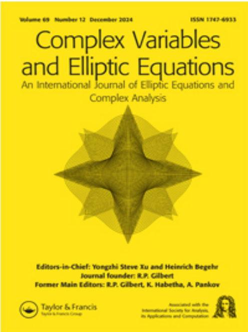

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On the singular problem involving fractional g-Laplacian

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ABSTRACT

In this paper, we show the existence of weak a solution to the equation

$$\begin{aligned} (-\Delta_g)^s u(x) &= f(x)u(x)^{-q(x)} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \end{aligned}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $q \in C^1(\bar{\Omega})$, and $(-\Delta_g)^s$ is the fractional g-Laplacian with g is the antiderivative of a Young function and f in suitable Orlicz space. This includes the mixed fractional $(p,q)-$Laplacian as a special case. The solution so obtained is also shown to be locally Hölder continuous.

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1. Introduction

Nonlocal problems have been a subject of immense interest in mathematics recently. Various studies have been published to verify if the results of the Laplace operator can be suitably generalized for problems involving fractional Laplacian $(-\Delta)^s$ and its generalization. Here and throughout the paper, s is understood to be a number in $(0,1)$, unless specified otherwise. Continuing with the spirit of recent developments in the study of nonlocal operators, in this article, we consider the following problem

$$\begin{aligned} (-\Delta_g)^s u(x) &= f(x)u(x)^{-q(x)} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{1}$$

with Ω being a smooth bounded domain in $\mathbb{R}^N$ and q is a non-negative C^1 function in $\overline{\Omega}$, and the fractional g-Laplacian operator is defined as

$$(-\Delta_g)^s u(x) := \int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right) \frac{dy}{|x-y|^{N+s}}$$

with $g : [0,\infty) \to \mathbb{R}$ is a right continuous function satisfying the following assumptions:

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(H_a) $g(0)=0$; $g(t)>0$ for $t>0$ and $\lim_{t\to+\infty} g(t)=\infty$.
(H_b) g is convex on $(0,\infty)$.
(H_c) g' is nondecreasing on $(0,\infty)$, and hence on $\mathbb{R}\setminus\{0\}$.

Given $g:\mathbb{R}\to\mathbb{R}$, we define $G:[0,\infty)\to[0,\infty)$, called the N-function or Young's function by

$$G(t):=\int_0^t g(\tau)\,\mathrm{d}\tau.$$

We also assume the following additional conditions on G and g:

(H_e) $g=G'$ is absolutely continuous, so it is differentiable almost everywhere.
(H_f) $\int_0^1 \frac{G^{-1}(\tau)}{\tau^{\frac{N+s}{N}}}\,\mathrm{d}\tau<\infty$ and $\int_1^\infty \frac{G^{-1}(\tau)}{\tau^{\frac{N+s}{N}}}d\tau=\infty$.
(H_g) There exist p^+,p^- such that

$$1<p^- -1\le \frac{tg'(t)}{g(t)}\le p^+ -1\le\infty \quad t>0.$$

Note that we will always be assuming conditions (H_a)–(H_g) on g and G throughout the whole paper until otherwise specified. In literature, G is known as a Young function or an N-function.

Remark 1.1: The following examples of G fits our framework:

(i) $G_p(t):=\frac{1}{p}t^p$, where $p\ge 2$.
(ii) If one takes $G_{p_1,p_2}(t):=\frac{1}{p_1}t^{p_1}+\frac{1}{p_2}t^{p_2}$, where $p_1,p_2\ge 2$. One gets,

$$(-\Delta_{g_{p_1,p_2}})^s=(-\Delta_{p_1})^s+(-\Delta_{p_2})^s.$$

(iii) For $a,b,c>0$ and $g(t)=t^a\log(b+ct)$ we get,

$$G(t)=\frac{t^{1+a}}{(1+a)^2}\left[H_1^2\left(1+a,1,2+a,-\frac{ct}{b}\right)+(1+a)\log(b+ct)-1\right]$$

with $p^-=1+a$, $p^+=2+a$, where H_1^2 is a hyper geometric function.

Before we start with the preliminaries, let us briefly recall some related literature concerning the singular problems. Singular problems have a long history starting from the seminal work of Crandall–Rabinowitz–Tartar [1], where for a suitably regular f, the problem $-\Delta u=f(x)u^{-\delta}$ was considered in a bounded domain and is shown to admit a classical solution irrespective of the sign of $\delta>0$, subject to Dirichlet boundary condition. The classical solution so obtained was shown to be the weak solution provided $0<\delta<3$ in another celebrated work of Lazer–Mckenna [2]. Singularly perturbed problems were also studied in [3, 4] and the reference therein. The case of $f\in L^p(\Omega)$, $p\ge 1$ was first treated in Boccardo–Orsina [5], who showed the existence and regularity results for different cases of m and δ. One may find the p-Laplace generalization of Boccardo–Orsina's work in Scuinzi et al. [6], where the delicate issue of uniqueness was also addressed. Anisotropic Laplacians with singular nonlinearities have also been dealt with in several papers, see [7–9] to name a few. In [10], the fractional problem given by

