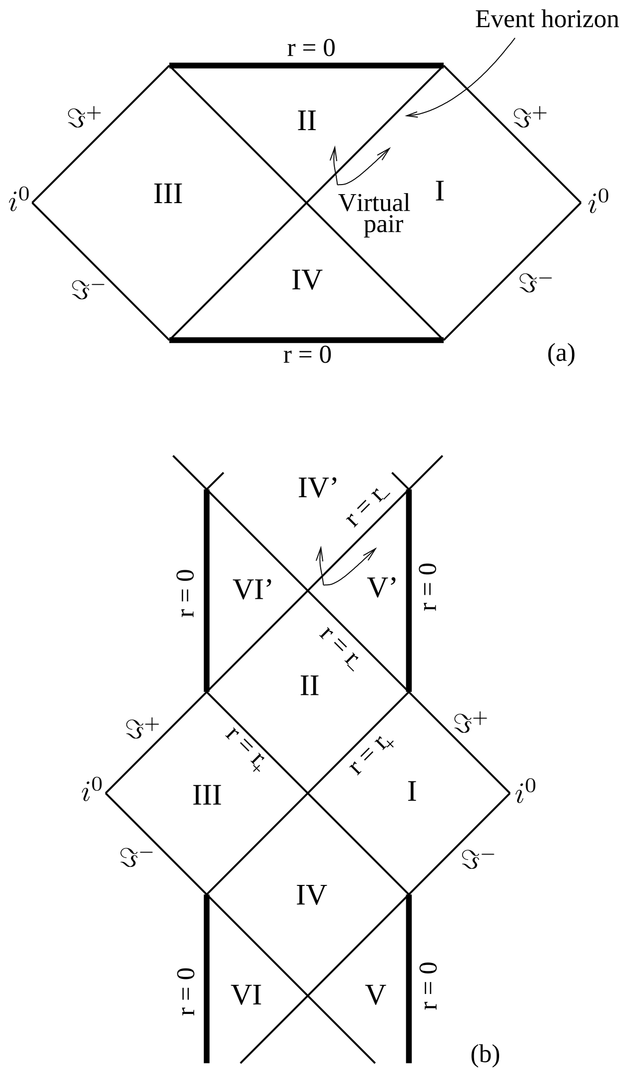

Figure 5.1. (a) Conformal diagram of the maximally extended Schwarzschild spacetime. In this diagram, the regions I and III represent spacetime surrounding the regions II (black hole) and IV (white hole). However, the regions I and III are causally separated. If a particle-antiparticle pair is spontaneously created near the event horizon of the hole in region I, it is possible that either a particle or an antiparticle is swallowed by the hole such that the other one is free to escape to the infinity at $\Im^+$. (b) Maximally extended Reissner-Nordström spacetime. Similarly as in the Schwarzschild spacetime, a virtual pair created near the inner horizon $r = r_-$ may avoid annihilation if the particle and the antiparticle are separated by the horizon.

is not considered astrophysically relevant. The answer to this question lies in the fact that Reissner-Nordström spacetime provides an explicit example of a spacetime geometry which contains *two horizons*, of which one is hidden from the outside observer. The problem we are interested in is the following: Does only one of the horizons emit Hawking radiation, as it is generally believed, or do both of the horizons radiate? This is an intriguing question, and if one is able to show that both of the horizons radiate, then this result may be seen to support the idea that *all* horizons of spacetime emit radiation. The main object of interest in this chapter is therefore not the Reissner-Nordström black hole itself, but the general semiclassical properties of gravity. One may also hope that our results are qualitatively the same for the more realistic Kerr black hole solution. Indeed, this might be the case because the causal structures of the Reissner-Nordström and the Kerr spacetimes are very similar. In that case the radiation of the inner horizon may even have astrophysical relevance since there is not necessarily mass inflation in the Kerr spacetime.

If the inner horizon of the Reissner-Nordström black hole really radiates, it is expected that this effect takes place during a very short time only. To see that this is the case, suppose that we begin with the purely classical Reissner-Nordström solution with two horizons, and apply the results of quantum field theory in the Reissner-Nordström spacetime. If, in the semiclassical limit, we find that the inner horizon radiates, then the backscattered part of that radiation is enough to trigger the mass inflation, and eventually the inner horizon disappears. In other words, if one is able to show that the inner horizon of the Reissner-Nordström black hole radiates, then it turns out that semiclassically the full Reissner-Nordström spacetime, even as a mathematical solution, is unstable. The radiation process of the inner horizon, however, should last as long as the inner horizon exists. We shall not discuss the backscattering effect and its consequences in more detail but we shall confine ourselves merely to the qualitative aspects discussed above. This approach is justified because we are interested in the radiation effects of the inner horizon of the Reissner-Nordström spacetime when the effects of mass inflation are still negligible.

In brief, the key points of our discussion can be expressed as follows. When the ideas of Hawking's original work are utilized in Rindler spacetime, one recalls that the so-called Unruh effect, which is closely related to the Hawking effect, can be obtained by simply comparing the solutions to the Klein-Gordon equation for massless particles from the points of view of inertial and uniformly accelerated observers. As a preliminary we show this in Sec. 5.2. In curved spacetime, however, the concept of inertial observer is replaced by the concept of freely falling observer. Inspired by this analogy we proceed to calculate the effective temperature of black hole horizons by comparing the solutions to the massless Klein-Gordon equation from the points of view of an observer in a radial free fall and an observer at rest with respect to the horizon. First, in Sec. 5.3 we perform, as an example, an analysis of the radiation of the outer horizon of a Reissner-Nordström black hole. After reproducing the well-known results, we proceed, in Sec. 5.4, to calculate the temperature of the radiation emitted by the inner horizon. In contrast to Wu and Cai, we find that the effective temperature for particles radiating from the inner horizon towards the singularity is not negative but *positive*: The inner horizon emits particles with positive energy and temperature. This radiation process is a local phenomena which takes place in the vicinity of the inner horizon, and it seems impossible to observe the effects of the radiation from outside of the Reissner-Nordström black hole. We close our discussion in Sec. 5.5 with some concluding remarks.

5.2 Preliminaries: The Unruh Effect

As a starting point of our analysis, let us consider the thermal radiation of the Rindler horizon found by Unruh in 1976 [64]. Rindler horizons are such horizons of spacetime that appear in the rest frame of a uniformly accelerated observer. In general, the equation of the world line of a uniformly accelerated observer in flat two-dimensional Minkowski spacetime is (unless otherwise stated we shall always have $c = G = \hbar = k_B = 1$)

$$X^2 - T^2 = \frac{1}{a^2}, \tag{5.1}$$

where a is the proper acceleration of the observer, and X and T, respectively, are the Minkowskian space and time coordinates [53]. The worldline of the observer may be written in the parametrized form:

$$T(\eta) = \frac{1}{a}\sinh(a\eta), \tag{5.2a}$$

$$X(\eta) = \frac{1}{a}\cosh(a\eta). \tag{5.2b}$$

In this expression, η is the proper time of the observer. The worldline of a uniformly accelerated observer has been drawn in Fig. 5.2, asymptotically approaching the Rindler horizon of the accelerated observer. From the figure, we can also see the four regions of Rindler spacetime, labelled as I, II, III and IV. Since this diagram is very similar to the Kruskal diagram of Schwarzschild spacetime, one would expect Rindler spacetime to have physical properties similar to those of Schwarzschild spacetime. In fact, it is easy to see that the causal features of the regions II and IV are, respectively, similar to those of a black and a white hole.

The simplest way to obtain the Unruh effect is probably the following. Define the Rindler coordinates t and x such that

$$x := \frac{1}{a}, \tag{5.3a}$$

$$t := a\eta. \tag{5.3b}$$

In these coordinates, a uniformly accelerated observer remains at rest with respect to the x-coordinate and the metric of two-dimensional Minkowski spacetime is written as

$$ds^2 = -x^2dt^2 + dx^2. \tag{5.4}$$

Consider then Klein-Gordon equation of massless particles in the rest frame of an accelerated observer. In general, that equation may be written as

$$g^{\mu\nu}D_\mu D_\nu\phi = 0, \tag{5.5}$$

where D_μ denotes covariant derivative, and when spacetime metric is that of Eq. (5.4),

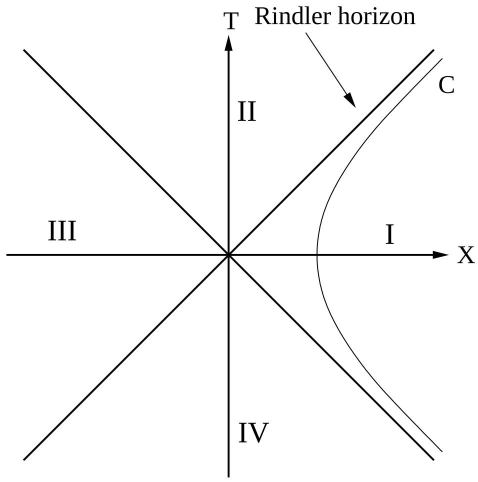

Figure 5.2. Rindler spacetime. The curve C represents the worldline of a uniformly accelerated observer.

Eq. (5.5) takes the form

$$\Big(-\frac{1}{x^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}\Big)\phi=0. \tag{5.6}$$

If one defines

$$x^* := \ln x, \tag{5.7}$$

Eq. (5.6) becomes

$$\Big(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^{*2}}\Big)\phi=0. \tag{5.8}$$

Orthonormal solutions to this equation are of the form

$$u_\omega = N_\omega e^{-i\omega U}, \tag{5.9}$$

where N_ω is an appropriate normalization constant, ω is taken to be positive, and we denote

$$U := t - x^*. \tag{5.10}$$

These solutions represent, from the point of view of an accelerated observer in the region I, particles with energy $\omega > 0$ propagating to the positive X-direction. In contrast, the corresponding solutions to the massless Klein-Gordon equation

$$\Big(-\frac{\partial^2}{\partial T^2}+\frac{\partial^2}{\partial X^2}\Big)\phi=0, \tag{5.11}$$

written from the point of view of an inertial observer at rest with respect to the Minkowski coordinates T and X, are of the form

$$u'_\omega = N_\omega e^{-i\omega\widetilde{u}}, \tag{5.12}$$

where

$$\widetilde{u} := T - X. \tag{5.13}$$

Again, for $\omega > 0$, these solutions represent particles with positive energy ω propagating to the positive X-direction.

The relationship between the solutions u_ω and u'_ω is given by the Bogolubov transformation

$$u_\omega = \sum_{\omega'} \left(A'_{\omega\omega'} u'_{\omega'} + B'_{\omega\omega'} u'^{*}_{\omega'} \right) \tag{5.14}$$

(see chapter 3 for details). It is easy to see from Eqs. (5.2) and (5.10) that

$$U = -\ln(-\widetilde{u}), \tag{5.15}$$

and therefore Eq. (5.14) may be written as

$$e^{i\omega \ln(-\widetilde{u})} = \sum_{\omega'} \left(A'_{\omega\omega'} e^{-i\omega'\widetilde{u}} + B'_{\omega\omega'} e^{i\omega'\widetilde{u}} \right), \tag{5.16}$$

where the Bogolubov coefficients $A'_{\omega\omega'}$ and $B'_{\omega\omega'}$ are expressible as Fourier integrals:

$$A'_{\omega\omega'} = \frac{1}{2\pi} \int_{-\infty}^{0} d\widetilde{u}\, e^{i\omega \ln(-\widetilde{u})} e^{i\omega'\widetilde{u}}, \tag{5.17a}$$

$$B'_{\omega\omega'} = \frac{1}{2\pi} \int_{-\infty}^{0} d\widetilde{u}\, e^{i\omega \ln(-\widetilde{u})} e^{-i\omega'\widetilde{u}}. \tag{5.17b}$$

The integration is performed from the negative infinity to zero because we are considering particles in the region I, and in that region $\widetilde{u} < 0$. It is straightforward to show by performing the integration in the complex plane that

$$\left|A'_{\omega\omega'}\right| = e^{\pi\omega} \left|B'_{\omega\omega'}\right| \tag{5.18}$$

(see Fig. 5.3). It follows from Eq. (3.13a) that between the Bogolubov coefficients there is a relationship:

$$\sum_{\omega'} \left(\left|A'_{\omega\omega'}\right|^2 - \left|B'_{\omega\omega'}\right|^2 \right) = 1. \tag{5.19}$$

One therefore finds that when the field is in vacuum from the point of view of an inertial observer, the number of particles with energy ω is, from the point of view of an accelerated observer,

$$n_\omega = \sum_{\omega'} \left|B'_{\omega\omega'}\right|^2 = \frac{1}{e^{2\pi\omega} - 1}. \tag{5.20}$$

This is the Planck spectrum at the temperature $T_0 = \frac{1}{2\pi}$, which is related to the temperature experienced by an observer situated at a given point in space by the Tolman relation [65]:

$$T = (g_{00})^{-\frac{1}{2}}\, T_0. \tag{5.21}$$

Hence, it follows that a uniformly accelerated observer observes particles coming out from the Rindler horizon with the blackbody spectrum corresponding to the characteristic temperature

$$T_U := \frac{1}{2\pi x} = \frac{a}{2\pi}, \tag{5.22}$$

even when, from the point of view of an inertial observer, the field is in vacuum. This result is known as the *Unruh effect*, and it is one of the most remarkable outcomes of quantum field theory.

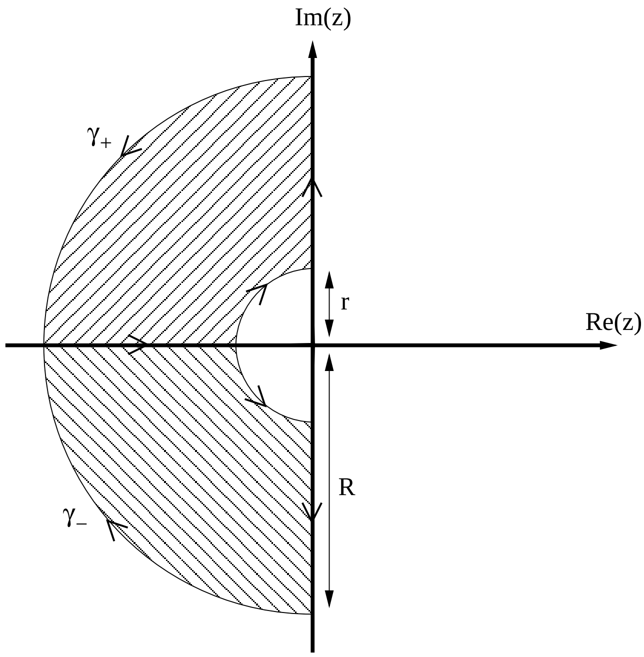

Figure 5.3. Integration contours in the complex plane. In this figure, γ_+ and γ_- are closed contours circulating the shaded regions in the upper and the lower half of the complex plane, respectively. When the integral in Eq. (5.17a) is calculated along the contour γ_+, one easily sees that, in the limit where $R \to \infty$ and $r \to 0$, the integrals along the arcs of the circles vanish. The analyticity of the functions under consideration in the shaded regions implies that the contour integral around γ_+ vanishes, and therefore the integral from negative infinity to zero along the real axis may be transformed into an integral from positive infinity to zero along the imaginary axis. Similar result holds for the integral in Eq. (5.17b) along the path γ_-, except that now the integral from negative infinity to zero along the real axis may be transformed to an integral from negative infinity to zero along the imaginary axis. The integrals along the imaginary axis lead directly to Eq. (5.18).

5.3 Reconsideration of the Hawking Effect for the Outer Horizon of a Reissner-Nordström Black Hole

The ideas inspired by the properties of Rindler spacetime can be easily utilized when one investigates the thermal properties of Reissner-Nordström horizons in the following manner. At first, one constructs a certain *geodesic system of coordinates* for the neighborhood of the horizon under scrutiny. The geodesic coordinates are constructed such that an observer in a *radial* free fall remains at rest with respect to those coordinates. Such an observer does not observe the horizon, and therefore one expects that no radiation effects should be experienced by him. Because of that one may view the particle vacuum of the freely falling observer as the vacuum that would exist in spacetime if there were no horizon at all. In this sense, the observer in a radial free fall is analogous to the inertial observer in flat spacetime, and we take this similarity as a starting point of our analysis.

To calculate the particle flux emitted by the horizon, one compares the solutions to the massless Klein-Gordon equation very close to the horizon in two different coordinate systems. One of these systems is the geodesic system of coordinates and the other one is a coordinate system at rest with respect to the horizon. The analysis of the Klein-Gordon modes must be performed infinitesimally close to the horizon for the very reason that only in that case one is able to solve the Klein-Gordon equation analytically (excluding, of course, the solutions at the asymptotic infinities). Then one can obtain the Bogolubov transformations between these solutions and infer the effective temperature of the radiation flux from the point of view of an observer at rest with respect to the horizon.

To see what all this really means consider, as an example, the outer horizon of a Reissner-Nordström black hole. The Reissner-Nordström metric can be written as

$$\begin{aligned} ds^2 = &-\Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)dt^2 + \Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)^{-1}dr^2 \\ &+ r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \end{aligned} \tag{5.23}$$

where M is the mass and Q is the electric charge of the hole. In addition to the physical singularity at $r = 0$, this metric has two coordinate singularities when

$$r = r_\pm := M \pm \sqrt{M^2 - Q^2}. \tag{5.24}$$

The two-surfaces where $r = r_+$ and $r = r_-$ are called, respectively, the outer and the inner horizons of the Reissner-Nordström black hole. It is well known that when $r > r_+$, and the backscattering effects are neglected, an observer at rest with respect to the coordinates r, θ, and φ observes thermal radiation emitted by the hole with a characteristic temperature

$$T_+ := \frac{\kappa_+}{2\pi\sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}} = \frac{\sqrt{M^2 - Q^2}}{2\pi\big(M^2 + \sqrt{M^2 - Q^2}\big)^2\sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}}, \tag{5.25}$$

where

$$\kappa_+ := \frac{r_+ - r_-}{2r_+^2} \tag{5.26}$$

is the surface gravity of the outer horizon. The factor $(1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1/2}$ is due to the "redshift" of the radiation. At asymptotic infinity the redshift factor is equal to one, but at the horizon $r = r_+$ it becomes infinitely large. In other words, an observer at rest very close to the horizon may measure an infinite temperature for the black hole radiation.

We shall now show how Eq. (5.25) may be obtained by means of the method we explained at the beginning of this section. As the first step, we separate the Klein-Gordon field ϕ of massless particles such that

$$\phi(t, r, \theta, \varphi) = \frac{1}{r} f(t, r)\, Y_{lm}(\theta, \varphi), \tag{5.27}$$

where Y_{lm} is the spherical harmonic satisfying the differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] Y_{lm} = -l(l+1)Y_{lm}, \tag{5.28}$$

where, as usual, the allowed values of l are $0, 1, 2, \dots$, and those of m are $0, \pm 1$, $\pm 2, \dots, \pm l$. In that case the massless Klein-Gordon equation, when written in terms of the coordinates t, r, θ, and φ, implies that

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V(r)\right] f = 0, \tag{5.29}$$

where we have defined the "tortoise coordinate" r_* such that

$$r_* := \int \frac{dr}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} = r - \frac{r_-^2}{r_+ - r_-} \ln|r - r_-| + \frac{r_+^2}{r_+ - r_-} \ln|r - r_+|, \tag{5.30}$$

and the "potential term"

$$V(r) := \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left[\frac{l(l+1)}{r^2} + \left(\frac{2M}{r^3} - \frac{2Q^2}{r^4}\right)\right]. \tag{5.31}$$

Very close to the horizon, where

$$\Delta := \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} \tag{5.32}$$

is infinitesimally small, the potential $V(r)$ vanishes, and the solutions to Eq. (5.29) corresponding to the particles with energy ω moving towards the horizon are of the form[1]

$$f(t,r) \sim e^{-i\omega V}, \tag{5.33}$$

and for the solutions coming outwards from the horizon we have

$$f(t,r) \sim e^{-i\omega U}, \tag{5.34}$$

where the coordinates V and U are the advanced and the retarded coordinates defined as

$$V := t + r_*, \tag{5.35a}$$
$$U := t - r_*. \tag{5.35b}$$

Therefore, from the point of view of the observer at rest very close to the horizon, the ingoing and the outcoming solutions to the massless Klein-Gordon equation are, respectively,

$$\phi_{\text{in}} \approx N_{\omega lm} Y_{lm} \frac{1}{r} e^{-i\omega V}, \tag{5.36a}$$
$$\phi_{\text{out}} \approx N_{\omega lm} Y_{lm} \frac{1}{r} e^{-i\omega U}, \tag{5.36b}$$

where $N_{\omega lm}$ is an appropriate normalization constant. Throughout this chapter, we shall always consider positive energy solutions only, and therefore the costant ω is taken to be positive.

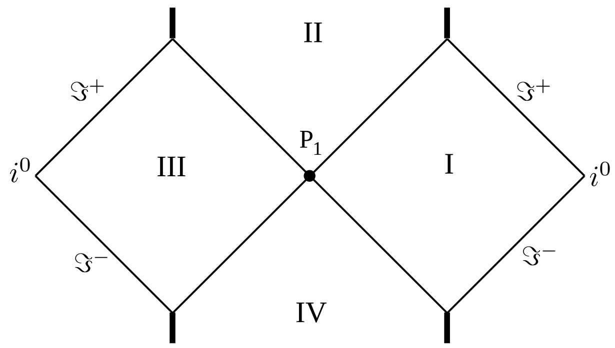

Figure 5.4. Part of Reissner-Nordström spacetime. The bifurcation point P_1 is situated at the intersection of the lines that separate the regions I, II, III and IV.

Consider now an observer in a radial free fall in the region I of the Reissner-Nordström spacetime, infinitesimally close to the point P_1 in Fig. 5.4. In other words, the observer is in a radial free fall just outside the outer horizon. First, let us intro-

[1]Note, though, that here the energy of the particles has not been "redshifted". Nevertheless, we shall continue to refer ω simply as 'energy'. The effects of the redshift are taken into account after we have obtained the effective temperature of the radiation.

duce in the Reissner-Nordström spacetime the coordinates (u, v) which are similar to the Kruskal coordinates in Schwarzschild spacetime. In general, these "Kruskal-type coordinates" may be defined in the different regions of the Reissner-Nordström spacetime such that

$$\begin{cases} u = \frac{1}{2}\big(e^{\alpha V} + e^{-\alpha U}\big), \\ v = \frac{1}{2}\big(e^{\alpha V} - e^{-\alpha U}\big), \end{cases} \qquad \text{(Region I, I', \dots)} \tag{5.37a}$$

$$\begin{cases} u = \frac{1}{2}\big(e^{\alpha V} - e^{-\alpha U}\big), \\ v = \frac{1}{2}\big(e^{\alpha V} + e^{-\alpha U}\big), \end{cases} \qquad \text{(Region II, II', \dots)} \tag{5.37b}$$

$$\begin{cases} u = -\frac{1}{2}\big(e^{\alpha V} + e^{-\alpha U}\big), \\ v = -\frac{1}{2}\big(e^{\alpha V} - e^{-\alpha U}\big), \end{cases} \qquad \text{(Region III, III', \dots)} \tag{5.37c}$$

$$\begin{cases} u = -\frac{1}{2}\big(e^{\alpha V} - e^{-\alpha U}\big), \\ v = -\frac{1}{2}\big(e^{\alpha V} + e^{-\alpha U}\big), \end{cases} \qquad \text{(Region IV, IV', \dots)} \tag{5.37d}$$

where α is an appropriate constant. When we study the physical properties of the outer horizon, the constant α must be chosen such that the metric on the two-surface $r = r_+$ is regular. The most natural choice is

$$\alpha = \kappa_+, \tag{5.38}$$

and this choice leads to the non-singular metric

$$ds^2 = \frac{1}{\kappa_+^2 r^2} e^{-2\kappa_+ r}(r - r_-)^{\frac{r_-^2}{r_+^2}+1}(-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{5.39}$$

It is now easy to construct a geodesic coordinate system for an infinitesimal neighborhood $\mathcal{U}(P_1)$ of the point P_1. By infinitesimal geodesic coordinate system we mean coordinates X^I ($I = 0, 1, 2, 3$) in $\mathcal{U}(P_1)$ such that, at the point P_1, the metric takes the form of that of flat spacetime, i.e.,

$$ds^2 = \eta_{IJ}\, dX^I dX^J, \tag{5.40}$$

where $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$ is the flat Minkowski metric, and the derivatives of the metric vanish. Let us define the coordinates

$$X^0 := l_+ v, \tag{5.41a}$$

$$X^1 := l_+ u, \tag{5.41b}$$

where

$$l_+ := \frac{1}{\kappa_+ r_+} e^{-\kappa_+ r_+}(r_+ - r_-)^{\frac{1}{2}\left(\frac{r_-^2}{r_+^2}+1\right)}. \tag{5.42}$$

By using these definitions one finds that at the point P_1 the metric can be written as

$$ds^2 = -\big(dX^0\big)^2 + \big(dX^1\big)^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{5.43}$$

and the derivatives of the metric with respect to X^0 and X^1 vanish (for details, see Appendix A). Therefore, the geodesic coordinates of the freely falling observer can be chosen to be X^0 and X^1. (Note that even though the above metric is not strictly of the form of Eq. (5.40), for the observer in a radial free fall these coordinates provide a geodesic coordinate system since in that case θ and φ are constants.)

