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AN AXIOMATIC THEORY OF NORMED MODULES VIA RIESZ SPACES

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ABSTRACT

We introduce and study an axiomatic theory of V-normed U-modules, where V is a Riesz space and U is an f-algebra; the spaces U and V also have some additional structure and are required to satisfy a compatibility condition. Roughly speaking, a V-normed U-module is a module over U that is endowed with a pointwise norm operator taking values in V. The aim of our approach is to develop a unified framework, which is tailored to the differential calculus on metric measure spaces, where U and V can take many different spaces of functions.

1. INTRODUCTION

1.1. General overview and motivations

In this paper, we introduce and study a class of structures named *V-normed U-modules*, where V is a Riesz space and U is an f-algebra (that is, a Riesz space together with a multiplication operation), which fulfil suitable compatibility requirements. Roughly speaking, a V-normed U-module is a module over U (thus in particular, a vector space) equipped with a 'pointwise norm' operator that takes values into the positive cone of V. Several structures of these kinds—where, typically, U and V are function spaces—have been investigated in the literature: we refer to them as 'functional' normed modules. Different theories of functional normed modules were developed in the last 35 years, with a variety of applications, for example, in analysis, geometry and mathematical finance. Before delving into a more precise description of our notion of V-normed U-module (in Subsection 1.2), let us provide a brief overview of various classes of spaces that are covered by our axiomatization.

- *Normed spaces*, which are $\mathbb{R}$-normed $\mathbb{R}$-modules.

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- *Lebesgue–Bochner spaces*, that is, spaces of p-integrable maps from a measure space to a normed space. These spaces are V-normed U-modules, where V is the space of p-integrable functions and U is the space of bounded measurable functions.
- More generally, *direct integrals of Banach spaces* [31] and different *spaces of measurable sections* of a measurable Banach bundle [34].
- *Random normed modules*, which were introduced by Guo [25, 26] after the work of Schweizer and Sklar [37] on probabilistic metric spaces. The theory of random normed modules has been thoroughly developed in a long series of works (mostly by Guo and his coauthors); see the survey paper [27]. Particular attention was devoted to the study of random conjugate spaces (see, for example, [29]), which have applications in mathematical finance and in the modelling of conditional risk measures [15].
- *Randomly normed L^0-modules*, which were developed by Haydon, Levy and Raynaud in [31] as a tool to study ultraproducts of Lebesgue–Bochner spaces. In this case, the Riesz spaces under consideration are Köthe function spaces, which are order-dense order-ideals in the space $L^0(\mu)$ of measurable functions on a given measure space. This theory and Guo's one—which were developed independently and concurrently—are fully consistent.
- *L^p-normed L^∞-modules* and their variants, which were introduced by Gigli [20], with the goal of developing an effective theory of measurable 1-forms and vector fields in the non-smooth setting of metric measure spaces. This approach was based on the work of Weaver [40] and on his definition of L^∞-module. Similar structures have been widely considered in the framework of Dirichlet forms (see, for example, [14, 33]) and in the investigation of 1-forms induced by Dirichlet spaces [5]. Gigli's theory is consistent with the above-mentioned notions of random normed modules (cf. with [20, Section 1.4] and [28]).
- *Normed A-modules* in the sense of [8, 9], where A is a suitable f-algebra. This approach, which is similar in spirit to the one that we adopt in this paper, has been applied to the study of mathematical models in finance.

Our interest in the language of normed modules is motivated by its applications in the differential calculus on metric measure spaces. Below, we briefly describe some important concepts and results from [20]. The goal of this description is 2-fold: to give a heuristic presentation of our definition of V-normed U-module and to expound the advantages of an axiomatic approach. However, we underline that our theory may be relevant even beyond the analysis of metric measure spaces.

On metric measure spaces $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, the study of Sobolev spaces $W^{1,p}(\mathrm{X})$ for $p \in (1, \infty)$ has been a fruitful field of research in the last decades (see, for example, [4, 10, 30, 38]). In order to develop a differential calculus modelled over $W^{1,p}(\mathrm{X})$, several notions of 'measurable (co)vector fields' were studied, for example, by [10] in the setting of doubling spaces supporting a Poincaré inequality. One of the objectives of [20] was to provide a meaningful notion of 'space of measurable 1-forms' for arbitrary metric measure spaces. This is encoded in the concept of *cotangent module*, which we are going to remind. It is proved in [20, Section 2.2.1] that it is possible to construct a vector space $L^p(T^*\mathrm{X})$ and a linear operator $\mathrm{d}\colon W^{1,p}(\mathrm{X}) \to L^p(T^*\mathrm{X})$ having the following features:

(i) The elements of $L^p(T^*\mathrm{X})$ can be multiplied by $L^\infty(\mathfrak{m})$-functions; to be precise, $L^p(T^*\mathrm{X})$ is a module over the commutative ring $L^\infty(\mathfrak{m})$.

(ii) There exists a map $|\cdot|\colon L^p(T^*\mathrm{X}) \to L^p(\mathfrak{m})^+$ that vanishes only at 0, that satisfies

$$|\omega + \eta| \leq |\omega| + |\eta|, \quad \text{for every } \omega, \eta \in L^p(T^*\mathrm{X}),$$

and that is compatible with the $L^\infty(\mathfrak{m})$-module structure, in the sense that $|f \cdot \omega| = |f||\omega|$ for every $f \in L^\infty(\mathfrak{m})$ and $\omega \in L^p(T^*\mathrm{X})$. The map $|\cdot|$ is said to be a *pointwise norm* operator. Moreover, the norm on $L^p(T^*\mathrm{X})$ induced by the pointwise norm via integration, that is,

$$\|\omega\|_{L^p(T^*\mathrm{X})} := \||\omega|\|_{L^p(\mathfrak{m})}, \quad \text{for every } \omega \in L^p(T^*\mathrm{X}),$$

is required to be complete.

(iii) The operator $\mathrm{d}\colon W^{1,p}(\mathrm{X}) \to L^p(T^*\mathrm{X})$, which is called the *differential*, satisfies

$$\|f\|_{W^{1,p}(\mathrm{X})} = \left(\|f\|^p_{L^p(\mathfrak{m})} + \||\mathrm{d}f|\|^p_{L^p(\mathfrak{m})}\right)^{1/p}, \quad \text{for every } f \in W^{1,p}(\mathrm{X}),$$

and has the property that the $L^\infty(\mathfrak{m})$-module generated by its image is dense in $L^p(T^*\mathrm{X})$.

Following [20, Definition 1.2.10], any couple $(\mathscr{M}, |\cdot|)$ verifying i) and ii) is called an $L^p(\mathfrak{m})$-*Banach* $L^\infty(\mathfrak{m})$-*module* (in fact, in [20] the term '$L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module' is used, but in this paper we need to distinguish between complete and non-complete modules). Nevertheless, a number of variants of this notion have been considered in [20] and in the subsequent literature:

- It might be convenient (and sometimes necessary) to drop the L^p-integrability assumption. Technically speaking, this is made precise by the notion of $L^0(\mathfrak{m})$-*Banach* $L^0(\mathfrak{m})$-*module*; see [20, Section 1.3]. For example, the notion of L^0-Banach L^0-module becomes essential in the construction of tensor products of $L^2(\mathfrak{m})$-Hilbert $L^\infty(\mathfrak{m})$-modules, cf. with [20, Section 1.5].
- The case $p = \infty$ has been studied as well. Indeed, $L^\infty(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules are fundamental in order to apply the lifting theory by von Neumann in the Banach module setting [13], which in turn allows us to provide 'fibrewise descriptions', that is, to show that any Banach module is the space of sections of some generalized Banach bundle [21]. At present, using fibres is the only way to provide an explicit characterization of duals and of pullbacks of Banach modules, which are useful objects for the applications in metric measure geometry.
- Under suitable curvature bounds (for example, in the setting of $\mathsf{RCD}(K,\infty)$ spaces), one is often interested in extending the differential calculus to codimension-one measures (for example, to perimeter measures). The functional-analytic framework that allows us to achieve this goal is based on the concept of $L^0(\mathrm{Cap})$-*Banach* $L^0(\mathrm{Cap})$-*module*, which was introduced in [11]. Here, Cap denotes the Sobolev capacity, which is an outer measure on X that is not Borel regular.

The aim of this work is to provide a unified theory of Banach modules, which covers—at least—all the notions of Banach modules discussed above. Indeed, albeit similar on some aspects, the several variants of Banach module often required different ad hoc definitions and proof strategies. Our goal is to introduce an 'axiomatic framework', where instead of function spaces we consider more general classes of Riesz spaces and f-algebras, as well as to obtain rather general existence results, which can be applied in all the specific cases we described above, whenever needed.

1.2. Main definitions

Let us now discuss the various objects we are going to introduce, also motivating the reasons behind our definitions. First, a key feature of all the 'functional' Banach modules from Section 1.1 is the possibility to multiply by *characteristic functions*. This is fundamental, for example, when constructing the cotangent module. Observe that in $L^\infty(\mathfrak{m})$ the characteristic functions of Borel sets are given exactly by the *idempotent elements*, that is, by those $f \in L^\infty(\mathfrak{m})$ satisfying $f^2 = f$. Moreover, two different function spaces appear in the definition of Banach module: the ring of functions that can be multiplied by the elements of the Banach module (for example, $L^\infty(\mathfrak{m})$), and the vector space of functions where the pointwise norm takes values (for example, $L^p(\mathfrak{m})$). These two function spaces must be related. For example, the compatibility requirement between the pointwise norm and the module structure uses the fact that $fg \in L^p(\mathfrak{m})$ whenever $f \in L^\infty(\mathfrak{m})$ and $g \in L^p(\mathfrak{m})$.

Taking all these features into account, we propose in Definition 2.23 the concept of

$$\textit{metric f-structure} \quad (\mathcal{U}, U, V).$$

Let us describe informally what a metric f-structure $(\mathcal{U}, U, V)$ is:

- $\mathcal{U}$ is an ambient *localizable* f-algebra (see Definition 2.15), which means that it is an f-algebra (that is, a Riesz space together with a compatible multiplication operation, see Definition 2.5) having plenty of *idempotent elements* (see (eq: idem)). This corresponds, for example, to the fact that simple functions are order-dense in $L^0(\mathfrak{m})$.
- (U, d_U) is a *metric f-algebra* (see Definition 2.22) that is an f-subalgebra of $\mathcal{U}$. This means that U is an f-algebra endowed with a complete distance d_U that verifies suitable compatibility conditions. For example, the space $L^\infty(\mathfrak{m})$ is an f-subalgebra of $L^0(\mathfrak{m})$, and together with (the distance induced by) its norm, $L^\infty(\mathfrak{m})$ is a metric f-algebra.
- (V, d_V) is a *metric Riesz space* (see Definition 2.20) that is also a Riesz subspace of $\mathcal{U}$ satisfying $UV = V$. For example, $L^p(\mathfrak{m})$ is a metric Riesz space and $L^\infty(\mathfrak{m}) \cdot L^p(\mathfrak{m}) = L^p(\mathfrak{m})$.

Our axiomatization of a metric f-structure is tailored to the kinds of Banach modules we are interested in. However, already in the framework of differential calculus on metric measure spaces, some important objects studied in the literature (for example, Lipschitz derivations [**39**] or local vector measures [**6**]) are not covered by our theory, roughly speaking because the f-algebra of bounded continuous functions is not localizable (as characteristic functions are typically not continuous).

As we discussed above, an example of metric f-structure is $(L^0(\mathfrak{m}), L^\infty(\mathfrak{m}), L^p(\mathfrak{m}))$. Taking into consideration the notion of $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-module from Section 1.1, one can think of the elements of U as those that can be multiplied by the elements of the Banach module, and the role of V is 'the space where the pointwise norm takes values', while $\mathcal{U}$ is an ambient space where both U and V can be embedded (which is convenient to formulate the requirement that $UV = V$). Having this discussion in mind, we propose in Definition 3.1 the concept of

$$V\textbf{-Banach}\,U\textbf{-module}\quad \mathscr{M}.$$

The definition of V-Banach U-module roughly states the following:

- $\mathscr{M}$ is a module over the commutative ring U endowed with a pointwise norm $|\cdot|\colon \mathscr{M} \to V^+$, which verifies the pointwise triangle inequality and is compatible with the module operations.
- $\mathscr{M}$ has the *gluing property*, which means every *admissible* sequence of *disjoint* elements $(v_n)_{n\in\mathbb{N}}$ of $\mathscr{M}$ can be 'glued together', thus obtaining a new element $\sum_{n\in\mathbb{N}} v_n \in \mathscr{M}$. The order structure of $(\mathcal{U}, U, V)$ comes into play here, that is, when declaring which sequences are admissible, see Definition 3.1 (ii). We also point out that, in general, $\sum_{n\in\mathbb{N}} v_n$ is just a formal series, which does not necessarily coincide with any kind of limit of finite sums.
- The distance $\mathsf{d}_{\mathscr{M}}(v,w) := \mathsf{d}_V(|v-w|, 0)$ on $\mathscr{M}$ is complete.

In the class of $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules with $p \in [1,\infty)$ we described in Section 1.1, we did not mention the gluing property, the reason being that in that specific framework it follows automatically from the other axioms. On the other hand, this is not always the case. For example, the gluing property has to be required when dealing with $L^\infty(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules (see [**20**, Example 1.2.5] or [**13**, Remark 2.22]). Moreover—different from what happens with $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules, where $|\sum_{n=1}^k v_n - \sum_{n\in\mathbb{N}} v_n| \to 0$ in $L^p(\mathfrak{m})$ – on $L^\infty(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules it is clear that the expression $\sum_{n\in\mathbb{N}} v_n$ might be only formal: in the space $L^\infty(\mathbb{R})$ itself (which is an $L^\infty(\mathbb{R})$-Banach $L^\infty(\mathbb{R})$-module), the elements $f_n := \mathbb{1}_{[n,n+1)}$ for $n \in \mathbb{Z}$ can be 'glued together', obtaining the constant function 1 ; however, 1 is not the limit in the $L^\infty(\mathbb{R})$-norm of the partial sums $\sum_{n=-k}^k f_k = \mathbb{1}_{[-k,k+1)}$ as $k \to \infty$. In this example, it is still true that the partial sums converge in some sense to the glued object (for example, in the weak * topology), but this needs not be the case for arbitrary $L^\infty(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules, which do not always have a predual.

We also mention that taking duals is very useful in differential calculus on metric measure spaces. For instance, the so-called *tangent module* $L^q(T\mathrm{X})$, which can be regarded as the space of 'q-integrable vector fields' on a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, is defined as the Banach module dual of the cotangent module $L^p(T^*\mathrm{X})$; see [**20**, Definition 2.3.1]. An important observation is that, according to

[20, Proposition 1.2.14 i)], the dual of an $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-module $\mathscr{M}$ is an $L^q(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-module $\mathscr{M}^*$, where p and q are conjugate exponents. This means that in our axiomatization when constructing the dual of a Banach module we have to change also the underlying metric f-structure. To address this issue, we propose in Definition 2.25 the concept of

$$\textbf{dual system of metric} f\textbf{-structures} \quad (\mathcal{U}, U, V, W, Z).$$

We omit the details here. However, the definition of dual system is given so that the *module dual* of a V-Banach U-module $\mathscr{M}$ is a W-Banach U-module, see Definition 3.15. More generally, the space $\mathrm{Hom}(\mathscr{M}, \mathscr{N})$ of all *homomorphisms* (Definition 3.11) from a V-Banach U-module $\mathscr{M}$ to a Z-Banach U-module $\mathscr{N}$ inherits a natural structure of W-Banach U-module (Theorem 3.12). An example of dual system of metric f-structures is $(L^0(\mathfrak{m}), L^\infty(\mathfrak{m}), L^p(\mathfrak{m}), L^q(\mathfrak{m}), L^1(\mathfrak{m}))$. When proving finer results about homomorphisms and dual modules, one often has to require a further regularity on the underlying f-algebras and Riesz spaces, namely, that they are *Dedekind complete* and they have the *countable sup property* (or *CSP*, for short); see Definition 2.4. The above assumptions amount to saying that every set that is bounded from above (resp. from below) has a supremum (resp. an infimum) and that such supremum (resp. infimum) can be expressed as a countable supremum (resp. a countable infimum) of elements of the given set. These properties are enjoyed, for example, by $L^p(\mathfrak{m})$ whenever $p \in \{0\} \cup [1, \infty]$ and $\mathfrak{m}$ is a σ-finite measure (Proposition 4.3), but they fail in $L^0(\mathrm{Cap})$ (Example 4.4). Dedekind completeness and CSP are also needed, for instance, to construct *local inverses* (Proposition 3.6) or to define the *support* of a metric f-structure (Definition 3.7).

1.3. Main results

Another objective of this work is to provide a rather complete toolbox of results and techniques concerning Banach modules over a metric f-structure, which we plan to apply in the future, as a 'black box', to many particular cases of interest. Our two main achievements are the following:

- Theorem 3.19: Given a metric f-structure $(\mathcal{U}, U, V)$, a vector space $\mathscr{V}$ and an even sublinear map $\psi \colon \mathscr{V} \to V^+$, there exists a unique couple $(\mathscr{M}_{\langle\psi\rangle}, T_{\langle\psi\rangle})$, where $\mathscr{M}_{\langle\psi\rangle}$ is a V-Banach U-module, while $T_{\langle\psi\rangle} \colon \mathscr{V} \to \mathscr{M}_{\langle\psi\rangle}$ is a linear operator with 'generating image' (in a suitable sense) such that $|T_{\langle\psi\rangle}\mathsf{v}| = \psi(\mathsf{v})$ for every $\mathsf{v} \in \mathscr{V}$. The uniqueness is formulated in categorical terms, that is, via a universal property (see also Corollary 3.22). This quite general existence result incorporates most of the existence results for Banach modules considered so far in the related literature. For example, the cotangent module $L^p(T^*\mathrm{X})$ and the differential d are given by $(L^p(T^*\mathrm{X}), \mathrm{d}) \cong (\mathscr{M}_{\langle\psi_p\rangle}, T_{\langle\psi_p\rangle})$, where the map $\psi_p \colon W^{1,p}(\mathrm{X}) \to L^p(\mathfrak{m})^+$ is defined as $\psi_p(f) := |Df|$. See Section 4.2.5 for this example, as well as for other relevant constructions of Banach modules induced by an even sublinear map.
- Theorem 3.16 is an existence criterion for homomorphisms of Banach modules. Indeed, given that the theory of V-Banach U-modules fits well in a categorical framework (see Definition 3.14), it is natural to couple Theorem 3.19 with an existence result for homomorphisms. For simplicity of presentation, let us state here only a corollary of Theorem 3.12: given a dual system $(\mathcal{U}, U, V, W, Z)$, a V-Banach U-module $\mathscr{M}$, a Z-Banach U-module $\mathscr{N}$, a 'generating' vector subspace $\mathscr{V}$ of $\mathscr{M}$ and a linear operator $T \colon \mathscr{V} \to \mathscr{N}$ satisfying $|Tv| \leq b|v|$ for some $b \in W^+$, there is a unique extension $\bar{T} \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ of T, which still satisfies $|\bar{T}v| \leq b|v|$.

Finally, we conclude the introduction by briefly mentioning other results we obtain in the paper:

- Using Theorems 3.19 and 3.12, we prove that each homomorphism of metric f-structures induces a *pushforward functor* (or, to be more precise, a 'direct image functor') in the categories of Banach modules; see Section 3.3.2.

- We prove a version of the *Hahn–Banach extension theorem* for V-Banach U-modules; see Theorem 3.30. It is used, for example, for studying module duals and embedding operators into the bidual; see Sections 3.4 and 3.4.1.
- We study *Hilbert modules*, for example, Banach modules whose pointwise norm verifies a pointwise parallelogram identity; see Definition 3.5. Among the several results we obtain, let us mention a Hilbert projection theorem and a Riesz representation theorem; see Section 3.5.
- We prove that V-Banach U-modules admit a *dimensional decomposition* (assuming Dedekind completeness and CSP of the metric f-structure); see Section 3.6.

In Sections 2 and 3 the whole treatment is at the level of 'abstract' Riesz spaces and f-algebras, without ever mentioning any kind of function spaces. The applications of our axiomatic theory to the various classes of Banach modules over spaces of functions are discussed in Section 4.

2. LOCALIZABLE *F*-ALGEBRAS AND METRIC *F*-STRUCTURES

In Section 2.1 we recall many useful definitions and results concerning Riesz spaces and f-algebras, which are quite standard and well-established; our presentation is essentially taken from [**18**, **17**] (see also [**1**, **2**]). In Section 2.2 we study the set of idempotent elements, while in Sections 2.3 and 2.4 we introduce the language of localizable f-algebras and of (dual systems of) metric f-structures, respectively.

2.1. Reminder on Riesz spaces and f-algebras

Let $(P, \leq)$ be a partially ordered set, and $S \neq \emptyset$ a subset of P. We recall the following definitions:

(i) We say that S is *upwards directed* if for every $p, p' \in S$ there exists $q \in S$ such that $p \leq q$ and $p' \leq q$. We say that S is *downwards directed* if for every $p, p' \in S$ there exists $q \in S$ such that $q \leq p$ and $q \leq p'$.
(ii) A sequence $(p_n)_{n\in\mathbb{N}} \subset P$ is said to be *non-decreasing* provided $p_n \leq p_{n+1}$ for every $n \in \mathbb{N}$, while it is said to be *non-increasing* provided $p_n \geq p_{n+1}$ for every $n \in \mathbb{N}$.
(iii) An element $p \in P$ is said to be an *upper bound* for S provided that $s \leq p$ holds for every $s \in S$. We say that p is the *supremum* of S, and we write $p = \sup S$, provided that $p \leq p'$ holds for any other upper bound $p' \in P$ for S. If $\sup S$ exists, then it is uniquely determined.
(iv) An element $q \in P$ is said to be a *lower bound* for S provided that $q \leq s$ holds for every $s \in S$. We say that q is the *infimum* of S, and we write $q = \inf S$, provided that $q' \leq q$ holds for any other lower bound $q' \in P$ for S. If $\inf S$ exists, then it is uniquely determined.
(v) We say that S is *order-bounded* provided that it has both an upper bound and a lower bound.
(vi) We say that P is *Dedekind σ-complete* provided that every countable non-empty subset of P with an upper bound has a supremum and every countable non-empty subset of P with a lower bound has an infimum.
(vii) P is *Dedekind complete* if every non-empty subset of P with an upper bound has a supremum or equivalently every non-empty subset of P with a lower bound has an infimum.

A map $\phi\colon P \to Q$ between partially ordered sets P and Q is said to be *order-preserving* provided

$$\phi(p) \leq \phi(q), \quad \text{for every } p, q \in P \text{with } p \leq q.$$

An order-preserving map $\phi\colon P \to Q$ is said to be *order-continuous* provided that it holds that

$$\exists \sup\{\phi(p) \mid p \in R\} = \phi(\bar{p}), \quad \text{whenever } R \subset P \text{ is upwards directed and } \exists\, \bar{p} := \sup R \in P,$$
$$\exists \inf\{\phi(q) \mid q \in S\} = \phi(\bar{q}), \quad \text{whenever } S \subset P \text{ is downwards directed and } \exists\, \bar{q} := \inf S \in P.$$

We say that an order-preserving map $\phi\colon P \to Q$ is *σ-order-continuous* provided it holds that

$$\exists \sup_{n\in\mathbb{N}} \phi(p_n) = \phi\Big(\sup_{n\in\mathbb{N}} p_n\Big), \quad \text{whenever } (p_n)_{n\in\mathbb{N}} \subset P \text{ is non-decreasing and } \sup_{n\in\mathbb{N}} p_n \text{ exists,}$$
$$\exists \inf_{n\in\mathbb{N}} \phi(q_n) = \phi\Big(\inf_{n\in\mathbb{N}} q_n\Big), \quad \text{whenever } (q_n)_{n\in\mathbb{N}} \subset P \text{ is non-increasing and } \inf_{n\in\mathbb{N}} q_n \text{ exists,}$$

where $\sup_{n\in\mathbb{N}} p_n$ stands for $\sup\{p_n\}_{n\in\mathbb{N}}$. Note that order-continuity implies σ-order-continuity.

A *lattice* is a partially ordered set $(P, \leq)$ such that $p \vee q := \sup\{p,q\}$ and $p \wedge q := \inf\{p,q\}$ exist for all $p,q \in P$. A set $S \subset P$ is called a *sublattice* of P if it is closed under $\vee$ and $\wedge$, that is,

$$p \vee q, p \wedge q \in S, \qquad \text{for every } p,q \in S.$$

A map $\phi\colon P \to Q$ between lattices P and Q is said to be a *lattice homomorphism* provided

$$\phi(p \vee q) = \phi(p) \vee \phi(q), \qquad \phi(p \wedge q) = \phi(p) \wedge \phi(q), \qquad \text{for every } p,q \in P.$$

For an arbitrary family $\{P_i\}_{i\in I}$ of partially ordered sets $P_i = (P_i, \leq_i)$, the product $P := \prod_{i\in I} P_i$ can be endowed with the following partial order: for any $(p_i)_{i\in I}, (q_i)_{i\in I} \in \prod_{i\in I} P_i$, we declare that $(p_i)_{i\in I} \leq (q_i)_{i\in I}$ if and only if $p_i \leq_i q_i$ for every $i \in I$. Observe that $(P, \leq)$ is a lattice if and only if $(P_i, \leq_i)$ is a lattice for every $i \in I$.

2.1.1. *The theory of Riesz spaces*

A *partially ordered linear space* $(U, \leq)$ is a vector space $U = (U, +, \cdot)$ over the field $\mathbb{R}$ of real numbers, together with a partial order $\leq$ on U such that the following properties are verified:

$$u + w \leq v + w, \quad \text{for every } u,v,w \in U \text{with } u \leq v,$$
$$\lambda u \geq 0, \quad \text{for every } \lambda \in \mathbb{R}^+ \text{and } u \in U \text{with } u \geq 0.$$

A *Riesz space* is a partially ordered linear space $U = (U, +, \cdot, \leq)$ that is a lattice. We define

$$u^+ := u \vee 0, \qquad u^- := (-u) \vee 0, \qquad |u| := (-u) \vee u,$$

for every $u \in U$. We have that $|u| \geq 0$ holds for every $u \in U$, with equality if and only if $u = 0$.

For a proof of the next result, we refer, for example, to [17, 352D] or [1, Theorem 1.3].

Proposition 2.1 (Basic properties of Riesz spaces) *Let U be a Riesz space. Then it holds that*

$$\lambda(u \vee v) = \lambda u \vee \lambda v, \quad \textit{for every } \lambda \in \mathbb{R} \textit{with } \lambda > 0 \textit{ and } u,v \in U, \tag{1a}$$

$$|\lambda u| = \lambda |u|, \quad \textit{for every } \lambda \in \mathbb{R}^+ \textit{and } u \in U, \tag{1b}$$

$$-u \vee v = (-u) \wedge (-v), \quad \textit{for every } u,v \in U, \tag{1c}$$

$$u + v \vee w = (u+v) \vee (u+w), \quad \textit{for every } u,v,w \in U, \tag{1d}$$

$$u + v \wedge w = (u+v) \wedge (u+w), \quad \textit{for every } u,v,w \in U, \tag{1e}$$

$$u \vee v + u \wedge v = u + v, \quad \textit{for every } u,v \in U, \tag{1f}$$

$$u = u^+ - u^-, \quad \textit{for every } u \in U, \tag{1g}$$

$$|u| = u^+ \vee u^- = u^+ + u^-, \quad \textit{for every } u \in U, \tag{1h}$$

$$u^+ \wedge u^- = 0, \quad \textit{for every } u \in U, \tag{1i}$$

$$(u+v)^+ \leq u^+ + v^+, \quad \textit{for every } u,v \in U, \tag{1j}$$

$$|u+v| \leq |u| + |v|, \quad \textit{for every } u,v \in U, \tag{1k}$$

$$u \wedge (v+w) \leq u \wedge v + u \wedge w, \quad \textit{for every } u,v,w \in U^+. \tag{1l}$$

A *Riesz subspace* of U is a linear subspace which is also a sublattice. A *homomorphism of Riesz spaces* $\phi\colon U \to V$ is a linear operator such that

$$\phi(u) \wedge \phi(v) = 0, \quad \text{for every } u,v \in U \text{such that } u \wedge v = 0.$$

By virtue of [17, 352G], each homomorphism of Riesz spaces $\phi\colon U \to V$ has the following property:

$$|\phi(u)| = \phi(|u|), \quad \text{for every } u \in U. \tag{2}$$

We denote by U^+ the *positive cone* of a Riesz space U, namely,

$$U^+ := \{u \in U \mid u \geq 0\}.$$

We recall from [1, Definition 1.22] that a Riesz subspace V of a given Riesz space U is said to be *super-order-dense* in U if for any $u \in U^+$ there exists a non-decreasing sequence $(u_n)_{n\in\mathbb{N}} \subset V^+$ such that $u = \sup_{n\in\mathbb{N}} u_n$. Moreover, a Riesz subspace V of a Riesz space U is said to be *solid* provided that $v \in V$ holds whenever $v \in U$, and there exists $u \in V$ such that $|v| \leq |u|$. We also recall from [18, Proposition 15B] the following result:

Proposition 2.2 *Any Dedekind σ-complete Riesz space U is Archimedean, that is, for any $u,v \in U$*

$$nu \leq v, \textit{ for every } n \in \mathbb{N} \quad \Longrightarrow \quad u \leq 0.$$

Definition 2.3 (Disjoint set) Let U be a Riesz space. Let S be a non-empty subset of U. Then we say that S is *disjoint* provided that it holds that

$$|u| \wedge |v| = 0, \quad \text{for every } u,v \in S \text{ such that } u \neq v.$$

When S is a finite disjoint set $\{u_1,\dots,u_n\} \subset U$, we say that the elements $u_1,\dots,u_n$ are pairwise disjoint.