$$\begin{aligned} (-\Delta)^s u(x) &= \lambda f(x)u(x)^{-\gamma} + Mu^p \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{2}$$

was first considered under the condition that $n > 2s$, $M \geq 0$, $0 < s < 1$, $1 < p < 2_s^* - 1$ and shown to admit a distributional solution for $f \in L^m(\Omega)$ and $\lambda > 0$ small. In [11], the authors studied the problem

$$\begin{aligned} (-\Delta_p)^s u(x) &= f(x)u(x)^{-\gamma} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

and proved the existence and uniqueness results. The first instance, to the best of our knowledge of studying variable exponent singularities was in Carmona–Martínez–Aparicio [12] where the rather surprising phenomenon of the restriction of the nonlinearity to 1 near the boundary of the domain for a weak solution was studied in contrast to the constant exponent where such restriction is imposed on the whole domain. Similar problems involving fractional p-Laplacian with variable exponent may be found in Garain–Mukherjee [13], Giacomoni–Mukherjee–Sreenadh [14] and Mukherjee–Sreenadh [15]. However, by considering the Orlicz setup, we can address a large class of interesting problems in one go.

In Section 2, we recall some preliminary results which are already known in literature. In Section 3, we state and prove our main results Theorems 3.2, 3.3, and 3.4.

2. Preliminaries

Let us start by introducing the reader to the functional setup related to the fractional Orlicz–Sobolev spaces. A detailed discussion can be found in [16–18]. Throughout the section, we shall assume Ω to be a bounded domain and $s \in (0,1)$. Throughout the rest of the article, C will stand for a generic constant, which may vary in each of its appearances. First, we define the modular functions:

$$M_{L^G(\Omega)}(f) := \int_\Omega G(|f(x)|)\,\mathrm{d}x \quad \text{and} \quad M_{W^{s,G}(\Omega)}(f) := \int_\Omega \int_\Omega G\left(\frac{|f(x)-f(y)|}{|x-y|^s}\right) \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}.$$

The Banach space

$$L^G(\Omega) := \left\{ f : \Omega \to \mathbb{R} \text{ measurable} \;\middle|\; \exists\, \lambda > 0 \text{ such that } M_{L^G(\Omega)}\left(\frac{f}{\lambda}\right) < \infty \right\}$$

is called the Orlicz space. This space is equipped with the norm

$$\|f\|_{L^G(\Omega)} := \inf\left\{ \lambda > 0 \;\middle|\; M_{L^G(\Omega)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

The infimum in the above definition is known to be achieved. The fractional Orlicz–Sobolev spaces are defined as

$$W^{s,G}(\Omega) := \left\{ f \in L^G(\Omega) \;\middle|\; \exists\, \lambda > 0 \text{ such that } M_{W^{s,G}(\Omega)}\left(\frac{f}{\lambda}\right) < \infty \right\}.$$

This space is equipped with the seminorm

$$\|f\|_{W^{s,G}(\Omega)} := \inf\left\{ \lambda > 0 \;\middle|\; M_{W^{s,G}(\Omega)}\left(\frac{f}{\lambda}\right) \le 1 \right\}.$$

However, we shall mainly be working with the spaces defined by

$$\hat{W}^{s,G}(\Omega) := \Bigg\{ f \in L^G_{loc}(\mathbb{R}^N) : \exists\, U \Subset \Omega \quad \text{s.t } \|f\|_{W^{s,G}(U)} + \int_{\mathbb{R}^N} g\left(\frac{|f(x)|}{1+|x|^s}\right) \frac{\mathrm{d}x}{(1+|x|)^{n+s}} < \infty \Bigg\}$$

and,

$$W_0^{s,G}(\Omega) := \left\{ f \in W^{s,G}(\mathbb{R}^N) \;\middle|\; f \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega \right\}.$$

It is clear that $W_0^{s,G}(\Omega) \subseteq \hat{W}^{s,G}(\Omega)$. We equip $W_0^{s,G}(\Omega)$ with the norm $\|\cdot\|_{W^{s,G}(\mathbb{R}^N)}$. Note that for $G(t) = t^p$; $1 < p < \infty$, $L^G(\Omega)$ and $W^{s,G}(\Omega)$ are well known Lebesgue space $L^p(\Omega)$ and the fractional Sobolev space $W^{s,p}(\Omega)$ respectively (see [19, p. 524]).

We now discuss some properties of these spaces which we shall use in the next section. We start by observing that the assumption (H_g) implies

$$2 < p^- \le \frac{tg(t)}{G(t)} \le p^+ < \infty, \quad t > 0. \tag{3}$$

To see this, note that assumption (H_g) implies $(tg(t))' \le p^+ G(t)'$. The following two lemmas will be used frequently in the rest of the article.