If the massless Klein-Gordon field is now separated such that

$$\phi(X^0, X^1, \theta, \varphi) = \frac{1}{r}\,\widetilde{f}(X^0, X^1)\, Y_{lm}(\theta, \varphi), \tag{5.44}$$

the Klein-Gordon equation, when written in terms of the coordinates X^0, X^1, θ, and φ, implies

$$\left[-\frac{\partial^2}{\partial(X^0)^2}+\frac{\partial^2}{\partial(X^1)^2}+\frac{1}{r}\left(\frac{\partial^2 r}{\partial(X^0)^2}-\frac{\partial^2 r}{\partial(X^1)^2}\right)-\frac{l(l+1)}{r^2}F_+(r)\right]\widetilde{f}(X^0, X^1) = 0, \tag{5.45}$$

where we have denoted

$$F_+(r) := \frac{1}{\kappa_+^2 l_+^2}\frac{1}{r^2}e^{-2\kappa_+ r}(r - r_-)^{\frac{r_-^2}{r_+^2}+1}. \tag{5.46}$$

It follows from Eqs. (5.41), (A.5) and (A.6) that

$$\frac{\partial r}{\partial X^0} = -\kappa_+ F_+(r) X^0, \tag{5.47a}$$

$$\frac{\partial r}{\partial X^1} = \kappa_+ F_+(r) X^0, \tag{5.47b}$$

and therefore

$$\frac{\partial^2 r}{\partial(X^0)^2} - \frac{\partial^2 r}{\partial(X^1)^2} = \kappa_+ F_+(r)\Big\{\kappa_+ F_+'(r)\big[\big(X^0\big)^2 - \big(X^1\big)^2\big] - 2\Big\}, \tag{5.48}$$

where the prime means derivative with respect to r. Using Eqs. (A.4)–(A.6) and (5.30), one finds

$$F_+'(r)\Big[\big(X^0\big)^2 - \big(X^1\big)^2\Big] = \frac{2}{\kappa_+} + \frac{1}{\kappa_+^2}\left(\frac{2M}{r^2} - \frac{2Q^2}{r^3}\right), \tag{5.49}$$

and therefore Eq. (5.45) takes the form

$$\left[-\frac{\partial^2}{\partial(X^0)^2} + \frac{\partial^2}{\partial(X^1)^2} - \widetilde{V}(r)\right]\widetilde{f}(X^0, X^1) = 0, \tag{5.50}$$

where the "potential term" is

$$\widetilde{V}(r) := \left[-\frac{2M}{r^3} + \frac{2Q^2}{r^4} + \frac{l(l+1)}{r^2} \right] F_+(r). \tag{5.51}$$

The function $F_+(r)$ has the property

$$F_+(r_+) = 1, \tag{5.52}$$

and therefore Eq. (5.50) takes, at the outer horizon of the Reissner-Nordström black hole, the form

$$\left[-\frac{\partial^2}{\partial(X^0)^2} + \frac{\partial^2}{\partial(X^1)^2} + \frac{2M}{r_+^3} - \frac{2Q^2}{r_+^4} - \frac{l(l+1)}{r_+^2} \right] \widetilde{f}(X^0, X^1) = 0. \tag{5.53}$$

So we see that, in contrast to Eq. (5.29), the "potential term" does not vanish at the horizon. For a macroscopic hole, however, the "potential term" may be neglected: For Reissner-Nordström black holes $r_+ \geq M$ and $0 \leq |Q| \leq M$, and so it follows that

$$\left| \frac{2M}{r_+^3} - \frac{2Q^2}{r_+^4} \right| \leq \frac{2}{M^2}, \tag{5.54}$$

which means that when, in Planck units, $M \gg 1$, the terms involving M and Q will vanish. Moreover, if the orbital angular momentum l of the Klein-Gordon particle is sufficiently small, we may neglect the term $l(l+1)/r_+^2$. In other words, we may write Eq. (5.53), in effect, as

$$\left[-\frac{\partial^2}{\partial(X^0)^2} + \frac{\partial^2}{\partial(X^1)^2} \right] \widetilde{f}(X^0, X^1) = 0. \tag{5.55}$$

For very small l, the ingoing and the outcoming positive energy solutions to the massless Klein-Gordon equation very close to the horizon $r = r_+$ are

$$\phi'_{\text{in}} \approx N_{\omega lm} Y_{lm} \frac{1}{r} e^{-i\omega\widetilde{v}}, \tag{5.56a}$$

$$\phi'_{\text{out}} \approx N_{\omega lm} Y_{lm} \frac{1}{r} e^{-i\omega\widetilde{u}}, \tag{5.56b}$$

where

$$\widetilde{v} = X^0 + X^1, \tag{5.57a}$$

$$\widetilde{u} = X^0 - X^1, \tag{5.57b}$$

and $N_{\omega lm}$ is the normalization constant corresponding to the fixed values of l, m, and $\omega > 0$. These solutions represent, from the point of view of the freely falling observer, particles with positive energy ω moving towards and out of the horizon, respectively. From Eqs. (5.35b), (5.41), and (5.57b), one easily finds that in the spacetime region I

$$U = -\kappa_+^{-1} \ln(-\widetilde{u}) + \kappa_+^{-1} \ln l_+. \tag{5.58}$$

Thus, in the spherically symmetric case, the Bogolubov transformation between the outcoming modes in Eqs. (5.36b) and (5.56b) can be written in the form

$$e^{i\omega\kappa_+^{-1} \ln(-\widetilde{u})}\, e^{-i\omega\kappa_+^{-1} \ln l_+} = \sum_{\omega'} \Big(A'_{\omega\omega'} e^{-i\omega'\widetilde{u}} + B'_{\omega\omega'} e^{i\omega'\widetilde{u}} \Big), \tag{5.59}$$

and, moreover, we can express the Bogolubov coefficients $A'_{\omega\omega'}$ and $B'_{\omega\omega'}$ as Fourier integrals such that

$$A'_{\omega\omega'} = \frac{1}{2\pi} e^{-i\omega\kappa_+^{-1} \ln l_+} \int_{-\infty}^{0} d\widetilde{u}\, e^{i\omega\kappa_+^{-1} \ln(-\widetilde{u})} e^{i\omega'\widetilde{u}}, \tag{5.60a}$$

$$B'_{\omega\omega'} = \frac{1}{2\pi} e^{-i\omega\kappa_+^{-1} \ln l_+} \int_{-\infty}^{0} d\widetilde{u}\, e^{i\omega\kappa_+^{-1} \ln(-\widetilde{u})} e^{-i\omega'\widetilde{u}}. \tag{5.60b}$$

As in the previous section, the integration is performed from the negative infinity to zero because we are considering particles in the region I, and in that region $\widetilde{u} < 0$.

The integrals in Eqs. (5.60) are similar to those found in Eqs. (5.17), and the integration in the complex plane gives

$$\big|A'_{\omega\omega'}\big| = e^{\pi\kappa_+^{-1}\omega} \big|B'_{\omega\omega'}\big|. \tag{5.61}$$

Therefore, by using Eq. (5.19), we find that when the field is in vacuum from the point of view of a freely falling observer, the number of the particles with energy ω observed by an observer at rest very close to the horizon is

$$n_\omega = \sum_{\omega'} \big|B'_{\omega\omega'}\big|^2 = \frac{1}{e^{2\pi\kappa_+^{-1}\omega} - 1}. \tag{5.62}$$

This is the Planck spectrum at the temperature

$$T = \frac{\kappa_+}{2\pi}, \tag{5.63}$$

which represents the temperature of the outer horizon experienced by an observer at rest with respect to the horizon when the redshift effects of the radiation are ignored. The redshift factor can be recovered by the Tolman relation (5.21), and as a result we find that, from the point of view of an observer at rest very close to the outer horizon, the Reissner-Nordström black hole emits radiation with a characteristic temperature

$$T_+ = (g_{00})^{-\frac{1}{2}} \frac{\kappa_+}{2\pi} = \frac{\kappa_+}{2\pi\Delta} = \frac{\sqrt{M^2 - Q^2}}{2\pi r_+^2 \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}}, \tag{5.64}$$

which is Eq. (5.25). In other words, we have shown that our method, which relies on the comparison of the solutions to the Klein-Gordon equation from the points of views of two observers close to the horizons, reproduces the familiar result which is usually obtained by means of the comparison of the solutions to the Klein-Gordon equation at $\Im^+$ and $\Im^-$. Encouraged by this welcome outcome of our analysis, we now proceed to apply our method for an analysis of the properties of the inner horizon of the Reissner-Nordström black hole.

5.4 Hawking Effect for the Inner Horizon of the Reissner-Nordström Black Hole

An analysis of the radiation emitted by the inner horizon of a Reissner-Nordström black hole can now be performed in a very similar way as that of the outer horizon. In essence, the ingoing and the outcoming solutions to the Klein-Gordon equation, when written in terms of the coordinates t, r, θ, and φ, can be obtained directly from Eqs. (5.36). However, since the radiation is now directed towards the singularity $r = 0$, the observer at rest with respect to the inner horizon must be situated *inside* the two-sphere $r = r_-$. Therefore, the roles of the ingoing and the outcoming modes interchange. More precisely, the solutions representing particles with positive energy ω are, from the point of view of the observer at rest very close to the inner horizon,

$$\phi_{\text{in}} \approx N_{\omega lm} Y_{lm} \, \frac{1}{r} \, e^{-i\omega U}, \tag{5.65a}$$

$$\phi_{\text{out}} \approx N_{\omega lm} Y_{lm} \, \frac{1}{r} \, e^{-i\omega V}. \tag{5.65b}$$

The solution ϕ_{in} represents a particle which moves towards the horizon, and therefore away from the singularity. The solution ϕ_{out}, in turn, represents a particle which moves out of the horizon, and therefore towards the singularity.

As it comes to the freely falling observer near the inner horizon, we cannot use the same geodesic coordinate system as we did in the previous section. This is a consequence of the fact that the Kruskal-type coordinates u and v of Eqs. (5.37) with the choice $\alpha = \kappa_+$ lead to the metric which is not regular at $r = r_-$. A remedy to this problem can be obtained by choosing

$$\alpha = \kappa_- := -\frac{r_+ - r_-}{2r_-^2} \tag{5.66}$$

and defining a new geodesic system of coordinates based on this choice. Consider now an observer in a radial free fall in the region VI' of Reissner-Nordström spacetime infinitesimally close to the point P_2 (see Fig. 5.5). When written in terms of the coordinates u and v, the spacetime metric takes the form

$$ds^2 = \frac{1}{\kappa_-^2 r^2} e^{-2\kappa_- r} (r_+ - r)^{\frac{r_+^2}{r_-^2}+1} (-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2). \tag{5.67}$$

In this expression, the coordinates u and v have been defined in such a way that in the regions V', IV', VI' and II of Fig. 5.1.(b), respectively, the coordinates u and v are given in terms of U and V by Eqs. (5.37a), (5.37b), (5.37c), and (5.37d). One may also easily check that in the region VI' the coordinates u and v are increasing functions of r and t, respectively. More precisely, when u is taken to be a constant, the coordinate v increases as a function of t, whereas the coordinate u increases as a function of r, when v is constant.

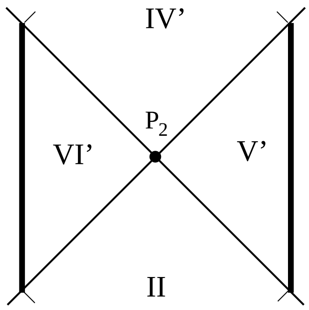

Figure 5.5. Part of Reissner-Nordström spacetime. The point P_2 is situated at the intersection of the lines that separate the regions V', IV', VI' and II.

Similarly as in the case of the outer horizon, we define a geodesic system of coordinates for an infinitesimal neighborhood of the point P_2 such that

$$X^0 := l_- v, \tag{5.68a}$$

$$X^1 := l_- u, \tag{5.68b}$$

where

$$l_- := \frac{1}{|\kappa_-| r_-} e^{-\kappa_- r_-} (r_+ - r_-)^{\frac{1}{2}\left(\frac{r_+^2}{r_-^2}+1\right)}. \tag{5.69}$$

When the remaining coordinates are chosen to be the spherical coordinates θ and φ, the metric is given by Eq. (5.43), and the derivatives of the metric vanish when $r = r_-$ (see Appendix A). Therefore, the coordinates X^0 and X^1 provide a geodesic system of coordinates. Furthermore, when the massless Klein-Gordon field is separated as in Eq. (5.44), one finds that the massless Klein-Gordon equation implies

$$\left[-\frac{\partial^2}{\partial(X^0)^2} + \frac{\partial^2}{\partial(X^1)^2} + \left(\frac{2M}{r^3} - \frac{2Q^2}{r^4} - \frac{l(l+1)}{r^2}\right) F_-(r)\right] \widetilde{f}(X^0, X^1) = 0, \tag{5.70}$$

where we have defined

$$F_-(r) := \frac{1}{\kappa_-^2 l_-^2} \frac{1}{r^2} e^{-2\kappa_- r} (r_+ - r)^{\frac{r_+^2}{r_-^2}+1}. \tag{5.71}$$

Again, one finds that

$$F_-(r_-) = 1, \tag{5.72}$$

and therefore Eq. (5.70) takes, at the point P_2, the form

$$\left[-\frac{\partial^2}{\partial(X^0)^2} + \frac{\partial^2}{\partial(X^1)^2} + \frac{2M}{r_-^3} - \frac{2Q^2}{r_-^4} - \frac{l(l+1)}{r_-^2} \right] \widetilde{f}(X^0, X^1) = 0. \tag{5.73}$$

The question about whether the terms involving r_- are negligibly small or not, is a very delicate one, and when the absolute value of the electric charge Q is very small, those terms will certainly *not* vanish. We may, however, consider a special case where $|Q|$ is "reasonably big". More precisely, we shall assume that there is a fixed positive number $\gamma \leq 1$ such that between $|Q|$ and M there is, in Planck units, the relationship:

$$|Q| = \gamma M. \tag{5.74}$$

In that case

$$r_- = (1 - \sqrt{1-\gamma^2})M, \tag{5.75}$$

and because

$$\sqrt{1-\gamma^2} = 1 - \frac{1}{2}\gamma^2 - \frac{1}{8}\gamma^4 - \frac{1}{16}\gamma^6 - \cdots < 1 - \frac{1}{2}\gamma^2, \tag{5.76}$$

we find that

$$\left| \frac{2M}{r_-^3} - \frac{2Q^2}{r_-^4} \right| \leq \frac{2M}{r_-^3} + \frac{2Q^2}{r_-^4} < \frac{48}{\gamma^6}\frac{1}{M^2}. \tag{5.77}$$

Hence, it follows that if γ is "reasonably big", and $M \gg 1$ in Planck units, the terms involving M and Q are negligible. The same line of reasoning implies that for "sufficiently small" l the term $l(l+1)/r_-^2$ may be neglected, and Eq. (5.73) may be written, in effect, in the form

$$\left[-\frac{\partial^2}{\partial(X^0)^2} + \frac{\partial^2}{\partial(X^1)^2} \right] \widetilde{f}(X^0, X^1) = 0. \tag{5.78}$$

One easily sees that in the region VI', the solutions corresponding to the particles with positive energy ω going in and out of the horizon are

$$\phi'_{\text{in}} \approx N_{\omega lm} Y_{lm} \frac{1}{r} e^{-i\omega\widetilde{u}}, \tag{5.79a}$$

$$\phi'_{\text{out}} \approx N_{\omega lm} Y_{lm} \frac{1}{r} e^{-i\omega\widetilde{v}}, \tag{5.79b}$$

where $\widetilde{v}$ and $\widetilde{u}$ are defined as in Eqs. (5.57).

It is now possible to write the Bogolubov transformation between the outcoming solutions (5.65b) and (5.79b) in the spherically symmetric case. One easily finds that in the region VI',

$$V = \kappa_-^{-1} \ln(-\widetilde{v}) - \kappa_-^{-1} \ln l_-. \tag{5.80}$$

Moreover, the Bogolubov transformation takes the form

$$e^{-i\omega\kappa_-^{-1}\ln(-\widetilde{v})}\,e^{i\omega\kappa_-^{-1}\ln l_-} = \sum_{\omega'}\Big(A'_{\omega\omega'}e^{-i\omega'\widetilde{v}} + B'_{\omega\omega'}e^{i\omega'\widetilde{v}}\Big), \tag{5.81}$$

and the Bogolubov coefficients can be expressed as

$$A'_{\omega\omega'} = \frac{1}{2\pi}e^{i\omega\kappa_-^{-1}\ln l_-}\int_{-\infty}^{0} d\widetilde{v}\,e^{-i\omega\kappa_-^{-1}\ln(-\widetilde{v})}e^{i\omega'\widetilde{v}}, \tag{5.82a}$$

$$B'_{\omega\omega'} = \frac{1}{2\pi}e^{i\omega\kappa_-^{-1}\ln l_-}\int_{-\infty}^{0} d\widetilde{v}\,e^{-i\omega\kappa_-^{-1}\ln(-\widetilde{v})}e^{-i\omega'\widetilde{v}}. \tag{5.82b}$$

It should be clearly noted, however, that so far our considerations have been based on the assumption that in the local neighborhood of the inner horizon one has a physically meaningful concept of particle. In other words, in our considerations we have interpreted the solutions (5.65) and (5.79) of the Klein-Gordon field as particle states which can be detected by certain observers located near the inner horizon, and between these solutions we have then written the Bogolubov transformation. Although such interpretation is indeed very natural, the situation may become problematic, since the time evolution of those particle states eventually forces us to deal with the singularity. In the Reissner-Nordström spacetime this issue is especially challenging, because in that case the boundary conditions of the Klein-Gordon field cannot be uniquely defined at the singularity [66]. This induces some loss of predictability in the evolution of our quantum states, because it is not clear how the solutions representing particles at the horizon evolve at "later times". Despite these ambiguities in the dynamics of the Klein-Gordon field, we shall not discuss these issues here in more detail. After all, in our analysis we are interested in the behaviour of the Klein-Gordon field at the inner horizon only, in which case it should be reasonable to interpret the field as particles propagating in spacetime—even if the ultimate fate of these quantum states remains unspecified. By accepting these assumptions, we proceed to calculate the effective temperature of the particle flux emitted by the inner horizon. As before, the Bogolubov coefficients yield the result

$$\big|A'_{\omega\omega'}\big| = e^{-\pi\kappa_-^{-1}\omega}\big|B'_{\omega\omega'}\big|. \tag{5.83}$$

Therefore, by using Eq. (5.19), we see that the number of particles with energy ω is, from the point of view of the observer at rest very close to the horizon $r = r_-$, when, from the point of view of the freely falling observer, the field is in vacuum,

$$n_\omega = \sum_{\omega'}\big|B'_{\omega\omega'}\big|^2 = \frac{1}{e^{-2\pi\kappa_-^{-1}\omega} - 1}. \tag{5.84}$$

From this distribution, one may infer that the temperature concerning the particle radiation emitted by the inner horizon is, when the redshift effects are ignored,

$$T = -\frac{\kappa_-}{2\pi}, \tag{5.85}$$

which is *positive*.

The result in Eq. (5.85) is in agreement with the findings of Ref. [62]. Since the temperature is positive, there are no interpretative problems concerning the thermodynamical properties of the radiation of the inner horizon. Again, the redshift factor can be recovered by using Eq. (5.21), and as the result one finds that the temperature, from the point of view of an observer at rest very close to the inner horizon, is

$$T_- := -\frac{\kappa_-}{2\pi\Delta} = \frac{\sqrt{M^2 - Q^2}}{2\pi r_-^2\sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}}. \tag{5.86}$$

As we can see, our expression for the temperature of the particles emitted by the inner horizon inside the inner horizon is very similar to Eq. (5.64), which gives the temperature of the particles emitted by the outer horizon outside the outer horizon. The only difference is that r_+ has been replaced by r_-. Furthermore, our result that the inner horizon emits particles inside the inner horizon with a positive temperature given by Eq. (5.86) has also a very interesting consequence: To maintain a local energy balance it is necessary that when particles with positive energy ω are emitted towards the singularity from the inner horizon, particles with energy $-\omega$ are emitted *away from the singularity* through the inner horizon.[2] The process is similar to the one which, according to the Hawking effect, takes place at the outer horizon of the Reissner-Nordström black hole: At the outer horizon negative energy particles go in and positive energy particles come out, and now we found that this is true at the inner horizon as well. According to our best knowledge this phenomenon, despite of its apparent triviality, has not been noticed before.

An intriguing question now arises: What happens to the negative energy particles which are emitted away from the singularity through the inner horizon?[3] An answer to this question appears to be rather difficult. Consider the conformal diagram of the maximally extended Reissner-Nordström spacetime of Fig. 5.6, where we have drawn the worldlines of the members of a virtual pair created at the inner horizon. The particle with positive energy, as we found in our analysis, remains inside the inner horizon, and finally meets with the black hole singularity (whatever that means). The particle with negative energy, in turn, enters the intermediate region between the horizons and one could—if the backscattering effects are neglected—speculate on the possibility that it travels across the intermediate region and finally comes out from the *white hole horizon*. A closer look, however, casts much doubt whether this

[2]Note that for the Klein-Gordon field, particles are their own antiparticles.

[3]For the sake of simplicity, at this point the word 'particle' shall be used to describe certain solutions to the Klein-Gordon equation also in the region IV'. More precisely, by 'particle' we mean here such a solution to the Klein-Gordon equation, which propagates along a null geodesic in the region IV', and which may be interpreted as an actual particle at the local neighborhood of the inner horizon, from the point of view of an observer at rest with respect to the horizon.

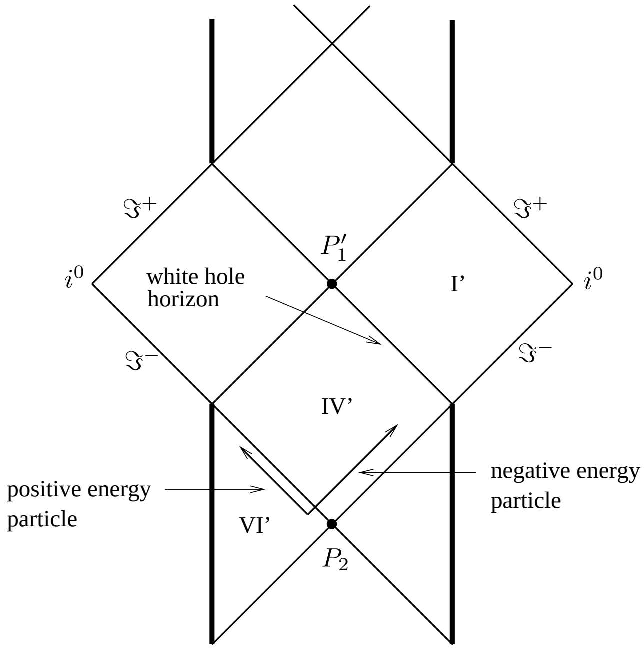

Figure 5.6. Worldlines of the members of a virtual pair created at the inner horizon. The particle with positive energy meets with the black hole singularity, whereas the particle with negative enters the intermediate region between the horizons.

could ever happen in the Reissner-Nordström spacetime. The heart of the problem lies at the ambiguity of the vacuum states in the Reissner-Nordström spacetime: It seems impossible to define a vacuum state that would give a finite energy-momentum tensor both at the inner horizon and at the outer horizon [67]. Indeed, this appears to be the case also in our analysis. To consider the radiation effects of the inner horizon, we introduced a vacuum state which was regular at the inner horizon but not at the outer horizon. As a consequence, the energy density of any matter or radiation travelling from the region VI' into the region IV' probably blows up as it approaches the outer horizon. This should prevent particles ever passing through the white hole horizon. Therefore, the conclusion seems to be that the radiation effects of the inner horizon cannot be observed from outside of the Reissner-Nordström black hole because the negative energy particles emitted by the inner horizon probably pile up along the white hole horizon without limit. It should be noted, however, that the situation may be different if we consider a Reissner-Nordström black hole which is not asymptotically flat but asymptotically de Sitter. It was proposed in Ref. [67] that for such black holes it may be possible to overcome the problems related to the vacuum states so that matter could actually travel through the interior regions of the black hole and finally come out of the corresponding white hole. Hence one may speculate on the possibility that in the asymptotically de Sitter Reissner-Nordström spacetime an observer situated outside of the black hole may observe a certain type of radiation coming out of the white hole which is due to the Hawking effect at the inner horizon of the black hole. However, because the vacuum states of the observers located at the neighbourhoods of the inner and the outer horizon would be completely different, it is not certain whether that radiation could be interpreted as a flux of (actual) particles.