Observe that if $\phi\colon U \to V$ is a homomorphism of Riesz spaces, then it holds that

$$\{\phi(u) \mid u \in S\} \subset V \text{ is disjoint}, \quad \text{for every } \emptyset \neq S \subset U \text{ disjoint}. \tag{3}$$

Indeed, if $u,v \in S$ and $\phi(u) \neq \phi(v)$, then $u \neq v$ and $|\phi(u)| \wedge |\phi(v)| = \phi(|u|) \wedge \phi(|v|) = 0$ by (2).
We also recall (see [2, p. 3] or [1, Definition 1.43]) the following notion:

Definition 2.4 (CSP) Let U be a Riesz space. Then we say that U has the *CSP* (or that U is a *CSP space*) if it holds that

$$\forall \emptyset \neq V \subset U \text{ such that } \exists \sup V, \quad \exists (v_n)_{n\in\mathbb{N}} \subset V : \quad \sup_{n\in\mathbb{N}} v_n = \sup V,$$
$$\forall \emptyset \neq \tilde{V} \subset U \text{ such that } \exists \inf V, \quad \exists (\tilde{v}_n)_{n\in\mathbb{N}} \subset \tilde{V} : \quad \inf_{n\in\mathbb{N}} \tilde{v}_n = \inf \tilde{V}.$$

2.1.2. *The theory of f-algebras*

Next, we recall the definition of *f-algebra*, which is—roughly speaking—a Riesz space endowed with a multiplication operation that verifies suitable compatibility properties.

DEFINITION 2.5 (f-algebra) An *f-algebra* $U = (U,+,\cdot,\leq,\times)$ is a Riesz space $(U,+,\cdot,\leq)$ together with a map $\times : U \times U \to U$—called a *multiplication*—such that the following properties hold:

$$u \times (v \times w) = (u \times v) \times w, \quad \text{for every } u,v,w \in U, \tag{4a}$$

$$(u+v) \times w = (u \times w) + (v \times w), \quad \text{for every } u,v,w \in U, \tag{4b}$$

$$\lambda(u \times v) = (\lambda u) \times v, \quad \text{for every } u,v \in U \text{ and } \lambda \in \mathbb{R}, \tag{4c}$$

$$u \times v = v \times u, \quad \text{for every } u,v \in U, \tag{4d}$$

$$u \times v \geq 0, \quad \text{for every } u,v \in U^+, \tag{4e}$$

$$(u \times w) \wedge v = 0, \quad \text{for every } u,v \in U \text{ with } u \wedge v = 0 \text{ and } w \in U^+, \tag{4f}$$

$$\exists\, \mathbf{1}_U \in U : \quad u \times \mathbf{1}_U = u, \quad \text{for every } u \in U. \tag{4g}$$

A homomorphism of *f-algebras* $\phi : U \to V$ is a homomorphism of Riesz spaces that is uniferent, *that is,* $\phi(\mathbf{1}_U) = \mathbf{1}_V$, and preserves the multiplication, *that is* $\phi(u \times v) = \phi(u) \times \phi(v)$ for all $u,v \in U$. An *f-subalgebra* of U is a Riesz subspace V of U closed under multiplication and with $\mathbf{1}_V = \mathbf{1}_U$.

REMARK 2.6 Some comments on Definition 2.5 are in order:

(i) The structure $(U,+,\cdot,\leq,\times)$ introduced in Definition 2.5 is usually called a *commutative f-algebra with multiplicative identity*. For the sake of brevity, we call it just an *f-algebra*.
(ii) It follows from (4a), (4b), (4d) and (4g) that the triple $(U,+,\times)$ is a commutative ring with identity $\mathbf{1}_U$. The field $\mathbb{R}$ can be viewed as a subring of U via the map $\mathbb{R} \ni \lambda \mapsto \lambda \mathbf{1}_U \in U$.
(iii) It follows from (4c) and (4g) that $\lambda u = (\lambda \mathbf{1}_U) \times u$ holds for every $\lambda \in \mathbb{R}$ and $u \in U$, and thus the multiplicative identity $\mathbf{1}_U$ can be unambiguously denoted by 1.

Given any $u,v \in U$, for the sake of brevity we will typically write uv instead of $u \times v$.

EXAMPLE 2.7 The real line $\mathbb{R} = (\mathbb{R},+,\cdot,\leq,\cdot)$ is an f-algebra.

PROPOSITION 2.8 (Basic properties of f-algebras) *Let U be an f-algebra. Then it holds that*

$$u^+u^- = 0, \quad \textit{for every } u \in U, \tag{5a}$$

$$(uv)^+ = uv^+, \quad \textit{for every } u \in U^+ \textit{ and } v \in U, \tag{5b}$$

$$|u-v| = |u+v|, \quad \textit{for every } u,v \in U \textit{ with } u \wedge v = 0, \tag{5c}$$

$$|u+v| = |u| + |v|, \quad \text{for every } u,v \in U \text{ with } |u| \wedge |v| = 0, \tag{5d}$$

$$|uv| = |u||v|, \quad \text{for every } u,v \in U, \tag{5e}$$

$$uv \leq uw, \quad \text{for every } u \in U^+ \text{ and } v,w \in U \text{ with } v \leq w. \tag{5f}$$

Proof. (5a) Given that $u^+ \wedge u^- = 0$ by (1i) and $u^- \geq 0$, we obtain that $u^+u^- \wedge u^- = 0$ by (4f). Since also $u^+ \geq 0$, by using again (4f) we can conclude that $u^+u^- = u^+u^- \wedge u^+u^- = 0$.

(5b) Since $v^+ \wedge v^- = 0$ by (1i) and $u \geq 0$, we deduce from (4f) that $uv^+ \wedge uv^- = 0$, so

$$(uv)^+ \overset{(1g)}{=} (uv^+ - uv^-)^+ \overset{(1d)}{=} uv^+ + (-uv^-) \vee (-uv^+) \overset{(1c)}{=} uv^+ - uv^+ \wedge uv^- = uv^+.$$

(5c) Note that (1d) yields $(u-v)^+ = (u-v) \vee 0 = u - u \wedge v = u$ and $(u-v)^- = v$. Then an application of (1h) gives $|u-v| = (u-v)^+ + (u-v)^- = u + v = |u+v|$, thus getting (5c).

(5d) Let us start by observing that

$$\begin{aligned}(u^+ + v^+) \wedge (u^- + v^-) &\overset{(1l)}{\leq} u^+ \wedge u^- + v^+ \wedge u^- + u^+ \wedge v^- + v^+ \wedge v^- \\ &\overset{(1i)}{=} v^+ \wedge u^- + u^+ \wedge v^- \leq 2\,|u| \wedge |v| = 0.\end{aligned}$$

Hence, (5c) yields $|u+v| = |(u^+ + v^+) - (u^- + v^-)| = |u^+ + v^+| + |u^- + v^-| = |u| + |v|$.

(5e) Given that $u^+ \wedge u^- = v^+ \wedge v^- = 0$ by (1i), we deduce from (4f) that $w \wedge w' = 0$ holds whenever $w, w' \in \{u^+v^+, u^+v^-, u^-v^+, u^-v^-\}$ satisfy $w \neq w'$. Then by applying (5d) we get

$$\begin{aligned}|uv| &= |(u^+ - v^-)(v^+ - v^-)| = |u^+v^+ - u^+v^- - u^-v^+ + u^-v^-| \\ &= u^+v^+ + u^+v^- + u^-v^+ + u^-v^- = (u^+ + u^-)(v^+ + v^-) = |u||v|.\end{aligned}$$

(5f) Since $w - v \geq 0$, we know from (4e) that $uw - uv = (w-v)u \geq 0$, as desired. □

PROPOSITION 2.9 *Let U be an f-algebra. Let $S \subset U$ be a given non-empty set. Then it holds*

$$S \text{ is disjoint} \quad \Longrightarrow \quad uv = 0, \text{ for every } u,v \in S \text{ such that } u \neq v.$$

If in addition the f-algebra U is Archimedean, then the converse implication is verified as well.

Proof. Let us prove the first part of the statement. Fix any $u,v \in S$ with $u \neq v$ and $|u| \wedge |v| = 0$. We can argue as in the proof of (5a): using (4f) twice, we first obtain that $|u||v| \wedge |v| = 0$ and then that $|u||v| = |u||v| \wedge |u||v| = 0$. Therefore, (5e) yields $|uv| = 0$ and thus accordingly $uv = 0$.

To prove the second part of the statement, assume that U is Archimedean. We aim to show that if there exist $u,v \in S$ such that $w := |u| \wedge |v| \neq 0$, then S is not disjoint. Notice that there exists $n \in \mathbb{N}$ such that $nw \not\leq 1$. Denote $w_+ := (nw-1)^+$ and $w_- := (nw-1)^-$, thus $w_+ \neq 0$. Observe also that $w_- = 1 - nw \wedge 1$ by (1c) and (1d) and that $w_- = (|u| \wedge |v|)^2 \leq |u||v|$. Therefore,

$$w_+ = w_+(w_- + nw \wedge 1) = w_+w_- + w_+(nw \wedge 1) \overset{(5a)}{=} w_+(nw \wedge 1) \leq nw(nw \wedge 1) \leq (nw)^2 \leq n^2|u||v|.$$

Given that $w_+ \neq 0$, we finally deduce that $|uv| = |u||v| \neq 0$, yielding the sought conclusion. □

2.2. Idempotent elements

Given an f-algebra U, we define the family of all *idempotent elements* of U as follows:

$$\mathrm{Idem}(U) := \{u \in U \mid u^2 = u\}, \tag{6}$$

where we adopt the shorthand notation

$$u^k := \underbrace{u \times \cdots \times u}_{k\text{times}} \in U, \quad \text{for every } u \in U \text{and } k \in \mathbb{N},$$

with the convention that $u^0 := 1$. Note that $0 \in \mathrm{Idem}(U)$ and $1 \in \mathrm{Idem}(U)$ for any f-algebra U.

The content of the next lemma is inspired by [17, 363X(g)]. Although elementary, we provide a proof for the reader's usefulness, since we did not find a suitable reference for it.

Lemma 2.10 (Properties of $\mathrm{Idem}(U)$) *Let U be an f-algebra. Then the following properties hold:*

(i) $uv \in \mathrm{Idem}(U)$ for every $u, v \in \mathrm{Idem}(U)$.
(ii) $u + v - 2uv \in \mathrm{Idem}(U)$ for every $u, v \in \mathrm{Idem}(U)$.
(iii) $1 - u \in \mathrm{Idem}(U)$ for every $u \in \mathrm{Idem}(U)$.
(iv) $0 \leq u \leq 1$ for every $u \in \mathrm{Idem}(U)$. In particular, $\mathrm{Idem}(U)$ is order-bounded in U.
(v) If $u, v \in \mathrm{Idem}(U)$ satisfy $uv = 0$, then $u + v \in \mathrm{Idem}(U)$ and $u + v = u \vee v$.
(vi) If $u \in U$ and $v \in \mathrm{Idem}(U)$, then $u - uv$ and v are disjoint.

Proof. (i) Trivially, it holds that $(uv)^2 = uvuv = u^2v^2 = uv$.

(ii) It follows from the observation that

$$\begin{aligned}(u + v - 2uv)^2 &= u^2 + uv - 2u^2v + vu + v^2 - 2uv^2 - 2u^2v - 2uv^2 + 4u^2v^2 \\ &= u + uv - 2uv + uv + v - 2uv - 2uv - 2uv + 4uv = u + v - 2uv.\end{aligned}$$

(iii) Just observe that $(1 - u)(1 - u) = 1 - 2u + u^2 = 1 - 2u + u = 1 - u$.

(iv) Given any $u \in U$, it holds $u = u^+ - u^-$ and $u^+u^- = 0$ by (1g) and (5a), respectively. Then

$$u^2 = (u^+ - u^-)(u^+ - u^-) = (u^+)^2 - u^+u^- - u^-u^+ + (u^-)^2 = (u^+)^2 + (u^-)^2 \geq 0,$$

where the last inequality follows from (4e). In particular, $u = u^2 \geq 0$ for every $u \in \mathrm{Idem}(U)$. Since $1 - u \in \mathrm{Idem}(U)$ by item (ii), we also have that $1 - u \geq 0$ or equivalently that $u \leq 1$.

(v) First, we may compute $(u + v)^2 = u^2 + 2uv + v^2 = u + v$, which shows that $u + v \in \mathrm{Idem}(U)$. Moreover, thanks to the fact that $u \leq u + v$ and $v \leq u + v$, we have that $u \vee v \leq u + v$. Conversely, it holds that $u(u + v) = u^2 \leq u(u \vee v)$ and $(1 - u)(u + v) = u + v - u^2 - uv = (1 - u)v \leq (1 - u)(u \vee v)$, thus accordingly $u + v = u(u + v) + (1 - u)(u + v) \leq u(u \vee v) + (1 - u)(u \vee v) = u \vee v$.

vi) First of all, we aim to show that $(1 - v) \wedge v = 0$. Item (iv) ensures that $(1 - v) \wedge v \geq 0$. Conversely, if $w \in U$ is a lower bound for $\{v, 1 - v\}$, then using (5f) we can estimate

$$w = vw + (1 - v)w \leq v(1 - v) + (1 - v)v = v - v^2 + v - v^2 = 0,$$

which yields $(1 - v) \wedge v = 0$. Hence, (4f) ensures that $|u - uv| \wedge v = (|u|(1 - v)) \wedge v = 0$. □

Observe that if $\phi\colon U \to V$ is a homomorphism of f-algebras, then it holds that

$$\phi(u) \in \mathrm{Idem}(V), \quad \text{for every } u \in \mathrm{Idem}(U). \tag{7}$$

Indeed, since ϕ preserves the multiplication, we have $\phi(u)^2 = \phi(u^2) = \phi(u)$ for every $u \in \mathrm{Idem}(U)$.

Remark 2.11 Given an f-algebra U and two elements $u,v \in \mathrm{Idem}(U)$, it holds that

$$u \leq v \qquad \Longleftrightarrow \qquad uv = u.$$

Indeed, if $uv = u$, then $u = uv \leq v$. On the other hand, if $u \leq v$, then $u = u^2 \leq uv \leq u$.

Definition 2.12 (Finite partition) Let U be an f-algebra. Then a given set $(u_i)_{i=1}^n \subset \mathrm{Idem}(U)$ is said to be a *finite partition* of an element $u \in \mathrm{Idem}(U)$ provided that it is disjoint and it satisfies

$$u_1 + \ldots + u_n = u.$$

We denote by $\mathcal{P}_f(u)$ the family of all finite partitions of u.

Thanks to Lemma 2.10 (v), a disjoint family $(u_i)_{i=1}^n \subset \mathrm{Idem}(U)$ belongs to $\mathcal{P}_f(u)$ if and only if

$$\sup\{u_1,\ldots,u_n\} = u.$$

Notice also that if $\phi\colon U \to V$ is a homomorphism of f-algebras, then it holds that

$$(\phi(u_i))_{i=1}^n \in \mathcal{P}_f(\phi(u)), \quad \text{for every } u \in \mathrm{Idem}(U) \text{ and } (u_i)_{i=1}^n \in \mathcal{P}_f(u). \tag{8}$$

Indeed, one has $\phi(u_i) \in \mathrm{Idem}(\phi(v))$ for every $i = 1,\ldots,n$ by (7), the elements $\phi(u_1),\ldots,\phi(u_n)$ are pairwise disjoint by (3) and $\phi(u_1) + \ldots + \phi(u_n) = \phi(u_1 + \ldots + u_n) = \phi(u)$ by the linearity of ϕ.

Definition 2.13 (Simple elements) The *simple elements* of an f-algebra U are defined as

$$\mathcal{S}(U) := \left\{ \sum_{i=1}^n \lambda_i u_i \;\middle|\; n \in \mathbb{N},\, (\lambda_i)_{i=1}^n \subset \mathbb{R},\, (u_i)_{i=1}^n \in \mathcal{P}_f(\mathbf{1}_U) \right\} \subset U.$$

The family of all *non-negative simple elements* of U is defined as $\mathcal{S}^+(U) := \mathcal{S}(U) \cap U^+$.

Lemma 2.14 *Let U be an f-algebra and $u \in \mathrm{Idem}(U)$. Then it holds that*

$$(u_i v_j)_{i,j} \in \mathcal{P}_f(u), \quad \textit{for every } (u_i)_{i=1}^n, (v_j)_{j=1}^m \in \mathcal{P}_f(u). \tag{9}$$

In particular, the space $\mathcal{S}(U)$ is an f-subalgebra of U. More precisely, it holds that

$$u + v = \sum_{i,j} (\lambda_i + \mu_j) u_i v_j, \quad uv = \sum_{i,j} \lambda_i \mu_j u_i v_j, \quad u \vee v = \sum_{i,j} (\lambda_i \vee \mu_j) u_i v_j, \quad u \wedge v = \sum_{i,j} (\lambda_i \wedge \mu_j) u_i v_j,$$

for every $u = \sum_{i=1}^n \lambda_i u_i \in \mathcal{S}(U)$ and $v = \sum_{j=1}^m \mu_j v_j \in \mathcal{S}(U)$.

Proof. Let us only check (9). Once (9) is established, the remaining part of the statement follows via elementary computations. Fix any $(u_i)_{i=1}^n, (v_j)_{j=1}^m \in \mathcal{P}_f(u)$. Lemma 2.10 (i) ensures that $u_i v_j \in \mathrm{Idem}(U)$ for every $i = 1,\ldots,n$ and $j = 1,\ldots,m$. Moreover, whenever $(i,j) \neq (i',j')$ we have that $(u_i v_j) \wedge (u_{i'} v_{j'}) \leq (u_i \wedge u_{i'}) \wedge (v_j \wedge v_{j'}) = 0$, thus $(u_i v_j)_{i,j}$ is a disjoint set. Finally, it holds that $\sum_{i,j} u_i v_j = (u_1 + \ldots + u_n)(v_1 + \ldots + v_m) = u$, which gives $(u_i v_j)_{i,j} \in \mathcal{P}_f(u)$. □

2.3. Localizable f-algebras

Let us now introduce the concept of *localizable f-algebra*, which is a Dedekind σ-complete f-algebra 'having plenty of idempotent elements'. Namely:

Definition 2.15 (Localizable f-algebra) Let U be a Dedekind σ-complete f-algebra whose multiplication map is σ-order-continuous on $U^+ \times U^+$. Then we say that U is *localizable* provided that the space of simple elements $\mathcal{S}(U)$ is super-order-dense in U. By a *homomorphism of localizable f-algebras* we mean a σ-order-continuous homomorphism of f-algebras.

Remark 2.16 On any Dedekind σ-complete f-algebra U, the sum operator $+\colon U \times U \to U$ is σ-order-continuous on $U^+ \times U^+$. Indeed, if $(u_n)_{n\in\mathbb{N}}, (v_m)_{m\in\mathbb{N}} \subset U^+$ are non-decreasing sequences, and we set $u := \sup_{n\in\mathbb{N}} u_n \in U^+$ and $v := \sup_{m\in\mathbb{N}} v_m \in U^+$, then for any $n, m \in \mathbb{N}$ we have that

$$u_n = (u_n + v_m) - v_m \leq (u_{n\vee m} + v_{n\vee m}) - v_m \leq \sup_{k\in\mathbb{N}}(u_k + v_k) - v_m.$$

Thanks to the arbitrariness of $n \in \mathbb{N}$, we deduce that $u \leq \sup_{k\in\mathbb{N}}(u_k + v_k) - v_m$. By arbitrariness of $m \in \mathbb{N}$, we conclude that $u + v \leq \sup_{k\in\mathbb{N}}(u_k + v_k)$. The converse inequality is trivial.

Lemma 2.17 *Let U be a localizable f-algebra. Then it holds that*

$$\sup_{n\in\mathbb{N}} u_n \in Idem(U), \qquad \inf_{n\in\mathbb{N}} u_n \in Idem(U), \quad \textit{for every } (u_n)_{n\in\mathbb{N}} \subset Idem(U).$$

Proof. Since the set $\mathrm{Idem}(U)$ is order-bounded by Lemma 2.10 iv), both $v := \sup_{n\in\mathbb{N}} u_n \in U^+$ and $w := \inf_{n\in\mathbb{N}} u_n \in U^+$ exist thanks to the Dedekind σ-completeness of U. Define $u'_1 := u_1$ and $u'_n := u_n - \sum_{k<n} u_n u'_k$ for every $n \geq 2$. By using items (iii), (v) and (vi) of Lemma 2.10 and an induction argument, one can show that the sequence $(u'_n)_{n\in\mathbb{N}}$ is disjoint and made of idempotent elements. Lemma 2.10 (v) also yields $\sup_{k\leq n} u'_k = \sup_{k\leq n} u_k$ for every $n \in \mathbb{N}$, so that accordingly

$$\sup_{n\in\mathbb{N}} u'_n = \sup_{n\in\mathbb{N}} \sup_{k\leq n} u_k = \sup_{n\in\mathbb{N}} u_n = v.$$

Now define $v_n := \sum_{k=1}^n u'_k$ for every $n \in \mathbb{N}$. Lemma 2.10 (v) ensures that $(v_n)_{n\in\mathbb{N}} \subset \mathrm{Idem}(U)$ and that $v_n = \sup_{k\leq n} u'_k$ for every $n \in \mathbb{N}$, and thus $v = \sup_{n\in\mathbb{N}} v_n$. Given that the sequence $(v_n)_{n\in\mathbb{N}}$ is non-decreasing by construction, the σ-order-continuity of the multiplication on $U^+ \times U^+$ guarantees that $v^2 = (\sup_{n\in\mathbb{N}} v_n)^2 = \sup_{n\in\mathbb{N}} v_n^2 = \sup_{n\in\mathbb{N}} v_n = v$, proving that $v \in \mathrm{Idem}(U)$. Finally, notice that we have $1 - w = \sup_{n\in\mathbb{N}}(1 - u_n) \in \mathrm{Idem}(U)$, so that $w \in \mathrm{Idem}(U)$ by Lemma 2.10 (iii). □

Definition 2.18 (Countable partition) Let U be a Dedekind σ-complete f-algebra. Then a disjoint family $(u_n)_{n\in\mathbb{N}} \subset \mathrm{Idem}(U)$ is said to be a *countable partition* of $u \in \mathrm{Idem}(U)$ provided

$$\sup_{n\in\mathbb{N}} u_n = u.$$

We denote by $\mathcal{P}(u)$ the family of all countable partitions of u. Observe that $\mathcal{P}_f(u) \subset \mathcal{P}(u)$.

Proposition 2.19 *Let U be a localizable f-algebra and $u \in Idem(U)$. Then it holds that*

$$(u_n v_m)_{n,m\in\mathbb{N}} \in \mathcal{P}(u), \quad \textit{for every } (u_n)_{n\in\mathbb{N}}, (v_m)_{m\in\mathbb{N}} \in \mathcal{P}(u).$$

Proof. Lemma 2.10 (i) ensures that $u_n v_m \in \mathrm{Idem}(U)$ for every $n, m \in \mathbb{N}$. Arguing as in Lemma 2.14, we see that $(u_n v_m)_{n\in\mathbb{N}}$ is a disjoint set. It remains to show that

$v := \sup_{n,m\in\mathbb{N}} u_n v_m = u$. Since $u_n v_m \leq 1$ for all $n,m \in \mathbb{N}$, we have $v \leq 1$. Let $w \in U$ be an upper bound for $(u_n v_m)_{n,m\in\mathbb{N}}$. Then $u_n(w - v_m) = u_n(w - u_n v_m) \geq 0$ for every $n,m \in \mathbb{N}$, so that $u_n(w-v_m)^+ \geq u_n(w-v_m)^-$ holds for every $n,m \in \mathbb{N}$ as a consequence of (5b), thus accordingly

$$(w-v_m)^+ = \sup_{n\in\mathbb{N}} u_n(w-v_m)^+ \geq \sup_{n\in\mathbb{N}} u_n(w-v_m)^- = (w-v_m)^-, \quad \text{for every } m \in \mathbb{N}.$$

This means that $w \geq v_m$ for every $m \in \mathbb{N}$, which gives $w \geq 1$. We conclude that $v = u$. □

Observe that if $\phi\colon U \to V$ a homomorphism of localizable f-algebras, then it holds that

$$\big(\phi(u_n)\big)_{n\in\mathbb{N}} \in \mathcal{P}(\phi(u)), \quad \text{for every } u \in \mathrm{Idem}(U) \text{and } (u_n)_{n\in\mathbb{N}} \in \mathcal{P}(u). \tag{10}$$

Indeed, the sequence $\big(\phi(u_n)\big)_{n\in\mathbb{N}} \subset V$ is made of idempotent elements by (7) and is a disjoint set by (3) and $\sup_{n\in\mathbb{N}} \phi(u_n) = \phi\big(\sup_{n\in\mathbb{N}} u_n\big) = \phi(u)$, thanks to the σ-order-continuity of ϕ.

2.4. Metric *f*-structures

For our purposes, the algebraic and order properties of a localizable f-algebra are not sufficient. Rather, we want to consider localizable f-algebras (and Riesz spaces) endowed with a well-behaved complete distance. In this regard, the first concept we introduce is that of *metric Riesz space*:

Definition 2.20 (Metric Riesz space) By a *metric Riesz space* we mean a couple (U, d_U) – where U is a Dedekind σ-complete Riesz space and d_U is a complete distance on U – such that

(i) The identity $\mathsf{d}_U(u,0) = \mathsf{d}_U(|u|,0)$ holds for every $u \in U$.
(ii) The distance d_U is *translation-invariant*, in the sense that

$$\mathsf{d}_U(u,v) = \mathsf{d}_U(u+w, v+w), \quad \text{for every } u,v,w \in U.$$

(iii) The distance-from-zero function $\mathsf{d}_U(\cdot,0)\colon U^+ \to \mathbb{R}^+$ is order-preserving.

A *homomorphism of metric Riesz spaces* is a Lipschitz homomorphism of Riesz spaces.

We remark that the notion of homomorphism in the above definition is not intended as a morphism in the categorical sense. In fact, in this paper we will not consider any category of metric Riesz spaces. The main reason is that the class of metric Riesz spaces includes the function spaces $L^0(\mu)$ (see Section 4.1). The latter are metrizable topological vector spaces, but —as far as we know—they are not endowed with a 'canonical' distance, and thus neither Lipschitz nor 1-Lipschitz homomorphisms of Riesz spaces seem to be an effective choice of morphism in the categorical sense. Arguably, one should introduce, for example, a more general concept of 'uniform Riesz space' (that is, a Riesz space equipped with a compatible uniform structure) in order to have a well-behaved category. A similar discussion applies to the notions of (homomorphisms of) metric f-algebras, metric f-structures and dual systems that we will introduce below.

Remark 2.21 Given a metric Riesz space (U, d_U), it holds that

$$\mathsf{d}_U(u+v,0) \leq \mathsf{d}_U(u,0) + \mathsf{d}_U(v,0), \quad \text{for every } u,v \in U. \tag{11}$$

Indeed, by using the translation invariance of d_U, we obtain that

$$\mathsf{d}_U(u+v,0) = \mathsf{d}_U(u,-v) \leq \mathsf{d}_U(u,0) + \mathsf{d}_U(0,-v) = \mathsf{d}_U(u,0) + \mathsf{d}_U(v,0).$$

Repeatedly applying (11), we get $\mathsf{d}_U\big(\sum_{i=1}^n u_i, 0\big) \leq \sum_{i=1}^n \mathsf{d}_U(u_i,0)$ for all $u_1,\dots,u_n \in U$.

In Definitions 2.22 and 2.23, given two metric spaces $(\mathrm{X},\mathsf{d}_{\mathrm{X}})$ and $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}})$, we will consider the distance $\mathsf{d}_{\mathrm{X}}\times\mathsf{d}_{\mathrm{Y}}$ on the Cartesian product $\mathrm{X}\times\mathrm{Y}$, which is given by

$$(\mathsf{d}_{\mathrm{X}}\times\mathsf{d}_{\mathrm{Y}})\big((x,y),(\tilde{x},\tilde{y})\big):=\mathsf{d}_{\mathrm{X}}(x,\tilde{x})+\mathsf{d}_{\mathrm{Y}}(y,\tilde{y}),\quad\text{for every }(x,y),(\tilde{x},\tilde{y})\in\mathrm{X}\times\mathrm{Y}.$$

Next, we introduce the family of *metric f-algebras*:

Definition 2.22 (Metric f-algebra) By a *metric f-algebra* we mean a couple (U,d_U), where

(i) U is a localizable f-algebra, and (U,d_U) is a metric Riesz space.
(ii) The multiplication $\times: U\times U\to U$ is continuous from $(U\times U,\mathsf{d}_U\times\mathsf{d}_U)$ to (U,d_U).
(iii) The family $\mathcal{S}(U)$ of all simple elements of U is dense in (U,d_U).
(iv) $\mathsf{d}_U(\varepsilon\mathbf{1}_U,0)\to 0$ as $\varepsilon\searrow 0$.

A *homomorphism of metric f-algebras* is a Lipschitz homomorphism of localizable f-algebras.

For some examples of metric f-algebras in the case of function spaces we are interested in, see Subsection 4.2.1. Having the notions of metric Riesz space and of metric f-algebra at our disposal, we can finally introduce *metric f-structures*:

Definition 2.23 (Metric f-structure) A *metric f-structure* is a triple $(\mathcal{U},U,V)$, where

(i) $\mathcal{U}$ is a localizable f-algebra.
(ii) $U=(U,\mathsf{d}_U)$ is a metric f-algebra such that U is a solid f-subalgebra of $\mathcal{U}$.
(iii) $V=(V,\mathsf{d}_V)$ is metric Riesz space such that V is a solid Riesz subspace of $\mathcal{U}$.
(iv) It holds $UV=V$, and the multiplication is continuous from $(U\times V,\mathsf{d}_U\times\mathsf{d}_V)$ to (V,d_V).
(v) Given any $(u_n)_{n\in\mathbb{N}}\in\mathcal{P}(\mathbf{1}_U)$ and $\varepsilon>0$, there exists $\delta>0$ such that for any $(v_n)_{n\in\mathbb{N}}\subset V^+$ with $\sum_{n\in\mathbb{N}}\mathsf{d}_V(u_nv_n,0)\leq\delta$ it holds that

$$(u_nv_n)_{n\in\mathbb{N}}\text{ is order-bounded in }V,\qquad \mathsf{d}_V\Big(\sup_{n\in\mathbb{N}}u_nv_n,0\Big)\leq\varepsilon.$$

We say that a metric f-structure $(\mathcal{U},U,V)$ is *Dedekind complete* (resp. *CSP*), provided that the spaces $\mathcal{U}$, U and V are Dedekind complete (resp. CSP).