Lemma 2.1: *Let G be an N-function, and let $g = G'$ satisfy (H_a)–(H_g). Then*

$$\lambda^{p^-} G(t) \le G(\lambda t) \le \lambda^{p^+} G(t) \quad \forall\, \lambda \ge 1, \quad \forall\, t > 0, \tag{4}$$

where p^+, p^- is the constant as defined in (H_g). The above inequality is equivalent to

$$\lambda^{p^-} G(t) \ge G(\lambda t) \ge \lambda^{p^+} G(t) \quad \forall\, 0 \le \lambda \le 1, \quad \forall\, t > 0.$$

Proof: For any $\lambda > 1$,

$$\log(\lambda^{p^-}) = \int_t^{\lambda t} \frac{p^-}{\tau}\,\mathrm{d}\tau \le \int_t^{\lambda t} \frac{g(\tau)}{G(\tau)}\,\mathrm{d}\tau \le \int_t^{\lambda t} \frac{p^+}{\tau}\,\mathrm{d}\tau = \log(\lambda^{p^+}).$$

This implies

$$\log(\lambda^{p^-}) \le \log\left(\frac{G(\lambda t)}{G(t)}\right) \le \log\left(\lambda^{p^+}\right).$$

The lemma follows. ∎

An immediate consequence of Lemma 2.1 is the following

Lemma 2.2: *When* $\|f\|_{W^{s,G}(\Omega)} \le 1$,

$$\|f\|^{p^+}_{W^{s,G}(\Omega)} \le M_{W^{s,G}(\Omega)}(f) \le \|f\|^{p^-}_{W^{s,G}(\Omega)},$$

and when $\|f\|_{W^{s,G}(\Omega)} \ge 1$,

$$\|f\|^{p^-}_{W^{s,G}(\Omega)} \le M_{W^{s,G}(\Omega)}(f) \le \|f\|^{p^+}_{W^{s,G}(\Omega)}.$$

Lemma 2.3: *Let G be an N-function satisfying* (H_a)–(H_g). *For any two real numbers a and b, we have*

$$(g(b) - g(a))(b - a) \ge C(G)\, G(|b - a|).$$

for some constant C depending on the N-function G.

Proof: By the symmetry of the inequality, it is enough to prove this lemma for the cases $0 < a \le b$ and $a < 0 < b$. In the first case, using Taylor's theorem with an integral form of reminder, we have

$$\begin{aligned} G(|b-a|) &= G(0) + g(0)|b-a| + \frac{1}{2}\int_0^{b-a} g'(t)(b-a-t)\,\mathrm{d}t \\ &= \frac{b-a}{2}\int_a^b g'(t-a)\frac{b-t}{b-a}\,\mathrm{d}t \le \frac{b-a}{2}\int_a^b g'(t)\,\mathrm{d}t \\ &= \frac{(b-a)(g(b)-g(a))}{2} \end{aligned}$$

the case $0 \ge a \ge b$ follows similarly.

Now suppose $a < 0 < b$. Using convexity of G, we get

$$\begin{aligned} G\left(\frac{|b-a|}{2}\right) &= G\left(\frac{b+(-a)}{2}\right) \le \frac{1}{2}(G(b) + G(-a)) \le \frac{1}{2}\left(\frac{bg(b)}{p^-} + \frac{(-a)g(-a)}{p^-}\right) \\ &\le \frac{1}{2p^-}(bg(b) + ag(a) - ag(b) - bg(a)) = \frac{1}{2p^-}(g(b)-g(a))(b-a). \end{aligned}$$

■

Definition 2.4: Let G be an N-function.

(1) The N-function $\overline{G}$ is called the conjugate of G, and is defined by

$$\overline{G}(t) := \int_0^t \overline{g}(\tau)\,\mathrm{d}\tau,$$

where

$$\overline{g}(t) := \sup\left\{\tau \,|\, g(\tau) \le t\right\}.$$

(2) The N-function G_*, defined by

$$G_*^{-1}(t) := \int_0^t \frac{G^{-1}(\tau)}{\tau^{\frac{N+s}{N}}}\, \mathrm{d}\tau,$$

is called the Sobolev conjugate of G.

(3) G is said to be essentially stronger than an N-function H, written as $H \prec\prec G$, if for any $k > 0$,

$$\lim_{t\to\infty} \frac{H(kt)}{G(t)} = 0.$$

Lemma 2.5 (Hölder Inequality): *Let G be an N-function, $N \geq 1$, and $\Omega \subseteq \mathbb{R}^N$. Then, we have for any $u, v : \Omega \to \mathbb{R}$,*

$$\int_\Omega |uv| \leq \|u\|_{L^G(\Omega)} \|v\|_{L^{\overline{G}}(\Omega)}$$

An N-function G is said to satisfy the Δ_2-condition if G is doubling, equivalently if the second inequality in Equation (4) is satisfied.