5.5 Concluding Remarks

In this chapter, we have found that in maximally extended Reissner-Nordström spacetime both the inner and the outer horizons emit thermal radiation, which, when the possible backscattering effects are neglected, obeys the normal blackbody spectrum. We obtained our results by means of an analysis which was similar to the normal derivation of the Unruh effect. More precisely, we considered the quantum-mechanical properties of the massless Klein-Gordon field in the vicinity of the horizons from the points of views of two different observers. One of these observers was at rest with respect to the Reissner-Nordström coordinates either just outside the outer horizon or inside the inner horizon, whereas another observer was in a free fall through the horizon. We found that an observer at rest observes particles even though, from the point of view of an observer in a free fall, the field is in vacuum. The observer at rest just outside the outer horizon observes outcoming particles with positive energy. Similarly, an observer at rest just inside the inner horizon observes particles with positive energy propagating towards the singularity.

The most remarkable result of our analysis is that, in contrast to common beliefs, the inner horizon is not a passive spectator but an active participant in the radiation processes of the Reissner-Nordström black hole [68]. Although this result is based on an almost trivial observation that both of the horizons of the Reissner-Nordström black hole emit particles, there may also be some element of surprise in it, and therefore the first question concerns the physical and the mathematical validity of our analysis. After all, we did not follow the usual route with an analysis based on a comparison of the solutions to the Klein-Gordon equation in the past and in the future null infinities. Since this kind of an analysis would have been impossible to perform when considering the radiation emitted by the inner horizon, we instead compared the solutions in the rest frames of two observers. Is this kind of approach valid?

The physical validity of this kind of approach has been considered, in the case of the Schwarzschild black hole, by Unruh, and similar arguments also apply here [64]. The best argument in favor of the validity of our analysis is probably given by the fact that exactly the same methods which were used in an analysis of the radiation of the inner horizon produced the well-known results for the radiation emitted by the outer horizon. Another problem is that we have simply ignored all possible backscattering effects. To consider such effects, one should perform a numerical analysis of the solutions to the Klein-Gordon equation. However, as mentioned in Sec. 5.1, it is expected that the backscattered particles will trigger the mass inflation, and therefore the radiation effects of the inner horizon of the Reissner-Nordström black hole are of very brief duration. As a consequence, the full Reissner-Nordström spacetime, even as a mathematical solution, is unstable at the semiclassical limit. Since the Reissner-Nordström spacetime is not considered astrophysically relevant, we do not expect our analysis to have direct astrophysical consequences. If, however, our results are qualitatively the same for more realistic Kerr black holes, the phenomena discussed in this chapter might possess some astrophysical significance. Nevertheless, the radiation effects of the inner horizon have much importance in their own right since they support the idea that all horizons of spacetime emit radiation.

Part III

Gravitational Entropy

Chapter 6

Interlude: Quantum-Mechanical Models of Black Holes

In this chapter, we shall deviate slightly from the main subjects of this thesis and discuss quantum-mechanical models of black holes. These models will be needed in the analysis of the chapter 7, where microscopic black holes will be taken as the building blocks of spacetime. Our main interest lies in the model developed by Louko and Mäkelä [69] for the Schwarzschild black hole. After introducing this model, we shall return back to the thermodynamics of black holes.

6.1 Hamiltonian Dynamics of the Schwarzschild Black Hole

When constructing quantum-mechanical models of black holes, the first task is to consider the Hamiltonian dynamics of spacetimes containing a black hole. The spacetimes containing a Schwarzschild black hole have been analyzed by Kuchař [70], and in what follows, we shall give a brief review of his results.

The starting point in Kuchař's work was the ADM formulation of general relativity, which was briefly discussed in Sec. 2.2. The general spherically symmetric ADM line element can be written as

$$ds^2 = -N^2(r,t)dt^2 + \Lambda^2(r,t)\big(dr + N^r(r,t)\,dt\big)^2 + R^2(r,t)d\Omega^2, \tag{6.1}$$

where $d\Omega^2$ is the metric on the unit two-sphere and Λ^2 and R^2 denote two independent components of the spatial metric q_{ab}. On each spacelike hypersurface of constant t, denoted by Σ_t, the metric is therefore given by the functions Λ and R, which we take to be the configuration variables of the theory. Both of these variables are chosen to remain positive, and therefore R is the radius of curvature of the two-sphere $r =$ constant. For the Schwarzschild solution, R simply represents the radial Schwarzschild coordinate. The function N is clearly the lapse function, and due to the spherical symmetry only the radial component N^r of the shift vector survives. In general, Eq. (6.1) describes the metric of any (globally hyperbolic) spherically symmetric spacetime, including, of course, the Schwarzschild spacetime. However, one should note that an eternal black hole has two spatial infinities rather than just one, corresponding to both the left and the right infinities of the conformal diagram. Hence, the coordinate r ranges from $-\infty$ to ∞.

From the metric (6.1), one can calculate the exterior curvature K_{ab} and the curvature scalar $\mathcal{R}$ of the hypersurface Σ_t. When these quantities are substituted into Eq. (2.19), one obtains the ADM action of spherically symmetric spacetimes. Obviously, the Hamiltonian form of this action reads

$$S = \int dt \int_{-\infty}^{\infty} dr \left(p_\Lambda \dot{\Lambda} + p_R \dot{R} - N\mathcal{H} - N^r \mathcal{H}_r\right), \tag{6.2}$$

where p_Λ and p_R are the canonical momenta of the variables Λ and R, and $\mathcal{H}$ and $\mathcal{H}^r$ are the super-Hamiltonian and the supermomentum, respectively. The limits where $r \to -\infty$ and $r \to \infty$ correspond, respectively, to the left and the right spatial infinities of the conformal diagram. For the explicit form of the canonical momenta and the constraint functions, the reader is encouraged to peruse Ref. [70]. This action, however, must be supplemented with the appropriate ADM boundary terms. In Ref. [70] Kuchař adopts certain *falloff conditions* which determine the asymptotic behaviour of the metric and the field variables. These conditions ensure that the spacetime is asymptotically flat, and that the radial coordinate coincides with the Minkowski radial coordinate at the both infinities. It follows from the theorem due to Birkhoff [71] that the only asymptotically flat spherically symmetric vacuum solution of Einstein's field equation is the Schwarzschild solution. Hence, if the falloff conditions are employed, the action (6.2) describes a Schwarzschild black hole, and it should be supplemented with the boundary terms appearing in the maximally extended Schwarzschild spacetime. That is, the boundary contributions are written as

$$S_{\partial\Sigma} = -\int dt\big(N_+ M_+ + N_- M_-\big), \tag{6.3}$$

where $N_-(t)$ and $N_+(t)$ are the asymptotic values of the lapse function at the left and at the right hand side spacelike infinities of the Kruskal manifold, respectively. It is required that $N_\pm$ are some prescribed functions of t which cannot be varied. The quantities $M_\pm(t)$, in turn, are both equal to the Schwarzschild mass.

The highlight of this story is that the Hamiltonian theory given above can be reduced to its true degrees of freedom. It was shown by Kuchař that by cleverly chosen canonical transformation, followed by the reduction by solving the constraints and substituting the solutions back into the action, one is left with the reduced action

$$S = \int dt \left(\mathbf{p}\dot{\mathbf{m}} - (N_+ + N_-)\mathbf{m}\right), \tag{6.4}$$

where the new canonical variables $(\mathbf{m}, \mathbf{p})$ depend on t only. This means that the Schwarzschild black hole has, as expected, only one degree of freedom, $\mathbf{m}$, which is the mass of the black hole. Thus, the reduced theory is no more a theory of fields but it is a theory of finite number of degrees of freedom. The variable $\mathbf{m}$ is obviously a positive quantity, but the corresponding canonical momentum $\mathbf{p}$ takes the values within the interval $(-\infty, \infty)$.

The dynamical content of the theory is expressed by the Hamiltonian equations of motion:

$$\dot{\mathbf{m}} = 0, \tag{6.5a}$$

$$\dot{\mathbf{p}} = -N_+ - N_-. \tag{6.5b}$$

It follows from the no-hair theorems that all the information about the spacetime geometry is carried by the variable $\mathbf{m}$. However, it can be easily seen from Eq. (6.5b) that

$$\mathbf{p} = -\big(T_+ - T_-\big), \tag{6.6}$$

where T_+ and T_- are, respectively, the asymptotic Minkowski time coordinates on a hypersurface Σ_t at the right and at the left hand side infinities (here we use the typical convention, where the Minkowski time at the right (left) hand side increases (decreases) towards the future). Therefore, although the variable $\mathbf{p}$ does not contain any information about the local spacetime geometry, it does contain information of how the spacelike hypersurfaces Σ_t are "anchored" at the two infinities.

6.2 Hamiltonian Throat Theory

As we proceed towards the quantum theory of the Schwarzschild black hole, we impose certain restrictions on the reduced action (6.4). To begin with, we shall confine ourselves to a special case, where $N_+ = 1$ and $N_- = 0$. In other words, we make the parameter time t to coincide, up to some additive constant, with the Minkowski time at the right hand side asymptotic infinity and freeze the time evolution of the hypersurfaces Σ_t at the left hand side asymptotic infinity. The latter restriction follows from the requirement that our theory should describe physics accessible to observers at one infinity only. It follows now that

$$S = \int dt\,(\mathbf{p}\dot{\mathbf{m}} - \mathbf{m}). \tag{6.7}$$

Besides the restrictions mentioned above, we shall also require that $|\mathbf{p}| < \pi\mathbf{m}$. As a corollary, the asymptotic Minkowski time, as well as the parameter time t, take their values within an interval of length $2\pi\mathbf{m}$ at the right hand side infinity. In the case of the parameter time t, these values are centered around a certain instant of time, which we shall denote here by t_0. Therefore, the condition $|\mathbf{p}| < \pi\mathbf{m}$ confines the values of t between the interval $-\pi\mathbf{m} < t - t_0 < \pi\mathbf{m}$.[1] Although at this point this restriction may seem completely arbitrary, we shall soon see how it follows naturally from the geometrical interpretation of the forthcoming theory.

[1]Note that the time instances $t = t_0 - \pi\mathbf{m}$ and $t = t_0 + \pi\mathbf{m}$ correspond now to the values $\mathbf{p} = \pi\mathbf{m}$ and $\mathbf{p} = -\pi\mathbf{m}$, respectively.

Let us now perform a transformation from the canonical variables $(\mathbf{m}, \mathbf{p})$ to a new pair of variables (a, p_a) such that

$$
\begin{aligned}
|\mathbf{p}| &= \int_a^{2\mathbf{m}} \frac{dx}{\sqrt{2\mathbf{m}x^{-1} - 1}} \\
&= \sqrt{2\mathbf{m}a - a^2} + \mathbf{m}\arcsin(1 - a/\mathbf{m}) + \frac{1}{2}\pi\mathbf{m}, \qquad (6.8a) \\
p_a &= \operatorname{sgn}(\mathbf{p})\sqrt{2\mathbf{m}a - a^2}. \qquad (6.8b)
\end{aligned}
$$

One can show that this transformation is canonical, and the action becomes

$$
S = \int dt\,(p_a\dot{a} - H), \tag{6.9}
$$

where the Hamiltonian is

$$
H = \frac{1}{2}\Big(\frac{p_a^2}{a} + a\Big). \tag{6.10}
$$

One easily sees that this Hamiltonian equals $\mathbf{m}$. Moreover, from Eq. (2.6a) one finds that the equation of motion for a takes the form

$$
\dot{a} = \frac{p_a}{a} = \operatorname{sgn}(\mathbf{p})\sqrt{2\mathbf{m}a^{-1} - 1} \tag{6.11}
$$

In other words, when $\mathbf{p} > 0$, the variable a increases with time, and when $\mathbf{p} < 0$, a decreases with time. It is also obvious from the transformations (6.8) that

$$
\begin{aligned}
a &\xrightarrow{\mathbf{p}\to\pm\pi\mathbf{M}} 0, \qquad (6.12a) \\
a &\xrightarrow{\mathbf{p}\to 0} 2\mathbf{M}. \qquad (6.12b)
\end{aligned}
$$

Hence we conclude that a begins from zero at $t = t_0 - \pi\mathbf{m}$, and increases until it reaches its maximum value $2\mathbf{m}$ at $t = t_0$. After that a begins to decrease and finally vanishes at $t = t_0 + \pi\mathbf{m}$.

We are now prepared to seek a geometrical interpretation of the variables a and p_a. First we note that the Kuchař's analysis was based on the properties of the spacelike hypersurfaces Σ_t at the asymptotic infinities only, leaving the hypersurfaces in other ways arbitrary. Therefore, we are allowed to interpret our configuration variables according to a suitable foliation Σ_t. Our choice of the foliation is now motivated by the following observation. We write the equation of motion for the variable a in the form

$$
\dot{a}^2 = \frac{2\mathbf{m}}{a} - 1. \tag{6.13}
$$

This equation is similar to the equation of a radial timelike geodesic going through the *bifurcation two-sphere*[2] in Kruskal spacetime, assuming that a is the curvature radius along the geodesic and the dot denotes the proper time derivative. This is

[2]The bifurcation two-sphere is located at the intersection of the past and the future horizons of the Kruskal diagram.

easily seen from the Schwarzschild metric. Inside the black hole, the metric of the radial timelike geodesic is

$$ds^2 = -d\tau^2 = \frac{dr^2}{1 - \frac{2M}{r}}, \tag{6.14}$$

and this equation leads directly to

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2M}{r} - 1. \tag{6.15}$$

Thus, if we can find a certain foliation Σ_t such that the proper time on the radial timelike geodesic inside the black hole equals the value of t at the infinity, the variable a becomes the radius of curvature along this geodesic. An interesting fact is that these kinds of foliations do exist (see Ref. [69] for details). It is even possible to make a choice such that the radius of curvature on the hypersurface Σ_t reaches its minimum value when it intersects the radial geodesic inside the hole. In this particular case the quantity a can be identified as the *throat radius* of the *Einstein-Rosen wormhole*[3]. From this point on, we shall therefore refer to a as the radius of the wormhole throat and to the corresponding theory as the Hamiltonian throat theory. The geometrical interpretations are illustrated in Fig. 6.1.

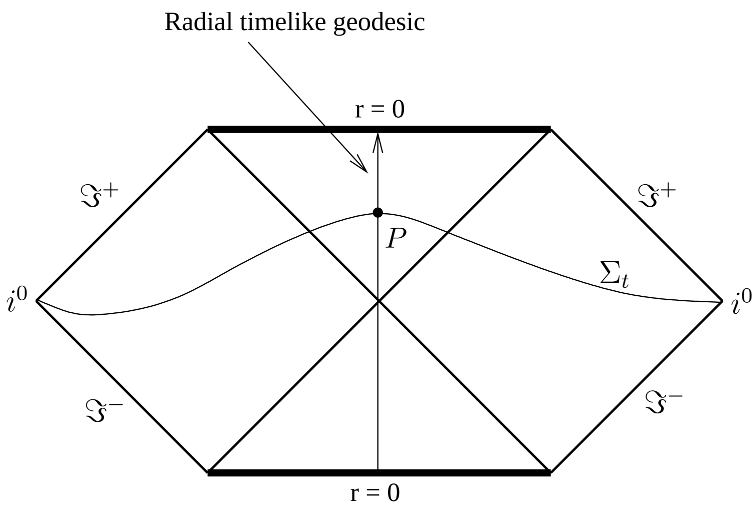

Figure 6.1. Geometrical interpretation of the configuration variable a. In this picture, the vertical line represents the radial timelike geodesic through the bifurcation two-sphere. If the curvature radius on Σ_t attains its minimum value at the intersection point P, then the variable a may be interpreted as the throat radius of the Einstein-Rosen wormhole Σ_t. The time interval $t - t_0$ equals the proper time τ elapsed on the radial geodesic assuming that $\tau = 0$ at the bifurcation two-sphere.

[3] A three-dimensional spacelike hypersurface in Kruskal spacetime which does not meet the singularities is called the Einstein-Rosen wormhole, or the Einstein-Rosen bridge. The minimum value of the curvature radius on this hypersurface is known as its throat radius.

The geometrical interpretation of the variable a gives now a natural explanation why the values of $\mathbf{p}$ were initially restricted between $-\pi\mathbf{m}$ and $\pi\mathbf{m}$: The radial timelike geodesic begins its life at the past singularity, goes through the bifurcation two-sphere, and finally ends its life at the future singularity such that the elapsed proper time along the geodesic equals $2\pi\mathbf{m}$. Therefore, the configuration variable a lives, in a certain sense, inside the hole only. That is to say, in the Hamiltonian throat theory, the spacetime dynamics is bounded inside the spacetime region containing the black hole (and the white hole). This feature is in harmony with the fact that the spacetime metric is static outside and non-stationary inside the Schwarzschild black hole.

6.3 Quantization

In the previous section, we stated that the Hamiltonian (6.10) equals the Schwarzschild mass. Therefore, the Hamiltonian may be seen as the energy of the spacetime from the point of view of an inertial observer at the asymptotic infinity. This gives us a reason to conclude that the corresponding Hamiltonian operator is indeed an energy operator (with respect to the asymptotic Minkowski frame). We now proceed to the quantization of the classical Hamiltonian throat theory.

Let the state vectors ψ of the quantum theory depend on the configuration variable a. These state vectors live in the Hilbert space $\mathbf{H} := L^2(\mathbb{R}^+; \mu da)$, where the inner product is defined as

$$\langle \psi_1, \psi_2 \rangle := \int_0^\infty \psi_1^* \psi_2 \, \mu da, \tag{6.16}$$

and $\mu(a)$ is a smooth positive weight function. We obtain the Hamiltonian operator $\hat{H}$ by substituting $p_a \to -i\frac{d}{da}$ in Eq. (6.10) such that the ordering of the factors yields a hermitian operator with respect to the inner product (6.16). We get:

$$\hat{H} = \frac{1}{2}\Big[-\frac{1}{\mu}\frac{d}{da}\Big(\frac{\mu}{a}\frac{d}{da}\Big)\Big]. \tag{6.17}$$

Naturally, the energy spectrum of $\hat{H}$ depends on the weight function $\mu(a)$. The general power law weight function has been studied in Ref. [69]. In that case, the results show that the energy spectrum is discrete and bounded from below. Hence there exists a certain ground state in energy. In physical sense, this indicates that one cannot extract an infinite amount of energy from the quantum black holes. In many cases the ground state is positive but in some situations the possibility of negative ground states cannot be excluded. However, in every case one can choose the *self-adjoint extension*[4] of $\hat{H}$ such that the ground state becomes positive. For large eigenvalues,

[4]A hermitian operator $\hat{A}$ is said to be *self-adjoint* if its domain $D(\hat{A}) \in \mathbf{H}$ is equal to the domain of its hermitian conjugate, $D(\hat{A}^\dagger) \in \mathbf{H}$. If there is a way to extend a hermitian operator $\hat{A}$ into a self-adjoint operator $\hat{A}'$, we say that $\hat{A}'$ is the self-adjoint extension of $\hat{A}$. If the extension is unique,

the *WKB analysis*[5] yields the asymptotic estimate

$$E_{\mathrm{WKB}} \sim \sqrt{2n}, \tag{6.18}$$

where n is an integer. The resulting area spectrum of the event horizon,

$$A \sim 32\pi n, \tag{6.19}$$

is proportional to the quantum number n. This kind of discrete area spectrum agrees with that proposed by Bekenstein in 1974. In brief, he proposed that the eigenvalues of the area of the event horizon of a black hole are of the form

$$A_n = n\gamma\, l_{\mathrm{Pl}}^2, \tag{6.20}$$

where n is an integer and γ is a pure number of order 1 [74]. In other words, Bekenstein's proposal states that the event horizon area of a black hole has an equal spacing in its spectrum.

Analogous results for the energy spectrum hold also in the presence of a negative cosmological constant, electric charge and angular momentum. The spacetimes with a negative cosmological constant are not asymptotically flat but the asymptotically flat infinities are replaced by asymptotically anti-de Sitter infinities [75, 76]. It turns out that the results are qualitatively independent of the existence of the negative cosmological constant. The Reissner-Nordström black holes have been analyzed in Refs. [69,77], and the Kerr-Newman black holes in Ref. [78]. Again, the energy spectra are discrete but, in contrast to the Schwarzschild black hole, the large eigenvalues of the area of the outer horizon do not follow the Bekenstein's proposal. However, if one considers the sum of the areas of the inner and the outer horizon, one concludes that this "total area" of the black hole is quantized in the manner proposed by Bekenstein.

---

the operator $\hat{A}$ is called essentially self-adjoint. For further information see, for instance, Ref. [72].

[5]The WKB approximation provides a method to find approximative solutions to differential equations [73].

Chapter 7

Spacetime Foam Model of the Schwarzschild Horizon

In this house, we OBEY the laws of thermodynamics!

Homer Simpson [79]

In the inspired final chapter of their classic book, Misner, Thorne, and Wheeler state that there are three levels of gravitational collapse [19]. The first two of them are the gravitational collapse of the whole universe during the final stages of its recontraction, and the gravitational collapse of a star when a black hole is formed. The third level of gravitational collapse is the quantum fluctuation of spacetime geometry at the Planck scale of distances. To rephrase Misner, Thorne, and Wheeler, "collapse at the Planck scale of distances is taking place everywhere and all the time in quantum fluctuations in the geometry and, one believes, the topology of space. In this sense, collapse is continually being done and undone,..."

The picture given by Misner, Thorne, and Wheeler of the Planck scale physics in these sentences is very charming. It immediately brings into one's mind mental images of tiny wormholes and black holes furiously bubbling as a sort of spacetime foam. This is indeed a wonderful picture but unfortunately the ideas it suggests have never been taken very far. Instead of developing models of spacetime where spacetime consists of tiny wormholes and black holes, researchers in the field of quantum gravity have gone in another direction and developed, among other things, a very successful theory of loop quantum gravity which treats spacetime as a spin network constituted of tiny loops.

The purpose of this chapter is, in a certain sense, to revive some aspects of the old picture of spacetime as a foam of tiny wormholes and black holes. Actually, there are reasons to believe that at the Planck length scale microscopic black holes might indeed play a role in the structure of spacetime. For instance, suppose that we want to localize a particle within a cube with an edge length equal to one Planck

length
$$l_{\rm Pl} := \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35}\ \rm m. \tag{7.1}$$
It follows from Heisenberg's uncertainty principle that in this case the momentum of the particle has an uncertainty $\Delta p \sim \hbar/l_{\rm Pl}$. In the ultrarelativistic limit the uncertainty in the energy of the particle is $\Delta E \sim c\Delta p$, which is around one Planck energy,
$$E_{\rm Pl} := \sqrt{\frac{\hbar c^5}{G}} \approx 2.0 \times 10^{9}\ \rm J. \tag{7.2}$$
In other words, we have enclosed one Planck energy inside a cube whose diameter is one Planck length. That amount of energy, however, is enough to shrink the spacetime region bounded by the cube into a black hole with a Schwarzschild radius equal to around one Planck length. So it seems possible that when probing the structure of spacetime at the Planck length scale one encounters Planck size black holes.

At the present stage of research, a construction of a precise mathematical model of spacetime as a whole made of tiny black holes is out of reach. However, it is possible to test the idea of spacetime as a foam of microscopic black holes in the context of an important specific problem of quantum gravity. The problem in question is the microscopic origin of black hole entropy. There is general agreement between the researchers working in the field of quantum gravity that black hole has an entropy which is, in natural units, one-quarter of its event horizon area. This result, sometimes known as the Bekenstein-Hawking entropy law, was introduced in Sec. 4.1. In addition to black holes, it is valid for cosmological, de Sitter, and Rindler horizons as well, and it seems to imply that in addition to the three classical ones, there is an enormous number of quantum-mechanical degrees of freedom in black holes. It is reasonable to expect that these additional degrees of freedom lie at the horizon of the black hole. During the recent ten years or so, several attempts have been made by experts in string theory and loop quantum gravity to identify the quantum-mechanical degrees of freedom of black holes and to provide a microscopic explanation for the black hole entropy [7].

To test the viability of the idea of spacetime as a foam of Planck size black holes we postulate, in this chapter, a specific spacetime foam model of the event horizon of a Schwarzschild black hole [80]. It follows from our model that the entropy of the Schwarzschild black hole is proportional to its horizon area, and we also present arguments suggesting that the constant of proportionality must be, in natural units, equal to one-quarter. In other words, it seems possible to obtain Eq. (4.2) from our model.

This chapter is organized as follows. In Sec. 7.1, we introduce our model, especially the postulates on which it is based. In short, the event horizon of a macroscopic Schwarzschild black hole is assumed to consist of Planck size, independent black holes. Each of the microscopic holes on the horizon is assumed to obey a sort of "Schrödinger equation" of black holes, which was published in Ref. [69]. The postulates of our model connect the quantum states of individual microscopic holes on the horizon with the area eigenvalues of the event horizon of the macroscopic Schwarzschild black hole.