A *homomorphism of metric f-structures* between two metric f-structures $(\mathcal{U}_1,U_1,V_1)$ and $(\mathcal{U}_2,U_2,V_2)$ is a homomorphism $\varphi:\mathcal{U}_1\to\mathcal{U}_2$ of f-algebras such that $\varphi|_{U_1}: U_1\to U_2$ is a homomorphism of metric f-algebras and $\varphi|_{V_1}: V_1\to V_2$ a is homomorphism of metric Riesz spaces.

In the case of function spaces, some examples of metric f-structures are listed in Subsection 4.2.2.

Example 2.24 If $U=(U,\mathsf{d}_U)$ is a metric f-algebra, then (U,U,U) is a metric f-structure.

As we discussed in Section 1, in order to study dual modules (and, more generally, spaces of homomorphisms) we also have to define the *dual systems* of metric f-structures:

Definition 2.25 (Dual system of metric f-structures) A quintuplet $(\mathcal{U},U,V,W,Z)$ is said to be a *dual system of metric f-structures* provided that the following conditions are verified:

(i) $(\mathcal{U},U,V)$, $(\mathcal{U},U,W)$ and $(\mathcal{U},U,Z)$ are metric f-structures.
(ii) It holds $Z=VW$ and the multiplication is continuous from $(V\times W,\mathsf{d}_V\times\mathsf{d}_W)$ to (Z,d_Z).

We say that $(\mathcal{U}, U, V, W, Z)$ is a *complete dual system of metric f-structures* if in addition:

(iii) $\mathcal{U}$, U, V, W, Z are Dedekind complete and the multiplication is order-continuous from $U^+ \times V^+$ to V^+, from $U^+ \times W^+$ to W^+, from $U^+ \times Z^+$ to Z^+ and from $V^+ \times W^+$ to Z^+.

Also, we say that a complete dual system $(\mathcal{U}, U, V, W, Z)$ is *CSP* if in addition $(\mathcal{U}, U, V)$, $(\mathcal{U}, U, W)$ and $(\mathcal{U}, U, Z)$ are CSP.
By a *homomorphism of dual systems* between two given dual systems of metric f-structures $(\mathcal{U}_1, U_1, V_1, W_1, Z_1)$ and $(\mathcal{U}_2, U_2, V_2, W_2, Z_2)$ we mean a map $\varphi\colon \mathcal{U}_1 \to \mathcal{U}_2$ that is a homomorphism of metric f-structures from $(\mathcal{U}_1, U_1, V_1)$ to $(\mathcal{U}_2, U_2, V_2)$, from $(\mathcal{U}_1, U_1, W_1)$ to $(\mathcal{U}_2, U_2, W_2)$ and from $(\mathcal{U}_1, U_1, Z_1)$ to $(\mathcal{U}_2, U_2, Z_2)$.

Some relevant examples of dual systems in the case of function spaces are presented in Section 4.2.3.

Example 2.26 Let $(\mathcal{U}, U, V)$ be a metric f-structure. Then $(\mathcal{U}, U, V, U, V)$ is a dual system of metric f-structures. If U is Dedekind complete and the multiplication map is order-continuous from $U^+ \times V^+$ to V^+, then $(\mathcal{U}, U, V, U, V)$ is a complete dual system of metric f-structures.

Remark 2.27 If $(\mathcal{U}, U, V, W, Z)$ is a dual system of metric f-structures, then $(\mathcal{U}, U, W, V, Z)$ is a dual system of metric f-structures as well. Moreover, if $(\mathcal{U}, U, V, W, Z)$ is a complete (resp. CSP complete) dual system, then $(\mathcal{U}, U, W, V, Z)$ is a complete (resp. CSP complete) dual system.

3. NORMED MODULES OVER A METRIC *F*-STRUCTURE

In Sections 3.1 and 3.2 we introduce the category of Banach modules over a metric f-structure; in the former we study the objects, while in the latter we study the morphisms. In Section 3.3 we prove some existence results concerning Banach modules and their homomorphisms, as well as some of their consequences. In Sections 3.4, 3.5 and 3.6 we study the Hahn–Banach theorem, the class of Hilbert modules and the dimensional decomposition of a Banach module, respectively.

3.1. Definitions and basic properties

First of all, let us give the definition of *normed/Banach module* over a metric f-structure:

Definition 3.1 (Normed module) Let $(\mathcal{U}, U, V)$ be a metric f-structure and $\mathscr{M}$ a module over U. Then we say that $\mathscr{M}$ is a *V-normed U-module* provided that it is endowed with a map $|\cdot|\colon \mathscr{M} \to V^+$ – called a *V-pointwise norm* operator on $\mathscr{M}$—such that the following properties are verified:

(i) Given any $u \in U$ and $v, w \in \mathscr{M}$, it holds that

$$|v| = 0 \quad \Longleftrightarrow \quad v = 0, \tag{12a}$$

$$|v + w| \leq |v| + |w|, \tag{12b}$$

$$|u \cdot v| = |u||v|. \tag{12c}$$

(ii) Gluing property. Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{P}(\mathbf{1}_U)$ and $(v_n)_{n\in\mathbb{N}} \subset \mathscr{M}$ be chosen so that the family $(|u_n \cdot v_n|)_{n\in\mathbb{N}}$ is order-bounded in V. Then there exists an element $v \in \mathscr{M}$

such that

$$u_n \cdot v = u_n \cdot v_n, \quad \text{for every } n \in \mathbb{N}. \tag{13}$$

We call $\mathrm{Adm}(\mathscr{M})$ the set of all families $(u_n, v_n)_{n\in\mathbb{N}}$ as above, while $\sum_{n\in\mathbb{N}} u_n \cdot v_n$ stands for the element $v \in \mathscr{M}$ satisfying (eq: glueing)—whose uniqueness follows from Lemma 3.2.

Moreover, we endow the space $\mathscr{M}$ with the distance $\mathsf{d}_{\mathscr{M}}$, which is defined as

$$\mathsf{d}_{\mathscr{M}}(v, w) := \mathsf{d}_V(|v - w|, 0), \quad \text{for every } v, w \in \mathscr{M}. \tag{14}$$

Whenever $(\mathscr{M}, \mathsf{d}_{\mathscr{M}})$ is a complete metric space, we say that $\mathscr{M}$ is a *V-Banach U-module*.

Some examples of functional normed/Banach modules that are covered by the above definition are presented in Section 4.2.4.

LEMMA 3.2 (Locality property) *Let $(\mathcal{U}, U, V)$ be a metric f-structure, and let $\mathscr{M}$ be a V-normed U-module. Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{P}(\mathbf{1}_U)$ and $v \in \mathscr{M}$ satisfy $u_n \cdot v = 0$ for every $n \in \mathbb{N}$. Then $v = 0$.*

Proof. Given that the multiplication map is σ-order-continuous on $\mathcal{U}^+ \times \mathcal{U}^+$, we deduce that

$$|v| = |v| \sup_{n\in\mathbb{N}} u_n = \sup_{n\in\mathbb{N}} u_n|v| = \sup_{n\in\mathbb{N}} |u_n \cdot v| = 0,$$

whence it follows that $v = 0$, as we claimed in the statement. □

Given any non-empty subset S of a V-normed U-module $\mathscr{M}$, we denote by $\mathscr{G}(S) \subset \mathscr{M}$ the family of those elements that can be obtained by gluing together elements of S. Namely, we set

$$\mathscr{G}(S) = \mathscr{G}_{\mathscr{M}}(S) := \left\{ \sum_{n\in\mathbb{N}} u_n \cdot v_n \;\middle|\; (u_n)_{n\in\mathbb{N}} \in \mathcal{P}(\mathbf{1}_U),\ (v_n)_{n\in\mathbb{N}} \subset S,\ (u_n, v_n)_{n\in\mathbb{N}} \in \mathrm{Adm}(\mathscr{M}) \right\}.$$

Observe that if S is a vector subspace of $\mathscr{M}$, then $\mathscr{G}(S)$ is a vector subspace of $\mathscr{M}$ as well.

PROPOSITION 3.3 *Let $(\mathcal{U}, U, V)$ be a metric f-structure. Then V is a V-Banach U-module, with the scalar multiplication $\cdot\colon U \times V \to V$ being given by the multiplication $\times$ in $\mathcal{U}$. Moreover, it holds*

$$\sum_{n\in\mathbb{N}} u_n v_n = \sup_{n\in\mathbb{N}} u_n v_n^+ - \sup_{n\in\mathbb{N}} u_n v_n^-, \quad \textit{for every } (u_n, v_n)_{n\in\mathbb{N}} \in \mathit{Adm}(V). \tag{15}$$

Proof. The fact that V is a U-module verifying item (i) of Definition 3.1 readily follows from the very definition of a metric f-algebra. Moreover, the distance on V defined as in (14) coincides with the original distance d_V itself, which is complete by assumption. It only remains to check the validity of the gluing property. To this aim, fix any $(u_n, v_n)_{n\in\mathbb{N}} \in \mathrm{Adm}(V)$. In particular, both sequences $(u_n v_n^+)_{n\in\mathbb{N}}$ and $(u_n v_n^-)_{n\in\mathbb{N}}$ are order-bounded, and thus the Dedekind σ-completeness of V yields existence of $w_+ := \sup_{n\in\mathbb{N}} u_n v_n^+ \in V^+$ and $w_- := \sup_{n\in\mathbb{N}} u_n v_n^- \in V^+$. We claim that

$$u_n(w_+ - w_-) = u_n v_n, \quad \text{for every } n \in \mathbb{N}. \tag{16}$$

To prove it, notice that $u_n w_+ = u_n v_n^+$ for every $n \in \mathbb{N}$: the inequality $\geq$ is trivial, while to get the converse one it suffices to observe that $(1 - u_n)w_+ + u_n v_n^+$ is an upper bound for $(u_m v_m^+)_{m\in\mathbb{N}}$. Similarly, one can show that $u_n w_- = u_n v_n^-$, whence it follows that $u_n(w_+ - w_-) = u_n v_n^+ - u_n v_n^-$, yielding (16). This proves the validity of the gluing property, as well as formula (15). □

PROPOSITION 3.4 (Continuity of normed module operations) *Let $(\mathcal{U}, U, V)$ be a metric f-structure and $\mathscr{M}$ a V-normed U-module. Then the following properties are verified:*

(i) The map $|\cdot|: \mathscr{M} \to V^+$ is 1-Lipschitz from $(\mathscr{M}, \mathsf{d}_{\mathscr{M}})$ to (V, d_V).
(ii) The map $+: \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is 1-Lipschitz from $(\mathscr{M} \times \mathscr{M}, \mathsf{d}_{\mathscr{M}} \times \mathsf{d}_{\mathscr{M}})$ to $(\mathscr{M}, \mathsf{d}_{\mathscr{M}})$.
(iii) The map $\cdot: U \times \mathscr{M} \to \mathscr{M}$ is continuous from $(U \times \mathscr{M}, \mathsf{d}_U \times \mathsf{d}_{\mathscr{M}})$ to $(\mathscr{M}, \mathsf{d}_{\mathscr{M}})$.

Proof. (i) Given that $|v| \leq |v-w| + |w|$ and $|w| \leq |w-v| + |v|$ hold for every $v, w \in \mathscr{M}$, we deduce that

$$\big||v| - |w|\big| \leq |v-w|, \quad \text{for every } v, w \in \mathscr{M}.$$

In particular, for any $v, w \in \mathscr{M}$ one has
$\mathsf{d}_V(|v|, |w|) = \mathsf{d}_V(\big||v| - |w|\big|, 0) \leq \mathsf{d}_V(|v-w|, 0) = \mathsf{d}_{\mathscr{M}}(v, w)$, which shows that $|\cdot|: \mathscr{M} \to V^+$ is a 1-Lipschitz mapping from $(\mathscr{M}, \mathsf{d}_{\mathscr{M}})$ to (V, d_V), as required.

(ii) For any $v, v', w, w' \in \mathscr{M}$ we have $|(v+w) - (v'+w')| \leq |v-v'| + |w-w'|$, thus accordingly

$$\begin{aligned} \mathsf{d}_{\mathscr{M}}(v+w, v'+w') &= \mathsf{d}_V(\big|(v+w) - (v'+w')\big|, 0) \leq \mathsf{d}_V(|v-v'| + |w-w'|, 0) \\ &\leq \mathsf{d}_V(|v-v'|, 0) + \mathsf{d}_V(|w-w'|, 0) = \mathsf{d}_{\mathscr{M}}(v, v') + \mathsf{d}_{\mathscr{M}}(w, w') \\ &= (\mathsf{d}_{\mathscr{M}} \times \mathsf{d}_{\mathscr{M}})((v, v'), (w, w')), \end{aligned}$$

for every $v, v', w, w' \in \mathscr{M}$. This proves that + is 1-Lipschitz from $(\mathscr{M} \times \mathscr{M}, \mathsf{d}_{\mathscr{M}} \times \mathsf{d}_{\mathscr{M}})$ to $(\mathscr{M}, \mathsf{d}_{\mathscr{M}})$.

(iii) Fix $(u_n)_{n \in \mathbb{N}} \subset U$ and $u \in U$ with $\lim_{n\to\infty} \mathsf{d}_U(u_n, u) = 0$. Fix $(v_n)_{n \in \mathbb{N}} \subset \mathscr{M}$ and $v \in \mathscr{M}$ with $\lim_{n\to\infty} \mathsf{d}_{\mathscr{M}}(v_n, v) = 0$. The continuity of $|\cdot|: \mathscr{M} \to V^+$ from (i) ensures that $|u_n| \to |u|$ in (U, d_U) and $|v_n - v| \to 0$ in (V, d_V). Hence, by letting $n \to \infty$ in $|u_n \cdot v_n - u \cdot v| \leq |u_n||v_n - v| + |u_n - u||v|$ we obtain $\lim_{n\to\infty} \mathsf{d}_{\mathscr{M}}(u_n \cdot v_n, u \cdot v) = 0$, which shows that $\cdot: U \times \mathscr{M} \to \mathscr{M}$ is continuous. □

Let $(\mathcal{U}, U, V)$ be a metric f-structure, and let $\mathscr{M}$ be a V-normed U-module. Then a given U-submodule $\mathscr{N}$ of $\mathscr{M}$ is said to be a *V-normed U-submodule* of $\mathscr{M}$, provided that it satisfies

$$\mathscr{G}_{\mathscr{M}}(\mathscr{N}) = \mathscr{N}.$$

In the case where $\mathscr{M}$ is a V-Banach U-module and $\mathscr{N}$ is $\mathsf{d}_{\mathscr{M}}$-closed in $\mathscr{M}$, we say that $\mathscr{N}$ is a *V-Banach U-submodule* of $\mathscr{M}$. These are some useful examples of V-Banach U-submodule:

- The 'localized' module $u \cdot \mathscr{M} := \{u \cdot v : v \in \mathscr{M}\}$ for every $u \in \mathrm{Idem}(U)$.
- The 'one-dimensional' module $U \cdot v := \{u \cdot v : u \in U\}$ for every $v \in \mathscr{M}$.
- The sum $\mathscr{N}_1 + \mathscr{N}_2 := \{v + w : v \in \mathscr{N}_1, w \in \mathscr{N}_2\}$ where $\mathscr{N}_1$, $\mathscr{N}_2$ are V-Banach U-submodules of $\mathscr{M}$. We say that $\mathscr{M}$ is the *direct sum* of $\mathscr{N}_1$ and $\mathscr{N}_2$, and we write

$$\mathscr{M} = \mathscr{N}_1 \oplus \mathscr{N}_2,$$

  if $\mathscr{M} = \mathscr{N}_1 + \mathscr{N}_2$ and $\mathscr{N}_1 \cap \mathscr{N}_2 = \{0\}$. In this case, the map $\mathscr{N}_1 \times \mathscr{N}_2 \ni (v, w) \mapsto v + w \in \mathscr{M}$ is bijective.

If $(\mathcal{U}, U, V)$ is a metric f-structure such that V is Dedekind complete, $\mathscr{M}$ is a V-Banach U-module and $\mathscr{N}$ is a V-Banach U-submodule of $\mathscr{M}$, then the *quotient module* $\mathscr{M}/\mathscr{N}$ is a V-Banach U-module if endowed with the following V-pointwise norm operator:

$$|v+\mathscr{N}| := \inf\{|v+w| \mid w \in \mathscr{N}\} \in V^+, \quad \text{for every } v+\mathscr{N} \in \mathscr{M}/\mathscr{N}.$$

Definition 3.5 (Hilbert module) Let $(\mathcal{U}, U, V, V, Z)$ be a dual system of metric f-structures. Then by a *V-Hilbert U-module* we mean a V-Banach U-module $\mathscr{H}$ such that

$$|v+w|^2 + |v-w|^2 = 2|v|^2 + 2|w|^2, \quad \text{for every } v, w \in \mathscr{H}. \tag{17}$$

We refer to (17) as the *pointwise parallelogram law* of $\mathscr{H}$.

The *pointwise scalar product* on $\mathscr{H}$ is defined as follows:

$$\mathscr{H} \times \mathscr{H} \ni (v,w) \mapsto v \cdot w := \frac{1}{2}\left(|v+w|^2 - |v|^2 - |w|^2\right) \in Z, \quad \text{for every } v, w \in \mathscr{H}.$$

One can readily check that the pointwise scalar product is *U-bilinear*, which means that

$$\mathscr{H} \ni v \mapsto v \cdot z \in Z, \quad \text{is } U\text{-linear,}$$
$$\mathscr{H} \ni w \mapsto z \cdot w \in Z, \quad \text{is } U\text{-linear,}$$

for any given $z \in \mathscr{H}$. We will study Hilbert modules more in detail in Section 3.5.

3.1.1. *Local invertibility*

Let $(\mathcal{U}, U, V)$ be a Dedekind complete CSP metric f-structure, and $\mathscr{M}$ a V-normed U-module. Then each $v \in \mathscr{M}$ is associated with the element $\chi_{\{v=0\}} \in \mathrm{Idem}(U)$, which we define as

$$\chi_{\{v=0\}} := \sup\{u \in \mathrm{Idem}(U) \mid u \cdot v = 0\} \in \mathrm{Idem}(U).$$

The idempotency of $\chi_{\{v=0\}}$ follows from Lemma 2.17 and the fact that U is Dedekind complete CSP. The gluing property of $\mathscr{M}$ ensures that $\chi_{\{v=0\}} \cdot v = 0$. We also define $\chi_{\{v\neq 0\}} := 1 - \chi_{\{v=0\}} \in \mathrm{Idem}(U)$, so that $v = \chi_{\{v\neq 0\}} \cdot v$. Similarly, we define $\chi_{\{v=w\}} := \chi_{\{v-w=0\}}$ and so on.

Proposition 3.6 (Local inverses) *Let U be a Dedekind complete CSP metric f-algebra. Let $u \in U^+$ be given. Then there exist a partition $(u_n)_{n\in\mathbb{N}}$ of $\chi_{\{u>0\}}$ and a sequence $(w_n)_{n\in\mathbb{N}} \subset U^+$ such that*

$$u_n(uw_n - 1) = 0, \quad \textit{for every } n \in \mathbb{N}.$$

Proof. First, recall that (U, U, U) is a metric f-structure (Example 2.24) and that U is a U-Banach U-module (Proposition 3.3). Since $\mathcal{S}(U)$ is super-order-dense in U (as guaranteed by the very definition of a localizable f-algebra), we can find a non-decreasing sequence $(s_n)_{n\in\mathbb{N}} \subset \mathcal{S}^+(U)$ such that $u = \sup_{n\in\mathbb{N}} s_n$. Note that, setting $b_n := \chi_{\{s_n>0\}}$ for every $n \in \mathbb{N}$, we have $\chi_{\{u>0\}} = \sup_{n\in\mathbb{N}} b_n$. Indeed, on the one hand $\chi_{\{s_n>0\}} \leq \chi_{\{u>0\}}$ for every $n \in \mathbb{N}$ and thus $\sup_n b_n \leq \chi_{\{u>0\}}$. On the other hand, if we denote $s := \chi_{\{u>0\}} - \sup_n b_n \in \mathrm{Idem}(U)$, then $s = 0$ in view of the fact that

$$su = s \sup_{n\in\mathbb{N}} s_n = \sup_{n\in\mathbb{N}} s s_n = 0.$$

Moreover, for any $n \in \mathbb{N}$ there exists a real number $\lambda_n > 0$ with $\lambda_n b_n \leq u$. More precisely, if s_n is written as $\sum_{i=1}^{k_n} \lambda_n^i u_n^i$ for some $k_n \in \mathbb{N}$, $(\lambda_n^i)_{i=1}^{k_n} \subset \mathbb{R} \cap (0, +\infty)$ and $(u_n^i)_{i=0}^{k_n} \in \mathcal{P}_f(\mathbf{1}_U)$, then we have that $b_n = u_n^1 \vee \dots \vee u_n^{k_n}$ and that $\lambda_n^1 \wedge \dots \wedge \lambda_n^{k_n}$ can be chosen as λ_n. Now we define $u_1 := b_1$ and $u_{n+1} := b_{n+1}(1 - b_n)$ for every $n \in \mathbb{N}$. Observe that $(u_n)_{n\in\mathbb{N}}$ is a partition of $\chi_{\{u>0\}}$. Next we consider the simple elements $t_j := \sum_{i=1}^{k_j} \frac{1}{\lambda_j^i} u_j^i$ for every $j \in \mathbb{N}$. Given any $n \in \mathbb{N}$, we have that the sequence $(u_n t_j)_{j=n}^\infty$ is non-increasing and satisfies $0 \leq u_n t_j \leq \frac{1}{\lambda_n} u_n$ for every $j \geq n$. Then the infimum $w_n := \inf_{j\geq n} u_n t_j \in U^+$ exists and, since $s_j t_j = b_j \geq u_n$ for every $j \geq n$, it holds that

$$\begin{aligned} u_n u w_n &= \lambda_n^{-1} u_n u - u_n u(\lambda_n^{-1} u_n - w_n) = \lambda_n^{-1} u_n u - u_n \Big(\sup_{j\geq n} s_j\Big)\Big(\lambda_n^{-1} u_n - \inf_{j\geq n} u_n t_j\Big) \\ &= \lambda_n^{-1} u_n u - \Big(\sup_{j\geq n} s_j\Big) \sup_{j\geq n}(\lambda_n^{-1} u_n - u_n t_j) = \lambda_n^{-1} u_n u - \sup_{j\geq n}(\lambda_n^{-1} u_n s_j - u_n s_j t_j) \\ &= \lambda_n^{-1} u_n u - \sup_{j\geq n}(\lambda_n^{-1} u_n s_j - u_n) = \lambda_n^{-1} u_n u - u_n(\lambda_n^{-1} u - 1) = u_n. \end{aligned}$$

Consequently, the statement is finally achieved. □

3.1.2. *Support of a metric f-structure*

Given a metric f-structure $(\mathcal{U}, U, V)$ that is Dedekind complete and CSP, we can define its *support*, which is the 'largest idempotent element where at least an element of V does not vanish'. Namely:

Definition 3.7 (Support) Let $(\mathcal{U}, U, V)$ be a Dedekind complete CSP metric f-structure. Then we define $\mathrm{S}(V)$ as

$$\mathrm{S}(V) := \sup\left\{\chi_{\{v\neq 0\}} \,\middle|\, v \in V^+\right\} \in \mathrm{Idem}(U).$$

We say that $\mathrm{S}(V)$ is the *support* of V or of the metric f-structure $(\mathcal{U}, U, V)$.

Let us check that the previous definition is well-posed. Since $\chi_{\{v\neq 0\}} \leq 1$ for every $v \in V^+$, the Dedekind completeness of V ensures that $\mathrm{S}(V)$ exists. Moreover, the countable representability assumption ensures the existence of a sequence $(v_n)_{n\in\mathbb{N}} \subset V^+$ such that $\mathrm{S}(V) = \sup_n \chi_{\{v_n\neq 0\}}$. Taking into account Lemma 2.17, it also follows that $\mathrm{S}(V) \in \mathrm{Idem}(U)$.

Remark 3.8 If $(\mathcal{U}, U, V, W, Z)$ is a CSP complete dual system of metric f-structures, then it holds that

$$\mathrm{S}(V) \wedge \mathrm{S}(W) \leq \mathrm{S}(Z).$$

In order to prove it, fix two sequences $(v_i)_{i\in\mathbb{N}} \subset V^+$ and $(w_j)_{j\in\mathbb{N}} \subset W^+$ with $\mathrm{S}(V) = \sup_i \chi_{\{v_i\neq 0\}}$ and $\mathrm{S}(W) = \sup_j \chi_{\{w_j\neq 0\}}$. Notice that $\chi_{\{v_i\neq 0\}} \wedge \chi_{\{w_j\neq 0\}} \leq \chi_{\{v_i w_j\neq 0\}} \leq \mathrm{S}(Z)$ for every $i, j \in \mathbb{N}$. Taking the supremum over $i, j \in \mathbb{N}$, we conclude that $\mathrm{S}(V) \wedge \mathrm{S}(W) \leq \mathrm{S}(Z)$, as we claimed.

Next we prove two technical results concerning the support of a metric f-structure.

Lemma 3.9 *Let $(\mathcal{U}, U, V)$ be a Dedekind complete CSP metric f-structure. Then there exists an element $h \in V^+ \cap U^+$ such that $h \leq 1$ and $\chi_{\{h>0\}} = S(V)$.*

Proof. Pick a sequence $(v_n)_{n\in\mathbb{N}} \subset V^+$ such that $v_n \leq 1$ for every $n \in \mathbb{N}$ and $\mathrm{S}(V) = \sup_n \chi_{\{v_n\neq 0\}}$. Define $u_1 := \chi_{\{v_1\neq 0\}} \in \mathrm{Idem}(U)$ and, recursively, $u_{n+1} := \chi_{\{v_{n+1}\neq 0\}}(1 - u_1)\dots(1 - u_n) \in \mathrm{Idem}(U)$ for every $n \in \mathbb{N}$. Notice that $(u_n)_{n\in\mathbb{N}}$ is a partition of $S(V)$. Recalling item (v) of Definition 2.23, we can find $\delta > 0$ such that $(u_n w_n)_{n\in\mathbb{N}}$ is order-bounded in V whenever $(w_n)_{n\in\mathbb{N}} \subset V^+$ is chosen so that $\sum_{n\in\mathbb{N}} \mathsf{d}_V(u_n w_n, 0) \leq \delta$. Now pick $(\lambda_n)_{n\in\mathbb{N}} \subset (0,1)$ with $\mathsf{d}_V(\lambda_n u_n v_n, 0) \leq \frac{\delta}{2^n}$ for all $n \in \mathbb{N}$. Therefore, the element $h := \sum_{n\in\mathbb{N}} \lambda_n u_n v_n \in V^+$ exists. Notice that $h \leq 1$ and $\chi_{\{h\neq 0\}} = S(V)$. In particular, $\chi_{\{h=0\}}|v| = (1 - S(V))\chi_{\{v=0\}}|v| = 0$ for every $v \in V$, whence the statement follows. □

Lemma 3.10 *Let $(\mathcal{U}, U, V)$ be a Dedekind complete CSP metric f-structure. Let $W \subset \mathcal{U}$ be a Dedekind complete CSP metric Riesz space such that $(\mathcal{U}, U, W)$ is a metric f-structure and $S(V) \leq S(W)$. Let $v \in V^+$ be given. Then there exists a partition $(u_n)_{n\in\mathbb{N}} \subset W \cap Idem(U)$ of $\chi_{\{v\neq 0\}}$ such that $u_n v \in W \cap U$ holds for every $n \in \mathbb{N}$.*

Proof. Thanks to Lemma 3.9, we can find an element $h \in W^+ \cap U^+$ such that $\chi_{\{h\neq 0\}} = S(W)$. We then define $\tilde{s}_i := \chi_{\{0<v\leq ih\}}$ and $\tilde{t}_j := \chi_{\{v\neq 0\}}\chi_{\{h\geq j^{-1}\mathbf{1}_U\}}$ for every $i, j \in \mathbb{N}$. We claim that

$$\sup_{i\in\mathbb{N}} \tilde{s}_i = \chi_{\{v\neq 0\}} = \sup_{j\in\mathbb{N}} \tilde{t}_j. \tag{18}$$

We prove the first equality, since the proof of the second one is similar. Clearly, $\sup_i \tilde{s}_i \leq \chi_{\{v\neq 0\}}$. For the converse inequality, we argue by contradiction: suppose that $s := \chi_{\{v\neq 0\}} - \sup_i \tilde{s}_i \neq 0$. Then $ish \leq v$ for every $i \in \mathbb{N}$, whence it follows (since $\mathcal{U}$ is Archimedean) that $sh = 0$, which leads to a contradiction. Therefore, the claim (18) is proved. Now let us define $s_1 := \tilde{s}_1$, $t_1 := \tilde{t}_1$ and, recursively, $s_{i+1} := \tilde{s}_{i+1} - s_1 \ldots s_i \tilde{s}_{i+1}$ and $t_{j+1} := \tilde{t}_{j+1} - t_1 \ldots t_j \tilde{t}_{j+1}$ for every $i, j \in \mathbb{N}$. Notice that (18) implies that $(s_i)_{i\in\mathbb{N}}$ and $(t_j)_{j\in\mathbb{N}}$ are partitions of $\chi_{\{v\neq 0\}}$. Moreover, $s_i v \leq ih \in W^+ \cap U^+$ and $t_j \leq jh \in W^+$ for every $i, j \in \mathbb{N}$, thus accordingly $s_i v \in W^+ \cap U^+$ and $t_j \in W^+$. Relabelling the family $\{s_i t_j : i, j \in \mathbb{N}\}$ as $\{u_n\}_{n\in\mathbb{N}}$, we finally obtain a partition $(u_n)_{n\in\mathbb{N}} \subset W \cap \mathrm{Idem}(U)$ of the element $\chi_{\{v\neq 0\}}$ satisfying $u_n v \in W \cap U$ for every $n \in \mathbb{N}$, as desired. □

3.2. Homomorphisms of normed modules

To begin with, we introduce the notion of a homomorphism between normed modules.