Lemma 2.6 ([18, Corollary 6.2]): *Let G be an N-function which satisfy the Δ_2-condition. Then there exists a constant $C = C(n, G, \Omega)$ such that for any $u \in W_0^{s,G}(\Omega)$,*

$$\int_\Omega G(u(x))\, \mathrm{d}x \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G\left(\frac{u(x) - u(y)}{|x-y|^s}\right) \frac{\mathrm{d}x\, \mathrm{d}y}{|x-y|^N}.$$

Lemma 2.7 ([20, Theorems 1 and 2] and Lemma 2.6): *Let G be an N-function, and let Ω be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity. Then we have the following:*

(1) *the embedding $W_0^{s,G}(\Omega) \to L^{G_*}(\Omega)$, is continuous.*
(2) *Moreover, for any N-function H, the embedding $W_0^{s,G}(\Omega) \to L^H(\Omega)$ is compact if $H \prec\prec G$.*

The above result is not optimal. For recent developments in this direction, we refer the reader to [21], especially to Sections 6 and 9 there.

Lemma 2.8 (Weak Harnack Inequality, [22, Theorem 3.2]): *If $u \in \hat{W}^{s,G}(B_{3^{-1}R})$ satisfies weakly*

$$\begin{cases} (-\Delta_g)^s u(x) \geq 0 & \text{if } x \in \Omega \\ u(x) \geq 0 & \text{if } x \in \mathbb{R}^N \end{cases}$$

then there exists $\sigma \in (0,1)$ such that

$$\inf_{B_{4^{-1}R}} u \geq \sigma R^s g^{-1}\left(\fint_{B_R \setminus B_{2^{-1}R}} g(R^{-s}|u|)\, \mathrm{d}x \right).$$

3. Main results

We begin this section by stating the definition of our weak solution.

Definition 3.1 (Weak solutions): The function $u \in \hat{W}^{s,G}(\Omega)$ is said to be a weak solution of Equation (1) if $u > 0$ in Ω, and for any $\varphi \in C_c^\infty(\Omega)$ one has, $\frac{f}{u^{q(\cdot)}} \in L^1_{\text{loc}}(\Omega)$ and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x)u(x)^{-q(x)}\phi(x)\,\mathrm{d}x. \quad (5)$$

The boundary condition is understood in the sense that

(1) if $q(x) \leq 1$ on $\Omega_\delta := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \delta\}$, then $u \in W_0^{s,G}(\Omega)$.
(2) Elsewhere one has, $\Phi(u) \in W_0^{s,G}(\Omega)$, where

$$\Phi(t) := \int_0^t G^{-1}\left(G(1)\tau^{q^*-1}\right)\,\mathrm{d}\tau.$$

Furthermore, we say that u is a subsolution (or supersolution) of Equation (1) if, for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \leq \ (\text{or } \geq)\ \int_\Omega f(x)u(x)^{-q(x)}\varphi(x)\,\mathrm{d}x. \quad (6)$$

We are now ready to state our main results.

Theorem 3.2: *Let there exist $\delta > 0$ such that $q(x) \leq 1$ on $\Omega_\delta := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \delta\}$ and $f \in L^{\overline{G_*}}(\Omega)$. Then Equation (1) has a weak solution in $W_0^{s,G}(\Omega)$ with $\text{essinf}_K u > 0$ for any $K \Subset \Omega$.*

Theorem 3.3: *Let g is sub-multiplicative and there exist $q^* > 1$, $\delta > 0$ such that $\|q\|_{L^\infty(\Omega_\delta)} \leq q^*$ and let*

$$H(t) := G_*\left(t^{\frac{p^-+q^*-1}{p^-q^*}}\right)$$

be an N-function such that $f \in L^{\overline{H}}(\Omega)$. Then Equation (1) has a weak solution $u \in W_{loc}^{s,G}(\Omega)$ with $\text{essinf}_K u > 0$ for any $K \Subset \Omega$ such that $\Phi(u) \in W_0^{s,G}(\Omega)$, where

$$\Phi(t) := \int_0^t G^{-1}\left(G(1)\tau^{q^*-1}\right)\,\mathrm{d}\tau.$$

Theorem 3.4: *Every weak solution of Equation (1) obtained through Theorems 3.2 and 3.3 belongs to $C^\alpha_{loc}(\Omega)$ for some $\alpha \in (0,1)$.*

In order to prove Theorems 3.2, 3.3, and 3.4, we first need to develop some results which are needed in the proves. The first result is a comparison principle.

Lemma 3.5 (Comparison Principle): *Let $u, v \in C(\mathbb{R}^N)$ with $[u]_{W^{s,G}(\mathbb{R}^N)}, [v]_{W^{s,G}(\mathbb{R}^N)} < \infty$, and $D \subseteq \mathbb{R}^N$ be a domain such that $|\mathbb{R}^N \setminus D| > 0$. If $v \geq u$ in $\mathbb{R}^N \setminus D$, and*

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \\ \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for $\varphi = (u-v)^+$, then $v \geq u$ in $\mathbb{R}^N$.