In Sec. 7.2, we proceed to calculate the entropy of the Schwarzschild black hole from the postulates of our model. An essential ingredient of our calculation of black hole entropy is our decision to apply *classical* statistics to our model. This decision of ours is motivated by Hawking's result that, at least in the semiclassical limit, black hole radiation spectrum is the continuous blackbody spectrum. If one attempted to apply any sort of quantum statistics to our model, a discrete radiation spectrum radically different from the one predicted by Hawking would follow even for macroscopic black holes. The continuous spectrum, however, is regained if classical, instead of quantum, statistics is applied to our model. We find that our model supplemented with classical statistics implies that at a very low temperature the entropy of the Schwarzschild black hole is, up to an additive constant, proportional to its horizon area. In Sec. 7.3 it is claimed, on the grounds of certain geometrical arguments, that it is natural to take the constant of proportionality to be equal to one-quarter. This result reproduces the Bekenstein-Hawking entropy law of Eq. (4.2).

Section 7.4 contains a critical analysis of our model. We list the reasonable objections against our model we have managed to find, and our answers to these objections.

7.1 The Model

The starting point of our model is Bekenstein's proposal of the year 1974, which states that the eigenvalues of the area of the event horizon of a black hole are of the form of Eq. (6.20). Bekenstein's proposal has been revived by several authors on various grounds [81]. One way to obtain Eq. (6.20) for Schwarzschild black holes is to consider the following eigenvalue equation for the Schwarzschild mass M of the Schwarzschild black hole (unless otherwise stated, we shall always use the natural units where $\hbar = c = G = k_B = 1$):

$$\left[-\frac{1}{2}a^{-s}\frac{d}{da}\left(a^{s-1}\frac{d}{da}\right) + \frac{1}{2}a\right]\psi(a) = M\psi(a). \tag{7.3}$$

This equation which, in a certain very restricted sense, may be viewed as a sort of "Schrödinger equation" of the black hole from the point of view of a distant observer at rest with respect to the hole, follows from the Hamiltonian (6.17) when we take $\mu = a^s$. As before, a is the throat radius of the Einstein-Rosen wormhole in a foliation where the time coordinate at the throat is the proper time of an observer in a free fall through the bifurcation two-sphere. Moreover, $\psi(a)$ is the wave function of the hole, and the inner product (6.16) between the black hole states is now determined by the real number s such that

$$\langle\psi_1|\psi_2\rangle = \int_0^\infty \psi_1^*(a)\psi_2(a)a^s da. \tag{7.4}$$

In other words, the Hilbert space is chosen to be the space $L^2(\mathbb{R}^+, a^s da)$.

In Sec. 6.3 we stated that the spectrum of M given by Eq. (7.3) is always discrete, bounded from below, and can be made positive by means of an appropriate choice of the self-adjoint extension. Moreover, it was shown in Ref. [69] that if $s = 2$, then for large n the WKB eigenvalues of the mass M are of the form

$$M_n^{\text{WKB}} = \sqrt{2n+1}. \tag{7.5}$$

Surprisingly, this WKB approximation for large n provides an excellent approximation for the mass eigenvalues even when n is small (i.e., of order 1). In the ground state where $n = 0$, the result given by Eq. (7.5) differs by around 1% from the real ground state mass eigenvalue, and the difference very rapidly goes to zero when n increases (see Appendix B). So we see that, because the event horizon area of a Schwarzschild black hole with mass M is

$$A = 16\pi M^2, \tag{7.6}$$

Eq. (7.3) implies, as an excellent approximation, the following spectrum for the horizon area:

$$A_n = 32\pi\Big(n + \frac{1}{2}\Big). \tag{7.7}$$

In other words, we recover Bekenstein's proposal of Eq. (6.20) with $\gamma = 32\pi$ as the constant of proportionality.

We are now prepared to state the postulates, or assumptions, of our model.

1. The event horizon of a macroscopic Schwarzschild black hole consists of independent, Planck size Schwarzschild black holes.

2. Each hole on the horizon obeys Eq. (7.3) with $s = 2$.

3. Holes in the ground state where $n = 0$ do not contribute to the horizon area.

4. When the hole is in the nth excited state, it contributes to the horizon an area which is proportional to n.

5. The total area of the horizon is proportional to the sum of the areas contributed by the holes on the horizon.

No doubt, it is rather daring to assume that the microscopic holes on the horizon are independent of each other. However, at this stage of research that assumption is necessary if we want to make progress. Moreover, one may speculate on the possibility that, when the density of black holes on the horizon is constant, the effects of all the rest of the holes on an individual hole on the horizon may cancel each other. Another bold assumption is contained in postulate 3. As such, postulate 3 views observable black holes as excitations of black holes in a ground state. When compared to postulates 1 and 3, postulates 2, 4, and 5 appear as rather natural. Postulates 4 and 5 imply that the horizon area is of the form

$$A = \alpha(n_1 + n_2 + \cdots + n_N), \tag{7.8}$$

where $n_1, n_2, \ldots, n_N$ are the quantum numbers associated with the area eigenstates of the holes on the horizon, and α is an unknown constant of proportionality. We shall consider the postulates of our model in more details in Sec. 7.4.

7.2 Entropy

When the concept of entropy of a black hole was introduced by Bekenstein and Hawking 30 years ago, all the derivations of the expression (4.2) for black hole entropy were performed by means of *semiclassical* arguments. When Hawking obtained an expression $T = \frac{1}{8\pi M}$ for black hole temperature yielding the expression (4.2) for black hole entropy, he treated spacetime classically and matter fields quantum-mechanically [5]. Later, in 1977, when Gibbons and Hawking calculated the black hole entropy by means of Euclidean path-integral methods [57], they simply used a semiclassical approximation to the path integral. In other words, the expression (4.2) for the black hole entropy is, to the greatest possible extent, a "semiclassical entropy" which we should obtain from a microscopic model of spacetime.

At this point, it is useful to recall what we actually mean by the very concept of entropy in quantum and classical physics. In quantum statistics, the entropy of a system in a given macrostate is the natural logarithm of the number of microstates corresponding to that macrostate, whereas in classical statistics entropy is the natural logarithm of the phase space volume corresponding to the given macrostate [82]. It may be shown that the entropy of quantum statistics reduces to the entropy of classical statistics when the spectra of observables are not assumed to be discrete but continuous. Which statistics—classical or quantum—should we use for spacetime itself?

The answer to this question depends on whether we consider the radiation spectrum of a black hole continuous or discrete. If the radiation spectrum is discrete, then the horizon area spectrum of the macroscopic hole, as well as the spectra of the microscopic holes on the horizon, is discrete, and we must use quantum statistics. Consequently, if the radiation spectrum is continuous, the area spectrum is also continuous and we must use classical statistics. According to Hawking, the radiation spectrum is the purely thermal, continuous blackbody spectrum. Therefore, the answer to our question is obvious: If we want to obtain for the black hole entropy an expression that is in agreement with Hawking's radiation law we must, of course, use *classical* statistics for the black hole itself.

This conclusion has nothing to do with the question about whether we really believe the black hole radiation spectrum to be continuous, as in Hawking's theory, or discrete as it is, for instance, according to Bekenstein's proposal. Curiously, it is possible to express an argument, based on Heisenberg's uncertainty principle, that when the effects of matter fields are so strong that they overshadow the quantum effects of spacetime, then the discrete spectrum predicted by Bekenstein's proposal reduces to Hawking's blackbody spectrum [83]. Actually, this is what we assume here. More precisely, we assume that when the effects of matter fields are absent, the black hole area spectrum is discrete and follows Bekenstein's proposal, but when the effects of matter fields are strong enough, the spectrum becomes continuous, and classical statistics may be applied to the black hole itself.

So let us calculate the classical entropy, or the natural logarithm of the classical phase space volume, of our model. Equation (7.3) may be deduced from the classical Hamiltonian (6.10),

$$H = \frac{1}{2a}\big(p^2 + a^2\big), \tag{7.9}$$

of the Schwarzschild black hole. We know that in this equation p is the canonical momentum conjugate to a, and that the numerical value of H is the Schwarzschild mass M of the hole. Therefore we find, using Eq. (7.5), that for a single black hole with "mass" M_1 and canonical coordinates a_1 and p_1 on the horizon,

$$\frac{1}{4a_1^2}\big(p_1^2 + a_1^2\big)^2 = M_1^2 = 2n_1 + 1. \tag{7.10}$$

However, since we are now considering classical statistics, n_1 is no longer an integer but may be any non-negative real number. Moreover, it is no longer possible to associate with M_1 any sensible physical meaning as the "mass" of a hole on the horizon. (A more detailed investigation of this issue is performed in Sec. 7.4.) M_1 is simply a parameter with the property that M_1^2 is proportional to the area contributed by a single microscopic hole to the total horizon area.

Equation (7.10) now implies that

$$\frac{1}{8a_1^2}\big(p_1^2 + a_1^2\big)^2 + \cdots + \frac{1}{8a_N^2}\big(p_N^2 + a_N^2\big)^2 = \frac{N}{2} + \frac{A}{\alpha}. \tag{7.11}$$

It can be shown (see Appendix C) that for fixed A and N this is a compact $(2N-1)$-dimensional hypersurface in a $2N$-dimensional phase space. When the metric of the phase space is defined as

$$ds^2 = \sum_{i=1}^{N}\big(dp_i^2 + da_i^2\big), \tag{7.12}$$

its $(2N-1)$-volume, in units of $(2\pi\hbar)^{N-1/2}$, is of the form

$$\Omega = C(N)\Big(N + \frac{2A}{\alpha}\Big)^{N-1/2}, \tag{7.13}$$

where $C(N)$ is a function that depends on N only. For fixed N, $C(N)$ is finite.

At this point, we should pause for a moment. In Sec. 7.1, we threw away with great pomp and ceremony the vacuum contribution to the black hole horizon area by stating that holes in the ground state do not contribute to the horizon area. Equation (7.11), however, involves the term $\frac{N}{2}$, which is equal to the vacuum contribution to the horizon area. Are we now quietly returning the vacuum contribution into our calculations?

The answer to this question is emphatically no. The reason why the term $\frac{N}{2}$ suddenly appears in Eq. (7.11) is that in Eq. (7.11) we calculate the *phase space volume* of our model, and the role of the term $\frac{N}{2}$ is to take into account the vacuum contribution to the phase space volume. In other words, although we assume that ground states do not contribute to the horizon area, they nevertheless contribute to the phase space volume. Actually, as we shall see in a moment, in very low temperatures most of the phase space volume is really occupied by the unobservable vacuum, or black holes in the ground state.

The entropy of the Schwarzschild black hole is the natural logarithm of the phase space volume Ω corresponding to a fixed horizon area A:

$$S = \ln\Omega = \ln C(N) + \Big(N - \frac{1}{2}\Big)\ln\Big(N + \frac{2A}{\alpha}\Big). \tag{7.14}$$

Using Eq. (7.6) and keeping N as a constant, we may obtain the temperature T of the Schwarzschild black hole:

$$\frac{1}{T} = \Big(\frac{\partial S}{\partial E}\Big)_{N,V} = \frac{N-\frac{1}{2}}{N}\frac{64\pi M}{\alpha + \frac{32\pi M^2}{N}} \tag{7.15}$$

or, since N is assumed to be very large,

$$T = \frac{\alpha}{64\pi M} + \frac{1}{2N}M. \tag{7.16}$$

It is interesting that if N, the number of microscopic holes on the horizon, is sufficiently small when compared to the mass M of the macroscopic hole then the temperature increases, instead of decreasing as in Hawking's theory of black hole radiation, as a function of M.

At this point, we introduce a new assumption to our theory. We assume that *most microscopic holes on the horizon are in the ground state*, where $n = 0$. More precisely, we assume that the average

$$\bar{n} := \frac{n_1 + \cdots + n_N}{N} = \frac{A}{N\alpha} \tag{7.17}$$

of the quantum numbers $n_1, \ldots, n_N$ has the property

$$\bar{n} \ll 1. \tag{7.18}$$

Since A is proportional to M^2, we find that this condition may be written as

$$\frac{M}{N} \ll \frac{1}{M}. \tag{7.19}$$

When the assumption (7.18) is employed, the second term on the right hand side of Eq. (7.16) vanishes, and the temperature becomes

$$T = \frac{\alpha}{64\pi M}. \tag{7.20}$$

The assumption (7.18) is sensible when we consider the thermodynamics of a macroscopic black hole at a *very low temperature*: At a very low temperature the constituents of any system tend to be as close to the ground state as possible. The same is true also for our model: We see that the minimum temperature is achieved when the second term on the right hand side of Eq. (7.16), and therefore $\bar{n}$, becomes as small as possible.

From Eq. (7.20) it now follows that, up to an additive constant which depends only on the number N of the microscopic holes on the horizon, the entropy of the Schwarzschild black hole is, at a very low temperature,

$$S = \frac{2A}{\alpha}. \tag{7.21}$$

In other words, we have obtained, up to an undetermined constant of proportionality, the Bekenstein-Hawking entropy law of Eq. (4.2). Here we have considered the Schwarzschild horizon only but our consideration may easily be generalized for other horizons as well.

7.3 Constant of Proportionality

It only remains to fix the constant α in Eq. (7.21). The constant α was defined in Eq. (7.8). From that definition it follows that, if we know how the total area of the horizon depends on the areas of the holes on the horizon, we may calculate α. A comparison of Eqs. (4.2) and (7.21) yields the result that, in natural units, we should have $\alpha = 8$, and our task is to obtain this value by means of geometrical arguments. Of course, nobody really knows what the "spacetime geometry" at the Planck length scale looks like, and therefore one should take such arguments with a pinch of salt. However, it is possible to express arguments which, when not understood as a real description of spacetime structure at the Planck length scale but rather as a sort of statistical average of the behavior of the microscopic holes on the horizon, may at least provide a heuristic aid to thought.

To begin with, consider Fig. 7.1. In that figure, we have drawn microscopic black holes on the horizon such that, at this stage, they do not overlap each other. A distant observer does not see a hole on the horizon as a sphere but as a *circle* with radius equal to its Schwarzschild radius R_S. The area of this circle is πR_S^2, which is one-quarter of the horizon area of the corresponding microscopic hole. Equation (7.7), together with postulate 3, therefore implies that the area covered by a microscopic hole in the nth excited state is $8\pi n$.

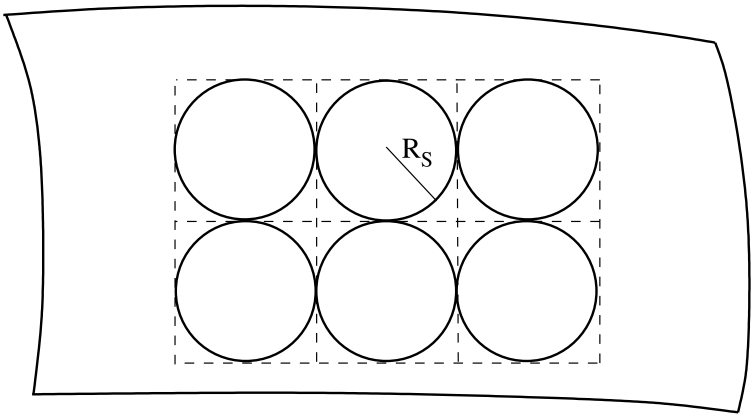

Figure 7.1. In our model, spacetime is constructed from Planck size black holes. A faraway observer sees each particular hole as a circle with a Schwarzschild radius R_S. When a hole on the horizon exactly fits inside a square, the area of the square is $\frac{4}{\pi}$ times the area of the corresponding circle.

When considering Fig. 7.1, however, one finds that the holes on the horizon do not cover the whole horizon but certain voids remain. A better way to cover the horizon is to use squares drawn around the holes such that each hole exactly fits inside the corresponding square. In that case, the area of each square is $\frac{4}{\pi}$ times the area of the corresponding circle. Summing the areas of the squares, we find that the horizon area is

$$A = 32(n_1 + n_2 + \cdots + n_N), \tag{7.22}$$

from which it follows that $\alpha = 32$. In other words, α is too large by a factor of 4.

A remedy for this problem may be found if we assume that the holes on the horizon overlap each other. Recall that we are obtaining an expression for the maximum entropy of a macroscopic Schwarzschild black hole. The maximum entropy may be gained if there is a maximum amount of microscopic holes on the horizon. That is because, if two black holes on the horizon come together to form a single black hole, the area of the resulting black hole is larger than is the sum of the areas of the original black holes. In other words, a larger amount of the horizon is covered by a smaller number of degrees of freedom. Since the maximum entropy is achieved when the number of degrees of freedom is as large as possible, it follows that when two black holes on the horizon overlap in the state of maximum entropy, they still must not form a single black hole.

The maximum overlap between two holes with equal sizes is achieved when the center points of the holes lie on each other's event horizons (see Fig. 7.2). In that case, the holes are not yet "swallowed" by each other, and the distance between the center points of the holes is their common Schwarzschild radius R_{S}. As one can see from Fig. 7.2, however, covering the horizon with black holes overlapping each other in this manner is equivalent to covering the horizon with holes with radius $\frac{1}{2}R_{\mathrm{S}}$ and not overlapping each other. Therefore one finds that the horizon area now takes the form

$$A = 8(n_1 + n_2 + \cdots + n_N), \tag{7.23}$$

which implies the desired result

$$\alpha = 8. \tag{7.24}$$

When this result is substituted in Eq. (7.21), we get

$$S = \frac{1}{4}A, \tag{7.25}$$

or, in SI units,

$$S = \frac{1}{4}\frac{k_B c^3}{\hbar G}A. \tag{7.26}$$

In other words, we have exactly recovered the Bekenstein-Hawking entropy of Eq. (4.2).

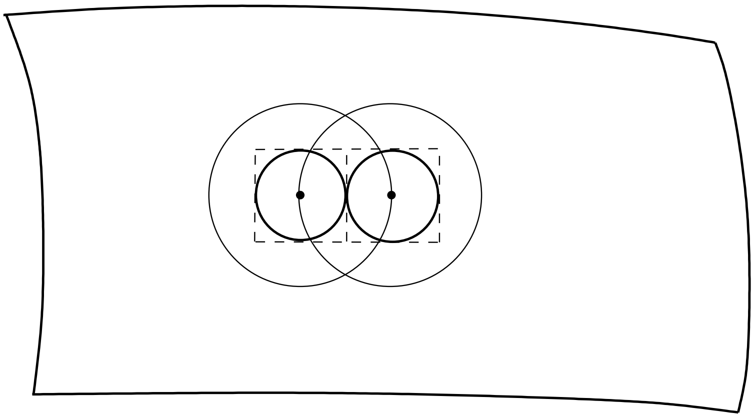

Figure 7.2. Two identical black holes on the horizon overlapping each other such that the center points of the holes lie on each other's event horizons. Covering the horizon with black holes overlapping each other in this manner is equivalent to covering the horizon with non-overlapping holes having radii that are one-half of the radii of the original holes.

However, it must again be strongly emphasized that the geometrical arguments given here should not be considered as a real description of the structure of spacetime at the Planck length scale. Rather, they should be viewed as mere heuristic aids to thought. The microscopic structure of spacetime is certainly very much more complicated than is the simple picture given in this section. Our geometrical arguments, however, have a certain statistical content which may be expressed as follows: The average number density of the microscopic black holes on the horizon is such that, if the holes were understood as classical objects, their center points would lie, in average, on each other's event horizons. Whether one accepts this argument or not is, at the present stage of research, a matter of taste but nevertheless it produces the desired value for the constant α.

7.4 Some Objections

As may have become clear to the reader, our model and the postulates on which it is based contain several ideas that are physically new. Always when expressing a new physical idea it is of vital importance to try to prove oneself wrong, and this is what we shall attempt to do in this section. More precisely, we list all the reasonable objections against our model we have managed to find, and our answers to these objections. Our objections are as follows.

1. It is very strange to think of a black hole horizon as being made of Planck size black holes. If black holes are made of Planck size black holes, then what are the Planck size black holes made of?

   Answer. In our model, we have assumed the Planck size black holes described by Eq. (7.3) to be fundamental constituents, or building blocks, of spacetime. So they are assumed to have no intrinsic structure. The idea that spacetime might be made of Planck size black holes in one way or another is supported, among other things, by the arguments based on Heisenberg's uncertainty principle. So far we have no precise model of the whole spacetime being made of Planck size black holes, and a construction of such a model should be viewed as a challenge for the future. We have only a model of a Schwarzschild horizon and, as one can see, our model reproduces the Bekenstein-Hawking entropy law.

2. Although this work intends to provide a model of a Schwarzschild horizon, the assumption that a horizon of spacetime is considered is nowhere explicitly used in the derivation of the Bekenstein-Hawking entropy law. Actually, it seems that arguments similar to those employed in this chapter may be used to imply that every spacelike two-surface of spacetime has an entropy that is proportional to its area, no matter whether that two-surface is part of a horizon or not. Is this not in contradiction with the known properties of gravitational entropy?

   Answer: It is true that an assumption that a horizon of spacetime is under scrutiny is nowhere used in our analysis. The primary reason for our choice to construct a model of a Schwarzschild horizon is that with a Schwarzschild horizon one may associate a well-defined concept of energy which, in turn, may be used when calculating the temperature of the horizon. It is also true that one could argue by means of reasoning similar to that used in this chapter that entropy is associated not only with spacetime horizons but with every spacelike two-surface of spacetime. To make this statement more precise, however, we should first clarify our terminology. Indeed, it may not be quite clear to the reader what we mean by the 'entropy of spacelike two-surface', and therefore it is probably worth clarifying this issue.

Our model is an example of a theory where the statistical origin of black hole entropy is assumed to stem from the microscopic structure of spacetime at the black hole horizon. A specific feature of this kind of approach is that the entropy may be understood to reside on the *spacelike two-surface* which determines the area of a given horizon. For instance, in our model the entropy of the Schwarzschild black hole was derived from the statistics of the spacelike two-sphere $r = 2M$. Hence one may, with a minor abuse of terminology, talk about the entropy of the spacelike two-sphere $r = 2M$ instead of the entropy of the Schwarzschild horizon. At this point, as well as later in this thesis, we shall frequently use this kind of terminology, mostly because it makes the expression of our ideas considerably easier. Obviously, one may now construct an arbitrary spacelike two-surface out of microscopic black holes in a similar way as in the case of a Schwarzschild horizon. Again, statistical arguments will suggest that the concept entropy should be attributed also to that two-surface. So we have arrived at the conclusion that in our model every finite spacelike two-surface, no matter whether that two-surface is a part of a horizon or not, possesses an entropy proportional to its area.

Although astonishing, this conclusion, however, does not contradict with the known physics. Actually, there seems to be very good grounds to believe that every piecewise smooth, spacelike two-surface of spacetime indeed carries a certain amount of entropy although that entropy seems to produce observable physical effects only for the observers having that two-surface as a part of a horizon. Provided that one accepts the view held, at least implicitly, by many authors that horizon entropy is indeed due to the microscopic structure of spacetime at the horizon, this result is a direct consequence of the well-established result that any finite part of a Rindler horizon has an entropy which, in natural units, is one-quarter of the area of that part. The line of reasoning producing this conclusion may be expressed as follows:

(a) Each finite part of a Rindler horizon possesses an entropy which is one-quarter of its area.

(b) Each finite spacelike two-plane is, from the point of view of an appropriate observer, a part of a Rindler horizon.

(c) Therefore, each finite spacelike two-plane possesses an entropy which is one-quarter of its area.

(d) Each piecewise smooth spacelike two-surface is a union of infinitesimal spacelike two-planes, each having an entropy equal to one-quarter of its area.

(e) Therefore, each piecewise smooth spacelike two-surface possesses an entropy which is one-quarter of its area.

In other words, an explicit assumption that one looks at the horizon is not needed when one attempts to obtain the Bekenstein-Hawking entropy law: There are good grounds to believe that the Bekenstein-Hawking entropy law holds, in a certain sense, for any piecewise smooth, spacelike two-surface. The possible connection between the area and entropy is further discussed in chapter 8.

3. How can the Planck size black holes on the horizon be independent of each other? Certainly there should be an extremely strong correlation between them.

   Answer. This is indeed what one might expect on classical grounds. However, we are now considering spacetime at very small length scales where quantum gravity reigns, and all bets are on. It is possible that the interactions between black holes effectively cancel each other in a somewhat similar way to the gravitational effects canceling each other from the point of view of an observer at the center of the Earth. This is an open question.

   In addition to these rather general remarks, it is possible to express a much more serious argument supporting the claim that the microscopic black holes might indeed be independent of each other. This argument is based on Jacobson's extremely important discovery that Einstein's equation may be viewed as a *thermodynamical equation of state* [10]. More precisely, Jacobson showed that, if one assumes that a local Rindler horizon always possesses an entropy which is one-quarter of its area, then Einstein's equation follows from the first law of thermodynamics. So one is inclined to think that Einstein's equation, with its prediction that macroscopic black holes attract each other, is probably not very fundamental at all but is merely a consequence of the statistics of the fundamental constituents of spacetime. There is not necessarily any interaction between these fundamental constituents at the microscopic level, but their statistical properties imply at the macroscopic level properties which might lead one to conclude the existence of a sort of effective interaction. As a familiar example, one may think of a classical ideal gas. The classical ideal gas has a certain pressure which might lead one to conclude (incorrectly) that between the particles of the gas there is a repulsive interaction which prevents one from compressing the gas. However, no such repulsive interaction really exists, and the pressure is simply a consequence of the statistical mechanics of the gas. Perhaps something similar happens with microscopic black holes. At the microscopic level, there is no interaction between the holes but their statistical properties imply, in a certain limit, Einstein's equation with all its predictions. Actually, this may well be the case, because it seems to us that our analysis could be generalized to show that not only black holes but also a local Rindler horizon possesses an entropy which is one-quarter of its area, and, as Jacobson has shown, a generalization of the Bekenstein-Hawking entropy law for local Rindler horizons implies Einstein's equation. So it is possible that the gravitational interaction between macroscopic black holes may be a direct consequence of the independence of microscopic black holes.