Definition 3.11 (Homomorphism of normed modules) Let $(\mathcal{U}, U, V, W, Z)$ be a dual system of metric f-structures, $\mathscr{M}$ a V-normed U-module and $\mathscr{N}$ a Z-normed U-module. Then we define

$$\mathrm{Hom}(\mathscr{M}, \mathscr{N}) := \left\{ T : \mathscr{M} \to \mathscr{N} \ U\text{-linear} \,\middle|\, \exists w \in W^+ : |Tv| \leq w|v|, \text{for every } v \in \mathscr{M} \right\}.$$

Next, we endow $\mathrm{Hom}(\mathscr{M}, \mathscr{N})$ with a U-module structure. Given any $T, S \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ and $u \in U$, we define the elements $T + S \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ and $u \cdot T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ as

$$(T + S)v := Tv + Sv, \quad \text{for every } v \in \mathscr{M},$$
$$(u \cdot T)v := u \cdot Tv, \quad \text{for every } v \in \mathscr{M},$$

respectively. One can readily check that the triple $(\mathrm{Hom}(\mathscr{M}, \mathscr{N}), +, \cdot)$ is a module over U.

In the case where W is Dedekind complete, for any given $T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ it holds that

$$\exists |T| := \inf \left\{ w \in W^+ \,\middle|\, |Tv| \leq w|v|, \text{for every } v \in \mathscr{M} \right\} \in W^+.$$

The space of homomorphisms between two normed modules inherits a normed module structure:

THEOREM 3.12 *Let $(\mathcal{U}, U, V, W, Z)$ be a complete dual system of metric f-structures. Let $\mathscr{M}$ be a V-normed U-module, and $\mathscr{N}$ a Z-normed U-module. Let $T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ be given. Then*

$$|Tv| \leq |T||v|, \quad \textit{for every } v \in \mathscr{M}.$$

Moreover, the space $(\mathrm{Hom}(\mathscr{M}, \mathscr{N}), |\cdot|)$ is a W-normed U-module and

$$\left|\sum_{n\in\mathbb{N}} u_n \cdot T_n\right| = \sum_{n\in\mathbb{N}} u_n|T_n|, \quad \textit{for every } (u_n, T_n)_{n\in\mathbb{N}} \in Adm(\mathrm{Hom}(\mathscr{M}, \mathscr{N})). \tag{19}$$

If in addition $\mathscr{N}$ is a Z-Banach U-module, then $\mathrm{Hom}(\mathscr{M}, \mathscr{N})$ is a W-Banach U-module.

Proof. VERIFICATION OF THE W-POINTWISE NORM AXIOMS. Given any $T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$, we define

$$\mathcal{F}_T := \left\{ w \in W^+ \;\middle|\; |Tv| \leq w|v|, \text{for every } v \in \mathscr{M} \right\} \neq \emptyset. \tag{20}$$

Since $\mathcal{F}_T$ is a sublattice of W^+—thus in particular it is downwards directed—we deduce that

$$|Tv| \leq \inf_{w\in\mathcal{F}_T} w|v| = |v| \inf_{w\in\mathcal{F}_T} w = |T||v|, \quad \text{for every } v \in \mathscr{M},$$

as a consequence of the order continuity of the multiplication from $V^+ \times W^+$ to Z^+. It readily follows that $|\cdot|\colon \mathrm{Hom}(\mathscr{M}, \mathscr{N}) \to W^+$ satisfies (12a) and (12b). It is also easy to check that the identity $|\lambda T| = |\lambda||T|$ holds for every $\lambda \in \mathbb{R}$ and $T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$. We now pass to the verification of (12c). For any $u \in U$ and $T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$, one has $|(u \cdot T)v| = |u||Tv| \leq |u||T||v|$ for every $v \in \mathscr{M}$, whence it follows that $|u \cdot T| \leq |u||T|$. On the other hand, we claim that also

$$|u||T| \leq |u \cdot T|, \quad \text{for every } u \in U \text{and } T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N}). \tag{21}$$

In the case where $u \in \mathrm{Idem}(U)$, the inequality stated in (21) follows from the observation that

$$u|T| = u|(1-u)\cdot T + u \cdot T| \leq u|(1-u)\cdot T| + u|u\cdot T| \leq u(1-u)|T| + |u\cdot T| = |u\cdot T|.$$

Moreover, if $u = \sum_{i=1}^k \lambda_i u_i \in \mathcal{S}^+(U)$ is given, then for any $j = 1, \ldots, k$ it holds that

$$u_j \sum_{i=1}^k \lambda_i |u_i \cdot T| \leq \sum_{i=1}^k \lambda_i u_i u_j |T| = \lambda_j u_j^2 |T| = |u_j u \cdot T| \leq u_j |u\cdot T|,$$

which implies that $u|T| = \sum_{i=1}^k \lambda_i u_i |T| = \sum_{i=1}^k \lambda_i |u_i \cdot T| \leq |u \cdot T|$, proving (21) for all $u \in \mathcal{S}^+(U)$. Given any $u \in U^+$, we can pick $(u_n)_{n\in\mathbb{N}} \subset \mathcal{S}^+(U)$ such that $\lim_{n\to\infty} \mathsf{d}_U(u_n, u) = 0$ and thus

$$|u||T| = \lim_{n\to\infty} |u_n||T| \leq \lim_{n\to\infty} |u_n \cdot T| = |u \cdot T|,$$

which proves (21) for all $u \in U^+$. Finally, given an arbitrary element $u \in U$ we have that

$$|u^+ \cdot T| \wedge |u^- \cdot T| \leq (u^+|T|) \wedge (u^-|T|) = (u^+ \wedge u^-)|T| \overset{(5a)}{=} 0,$$

so that $|u \cdot T| = |u^+ \cdot T| + |u^- \cdot T|$ by (5d) and thus $|u||T| = u^+|T| + u^-|T| \leq |u^+ \cdot T| + |u^- \cdot T| = |u \cdot T|$. This proves (21) for general $u \in U$. Therefore, the proof of the validity of (12c) is complete.

VERIFICATION OF THE GLUING PROPERTY. To prove that $(\mathrm{HOM}(\mathscr{M},\mathscr{N}),|\cdot|)$ is a W-normed U-module, it only remains to show the validity of the gluing property. Fix any $(u_n)_{n\in\mathbb{N}}\in\mathcal{P}(\mathbf{1}_U)$ and $(T_n)_{n\in\mathbb{N}}\subset\mathrm{HOM}(\mathscr{M},\mathscr{N})$ with $(|u_n\cdot T_n|)_{n\in\mathbb{N}}$ order-bounded in W. Since $|u_n\cdot T_n|=|u_n||T_n|$, this means that $(u_n,|T_n|)_{n\in\mathbb{N}}\in\mathrm{Adm}(W)$ and thus $w:=\sum_{n\in\mathbb{N}}u_n|T_n|\in W^+$ exists, cf. with Proposition 3.3. Given any $v\in\mathscr{M}$, we have that $|u_n\cdot T_nv|\leq|u_n\cdot T_n||v|$ for every $n\in\mathbb{N}$, which ensures that $(u_n,T_nv)_{n\in\mathbb{N}}\in\mathrm{Adm}(\mathscr{N})$, so it makes sense to consider $Tv:=\sum_{n\in\mathbb{N}}u_n\cdot T_nv\in\mathscr{N}$. The U-linearity of the resulting map $T\colon\mathscr{M}\to\mathscr{N}$ can be easily checked. Given that for any $n\in\mathbb{N}$ and $v\in\mathscr{M}$ one has that $u_n|Tv|=|u_n\cdot Tv|=|u_n\cdot T_nv|\leq u_n|T_n||v|=u_nw|v|$, we deduce that

$$|Tv|=\sup_{n\in\mathbb{N}}u_n|Tv|\leq\sup_{n\in\mathbb{N}}u_nw|v|=w|v|,\quad\text{for every } v\in\mathscr{M}.$$

This yields $T\in\mathrm{HOM}(\mathscr{M},\mathscr{N})$ and $|T|\leq w$. Note that $(u_n\cdot T)v=u_n\cdot Tv=u_n\cdot T_nv=(u_n\cdot T_n)v$ for every $v\in\mathscr{M}$, so that $T=\sum_{n\in\mathbb{N}}u_n\cdot T_n$. Finally, for any $n\in\mathbb{N}$ we have $u_n\cdot T_n=u_n\cdot T$ and thus $u_n|T_n|=u_n|T|$, which gives $w=\sum_{n\in\mathbb{N}}u_n|T_n|=\sup_{n\in\mathbb{N}}u_n|T|=|T|$. This proves (19).

COMPLETENESS. Suppose $(\mathscr{N},\mathsf{d}_{\mathscr{N}})$ is complete. Let $(T_n)_{n\in\mathbb{N}}\subset\mathrm{HOM}(\mathscr{M},\mathscr{N})$ be a Cauchy sequence. Until taking a not relabeled subsequence, we may assume that $\mathsf{d}_W(|T_{n+1}-T_n|,0)\leq 2^{-n}$ for every $n\in\mathbb{N}$. Define $S_n:=|T_1|+\sum_{k<n}|T_{k+1}-T_k|\in W^+$ for every $n\in\mathbb{N}$. If $n,m\in\mathbb{N}$ are such that $n<m$, then we have that $S_m-S_n=\sum_{k=n}^{m-1}|T_{k+1}-T_k|$, and thus accordingly

$$\mathsf{d}_W(|S_m-S_n|,0)\leq\sum_{k=n}^{m-1}\mathsf{d}_W(|T_{k+1}-T_k|,0)\leq\sum_{k=n}^{m-1}\frac{1}{2^k}\leq\frac{1}{2^{n-1}}.$$

This shows that $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in W^+, so that it d_W-converges to some $S\in W^+$. Notice that $(S_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence by construction, and thus in particular it holds

$$|T_n|=\big|(T_n-T_{n-1})+(T_{n-1}-T_{n-2})+\cdots+(T_2-T_1)+T_1\big|\leq S_n\leq S,\quad\forall n\in\mathbb{N}.\tag{22}$$

Since the multiplication is continuous from $V\times W$ to Z and $\mathsf{d}_W(|T_m-T_n|,0)\to 0$ as $n,m\to\infty$, for any fixed element $v\in\mathscr{M}$ we have that $\mathsf{d}_{\mathscr{N}}(T_mv,T_nv)\leq\mathsf{d}_Z(|T_m-T_n||v|,0)\to 0$ as $n,m\to\infty$, which shows that $(T_nv)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{N}$. We then define $Tv:=\lim_{n\to\infty}T_nv\in\mathscr{N}$. The resulting mapping $T\colon\mathscr{M}\to\mathscr{N}$ is U-linear, as it is a pointwise limit of U-linear maps. Also,

$$|Tv|=\lim_{n\to\infty}|T_nv|\overset{(22)}{\leq}S|v|,\quad\text{for every } v\in\mathscr{M},$$

whence it follows that $T\in\mathrm{HOM}(\mathscr{M},\mathscr{N})$ and $|T|\leq S$. Now set $P^n_m:=\sum_{k=n}^{m-1}|T_{k+1}-T_k|\in W^+$ for every $n,m\in\mathbb{N}$ with $n<m$. Arguing as we did before, we see that $(P^n_m)_{m\in\mathbb{N}}$ is d_W-Cauchy, and thus $\lim_{m\to\infty}P^n_m=P^n$ for some $P^n\in W^+$. Note that $P^n_m\leq P^n$ and $\mathsf{d}_W(P^n,0)\leq 2^{-n+1}$. Hence,

$$|(T-T_n)v|=\lim_{m\to\infty}|(T_m-T_n)v|\leq\lim_{m\to\infty}P^n_m|v|=P^n|v|,\quad\text{for every } v\in\mathscr{M},$$

which implies $|T-T_n|\leq P^n\to 0$ as $n\to\infty$. The completeness of $\mathrm{HOM}(\mathscr{M},\mathscr{N})$ follows. □

In the case where the dual system under consideration is CSP complete, the W-pointwise norm $|T|$ of any given $T\in\mathrm{HOM}(\mathscr{M},\mathscr{N})$ can be also characterized as follows:

Lemma 3.13 *Let $(\mathcal{U}, U, V, W, Z)$ be a CSP complete dual system of metric f-structures, $\mathscr{M}$ a V-normed U-module and $\mathscr{N}$ a Z-normed U-module. Then it holds that*

$$|T| = \sup\{|Tv| \mid v \in \mathscr{M}, |v| \leq 1\}, \quad \text{for every } T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N}). \tag{23}$$

Proof. On the one hand, $|Tv| \leq |T||v| \leq |T|$ for all $v \in \mathscr{M}$ with $|v| \leq 1$, and thus the right-hand side in (23) exists and defines an element $b \in W^+$ with $b \leq |T|$. On the other hand, we claim that

$$|Tv| \leq b|v|, \quad \text{for every } v \in \mathscr{M}. \tag{24}$$

In order to prove it, fix any $v \in \mathscr{M}$. By using Lemma 3.10 and Proposition 3.6, we can find a partition $(u_n)_{n\in\mathbb{N}}$ of $\chi_{\{v\neq 0\}}$ and a sequence $(w_n)_{n\in\mathbb{N}} \subset U^+$ such that $u_n|v| \in U$ and $u_n w_n |v| = u_n$ for every $n \in \mathbb{N}$. Letting $v_n := (u_n w_n) \cdot v \in \mathscr{M}$, we have $|v_n| = u_n \leq 1$ and thus $|Tv_n| \leq b$. Then

$$|Tv| = \sum_{n\in\mathbb{N}} (u_n w_n |v|)|Tv| = \sum_{n\in\mathbb{N}} |v||Tv_n| \leq \sum_{n\in\mathbb{N}} u_n b|v| = b|v|.$$

This proves (24) and accordingly that $b = |T|$, thus concluding the proof of the statement. □

Given a dual system of metric f-structures $(\mathcal{U}, U, V, W, Z)$, a V-Banach U-module $\mathscr{M}$ and a Z-Banach U-module $\mathscr{N}$, we define the *kernel* of a homomorphism $T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ as

$$\ker(T) := T^{-1}(\{0\}) = \{v \in \mathscr{M} \mid Tv = 0\}.$$

It can be readily checked that $\ker(T)$ is a V-Banach U-submodule of $\mathscr{M}$.

It is natural to introduce the categories of normed modules and of Banach modules:

Definition 3.14 (Category of normed modules) Let $(\mathcal{U}, U, V)$ be a metric f-structure. Then we define the category $\mathit{NormMod}_{(\mathcal{U},U,V)}$ of normed modules over $(\mathcal{U}, U, V)$ as follows:

(i) The objects of $\mathit{NormMod}_{(\mathcal{U},U,V)}$ are given by the V-normed U-modules.
(ii) For any two objects $\mathscr{M}$ and $\mathscr{N}$ of $\mathit{NormMod}_{(\mathcal{U},U,V)}$, the morphisms between $\mathscr{M}$ and $\mathscr{N}$ are given by those homomorphisms $T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ satisfying $|Tv| \leq |v|$ for every $v \in \mathscr{M}$.

Moreover, we denote by $\mathit{BanMod}_{(\mathcal{U},U,V)}$ the full subcategory of $\mathit{NormMod}_{(\mathcal{U},U,V)}$ whose objects are the V-Banach U-modules.

It would be interesting—but outside the scope of this work—to investigate which results of [**36**] are valid for Banach modules over a more general class of metric f-structures.

Let us also define the dual of a Banach module:

Definition 3.15 (Dual Banach module) Let $(\mathcal{U}, U, V, W, Z)$ be a complete dual system of metric f-structures, and $\mathscr{M}$ a V-normed U-module. Then the *dual* of $\mathscr{M}$ is the W-Banach U-module

$$\mathscr{M}^* := \mathrm{Hom}(\mathscr{M}, Z).$$

The duality pairing between $\mathscr{M}$ and $\mathscr{M}^*$ is given by

$$\mathscr{M}^* \times \mathscr{M} \ni (\omega, v) \mapsto \langle \omega, v \rangle := \omega(v) \in Z.$$

3.2.1. *On the existence of homomorphisms*

As we will see later (in Section 4.2), when working with normed modules over a function space, one often has to prove the existence of homomorphisms of normed modules verifying suitable properties. All these existence statements can be deduced from the following general result.

THEOREM 3.16 *Let $(\mathcal{U}_1, U_1, V_1)$ be a metric f-structure, and $(\mathcal{U}_2, U_2, V_2, W_2, Z_2)$ a dual system of metric f-structures. Let $\phi\colon (\mathcal{U}_1, U_1, V_1) \to (\mathcal{U}_2, U_2, V_2)$ be a homomorphism of metric f-structures. Let $\mathscr{M}$ be a V_1-normed U_1-module, and $\mathscr{N}$ a Z_2-Banach U_2-module. Let $\mathscr{V}$ be a vector subspace of $\mathscr{M}$ such that $\mathscr{G}(\mathscr{V})$ is dense in $\mathscr{M}$. Fix $T\colon \mathscr{V} \to \mathscr{N}$ linear such that for some $b \in W_2^+$ it holds*

$$|Tv| \le b\,\phi(|v|), \quad \text{for every } v \in \mathscr{V}. \tag{25}$$

Then there exists a unique linear operator $\bar{T}\colon \mathscr{M} \to \mathscr{N}$ such that $\bar{T}|_{\mathscr{V}} = T$ and

$$|\bar{T}v| \le b\,\phi(|v|), \quad \text{for every } v \in \mathscr{M}. \tag{26}$$

In particular, the map $\bar{T}\colon \mathscr{M} \to \mathscr{N}$ is continuous. Moreover, it holds that

$$\bar{T}(u \cdot v) = \phi(u) \cdot \bar{T}(v), \quad \text{for every } u \in U_1 \text{ and } v \in \mathscr{M}. \tag{27}$$

Proof. First of all, we define the operator $S\colon \mathscr{G}(\mathscr{V}) \to \mathscr{N}$ as

$$S\Big(\sum_{n\in\mathbb{N}} u_n \cdot v_n\Big) := \sum_{n\in\mathbb{N}} \phi(u_n) \cdot Tv_n, \quad \text{for every } \sum_{n\in\mathbb{N}} u_n \cdot v_n \in \mathscr{G}(\mathscr{V}).$$

Let us check that S is well-defined. Letting $z \in V_1^+$ be an upper bound for $(u_n|v_n|)_{n\in\mathbb{N}}$, we infer from (25) that $|\phi(u_n) \cdot Tv_n| = \phi(u_n)|Tv_n| \le b\,\phi(u_n|v_n|) \le b\,\phi(z) \in Z_2^+$ for every $n \in \mathbb{N}$. Given that $(\phi(u_n))_{n\in\mathbb{N}} \in \mathcal{P}(U_2)$ by (10), we deduce that $(\phi(u_n), Tv_n)_{n\in\mathbb{N}} \in \mathrm{Adm}(\mathscr{N})$ and thus accordingly $\sum_{n\in\mathbb{N}} \phi(u_n) \cdot Tv_n \in \mathscr{N}$ is well-defined. Also, if $\sum_{n\in\mathbb{N}} u_n \cdot v_n = \sum_{m\in\mathbb{N}} \tilde{u}_m \cdot \tilde{v}_m$, then

$$\begin{aligned}\big|\phi(u_n\tilde{u}_m) \cdot Tv_n - \phi(u_n\tilde{u}_m) \cdot T\tilde{v}_m\big| = \phi(u_n\tilde{u}_m)\big|T(v_n - \tilde{v}_m)\big| &\overset{(25)}{\le} b\,\phi(u_n\tilde{u}_m|v_n - \tilde{v}_m|)\\ &= b\,\phi(\big|(u_n\tilde{u}_m) \cdot v_n - (u_n\tilde{u}_m) \cdot \tilde{v}_m\big|) = 0\end{aligned}$$

holds for every $n, m \in \mathbb{N}$, which implies that $\sum_{n\in\mathbb{N}} \phi(u_n) \cdot v_n = \sum_{m\in\mathbb{N}} \phi(\tilde{u}_m) \cdot \tilde{v}_m$. All in all, the definition of S is well-posed. Observe that S is linear by construction. Moreover, one has that

$$|Sw| = \sum_{n\in\mathbb{N}} \phi(u_n)|Tv_n| \overset{(25)}{\le} b \sum_{n\in\mathbb{N}} \phi(u_n)\phi(|v_n|) = b\,\phi(|w|), \quad \forall w = \sum_{n\in\mathbb{N}} u_n \cdot v_n \in \mathscr{G}(\mathscr{V}). \tag{28}$$

Notice that S is the unique linear map from $\mathscr{G}(\mathscr{V})$ to $\mathscr{N}$ satisfying both $S|_{\mathscr{V}} = T$ and (28). It also follows from (28) that the map S is Cauchy-continuous: if a given sequence $(w_i)_{i\in\mathbb{N}} \subset \mathscr{G}(\mathscr{V})$ is $\mathsf{d}_{\mathscr{M}}$-Cauchy, then $\mathsf{d}_{V_2}(\phi(|w_i - w_j|), 0) \to 0$ as $i, j \to \infty$, and thus

$$\varlimsup_{i,j\to\infty} \mathsf{d}_{Z_2}(|Sw_i - Sw_j|, 0) \overset{(28)}{\le} \varlimsup_{i,j\to\infty} \mathsf{d}_{Z_2}(b\,\phi(|w_i - w_j|), 0) = 0,$$

which shows that $(Sw_i)_{i\in\mathbb{N}}$ is $\mathsf{d}_{\mathscr{N}}$-Cauchy. Therefore, S can be uniquely extended to a linear, continuous map $\bar{T}\colon \mathscr{M} \to \mathscr{N}$. Thanks to an approximation argument, we can deduce from

(28) that $\bar{T}$ verifies (26), whence the continuity of $\bar{T}$ immediately follows. Finally, we can estimate

$$\begin{aligned}\left|\bar{T}(u\cdot v)-\phi(u)\cdot\bar{T}(v)\right| &\leq \left|\bar{T}(u\cdot v)-\phi(u)\cdot\bar{T}(u\cdot v)\right|+\left|\phi(u)\cdot\bar{T}(u\cdot v)-\phi(u)\cdot\bar{T}(v)\right|\\ &=(1-\phi(u))\left|\bar{T}(u\cdot v)\right|+\phi(u)\left|\bar{T}(u\cdot v-v)\right|\\ &\leq b\,\phi(1-u)\phi(|u\cdot v|)+b\,\phi(u)\phi(|u\cdot v-v|)\\ &=2b\,\phi(u(1-u)|v|)=0,\end{aligned}$$

for every $v\in\mathscr{M}$ and $u\in\mathrm{Idem}(U_1)$, proving (27) when u is idempotent. By linearity, we deduce that (27) holds whenever $u\in\mathcal{S}(U_1)$. Thanks to the density of $\mathcal{S}(U_1)$ in U_1, as well as to the continuity of $\bar{T}$, ϕ and the scalar multiplications, we conclude that (27) is verified. □

Let us isolate a useful byproduct of the last part of the proof of Theorem 3.16:

Lemma 3.17 *Let $(\mathcal{U}_1,U_1,V_1)$ be a metric f-structure, and $(\mathcal{U}_2,U_2,V_2,W_2,Z_2)$ a dual system of metric f-structures. Let $\phi\colon(\mathcal{U}_1,U_1,V_1)\to(\mathcal{U}_2,U_2,V_2)$ be a homomorphism of metric f-structures. Let $\mathscr{M}$ be a V_1-normed U_1-module, and $\mathscr{N}$ a Z_2-normed U_2-module. Let $T\colon\mathscr{M}\to\mathscr{N}$ be a linear operator having the following property: there exists an element $b\in W_2^+$ such that*

$$|Tv|\leq b\,\phi(|v|),\quad\text{for every } v\in\mathscr{M}. \tag{29}$$

Then T is a continuous operator satisfying $T(u\cdot v)=\phi(u)\cdot T(v)$ for every $u\in U_1$ and $v\in\mathscr{M}$.

As an immediate consequence of Lemma 3.17, we obtain a criterion to detect homomorphisms:

Corollary 3.18 Let $(\mathcal{U},U,V,W,Z)$ be a dual system of metric f-structures, $\mathscr{M}$ a V-normed U-module and $\mathscr{N}$ a Z-normed U-module. Let $T\colon\mathscr{M}\to\mathscr{N}$ be a linear operator such that

$$|Tv|\leq w|v|,\quad\text{for every } v\in\mathscr{M},$$

for some $w\in W^+$. Then T is U-linear and continuous, and thus in particular $T\in\mathrm{Hom}(\mathscr{M},\mathscr{N})$.

Proof. Apply Lemma 3.17 with $(\mathcal{U}_1,U_1,V_1):=(\mathcal{U},U,V)$, $(\mathcal{U}_2,U_2,V_2,W_2,Z_2):=(\mathcal{U},U,V,W,Z)$ and $\phi:=\mathrm{id}_{\mathcal{U}}$. □

3.3. Constructions of normed modules

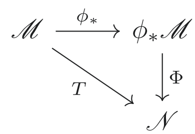
3.3.1. Banach module induced by an even sublinear map

We fix some terminology. Consider a vector space $\mathscr{V}$, a Riesz space U and a map $\psi\colon\mathscr{V}\to U^+$. Then we say that ψ is *positively homogeneous* if $\psi(\lambda\mathsf{v})=\lambda\,\psi(\mathsf{v})$ for every $\mathsf{v}\in\mathscr{V}$ and $\lambda\in\mathbb{R}^+$, while we say that ψ is *subadditive* if $\psi(\mathsf{v}+\mathsf{w})\leq\psi(\mathsf{v})+\psi(\mathsf{w})$ for every $\mathsf{v},\mathsf{w}\in\mathscr{V}$. The map ψ is said to be *sublinear* provided that it is both positively homogeneous and subadditive. Finally, we say that the map ψ is *even* provided that it satisfies $\psi(-\mathsf{v})=\psi(\mathsf{v})$ for every $\mathsf{v}\in\mathscr{V}$.

Theorem 3.19 (Banach module generated by an even sublinear map) *Let $(\mathcal{U},U,V)$ be a metric f-structure. Let $\mathscr{V}$ be a vector space, and $\psi\colon\mathscr{V}\to V^+$ an even, sublinear mapping. Then there exists a unique couple $(\mathscr{M}_{\langle\psi\rangle},T_{\langle\psi\rangle})$—where $\mathscr{M}_{\langle\psi\rangle}$ is a V-Banach U-module and the operator $T_{\langle\psi\rangle}\colon\mathscr{V}\to\mathscr{M}_{\langle\psi\rangle}$ is linear—such that the following properties are verified:*

(i) $|T_{\langle\psi\rangle}\mathsf{v}|=\psi(\mathsf{v})$ *for every* $\mathsf{v}\in\mathscr{V}$.
(ii) $\mathscr{G}(T_{\langle\psi\rangle}(\mathscr{V}))$ *is dense in* $\mathscr{M}_{\langle\psi\rangle}$.

Uniqueness is up to unique isomorphism: given another couple $(\mathscr{M}, T)$ with the same properties, there exists a unique isomorphism of V-Banach U-modules $\Phi\colon \mathscr{M}_{\langle\psi\rangle} \to \mathscr{M}$ such that



is a commutative diagram.