Proof: We need to show that $v \geq u$ in D. The two integrals can be shown to be finite using Hölder's inequality and the assumptions on g. Then we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)-g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\right]\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \geq 0.$$

This, and the identity

$$g(t_2)-g(t_1) = (t_2-t_1)\int_0^1 g'((t_2-t_1)\tau + t_1)\,\mathrm{d}\tau,$$

gives

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big(v(x)-v(y)-u(x)+u(y)\big)\,Q(x,y)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \geq 0, \tag{7}$$

where

$$Q(x,y) := \int_0^1 g'\left(\frac{(v(x)-v(y)-u(x)+u(y))\tau + u(x)-u(y)}{|x-y|^s}\right)\mathrm{d}\tau.$$

From the assumption on g, we know $g' \geq 0$. So $Q(x,y) \geq 0$, and $Q(x,y) = 0$ if and only if the integrand is identically zero. Again this happens if and only if $v(x) = v(y)$ and $u(x) = u(y)$.

Choose $\varphi = (u-v)^+$ and $\psi := u - v$. Equation (7), then, becomes

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} Q(x,y)\frac{(\varphi(x)-\varphi(y))(\psi(y)-\psi(x))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \geq 0. \tag{8}$$

We can see that, after choosing $\varphi := (u-v)^+$, and using the fact that $\psi^+(y)\psi^-(y) = 0$,

$$\begin{aligned}&(\varphi(x)-\varphi(y))(\psi(y)-\psi(x)) \\ &\quad = -(\psi^+(x)-\psi^+(y))^2 - \psi^-(y)\psi^+(x) - \psi^-(x)\psi^+(y) \leq 0.\end{aligned}$$

This along with Equation (8), and the fact that $Q(x,y) \geq 0$ implies $Q(x,y) = 0$ or $-(\psi^+(x)-\psi^+(y))^2 - \psi^-(y)\psi^+(x) - \psi^-(x)\psi^+(y) = 0$ almost everywhere. In both the cases, we must have $\psi^+(x) = \psi^+(y)$ for a.e. (x,y). Since $(u-v)^+ = 0$ on $\mathbb{R}^N \setminus D$, by continuity of u, v, we conclude that $\psi^+ = 0$ on $\mathbb{R}^N$. This implies $v \geq u$ on $\mathbb{R}^N$. ■

Lemma 3.6: *Let g be sub-multiplicative, that is, there is a constant $C > 0$ for which $Cg(t_1 t_2) \leq g(t_1)g(t_2)$ for any $t_1, t_2 > 0$. Let F and u be such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega F\varphi\,\mathrm{d}x,$$

for any $\varphi \in W_0^{s,G}(\Omega)$. Then for any convex and Lipschitz function Φ, we have

$$\int_\Omega F(x)g(\Phi'(u(x)))\Phi(u)\,\mathrm{d}x \geq C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|\Phi(u(x))-\Phi(u(y))|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}.$$

Proof: First, note that, by density argument, we can assume Φ to be C^1. Choose $\varphi = g(\Phi'(u))\psi$. Then we have

$$\begin{aligned}
2\iint_{\{u(x)>u(y)\}} & g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{g(\Phi'(u(x)))\psi(x)-g(\Phi'(u(y)))\psi(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{g(\Phi'(u(x)))\psi(x)-g(\Phi'(u(y)))\psi(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_\Omega F(x)g(\Phi'(u(x)))\psi(x)\,\mathrm{d}x.
\end{aligned}$$

Set $u(x) = a$, $u(y) = b$, $\psi(x) = A$ and $\psi(y) = B$. Then the integrand in the LHS becomes

$$g\left(\frac{a-b}{|x-y|^s}\right)\frac{g(\Phi'(a))A-g(\Phi'(b))B}{|x-y|^{N+s}}.$$

Using the convexity of Φ, we have

$$\Phi(a)-\Phi(b) \leq \Phi'(a)(a-b) \quad \text{and} \quad \Phi(a)-\Phi(b) \geq \Phi'(b)(a-b).$$

We then have

$$\begin{aligned}
g&\left(\frac{a-b}{|x-y|^s}\right)\frac{g(\Phi'(a))A-g(\Phi'(b))B}{|x-y|^{N+s}} \\
&\geq g\left(\frac{a-b}{|x-y|^s}\right)\frac{g\left(\frac{\Phi(a)-\Phi(b)}{a-b}\right)A-g\left(\frac{\Phi(a)-\Phi(b)}{a-b}\right)B}{|x-y|^{N+s}} \\
&= g\left(\frac{a-b}{|x-y|^s}\right)g\left(\frac{\Phi(a)-\Phi(b)}{a-b}\right)\frac{A-B}{|x-y|^{N+s}} \\
&\geq Cg\left(\frac{\Phi(a)-\Phi(b)}{|x-y|^s}\right)\frac{A-B}{|x-y|^{N+s}}.
\end{aligned}$$