4. Is it not absurd that black holes in the ground state where $n = 0$ do not contribute to the horizon area?

   Answer. This is probably the most serious objection against our model. However, the idea that vacuum states do not produce measurable effects (except in very special circumstances) is not very unfamiliar in physics. In quantum field theories in flat spacetime, the field is decomposed into its Fourier components, and the possible energies of each component are of the form

   $$E_{\vec{k},n} = \left(n + \frac{1}{2}\right)\hbar\omega_{\vec{k}}, \tag{7.27}$$

   where $n = 0, 1, 2, \dots$ and $\omega_{\vec{k}}$ is the angular frequency of the component corresponding to the wave vector $\vec{k}$. The vacuum energy where $n = 0$ is never observed, but only the energies that are above the vacuum energy may be measured. In other words, everything one can observe is an excitation of the vacuum but the vacuum itself cannot be observed (except by means of the Casimir effect). It is not entirely impossible that this feature of nature might pertain also to quantum gravity. In quantum gravity, however, energy is replaced by area: As one can see from Eq. (7.7), the possible areas of the microscopic holes on the horizon are exactly of the same form as are the energies of the Fourier components of a quantized field. One is therefore tempted to draw an analogy between the quantization of energy in ordinary quantum field theories, and the quantization of area in our model: In the same way as everything one observes in ordinary quantum field theories is an excitation of the vacuum state of energy, in our model everything one observes is an excitation of the vacuum state of area. This analogy may be viewed as the main justification for postulate 3.

5. It follows from postulate 5 that the squares M_i^2 of the masses of the individual holes on the horizon sum up, essentially, to the square M^2 of the total mass of the hole. Is this not wrong because certainly the masses themselves should sum up to the total mass of the hole?

   Answer. The concept of mass is very problematic already in classical general relativity. Actually, the concept of mass may be properly defined only when spacetime is asymptotically flat, and in that case the relevant mass concept is the ADM mass, which measures the mass-energy included by the whole spacetime. When we try to define the mass-energy included by a specific part of curved spacetime, however, we run into grave difficulties because the energy-momentum tensor of the gravitational field, in general, is not well defined. So it is not clear what we are talking about when we say that the mass of a certain object in curved spacetime is such and such, unless that is the only object in an asymptotically flat spacetime and we may define its ADM mass.

In particular, the concept of the mass of an object at the event horizon of a black hole is very problematic. To see how problematic it really is, consider the four-momentum of a particle having a mass m in flat spacetime:

$$p^\mu = m\dot{x}^\mu, \tag{7.28}$$

where the overdot means the proper time derivative. When spacetime is static, i.e., the metric tensor does not depend on the time parameter x^0, the quantity

$$p_0 = m\, g_{0\mu}\dot{x}^\mu \tag{7.29}$$

is conserved along geodesics and also along curves having timelike Killing vectors as their tangents, and it offers the best attainable definition for the concept of energy of a particle in curved spacetime. Since the Schwarzschild metric may be written as

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2, \tag{7.30}$$

we find that

$$p_0 = m\left(1 - \frac{2M}{r}\right)\dot{t}. \tag{7.31}$$

For a particle at rest with respect to the Schwarzschild coordinates we have

$$\dot{t} = \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1/2}, \tag{7.32}$$

and therefore

$$p_0 = m\left(1 - \frac{2M}{r}\right)^{1/2}. \tag{7.33}$$

Hence we see that, when an object lies at the horizon where $r = 2M$, its energy, according to the above mentioned definition, is zero. Of course, it is impossible to keep a particle still at the horizon, but nevertheless our considerations provide an example of the problems of the mass-energy concept at the event horizon of a black hole, even at the classical level. At the quantum level, one may expect even deeper problems. Therefore, if we consider the event horizon of a black hole as a system made of tiny black holes, the mass parameters M_i of the holes do not represent mass in any ordinary sense. Consequently, the sum of the parameters M_i is not the Schwarzschild mass M of the macroscopic hole either. However, each hole on the horizon covers a region with an area proportional to M_i^2 on the horizon. Therefore, it is natural to postulate that the sum of the quantities M_i^2 equals, up to a constant of proportionality, the square M^2 of the Schwarzschild mass of the hole.

6. The entropy of the hole has been calculated by using classical statistics right from the beginning. Would it not have been more appropriate to perform the calculations by using quantum statistics first and then take the classical limit?

   Answer. This question has been considered in detail already in Sec. 7.2. Still, it is appropriate to repeat here the essential points. It follows from Eq. (7.8), which, in turn, is a direct consequence of our postulates, that the radiation spectrum of a black hole is discrete. However, according to Hawking's radiation law the black hole radiation spectrum is a continuous blackbody spectrum. It is impossible to obtain Hawking's continuous spectrum from our model unless one assumes right from the beginning that the quantum numbers $n_1, n_2, \ldots, n_N$ may take any values, and classical statistics applies. It follows from Eq. (7.8) that the possible frequencies of the quanta of radiation are, for $\alpha = 8$, integer multiples of the fundamental frequency

   $$v_0 := \frac{c^3}{8\pi^2 G}\frac{1}{M}. \tag{7.34}$$

   Even for macroscopic black holes (i.e., when M is of the order of ten solar masses) this quantity is fairly big: It is of the order of 0.3 kHz which is about the same as is the resolving power of an ordinary portable radio receiver. So if one wants Hawking's continuous spectrum out from the discrete spectrum predicted by Eq. (7.8), the stationary quantum states of the black holes must become very much mixed with each other such that classical statistics, in effect, may be applied. A mechanism for this kind of mixing through the interaction between the hole and matter fields was proposed in Ref. [83].

7. The expression of Eq. (7.14) for the black hole entropy is not exactly one-quarter of the horizon area, even for big N, but it contains an additive constant which depends only on N. The presence of that term should produce physical effects different from those obtainable from the Bekenstein-Hawking entropy law.

   Answer. Actually, the quantity N represents the *particle number* in our model, the "particles" being now the microscopic black holes on the horizon. In all thermodynamical systems the physically observable thermodynamical quantities are related to those partial derivatives of entropy where the particle number N is kept constant during the differentiation. For instance, the temperature T of a system is the inverse of the partial derivative of entropy S with respect to energy E such that the particle number N and the volume V of that system are kept constant:

   $$\frac{1}{T} := \left(\frac{\partial S}{\partial E}\right)_{V,N}. \tag{7.35}$$

   Moreover, the pressure p of a system is its temperature times the partial derivative of entropy with respect to the volume V such that energy E and particle number N are kept constant:

$$p := T\Big(\frac{\partial S}{\partial V}\Big)_{E,N}. \tag{7.36}$$

The only quantity that in any thermodynamical system is related to the partial derivative of entropy with respect to the particle number N is the *chemical potential*

$$\mu := -T\Big(\frac{\partial S}{\partial N}\Big)_{E,V} \tag{7.37}$$

of the system. As long as one is not interested in the chemical potential of the Schwarzschild black hole, the physical predictions obtainable from our expression for black hole entropy are identical to those obtainable from the Bekenstein-Hawking entropy law. When we come to the chemical potential of the Schwarzschild black hole, in turn, nobody has the slightest idea about what that might be for the very simple reason that nobody knows what the "particles" constituting a black hole would be. So we see that our expression for the black hole entropy does not contradict the existing knowledge of black hole thermodynamics. In general, it seems possible, at least in principle, to use different chemical potentials to distinguish different derivations of the Bekenstein-Hawking entropy law. More precisely, even if a certain derivation of the Bekenstein-Hawking entropy law yielded an expression for the temperature of the horizon consistent with that law, the number of degenerate states corresponding to the same horizon area A is not necessarily $e^{\frac{1}{4}A}$ but an expression for the number of degenerate states may contain a model-dependent prefactor, which is a function of the chemical potential of the system.

7.5 Concluding Remarks

In this chapter, we have considered a spacetime foam model of the Schwarzschild horizon where the horizon of a macroscopic Schwarzschild black hole consists of Planck size Schwarzschild black holes. Using this model, we found that the entropy of a macroscopic Schwarzschild black hole is, up to an additive constant, proportional to its horizon area. It seems that our derivation of this result is valid for any horizon but here we have restricted our consideration to Schwarzschild horizons only. Certain heuristic arguments may be employed to imply that the constant of proportionality is, in natural units, equal to one-quarter.

The key points in our derivation were an assumption that when a microscopic black hole on the horizon is in a ground state, it does not contribute to the horizon area, and our decision to apply *classical* statistics to our spacetime foam model. An assumption that a hole in a ground state, even if its Schwarzschild mass were non-zero, does not contribute to the horizon area, may be motivated by an analogy with ordinary quantum field theories: In ordinary quantum field theories the vacuum state of energy is not observed, whereas in our model the vacuum state of energy is replaced by the vacuum state of area. As it comes to our decision to apply classical, instead of quantum statistics to our model, that decision may be considered justified on the grounds that according to Hawking the radiation spectrum of a black hole is a continuous blackbody spectrum. If we want to obtain a continuous spectrum for that radiation, we must assume that the spectra of observables of a macroscopic black hole are continuous, and therefore we must use classical statistics for the black hole itself.

In our model, we assumed at first that the tiny holes on the horizon obey Bekenstein's proposal. In other words, we assumed that when the effects of matter fields are neglected, the horizon area spectra of the holes on the horizon have an equal spacing. That spacing was obtained from Eq. (7.3), the "Schrödinger equation" of the Schwarzschild black hole. When the effects of matter fields are assumed to be strong enough, however, the discrete area spectrum washes out into the continuum, and Hawking's continuous blackbody spectrum is recovered. In other words, we considered the implications of Eq. (7.3) in a limit where the uncertainties in the area eigenvalues are of the same order of magnitude as is the spacing between the area eigenvalues of nearby states. The fact that we used Eq. (7.3) in our derivation of an expression for the black hole entropy may be viewed as an argument supporting the physical validity of that equation as well as of Bekenstein's proposal. In this chapter, we do not express opinions about whether the radiation spectrum really is continuous or discrete. We only showed that if the radiation spectrum is assumed to be continuous, then the Bekenstein-Hawking entropy law follows from our model.

Taken as a whole, our model may be viewed as an attempt to understand the microscopic structure of spacetime. No doubt, the picture provided by our model is very different from those provided by, for instance, string theory and loop quantum gravity. An advantage of our model, however, is that it takes seriously the possibility suggested, among other things, by Heisenberg's uncertainty principle that spacetime at the Planck length scale might consist of Planck size black holes. So far we have no precise model of spacetime itself but only of its Schwarzschild horizon. However, our results with the derivation of the Bekenstein-Hawking entropy law are encouraging, and it will be very interesting to see whether the ideas presented in this chapter may be worked out into a precise mathematical and physical model of the microscopic structure of spacetime.

Chapter 8

Gravitation and Thermodynamics: The Einstein Equation of State Revisited

8.1 Introduction

Ever since the discovery of the Bekenstein-Hawking entropy law, it has become increasingly clear that there is a deep connection between gravitation and thermodynamics. However, even today it is not properly understood what exactly this connection may be. The most surprising point of view on these matters was probably provided by Jacobson in 1995, when he discovered that Einstein's field equation is actually a thermodynamical equation of state of spacetime and matter fields [10]. As it was briefly mentioned in the previous chapter, the key point in his analysis was to require that the first law of thermodynamics, which implies the fundamental thermodynamical relation

$$\delta Q = T\,dS, \tag{8.1}$$

holds for all local Rindler horizons, and that the entropy S of a finite part of the Rindler horizon is one-quarter of its area. Jacobson considered an observer very close to his local Rindler horizon (which means that the proper acceleration a of the observer is extremely large). For the temperature T in Eq. (8.1), Jacobson took the Unruh temperature

$$T_{\mathrm{U}} = \frac{a}{2\pi} \tag{8.2}$$

experienced by the observer, and the heat flow δQ through the past Rindler horizon was defined to be the boost-energy current carried by matter. Jacobson was able to show that, under the assumptions mentioned above, the heat flow through the horizon causes a change in the horizon area in such a way that Einstein's field equation is satisfied. In other words, he was able to derive Einstein's field equation by assuming the first law of thermodynamics and the proportionality of entropy to the area of the horizon. Viewed in this way, Einstein's field equation is nothing more than a thermodynamical equation of a state [84, 85].

The purpose of this chapter is to investigate whether there are some other (possibly more general) principles of nature that would imply Einstein's field equation. In Sec. 7.4 we have seen that there are reasons to believe that the concept of gravitational entropy should be extended from horizons to arbitrary spacelike two-surfaces with finite areas (a more detailed analysis can be found in Ref. [86]). This conclusion was motivated by the view that the gravitational entropy associated with a horizon stems from the microscopic degrees of freedom of spacetime at the horizon: One expects that similar microscopic degrees of freedom of spacetime are present at any spacelike two-surface, no matter whether that two-surface is part of a horizon or not, and therefore every spacelike two-surface should possess certain (statistical) notion of entropy. Apart from this rather heuristic argumentation, there are also more serious reasons supporting this claim. In Ref. [9] it was proposed that an accelerated two-plane may be associated with an entropy which is, in natural units, one-half of the area of that plane. This proposal is, in some sense, related to the well-known result that the entropy associated with a spacetime horizon is one-quarter of the area of the horizon. The reason for the difference in the constant of proportionality is still unclear, but it may result from the fact that a spacetime horizon is, according to observers having that surface as a horizon, only a one-sided surface, whereas an accelerated spacelike two-surface has two sides [87].

In this chapter we shall find that Einstein's field equation can be derived from a hypothesis which is closely related to this proposal. Our derivation will be based on a consideration of a very small, spacelike two-plane in a uniformly accelerating motion in the direction perpendicular to the plane. When the plane moves in spacetime, matter will flow through the plane. Since the matter has, from the point of view of an observer at rest with respect to the plane, a certain non-zero temperature, it also has a certain entropy content. In other words, entropy flows through the plane. Since the plane is in an accelerating motion, the entropy flow through the plane (amount of entropy flown through the plane in unit time) is not constant, but it will change as a function of the proper time of an observer moving along with the plane.

The change in the entropy flow through the plane has two parts. One of these parts is due to the simple fact that the plane moves from one point to another in spacetime, and the entropy densities in the different points of spacetime may be different. This part has nothing to do with the acceleration of the plane. Another part in the change of the entropy flow, however, is caused by the change in the velocity of the plane with respect to the matter fields: When the velocity of the plane with respect to the matter fields changes, so does the entropy flow through the plane. This part in the entropy flow is caused by the acceleration of the plane, and it is this part in the change of the entropy flow, where we shall focus our attention. For the sake of brevity and simplicity we shall call that part as the change in the *acceleration entropy flow*.

When the accelerating plane moves in curved spacetime, its area will change. The area change may be calculated by following the world lines of the points of the plane and parametrizing those worldlines by means of the proper time τ measured along them. To consider the behaviour of those worldlines, we must first choose appropriate initial conditions for their congruence. When the plane lies at a certain point $\mathcal{P}$ of spacetime, we take the proper time $\tau = 0$ at each worldline, and assume that in the local neighbourhood of the point $\mathcal{P}$ the future pointing tangent vectors of the worldlines of the points of the plane are parallel to each other. This implies that at $\mathcal{P}$ the rate of change of the area A of the plane is zero, i.e., $\frac{dA}{d\tau}|_{\tau=0} = 0$. If spacetime is curved, however, $\frac{dA}{d\tau}$ will become non-zero when we move away from the point $\mathcal{P}$. We shall call the quantity $-\frac{dA}{d\tau}$ as the *shrinking speed* of the area of the plane.

Under the assumptions that the tangent vectors of the worldlines of the points of the plane are initially parallel to each other, which implies $\frac{dA}{d\tau}|_{\tau=0} = 0$, and that the boost energy flow through the plane is exactly the heat flow, we express the following hypothesis concerning the rates of changes in the acceleration entropy flow through an accelerating plane, and in the shrinking speed of the area of the plane:

If the temperature of the matter flowing through an accelerating, spacelike two-plane is equal to the Unruh temperature measured by an observer at rest with respect to the plane, then the rate of change in the acceleration entropy flow through the plane is, in natural units, exactly one-half of the rate of change in the shrinking speed of the area of the plane.

Using this hypothesis, and this hypothesis only, together with Eq. (8.1), we shall obtain Einstein's field equation with a vanishing cosmological constant. Our hypothesis may be expressed by means of a formula

$$\frac{d^2 S_a}{d\tau^2}\Big|_{\tau=0} = -\frac{1}{2}\frac{d^2 A}{d\tau^2}\Big|_{\tau=0}, \tag{8.3}$$

where $\frac{dS_a}{d\tau}$ denotes the acceleration entropy flow, and $-\frac{dA}{d\tau}$ the shrinking speed of the area. Because the Unruh temperature T_{U} of Eq. (8.2) represents, in some sense, the temperature of spacetime from the point of view of an observer moving with a constant proper acceleration a, we may view Eq. (8.3) as an equation which holds, when matter and spacetime are, from the point of view of an accelerating observer, in a thermal equilibrium with each other. When we calculate the rate of change in the acceleration entropy flow through the plane, we must use Eq. (8.1). More precisely, we first calculate the rate of change in the flow of heat through the plane and then, using Eq. (8.1) and identifying T as the Unruh temperature T_{U} of Eq. (8.2), we calculate the rate of change in the acceleration entropy flow. We have assumed that the boost energy flow through our accelerating plane is exactly the heat flow for the simple reason that it makes the calculation of the flow of entropy very easy: We just calculate the boost energy flow, and then use Eq. (8.1). If there were other forms of energy, except heat, flowing through our plane, it would not be quite clear what we actually mean by the concept of entropy flow, and our analysis would become much more complicated. It is most gratifying that Einstein's field equation follows from our hypothesis even with this rather restrictive assumption, regardless of what kind of matter we happen to have.

We begin our investigations in Sec. 8.2 by considering the trajectory of our plane. We shall assume that at a certain point $\mathcal{P}$ of spacetime we have an orthonormal geodesic frame of reference, where all components of the energy-momentum tensor $T_{\mu\nu}$ of matter are fixed and finite. We shall then assume that in this frame of reference we have a very small spacelike two-plane, which moves, at the point $\mathcal{P}$, with a velocity very close to the speed of light to the direction of its normal and, at the same time, is in a uniformly accelerating motion to the opposite direction. In order to make our analysis sufficiently local, the proper acceleration of the plane is taken to be very large. For sufficiently large values of the proper acceleration, one may view the local neighbourhood of the point $\mathcal{P}$ as a region of spacetime which possesses the ordinary properties of the Rindler spacetime, including the Unruh temperature $T_{\rm U}$ of Eq. (8.2).

The motivation for our decision to consider a plane moving with a very high speed in a chosen frame of reference becomes obvious in Sec. 8.3, where we consider the flow of heat through our accelerating plane. It is fairly easy to show that if matter consists of a gas of non-interacting massless particles, i.e., of massless radiation, then the flow of boost energy through the plane is exactly the heat flow through the plane. Unfortunately, if the particles of the matter fields are massive, the situation becomes more complicated, because in that case other forms of energy, except heat, (mass-energy, for instance) are carried through the plane. However, if the plane moves with an enormous velocity with respect to the matter fields, then the kinetic energies of the particles of the fields vastly exceed, in the rest frame of the plane, all the other forms of energy. In this limit we may consider matter, in effect, as a gas of non-interacting massless particles, and the boost energy flow is exactly the heat flow. We identify that part in the rate of change in the heat flow, which is due to mere acceleration of the plane, and using Eq. (8.1) we calculate the rate of change in the acceleration entropy flow.

In Sec. 8.4 we shall focus our attention to the change in the area of our accelerating two-plane. As the final result of Sec. 8.4 we shall get the rate of change in the shrinking speed of the plane.

After obtaining an expression for the rate of change in the acceleration entropy flow in Sec. 8.3, and for the rate of change in the shrinking speed in Sec. 8.4, we are finally able to obtain, in Sec. 8.5, Einstein's field equation by means of our hypothesis. We fix the proper acceleration a of the plane in such a way that the Unruh temperature $T_{\rm U}$ of Eq. (8.2) measured by an observer at rest with respect to the accelerating plane is the same as the temperature of the matter flowing through the plane. Einstein's field equation is a straightforward consequence of our hypothesis in the limit, where the plane moves with a velocity very close to the speed of light with respect to the matter fields. The field equation, which will be obtained in Sec. 8.5, however, will still contain an unspecified cosmological constant.

To fix the cosmological constant we consider, in Sec. 8.6, the special case, where spacetime is filled with isotropic electromagnetic (or any massless) radiation in thermal equilibrium, and our accelerating plane is, at a certain point $\mathcal{P}$ of spacetime, at rest with respect to that radiation. Since the particles of the radiation field are massless and non-interacting, the heat flow is exactly the boost energy flow, and there is no need to assume that the plane would move with an enormous speed with respect to the matter fields. Nevertheless, our hypothesis again implies Einstein's field equation in a form valid in the special case under consideration. That equation, however, does not include the cosmological constant, and therefore we may draw an important conclusion that the cosmological constant must vanish.

We close our discussion in Sec. 8.7 with some concluding remarks.

8.2 Trajectory of the Plane

It is now time to specify our thermodynamical system in detail. Take a spacetime point $\mathcal{P}$ and define an orthonormal geodesic system of coordinates t, x, y, z at the local neighbourhood of that point. The origin of the coordinates is taken to lie at $\mathcal{P}$. Consider then a uniformly accelerated observer with a proper acceleration a travelling through $\mathcal{P}$ in the direction of the positive z-axis. We denote the velocity of that observer at $\mathcal{P}$ by $v > 0$. Furthermore, we assume that the acceleration of that observer is directed (in space) towards the negative z-axis. With the accelerating observer we shall now associate a small accelerated two-plane in the following way. In the local neighbourhood surrounding the observer, it is possible to define the concept of a two-plane. We consider a small two-plane which always remains at rest with respect to the observer. This means that at every point of the world line of the observer, we visualize a certain spacelike two-plane, constantly moving along with the observer. We assume that this two-plane is perpendicular to the z-axis, which means that the acceleration is directed perpendicular to the plane.

There are obvious physical reasons to require that the proper acceleration of the plane must be very large. When spacetime is curved, one may associate the ordinary Rindler wedge of the accelerating observer with the local neighbourhood of the point $\mathcal{P}$ only. Hence, if we want to employ the properties of Rindler spacetime in our calculations, we must analyze the thermodynamics of the plane in the limit where the proper acceleration a becomes very large. However, we shall not specify the actual magnitude of the proper acceleration in more detail. When the curvature of spacetime is reasonable large, one may always make the analysis sufficiently local by increasing the value of a. Only in very special circumstances, that is, when the effects of the curvature on the metric of spacetime become significant at the Planck scale of distances, our arguments probably fail to hold. In all what follows, we shall therefore always assume that the proper acceleration a is sufficiently large.

The equation for the worldline of our plane may be now written as

$$\big(z - z_0\big)^2 - \big(t - t_0\big)^2 = \frac{1}{a^2}, \tag{8.4}$$

where z_0 and t_0 are constants depending on the values of a and v at the point $\mathcal{P}$. In the (flat) tangent space of the point $\mathcal{P}$, these constants have solid geometrical interpretations (see Fig. 8.1). Equation (8.4) gives the equation of the worldline of the plane in an immediate vicinity of the point $\mathcal{P}$ with respect to the orthonormal geodesic coordinates t, x, y, and z. If we solve z from Eq. (8.4) and differentiate z with respect to the time coordinate t, we find that the velocity of the plane is, as a function of the time t,

$$\frac{dz}{dt} = \frac{a\,(t_0 - t)}{\sqrt{1 + a^2(t_0 - t)^2}}. \tag{8.5}$$

Hence, at the point $\mathcal{P}$, the velocity of the plane is

$$v = \frac{at_0}{\sqrt{1 + a^2 t_0^2}}. \tag{8.6}$$

It is convenient to write the velocity v by means of a new parameter $\epsilon \in (0, 1)$ such that

$$v = \frac{1-\epsilon}{1+\epsilon}, \tag{8.7}$$

and it follows from Eq. (8.6) that the constant t_0 may be expressed in terms of ϵ and a as

$$t_0 = \frac{1-\epsilon}{a\sqrt{2\epsilon}}. \tag{8.8}$$

As one may observe, for fixed a the quantity t_0 goes to infinity when ϵ goes to zero.