Proof. EXISTENCE. Let us denote by $\bar{\mathcal{F}}$ the family of all sequences $(u_n, \mathsf{v}_n)_{n\in\mathbb{N}}$ such that $(u_n)_{n\in\mathbb{N}} \in \mathcal{P}(\mathbf{1}_U)$, $(\mathsf{v}_n)_{n\in\mathbb{N}} \subset \mathscr{V}$ and $\big(u_n\psi(\mathsf{v}_n)\big)_{n\in\mathbb{N}}$ are order-bounded in V. We introduce a relation $\sim$ on the set $\bar{\mathcal{F}}$: given any $(u_n, \mathsf{v}_n)_n, (\tilde{u}_m, \tilde{\mathsf{v}}_m)_m \in \bar{\mathcal{F}}$, we declare that $(u_n, \mathsf{v}_n)_n \sim (\tilde{u}_m, \tilde{\mathsf{v}}_m)_m$ if and only if

$$u_n\tilde{u}_m\psi(\mathsf{v}_n - \tilde{\mathsf{v}}_m) = 0, \quad \text{for every } n, m \in \mathbb{N}.$$

One can readily check that $\sim$ is an equivalence relation on $\bar{\mathcal{F}}$: reflexivity follows from $\psi(0) = 0$, symmetry from the symmetry of ψ and transitivity from the subadditivity of ψ. We then define

$$\mathcal{F} := \bar{\mathcal{F}}/\sim.$$

For any $(u_n, \mathsf{v}_n)_n \in \bar{\mathcal{F}}$, we denote by $[u_n, \mathsf{v}_n]_n \in \mathcal{F}$ its equivalence class with respect to $\sim$. We set

$$[u_n, \mathsf{v}_n]_n + [\tilde{u}_m, \tilde{\mathsf{v}}_m]_m := [u_n\tilde{u}_m, \mathsf{v}_n + \tilde{\mathsf{v}}_m]_{n,m}, \quad \forall [u_n, \mathsf{v}_n]_n, [\tilde{u}_m, \tilde{\mathsf{v}}_m]_m \in \mathcal{F},$$

$$u \cdot [\tilde{u}_m, \tilde{\mathsf{v}}_m]_m := [u_i\tilde{u}_m, \lambda_i\tilde{\mathsf{v}}_m]_{i,m}, \quad \forall u = \sum_{i=1}^{n} \lambda_i u_i \in \mathcal{S}(U), [\tilde{u}_m, \tilde{\mathsf{v}}_m]_m \in \mathcal{F},$$

$$\big|[u_n, \mathsf{v}_n]_n\big| := \sup_{n\in\mathbb{N}} u_n\psi(\mathsf{v}_n), \quad \forall [u_n, \mathsf{v}_n]_n \in \mathcal{F}.$$

Routine verifications show the well-posedness of the resulting operations

$$+\colon \mathcal{F} \times \mathcal{F} \to \mathcal{F}, \qquad \cdot\colon \mathcal{S}(U) \times \mathcal{F} \to \mathcal{F}, \qquad |\cdot|\colon \mathcal{F} \to V^+. \tag{30}$$

We also define the map $\tilde{T}\colon \mathscr{V} \to \mathcal{F}$ as $\tilde{T}(\mathsf{v}) := [\mathbf{1}_U, \mathsf{v}]$ for all $\mathsf{v} \in \mathscr{V}$ and the distance $\mathsf{d}_{\mathcal{F}}$ on $\mathcal{F}$ as

$$\mathsf{d}_{\mathcal{F}}(w, \tilde{w}) := \mathsf{d}_V(|w - \tilde{w}|, 0), \quad \text{for every } w, \tilde{w} \in \mathcal{F}.$$

Next we denote by $(\mathscr{M}_{\langle\psi\rangle}, \mathsf{d}_{\mathscr{M}_{\langle\psi\rangle}})$ the metric completion of $(\mathcal{F}, \mathsf{d}_{\mathcal{F}})$ and by $\iota\colon \mathcal{F} \to \mathscr{M}_{\langle\psi\rangle}$ the canonical isometric embedding map. Define also $T_{\langle\psi\rangle}\colon \mathscr{V} \to \mathscr{M}_{\langle\psi\rangle}$ as $T_{\langle\psi\rangle} := \iota \circ \tilde{T}$. With a slight abuse of notation, we regard $\mathcal{F}$ as a subset of $\mathscr{M}_{\langle\psi\rangle}$. Standard verifications show that the operations in (30) are Cauchy-continuous, so they can be uniquely extended to continuous maps

$$+\colon \mathscr{M}_{\langle\psi\rangle} \times \mathscr{M}_{\langle\psi\rangle} \to \mathscr{M}_{\langle\psi\rangle}, \qquad \cdot\colon U \times \mathscr{M}_{\langle\psi\rangle} \to \mathscr{M}_{\langle\psi\rangle}, \qquad |\cdot|\colon \mathscr{M}_{\langle\psi\rangle} \to V^+.$$

By an approximation argument, one can show that $\mathscr{M}_{\langle\psi\rangle}$ is a U-module, $|\cdot|\colon \mathscr{M}_{\langle\psi\rangle} \to V^+$ verifies the V-pointwise norm axioms and

$$\mathsf{d}_{\mathscr{M}_{\langle\psi\rangle}}(w, \tilde{w}) := \mathsf{d}_V(|w - \tilde{w}|, 0), \quad \text{for every } w, \tilde{w} \in \mathscr{M}_{\langle\psi\rangle}.$$

In order to prove that $\mathscr{M}_{\langle\psi\rangle}$ is a V-Banach U-module, it only remains to check the validity of the gluing property. Fix any $(u_n)_{n\in\mathbb{N}} \in \mathcal{P}(\mathbf{1}_U)$ and $(w_n)_{n\in\mathbb{N}} \subset \mathscr{M}_{\langle\psi\rangle}$ such that $(u_n|w_n|)_{n\in\mathbb{N}}$ is order-bounded in V. In view of Definition 2.23 v), for any $k \in \mathbb{N}$ we can find $(w_n^k)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $(u_n|w_n - w_n^k|)_{n\in\mathbb{N}}$ is order-bounded in V—thus $(u_n|w_n^k|)_{n\in\mathbb{N}}$ is order-bounded in V—and

$$\sum_{n\in\mathbb{N}} \mathsf{d}_V(u_n|w_n - w_n^k|, 0) \leq \frac{1}{2^k}, \qquad \mathsf{d}_V\left(\sup_{n\in\mathbb{N}} u_n|w_n - w_n^k|, 0\right) \leq \frac{1}{2^k}.$$

Writing w_n^k in the form $[\tilde{u}_{n,j}^k, \mathsf{v}_{n,j}^k]_j$, we define the element $z^k \in \mathcal{F}$ as $z^k := [u_n\tilde{u}_{n,j}^k, \mathsf{v}_{n,j}^k]_{n,j}$. Then

$$\begin{aligned} \mathsf{d}_{\mathscr{M}_{\langle\psi\rangle}}(z^k, z^{k+1}) &= \mathsf{d}_V(|z^k - z^{k+1}|, 0) = \mathsf{d}_V\left(\sup_{n\in\mathbb{N}} u_m|w_n^k - w_n^{k+1}|, 0\right) \\ &\leq \mathsf{d}_V\left(\sup_{n\in\mathbb{N}} u_m|w_n^k - w_n|, 0\right) + \mathsf{d}_V\left(\sup_{n\in\mathbb{N}} u_m|w_n - w_n^{k+1}|, 0\right) \leq \frac{3}{2^{k+1}}, \end{aligned}$$

for every $k \in \mathbb{N}$. This implies that $(z^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\mathscr{M}_{\langle\psi\rangle}, \mathsf{d}_{\mathscr{M}_{\langle\psi\rangle}})$, so that it converges to some element $z \in \mathscr{M}_{\langle\psi\rangle}$. For any $n \in \mathbb{N}$, we deduce that

$$\mathsf{d}_{\mathscr{M}_{\langle\psi\rangle}}(u_n \cdot z, u_n \cdot w_n) = \lim_{k\to\infty} \mathsf{d}_{\mathscr{M}_{\langle\psi\rangle}}(u_n \cdot z^k, u_n \cdot w_n) = \lim_{k\to\infty} \mathsf{d}_V(u_n|w_n - w_n^k|, 0) \leq \lim_{k\to\infty} \frac{1}{2^k} = 0,$$

so that $u_n \cdot z = u_n \cdot w_n$. This shows that $z = \sum_{n\in\mathbb{N}} u_n \cdot w_n$, and thus the gluing property is proved.

To conclude, let us check that $(\mathscr{M}_{\langle\psi\rangle}, T_{\langle\psi\rangle})$ verifies (i) and (ii). The mapping $T_{\langle\psi\rangle}$ is linear and satisfies $|T_{\langle\psi\rangle}\mathsf{v}| = |[1, \mathsf{v}]| = \psi(\mathsf{v})$ for every $\mathsf{v} \in \mathscr{V}$, so that (i) is proved. Finally, $\mathscr{G}(T_{\langle\psi\rangle}(\mathscr{V})) = \mathcal{F}$ and $\sum_{n\in\mathbb{N}} u_n \cdot T_{\langle\psi\rangle}\mathsf{v}_n = [u_n, \mathsf{v}_n]_n$ for all $[u_n, \mathsf{v}_n]_n \in \mathcal{F}$. Being $\mathcal{F}$ dense in $\mathscr{M}_{\langle\psi\rangle}$, (ii) is also proved.

Uniqueness. It is a consequence of Corollary 3.22. Indeed, initial objects are colimits (of empty diagrams), and thus in particular they are unique up to a unique isomorphism (cf. with [35]). □

Proposition 3.20 *Let $(\mathcal{U}_1, U_1, V_1)$ be a metric f-structure, and let $(\mathcal{U}_2, U_2, V_2, W_2, Z_2)$ be a dual system of metric f-structures. Let $\phi\colon (\mathcal{U}_1, U_1, V_1) \to (\mathcal{U}_2, U_2, V_2)$ be a homomorphism of metric f-structures. Let $\mathscr{V}$ be a vector space, and $\psi\colon \mathscr{V} \to V_1^+$ an even, sublinear mapping. Let $\mathscr{N}$ be a Z_2-Banach U_2-module. Let $S\colon \mathscr{V} \to \mathscr{N}$ be a linear map such that for some $b \in W_2^+$ it holds*

$$|S\mathsf{v}| \leq b\,(\phi \circ \psi)(\mathsf{v}), \quad \textit{for every } \mathsf{v} \in \mathscr{V}.$$

Then there exists a unique linear operator $\Phi\colon \mathscr{M}_{\langle\psi\rangle} \to \mathscr{N}$ such that $\Phi \circ T_{\langle\psi\rangle} = S$ and

$$|\Phi v| \leq b\,\phi(|v|), \quad \textit{for every } v \in \mathscr{M}_{\langle\psi\rangle}.$$

In particular, the map $\Phi\colon \mathscr{M}_{\langle\psi\rangle} \to \mathscr{N}$ is continuous. Moreover, it holds that

$$\Phi(u \cdot v) = \phi(u) \cdot \Phi(v), \quad \textit{for every } u \in U_1 \textit{ and } v \in \mathscr{M}_{\langle\psi\rangle}.$$

Proof. Since $T_{\langle\psi\rangle}$ is linear, we have that $\mathscr{W} := T_{\langle\psi\rangle}(\mathscr{V})$ is a vector subspace of $\mathscr{M}_{\langle\psi\rangle}$. Item (ii) of Theorem 3.19 ensures that $\mathscr{G}(\mathscr{W})$ is dense in $\mathscr{M}_{\langle\psi\rangle}$. Moreover, let us define $T\colon \mathscr{W} \to \mathscr{N}$ as

$$T(T_{\langle\psi\rangle}\mathsf{v}) := S\mathsf{v}, \quad \text{for every } \mathsf{v} \in \mathscr{V}.$$

Since $|S\mathsf{v}| \le b(\phi \circ \psi)(\mathsf{v}) = b\,\phi(|T_{\langle\psi\rangle}\mathsf{v}|)$ by item (i) of Theorem 3.19, we deduce that T is well-defined. Notice also that T is linear and satisfies $|Tv| \le b\,\phi(|v|)$ for every $v \in \mathscr{W}$. Therefore, the statement follows from Theorem 3.16 applied to the operator T. □

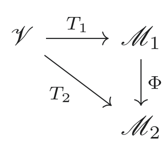
Corollary 3.21 Let $(\mathcal{U}, U, V)$ be a metric f-structure. Let $\mathscr{V}$ be a vector space, and $\psi\colon \mathscr{V} \to V^+$ an even, sublinear mapping. Let $\mathscr{N}$ be a V-Banach U-module, and $S\colon \mathscr{V} \to \mathscr{N}$ a linear operator such that $|S\mathsf{v}| \le \psi(\mathsf{v})$ for every $\mathsf{v} \in \mathscr{V}$. Let $(\mathscr{M}_{\langle\psi\rangle}, T_{\langle\psi\rangle})$ be as in Theorem 3.19. Then there exists a unique homomorphism of V-Banach U-modules $\Phi\colon \mathscr{M}_{\langle\psi\rangle} \to \mathscr{N}$ with $\Phi \circ T_{\langle\psi\rangle} = S$. Moreover, it holds that $|\Phi v| \le |v|$ for every $v \in \mathscr{M}_{\langle\psi\rangle}$.

Proof. Use Proposition 3.20 with $(\mathcal{U}_1, U_1, V_1) := (\mathcal{U}, U, V)$, $(\mathcal{U}_2, U_2, V_2, W_2, Z_2) := (\mathcal{U}, U, V, U, V)$, $\phi := \mathrm{id}_{\mathcal{U}}$ and $b := \mathbf{1}_U$. □

Next, we aim to interpret the couple $(\mathscr{M}_{\langle\psi\rangle}, T_{\langle\psi\rangle})$ given by Theorem 3.19 in categorical terms. Given $(\mathcal{U}, U, V)$, $\mathscr{V}$ and ψ as in Theorem 3.19, we denote by $\mathbf{C}_\psi$ the category defined as follows:

(i) The objects of $\mathbf{C}_\psi$ are the couples $(\mathscr{M}, T)$, with $\mathscr{M}$ being a V-Banach U-module and $T\colon \mathscr{V} \to \mathscr{M}$ a linear map satisfying $|T\mathsf{v}| \le \psi(\mathsf{v})$ for every $\mathsf{v} \in \mathscr{V}$.

(ii) The morphisms between two objects $(\mathscr{M}_1, T_1)$ and $(\mathscr{M}_2, T_2)$ of $\mathbf{C}_\psi$ are given by those morphisms $\Phi\colon \mathscr{M}_1 \to \mathscr{M}_2$ in the category $BanMod_{(\mathcal{U},U,V)}$ for which



is a commutative diagram.

Corollary 3.22 Let $(\mathcal{U}, U, V)$ be a metric f-structure. Let $\mathscr{V}$ be a vector space, and $\psi\colon \mathscr{V} \to V^+$ an even, sublinear mapping. Then $(\mathscr{M}_{\langle\psi\rangle}, T_{\langle\psi\rangle})$ is the initial object of the category $\mathbf{C}_\psi$, meaning that for any object $(\mathscr{M}, T)$ of $\mathbf{C}_\psi$ there exists a unique morphism $\Phi\colon (\mathscr{M}_{\langle\psi\rangle}, T_{\langle\psi\rangle}) \to (\mathscr{M}, T)$.

Proof. Let $(\mathscr{M}, T)$ be an arbitrary object of $\mathbf{C}_\psi$. Then Corollary 3.21 ensures that there exists a unique morphism $\Phi\colon (\mathscr{M}_{\langle\psi\rangle}, T_{\langle\psi\rangle}) \to (\mathscr{M}, T)$ in $\mathbf{C}_\psi$ such that $\Phi \circ T_{\langle\psi\rangle} = T$, as desired. □

3.3.2. *Pushforward of a normed module*

As a consequence of Theorem 3.19, we can prove that each homomorphism of metric f-structures induces a ‘pushforward functor’ between the respective categories of Banach modules:

Theorem 3.23 (Pushforward of a normed module) *Let $(\mathcal{U}_1, U_1, V_1)$ and $(\mathcal{U}_2, U_2, V_2)$ be metric f-structures. Let $\phi\colon \mathcal{U}_1 \to \mathcal{U}_2$ be a homomorphism of metric f-structures. Let $\mathscr{M}$ be a V_1-normed U_1-module. Then there exists a unique couple $(\phi_*\mathscr{M}, \phi_*)$—where $\phi_*\mathscr{M}$ is a V_2-Banach U_2-module and the operator $\phi_*\colon \mathscr{M} \to \phi_*\mathscr{M}$ is linear—such that the following properties are verified:*

(i) $|\phi_* v| = \phi(|v|)$ for every $v \in \mathscr{M}$.

(ii) $\mathscr{G}(\phi_*(\mathscr{M}))$ is dense in $\phi_*\mathscr{M}$.

Uniqueness is up to unique isomorphism: given another couple $(\mathscr{N}, T)$ with the same properties, there exists a unique isomorphism of V_2-Banach U_2-modules $\Phi\colon \phi_*\mathscr{M} \to \mathscr{N}$ such that

$$\begin{array}{ccc} \mathscr{V} & \xrightarrow{T_{\langle\psi\rangle}} & \mathscr{M}_{\langle\psi\rangle} \\ & {\scriptstyle T}\searrow & \downarrow{\scriptstyle \Phi} \\ & & \mathscr{M} \end{array}$$

is a commutative diagram.

Proof. Define $\psi\colon \mathscr{M} \to V_2^+$ as $\psi(v) := \phi(|v|)$ for every $v \in \mathscr{M}$. Notice that ψ is even and sublinear, and thus it makes sense to consider the V_2-Banach U_2-module $\phi_*\mathscr{M} := \mathscr{M}_{\langle\psi\rangle}$ and the linear operator $\phi_* := T_{\langle\psi\rangle}\colon \mathscr{M} \to \phi_*\mathscr{M}$. Observe that $|\phi_* v| = \psi(v) = \phi(|v|)$ for every $v \in \mathscr{M}$, which shows (i), while (ii) is only a rephrasing of Theorem 3.19 (ii). This proves the existence part of the statement. Finally, the uniqueness part follows from the uniqueness stated in Theorem 3.19. □

Proposition 3.24 *Let $(\mathcal{U}_1, U_1, V_1)$ be a metric f-structure, and let $(\mathcal{U}_2, U_2, V_2, W_2, Z_2)$ be a dual system of metric f-structures. Let $\phi\colon (\mathcal{U}_1, U_1, V_1) \to (\mathcal{U}_2, U_2, V_2)$ be a homomorphism of metric f-structures. Fix a V_1-normed U_1-module $\mathscr{M}$ and a Z_2-Banach U_2-module $\mathscr{N}$. Let $T\colon \mathscr{M} \to \mathscr{N}$ be a linear operator such that there exists an element $b \in W_2^+$ satisfying*

$$|Tv| \leq b\,\phi(|v|), \quad \text{for every } v \in \mathscr{M}. \tag{31}$$

Then there exists a unique homomorphism $\hat{T} \in \mathrm{Hom}(\phi_*\mathscr{M}, \mathscr{N})$ such that

$$\hat{T}(\phi_* v) = Tv, \quad \text{for every } v \in \mathscr{M}. \tag{32}$$

Moreover, it holds that $|\hat{T}w| \leq b|w|$ for every $w \in \phi_*\mathscr{M}$.

Proof. Recall from (the proof of) Theorem 3.23 that, letting $\psi\colon \mathscr{M} \to V_2^+$ be $\psi(v) := \phi(|v|)$, it holds $(\phi_*\mathscr{M}, \phi_*) \cong (\mathscr{M}_{\langle\psi\rangle}, T_{\langle\psi\rangle})$. The statement then follows from Proposition 3.20. □

Corollary 3.25 Let $(\mathcal{U}_1, U_1, V_1), (\mathcal{U}_2, U_2, V_2)$ be metric f-structures, and $\phi\colon \mathcal{U}_1 \to \mathcal{U}_2$ a homomorphism of metric f-structures. Let $\mathscr{M}, \mathscr{N}$ be V_1-normed U_1-modules. Let $T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ be given. Then there exists a unique homomorphism $\phi_*T \in \mathrm{Hom}(\phi_*\mathscr{M}, \phi_*\mathscr{N})$ such that

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{T} & \mathscr{N} \\ {\scriptstyle \phi_*}\downarrow & & \downarrow{\scriptstyle \phi_*} \\ \phi_*\mathscr{M} & \xrightarrow[\phi_* T]{} & \phi_*\mathscr{N} \end{array}$$

is a commutative diagram. Moreover, if both U_1 and U_2 are Dedekind complete, then it holds that

$$|\phi_* T| \leq \phi(|T|).$$

Proof. Define $\tilde{T} := \phi_* \circ T\colon \mathscr{M} \to \phi_*\mathscr{N}$. Let $w \in U_1^+$ such that $|Tv| \leq w|v|$ for every $v \in \mathscr{M}$,

$$|\tilde{T}v| = |\phi_*(Tv)| = \phi(|Tv|) \leq \phi(w|v|) = \phi(w)\phi(|v|), \quad \text{for every } v \in \mathscr{M}. \tag{33}$$

Therefore, the existence and the uniqueness of ϕ_*T follow from Proposition 3.24 applied to $\tilde{T}$. Finally, suppose that U_1 and U_2 are Dedekind complete. Then we can choose $w := |T|$ in

(33), so that we have $|(\phi_* T)w| \le \phi(|T|)|w|$ for every $w \in \phi_*\mathscr{M}$, whence it follows that $|\phi_* T| \le \phi(|T|)$. □

Combining Theorem 3.23 and Corollary 3.25, any homomorphism $\phi\colon (\mathcal{U}_1, U_1, V_1) \to (\mathcal{U}_2, U_2, V_2)$ induces a functor

$$\phi_*\colon \mathit{BanMod}_{(\mathcal{U}_1,U_1,V_1)} \to \mathit{BanMod}_{(\mathcal{U}_2,U_2,V_2)}.$$

We point out that, even though we chose the term 'pushforward' by analogy with [**20**, Section 1.6], from the categorical perspective the correct term would be the *direct image functor*.

Remark 3.26 Let $(\mathcal{U}_1, U_1, V_1, V_1, Z_1)$, $(\mathcal{U}_2, U_2, V_2, V_2, Z_2)$ be dual systems of metric f-structures. Let $\phi\colon \mathcal{U}_1 \to \mathcal{U}_2$ be a homomorphism of dual systems. Let $\mathscr{H}$ be a V_1-Hilbert U_1-module. Then

$$\phi_*\mathscr{H}, \quad \text{is a } V_2\text{-Hilbert } U_2\text{-module,} \tag{34a}$$

$$\phi_* v \cdot \phi_* w = \phi(v \cdot w), \quad \text{for every } v, w \in \mathscr{H}. \tag{34b}$$

To prove (34a), notice that Theorem 3.23 (i) implies that the elements of $\mathscr{G}(\phi_*(\mathscr{H}))$—thus all the elements of $\phi_*\mathscr{H}$, thanks to Theorem 3.23 (ii)—satisfy the pointwise parallelogram law. Also,

$$\phi_* v \cdot \phi_* w = \frac{1}{2}\left(|\phi_*(v+w)|^2 - |\phi_* v|^2 - |\phi_* w|^2\right) = \phi\left(\frac{1}{2}\left(|v+w|^2 - |v|^2 - |w|^2\right)\right) = \phi(v \cdot w)$$

hold for every $v, w \in \mathscr{H}$, which shows that (34b) is verified.

Theorem 3.27 *Let $(\mathcal{U}_1, U_1, V_1, W_1, Z_1)$ and $(\mathcal{U}_2, U_2, V_2, W_2, Z_2)$ be CSP complete dual systems of metric f-structures. Let $\phi\colon \mathcal{U}_1 \to \mathcal{U}_2$ be a homomorphism of dual systems. Let $\mathscr{M}$ be a V_1-Banach U_1-module. Then there exists a unique homomorphism $I_\phi \in \mathrm{Hom}(\phi_*\mathscr{M}^*, (\phi_*\mathscr{M})^*)$ such that*

$$\langle I_\phi(\phi_*\omega), \phi_* v\rangle = \phi(\langle \omega, v\rangle), \quad \textit{for every } \omega \in \mathscr{M}^* \textit{ and } v \in \mathscr{M}. \tag{35}$$

Moreover, it holds that

$$|I_\phi(\eta)| = |\eta|, \quad \textit{for every } \eta \in \phi_*\mathscr{M}^*. \tag{36}$$

Proof. Given any $\omega \in \mathscr{M}^*$, we define the operator $\tilde{i}_\phi(\omega)\colon \mathscr{M} \to Z_2$ as

$$\tilde{i}_\phi(\omega)v := \phi(\langle \omega, v\rangle), \quad \text{for every } v \in \mathscr{M}.$$

Note that $\tilde{i}_\phi(\omega)$ is linear and satisfies $|\tilde{i}_\phi(\omega)v| \le \phi(|\omega|)\phi(|v|)$ for all $v \in \mathscr{M}$. Hence, we know from Proposition 3.24 that there is a unique element $i_\phi(\omega) \in (\phi_*\mathscr{M})^*$ such that $|i_\phi(\omega)| \le \phi(|\omega|)$ and

$$\langle i_\phi(\omega), \phi_* v\rangle = \tilde{i}_\phi(\omega)v = \phi(\langle \omega, v\rangle), \quad \text{for every } v \in \mathscr{M}.$$

Since the resulting operator $i_\phi\colon \mathscr{M}^* \to (\phi_*\mathscr{M})^*$ is linear, by applying Proposition 3.24 again we deduce that there exists a unique $\mathrm{I}_\phi \in \mathrm{Hom}(\phi_*\mathscr{M}^*, (\phi_*\mathscr{M})^*)$ with $|\mathrm{I}_\phi| \le 1$ such that (35) holds.

It remains to check (36). Thanks to Proposition 3.4 (i) and Theorem 3.23 (ii), it suffices to prove (36) for $\eta \in \mathscr{G}(\phi_*(\mathscr{M}^*))$, say $\eta = \sum_{n\in\mathbb{N}} u_n \cdot \phi_*\omega_n$ with $(u_n)_n \in \mathcal{P}(\mathbf{1}_{U_2})$ and $(\omega_n)_n \subset \mathscr{M}^*$. Given any $n \in \mathbb{N}$, we deduce from Lemma 3.13 that $|\omega_n| = \sup\{\langle\omega_n, v\rangle : v \in \mathscr{M}, |v| \leq 1\}$, so we can pick a sequence $(v_n^i)_{i\in\mathbb{N}} \subset \mathscr{M}$ such that $|v_n^i| \leq 1$ for all $i \in \mathbb{N}$ and $|\omega_n| = \sup_i \langle\omega_n, v_n^i\rangle$. Then

$$\phi(|\omega_n|) = |\phi_*\omega_n| = \sup_{i\in\mathbb{N}} \phi(\langle\omega_n, v_n^i\rangle) \overset{(23)}{=} \sup_{i\in\mathbb{N}} \langle \mathrm{I}_\phi(\phi_*\omega_n), \phi_* v_n^i\rangle. \tag{37}$$

Multiplying by u_n and summing over n, we deduce (using Lemma 3.13 again and $|\phi_* v_n^i| \leq 1$) that

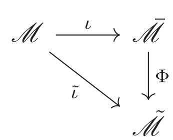
$$\begin{aligned} |\eta| &= \sum_{n\in\mathbb{N}} u_n|\phi_*\omega_n| \overset{(37)}{=} \sum_{n\in\mathbb{N}} u_n \sup_{i\in\mathbb{N}} \langle \mathrm{I}_\phi(\phi_*\omega_n), \phi_* v_n^i\rangle = \sum_{n\in\mathbb{N}} \sup_{i\in\mathbb{N}} \langle u_n \cdot \mathrm{I}_\phi(\phi_*\omega_n), \phi_* v_n^i\rangle \\ &= \sum_{n\in\mathbb{N}} \sup_{i\in\mathbb{N}} \langle \mathrm{I}_\phi(u_n\cdot\eta), \phi_* v_n^i\rangle \leq \sum_{n\in\mathbb{N}} |\mathrm{I}_\phi(u_n\cdot\eta)| = \sum_{n\in\mathbb{N}} u_n|\mathrm{I}_\phi(\eta)| = |\mathrm{I}_\phi(\eta)|. \end{aligned}$$

Since $|\mathrm{I}_\phi| \leq 1$ yields the converse inequality $|\mathrm{I}_\phi(\eta)| \leq |\eta|$, the statement is finally achieved. □

3.3.3. *Completion of a normed module*

It follows from Theorem 3.23 that each V-normed U-module can be 'completed' to a V-Banach U-module, and much like the metric, completion of a normed space has a Banach space structure:

THEOREM 3.28 (Completion of a normed module) *Let $(\mathcal{U}, U, V)$ be a metric f-structure, and let $\mathscr{M}$ be a V-normed U-module. Then there exists a unique couple $(\bar{\mathscr{M}}, \iota)$ such that $\bar{\mathscr{M}}$ is a V-Banach U-module and $\iota\colon \mathscr{M} \to \bar{\mathscr{M}}$ is a U-linear map with a dense range satisfying $|\iota v| = |v|$ for every $v \in \mathscr{M}$. Uniqueness is up to unique isomorphism: given another couple $(\tilde{\mathscr{M}}, \tilde{\iota})$ with the same properties, there is a unique isomorphism $\Phi\colon \bar{\mathscr{M}} \to \tilde{\mathscr{M}}$ of V-Banach U-modules such that*



is a commutative diagram. We say that the couple $(\bar{\mathscr{M}}, \iota)$, or just $\bar{\mathscr{M}}$, is the completion of $\mathscr{M}$.

Moreover, if $\mathscr{M}, \mathscr{N}$ are V-normed U-modules and $T \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ is given, then there exists a unique homomorphism $\bar{T} \in \mathrm{Hom}(\bar{\mathscr{M}}, \bar{\mathscr{N}})$ such that $\bar{T}|_{\mathscr{M}} = T$, where we regard $\mathscr{M}$ and $\mathscr{N}$ as subsets of $\bar{\mathscr{M}}$ and $\bar{\mathscr{N}}$, respectively. If in addition U is Dedekind complete, then it holds $|\bar{T}| = |T|$.

Proof. The identity mapping $\mathrm{id}_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{U}$ is a homomorphism of metric f-structures from $(\mathcal{U}, U, V)$ to itself, and thus we can define $\bar{\mathscr{M}} := (\mathrm{id}_{\mathcal{U}})_*\mathscr{M}$ and $\iota := (\mathrm{id}_{\mathcal{U}})_*\colon \mathscr{M} \to \bar{\mathscr{M}}$. By Theorem 3.23, to prove the first part of the statement amounts to showing that ι is U-linear and that $\iota(\mathscr{M})$ is dense in $\bar{\mathscr{M}}$. The former property follows from Corollary 3.18. About the latter, recall that $\mathscr{G}(\iota(\mathscr{M}))$ is dense in $\bar{\mathscr{M}}$, and thus it only remains to show that $\iota(\mathscr{M}) = \mathscr{G}(\iota(\mathscr{M}))$. The inclusion $\iota(\mathscr{M}) \subset \mathscr{G}(\iota(\mathscr{M}))$ is trivial. Conversely, fix $w = \sum_{n\in\mathbb{N}} u_n \cdot \iota v_n \in \mathscr{G}(\iota(\mathscr{M}))$. Since $(u_n, \iota v_n)_{n\in\mathbb{N}} \in \mathrm{Adm}(\bar{\mathscr{M}})$, we have that $(u_n, v_n)_{n\in\mathbb{N}} \in \mathrm{Adm}(\mathscr{M})$, and thus it makes sense to consider $v := \sum_{n\in\mathbb{N}} u_n \cdot v_n \in \mathscr{M}$. We claim that $\iota v = w$, whence it follows that $w \in \iota(\mathscr{M})$ and thus $\mathscr{G}(\iota(\mathscr{M})) \subset \iota(\mathscr{M})$. Given that

$$u_n \cdot \iota v = \iota(u_n \cdot v) = \iota(u_n \cdot v_n) = u_n \cdot \iota v_n = u_n \cdot w, \quad \text{for every } n \in \mathbb{N},$$

we deduce from Lemma 3.2 that $\iota v = w$. Therefore, the first part of the statement is proved.

About the second part of the statement, observe that $\bar{T} := (\mathrm{id}_{\mathcal{U}})_* T \in \mathrm{Hom}(\bar{\mathscr{M}}, \bar{\mathscr{N}})$ is the unique homomorphism extending T. Note also that if $\bar{u} \in U^+$ satisfies $|Tv| \leq \bar{u}|v|$ for all $v \in \mathscr{M}$, then $|\bar{T}v| \leq \bar{u}|v|$ for every $v \in \bar{\mathscr{M}}$ by approximation. Finally, borrowing the notation from the proof of Theorem 3.12 (see (20)), we get that $\mathcal{F}_T = \mathcal{F}_{\bar{T}}$, so that (assuming that U is Dedekind complete) we conclude that $|\bar{T}| = |T|$. □

3.4. Hahn–Banach extension theorem

The aim of this section is to obtain a normed module version of the Hahn–Banach extension theorem as well as to investigate some of its basic consequences.

Let $(\mathcal{U}, U, V)$ be a metric f-structure, and $\mathscr{M}$ a module over U. Then we say that a given map $p\colon \mathscr{M} \to V^+$ is *U-sublinear* if it is subadditive (that is, $p(v+w) \leq p(v) + p(w)$ for every $v, w \in \mathscr{M}$) and *positively U-homogeneous*, which means that $p(u \cdot v) = u\,p(v)$ for all $u \in U^+$ and $v \in \mathscr{M}$.