This, after taking $\psi = \Phi(u)$ (note that Φ is assumed to be C^1), gives

$$\int_\Omega F(x)g(\Phi'(u(x)))\Phi(u)\,\mathrm{d}x \geq C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|\Phi(u(x))-\Phi(u(y))|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}.$$

■

Lemma 3.7: *Let $f \in L^\infty(\Omega)$ with $f \geq 0$, and f is not identically zero. Then the problem*

$$\begin{cases} (-\Delta_g)^s u = f, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{9}$$

has a unique solution $u \in W_0^{s,G}(\Omega) \cap L^\infty(\Omega)$.

Proof: The existence, uniqueness, and continuity follows from [18, Theorem 6.16], Lemma 2.8, and the fact that $f \geq 0$, so that $(-\Delta_g)^s u \geq 0$ on Ω, using Lemma 3.5. It remains to show that $u \in L^\infty(\Omega)$. For this, we shall assume, without loss of generality, that $\Omega \subseteq B(0,1)$ and fix $\alpha > 1$.

Let us consider

$$v_\alpha(x) = \begin{cases} \alpha(1 - |x|), & \text{when } |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for since $\alpha > 1$, for any $0 < \lambda < 1$ we have, using Lemma 2.1 and Equation (3), the estimate $g(\alpha\lambda t) > \frac{p^-\alpha^{p^--1}\lambda^{p^+-1}G(t)}{t}$ when $t > 0$. Again, for $x \in \Omega \subseteq B(0,1) \subseteq B(x, 1 + |x|)$ we get

$$\begin{aligned} (-\Delta_g)^s v_\alpha(x) &\geq \int_{|y|>1} g\left(\frac{v_\alpha(x) - v_\alpha(y)}{|x-y|^s}\right) \frac{dy}{|x-y|^{N+s}} \\ &= \int_{|y|>1} g\left(\frac{v_\alpha(x)}{|x-y|^s}\right) \frac{dy}{|x-y|^{N+s}} \\ &\geq p^-\alpha^{p^--1}(1-|x|)^{p^+-1} \int_{|y|>1} G\left(\frac{1}{|x-y|^s}\right) \frac{dy}{|x-y|^N} \\ &= p^-\alpha^{p^--1}(1-|x|)^{p^+-1} \int_{|y|>1} G\left(\frac{1}{(1+|y|)^s}\right) \frac{dy}{(1+|y|)^N} \to \infty \end{aligned}$$

uniformly as $\alpha \to \infty$. Thus, as f is bounded, we can choose α large enough to get $(-\Delta)_g^s v_\alpha > (-\Delta)_g^s u$. Applying Lemma 3.5 we get $u \leq v_\alpha$ in $\mathbb{R}^N$. Thus, u is bounded. ■

We consider the following approximated problem of Equation (1), where we used the notation, $f_n = \min\{f, n\}$ for all $n \in \mathbb{N}$, and assumed $q > 0$ is C^1,

$$\begin{aligned} (-\Delta_g)^s u(x) &= \frac{f_n(x)}{\left(u(x) + \frac{1}{n}\right)^{q(x)}} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{10}$$

Lemma 3.8: *For a fixed $n \in \mathbb{N}$, Equation (10) has a weak solution $u_n \in C^{\alpha(n)}(\Omega)$ where $\alpha(n) \in (0,1) \quad \forall\, n \in \mathbb{N}$.*

Proof: Note that $\frac{f_n(x)}{(u^+(x)+\frac{1}{n})^{q(x)}} \in L^\infty(\Omega)$. Hence by Lemma 3.7, there exists a unique solution $w \in W_0^{s,G}(\Omega) \cap L^\infty(\Omega)$ to the problem

$$(-\Delta_g)^s w(x) = \frac{f_n(x)}{(u^+(x) + \frac{1}{n})^{q(x)}} \quad \text{in } \Omega,$$
$$w > 0 \quad \text{in } \Omega,$$
$$w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

This allows us to define the operator $S : W_0^{s,G}(\Omega) \to W_0^{s,G}(\Omega)$ by $S(u) = w$ the solution of

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{w(x)-w(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \frac{f_n(x)\varphi(x)}{(u(x)^+ + \frac{1}{n})^{q(x)}}\,\mathrm{d}x.$$

Multiplying both sides of this equation by w, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{w(x)-w(y)}{|x-y|^s}\right)\frac{(w(x)-w(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_\Omega \frac{f_n(x)w(x)}{(u(x)^+ + \frac{1}{n})^{q(x)}}\,\mathrm{d}x \le n^{1+\|q\|_{L^\infty(\Omega)}}\|w\|_{L^1(\Omega)}.$$