Now, what shall be the role of the parameter ϵ in our analysis? We see from Eq. (8.7) that ϵ describes the velocity of our plane at $\mathcal{P}$ with respect to the given system of coordinates. In the limit, where $\epsilon = 1$, the plane is at rest at the point $\mathcal{P}$. On the other hand, when ϵ takes its values within the interval $(0, 1)$, the plane has initially a certain velocity relative to the positive z-axis such that in the limit where $\epsilon \to 0$, the velocity becomes close to 1, the speed of light in the natural units. Obviously, for sufficiently small ϵ, the plane moves with relativistic speeds with respect to all matter fields, regardless of the properties of matter at $\mathcal{P}$. Similar results hold also vice versa: As ϵ approaches zero, the velocity of the flow of the matter fields across the plane approaches the speed of light. We have previously argued that under these circumstances the flow of heat vastly dominates other forms of energy transfer (the demonstration of this claim will be given in Sec. 8.3). Therefore, we shall henceforth always assume that the parameter ϵ becomes very small. Only in this limit, we may always interpret the energy flow through the plane as heat. As we shall soon see, in this limit the calculations also turn out relatively simple.

So far we have managed to find an appropriate parameter which determines the velocity of the matter flux across the accelerating two-plane. It is now time to formulate our ideas by using this parameter. We denote the future pointing unit tangent vector of the observer's worldline by ξ^μ and a spacelike unit normal vector of the plane by η^μ. Because the observer, together with the plane, is assumed to move in the direction perpendicular to the plane, the vectors ξ^μ and η^μ are orthogonal.

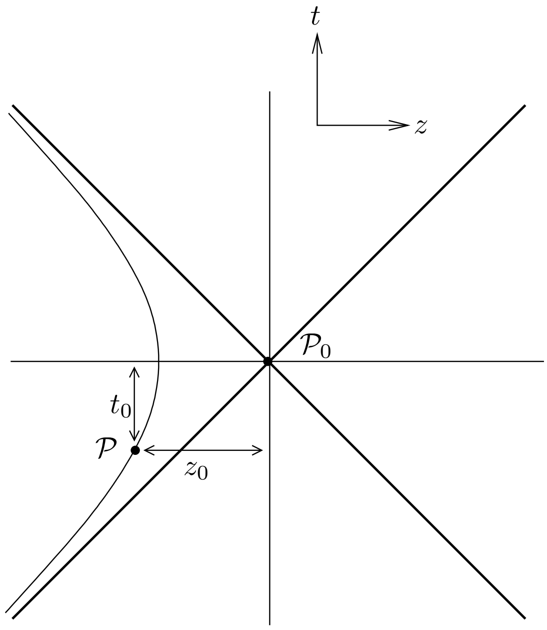

Figure 8.1. Geometrical interpretations of the constants t_0 and z_0. In this figure, the worldline of the accelerated two-plane (or, equivalently, the worldline of the accelerated observer) going through $\mathcal{P}$ is drawn in the frame of reference equipped with the geodesic coordinates t and z. The origin of the coordinates t and z should lie at the point $\mathcal{P}$. The past and the future Rindler horizons of the plane are the thick lines which intersect at the point $\mathcal{P}_0$. The constant t_0 is then the value of the coordinate t at the point $\mathcal{P}_0$, whereas the constant z_0 is the value of the coordinate z at $\mathcal{P}_0$.

Moreover, we choose η^μ in such a way that the observer is accelerated in the direction of the vector $-\eta^\mu$. Since the observer is assumed to move, at the point $\mathcal{P}$, with the velocity v to the direction of the positive z-axis, the non-zero components of the vectors ξ^μ and η^μ are

$$\xi^0 = \cosh\phi, \tag{8.9a}$$

$$\xi^3 = \sinh\phi, \tag{8.9b}$$

$$\eta^0 = \sinh\phi, \tag{8.9c}$$

$$\eta^3 = \cosh\phi, \tag{8.9d}$$

where

$$\phi := \text{arsinh}\Big(\frac{v}{\sqrt{1-v^2}}\Big) \tag{8.10}$$

is the boost angle. Using Eq. (8.7) we find:

$$\xi^\mu = \frac{1}{2}\left(\frac{k^\mu}{\sqrt{\epsilon}} + \sqrt{\epsilon}\, l^\mu\right), \tag{8.11a}$$

$$\eta^\mu = \frac{1}{2}\left(\frac{k^\mu}{\sqrt{\epsilon}} - \sqrt{\epsilon}\, l^\mu\right), \tag{8.11b}$$

where $k^\mu := (1,0,0,1)$ and $l^\mu := (1,0,0,-1)$ are null vectors. This means that when the parameter ϵ becomes small, the world line of the observer seems to lie very close to the null geodesic generated by the null vector k^μ. In the limit, where the proper acceleration a goes to infinity, the null vector k^μ becomes a generator of the past Rindler horizon of the observer moving with the plane, whereas the null vector l^μ becomes a generator of the future Rindler horizon (see Fig. 8.2).

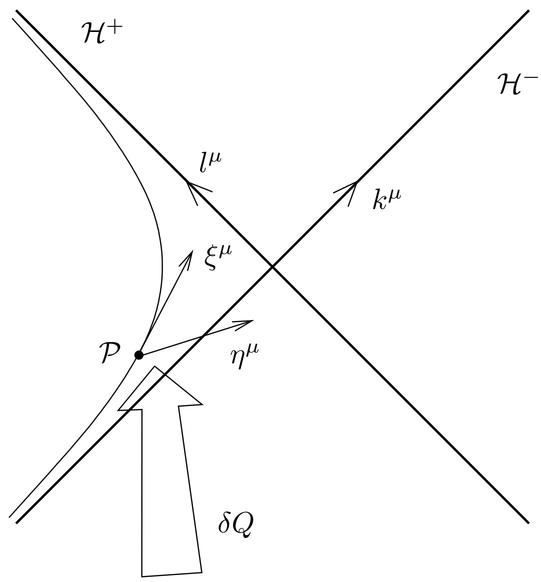

Figure 8.2. The worldline of the accelerated spacelike two-plane. ξ^μ is the future pointing unit tangent vector of the worldline, and η^μ is the spacelike unit normal vector of the plane. As one may observe, the worldline of the plane lies close to its past Rindler horizon $\mathcal{H}^-$, which is generated by the null vector k^μ, whereas its future Rindler horizon $\mathcal{H}^+$ is generated by the null vector l^μ. The large arrow represents the heat that flows through the past horizon.

8.3 Flow of Heat

Consider now the boost energy flow through our accelerating two-plane. At the point $\mathcal{P}$ the amount of boost energy flown through the plane during an infinitesimal proper time interval $d\tau$ to the direction of the spacelike unit normal vector η^μ of the plane is

$$\delta Q(\mathcal{P}) = A(\mathcal{P})\, T_{\mu\nu}(\mathcal{P})\, \xi^\mu(\mathcal{P})\, \eta^\nu(\mathcal{P})\, d\tau. \tag{8.12}$$

In this equation, $A(\mathcal{P})$ and $T_{\mu\nu}(\mathcal{P})$, respectively, are the area of the plane and the energy-momentum tensor of matter at the point $\mathcal{P}$. The boost energy flow through the plane (boost energy transferred through the plane in unit time) is therefore, at the point $\mathcal{P}$:

$$\frac{\delta Q(\mathcal{P})}{d\tau} = A(\mathcal{P})\, T_{\mu\nu}(\mathcal{P})\, \xi^\mu(\mathcal{P})\, \eta^\nu(\mathcal{P}). \tag{8.13}$$

We have denoted the boost energy flow by $\frac{\delta Q}{d\tau}$ for a very good reason: We shall see in a moment that when the plane moves with respect to the matter fields with a very great velocity, the boost energy flow is, in effect, the heat energy flow. Because of that we shall henceforth always talk about the heat flow, instead of the boost energy flow.

After a very short elapsed proper time interval τ, the plane has been moved from the point $\mathcal{P}$ to the point $\mathcal{P}'$. At the point $\mathcal{P}'$ the heat flow is

$$\frac{\delta Q(\mathcal{P}')}{d\tau} = A(\mathcal{P}')\, T_{\mu\nu}(\mathcal{P}')\, \xi^\mu(\mathcal{P}')\, \eta^\nu(\mathcal{P}'). \tag{8.14}$$

Since τ is assumed to be very small, we may write the energy-momentum tensor of matter at the point $\mathcal{P}'$ as a very good approximation:

$$T_{\mu\nu}(\mathcal{P}') = T_{\mu\nu}(\mathcal{P}) + \frac{dT_{\mu\nu}(\mathcal{P})}{d\tau}\tau = T_{\mu\nu}(\mathcal{P}) + T_{\mu\nu,\alpha}(\mathcal{P})\, \xi^\alpha(\mathcal{P})\, \tau, \tag{8.15}$$

where we have neglected the terms non-linear in τ. In the same way we may write the vector fields ξ^μ and η^μ at the point $\mathcal{P}'$:

$$\xi^\mu(\mathcal{P}') = \xi^\mu(\mathcal{P}) + a\tau\, \eta^\nu(\mathcal{P}), \tag{8.16a}$$

$$\eta^\mu(\mathcal{P}') = a\tau\, \xi^\mu(\mathcal{P}) + \eta^\mu(\mathcal{P}), \tag{8.16b}$$

where we have, again, neglected the terms non-linear in τ. In our hypothesis we have chosen the initial conditions for the plane such that

$$\frac{dA(\mathcal{P})}{d\tau} = 0, \tag{8.17}$$

and so we find that the change in the flow of heat through the plane during the proper time interval τ is

$$\begin{aligned}\frac{\delta Q(\mathcal{P}')}{d\tau} - \frac{\delta Q(\mathcal{P})}{d\tau} &= A(\mathcal{P})\, T_{\mu\nu,\alpha}(\mathcal{P})\, \xi^\mu(\mathcal{P})\, \eta^\nu(\mathcal{P})\, \xi^\alpha(\mathcal{P})\, \tau \\ &\quad + a\tau\, A(\mathcal{P})\, T_{\mu\nu}(\mathcal{P}) \left[\xi^\mu(\mathcal{P})\, \xi^\nu(\mathcal{P}) + \eta^\mu(\mathcal{P})\, \eta^\nu(\mathcal{P})\right].\end{aligned} \tag{8.18}$$

Hence the rate of change in the heat flow at the point $\mathcal{P}$ may be written in the form

$$\frac{\delta^2 Q}{d\tau^2} = \frac{\delta^2 Q_t}{d\tau^2} + \frac{\delta^2 Q_a}{d\tau^2}, \tag{8.19}$$

where we have defined:

$$\frac{\delta^2 Q_t}{d\tau^2} := A\, T_{\mu\nu,\alpha}\, \xi^\mu \eta^\nu \xi^\alpha, \tag{8.20a}$$

$$\frac{\delta^2 Q_a}{d\tau^2} := aA\, T_{\mu\nu}(\xi^\mu \xi^\nu + \eta^\mu \eta^\nu). \tag{8.20b}$$

To simplify the notations we have dropped references to the point $\mathcal{P}$. All quantities have been calculated at that point.

The first term on the right hand side of Eq. (8.19) is now due to the simple fact that the tensor $T_{\mu\nu}$ is different in different points of spacetime. That term has nothing to do with the acceleration of the plane. The second term, in turn, is a consequence of the change in the velocity of the plane with respect to the matter fields. In other words, it is due to the acceleration of the plane. We shall therefore call that term as the rate of change in the *acceleration heat flow*. If the components of the tensor $T_{\mu\nu}$ are assumed to be fixed and finite at the point $\mathcal{P}$ in a frame of reference, where the plane moves with a velocity very close to the speed of light, we find, using Eqs. (8.11), that the rate of change in the acceleration heat flow is

$$\frac{\delta^2 Q_a}{d\tau^2} = \frac{a}{2\epsilon}\, A T_{\mu\nu} k^\mu k^\nu + \mathcal{O}(\epsilon), \tag{8.21}$$

where $\mathcal{O}(\epsilon)$ denotes the terms, which are of the order ϵ, or higher.

At this point we should check, whether Eq. (8.21) really gives the rate of change in the flow of *heat* through our two-plane, regardless of what kind of matter we have. At the point $\mathcal{P}$ we have defined an orthonormal geodesic frame of reference such that the plane moves to the direction of the positive z-axis with a velocity v very close to that of light. This means that the matter fields move, in the rest frame of the plane, with enormous speeds through the plane. In this limit the kinetic energies of the particles of the matter fields vastly exceed the other forms of energy, and we may consider matter, in effect, as a gas of non-interacting massless particles. The energy density of such a gas is

$$\rho = T_{\mu\nu} \xi^\mu \xi^\nu, \tag{8.22}$$

and its pressure is[1]

$$p = T_{\mu\nu} \eta^\mu \eta^\nu. \tag{8.23}$$

According to the first law of thermodynamics the entropy S of a system depends on the energy E, pressure p, and the volume V of the system such that

$$T\, dS = dE + p\, dV. \tag{8.24}$$

[1]To be quite precise, Eq. (8.23) gives the pressure in a direction perpendicular to our plane. In that direction the momenta of the particles of the field vastly exceed the momenta in the other directions. So our system may, in effect, be considered as a one-dimensional gas of non-interacting massless particles, where there exists pressure in just one spatial direction. Nevertheless, we may apply Eq.(8.24), the first law of thermodynamics even for this kind of gas.

Since the energy density ρ of our gas does not depend on the volume V, we have

$$T\,dS = (\rho + p)\,dV, \tag{8.25}$$

and using Eqs. (8.22) and (8.23) we find that the entropy density (entropy per unit volume) of the gas is:

$$s = \frac{1}{T}\,(T_{\mu\nu}\,\xi^\mu\xi^\nu + T_{\mu\nu}\,\eta^\mu\eta^\nu). \tag{8.26}$$

Eq. (8.26) gives the entropy density of the gas at the spacetime point $\mathcal{P}$, in the rest frame of the plane. Since the proper acceleration of the plane is a we find, using Eq. (8.1), that the the change in the flow of heat caused by the mere acceleration of the plane during a very short proper time interval τ is

$$\frac{\delta Q_a(\mathcal{P}')}{d\tau} - \frac{\delta Q_a(\mathcal{P})}{d\tau} = a\tau A\,T s, \tag{8.27}$$

where all quantities on the right hand side have been calculated at the point $\mathcal{P}$. Using Eqs. (8.11) and (8.26) we therefore observe that the rate of change in the acceleration heat flow is

$$\frac{\delta^2 Q_a}{d\tau^2} = \frac{a}{2\epsilon} A\,T_{\mu\nu}\,k^\mu k^\nu + \mathcal{O}(\epsilon), \tag{8.28}$$

which is exactly Eq. (8.21).

By means of the rate of change in the acceleration heat flow we define the rate of change in the *acceleration entropy flow* as:

$$\frac{d^2 S_a}{d\tau^2} := \frac{1}{T}\frac{\delta^2 Q_a}{d\tau^2}. \tag{8.29}$$

The rate of change in the acceleration entropy flow is that part of the rate of change in the entropy flow through the plane, which caused by the mere acceleration of the plane, i.e., the change in the velocity of the plane with respect to the matter fields. If the absolute temperature T of the matter flowing through the plane is, from the point of view of an observer at rest with respect to the plane, the same as the Unruh temperature T_{U} we find, using Eqs. (8.2), (8.21) and (8.29), that

$$\frac{d^2 S_a}{d\tau^2} = \frac{\pi}{\epsilon} A\,T_{\mu\nu}\,k^\mu k^\nu + \mathcal{O}(\epsilon). \tag{8.30}$$

8.4 Change of Area

The next task is to determine the rate of change in the shrinking speed of the area A. To this end, consider the congruence of the timelike world lines of the points of our plane. We follow these worldlines, and we consider how the area of the spacelike two-surface, where the proper time τ measured along these worldlines is constant, will change as a function of τ. When the plane is at the point $\mathcal{P}$ of spacetime, $\tau = 0$ and the rate of change in the area, $\frac{dA}{d\tau}$, is assumed to vanish. Unfortunately, the proper time τ is not the best possible parameter for those world lines. Because of that we shall construct another parametrization for the elements of our congruence.

The starting point of our construction is that we pick up a time orthogonal, or Gaussian normal system of coordinates for spacetime such that, at the spacetime point $\mathcal{P}$ considered above, our coordinates coincide with the orthonormal geodesic coordinates t, x, y, and z defined in the local neighbourhood of that point. In these coordinates the line element of spacetime takes everywhere the form

$$ds^2 = -dt^2 + q_{mn}dx^m dx^n, \tag{8.31}$$

where $m, n = 1, 2, 3$, and q_{mn} is the metric on the spacelike hypersurface where $t =$ constant such that at the point $\mathcal{P}$ q_{mn} takes its flat space value δ_{mn}. When we move away from the point $\mathcal{P}$, however, q_{mn} will deviate very slightly from its flat space value.

The idea of our parametrization is now to use the time coordinate t of Eq. (8.31), instead of the proper time τ measured along the worldlines, as the parameter of the elements of our congruence. More precisely, for every worldline of the congruence we find in which way the coordinates x^μ ($\mu = 0, 1, 2, 3$) of the points of that worldline depend on t. After finding this dependence of the coordinates x^μ on the parameter t, we define the future pointing tangent vector field

$$w^\mu := \frac{dx^\mu}{dt} \tag{8.32}$$

for the elements of our congruence. It follows from the chain rule that between the vector field w^μ and the future pointing unit tangent vector field ξ^μ of the congruence there is the relationship:

$$w^\mu = \xi^\mu \frac{d\tau}{dt}. \tag{8.33}$$

Because

$$w^\alpha w_{\mu;\nu;\alpha} = w^\alpha w_{\mu;\alpha;\nu} - w^\alpha(w_{\mu;\alpha;\nu} - w_{\mu;\nu;\alpha}), \tag{8.34}$$

we get, using the product rule of covariant differentiation, and the basic properties of the Riemann tensor $R^\alpha_{\ \beta\mu\nu}$:

$$w^\alpha w_{\mu;\nu;\alpha} = (w^\alpha w_{\mu;\alpha})_{;\nu} - w^\alpha_{\ ;\nu} w_{\mu;\alpha} + R_{\beta\mu\nu\alpha} w^\alpha w^\beta. \tag{8.35}$$

If we contract the indices μ and ν, use the symmetry properties of the Riemann tensor, and rename the indices, we finally get:

$$\frac{d\theta}{dt} = C^\mu_{\ ;\mu} - w^\mu_{\ ;\nu} w^\nu_{\ ;\mu} + R_{\mu\nu} w^\mu w^\nu, \tag{8.36}$$

where $R_{\mu\nu}$ is the Ricci tensor of spacetime. In Eq. (8.36) we have defined:

$$\theta := w^\mu_{\ ;\mu}, \tag{8.37a}$$

$$C^\mu := w^\alpha w^\mu_{\ ;\alpha}. \tag{8.37b}$$

Equation (8.36) is the key equation in our derivation of Einstein's field equation from thermodynamical considerations.

To see what all this means consider, as an example, the special case where spacetime is everywhere flat, and all points of all elements of the congruence are accelerated with a constant proper acceleration a to the direction of the negative z-axis. In that case it follows from Eq. (8.4) that the only coordinates of the points of the elements of the congruence having explicit dependence on t are the coordinates x^0 and x^3 such that

$$x^0 = t, \tag{8.38a}$$

$$x^3 = z = -\sqrt{\frac{1}{a^2} + (t - t_0)^2} + z_0, \tag{8.38b}$$

and therefore the only non-zero components of the vector field w^μ are

$$w^0 = 1, \tag{8.39a}$$

$$w^3 = \frac{a(t_0 - t)}{\sqrt{1 + a^2(t - t_0)^2}}, \tag{8.39b}$$

and the only non-zero component to the vector field C^μ is

$$C^3 = -\frac{2a^3(t - t_0)^2 + a}{\left[1 + a^2(t - t_0)^2\right]^{3/2}}. \tag{8.40}$$

One easily finds that the quantity θ vanishes identically, as well as do the first two terms on the right hand side of Eq. (8.36). Moreover, since $R_{\mu\nu} \equiv 0$ in flat spacetime, we find that Eq. (8.36) is indeed satisfied by the tangent vector field w^μ of our congruence.

When spacetime is flat, there is no change in the area of our accelerated plane during the course of time. However, when spacetime is curved, the area will change. It follows from the considerations made above that when spacetime is curved, Eq. (8.36) is written, at the point $\mathcal{P}$ where $\tau = t = 0$, in the form

$$\frac{d\theta}{dt} = R_{\mu\nu}\, w^\mu w^\nu \tag{8.41}$$

or, if we use Eq. (8.33) and the chain rule,

$$\frac{d\theta}{d\tau} = R_{\mu\nu}\, \xi^\mu \xi^\nu\, \frac{d\tau}{dt}. \tag{8.42}$$

Since θ vanishes at the point $\mathcal{P}$, we therefore find that after a very short elapsed proper time τ we have:

$$\theta = R_{\mu\nu}\, \xi^\mu \xi^\nu\, \frac{d\tau}{dt}\, \tau, \tag{8.43}$$

where we have neglected the terms non-linear in τ.

The change in the area of the plane may be calculated by means of θ. It follows from the definition of w^μ that $w^0 \equiv 1$, and therefore $w^0{}_{;0}$ vanishes identically in the Gaussian normal system of coordinates of Eq. (8.31). So we have

$$\theta = w^1{}_{;1} + w^2{}_{;2} + w^3{}_{;3}. \tag{8.44}$$

In the immediate vicinity of the point $\mathcal{P}$, where we have an orthonormal geodesic system of coordinates, the change in the area A of our plane during an infinitesimal time interval dt is

$$dA = (w^1{}_{;1} + w^2{}_{;2})A\,dt, \tag{8.45}$$

and hence it follows that

$$dA = \theta A\,dt, \tag{8.46}$$

provided that we are able to show that $w^3{}_{;3}$ vanishes when ϵ becomes small, i.e., that

$$\lim_{\epsilon \to 0} |w^3{}_{;3}| = 0. \tag{8.47}$$

We shall now demonstrate Eq. (8.47).

Assume that every point of the worldline of the plane has a neighbourhood in the rest frame of the plane, where the vector field w^μ is smooth and timelike for every $\epsilon > 0$, and that the functions obtained as the limits of w^μ and $w^\mu{}_{;\alpha}$, when ϵ goes to zero, are continuous. At the point $\mathcal{P}$ the covariant derivative $w^3{}_{;3}$ vanishes. On the other hand, in the region near to that point, the metric of spacetime differs slightly from the metric of flat spacetime, and $w^3{}_{;3}$ becomes non-zero. Therefore, consider a situation where the plane has been accelerated during a very short time interval $t < t_0$. If we neglect the terms non-linear in t, the z-coordinate along the plane is vt. It follows from our assumptions that there exists a fixed number $L > 0$ such that the vector field w^μ is smooth and timelike within the interval, where $vt - \gamma^{-1}L \leq z \leq vt + \gamma^{-1}L$. The factor

$$\gamma^{-1} := \left[1 - \frac{a^2(t_0 - t)^2}{1 + a^2(t_0 - t)^2}\right]^{1/2} \tag{8.48}$$

is due to the Lorentz contraction. Furthermore, for small L we may write as a very good approximation:

$$w^3_- = w^3 - w^3{}_{;3}\,\gamma^{-1}L, \tag{8.49a}$$

$$w^3_+ = w^3 + w^3{}_{;3}\,\gamma^{-1}L, \tag{8.49b}$$

In these equations w^3 and $w^3{}_{;3}$ have been calculated at the point, where $z = vt$. Moreover, w^3_- and w^3_+ are, respectively, the values of the z-component of the vector w^μ at the points, where $z = vt - \gamma^{-1}L$ and $z = vt + \gamma^{-1}L$, in a basis which has been parallel transported from the point, where $z = vt$, to the points, where

$z = vt \pm \gamma^{-1}L$. Because $w^0 \equiv 1$, and the vector w^μ is timelike, we must have $|w^3_\pm| < 1$. Hence, by using Eq. (8.49a) for negative values of $w^3_{;3}$ and Eq. (8.49b) for positive values of $w^3_{;3}$, one concludes, by means of Eq. (8.39b), that

$$|w^3_{;3}| < \frac{\gamma}{L}\left[1 - \frac{a(t_0 - t)}{\sqrt{1 + a^2(t - t_0)^2}}\right]. \tag{8.50}$$

It follows easily from Eq. (8.8) that when ϵ approaches to zero, the right hand side of this equation falls to zero as $\sqrt{\epsilon/2}$. This proves Eq. (8.47), and therefore Eq. (8.46) holds. Using Eq. (8.43), the chain rule and Eq. (8.11) we find that the change in the area of the accelerated plane during an infinitesimal proper time interval $d\tau$ is

$$dA = A\,R_{\mu\nu}\,\xi^\mu\xi^\nu\,\tau\,d\tau = A\,R_{\mu\nu}\left(\frac{k^\mu k^\nu}{4\epsilon} + \mathcal{O}(1)\right)\tau\,d\tau, \tag{8.51}$$

where $\mathcal{O}(1)$ denotes the terms of the order ϵ^0 or higher. Clearly, this equation implies $\frac{dA}{d\tau}|_{\tau=0} = 0$ as required by our hypothesis. Hence we find that at the point $\mathcal{P}$:

$$\frac{d^2A}{d\tau^2} = \frac{1}{4\epsilon}\,A\,R_{\mu\nu}\,k^\mu k^\nu + \mathcal{O}(1). \tag{8.52}$$

It is obvious that since ϵ is assumed to be very small, the leading term is the one which is proportional to $1/\epsilon$. We shall see in the next section that when matter flows through our accelerating plane, the plane will shrink. The negative of the right hand side of Eq. (8.52) therefore gives the rate of change in the shrinking speed of the plane.