Lemma 3.29 (One-dimensional dominated extension) *Let $(\mathcal{U}, U, V)$ be a Dedekind complete CSP metric f-structure, $\mathscr{M}$ a V-normed U-module and $\mathscr{N} \subsetneq \mathscr{M}$ a V-normed U-submodule of $\mathscr{M}$. Fix any $z \in \mathscr{M}$ such that $u \cdot z \notin \mathscr{N}$ for every $u \in Idem(U) \setminus \{0\}$, with $u \leq \chi_{\{z \neq 0\}}$. Let $f\colon \mathscr{N} \to V$ be a U-linear map, and $p\colon \mathscr{M} \to V^+$ a U-sublinear map with $f \leq p$ on $\mathscr{N}$. Define $\mathscr{N}_{+z} := \mathscr{N} + U \cdot z$. Then there exists a U-linear map $\bar{f}\colon \mathscr{N}_{+z} \to V$ with $\bar{f}|_{\mathscr{N}} = f$ such that $\bar{f} \leq p$ on $\mathscr{N}_{+z}$.*

Proof. Given any $v, w \in \mathscr{N}$, we can estimate

$$f(v) - f(w) = f(v-w) \leq p(v-w) = p(v+z-(w+z)) \leq p(v+z) + p(-w-z),$$

whence it follows that $-p(-w-z) - f(w) \leq p(v+z) - f(v)$ for every $v, w \in \mathscr{N}$. Substituting $w = 0$, we obtain that $p(z) \leq p(v+z) - f(v)$ for every $v \in \mathscr{N}$, and thus the Dedekind completeness of V ensures that $b := \inf\{p(v+z) - f(v) \,:\, v \in \mathscr{N}\} \in V$ exists. Then we have that

$$-p(-w-z) - f(w) \leq b \leq p(v+z) - f(v), \quad \text{for every } v, w \in \mathscr{N}. \tag{38}$$

Substituting $v = w = 0$ in (38) and multiplying by $\chi_{\{z=0\}}$, we deduce that $\chi_{\{z=0\}} \cdot b = 0$, so that

$$\chi_{\{z=0\}} \leq \chi_{\{b=0\}}. \tag{39}$$

Next we claim that for any $v, \tilde{v} \in \mathscr{N}$ and $u, \tilde{u} \in U$ it holds that

$$v + u \cdot z = \tilde{v} + \tilde{u} \cdot z \quad \Longrightarrow \quad v = \tilde{v} \text{ and } u \cdot b = \tilde{u} \cdot b. \tag{40}$$

To prove it, suppose that $(u - \tilde{u}) \cdot z = \tilde{v} - v$. Pick a partition $(u_n)_{n\in\mathbb{N}}$ of $\chi_{\{u \neq \tilde{u}\}}$ and $(w_n)_{n\in\mathbb{N}} \subset U$ such that $u_n(u-\tilde{u})w_n = u_n$ for every $n \in \mathbb{N}$. Multiplying $(u-\tilde{u}) \cdot z = \tilde{v} - v$ by $u_n w_n$, we obtain

$$u_n \cdot z = (u_n(u-\tilde{u})w_n) \cdot z = (u_n w_n) \cdot (\tilde{v} - v) \in \mathscr{N}, \quad \text{for every } n \in \mathbb{N}.$$

Hence, the gluing property of $\mathscr{N}$ ensures that $\chi_{\{u \neq \tilde{u}\}} \cdot z \in \mathscr{N}$, and thus accordingly $\chi_{\{u \neq \tilde{u}\}} \leq \chi_{\{z=0\}}$. This implies that $\tilde{v} - v = u \cdot z - \tilde{u} \cdot z = 0$ and (recalling (39)) that $u \cdot b = \tilde{u} \cdot b$, proving (40). Therefore, the map $\bar{f}\colon \mathscr{N}_{+z} \to V$ defined as follows is well-posed:

$$\bar{f}(v + u \cdot z) := f(v) + u \cdot b, \quad \text{for every } v \in \mathscr{N} \text{ and } u \in U.$$

It is immediate to check that $\bar{f}$ is a U-linear extension of f. It only remains to show that $\bar{f} \leq p$ on $\mathscr{N}_{+z}$. To this aim, fix $v \in \mathscr{N}$ and $u \in U \setminus \{0\}$. Pick a partition $(u_n)_{n\in\mathbb{N}}$ of $\{u > 0\}$, a partition $(\tilde{u}_n)_{n\in\mathbb{N}}$ of $\{u<0\}$ and $(w_n)_{n\in\mathbb{N}},(\tilde{w}_n)_{n\in\mathbb{N}}\subset U^+$ such that $u_n u^+ w_n = u_n$ and $\tilde{u}_n u^- \tilde{w}_n = \tilde{u}_n$ for all $n\in\mathbb{N}$. It follows from (38) that $p(w_n\cdot v+z)-f(w_n\cdot v)\geq b$ and $-p(\tilde{w}_n\cdot v-z)+f(\tilde{w}_n\cdot v)\leq b$. Multiplying by $u_n u^+$ and $\tilde{u}_n u^-$, respectively, we obtain $u_n\cdot(f(v)+u\cdot b)\leq u_n\cdot p(v+u\cdot z)$ and

$$\tilde{u}_n\cdot(f(v)+u\cdot b)=\tilde{u}_n\cdot(f(v)-u^-\cdot b)\leq \tilde{u}_n\cdot p(v-u^-\cdot z)=\tilde{u}_n\cdot p(v+u\cdot z),$$

respectively. Using the gluing property, we conclude that $\bar{f}(v+u\cdot z)=f(v)+u\cdot b\leq p(v+u\cdot z)$. □

THEOREM 3.30 (Hahn–Banach theorem for normed modules) *Let $(\mathcal{U},U,V)$ be a Dedekind complete CSP metric f-structure. Let $\mathscr{M}$ be a V-normed U-module, and let $\mathscr{N}\subsetneq\mathscr{M}$ be a V-normed U-submodule of $\mathscr{M}$. Let $f\colon\mathscr{N}\to V$ be a U-linear map, and $p\colon\mathscr{M}\to V^+$ a U-sublinear map with $f\leq p$ on $\mathscr{N}$. Then there exists a U-linear map $\bar{f}\colon\mathscr{M}\to V$ such that $\bar{f}|_{\mathscr{N}}=f$ such that $\bar{f}\leq p$ on $\mathscr{M}$.*

Proof. Let us denote by $\mathcal{F}$ the family of all couples $(\mathscr{Q},g)$, where $\mathscr{Q}$ is a V-normed U-submodule of $\mathscr{M}$ containing $\mathscr{N}$ and $g\colon\mathscr{Q}\to V$ is a U-linear extension of f satisfying $q\leq p$ on $\mathscr{Q}$. Clearly $\mathcal{F}$ is non-empty, as it contains $(\mathscr{N},f)$. We endow $\mathcal{F}$ with the partial order $\preceq$ defined as follows: given $(\mathscr{Q},g),(\tilde{\mathscr{Q}},\tilde{g})\in\mathcal{F}$, we declare that $(\mathscr{Q},g)\preceq(\tilde{\mathscr{Q}},\tilde{g})$ provided $\mathscr{Q}\subset\tilde{\mathscr{Q}}$ and $\tilde{g}|_{\mathscr{Q}}=g$. It is easy to check that any totally ordered subset $\mathcal{C}$ of $(\mathcal{F},\preceq)$ has an upper bound, namely, $(\mathscr{Q},g_0)$, where $\mathscr{Q}_0:=\bigcup_{(\mathscr{Q},g)\in\mathcal{F}}\mathscr{Q}$ and $g_0\colon\mathscr{Q}_0\to V$ is given by $g_0(v):=g(v)$ for every $(\mathscr{Q},g)\in\mathcal{F}$ with $v\in\mathscr{Q}$. Hence, an application of Zorn's lemma yields the existence of a maximal element $(\mathscr{N}_0,f_0)$ of $(\mathcal{F},\preceq)$. In order to conclude, we aim to show that $\mathscr{N}_0=\mathscr{M}$. We argue by contradiction: suppose that $\mathscr{M}\setminus\mathscr{N}_0\neq\emptyset$. Fix any $\tilde{z}\in\mathscr{M}\setminus\mathscr{N}_0$. The fact that U is Dedekind complete and CSP ensures that

$$\exists\, q:=\sup\left\{u\in\mathrm{Idem}(U)\;\middle|\;u\cdot\tilde{z}\in\mathscr{N}_0\right\}\in\mathrm{Idem}(U).$$

Moreover, the gluing property implies that $q\cdot\tilde{z}\in\mathscr{N}_0$. Then we define $z:=(1-q)\cdot\tilde{z}\in\mathscr{M}$. Observe that $u\cdot z\notin\mathscr{N}_0$ holds for every $u\in\mathrm{Idem}(U)\setminus\{0\}$, with $u\leq 1-q=\chi_{\{z\neq 0\}}$. Therefore, Lemma 3.29 yields the existence of a map $\bar{f}\colon\mathscr{N}_0+U\cdot z\to V$ such that $(\mathscr{N}_0+U\cdot z,\bar{f})\in\mathcal{F}$ and $(\mathscr{N}_0,f_0)\preceq(\mathscr{N}_0+U\cdot z,\bar{f})$, which leads to a contradiction with the maximality of $(\mathscr{N}_0,f_0)$. □

COROLLARY 3.31 Let $(\mathcal{U},U,V,W,Z)$ be a CSP complete dual system of metric f-structures. Let $\mathscr{M}$ be a V-Banach U-module. Let $v\in\mathscr{M}$ satisfy $|v|\in Z\cap U$ and $\chi_{\{v\neq 0\}}\in W$. Then there exists an element $\omega\in\mathscr{M}^*$ such that $\langle\omega,v\rangle=|v|$ and $|\omega|=\chi_{\{v\neq 0\}}$.

Proof. Notice that $U\cdot v$ is a V-Banach U-submodule of $\mathscr{M}$. We define the map $T\colon U\cdot v\to Z$ as

$$T(u\cdot v):=u|v|,\quad \text{for every } u\in U.$$

Clearly, T is well-posed and U-linear. Moreover, we have $|T(u\cdot v)|=\chi_{\{v\neq 0\}}|u\cdot v|$ for every $u\in U$, and thus $T\in\mathrm{HOM}(U\cdot v,Z)$ and $|T|\leq\chi_{\{v\neq 0\}}$. Now let us define $p\colon\mathscr{M}\to Z^+$ as $p(w):=\chi_{\{v\neq 0\}}|w|$ for every $w\in\mathscr{M}$. It can be readily checked that p is U-sublinear. Since $T\leq p$ on $U\cdot v$, an application of Theorem 3.30 yields a U-linear map $\omega\colon\mathscr{M}\to Z$ satisfying $\omega|_{U\cdot v}=T$ and $\omega\leq p$ on $\mathscr{M}$. The latter gives $|\omega(w)|\leq\chi_{\{v\neq 0\}}|w|$ for every $w\in\mathscr{M}$, which shows that $\omega\in\mathscr{M}^*$ and $|\omega|\leq\chi_{\{v\neq 0\}}$. Notice also that $\langle\omega,v\rangle=|v|$ by construction. Therefore, in order to conclude it suffices to check that $|\omega|\geq\chi_{\{v\neq 0\}}$. To this aim, pick a partition $(u_n)_{n\in\mathbb{N}}$ of $\{v\neq 0\}$ and a sequence $(w_n)_{n\in\mathbb{N}}\subset U^+$ such that $u_n w_n|v|=u_n$ holds for every $n \in \mathbb{N}$. Now fix any $g \in W^+$ satisfying $|\langle \omega, w\rangle| \leq g|w|$ for every $w \in \mathscr{M}$. Then for any $n \in \mathbb{N}$ we can estimate

$$u_n g = u_n w_n |v| g = g|(u_n w_n) \cdot v| \geq |\langle \omega, (u_n w_n) \cdot v\rangle| = u_n w_n |v| = u_n,$$

which implies that $g \geq \chi_{\{v \neq 0\}}$ thanks to the arbitrariness of $n \in \mathbb{N}$. The statement is achieved. □

3.4.1. *Reflexive Banach modules*

Let $(\mathcal{U}, U, V, W, Z)$ be a complete dual system of metric f-structures, and let $\mathscr{M}$ be a V-Banach U-module. Then we denote the *bidual* of $\mathscr{M}$ by $\mathscr{M}^{**} := (\mathscr{M}^*)^*$. Here, we are considering the dual of $\mathscr{M}^*$ with respect to the dual system $(\mathcal{U}, U, W, V, Z)$ (recall Remark 2.27), so that $\mathscr{M}^{**}$ is a V-Banach U-module.

Definition 3.32 (Embedding into the bidual) Let $(\mathcal{U}, U, V, W, Z)$ be a complete dual system of metric f-structures. Let $\mathscr{M}$ be a V-Banach U-module. Then we define $\mathrm{J}_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}^{**}$ as

$$\langle \mathrm{J}_{\mathscr{M}}(v), \omega\rangle := \langle \omega, v\rangle, \quad \text{for every } v \in \mathscr{M} \text{ and } \omega \in \mathscr{M}^*.$$

Notice that the map $\mathscr{M} \times \mathscr{M}^* \ni (v, \omega) \mapsto \langle \mathrm{J}_{\mathscr{M}}(v), \omega\rangle \in Z$ is U-bilinear. Moreover, one has

$$|\langle \mathrm{J}_{\mathscr{M}}(v), \omega\rangle| \leq |\omega||v|, \quad \text{for every } v \in \mathscr{M} \text{ and } \omega \in \mathscr{M}^*.$$

It follows that $\mathrm{J}_{\mathscr{M}}(v) \in \mathscr{M}^{**}$ for every $v \in \mathscr{M}$, $\mathrm{J}_{\mathscr{M}} \in \mathrm{Hom}(\mathscr{M}, \mathscr{M}^{**})$ and $|\mathrm{J}_{\mathscr{M}}| \leq 1$. Under suitable assumptions, the homomorphism $\mathrm{J}_{\mathscr{M}}$ actually preserves the pointwise norm:

Proposition 3.33 *Let $(\mathcal{U}, U, V, W, Z)$ be a CSP complete dual system of metric f-structures. Suppose that $\mathrm{S}(V) \leq \mathrm{S}(W)$. Let $\mathscr{M}$ be a V-Banach U-module. Then it holds that*

$$\left|J_{\mathscr{M}}(v)\right| = |v|, \quad \textit{for every } v \in \mathscr{M}.$$

Proof. Let $v \in \mathscr{M}$ be fixed. We aim to show that $|\mathrm{J}_{\mathscr{M}}(v)| \geq |v|$. In view of Remark 3.8, we know that $\mathrm{S}(V) \leq \mathrm{S}(Z)$. Hence, applying Lemma 3.10 we obtain a partition $(u_n)_{n\in\mathbb{N}}$ of $\{v \neq 0\}$ such that $u_n|v| \in Z \cap U$ and $\chi_{\{u_n \cdot v \neq 0\}} \in W$ for every $n \in \mathbb{N}$. Using Corollary 3.31, we can find a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \mathscr{M}^*$ such that $\langle \omega_n, u_n \cdot v\rangle = u_n|v|$ and $|\omega_n| = \chi_{\{u_n \cdot v \neq 0\}}$ for all $n \in \mathbb{N}$. Then

$$u_n \langle \mathrm{J}_{\mathscr{M}}(v), \omega_n\rangle = \langle \omega_n, u_n \cdot v\rangle = u_n|v| = u_n|\omega_n||v|, \quad \text{for every } n \in \mathbb{N}.$$

This implies that $u_n|\mathrm{J}_{\mathscr{M}}(v)| \geq u_n|v|$ for every $n \in \mathbb{N}$, whence it follows that $|\mathrm{J}_{\mathscr{M}}(v)| \geq |v|$. □

In view of Proposition 3.33, it is then natural to give the following definition of reflexivity:

Definition 3.34 (Reflexive Banach module) Let $(\mathcal{U}, U, V, W, Z)$ be a CSP complete dual system of metric f-structures. Suppose that $\mathrm{S}(V) = \mathrm{S}(W)$. Then we say that a V-Banach U-module $\mathscr{M}$ is *reflexive* provided that the embedding operator $\mathrm{J}_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}^{**}$ is surjective.

3.5. Hilbert modules

In this section we investigate the properties of Hilbert modules. Among others, we will prove a Cauchy–Schwarz inequality (Lemma 3.35) and a Riesz-representation-type result (Theorem 3.38), we will study orthogonal projections (Theorem 3.36 and Proposition 3.37) and we will show that Hilbert modules are reflexive (Proposition 3.39).

Let $(\mathcal{U}, U, V, V, Z)$ be a dual system of metric f-structures, and $\mathscr{H}$ a V-Hilbert U-module. Then

$$v \cdot w = \frac{1}{4}|v+w|^2 - \frac{1}{4}|v-w|^2, \quad \text{for every } v, w \in \mathscr{H}. \tag{41}$$

Indeed, recalling the definition of the pointwise parallelogram law, we obtain that

$$\frac{1}{4}|v+w|^2 = \frac{1}{4}|v|^2 + \frac{1}{4}|w|^2 + \frac{1}{2}v \cdot w,$$
$$\frac{1}{4}|v-w|^2 = \frac{1}{4}|v|^2 + \frac{1}{4}|w|^2 - \frac{1}{2}v \cdot w.$$

Subtracting the second identity from the first one, we get (41).

Lemma 3.35 (Cauchy–Schwarz inequality) *Let $(\mathcal{U}, U, V, V, Z)$ be a dual system of metric f-structures, and $\mathscr{H}$ a V-Hilbert U-module. Then*

$$|v \cdot w| \le |v||w|, \quad \textit{for every } v, w \in \mathscr{H}. \tag{42}$$

Proof. Using (41) and the fact that $|v| - |w| \le |v+w| \le |v| + |w|$, we obtain that

$$\begin{aligned} v \cdot w &= \frac{1}{4}|v+w|^2 - \frac{1}{4}|v-w|^2 \le \frac{1}{4}\big((|v|+|w|)^2 - (|v|-|w|)^2\big) \\ &= \frac{1}{4}\big(|v|^2 + |w|^2 + 2|v||w| - |v|^2 - |w|^2 + 2|v||w|\big) = |v||w|. \end{aligned}$$

We also have that $-(v \cdot w) = (-v) \cdot w \le |-v||w| = |v||w|$. Therefore, (eq: CS) is proved. □

Given a V-Hilbert U-module $\mathscr{H}$ and a V-Hilbert U-submodule $\mathscr{N}$ of $\mathscr{H}$, we define the *orthogonal complement* of $\mathscr{N}$ in $\mathscr{H}$ as

$$\mathscr{N}^\perp := \big\{v \in \mathscr{H} \;\big|\; v \cdot w = 0, \text{ for every } w \in \mathscr{N}\big\}.$$

One can readily check that $\mathscr{N}^\perp$ is a V-Hilbert U-submodule of $\mathscr{H}$.

Theorem 3.36 (Hilbert projection theorem for Hilbert modules) *Let $(\mathcal{U}, U, V, V, Z)$ be a CSP complete dual system of metric f-structures. Let $\mathscr{H}$ be a V-Hilbert U-module satisfying*

$$\mathsf{d}_{\mathscr{H}}(v,0)^2 \le \mathsf{d}_Z(|v|^2, 0), \quad \textit{for every } v \in \mathscr{H}. \tag{43}$$

Let $C \neq \emptyset$ be a closed, convex subset of $\mathscr{H}$ such that $\mathscr{G}(C) = C$. Let $v \in \mathscr{H}$ be fixed. We define

$$|v - C| := \inf\big\{|v-w| \;\big|\; w \in C\big\} \in V^+, \qquad \mathsf{d}_{\mathscr{H}}(v, C) := \inf\big\{\mathsf{d}_{\mathscr{H}}(v,w) \;\big|\; w \in C\big\} \in \mathbb{R}^+.$$

Then it holds that

$$\mathsf{d}_V(|v-C|, 0) = \mathsf{d}_{\mathscr{H}}(v, C). \tag{44}$$

Moreover, there exists a unique $P_C(v) \in C$, the orthogonal projection of v onto C, such that

$$|v - C| = |v - P_C(v)|.$$

Proof. Applying Lemma 3.9, we can find an element $h \in V^+$ such that $h \leq 1$ and $\chi_{\{h=0\}} V = \{0\}$. By our assumptions, there exists a sequence $(\tilde{w}_k)_{k\in\mathbb{N}} \subset C$ such that $|v - C| = \inf_{k\in\mathbb{N}} |v - \tilde{w}_k|$. One can readily check that for any $n \in \mathbb{N}$ there exists a partition $(u_k^n)_{k\in\mathbb{N}}$ of $\chi_{\{h\neq 0\}}$ such that

$$u_k^n|v - C| \leq u_k^n|v - \tilde{w}_k| \leq u_k^n\left(|v - C| + \frac{1}{n}h\right), \quad \text{for every } k \in \mathbb{N}.$$

In particular, it holds $(u_k^n, \tilde{w}_k)_{k\in\mathbb{N}} \in \mathrm{Adm}(\mathscr{H})$, and thus it makes sense to set $\omega_n := \sum_{k\in\mathbb{N}} u_k^n \cdot \tilde{w}_k \in \mathscr{H}$. Given that C is closed under the gluing operation, we have that $(w_n)_{n\in\mathbb{N}} \subset C$. Notice also that

$$|v - C| \leq |v - w_n| \leq |v - C| + \frac{1}{n}h, \quad \text{for every } n \in \mathbb{N}. \tag{45}$$

Since $\lim_n \mathsf{d}_V(n^{-1}h, 0) = 0$, we deduce from (45) that $\lim_n \mathsf{d}_V(|v - C|, |v - w_n|) = 0$, and thus

$$\mathsf{d}_{\mathscr{H}}(v, C) \leq \lim_{n\to\infty} \mathsf{d}_{\mathscr{H}}(v, w_n) = \lim_{n\to\infty} \mathsf{d}_V(|v - w_n|, 0) = \mathsf{d}_V(|v - C|, 0).$$

On the other hand, we have $|v - C| \leq |v - w|$, and thus $\mathsf{d}_V(|v - C|, 0) \leq \mathsf{d}_{\mathscr{H}}(v, w)$, for every $w \in C$. By taking the infimum over $w \in C$, we get $\mathsf{d}_V(|v - C|, 0) \leq \mathsf{d}_{\mathscr{H}}(v, C)$. All in all, (44) is proved.

The Hilbertianity of $\mathscr{H}$ and the convexity of C ensure that for any $n, m \in \mathbb{N}$ it holds that

$$\begin{aligned} |w_n - w_m|^2 &= 2|v - w_n|^2 + 2|v - w_m|^2 - 4\left|v - \frac{w_n + w_m}{2}\right|^2 \\ &\leq 2|v - w_n|^2 + 2|v - w_m|^2 - 4|v - C|^2 \\ &\overset{(45)}{\leq} 2\left(\frac{1}{n^2} + \frac{1}{m^2}\right)h^2 + 4\left(\frac{1}{n} + \frac{1}{m}\right)|v - C|h, \end{aligned} \tag{46}$$

whence it follows that $\mathsf{d}_V(|w_n - w_m|, 0) \leq \sqrt{\mathsf{d}_Z(|w_n - w_m|^2, 0)} \to 0$ as $n, m \to \infty$. Hence, $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{H}$, and thus $\lim_n \mathsf{d}_{\mathscr{H}}(w_n, \bar{w}) = 0$ holds for some $\bar{w} \in C$. In particular, we have that $\mathsf{d}_V(|v - C|, |v - \bar{w}|) = \lim_n \mathsf{d}_V(|v - C|, |v - w_n|) = 0$, so that $|v - C| = |v - \bar{w}|$. To prove that $\bar{w}$ is the unique element of C with this property, fix any $\tilde{w} \in C$ with $|v - \tilde{w}| = |v - C|$. Then

$$0 \leq |\tilde{w} - \bar{w}|^2 = 2|v - \tilde{w}|^2 + 2|v - \bar{w}|^2 - 4\left|v - \frac{\tilde{w} + \bar{w}}{2}\right|^2 \leq 2|v - C|^2 + 2|v - C|^2 - 4|v - C|^2 = 0,$$

which forces the identity $\tilde{w} = \bar{w}$. All in all, the statement is finally achieved. □

The choice of the terminology 'orthogonal projection' is justified by the following result:

Proposition 3.37 *Let $(\mathcal{U}, U, \mathcal{V}, V, Z)$ be a CSP complete dual system of metric f-structures. Let $\mathscr{H}$ be a V-Hilbert U-module satisfying (43). Let $\mathscr{N}$ be a V-Hilbert U-submodule of $\mathscr{H}$. Then:*

(i) $v - P_{\mathscr{N}}(v) \in \mathscr{N}^{\perp}$ for every $v \in \mathscr{H}$.
(ii) It holds that $\mathscr{N} \oplus \mathscr{N}^{\perp}$.
(iii) The map $P_{\mathscr{N}}\colon \mathscr{H} \to \mathscr{N}$ belongs to $\mathrm{Hom}(\mathscr{H}, \mathscr{N})$.
(iv) $v = P_{\mathscr{N}}(v) + P_{\mathscr{N}^{\perp}}(v)$ and $|v|^2 = |P_{\mathscr{N}}(v)|^2 + |P_{\mathscr{N}^{\perp}}(v)|^2$ for every $v \in \mathscr{H}$.

Proof. Fix $w \in \mathscr{N}$. Denote $\bar{w} := P_{\mathscr{N}}(v)$. Pick a partition $(u_n)_{n\in\mathbb{N}}$ of $\mathbf{1}_U$ such that $u_n(v-\bar{w}) \cdot w \in U$ for every $n \in \mathbb{N}$. Since $\bar{w} - u \cdot w \in \mathscr{N}$ for every $u \in U$, we have that $|v - \bar{w}| \leq |v - \bar{w} + u \cdot w|$, and thus

$$|v-\bar{w}|^2 \leq |v-\bar{w}+u\cdot w|^2 = |v-\bar{w}|^2 + u|w|^2 - 2u(v-\bar{w})\cdot w. \tag{47}$$

Fix any $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. Substituting $u = -\varepsilon u_n(v-\bar{w})\cdot w$ into (47), multiplying both sides by $\varepsilon^{-1}u_n$ and rearranging the various terms, we obtain that

$$2u_n|(v-\bar{w})\cdot w|^2 \leq \varepsilon u_n|(v-\bar{w})\cdot w|^2|w|^2.$$

Letting $\varepsilon \searrow 0$, we get $u_n|(v-\bar{w})\cdot w| = 0$ for every $n \in \mathbb{N}$, and thus $v - \bar{w} \in \mathscr{N}^\perp$. Then (i) is proved.

To prove (ii), we aim to show that $\mathscr{N} + \mathscr{N}^\perp = \mathscr{H}$ and $\mathscr{N} \cap \mathscr{N}^\perp = \{0\}$. For the former, just observe that any $v \in \mathscr{H}$ can be written as $(v - P_{\mathscr{N}}(v)) + P_{\mathscr{N}}(v)$, where $v - P_{\mathscr{N}}(v) \in \mathscr{N}^\perp$ by (i) and $P_{\mathscr{N}}(v) \in \mathscr{N}$. For the latter, note that if $v \in \mathscr{N} \cap \mathscr{N}^\perp$, then $|v|^2 = v \cdot v = 0$ and thus $v = 0$.

Let us now pass to the verification of (iii). Given any $v, w \in \mathscr{H}$, we deduce from (i) that

$$\underbrace{\big(v+w-P_{\mathscr{N}}(v+w)\big)}_{\in\mathscr{N}^\perp} + \underbrace{P_{\mathscr{N}}(v+w)}_{\in\mathscr{N}} = v+w = \underbrace{\big(v-P_{\mathscr{N}}(v)+w-P_{\mathscr{N}}(w)\big)}_{\in\mathscr{N}^\perp} + \underbrace{P_{\mathscr{N}}(v)+P_{\mathscr{N}}(w)}_{\in\mathscr{N}},$$

and thus accordingly (ii) implies that $P_{\mathscr{N}}(v+w) = P_{\mathscr{N}}(v) + P_{\mathscr{N}}(w)$. Similarly, for any element $u \in U$ we have that $(u\cdot v - P_{\mathscr{N}}(u\cdot v)) + P_{\mathscr{N}}(u\cdot v) = u \cdot v = u\cdot(v - P_{\mathscr{N}}(v)) + u \cdot P_{\mathscr{N}}(v)$, which forces the identity $P_{\mathscr{N}}(u\cdot v) = u \cdot P_{\mathscr{N}}(v)$. All in all, we showed that $P_{\mathscr{N}}$ is a U-linear map. We also have that $|v|^2 = |v - P_{\mathscr{N}}(v)|^2 + |P_{\mathscr{N}}(v)|^2 \geq |P_{\mathscr{N}}(v)|^2$ for every $v \in \mathscr{H}$, and thus $P_{\mathscr{N}} \in \mathrm{Hom}(\mathscr{H}, \mathscr{N})$.