Applying Equation (3), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|w(x)-w(y)|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}$$
$$\le \frac{1}{p^-}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{|w(x)-w(y)|}{|x-y|^s}\right)\frac{|w(x)-w(y)|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \frac{1}{p^-}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{(w(x)-w(y))}{|x-y|^s}\right)\frac{(w(x)-w(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\le \frac{n^{1+\|q\|_{L^\infty(\Omega)}}}{p^-}\|w\|_{L^1(\Omega)}.$$

Assume $\|w\|_{W_0^{s,G}(\Omega)} > 1$,

$$\frac{1}{\|w\|^{p^-}_{W_0^{s,G}(\Omega)}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|w(x)-w(y)|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}$$
$$\ge \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|w(x)-w(y)|}{\|w\|_{W_0^{s,G}(\Omega)}|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} = 1.$$

So, we have

$$\|w\|^{p^-}_{W^{s,G}_0(\Omega)} \leq \frac{n^{1+\|q\|_{L^\infty(\Omega)}}}{p^-}\|w\|_{L^1(\Omega)},$$

and consequently, by Lemma 2.7,

$$\|w\|^{p^- -1}_{W^{s,G}_0(\Omega)} \leq Cn^{1+\|q\|_{L^\infty(\Omega)}}$$

provided $\|w\|_{W^{s,G}_0(\Omega)} > 1$. Setting $R := \max\left\{1, (Cn^{1+\|q\|_{L^\infty(\Omega)}})^{\frac{1}{p^- -1}}\right\}$, we can see that S maps the ball of radius R in the metric space $W^{s,G}_0(\Omega)$, into itself. The proof will now be complete if we show that S is continuous and compact.

Proof of continuity of S: Assume that $u_i \to u$ in $W^{s,G}_0(\Omega)$. Set $w_i = S(u_i)$ and $w = S(u)$. So that we have for any $\varphi \in W^{s,G}_0(\Omega)$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{w_i(x)-w_i(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \frac{f_n(x)\varphi(x)}{\left(u_i(x)^+ + \frac{1}{n}\right)^{q(x)}}\,\mathrm{d}x \quad \text{and} \quad (11)$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{w(x)-w(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \frac{f_n(x)\varphi(x)}{\left(u(x)^+ + \frac{1}{n}\right)^{q(x)}}\,\mathrm{d}x. \quad (12)$$

We have to show that $w_i \to w$ in $W^{s,G}_0(\Omega)$. By Lemma 2.7, passing to a subsequence, $u_i \to u$ in $L^{G_*}(\Omega)$ and $u_i \to u$ a.e. in Ω. Set $\overline{w_i} := w_i - w$. Subtracting Equation (??) from Equation (11), with the choice $\varphi = \overline{w_i}$, and then applying Lemma 2.3 for $a = \frac{w(x)-w(y)}{|x-y|^s}$ and, $b = \frac{w_i(x)-w_i(y)}{|x-y|^s}$, we get

$$C(G)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|\overline{w_i}(x)-\overline{w_i}(y)|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq \int_\Omega f_n(x)\left(\frac{1}{\left(u_i(x)^+ + \frac{1}{n}\right)^{q(x)}} - \frac{1}{\left(u(x)^+ + \frac{1}{n}\right)^{q(x)}}\right)(w_i(x)-w(x))\,\mathrm{d}x.$$

We apply Lemma 2.2 on the left-hand side and Hölder inequality on the right-hand side of this equation to get,

$$C(G)\min\left\{\|w_i - w\|^{p^+}_{W^{s,G}}, \|w_i - w\|^{p^-}_{W^{s,G}}\right\}$$
$$\leq C\left\|f_n(x)\left(\frac{1}{\left(u_i(x)^+ + \frac{1}{n}\right)^{q(x)}} - \frac{1}{\left(u(x)^+ + \frac{1}{n}\right)^{q(x)}}\right)\right\|_{L^{G'_*}}\|w_i - w\|_{L^{G_*}}$$
$$\leq C\left\|f_n(x)\left(\frac{1}{\left(u_i(x)^+ + \frac{1}{n}\right)^{q(x)}} - \frac{1}{\left(u(x)^+ + \frac{1}{n}\right)^{q(x)}}\right)\right\|_{L^{G'_*}}\|w_i - w\|_{W^{s,G}},$$

where the last inequality follows from Lemma 2.6. This gives

$$\min\left\{\|w_i - w\|^{p^+ -1}_{W^{s,G}}, \|w_i - w\|^{p^- -1}_{W^{s,G}}\right\}$$

$$\leq C\left\|f_n(x)\left(\frac{1}{\left(u_i(x)^+ + \frac{1}{n}\right)^{q(x)}} - \frac{1}{\left(u(x)^+ + \frac{1}{n}\right)^{q(x)}}\right)\right\|_{L^{G'_*}}.$$

Now observe that

$$\left|f_n(x)\left(\frac{1}{\left(u_i(x)^+ + \frac{1}{n}\right)^{q(x)}} - \frac{1}{\left(u(x)^+ + \frac{1}{n}\right)^{q(x)}}\right)\right| \leq 2n^{q(x)+1} \leq 2n^{\|q\|_{L^\infty}+1}.$$

Hence, as $u_i \to u$ pointwise a.e., by DCT it follows that $w_i \to w$ in $W_0^{s,G}$. Thus S is continuous.