8.5 Einstein's Field Equation

We are now prepared to obtain Einstein's field equation by means of thermodynamical arguments. We shall assume that the temperature of the matter flowing through our accelerating plane is, from the point of view of an observer at rest with respect to the plane, exactly the Unruh temperature T_{U} measured by that observer. Our hypothesis (8.3), together with Eqs. (8.30) and (8.52) then implies, at the point $\mathcal{P}$:

$$\frac{\pi}{\epsilon}A\,T_{\mu\nu}\,k^\mu k^\nu + \mathcal{O}(\epsilon) = -\frac{1}{8\epsilon}A\,R_{\mu\nu}\,k^\mu k^\nu + \mathcal{O}(1). \tag{8.53}$$

In the limit, where $\epsilon \to 0$, i.e., when the velocity of the plane becomes close to the speed of light one finds that, in the leading order for small ϵ,

$$R_{\mu\nu}\,k^\mu k^\nu = -8\pi\,T_{\mu\nu}\,k^\mu k^\nu. \tag{8.54}$$

Because k^μ may be chosen to an arbitrary null vector, we must have

$$R_{\mu\nu} + f\,g_{\mu\nu} = -8\pi\,T_{\mu\nu}, \tag{8.55}$$

where f is some function of the spacetime coordinates. It follows from the Bianchi identity

$$\left(R^{\mu}_{\ \nu} - \frac{1}{2}R\,\delta^{\mu}_{\ \nu}\right)_{;\mu} = 0 \tag{8.56}$$

that $f = -\frac{1}{2}R + \Lambda$ for some constant Λ, and hence we arrive at the equation

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda\,g_{\mu\nu} = -8\pi\,T_{\mu\nu}, \tag{8.57}$$

which is Einstein's field equation with the cosmological constant Λ.

8.6 Cosmological Constant

It is a common feature of all attempts made so far to obtain Einstein's field equation by means of thermodynamical considerations that they leave completely unspecified the cosmological constant Λ [88]. This is a pity, because cosmological constant presents one of the most intriguing problems of modern physics: Why is the cosmological constant so close to zero? Indeed, according to the astronomical observations the absolute value of Λ is, in SI units, certainly less than $10^{-35}1/\mathrm{s}^2$, which corresponds to the vacuum energy density

$$\rho_{\text{vac}} \sim \frac{\Lambda c^2}{G} \sim 10^{-8}\mathrm{J/m^3}. \tag{8.58}$$

This energy density, in turn, corresponds to the mass density, where we have 1 kilogram of matter inside a cube whose edge length is about the same as is the distance between the Earth and the Moon. Is there some perennial physical principle, which constrains the cosmological constant to be so incredibly small?

It is interesting that this question may be addressed by means of our hypothesis contained in Eq. (8.3). At the same time, our investigations will act as a consistency check of our analysis.

Consider a spacetime filled with isotropic electromagnetic radiation in thermal equilibrium and at rest with respect to our plane at the point $\mathcal{P}$. In other words, we shall assume that at the point $\mathcal{P}$ the boost energy flow of the radiation through the plane is zero. The energy density ρ of the radiation is given by Eq. (8.22), and the pressure p is given by Eq. (8.23). It is a specific property of this kind of radiation that between the energy density and the pressure of the radiation there is the relationship:

$$p = \frac{1}{3}\rho, \tag{8.59}$$

and that the tensor $T_{\mu\nu}$ is *traceless*, i.e.,

$$T^{\alpha}_{\ \alpha} = 0. \tag{8.60}$$

Using Eqs. (8.20b), (8.22) and (8.23), together with Eq. (8.59) we find that the rate of change in the acceleration heat flow through the plane is

$$\frac{\delta^2 Q_a}{d\tau^2} = \frac{4}{3} a A \rho = \frac{4}{3} a A\, T_{\mu\nu}\, \xi^\mu \xi^\nu. \tag{8.61}$$

The same result may also be obtained by means of the well-known fact that the entropy density of electromagnetic radiation in thermal equilibrium is [89]

$$s = \frac{4}{3T}\rho. \tag{8.62}$$

Eq. (8.61) follows straightforwardly from Eq. (8.62) and the relation $\delta Q = T\, dS$. So we see that Eq. (8.61) indeed gives the rate of change in the flow of *heat* through our accelerating plane. According to the definition (8.29), the rate of change in the acceleration entropy flow corresponding to the acceleration heat flow of Eq. (8.61) is

$$\frac{d^2 S_a}{d\tau^2} = \frac{8\pi}{3} A\, T_{\mu\nu}\, \xi^\mu \xi^\nu, \tag{8.63}$$

where we have, again, identified the absolute temperature T as the Unruh temperature $T_{\rm U}$ of Eq. (8.2).

Consider now the change in the area A of the spacelike two-plane. Since spacetime is assumed to be filled with isotropic, massless radiation in thermal equilibrium, and to be at rest with respect to the plane at the point $\mathcal{P}$, spacetime may be assumed to be isotropic in the neighborhood of the point $\mathcal{P}$. In other words, spacetime expands and contracts in the same ways in all spatial directions. In that case $w^3{}_{;3}$ no more vanishes, but $w^1{}_{;1}$, $w^2{}_{;2}$ and $w^3{}_{;3}$ are equals. So it follows from Eqs. (8.43), (8.44) and (8.45) that

$$\frac{d^2 A}{d\tau^2} = \frac{2}{3} A R_{\mu\nu} \xi^\mu \xi^\nu. \tag{8.64}$$

Using Eqs. (8.3), (8.63) and (8.64) we therefore find:

$$R_{\mu\nu}\, \xi^\nu \xi^\nu = -8\pi\, T_{\mu\nu}\, \xi^\mu \xi^\nu. \tag{8.65}$$

Since ξ^μ is an arbitrary, future directed timelike unit vector field, we must have

$$R_{\mu\nu} = -8\pi T_{\mu\nu}, \tag{8.66}$$

which is exactly Einstein's field equation

$$R_{\mu\nu} = -8\pi (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha_{\ \alpha}), \tag{8.67}$$

or

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} \tag{8.68}$$

in the special case, where the tensor $T_{\mu\nu}$ is traceless, i.e., Eq. (8.60) holds. So we have obtained Einstein's field equation from our hypothesis in the special case, where matter consists of electromagnetic (or any massless) radiation in thermal equilibrium. In doing so, we have not assumed that the plane would move with a speed close to that of light with respect to the matter fields. Instead, the plane was assumed to be, at the point $\mathcal{P}$, at rest with respect to the radiation.

When we compare Eqs. (8.57) and (8.68), we find that, in contrast to Eq. (8.57), Eq. (8.68) does not involve the cosmological constant Λ. In other words, the cosmological constant must vanish:

$$\Lambda = 0 \tag{8.69}$$

Our thermodynamical approach to gravity therefore makes a precise prediction, which is consistent with the present observations, which imply that the cosmological constant, although not necessarily exactly zero, must nevertheless be extremely small [90]. This result may be viewed as an argument supporting the idea of a thermodynamical origin of gravitation.

When compared to the earlier thermodynamical approaches to gravity, where the value of the cosmological constant has been left unspecified, it may be surprising that in our approach the cosmological constant becomes fixed. There is, however, a natural explanation for this difference. The primary reason why we were able to fix the cosmological constant was our choice to investigate the thermodynamical properties of an accelerating two-plane, instead of the thermodynamical properties of a null surface. Indeed, in Secs. 8.2-8.5 we first considered the two-plane in the case where it moved, in the coordinate system under investigation, very close to a certain null surface, with a very great speed with respect to the matter fields, and this consideration brought along an arbitrary cosmological constant. However, because the object under investigation was an accelerating two-plane, instead of a null surface, we were also able to consider an entirely different physical situation, where the two-plane no longer moved close to any null surface, but instead it was assumed to be at rest with respect to a specific matter field. Considering this specific case we found that the cosmological constant must vanish. In the other thermodynamical approaches to gravity made so far, in turn, one has always considered a null surface of spacetime, and therefore the cosmological constant cannot be fixed by means of arguments similar to those used in this section. Actually, it is interesting that in our approach an arbitrary cosmological constant appears in the very situation where the two-plane moves very close to a certain null surface. This suggests that the appearance of an unspecified cosmological constant is a characteristic feature of the thermodynamical theories of null surfaces.

8.7 Concluding Remarks

In this chapter we have obtained Einstein's field equation with a vanishing cosmological constant by means of very simple thermodynamical arguments concerning the properties of a very small spacelike two-plane in a uniformly accelerating motion.

Our derivation was based on a hypothesis that when matter flows through the plane, and the temperature of the matter is the same as the Unruh temperature measured by an observer at rest with respect to the plane, then the rate of change in the flow of entropy caused by the mere acceleration of the plane is, in natural units, exactly one-half of the rate of change in the shrinking speed of the area of the plane. From this hypothesis we obtained, by means of the fundamental thermodynamical relation $\delta Q = T\,dS$, Einstein's field equation with a vanishing cosmological constant.

When spacetime is filled with isotropic, massless, non-self-interacting radiation field (electromagnetic field, for instance) in thermal equilibrium, it is very easy to obtain Einstein's field equation by means of thermodynamical arguments, because it turns out that in this case the boost energy flow through the plane is exactly the heat flow of the radiation. However, if the fields are massive, or self-interacting, the situation becomes more complicated because the boost energy flow involves other forms of energy, except heat, as well (mass-energy, for instance). In that case we may consider the situation, where the plane moves with respect to the matter fields with a velocity very close to that of light. When the plane moves with respect to the matter fields with an enormous velocity, it turns out that the amount of heat vastly exceeds the amounts of other forms of energy carried by matter through the plane. Our derivation of Einstein's field equation for general matter fields brought along an unspecified cosmological constant.

To fix the cosmological constant we then derived Einstein's field equation in the special case where matter consists of massless radiation in thermal equilibrium and at rest with respect to our plane. The resulting equation, however, did not involve the cosmological constant, and therefore we concluded that the cosmological constant must vanish.

Our derivation of Einstein's field equation by means of purely thermodynamical arguments provides support for the idea, earlier expressed by Jacobson, that Einstein's field equation may actually be understood as a thermodynamical equation of state of spacetime and matter fields. Indeed, our thermodynamical approach to gravitation even seems to explain the incredible smallness of the cosmological constant.

Although our thermodynamical derivation of Einstein's field equation bears a lot of similarities with Jacobson's derivation, it should be strongly emphasized the radical difference between these two derivations: Jacobson considered the boost energy flow through a *horizon* of spacetime, whereas we considered the boost energy flow through an accelerating, spacelike two-plane. Horizons of spacetime are null hypersurfaces of spacetime, and therefore they are created, when all points of a spacelike two-surface move along certain *null* curves of spacetime. In contrast, our spacelike two-plane was assumed to move in spacetime with a speed less than that of light, and therefore all of its points move along *timelike* curves of spacetime. Because of that our two-plane should not be considered as a part of any horizon of spacetime. Nevertheless, we found that if the entropy carried by matter through the plane is connected with the change in its area in a certain manner, then Einstein's field equation follows. The fact that an assumption of a simple proportionality between the rates of changes in the entropy flow and in the shrinking speed of the plane yields Einstein's field equation even when that two-plane is not a part of any horizon of spacetime strongly suggests that one may associate meaningfully the concept of gravitational entropy not only with horizons, but also with arbitrary spacelike two-surfaces of spacetime. It is still uncertain what the consequences of such a possibility may be, but they will most likely have some influence on our views of the nature of gravitational entropy.

Epiloque

The time has come to make some final comments on the issues discussed in this thesis. It seems that today the research on quantum gravity focuses heavily on the string theory and few other approaches. Most of these approaches are based, in one way or another, on a quantization of some classical field. The old quantization methods have certainly given useful insight into many issues, but whether they are able to yield a completely satisfactory theory of quantum gravity, only time will tell. Personally, I find myself more attracted to new and fresh ideas.

In this work, we have learned that the connection between gravitation and thermodynamics may be much deeper than previously expected. The Hawking effect seems to be present at the inner horizons of charged black holes, and this supports the idea that all spacetime horizons emit radiation. Moreover, we have seen that it is possible to construct a microscopic model of a Schwarzschild horizon, where spacetime is made of microscopically small black holes. From this model one can obtain the Bekenstein-Hawking entropy law. The most surprising finding in this work, however, is probably the possibility that with any spacelike two-surface one may associate a certain statistical entropy. Although these issues surely need more investigation, we were able to show that in certain circumstances such entropy, together with the first law of thermodynamics, implies Einstein's field equation. This result supports the view that Einstein's field equation is actually a thermodynamical equation of state. Consequently, it may not be appropriate to quantize general relativity as if it were an ordinary field theory. Rather, this view suggests to seek (still unknown) microscopic entities whose statistical properties would imply Einstein's equations.

In this thesis, the picture given about the possible structure of spacetime is very different than, for instance, in loop quantum gravity and string theory. However, in the light of this work, I feel that it is at least worth of an effort to see where these kinds of ideas will lead us. No doubt, it requires a great deal of foolhardiness to step outside the familiar paradigms and try to develop an entirely novel approach to the problem of quantum gravity. In the end, however, it may well be that a sounder comprehension of gravitation and spacetime requires completely revised physical principles. It is tempting to follow the well-established enterprises of quantum gravity. Yet, it is my humble wish that the scientists of tomorrow would find courage to follow their own ideas and boldly go where no man has gone before.

Appendix A

Infinitesimal Geodesic Coordinates Near the Horizons of the Reissner-Nordström Spacetime

Consider geodesic coordinates near the outer horizon of a Reissner-Nordström black hole. More precisely, consider a geodesic coordinate system in the region I infinitesimally close to the point P_1 of Fig. 5.4. By using the definitions (5.41) such that $\alpha = \kappa_+$ in Eqs. (5.37), the metric in the region I of Reissner-Nordström spacetime can be written as

$$ds^2 = \frac{1}{l_+^2}\frac{1}{\kappa_+^2 r^2} e^{-2\kappa_+ r}(r - r_-)^{\frac{r_-^2}{r_+^2}+1}\left(-\left(dX^0\right)^2 + \left(dX^1\right)^2\right)$$
$$+ r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{A.1}$$

Especially, on the two-surface $r = r_+$ the metric takes a very simple form

$$ds^2 = -\left(dX^0\right)^2 + \left(dX^1\right)^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{A.2}$$

Thus, the coordinates X^0 and X^1 provide an infinitesimal geodesic coordinate system for the freely falling observer if only the derivatives of the metric with respect to X^0 and X^1 vanish at the point P_1. According to Eq. (A.1) the components of the metric depend merely on r. Moreover, in order to show that the derivatives of the metric vanish, it is sufficient to show that

$$\frac{\partial r}{\partial u} = \frac{\partial r}{\partial v} = 0 \tag{A.3}$$

at the point P_1.

Let us first note that the relationship between the coordinates u, v, and r can be expressed in an implicit form such that

$$u^2 - v^2 = e^{2\kappa_+ r_*}. \tag{A.4}$$

By differentiating both sides with respect to u, one gets

$$2u = 2\kappa_+ \, e^{2\kappa_+ r_*} \frac{dr_*}{dr} \frac{\partial r}{\partial u}, \tag{A.5}$$

and similarly, differentiation with respect to v gives

$$-2v = 2\kappa_+ \, e^{2\kappa_+ r_*} \frac{dr_*}{dr} \frac{\partial r}{\partial v}. \tag{A.6}$$

Since all equations (5.37) must be satisfied at P_1, one finds that $u = v = 0$ at P_1. However, it is easy to see that when $r = r_+$,

$$2\kappa_+ \, e^{2\kappa_+ r_*} \frac{dr_*}{dr} = 2\kappa_+ r_+^2 \, e^{2\kappa_+ r_+} (r_+ - r_-)^{-\left(\frac{r_-^2}{r_+^2}+1\right)} \neq 0. \tag{A.7}$$

Hence, Eq. (A.3) is satisfied at the point P_1.

Next, let us concentrate on the geodesic coordinates near the inner horizon. Consider a geodesic coordinate system in the region VI', infinitesimally close to the point P_2 of Fig. 5.5. In this case, we choose $\alpha = \kappa_-$ for the Kruskal-type coordinates. By using the definitions (5.68) and (5.69) the metric in the region VI' can be written as

$$\begin{aligned} ds^2 = \frac{1}{l_-^2} \frac{1}{\kappa_-^2 r^2} e^{-2\kappa_- r} (r_+ - r)^{\frac{r_+^2}{r_-^2}+1} \left(-\left(dX^0\right)^2 + \left(dX^1\right)^2 \right) \\ + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2). \end{aligned} \tag{A.8}$$

When $r = r_-$, the metric has the form of that of Eq. (A.2), and, therefore, the coordinates X^0 and X^1 provide an infinitesimal geodesic coordinate system if the derivatives of the metric vanish at the point P_2. Again, one easily sees that it is sufficient to show that Eq. (A.3) holds also at P_2.

To show this, we proceed as before. The relationship between the coordinates u, v, and r can now be expressed as

$$u^2 - v^2 = e^{2\kappa_- r_*}. \tag{A.9}$$

Differentiating both sides with respect to u and v gives, respectively,

$$2u = 2\kappa_- \, e^{2\kappa_- r_*} \frac{dr_*}{dr} \frac{\partial r}{\partial u}, \tag{A.10}$$

$$-2v = 2\kappa_- \, e^{2\kappa_- r_*} \frac{dr_*}{dr} \frac{\partial r}{\partial v}. \tag{A.11}$$

Similarly as before, we have $u = v = 0$ at P_2. Furthermore, one easily finds that, when $r = r_-$,

$$2\kappa_-\, e^{2\kappa_- r_*} \frac{dr_*}{dr} = 2\kappa_- r_-^2\, e^{2\kappa_- r_-} (r_+ - r_-)^{-\left(\frac{r_+^2}{r_-^2}+1\right)} \neq 0. \qquad \text{(A.12)}$$

Hence, Eq. (A.3) is satisfied also at the point P_2.

Appendix B

Mass Eigenvalues

During the preparation of the paper I, we studied the mass eigenvalues of Eq. (7.3), when $s = 2$, for small n in two different ways. The first method we used is perturbation theory. The second method is to solve Eq. (7.3) numerically. Both of these methods, of which the second one is more reliable, seem to imply that Eq. (7.5) provides an excellent approximation for the mass eigenvalues even for small n.

B.1 Perturbation Theoretic Approach

As a starting point we consider Eq. (7.3) when $s = 2$:

$$\left[-\frac{1}{2a}\left(\frac{d^2}{da^2}+\frac{1}{a}\frac{d}{da}\right)+\frac{1}{2}a\right]\psi(a) = M\psi(a). \tag{B.1}$$

We write the Hamiltonian of Eq. (B.1) in the form

$$\hat{H} = -\frac{1}{2a}\frac{d^2}{da^2}-\frac{1}{2a^2}\frac{d}{da}+\frac{1}{2}a =: \hat{H}_0+\hat{H}', \tag{B.2}$$

where

$$\hat{H}_0 := -\frac{1}{2a}\frac{d^2}{da^2}+\frac{1}{2}a \tag{B.3}$$

and

$$\hat{H}' := -\frac{1}{2a^2}\frac{d}{da}. \tag{B.4}$$

If we neglect the term $\hat{H}'$ in Eq. (B.1) and denote

$$u := a - M, \tag{B.5}$$

we see that the resulting differential equation takes the form

$$\left(-\frac{1}{2}\frac{d^2}{du^2}+\frac{1}{2}u^2\right)\psi(u) = \frac{1}{2}M^2\psi(u). \tag{B.6}$$

This equation is similar to that of a one-dimensional harmonic oscillator, and therefore members of one set of solutions to Eq. (B.6) are of the form

$$\psi_n^{(0)}(a) = N_n H_n(a - M_n^{(0)}) e^{-\frac{1}{2}(a - M_n^{(0)})^2}, \tag{B.7}$$

where H_n denotes the Hermite polynomial of order n, N_n is a normalization constant, and the eigenvalues $M_n^{(0)}$ are

$$M_n^{(0)} = \sqrt{2n+1}. \tag{B.8}$$

One can see that the eigenvalues of Eq. (B.8) are the same as the WKB eigenvalues in Eq. (7.5).

Now we consider the term $\hat{H}'$ as a small perturbation of the unperturbed Hamiltonian $\hat{H}_0$. Here we must use somewhat "non-standard" methods for calculating the first-order perturbation because the Hamiltonian $\hat{H}_0$ is not a Hermitian operator with respect to the inner product (7.4). Usually, when using perturbation theory, one writes the Hamiltonian in the form $\hat{H} = \hat{H}_0 + \lambda \hat{H}'$, where the perturbation parameter $\lambda \in [0,1]$. During the calculation of first-order perturbation one must operate with $\hat{H}_0$ on the bra-vectors $\langle \psi_n^{(0)}|$. The problem is that if $\hat{H}_0$ is not a Hermitian operator, it cannot be transported from the right to the left hand side inside the Dirac brackets, and the first-order perturbation cannot be calculated in a standard way. However, if we assume that our perturbation expansion is viable when $\lambda = 1$, we can set right from the beginning that $\lambda = 1$. In this case the operator $\hat{H}_0 + \lambda \hat{H}$ is indeed a Hermitian operator and one can operate with this operator on the bra-vectors in the usual manner. Using this trick, one can calculate the first-order perturbation. Let us begin with the perturbation expansions

$$|\Phi_n\rangle = |\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \cdots, \tag{B.9}$$

$$M_n = M_n^{(0)} + \lambda M_n^{(1)} + \lambda^2 M_n^{(2)} + \cdots, \tag{B.10}$$

where the vectors $|\Phi_n\rangle$ are eigenstates of the Hamiltonian $\hat{H}$ and the numbers M_n are the corresponding eigenvalues. The symbols $|\psi_n^{(k)}\rangle$ and $M_n^{(k)}$ denote corrections of order k to the unperturbed eigenstate $|\psi_n^{(0)}\rangle$ and to the eigenvalue $M_n^{(0)}$. When these expansions are inserted into the equation $\langle \psi_n^{(0)}|\hat{H}|\Phi_n\rangle = \langle \psi_n^{(0)}|M_n|\Phi_n\rangle$, we get

$$\begin{aligned}
&\langle \psi_n^{(0)}|(\hat{H}_0 + \lambda \hat{H}')|\psi_n^{(0)}\rangle + \lambda \langle \psi_n^{(0)}|(\hat{H}_0 + \lambda \hat{H}')|\psi_n^{(1)}\rangle + \\
&\qquad \lambda^2 \langle \psi_n^{(0)}|(\hat{H}_0 + \lambda \hat{H}')|\psi_n^{(2)}\rangle + \cdots \\
&= \Big(M_n^{(0)} + \lambda M_n^{(1)} + \lambda^2 M_n^{(2)} + \cdots \Big) \times \\
&\qquad \Big(\langle \psi_n^{(0)}|\psi_n^{(0)}\rangle + \lambda \langle \psi_n^{(0)}|\psi_n^{(1)}\rangle + \lambda^2 \langle \psi_n^{(0)}|\psi_n^{(2)}\rangle + \cdots \Big)
\end{aligned}$$

$$\Leftrightarrow \langle(\hat{H}_0 + \lambda\hat{H}')\psi_n^{(0)}|\psi_n^{(0)}\rangle + \lambda\langle(\hat{H}_0 + \lambda\hat{H}')\psi_n^{(0)}|\psi_n^{(1)}\rangle +$$
$$\lambda^2\langle(\hat{H}_0 + \lambda\hat{H}')\psi_n^{(0)}|\psi_n^{(2)}\rangle + \cdots$$
$$= \Big(M_n^{(0)} + \lambda M_n^{(1)} + \lambda^2 M_n^{(2)} + \cdots \Big) \times$$
$$\Big(1 + \lambda\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle + \lambda^2\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle + \cdots\Big)$$
$$\Leftrightarrow \lambda\Big(\langle\psi_n^{(0)}|\hat{H}'|\psi_n^{(0)}\rangle - M_n^{(1)}\Big) +$$
$$\lambda^2\Big(\langle\hat{H}'\psi_n^{(0)}|\psi_n^{(1)}\rangle - M_n^{(1)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle - M_n^{(2)}\Big) + \cdots = 0. \quad \text{(B.11)}$$

Hence the first-order perturbation is of the standard form

$$M_n^{(1)} = \langle\psi_n^{(0)}|\hat{H}'|\psi_n^{(0)}\rangle. \quad \text{(B.12)}$$

Let us now calculate the first-order perturbation explicitly by using the definition (7.4) of the inner product. We get

$$M_n^{(1)} = \langle\psi_n^{(0)}|\hat{H}'|\psi_n^{(0)}\rangle = \int_0^\infty \psi_n^{(0)*}(a)\Big(-\frac{1}{2a^2}\frac{d}{da}\psi_n^{(0)}(a)\Big)a^2\,da$$
$$= -\frac{1}{2}N_n^2\int_0^\infty H_n\big(a - M_n^{(0)}\big)\,e^{-(a-M_n^{(0)})^2}\Big[H_n'\big(a - M_n^{(0)}\big) +$$
$$\big(M_n^{(0)} - a\big)\cdot H_n\big(a - M_n^{(0)}\big)\Big]\,da, \quad \text{(B.13)}$$

where $H_n'\big(a - M_n^{(0)}\big) = \frac{d}{da}H_n\big(a - M_n^{(0)}\big)$, and the coefficients N_n are determined by the requirement

$$\int_0^\infty \psi_n^{(0)*}(a)\psi_n^{(0)}(a)a^2\,da = 1. \quad \text{(B.14)}$$

It turns out that the integral in Eq. (B.13) can be written in a very simple form:

$$M_n^{(1)} = \frac{1}{4}N_n^2 H_n^2\big(M_n^{(0)}\big)\,e^{-M_n^{(0)\,2}}. \quad \text{(B.15)}$$

Integrals in Eq. (B.14) can be solved analytically by means of the error function $\mathrm{erf}(x)$, but unfortunately a simple general formula for the coefficient N_n seems to be out of reach. However, one easily finds that

$$N_0^2 = \Big[\frac{1}{2}\,e^{-1} + \frac{3}{4}\sqrt{\pi}\big[1 + \mathrm{erf}(1)\big]\Big]^{-1}, \quad \text{(B.16)}$$

and therefore, according to Eq. (B.15), we have

$$M_0^{(1)} = \frac{1}{4}\Big[\frac{1}{2} + \frac{3}{4}\,e\sqrt{\pi}\big[1 + \mathrm{erf}(1)\big]\Big]^{-1} \approx 0.0349228. \quad \text{(B.17)}$$

First-order perturbations in the cases where $n = 1, 2, 3, \ldots$ can also be evaluated analytically but the integration in Eq. (B.14) becomes more complicated when n increases. Therefore, one might find some mathematical programs (like MATHEMATICA) to be very useful tools when one tries to calculate the coefficients N_n analytically. In Table B.1 we give the numerical values of first-order perturbations in the cases where $n = 0, 1, \ldots, 10$.