Finally, we prove (iv). Given any $w \in \mathscr{N}^\perp$, we have that $|w-v|^2 = |w|^2 + |v|^2$ for every $v \in \mathscr{N}$ and thus $|w - \mathscr{N}|^2 = |w|^2$, which implies that $P_{\mathscr{N}}(w) = 0$. Using also (iii), we deduce that

$$\big|v - (v - P_{\mathscr{N}}(v))\big|^2 = \big|P_{\mathscr{N}}(v)\big|^2 = \big|P_{\mathscr{N}}(v) - P_{\mathscr{N}}(w)\big|^2 = \big|P_{\mathscr{N}}(v-w)\big|^2 \leq |v-w|^2$$

for every $v \in \mathscr{H}$. Hence, $\big|v - (v - P_{\mathscr{N}}(v))\big| = |v - \mathscr{N}^\perp|$, which implies $P_{\mathscr{N}^\perp}(v) = v - P_{\mathscr{N}}(v)$. In particular, one has $|v|^2 = \big|P_{\mathscr{N}}(v) + P_{\mathscr{N}^\perp}(v)\big|^2 = |P_{\mathscr{N}}(v)|^2 + |P_{\mathscr{N}^\perp}(v)|^2$, so (iv) is also proved. □

Theorem 3.38 (Riesz representation theorem for Hilbert modules) *Let $(\mathcal{U}, U, \mathcal{V}, V, Z)$ be a CSP complete dual system of metric f-structures. Let $\mathscr{H}$ be a V-Hilbert U-module satisfying (43). We define the operator $R_{\mathscr{H}} \colon \mathscr{H} \to \mathscr{H}^*$ as*

$$\langle R_{\mathscr{H}}(w), v\rangle := v \cdot w, \quad \textit{for every } v, w \in \mathscr{H}.$$

Then $R_{\mathscr{H}}$ is an isomorphism of V-Banach U-modules, $\mathscr{H}^*$ is a V-Hilbert U-module and

$$R_{\mathscr{H}}(v) \cdot R_{\mathscr{H}}(w) = v \cdot w, \quad \textit{for every } v, w \in \mathscr{H}. \tag{48}$$

Proof. The properties of the pointwise scalar product and the Cauchy–Schwartz inequality ensure that $\mathrm{R}_{\mathscr{H}}(w) \in \mathscr{H}^*$ and $|\mathrm{R}_{\mathscr{H}}(w)| \leq |w|$ for all $w \in \mathscr{H}$. Then $\mathrm{R}_{\mathscr{H}} \in \mathrm{Hom}(\mathscr{H}, \mathscr{H}^*)$ and $|\mathrm{R}_{\mathscr{H}}| \leq 1$ (recall Example 2.26). To conclude, it remains to prove that $\mathrm{R}_{\mathscr{H}}$ is surjective and that it holds that $|\mathrm{R}_{\mathscr{H}}(w)| \geq |w|$ for every $v \in \mathscr{H}$. To this aim, fix any $\eta \in \mathscr{H}^* \setminus \{0\}$. We know that $\ker(\eta)$ is a V-Banach U-submodule of $\mathscr{H}$ with $\ker(\eta) \neq \mathscr{H}$, so that there exists $\tilde{w} \in \ker(\eta)^\perp \setminus \{0\}$. We can find a partition $(u_n)_{n\in\mathbb{N}}$ of $\chi_{\{\tilde{w}\neq 0\}}$ and a sequence $(a_n)_{n\in\mathbb{N}} \subset U^+$ such that $u_n|\tilde{w}|, u_n\langle\eta, \tilde{w}\rangle \in U$ and $u_n a_n|\tilde{w}| = u_n$ for every $n \in \mathbb{N}$. Then we define $w_n := (u_n\langle\eta,\tilde{w}\rangle a_n^2) \cdot \tilde{w} \in \mathscr{H}$ for every $n \in \mathbb{N}$. Notice that $|w_n| = u_n a_n|\langle\eta,\tilde{w}\rangle|$ and

$\langle \eta, w_n \rangle = |w_n|^2$. In particular, $|w_n| \leq u_n |\eta|$ for all $n \in \mathbb{N}$, and thus it makes sense to define $w := \sum_{n\in\mathbb{N}} u_n \cdot w_n \in \mathscr{H}$. It holds that $|w| \leq |\eta|$ and $\langle \eta, w \rangle = |w|^2$. Pick a partition $(\tilde{u}_k)_{k\in\mathbb{N}}$ of $\chi_{\{w\neq 0\}}$ and $(b_k)_{k\in\mathbb{N}} \subset U^+$ such that $\tilde{u}_k |w|^2 \in U$ and $\tilde{u}_k b_k |w|^2 = \tilde{u}_k$ for every $k \in \mathbb{N}$. Given any element $v \in \mathscr{H}$, we have that for every $k \in \mathbb{N}$ it holds that

$$\tilde{u}_k \cdot (v \cdot w) = (\tilde{u}_k \cdot v - (\tilde{u}_k \langle \eta, v \rangle b_k) \cdot w) \cdot w + \tilde{u}_k \langle \eta, v \rangle b_k |w|^2. \tag{49}$$

Observe that $\tilde{u}_k \cdot v - (\tilde{u}_k \langle \eta, v\rangle b_k) \cdot w \in \ker(\eta)$, as a consequence of the following computation:

$$\langle \eta, \tilde{u}_k \cdot v - (\tilde{u}_k \langle \eta, v \rangle b_k) \cdot w \rangle = \tilde{u}_k \langle \eta, v \rangle - \tilde{u}_k \langle \eta, v \rangle b_k \langle \eta, w \rangle = \tilde{u}_k \langle \eta, v \rangle - \tilde{u}_k \langle \eta, v \rangle b_k |w|^2 = 0.$$

It holds $w \in \ker(\eta)^\perp$, whence it follows that $(\tilde{u}_k \cdot v - (\tilde{u}_k \langle \eta, v \rangle b_k) \cdot w) \cdot w = 0$, and thus (49) yields

$$\tilde{u}_k \cdot (v \cdot w) = \tilde{u}_k \langle \eta, v \rangle b_k |w|^2 = \tilde{u}_k \langle \eta, v \rangle.$$

Thanks to the arbitrariness of $k \in \mathbb{N}$, we finally conclude that $\langle \eta, v \rangle = v \cdot w$ for every $v \in \mathscr{H}$, which means that $\eta = \mathrm{R}_{\mathscr{H}}(w)$ and $|\mathrm{R}_{\mathscr{H}}(w)| = |\eta| \geq |w|$. This completes the proof. □

Proposition 3.39 *Let $(\mathcal{U}, U, V, V, Z)$ be a CSP complete dual system of metric f-structures. Let $\mathscr{H}$ be a V-Hilbert U-module satisfying (43). Then it holds that $\mathscr{H}$ is reflexive.*

Proof. Given any $v \in \mathscr{H}$ and $\omega \in \mathscr{H}^*$, we have that

$$\begin{aligned} \langle \mathrm{R}_{\mathscr{H}^*}(\mathrm{R}_{\mathscr{H}}(v)), \omega \rangle &= \omega \cdot \mathrm{R}_{\mathscr{H}}(v) = \mathrm{R}_{\mathscr{H}}(\mathrm{R}^{-1}_{\mathscr{H}}(\omega)) \cdot \mathrm{R}_{\mathscr{H}}(v) \overset{(48)}{=} \mathrm{R}^{-1}_{\mathscr{H}}(\omega) \cdot v \\ &= \langle \mathrm{R}_{\mathscr{H}}(\mathrm{R}^{-1}_{\mathscr{H}}(\omega)), v \rangle = \langle \omega, v \rangle = \langle \mathrm{J}_{\mathscr{H}}(v), \omega \rangle. \end{aligned}$$

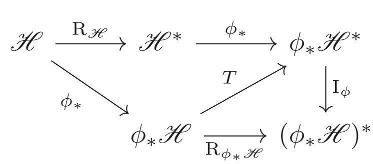
This shows that $\mathrm{J}_{\mathscr{H}} = \mathrm{R}_{\mathscr{H}^*} \circ \mathrm{R}_{\mathscr{H}}$. Since both $\mathrm{R}_{\mathscr{H}}$ and $\mathrm{R}_{\mathscr{H}^*}$ are surjective by Theorem 3.38, we conclude that $\mathrm{J}_{\mathscr{H}}$ is surjective, and thus $\mathscr{H}$ is reflexive, yielding the sought conclusion. □

Proposition *Let $(\mathcal{U}_1, U_1, V_1, V_1, Z_1)$, $(\mathcal{U}_2, U_2, V_2, V_2, Z_2)$ be CSP complete dual systems of metric f-structures. Let $\phi \colon \mathcal{U}_1 \to \mathcal{U}_2$ be a homomorphism of dual systems. Let $\mathscr{H}$ be a V_1-Hilbert U_1-module satisfying (43). Then $\mathrm{I}_\phi \colon \phi_*\mathscr{H}^* \to (\phi_*\mathscr{H})^*$ is an isomorphism of V_2-Banach U_2-modules.*

Proof. By Theorem 3.27, it suffices to check that $\mathrm{I}_\phi \colon \phi_*\mathscr{H}^* \to (\phi_*\mathscr{H})^*$ is invertible. Recall from Remark 3.26 that $\phi_*\mathscr{H}$ is a V_2-Hilbert U_2-module and $\phi_* v \cdot \phi_* w = \phi(v \cdot w)$ for all $v, w \in \mathscr{H}$. Then

$$\langle (\mathrm{I}_\phi \circ \phi_* \circ \mathrm{R}_{\mathscr{H}})(v), \phi_* w \rangle = \phi(\langle \mathrm{R}_{\mathscr{H}}(v), w \rangle) = \phi(v \cdot w) = \phi_* v \cdot \phi_* w = \langle (\mathrm{R}_{\phi_*\mathscr{H}} \circ \phi_*)(v), \phi_* w \rangle$$

for every $v, w \in \mathscr{H}$. Moreover, $\phi_* \circ \mathrm{R}_{\mathscr{H}} \colon \mathscr{H} \to \phi_*\mathscr{H}^*$ is linear and satisfies $|\phi_*(\mathrm{R}_{\mathscr{H}}(v))| = |v|$ for every $v \in \mathscr{H}$, and thus Proposition 3.24 gives an element $T \in \mathrm{Hom}(\phi_*\mathscr{H}, \phi_*\mathscr{H}^*)$ such that



is a commutative diagram. Hence, I_ϕ is invertible and $\mathrm{I}_\phi^{-1} = T \circ \mathrm{R}^{-1}_{\phi_*\mathscr{H}}$, concluding the proof. □

3.6. Dimensional decomposition

Given a commutative ring R and a non-empty subset S of an R-module M, we denote by $\langle S\rangle_R$ the R-submodule of M generated (in the algebraic sense) by S. Namely, we define

$$\langle S\rangle_R = \left\{ \sum_{i=1}^{n} r_i \cdot v_i \,\middle|\, n \in \mathbb{N},\, (r_i)_{i=1}^n \subset R,\, (v_i)_{i=1}^n \subset S \right\}.$$

Definition 3.41 (Independence, generators, local basis) Let $(\mathcal{U}, U, V)$ be a metric f-structure, and $\mathscr{M}$ a V-Banach U-module. Let $v_1, \dots, v_n \in \mathscr{M}$ and $u \in \mathrm{Idem}(U)$ be given. Then we say that

(i) $v_1, \dots, v_n$ are *independent of* u if for any $u_1, \dots, u_n \in U$ it holds that

$$\sum_{i=1}^{n} (uu_i) \cdot v_i = 0 \quad \Longleftrightarrow \quad uu_i = 0, \text{ for every } i = 1, \dots, n.$$

(ii) $v_1, \dots, v_n$ *generate* $\mathscr{M}$ *on* u provided $\mathscr{G}(\langle u \cdot S\rangle_U) = u \cdot \mathscr{M}$, where $S := \{v_1, \dots, v_n\}$.

(iii) $v_1, \dots, v_n$ form a *local basis of* $\mathscr{M}$ *on* u provided that they are independent of u and they generate $\mathscr{M}$ on u.

For brevity, in the case where $u = \mathbf{1}_U$ we do not specify 'on $\mathbf{1}_U$' in the above terminology.

In order to provide a well-defined notion of local dimension, we first need to show that two local bases on the same idempotent element must have the same cardinality:

Lemma 3.42 *Let $(\mathcal{U}, U, V)$ be a Dedekind complete CSP metric f-structure, and $\mathscr{M}$ a V-Banach U-module. Let $v_1, \dots, v_n \in \mathscr{M}$ and $w_1, \dots, w_m \in \mathscr{M}$ be local bases of $\mathscr{M}$ on $u \in \mathrm{Idem}(U)$. Then it holds that $n = m$.*

Proof. Observe that it suffices to check that if $v_1, \dots, v_n$ generate $\mathscr{M}$ on u and $w_1, \dots, w_m$ are independent of u, then $n \geq m$. Moreover, thanks to a finite induction argument, it is enough to show that if $k \leq m$ and $w_1, \dots, w_{k-1}, v_k, \dots, v_n$ generate $\mathscr{M}$ on some $u_0 \in \mathrm{Idem}(U)\setminus\{0\}$ with $u_0 \leq u$, then there exists $u_1 \in \mathrm{Idem}(U)\setminus\{0\}$ with $u_1 \leq u_0$ such that $w_1, \dots, w_k, v_{k+1}, \dots, v_n$ generate $\mathscr{M}$ on u_1 (up to reordering $v_k, \dots, v_n$). First of all, we can find elements $\tilde{u} \in \mathrm{Idem}(U)\setminus\{0\}$ with $\tilde{u} \leq u_0$ and $\tilde{u}_1, \dots, \tilde{u}_n \in U$ such that

$$\tilde{u} \cdot w_k = \sum_{i=1}^{k-1} \tilde{u}_i \cdot w_i + \sum_{i=k}^{n} \tilde{u}_i \cdot v_i. \tag{50}$$

Since $w_1, \dots, w_k$ are independent of $\tilde{u}$, it cannot happen that $\tilde{u}_k = \dots = \tilde{u}_n = 0$. Hence, until reordering $v_k, \dots, v_n$, we can assume that $z := \chi_{\{\tilde{u}_k \neq 0\}}\tilde{u} \neq 0$. Now take a partition $(z_j)_{j\in\mathbb{N}}$ of z and elements $(\tilde{z}_j)_{j\in\mathbb{N}} \subset U$ such that $z_j(\tilde{u}_k\tilde{z}_j - 1) = 0$ for every $j \in \mathbb{N}$. There exists $j_0 \in \mathbb{N}$ such that $z_{j_0} \neq 0$, so that multiplying both sides of (50) by $z_{j_0}\tilde{z}_{j_0}$ we obtain that

$$z_{j_0} \cdot v_k = (z_{j_0}\tilde{z}_{j_0}\tilde{u}_k) \cdot v_k = (z_{j_0}\tilde{z}_{j_0}) \cdot w_k - \sum_{i=1}^{k-1} (z_{j_0}\tilde{z}_{j_0}\tilde{u}_i) \cdot w_i - \sum_{i=k+1}^{n} (z_{j_0}\tilde{z}_{j_0}\tilde{u}_i) \cdot v_i.$$

This implies that, letting $u_1 := z_{j_0}$, it holds that $w_1, \dots, w_k, v_{k+1}, \dots, v_n$ generate $\mathscr{M}$ on u_1. □

In view of Lemma 3.42, the following definition is thus well-posed:

Definition 3.43 (Local dimension) Let $(\mathcal{U}, U, V)$ be a Dedekind complete CSP metric f-structure, and let $\mathscr{M}$ be a V-Banach U-module. Then we say that $\mathscr{M}$ *has local dimension* $n \in \mathbb{N}$ *on* $u \in \mathrm{Idem}(U)$ if there exists a local basis $v_1, \dots, v_n$ of $\mathscr{M}$ on u.

Finally, we can show that each Banach module admits a (unique) *dimensional decomposition*:

THEOREM 3.44 (Dimensional decomposition) *Let $(\mathcal{U}, U, V)$ be a Dedekind complete CSP metric f-structure, and $\mathscr{M}$ a V-Banach U-module. Then there exists a unique partition $(D_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of $\mathbf{1}_U$ such that*

(i) $\mathscr{M}$ has local dimension n on D_n, for every $n \in \mathbb{N}$.
(ii) Given any $n \in \mathbb{N}$ and $u \in Idem(U)\setminus\{0\}$ such that $u \leq D_\infty$, it holds that $\mathscr{M}$ does not have local dimension n on u.

We say that $(D_n)_{n\in\mathbb{N}\cup\{\infty\}} \subset Idem(U)$ is the dimensional decomposition of $\mathscr{M}$.

Proof. Thanks to the countable representability assumption, it makes sense to define

$$D_n := \sup\left\{u \in \mathrm{Idem}(U) \,\middle|\, \mathscr{M} \text{ has local dimension } n \text{ on } u\right\} \in \mathrm{Idem}(U), \quad \text{for every } n \in \mathbb{N},$$

as well as $D_\infty := \mathbf{1}_U - \sup_{n\in\mathbb{N}} D_n \in \mathrm{Idem}(U)$. We know from Lemma 3.42 that D_n and D_m are disjoint whenever $n, m \in \mathbb{N}$ and $n \neq m$, and thus it follows that $(D_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is a partition of $\mathbf{1}_U$. In order to conclude, it suffices to check that $\mathscr{M}$ has local dimension n on D_n for every $n \in \mathbb{N}$. Using again the Dedekind completeness and CSP assumptions, we can construct a partition $(u^j)_{j\in\mathbb{N}}$ of D_n such that $\mathscr{M}$ has local dimension n on each u^j. For any $j \in \mathbb{N}$, take a local basis $v_1^j, \ldots, v_n^j \in u_j \cdot \mathscr{M}$ of $\mathscr{M}$ on u^j. Thanks to Lemma 3.10 and until further refining the partition $(u^j)_{j\in\mathbb{N}}$, it is not restrictive to require also that $|v_i^j| \in U$ for every $i = 1, \ldots, n$ and $j \in \mathbb{N}$. Fix an element h as in Lemma 3.9. The dependence of $v_1^j, \ldots, v_n^j$ on u^j ensures that $\chi_{\{|v_i^j|>0\}} = u^j$ for every $i = 1, \ldots, n$, and thus we can find a partition $(u_{i,k}^j)_{k\in\mathbb{N}}$ of u^j and $(z_{i,k}^j)_{k\in\mathbb{N}} \subset U^+$ such that

$$u_{i,k}^j(z_{i,k}^j|v_i^j| - 1) = 0, \quad \text{for every } k \in \mathbb{N}.$$

Define $v_{i,k}^j := (u_{i,k}^j z_{i,k}^j h) \cdot v_i^j \in \mathscr{M}$ for every $k \in \mathbb{N}$. Since $|v_{i,k}^j| = u_{i,k}^j z_{i,k}^j h |v_i^j| = u_{i,k}^j h$ and $(u_{i,k}^j)_{j,k}$ is a partition of D_n, for any $i = 1, \ldots, n$ it holds that $(u_{i,k}^j, v_{i,k}^j)_{j,k} \in \mathrm{Adm}(\mathscr{M})$, and thus

$$\exists\, v_i := \sum_{j,k\in\mathbb{N}} u_{i,k}^j \cdot v_{i,k}^j \in \mathscr{M}.$$

It is then easy to check that $v_1, \ldots, v_n$ is local basis of $\mathscr{M}$ on D_n, whence the statement follows. □

REMARK 3.45 At the level of functional Banach modules (cf. with Section 4.2.4), a dimensional decomposition result has been first obtained in [**31**, Theorem 6.5] (see also Lemma 6.4 therein) and a similar result was provided later in [**20**, Proposition 1.4.5]. Taking into account [**31**, Theorem 3.7] together with Remark 4.8, it might be possible to obtain a version of Theorem 3.44 as a consequence of [**31**, Theorem 3.7].

4. APPLICATIONS TO FUNCTIONAL NORMED MODULES

In Section 4.1 we introduce the various spaces of functions that are typically used in metric measure geometry. In Section 4.2 we explain the relation between the 'functional' Banach modules and the axiomatic Banach modules we introduced in this paper. In Section 4.3 we obtain a side result, which states that every localizable f-algebra can be in fact realized as a space of functions.

4.1. Function spaces

A *measurable space* (X,Σ) is a set $\mathrm{X} \neq \emptyset$ together with a σ-algebra Σ. Whenever X is a topological space, we implicitly assume that Σ is the Borel σ-algebra of X. We define

$$\mathcal{L}^0(\Sigma) := \big\{ f\colon \mathrm{X} \to \mathbb{R} \,\big|\, f \text{ is measurable} \big\}.$$

Then $\mathcal{L}^0(\Sigma)$ is an f-algebra if endowed with the following operations: for any $f, g \in \mathcal{L}^0(\Sigma)$, we set

$$\begin{aligned} (f+g)(x) &:= f(x)+g(x), && \text{for every } x \in \mathrm{X}, \\ (\lambda f)(x) &:= \lambda f(x), && \text{for every } \lambda \in \mathbb{R} \text{ and } x \in \mathrm{X}, \\ f &\leq g, && \text{if and only if } f(x) \leq g(x) \text{for every } x \in \mathrm{X}, \\ (fg)(x) &:= f(x)g(x), && \text{for every } x \in \mathrm{X}. \end{aligned}$$

The space $\mathcal{L}^\infty(\Sigma)$ of all bounded measurable real-valued functions on (X,Σ), which is given by

$$\mathcal{L}^\infty(\Sigma) := \left\{ f \in \mathcal{L}^0(\Sigma) \,\middle|\, \|f\|_{\mathcal{L}^\infty(\Sigma)} := \sup_{x\in \mathrm{X}} |f|(x) < +\infty \right\},$$

is an f-subalgebra of $\mathcal{L}^0(\Sigma)$. Moreover, $(\mathcal{L}^\infty(\Sigma), \|\cdot\|_{\mathcal{L}^\infty(\Sigma)})$ is a Banach space. Given any $E \in \Sigma$, we denote by $\mathbb{1}_E \in \mathcal{L}^\infty(\Sigma)$ the *characteristic function* of E, which is defined as

$$\mathbb{1}_E(x) := \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \in \mathrm{X}\setminus E. \end{cases}$$

The space $\mathsf{Sf}(\Sigma) \subset \mathcal{L}^\infty(\Sigma)$ of all *simple functions* on (X,Σ) is then defined in the following way:

$$\mathsf{Sf}(\Sigma) := \left\{ \sum_{i=1}^n \lambda_i \mathbb{1}_{E_i} \,\middle|\, n \in \mathbb{N},\, (\lambda_i)_{i=1}^n \subset \mathbb{R},\, (E_i)_{i=1}^n \subset \Sigma \text{ partition of } \mathrm{X} \right\}.$$

REMARK 4.1 (Density of simple functions) Given any $f \in \mathcal{L}^0(\Sigma)^+$, there exists $(f_n)_{n\in\mathbb{N}} \subset \mathsf{Sf}(\Sigma)$ such that $0 \leq f_n \leq f_{n+1}$ for every $n \in \mathbb{N}$ and $f = \sup_{n\in\mathbb{N}} f_n$. If in addition $f \in \mathcal{L}^\infty(\Sigma)$, then the sequence $(f_n)_{n\in\mathbb{N}}$ can also be chosen so that $\|f - f_n\|_{\mathcal{L}^\infty(\Sigma)} \to 0$, so that $\mathsf{Sf}(\Sigma)$ is dense in $\mathcal{L}^\infty(\Sigma)$. For example, the functions $f_n := \sum_{i=0}^{n2^n-1} i2^{-n} \mathbb{1}_{\{i2^{-n} \leq f < (i+1)2^{-n}\}} \in \mathcal{L}^\infty(\Sigma)^+$ do the job.

It is immediate to verify that

$$\begin{aligned} \mathrm{Idem}(\mathcal{L}^0(\Sigma)) &= \mathrm{Idem}(\mathcal{L}^\infty(\Sigma)) = \{\mathbb{1}_E \mid E \in \Sigma\}, \\ \mathcal{S}(\mathcal{L}^0(\Sigma)) &= \mathcal{S}(\mathcal{L}^\infty(\Sigma)) = \mathsf{Sf}(\Sigma). \end{aligned} \tag{51}$$

It then follows from (51) and Remark 4.1 that $\mathcal{L}^0(\Sigma)$ and $\mathcal{L}^\infty(\Sigma)$ are localizable f-algebras.

4.1.1. Enhanced measurable spaces

By an *enhanced measurable space* we mean a triple $(\mathrm{X},\Sigma,\mathcal{N})$, where (X,Σ) is a measurable space and $\mathcal{N} \subset \Sigma$ is a σ*-ideal*, meaning that it verifies the following conditions:

(i) $\emptyset \in \mathcal{N}$.
(ii) If $N \in \mathcal{N}$ and $N' \in \Sigma$ satisfy $N' \subset N$, then $N' \in \mathcal{N}$.
(iii) $\bigcup_{n\in\mathbb{N}} N_n \in \mathcal{N}$ for every countable family $(N_n)_{n\in\mathbb{N}} \subset \mathcal{N}$.

The σ-ideal $\mathcal{N}$ induces an equivalence relation on $\mathcal{L}^0(\Sigma)$: given any $f, g \in \mathcal{L}^0(\Sigma)$, we declare that

$$f \sim_{\mathcal{N}} g \qquad \Longleftrightarrow \qquad \{f \neq g\} := \{x \in \mathrm{X} \mid f(x) \neq g(x)\} \in \mathcal{N}.$$

When $f \sim_{\mathcal{N}} g$, we say that $f = g$ holds $\mathcal{N}$*-almost everywhere* or $\mathcal{N}$*-a.e.* for short. We define

$$L^0(\mathcal{N}) := \mathcal{L}^0(\Sigma)/\sim_{\mathcal{N}}, \qquad L^\infty(\mathcal{N}) := \mathcal{L}^\infty(\Sigma)/\sim_{\mathcal{N}}.$$

We denote by $[f]_{\mathcal{N}} \in L^0(\mathcal{N})$ the equivalence class of $f \in \mathcal{L}^0(\Sigma)$. For brevity, we set

$$\mathbb{1}_E^{\mathcal{N}} := [\mathbb{1}_E]_{\mathcal{N}}, \quad \text{for every } E \in \Sigma.$$

Passing to the quotient, $L^0(\mathcal{N})$ and $L^\infty(\mathcal{N})$ inherit an f-algebra structure from $\mathcal{L}^0(\Sigma)$ and $\mathcal{L}^\infty(\Sigma)$, respectively. Moreover, $L^\infty(\mathcal{N})$ becomes a Banach space if endowed with the quotient norm

$$\|f\|_{L^\infty(\mathcal{N})} := \inf_{\bar{f} \in [f]_{\mathcal{N}}} \|\bar{f}\|_{\mathcal{L}^\infty(\Sigma)}, \quad \text{for every } f \in L^\infty(\mathcal{N}).$$

The map $\mathcal{L}^\infty(\Sigma) \ni f \mapsto [f]_{\mathcal{N}} \in L^\infty(\mathcal{N})$ is linear 1-Lipschitz. Note that $\{\emptyset\}$ is a σ-ideal of Σ and

$$L^0(\{\emptyset\}) = \mathcal{L}^0(\Sigma), \qquad \big(L^\infty(\{\emptyset\}), \|\cdot\|_{L^\infty(\{\emptyset\})}\big) = \big(\mathcal{L}^\infty(\Sigma), \|\cdot\|_{\mathcal{L}^\infty(\Sigma)}\big).$$

Example 4.2 (of enhanced measurable space) Let (X, Σ) be a measurable space. Consider the restriction $\mu \colon \Sigma \to [0, +\infty]$ of an outer measure on X. Then the set $\mathcal{N}_\mu$ of μ*-null sets*, given by

$$\mathcal{N}_\mu := \{N \in \Sigma \mid \mu(N) = 0\},$$

is a σ-ideal, and thus $(\mathrm{X}, \Sigma, \mathcal{N}_\mu)$ is an enhanced measurable space. In this case, we abbreviate $L^0(\mathcal{N}_\mu)$ and $L^\infty(\mathcal{N}_\mu)$ to $L^0(\mu)$ and $L^\infty(\mu)$, respectively. Similarly for $\sim_\mu$, $[\cdot]_\mu$, and $\mathbb{1}_E^\mu$.

4.1.2. σ-Finite measure spaces

A *measure space* $(\mathrm{X}, \Sigma, \mu)$ is a measurable space (X, Σ) together with a σ-additive measure μ. We assume that the measure μ is σ-finite. Given any exponent $p \in [1, \infty)$, we define

$$\mathcal{L}^p(\mu) := \left\{ f \in \mathcal{L}^0(\Sigma) \,\middle|\, \|f\|_{\mathcal{L}^p(\mu)} := \int |f|^p \,\mathrm{d}\mu < +\infty \right\}.$$

It holds that $\mathcal{L}^p(\mu)$ is a vector subspace of $\mathcal{L}^0(\Sigma)$ and $\big(\mathcal{L}^p(\mu), \|\cdot\|_{\mathcal{L}^p(\mu)}\big)$ is a complete seminormed space. The *p-Lebesgue space* $\big(L^p(\mu), \|\cdot\|_{L^p(\mu)}\big)$ on $(\mathrm{X}, \Sigma, \mu)$ is the Banach space defined as

$$L^p(\mu) := \{[f]_\mu \in L^0(\mu) \mid f \in \mathcal{L}^p(\mu)\} = \mathcal{L}^p(\mu)/\sim_\mu$$

together with the quotient norm $\|\cdot\|_{L^p(\mu)}$, which is given as follows: for any $f \in L^p(\mu)$, one has

$$\|f\|_{L^p(\mu)} := \|\bar{f}\|_{\mathcal{L}^p(\mu)}, \quad \text{for some (thus, for any) representative } \bar{f} \in [f]_\mu.$$

Observe that $L^p(\mu)$ is also a Riesz subspace of $L^0(\mu)$ for every exponent $p \in [1, \infty)$.

We endow the space $L^0(\mu)$ with the following distance: fix a finite measure $\tilde{\mu}$ on (X, Σ) such that $\mu \ll \tilde{\mu} \ll \mu$, whose existence is guaranteed by the σ-finiteness of μ ; then we define

$$\mathsf{d}_{L^0(\mu)}(f, g) := \int |f - g| \wedge 1 \,\mathrm{d}\tilde{\mu} \quad \text{for every } f, g \in L^0(\mu).$$

Whenever μ is finite already, we implicitly choose $\tilde{\mu} := \mu$. The distance $\mathsf{d}_{L^0(\mu)}$ is complete, and the inclusion $L^p(\mu) \hookrightarrow L^0(\mu)$ is continuous with a dense image for every $p \in [1, \infty]$. Notice also that the distance $\mathsf{d}_{L^0(\mu)}$ is not canonical (since it depends on the chosen auxiliary measure $\tilde{\mu}$), but the induced topology is independent of the specific $\tilde{\mu}$. Given any $(f_n)_{n\in\mathbb{N}} \subset L^0(\mu)$ and $f \in L^0(\mu)$, it holds that $\lim_{n\to\infty} \mathsf{d}_{L^0(\mu)}(f_n,f) = 0$ if and only if there exists a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ satisfying $f(x) = \lim_{i\to\infty} f_{n_i}(x)$ for μ-a.e. $x \in \mathrm{X}$.