However, the perturbation method has two weak points. First, we cannot be sure that our perturbation expansion converges. We can only hope that first-order perturbations provide a good approximation to the difference between the real and the WKB eigenvalues of the Schwarzschild mass M. Second, it is hard to discover to what self-adjoint extension our perturbation approach is connected (for details see Ref. [69]). For these reasons our results for first-order perturbations can be thought of only as suggestive and we need a more reliable method for the study of the eigenvalues of M. This method will be discussed in the next section.

n	$M_n^{(1)}$	M_n	M_n^{WKB}
0	0.0349228	1.01	1
1	0.0093795	1.74	1.7320...
2	0.00513602	2.24	2.2360...
3	0.00346	2.65	2.6457...
4	0.00257746	3.01	3
5	0.00203793	3.32	3.3166...
6	0.00167622	3.61	3.6055...
7	0.001418	3.88	3.8729...
8	0.00122503	4.13	4.1231...
9	0.00107574	4.36	4.3588...
10	0.000957045	4.59	4.5825...

Table B.1. Numerical values of $M_n^{(1)}$, M_n, and M_n^{WKB}. The results are given with the best possible precision our numerical analysis can offer. One may observe that WKB eigenvalues for the mass M provide an excellent approximation to the exact mass eigenvalues even when n is small.

B.2 Numerical Analysis

The numerical analysis of the mass eigenvalues was performed with FEMLAB, and the results can be found in Table B.1. In our numerical analysis we solve eigenvalues for the differential equation

$$\left(-\frac{9}{8}\frac{d^2}{dx^2} - \frac{9}{32}\frac{1}{x^2} + \frac{1}{2}x^{2/3} \right)\psi(x) = M\psi(x) \qquad \text{(B.18)}$$

with the boundary conditions $\psi(0) = \psi(\infty) = 0$ (of course, the boundary condition $\psi(\infty) = 0$ must be put in as $\psi(a) = 0$, where a is a “sufficiently” large number).

It may be shown that the mass spectrum given by this equation is identical to the mass spectrum given by Eq. (B.1) [69]. One can also study which self-adjoint extension the numerical routine has chosen when calculating the mass eigenvalues: The eigenfunctions corresponding to M_n in Table B.1 seem to behave, within the limits of the precision of the numerical computing, like $\psi(x) \sim \sqrt{x}$ when $x \to 0$. This observation fixes the self-adjoint extension for our eigenvalues.

Finally, it should also be noted that, due to a limited accuracy of numerical computing, some of the eigenvalues M_n might be around 1% smaller than the values in Table B.1. This does not cause any problems and actually the mass eigenvalues are brought even closer to the WKB estimate of Eq. (7.5).

Appendix C

Phase Space Volume

In chapter 7 we have introduced a microscopic model of the Schwarzschild horizon such that the phase space volume corresponding to a fixed horizon area A is determined by the condition (7.11):

$$\frac{1}{8a_1^2}\left(p_1^2+a_1^2\right)^2+\ldots+\frac{1}{8a_N^2}\left(p_N^2+a_N^2\right)^2=\frac{N}{2}+\frac{A}{\alpha}. \tag{C.1}$$

This equation describes a closed, compact $(2N-1)$-dimensional hypersurface Σ in the $2N$-dimensional phase space spanned by the canonical coordinates a_i and p_i. To calculate its volume we choose coordinates $\lambda_1, \lambda_2, \ldots, \lambda_{N-1} \in [0,1]$ and $\varphi_1, \varphi_2, \ldots, \varphi_N \in [0, 2\pi]$ on the hypersurface Σ such that

$$\begin{aligned}
a_1 &= L(\,\lambda_1+\lambda_1\cos\varphi_1), && \text{(C.2a)}\\
p_1 &= L\,\lambda_1\sin\varphi_1, && \text{(C.2b)}\\
a_2 &= L\,(\lambda_2+\lambda_2\cos\varphi_2), && \text{(C.2c)}\\
p_2 &= L\,\lambda_2\sin\varphi_2, && \text{(C.2d)}\\
&\vdots\\
a_N &= L\,(\lambda_N+\lambda_N\cos\varphi_N), && \text{(C.2e)}\\
p_N &= L\,\lambda_N\sin\varphi_N. && \text{(C.2f)}
\end{aligned}$$

Here $L=\sqrt{N+\frac{2A}{\alpha}}$, and $\lambda_N=\lambda_N(\lambda_1,\lambda_2,\ldots,\lambda_{N-1})$ can be calculated from Eq. (C.1). If one substitutes the coordinates a_i and p_i in terms of $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $\varphi_1, \varphi_2, \ldots, \varphi_N$ into Eq. (C.1), one gets

$$\lambda_N=\sqrt{1-\lambda_1^2-\lambda_2^2-\cdots-\lambda_{N-1}^2}. \tag{C.3}$$

Equation (C.3) represents a hypersphere in the N-space spanned by the coordinates λ_i. So it is natural to perform another coordinate transformation from the coordinates

λ_i to the generalized spherical coordinates $\theta_1, \ldots, \theta_{N-2} \in [0, \frac{\pi}{2}]$ and $\phi \in [0, \frac{\pi}{2}]$:

$$\lambda_1 = \cos\phi \prod_{i=1}^{N-2} \sin\theta_i, \tag{C.4a}$$

$$\lambda_2 = \sin\phi \prod_{i=1}^{N-2} \sin\theta_i, \tag{C.4b}$$

$$\lambda_3 = \cos\theta_1 \prod_{i=1}^{N-3} \sin\theta_i, \tag{C.4c}$$

$$\lambda_4 = \cos\theta_2 \prod_{i=1}^{N-4} \sin\theta_i, \tag{C.4d}$$

$$\vdots$$

$$\lambda_N = \cos\theta_{N-2}. \tag{C.4e}$$

Generally, the volume of the any (smooth) N-dimensional hypersurface S can be evaluated from the integral

$$\int_S \sqrt{g}\, d^N x, \tag{C.5}$$

where g is the determinant of the metric. We now define the metric of the phase space as

$$ds^2 = \sum_{i=0}^{N} \left(dp_i^2 + da_i^2\right). \tag{C.6}$$

Then the position vector of a given point on the hypersurface Σ can formally be written as

$$\begin{aligned} \vec{r} = {} & a_1(\varphi_1, \ldots, \varphi_N, \theta_1, \ldots, \theta_{N-2}, \phi)\, \hat{e}_1 + \\ & p_1(\varphi_1, \ldots, \varphi_N, \theta_1, \ldots, \theta_{N-2}, \phi)\, \hat{e}_2 + \\ & \vdots \\ & a_N(\varphi_1, \ldots, \varphi_N, \theta_1, \ldots, \theta_{N-2}, \phi)\, \hat{e}_{2N-1} + \\ & p_N(\varphi_1, \ldots, \varphi_N, \theta_1, \ldots, \theta_{N-2}, \phi)\, \hat{e}_{2N}, \end{aligned} \tag{C.7}$$

where the $\hat{e}_i$'s $(i = 1, \ldots, 2N)$ are orthogonal unit vectors in our $2N$-dimensional phase space. From Eqs. (C.2) it follows that all non-vanishing components of the metric on the hypersurface, defined by the relation

$$g_{ij} = \vec{b}_i \cdot \vec{b}_j, \tag{C.8}$$

where $\vec{b}_i = \frac{\partial \vec{r}}{\partial x^i}$ $(i = 1, \ldots, 2N-1)$ is the tangent vector of the coordinate curve corresponding to the coordinate x^i, are proportional to L^2, and otherwise contain only products of the sines and the cosines of the angles ϕ, θ_i, and φ_i. Therefore the phase space volume Ω corresponding to the fixed area A has, in units of $(2\pi\hbar)^{N-1/2}$, the form

$$\Omega = \frac{L^{2N-1}}{(2\pi)^{N-1/2}} \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta_1 \cdots \int_0^{\pi/2} d\theta_{N-2} \int_0^{2\pi} d\varphi_1 \cdots \int_0^{2\pi} d\varphi_N \sqrt{\tilde{g}}, \tag{C.9}$$

where $\tilde{g}$ is the determinant of the metric whose components $\tilde{g}_{ij}$ are obtained from the components g_{ij} such that we simply divide each g_{ij} by L^2.

The $(2N-1)$-dimensional integral in Eq. (C.9) is, unfortunately, very hard to calculate. However, one can easily see that this integral converges. From Eqs. (C.2) and (C.4) it follows that $\tilde{g}$ is everywhere finite (it contains only products of the sines and the cosines of the angles ϕ, θ_i, and φ_i). Furthermore, the integration is performed over a bounded region because $\varphi_i \in [0, 2\pi]$ and $\theta_1, \phi \in [0, \frac{\pi}{2}]$. Therefore, the $(2N-1)$-dimensional integral in Eq. (C.9) converges. Its value, however, depends only on N and we can denote

$$\frac{1}{(2\pi)^{N-1/2}} \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta_1 \cdots \int_0^{\pi/2} d\theta_{N-2} \int_0^{2\pi} d\varphi_1 \cdots \int_0^{2\pi} d\varphi_N \sqrt{\tilde{g}} =: C(N). \tag{C.10}$$

So we can write the phase space volume Ω as

$$\Omega = C(N) \left(N + \frac{2A}{\alpha} \right)^{N-1/2}, \tag{C.11}$$

which is Eq. (7.13).

Bibliography

[1] In the episode *Spectre of the Gun* of *Star Trek* (CBS Studios Inc.).

[2] A. Einstein, Preuss. Akad. Wiss. Berlin, Sitzungsber., 844 (1915).

[3] K. Schwarzschild , Deut. Akad. Wiss. Berlin, Sitzungsber., 189 (1916).

[4] R. P. Feynman, Rev. Mod. Phys **20**, 327 (1948).

[5] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[6] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973). See also J. D. Bekenstein, Nuovo Cimento Lett. **4**, 737 (1972).

[7] See, for example, A. Ashtekar, J. Baez, A. Corichi and K. Krasnov, Phys. Rev. Lett. **80**, 904 (1998), A. Ashtekar, J. Baez and K. Krasnov, Adv. Theor. Math. Phys. **4**, 1 (2000), A. Strominger, and C. Vafa, Phys. Lett. **B379**, 99 (1996), G. T. Horowitz in *Black holes and Relativistic Stars*, ed. by R. M. Wald (University of Chicago Press, 1998). A comprehensive review of the present status of loop quantum gravity and string theory has recently been written by L. Smolin, hep-th/0303185. See also T. Padmanabhan and A. Patel, hep-th/0305165, gr-qc/0309053.

[8] A good survey of black hole entropy can be found in C. Rovelli, gr-qc/9803024.

[9] J. Mäkelä, gr-qc/0605098. See also J. Mäkelä, gr-qc/0506087.

[10] T. Jacobson, Phys. Rev. Lett. **75**, 1260 (1995). See also T. Jacobson, Found. Phys. **33**, 323 (2003) and T. Jacobson in *General relativity and Relativistic Astrophysics: Eighth Canadian Conference* (AIP Conference Proceedings 493, edited by C. P. Burgess and R. C. Myers (AIP, 1999)).

[11] See, for instance,
www.worldofquotes.com/author/Steven-Wright/1/index.html
(dated 1.2.2007).

[12] J. Michell, Trans. Royal Soc. London **74**, 35 (1784).

[13] P. S. Laplace, *Exposition du system du monde* (1795).

[14] The first calculation of complete gravitational collapse was performed by J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).

[15] An introduction to the gravitational collapse of stars can be found, for instance, in H. Ohanian and R. Ruffini, *Gravitation and Spacetime*, second edition (W. W. Norton & Company, New York, 1994).

[16] S. Chandrasekhar, Mon. Nat. Roy. Astron. Soc. **95**, 207 (1935).

[17] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55**, 374 (1939).

[18] M. J. Rees, in *Structure and Properties of Nearby Galaxies*, ed. by E. M. Berkhuijsen and R. Wielebinski (International Astronomical Union, 1978).

[19] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, New York, 1973).

[20] R. Penrose, Rev. Nuovo Cim. **1**, 252 (1969).

[21] The proper definition of asymptotical flatness was introduced by R. Penrose in *Relativity, Groups and Topology*, ed. by C. M. DeWitt, Les Houches Summer School, 1963 (Gordon and Breach, New York, 1963).

[22] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, New York, 1973).

[23] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).

[24] The method of the conformal compactification of spacetime was developed by Penrose. For an introduction, see Ref. [19]. For a more detailed presentation, see R. Penrose in Ref. [21].

[25] The proof is given, for instance, in Ref. [23].

[26] S. W. Hawking, Proc. Roy. Soc. Lond. A **300**, 182 (1967).

[27] R. Penrose, Phys. Rev. Lett. **14**, 57 (1965).

[28] S. W. Hawking and R. Penrose, Proc. Roy. Soc. Lond. A **314**, 529 (1970).

[29] D. Gannon, J. Math. Phys. **16**, 2364 (1975).

[30] D. Gannon, Gen. Rel. and Grav. **7**, 219 (1976).

[31] An introduction to cosmological models can be found in L. Bergström and A. Goobar, *Cosmology and Particle Astrophysics* (John Wiley and Sons, Chichester, 1999). For a more comprehensive review see, for instance, E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, 1990).

[32] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon Press, Oxford, 1983).

[33] H. Reissner, Ann. Physik **50**, 106 (1916).

[34] G. Nordström, Proc. Kon. Ned. Akad. Wet. **20**, 1238 (1918).

[35] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[36] E. T. Newman et al., J. Math. Phys. **6**, 918 (1965).

[37] For charged black holes, the most satisfactory mechanism is probably given by the gravitational collapse of charged thin spherical shell. For details, see E. Hawkins, Phys. Rev. D **49**, 6556 (1994), Erratum-ibid, Phys.Rev. D **50**, 7744 (1994).

[38] See, for instance, M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, Cambridge, 1996).

[39] M. J. Rees, R. Ruffini and J. A. Wheeler, *Black Holes, Gravitational waves and Cosmology* (Gordon and Breach, New York, 1974).

[40] W. Israel, Phys. Rev. **164**, 1776 (1967).

[41] W. Israel, Commun. Math. Phys. **8**, 245 (1968).

[42] B. Carter, Phys. Rev. Lett. **26**, 331 (1971).

[43] D. C. Robinson, Phys. Rev. Lett. **34**, 905 (1975).

[44] P. O. Mazur, J. Phys. A **15**, 3173 (1982).

[45] J. D. Bekenstein, Phys. Rev. D **5**, 1239 (1972).

[46] J. B. Hartle in *Magic without Magic*, ed. J. Klauder (Freeman, San Francisco, 1972).

[47] C. Teitelboim, Phys. Rev. D **5**, 2941 (1972).

[48] S. W. Hawking, Commun. Math. Phys. **25**, 152 (1972).

[49] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).

[50] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (The University of Chicago Press, Chicago, 1994).

[51] R. Arnowitt, S. Deser and C. W. Misner in *Gravitation: An Introduction to the Current Research*, ed. by L. Witten (Wiley, New York, 1962).

[52] T. Regge and C. Teitelboim, Ann. Physics **88**, 286 (1974).

[53] N. D. Birrell and P. C. W. Davies, *Quantum Field Theory in Curved Space* (Cambridge University Press, Cambridge, 1982).

[54] N. N. Bogolubov, Zh. Eksp. Teor. Fiz. **34**, 58 (1958) (Sov. Phys. JETP **7**, 51 (1958)).

[55] T. Damour and R. Ruffini, Phys. Rev. D **14**, 332 (1976).

[56] D. N. Page, Phys. Rev. D **13**, 198 (1976). See also D. N. Page, Phys. Rev. D **14**, 3260 (1976).

[57] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977). See also G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977) and S. W. Hawking, Phys Rev. D **18**, 1747 (1978).

[58] For a review see S. W. Hawking in *General Relativity, An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press,New York, 1979).

[59] M. Visser, Int. J. Mod. Phys. D **12**, 649 (2003).

[60] J. D. Bekenstein, Phys. Rev. D **9**, 3292 (1974). See also J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973)

[61] S. Q. Wu and X. Cai, Nuovo Cim. B **115**, 143 (2000). The problem of negative temperature has been further considered by Wu in S. Q. Wu, Phys Lett. B **608**, 251 (2005) and S. Q. Wu, hep-th/0411002.

[62] T. Padmanabhan, Gen. Rel. Grav. **34**, 2029 (2002). See also T. Padmanabhan, Class. Quant. Grav. **21**, 4485 (2004), Phys. Rept. **406**, 49 (2005), and gr-qc/0412068, T. Padmanabhan and A. Patel gr-qc/0309053, D. Kothawala, S. Sarkar and T. Padmanabhan, gr-qc/0701002, and T. Padmanabhan and A. Paranjape, gr-qc/0701003.

[63] E. Poisson and W. Israel, Phys. Rev. D **41**, 1796 (1990). See also E. Poisson and W. Israel, Phys. Rev. Lett. **63**, 1663 (1989), and Phys. Lett. B **233**, 74 (1989).

[64] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).

[65] For the information about the changes caused by general relativity in the usual thermodynamical relations see, for example, L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press Ltd., Oxford, 1980).

[66] G. T. Horowitz and D. Marolf, Phys. Rev. D **52**, 5670 (1995).

[67] F. Mellor and I. Moss, Class. Quant. Grav. **6**,1379 (1989), F. Mellor and I. Moss, Phys. Lett. B **222**, 361 (1989), P. C. W. Davies and I. Moss, Class. Quant. Grav. **6**, L173 (1989).

[68] Some speculation about the possibility that inner horizons might play an important role in black hole radiation and entropy have been expressed, for instance, in J. Mäkelä and P. Repo, Phys. Rev. D **57**, 4899 (1998) and J. Mäkelä, P. Repo, M. Luomajoki and J. Piilonen, Phys. Rev. D **64**, 024018 (2001).

[69] J. Louko and J. Mäkelä, Phys. Rev. D **54**, 4982 (1996).

[70] K. Kuchař, Phys. Rev. D **50**, 3961 (1994). See also, T. Thiemann and H. A. Kastrup, Nucl. Phys. B **399**, 211 (1993), H. A. Kastrup and T. Thiemann, Nucl. Phys. B **425**, 665 (1994).

[71] G. D. Birkhoff, *Relativity and Modern Physics* (MA: Harvard University Press, Cambridge, 1923).

[72] A. Z. Capri, *Nonrelativistic Quantum Mechanics* (World Scientific Publishing Co. Pte. Ltd., Singapore, 2002).

[73] See, for instance, A. Messiah, *Quantum Mechanics, Vol 1* (The North-Holland Physics Publishing Company, Amsterdam, 1991).

[74] J. D. Bekenstein, Lett. Nuovo Cimento, **11**, 467 (1974).

[75] K. Lake, Phys. Rev. D **19**, 421 (1979).

[76] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. D **49**, 975 (1994).

[77] J. Mäkelä and P. Repo, Phys. Rev. D **57**, 4899 (1998). See also J. Mäkelä, Found. Phys. **32**, 1809 (2002).

[78] J. Mäkelä, P. Repo, M. Luomajoki and J. Piilonen, Phys. Rev. D **64**, 024018 (2001). See also J. Mäkelä, Found. Phys. **32**, 1809 (2002).

[79] In the episode *The PDA Dispands* of *The Simpsons* (Fox Broadcasting Company).

[80] As itself, our idea to construct Schwarzschild horizon from microscopic black holes is not quite new. Similar ideas have been expressed by R. Garattini, Phys. Lett. B **459**, 461 (1999), Nucl. Phys. Proc. Suppl. **88**, 297 (2000), Int. J. Mod. Phys. D **4**, 635 (2002), Entropy **2**, 26 (2000), gr-qc/0111068. See also F. Scardigli, Class. Quant. Grav. **14**, 1781 (1997).

[81] See, for example, J. D. Bekenstein and V. F. Mukhanov, Phys. Lett. B **360**, 7 (1995), P. C. Mazur, Phys. Rev. Lett. **57**, 929 (1987), Y. Peleg. Phys. Lett. B **356**, 462 (1995), A. Barvinsky and G. Kunstatter, Phys. Lett. B **389**, 231 (1996), H. A. Kastrup, Phys. Lett. B **389**, 75 (1996), S. Hod, Phys. Rev. Lett. **81**, 4293 (1998), S. Hod, Gen. Rel, Grav. **31**, 1639 (1999), H. A. Kastrup, Ann. Phys. (Leibzig) **9**, 503 (2000), M. Bojowald and H. A. Kastrup, Class. Quant. Grav. **17**, 3009 (2000), A. Barwinsky, S. Das and G. Kunstatter, Phys. Lett. B **517**, 415 (2001), C. Vaz, Phys. Rev. D **61**, 064017 (2000). A. Barvinsky,

S. Das and G. Kunstatter, Class. Quant. Grav. **18**, 4845 (2001), J. D. Bekenstein, Int. J. Mod. Phys A **17S1**, 21 (2002), J. D. Bekenstein and G. Gour, Phys. Rev. D **66**, 024005 (2002).

[82] See, for example, L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Butterworth-Heinemann, 1999).

[83] J. Mäkelä, Phys. Lett. B **390**, 115 (1997).

[84] Curiously, the idea that gravitation may be only a statistical effect was proposed already by A. Eddington in *Space, Time and Gravitation* (Syndics of the Cambridge University Press, London, 1920).

[85] Jacobson's analysis has been recently reconsidered by S. C. Tiwari, gr-qc/0612099.

[86] J. Mäkelä and A. Peltola, gr-qc/0406032.

[87] Somewhat similar problems also arise in the loop quantum gravity, when one tries to define the concept of area for "one-sided" two-surfaces (i.e., for horizons). This problem has been discussed, for instance, by T. Thiemann in *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, 2004) and C. Rovelli in *Quantum Gravity* (Cambridge University Press, Cambridge, 2004).

[88] Besides in Jacobson's approach, an unspecified cosmological constant appears also in the article T. Padmanabhan and A. Paranjape, Phys. Rev. D **75**, 064004 (2007).

[89] See, for instance, F. Mandl, *Statistical Physics*, 2nd edn. (John Wiley & Sons Ltd., Chichester, 1988).

[90] The role of the cosmological constant in the present observations of dark energy has been recently discussed, for instance, by R. Bean, S. Carroll and M. Trodden, astro-ph/0510059.