The following result is well-known:

Proposition 4.3 *Let (X,Σ,μ) be a σ-finite measure space. Then the space $L^0(\mu)$ is Dedekind complete and CSP. In particular, $L^p(\mu)$ is Dedekind complete and CSP for every $p \in [1,\infty]$.*

Proof. Thanks to [16, 241Y(e)] we know that $L^0(\mu)$ is CSP, which implies that $L^p(\mu)$ is CSP for all $p \in [1,\infty]$. The Dedekind completeness of $L^0(\mu)$ (and thus of $L^p(\mu)$ for all $p \in [1,\infty]$) follows from [16, 211L(c), 211L(d) and 241G]. □

In fact, if $p \in [1,\infty)$, then $L^p(\mu)$ is Dedekind complete for any (possibly non- σ-finite) measure space (X,Σ,μ) ; cf. with [16, 244L].

4.1.3. Submodular outer measures

Given a metric space (X,d), we denote by $\mathscr{B}(\mathrm{X})$ its Borel σ-algebra. If μ is an outer measure on the set X, then we say that:

- μ is *boundedly finite* if $\mu(B) < +\infty$ whenever $B \in \mathscr{B}(\mathrm{X})$ is bounded.
- μ is *submodular* (see, for example, [12, p. 16]) if it verifies

$$\mu(E\cup F) + \mu(E\cap F) \le \mu(E) + \mu(F) \quad \text{for every } E,F \in \mathscr{B}(\mathrm{X}).$$

The integral of a Borel function $f: \mathrm{X} \to [0,+\infty]$ with respect to an outer measure μ on X can be defined via Cavalieri's formula, in the following way:

$$\int_E f\,\mathrm{d}\mu := \int_0^{+\infty} \mu(\{x\in E : f(x) > t\})\,\mathrm{d}t \quad \text{for every } E \in \mathscr{B}(\mathrm{X}).$$

Observe that the above integral is well-defined because $[0,+\infty) \ni t \mapsto \mu(\{x\in E : f(x) > t\})$ is a non-increasing function. As proved in [12, Chapter 6] (see also [11, Theorem 1.5]), a given outer measure μ is submodular if and only if the associated integral is subadditive, meaning that

$$\int_{\mathrm{X}} f+g\,\mathrm{d}\mu \le \int_{\mathrm{X}} f\,\mathrm{d}\mu + \int_{\mathrm{X}} g\,\mathrm{d}\mu \quad \text{for every } f,g: \mathrm{X} \to [0,+\infty] \text{Borel.}$$

Two classes of boundedly finite, submodular outer measures are particularly relevant to us:

- The outer measure induced (via Carathéodory construction) by a boundedly finite Borel measure on X.
- The Sobolev p-capacity Cap_p on a metric measure space, see, for example, [32].

Given any boundedly finite, submodular outer measure μ on (X,d), we introduce a distance $\mathsf{d}_{L^0(\mu)}$ as follows: we fix an increasing sequence $(B_n)_n$ of bounded open subsets of X with the property that each bounded subset B of X is contained in B_n for some $n \in \mathbb{N}$, and we define

$$\mathsf{d}_{L^0(\mu)}(f,g) := \sum_{n\in\mathbb{N}} \frac{1}{2^n(\mu(B_n)\vee 1)} \int_{B_n} |f-g|\wedge 1\,\mathrm{d}\mu \quad \text{for every } f,g \in L^0(\mu).$$

The submodularity of μ guarantees that $\mathsf{d}_{L^0(\mu)}$ is a distance. Arguing as in [12, Proposition 1.10], one can see that $(f_i)_{i\in\mathbb{N}} \subset L^0(\mu)$ satisfies $\mathsf{d}_{L^0(\mu)}(f_i,f) \to 0$ for some $f \in L^0(\mu)$ if and only if

$$\lim_{i\to\infty} \mu\big(B \cap \{|f_i - f| > \varepsilon\}\big) = 0 \quad \text{for every } \varepsilon > 0 \text{ and } B \in \mathscr{B}(\mathrm{X}) \text{ bounded.}$$

In particular, arguing as in [11, Proposition 1.12] one can see that if $\mathsf{d}_{L^0(\mu)}(f_i,f) \to 0$, then there exists a subsequence $(i_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ such that $f(x) = \lim_j f_{i_j}(x)$ holds for μ-a.e. $x \in \mathrm{X}$. The converse implication—different from what happens with boundedly finite Borel measures—might fail (see, for example, [11, Remark 1.13]).

Example 4.4 Consider the real line $\mathbb{R}$ equipped with the Sobolev 2-capacity Cap_2. Since all singletons have positive capacity, we have $\mathcal{N}_{\mathrm{Cap}_2} = \{\emptyset\}$, and thus $L^0(\mathrm{Cap}_2) = \mathcal{L}^0(\mathscr{B}(\mathbb{R}))$. Then

$$L^0(\mathrm{Cap}_2) \quad \text{is neither Dedekind complete nor CSP.}$$

Indeed, the set $\{\mathbb{1}_{\{t\}} : t \in \mathbb{R}\} \subset L^0(\mathrm{Cap}_2)$ has an upper bound and $\sup_{t\in\mathbb{R}} \mathbb{1}_{\{t\}} = \mathbb{1}_{\mathbb{R}}$, but whenever $C \subset \mathbb{R}$ is a countable set we have $\sup_{t\in C} \mathbb{1}_{\{t\}} = \mathbb{1}_C \neq \mathbb{1}_{\mathbb{R}}$. Also, since $L^0(\mathrm{Cap}_2)$ is the space of (Cap_2-a.e. equivalence classes of) Borel functions from $\mathbb{R}$ to $\mathbb{R}$, we have that $L^0(\mathrm{Cap}_2)$ is not even Dedekind complete: given any $Q \subset \mathbb{R}$ that is not Borel, we have that $\{\mathbb{1}_{\{t\}} : t \in Q\}$ has an upper bound, but does not admit a supremum (which ought to be $\mathbb{1}_Q$) in $L^0(\mathrm{Cap}_2)$.

4.2. Normed modules over function spaces

4.2.1. Examples of metric f-algebras

Some examples of metric f-algebras:

- Let $(\mathrm{X},\Sigma,\mathcal{N})$ be an enhanced measurable space. Then $\big(L^\infty(\mathcal{N}),\|\cdot\|_{L^\infty(\mathcal{N})}\big)$ is a metric f-algebra. If $\mathcal{N} = \mathcal{N}_\mu$ for some σ-finite measure μ on (X,Σ), then $L^\infty(\mu)$ is Dedekind complete and CSP.
- Let (X,Σ,μ) be a σ-finite measure space. Then $\big(L^0(\mu),\mathsf{d}_{L^0(\mu)}\big)$ is a Dedekind complete CSP metric f-algebra.
- Let μ be a boundedly finite, submodular outer measure on some metric space (X,d). Then $\big(L^0(\mu),\mathsf{d}_{L^0(\mu)}\big)$ is a metric f-algebra.

Notice also that if (X,Σ,μ) is a σ-finite measure space and $p \in [1,\infty)$, then $\big(L^p(\mu),\|\cdot\|_{L^p(\mu)}\big)$ is a Dedekind complete CSP metric Riesz space.

4.2.2. Examples of metric f-structures

Below we list some important examples of metric f-structures:

- Let $(\mathrm{X},\Sigma,\mathcal{N})$ be an enhanced measurable space. Then

$$\big(L^\infty(\mathcal{N}),L^\infty(\mathcal{N}),L^\infty(\mathcal{N})\big)$$

  is a metric f-structure. It is Dedekind complete and CSP if $\mathcal{N} = \mathcal{N}_\mu$ for some σ-finite measure μ on (X,Σ).
- Let (X,Σ,μ) be a σ-finite measure space and $p \in [1,\infty)$. Then

$$\big(L^0(\mu),L^0(\mu),L^0(\mu)\big), \quad \big(L^0(\mu),L^\infty(\mu),L^p(\mu)\big)$$

  are Dedekind complete CSP metric f-structures.

- Let μ be a boundedly finite, submodular outer measure on some metric space (X, d). Then
$$(L^0(\mu), L^0(\mu), L^0(\mu))$$
is a metric f-structure.

Let us point out that the example of $(L^0(\mu), L^0(\mu), L^0(\mu))$ is one of the reasons why we consider *metric* rather than just *normed* f-structures (that is, where the distance is induced by a norm).

4.2.3. *Examples of dual systems*

Some examples of dual systems of metric f-structures:

- Let $(\mathrm{X}, \Sigma, \mathcal{N})$ be an enhanced measurable space. Then
$$(L^\infty(\mathcal{N}), L^\infty(\mathcal{N}), L^\infty(\mathcal{N}), L^\infty(\mathcal{N}), L^\infty(\mathcal{N}))$$
is a dual system of metric f-structures. It is a CSP complete dual system if $\mathcal{N} = \mathcal{N}_\mu$ for some σ-finite measure μ.
- Let $(\mathrm{X}, \Sigma, \mu)$ be a σ-finite measure space. Fix $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then
$$(L^0(\mu), L^0(\mu), L^0(\mu), L^0(\mu), L^0(\mu)), \quad (L^0(\mu), L^\infty(\mu), L^p(\mu), L^q(\mu), L^r(\mu))$$
are Dedekind complete CSP metric f-structures.
- Let μ be a boundedly finite, submodular outer measure on some metric space (X, d). Then
$$(L^0(\mu), L^0(\mu), L^0(\mu), L^0(\mu), L^0(\mu))$$
is a dual system of metric f-structures.

4.2.4. *Examples of functional normed modules*

Below we list some classes of Banach modules that have been studied in the literature and fall into the category of Banach modules over a metric f-structure:

- $L^p(\mu)$-Banach $L^\infty(\mu)$-modules, where $(\mathrm{X}, \Sigma, \mu)$ is a σ-finite measure space and $p \in [1, \infty]$, which have been introduced in [**20**, Definition 1.2.10]. We point out that in [**20**] the terminology is different: every $L^p(\mu)$-normed $L^\infty(\mu)$-module is assumed to be complete (thus a Banach module with our definitions), and non-complete normed modules are not considered.
- $L^0(\mu)$-Banach $L^0(\mu)$-modules, where $(\mathrm{X}, \Sigma, \mu)$ is a σ-finite measure space. The definition was given in [**19**, Definition 2.6], but the concept appeared previously in [**20**, Section 1.3].
- $L^0(\mu)$-Banach $L^0(\mu)$-modules, where μ is a boundedly-finite, submodular outer measure on a metric space (X, d), see [**7**, Definition 2.1]. In the particular case where μ is the Sobolev 2-capacity on a metric measure space, it appeared previously in [**11**, Definition 3.1].
- $\mathcal{L}^\infty(\Sigma)$-Banach $\mathcal{L}^\infty(\Sigma)$-modules, where (X, Σ) is a measurable space. See [**13**, Definition 3.1].
- $L^\infty(\mathcal{N})$-Banach $L^\infty(\mathcal{N})$-modules, where $(\mathrm{X}, \Sigma, \mathcal{N})$ is an enhanced measurable space, which were introduced in [**21**, Definition 4.3].

Notice that, given a σ-finite measure space $(\mathrm{X}, \Sigma, \mu)$ and an $L^\infty(\mu)$-Banach $L^\infty(\mu)$-module $\mathscr{M}$, two different notions of duals of $\mathscr{M}$ have been considered (corresponding to two different underlying dual systems of metric f-structures):

- The dual of $\mathscr{M}$ in the sense of [**20**, Definition 1.2.6] is an $L^1(\mu)$-Banach $L^\infty(\mu)$-module, since the dual system under consideration is $(L^0(\mu), L^\infty(\mu), L^\infty(\mu), L^1(\mu), L^1(\mu))$.
- The dual of $\mathscr{M}$ in the sense of [**21**, Definition 4.15] is an $L^\infty(\mu)$-Banach $L^\infty(\mu)$-module, since the dual system under consideration is $(L^\infty(\mu), L^\infty(\mu), L^\infty(\mu), L^\infty(\mu), L^\infty(\mu))$.

4.2.5. Some applications

We list some examples of known constructions that follow from Theorems 3.19, 3.16 and 3.23:

- COTANGENT MODULE. Let $(\mathrm{X},\mathsf{d},\mu)$ be a metric measure space and $p\in(1,\infty)$. We denote by $W^{1,p}(\mathrm{X})$ the p-Sobolev space of $(\mathrm{X},\mathsf{d},\mu)$ and by $|Df|\in L^p(\mu)$ the minimal weak upper gradient of $f\in W^{1,p}(\mathrm{X})$ (for example, in the sense of [4, 10, 38]; all these approaches are equivalent by [3]). Consider the metric f-structure $(L^0(\mu),L^\infty(\mu),L^p(\mu))$, as well as the map $\psi_p\colon W^{1,p}(\mathrm{X})\to L^p(\mathfrak{m})^+$ given by $\psi_p(f):=|Df|$ for every $f\in W^{1,p}(\mathrm{X})$. Then the *cotangent module* $L^p(T^*\mathrm{X})$ and the *differential* operator $\mathrm{d}\colon W^{1,p}(\mathrm{X})\to L^p(T^*\mathrm{X})$ are

$$(L^p(T^*\mathrm{X}),\mathrm{d})\cong(\mathscr{M}_{\langle\psi_p\rangle},T_{\langle\psi_p\rangle}).$$

  The cotangent module (for $p=2$) has been introduced in [20, Definition 2.2.1] and refined in [19, Theorem/Definition 2.8]. Many other generalizations appeared later: for example, one can drop the L^p-integrability assumption (see [23, Proposition 4.18]), one can construct the capacitary tangent module on an $\mathsf{RCD}(K,\infty)$ space (see [11, Theorem 3.6]) or one can consider the cotangent modules induced by axiomatic classes of Sobolev-type spaces (see [22, Theorem 3.2]). Concerning the latter notion, we point out that—thanks to Theorem 3.19—the strong locality assumption on the D-structure in [22, Theorem 3.2] can be removed.
- PULLBACK MODULE. Let $(\mathrm{X},\Sigma_\mathrm{X},\mu_\mathrm{X})$, $(\mathrm{Y},\Sigma_\mathrm{Y},\mu_\mathrm{Y})$ be σ-finite measure spaces and $\varphi\colon\mathrm{X}\to\mathrm{Y}$ a map of bounded compression, that is, φ is measurable and there exists a constant $C>0$ such that $\varphi_\#\mu_\mathrm{X}\le C\mu_\mathrm{Y}$. Notice that φ induces a homomorphism of metric f-structures

$$\boldsymbol{\varphi}\colon(L^0(\mu_\mathrm{Y}),L^\infty(\mu_\mathrm{Y}),L^p(\mu_\mathrm{Y}))\to(L^0(\mu_\mathrm{X}),L^\infty(\mu_\mathrm{X}),L^p(\mu_\mathrm{X}))$$

  for every exponent $p\in[1,\infty)$ via pre-composition. Namely, given any $f\in L^0(\mu_\mathrm{Y})$ we define

$$\boldsymbol{\varphi}(f):=[\bar f\circ\varphi]_{\mu_\mathrm{X}},\quad\text{for any }\bar f\in\mathcal{L}^0(\Sigma_\mathrm{Y})\text{with }[\bar f]_{\mu_\mathrm{Y}}=f.$$

  Let $\mathscr{M}$ be an $L^p(\mu_\mathrm{Y})$-Banach $L^\infty(\mu_\mathrm{Y})$-module. Then the *pullback module* is given by

$$(\varphi^*\mathscr{M},\varphi^*)\cong(\boldsymbol{\varphi}_*\mathscr{M},\boldsymbol{\varphi}_*).$$

  The pullback module was introduced in [20, Definition 1.6.2], [19, Theorem/Definition 2.23] and has many generalizations: for example, for L^0-Banach L^0-modules and under the weaker assumption $\varphi_\#\mu_\mathrm{X}\ll\mu_\mathrm{Y}$ (see [24, Theorem/Definition 3.2]) or for L^∞-Banach L^∞-modules (see [21, Theorem/Definition 4.11]).
- L^0-COMPLETION. Let (X,Σ,μ) be a σ-finite measure space, and let $\mathscr{M}$ be an $L^p(\mu)$-Banach $L^\infty(\mu)$-module, for some exponent $p\in[1,\infty]$. Then the L^0*-completion* of $\mathscr{M}$ (in the sense of [19, Theorem/Definition 1.7]) is given by

$$(\bar{\mathscr{M}},\iota)\cong(\mathscr{M}_{\langle\psi_\mathscr{M}\rangle},T_{\langle\psi_\mathscr{M}\rangle}),$$

  where we define $\psi_\mathscr{M}\colon\mathscr{M}\to L^0(\mu)^+$ as $\psi_\mathscr{M}(v):=|v|$ for every $v\in\mathscr{M}$.
- VON NEUMANN LIFTING. Let (X,Σ,μ) be a complete σ-finite measure space, ℓ a von Neumann lifting of μ and $\mathscr{M}$ an $L^\infty(\mu)$-Banach $L^\infty(\mu)$-module. Then the *von Neumann lifting* of $\mathscr{M}$ (see [13, Theorem 3.5]) is given by

$$(\ell\mathscr{M},\ell)\cong(\mathscr{M}_{\langle\psi_\ell\rangle},T_{\langle\psi_\ell\rangle}),$$

  where we define $\psi_\ell\colon\mathscr{M}\to\mathcal{L}^\infty(\Sigma)^+$ as $\psi_\ell(v):=\ell(|v|)$ for every $v\in\mathscr{M}$.

We point out that, in addition to the objects we discussed above, also the associated existence results for homomorphisms (for example, the universal property of pullback modules [**20**, Proposition 1.6.3] or the lifting of a homomorphism [**21**, Proposition 4.14]) can be deduced from Proposition 3.20.

4.3. The Realization Theorem

In this conclusive section of the paper, we show (see Theorem 4.7) that any localizable f-algebra can be 'realized' as a space of functions.

A *Boolean ring* is a ring $(R,+,\cdot)$ such that $r^2=r$ for every $r\in R$. In particular, $r=-r$ and $rs=sr$ for every $r,s\in R$. A *Boolean algebra* is a a Boolean ring $(A,+,\cdot)$ with a multiplicative identity 1_A. A ring homomorphism $\phi\colon A\to B$ between two Boolean algebras A and B is said to be a *Boolean homomorphism*, provided that it is also *uniferent*, meaning that $\phi(1_A)=1_B$.

Given a set $\mathrm{X}\neq\emptyset$ and an algebra Σ of subsets of X, the triple $(\Sigma,\Delta,\cap)$ is a Boolean algebra with zero $\emptyset$ and identity X. The following fundamental result—which is known as the *Stone's Representation Theorem for Boolean algebras*—states that in fact any Boolean algebra can be expressed as an algebra of sets. We will employ it in the proof of Proposition 4.6.

Theorem 4.5 (Stone's Theorem) *Let A be a Boolean algebra. Then there exist a set X and an algebra Σ of subsets of X such that $(A,+,\cdot)$ and $(\Sigma,\Delta,\cap)$ are isomorphic as Boolean algebras.*

Let U be a given f-algebra. Then we define the operations $\boxplus\colon \mathrm{Idem}(U)\times\mathrm{Idem}(U)\to\mathrm{Idem}(U)$ and $\boxtimes\colon \mathrm{Idem}(U)\times\mathrm{Idem}(U)\to\mathrm{Idem}(U)$ on $\mathrm{Idem}(U)$ as

$$u\boxplus v:=u+v-2uv,\qquad u\boxtimes v:=uv,\qquad \text{for every } u,v\in\mathrm{Idem}(U).$$

Their well-posedness follows from items (i) and (ii) of Lemma 2.10. It is easy to check that the triple $(\mathrm{Idem}(U),\boxplus,\boxtimes)$ is a Boolean algebra with zero element 0 and multiplicative identity 1.

Proposition 4.6 *Let U be a Dedekind σ-complete f-algebra whose multiplication map is σ-order-continuous on $U^+\times U^+$. Then the space $(\mathrm{Idem}(U),\boxplus,\boxtimes)$ is Boolean isomorphic to a σ-algebra.*

Proof. Thanks to Stone's Representation Theorem 4.5, we can find a set $\mathrm{X}\neq\emptyset$, an algebra Σ of subsets of X and a Boolean isomorphism $\mathrm{I}\colon(\mathrm{Idem}(U),\boxplus,\boxtimes)\to(\Sigma,\Delta,\cap)$. We claim that

$$\mathrm{I}\Big(\sup_{n\in\mathbb{N}}u_n\Big)=\bigcup_{n\in\mathbb{N}}\mathrm{I}(u_n),\qquad \text{for every } (u_n)_{n\in\mathbb{N}}\subset\mathrm{Idem}(U). \tag{52}$$

Call $u:=\sup_{n\in\mathbb{N}}u_n$. Recall that $u\in\mathrm{Idem}(U)$ by Lemma 2.17. Given any $n\in\mathbb{N}$, it holds that $u_n\leq u$, and thus Remark 2.11 gives $\mathrm{I}(u_n)\cap\mathrm{I}(u)=\mathrm{I}(u_nu)=\mathrm{I}(u_n)$, which yields $\bigcup_{n\in\mathbb{N}}\mathrm{I}(u_n)\subset\mathrm{I}(u)$. Conversely, pick any set $E\in\Sigma$ with $\mathrm{I}(u_n)\subset E$ for every $n\in\mathbb{N}$. Calling $v:=\mathrm{I}^{-1}(E)$, we have that $\mathrm{I}(u_nv)=\mathrm{I}(u_n)\cap\mathrm{I}(v)=\mathrm{I}(u_n)$, so that $u_nv=v_n$. Hence, it holds that

$$u=\sup_{n\in\mathbb{N}}u_n=\sup_{n\in\mathbb{N}}u_nv=v\sup_{n\in\mathbb{N}}u_n=uv,$$

thus $\mathrm{I}(v)\cap\mathrm{I}(u)=\mathrm{I}(uv)=\mathrm{I}(u)$. We obtain that $\mathrm{I}(u)\subset\mathrm{I}(v)=E$, whence (52) follows. We deduce that Σ is a σ-algebra, so that (X,Σ) is a measurable space. This completes the proof. □

Theorem 4.7 (Realization Theorem) *Let U be a localizable f-algebra. Then there exists a measurable space (X,Σ) such that U is isomorphic (as an f-algebra) to an f-subalgebra of $\mathcal{L}^0(\Sigma)$.*

More precisely, the measurable space (X,Σ) can be chosen so that $(\Sigma,\Delta,\cap)$ is isomorphic (as a Boolean algebra) to $(\mathrm{Idem}(U),\boxplus,\boxtimes)$.

Proof. Proposition 4.6 yields a measurable space (X,Σ) such that $(\Sigma,\Delta,\cap)$ and $(\mathrm{Idem}(U),\boxplus,\boxtimes)$ are isomorphic as Boolean algebras. We introduce the mapping $\iota\colon \mathcal{S}^+(U)\to\mathcal{L}^0(\Sigma)$ as follows: given any simple element $u=\sum_{i=1}^k \lambda_i u_i\in\mathcal{S}^+(U)$, we define the function $\iota(u)\colon \mathrm{X}\to[0,+\infty]$ as

$$\iota(u)(x):=\sum_{i=1}^k \lambda_i \mathbb{1}_{\mathrm{I}(u_i)}(x),\quad \text{for every } x\in\mathrm{X},$$

where $\mathrm{I}\colon \mathrm{Idem}(U)\to\Sigma$ is some fixed Boolean isomorphism. Observe that $\iota(u)$ belongs to $\mathsf{Sf}(\Sigma)$. In order to extend the mapping ι to U^+, we first need to prove the following two auxiliary results:

(a) If $u\in U^+$ and $(u_n)_{n\in\mathbb{N}}\subset\mathcal{S}^+(U)$ is a non-decreasing sequence satisfying $u=\sup_{n\in\mathbb{N}}u_n$, then it holds that $\sup_{n\in\mathbb{N}}\iota(u_n)(x)<+\infty$ for every $x\in\mathrm{X}$.
(b) If $u\in U^+$ and $(u_n)_{n\in\mathbb{N}},(v_n)_{n\in\mathbb{N}}\subset\mathcal{S}^+(U)$ are non-decreasing sequences with $u=\sup_{n\in\mathbb{N}}u_n$ and $u=\sup_{n\in\mathbb{N}}v_n$, then it holds that $\sup_{n\in\mathbb{N}}\iota(u_n)(x)=\sup_{n\in\mathbb{N}}\iota(v_n)(x)$ for every $x\in\mathrm{X}$.

To prove (a), we argue by contradiction: suppose that $\sup_{n\in\mathbb{N}}\iota(u_n)(x_0)=+\infty$ for some $x_0\in\mathrm{X}$. For any $n\in\mathbb{N}$, we can find $\lambda_n\in[0,+\infty)$ and $\tilde u_n\in\mathrm{Idem}(U)$ with $\lambda_n\tilde u_n=\tilde u_n u_n$ and $x_0\in\mathrm{I}(\tilde u_n)$. One has $\lambda_n=\iota(u_n)(x)\to+\infty$ as $n\to\infty$. Define $E:=\bigcap_{n\in\mathbb{N}}\mathrm{I}(\tilde u_n)\in\Sigma$ and $w:=\mathrm{I}^{-1}(E)\in\mathrm{Idem}(U)$. Notice that $x_0\in E$ and $w\leq\tilde u_n$ for every $n\in\mathbb{N}$. Given $k\in\mathbb{N}$, there is $n_k\in\mathbb{N}$ with $\lambda_{n_k}\geq k$, and thus

$$kw\leq\lambda_{n_k}\tilde u_{n_k}=\tilde u_{n_k}u_{n_k}\leq u.$$

Since U is Archimedean by Proposition 2.2, we deduce that $w=0$ and thus $E=\emptyset$. This leads to a contradiction with the fact that $x_0\in E$ so that (a) is proved. We pass to the verification of (b). Fix any point $x\in\mathrm{X}$. For any $n\in\mathbb{N}$, we can pick $\lambda_n,\mu_n\in[0,+\infty)$ and $\tilde u_n,\tilde v_n\in\mathrm{Idem}(U)$ such that $\lambda_n\tilde u_n=\tilde u_n u_n$, $\mu_n\tilde v_n=\tilde v_n v_n$ and $x\in\mathrm{I}(\tilde u_n)\cap\mathrm{I}(\tilde v_n)$. Setting $E:=\bigcap_{n\in\mathbb{N}}\mathrm{I}(\tilde u_n)\cap\mathrm{I}(\tilde v_n)\in\Sigma$, we have $x\in E$, and thus $w:=\mathrm{I}^{-1}(E)\neq 0$. By the σ-order continuity of the multiplication, we obtain

$$\begin{aligned}\Big(\sup_{n\in\mathbb{N}}\iota(u_n)(x)\Big)w&=\Big(\sup_{n\in\mathbb{N}}\lambda_n\Big)w=\sup_{n\in\mathbb{N}}\lambda_n w=\sup_{n\in\mathbb{N}}\lambda_n\tilde u_n w=\sup_{n\in\mathbb{N}}u_n\tilde u_n w=\Big(\sup_{n\in\mathbb{N}}u_n\Big)w=uw\\&=\Big(\sup_{n\in\mathbb{N}}v_n\Big)w=\sup_{n\in\mathbb{N}}\mu_n\tilde v_n w=\Big(\sup_{n\in\mathbb{N}}\mu_n\Big)w=\Big(\sup_{n\in\mathbb{N}}\iota(v_n)(x)\Big)w,\end{aligned}$$

where we used that $\tilde u_n w=\tilde v_n w=w$. This yields $\sup_{n\in\mathbb{N}}\iota(u_n)(x)=\sup_{n\in\mathbb{N}}\iota(v_n)(x)$, proving (b).

We now define the function $\iota(u)\colon\mathrm{X}\to[0,+\infty)$ for any $u\in U^+$ in the following way: given any non-decreasing sequence $(u_n)_{n\in\mathbb{N}}\subset\mathcal{S}^+(U)$ such that $u=\sup_{n\in\mathbb{N}}u_n$—whose existence is guaranteed by the assumption that the f-algebra U is localizable—we define

$$\iota(u)(x):=\sup_{n\in\mathbb{N}}\iota(u_n)(x),\quad\text{for every } x\in\mathrm{X}.$$

The properties (a) and (b) ensure that $\iota(u)$ is well-posed. Notice that $\iota(u)\in\mathcal{L}^0(\Sigma)$, as a countable supremum of elements of $\mathcal{L}^0(\Sigma)$. The σ-order continuity of the sum and multiplication maps gives

$$\iota(u)+\iota(v)=\iota(u+v),\qquad \iota(uv)=\iota(u)\iota(v),\qquad\text{for every } u,v\in U^+. \tag{53}$$

Finally, we extend ι to a mapping $\iota\colon U\to\mathcal{L}^0(\Sigma)$ by setting

$$\iota(u):=\iota(u^+)-\iota(u^-),\quad\text{for every } u\in U.$$

Using (53), one can easily show that ι is a homomorphism of f-algebras. In order to conclude, it only remains to check that ι is injective. To this aim, fix any $u \in U$ such that $\iota(u) = 0$. We want to show that $u = 0$. Since $u^+u^- = 0$ by (5a), we deduce that $\iota(u^+)\iota(u^-) = \iota(u^+u^-) = 0$, which yields $\iota(u^+) = \iota(u^-) = 0$. Hence, it suffices to prove the implication $\iota(u) = 0 \Longrightarrow u = 0$ in the case where $u \in U^+$. Choose a non-decreasing sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{S}^+(U)$ such that $u = \sup_{n\in\mathbb{N}} u_n$. We have that $\sup_{n\in\mathbb{N}} \iota(u_n) = 0$, whence it follows that $u_n = 0$ for every $n \in \mathbb{N}$ and thus $u = 0$. □

Remark 4.8 Several variants of 'realization theorems'—regarding even more general structures, such as Banach/Orlicz lattices—can be found in the literature (see [**31**]). An instance of such a result is [**31**, Theorem 3.7], which provides the following characterization: a Dedekind complete vector lattice E is isomorphic to an order-dense order-ideal in some space $L^0(\mu)$ (see the definition of Köthe function space in [**31**, p. 17]) if and only if the Boolean algebra $\mathfrak{B}(E)$ consisting of the band projections on E is a measure algebra; we refer to [**31**] for the precise definitions of the involved concepts. It would be interesting—but currently unclear and outside the scope of this paper—to understand whether Theorem 4.7 can be actually deduced from [**31**, Theorem 3.7].

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