Proof of compactness of S: Assume that u_i is a bounded sequence in $W_0^{s,G}(\Omega)$. As before, denote $w_i := S(u_i)$. We wish to show that w_i has a convergent subsequence in $W_0^{s,G}(\Omega)$. From Equation (11), Lemmas 2.5 and 2.2, we get

$$\begin{aligned}
\min\left\{\|w_i\|^{p^+}_{W^{s,G}}, \|w_i\|^{p^-}_{W^{s,G}}\right\} &\leq C(G)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{w_i(x)-w_i(y)}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \\
&\leq C(G)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{w_i(x)-w_i(y)}{|x-y|^s}\right)\frac{(w_i(x)-w_i(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= C(G)\int_\Omega \frac{f_n(x)w_i(x)}{\left(u_i(x)^+ + \frac{1}{n}\right)^{q(x)}}\,\mathrm{d}x \leq n^{1+\|q\|_{L^\infty(\Omega)}}\|w_i\|_{L^1(\Omega)} \\
&\leq Cn^{1+\|q\|_{L^\infty(\Omega)}}\|w_i\|_{W^{s,G}(\Omega)}.
\end{aligned}$$

This shows that w_i is a bounded sequence in $W_0^{s,G}(\Omega)$. From the boundedness of the two sequences, u_i, w_i, we conclude that there exists $u, w \in W_0^{s,G}(\Omega)$ such that $u_i \rightharpoonup u$ and $w_i \rightharpoonup w$ in $W_0^{s,G}(\Omega)$. We now want to show $S(u) = w$, that is for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{w(x)-w(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \frac{f_n(x)\varphi(x)}{\left(u(x)^+ + \frac{1}{n}\right)^{q(x)}}\,\mathrm{d}x. \tag{13}$$

Note that we already know

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{w_i(x)-w_i(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \frac{f_n(x)\varphi(x)}{\left(u_i(x)^+ + \frac{1}{n}\right)^{q(x)}}\,\mathrm{d}x. \tag{14}$$

By DCT, it is seen easily that the right-hand side of Equation (14) converges to the right-hand side of Equation (13). It remains to show the convergence of the left-hand side. Note that, under the hypotheses on g, we get that g is a bijection of $\mathbb{R}$ onto itself. In this case $\overline{g}$, as defined in Definition 2.4 becomes g^{-1}. We have, by the definition of conjugate N-function, change of variable, hypothesis H_g,

$$\overline{G}(g(t)) = \int_0^{g(t)} g^{-1}(\tau)\,\mathrm{d}\tau = \int_0^t \tau g'(\tau)\,\mathrm{d}\tau \equiv \int_0^t g(\tau)\,\mathrm{d}\tau = G(t).$$

Using this and the fact that w_i's are bounded in $W_0^{s,G}(\Omega)$, we have that $g\left(\frac{|w_i(x)-w_i(y)|}{|x-y|^s}\right)$ is a bounded sequence in $L^{\overline{G}}\left(\frac{1}{|x-y|^N}, \mathbb{R}^N \times \mathbb{R}^N\right)$ hence it has a weakly convergent subsequence. Thus we conclude that, up to a subsequence,

$$g\left(\frac{|w_i(x)-w_i(y)|}{|x-y|^s}\right) \rightharpoonup g\left(\frac{|w(x)-w(y)|}{|x-y|^s}\right)$$

weakly in $L^{\overline{G}}\left(\frac{1}{|x-y|^N}, \mathbb{R}^N \times \mathbb{R}^N\right)$. Now, since $\frac{|\varphi(x)-\varphi(y)|}{|x-y|^s} \in L^G\left(\frac{1}{|x-y|^N}, \mathbb{R}^N \times \mathbb{R}^N\right)$,

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{|w_i(x)-w_i(y)|}{|x-y|^s}\right)\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{|w(x)-w(y)|}{|x-y|^s}\right)\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

Since the solution so obtained is in $u_n \in W_0^{s,G}(\Omega) \cap L^\infty(\Omega)$ and hence it is $C^{\alpha(n)}(\Omega)$ where $\alpha(n) \in (0,1)$, $\forall\, n \in \mathbb{N}$ by Theorem 1.1 of Bonder et al. [22]. ■

Lemma 3.9: *Assume g to be convex on $(0,1)$. The sequence of functions $\{u_n\}_n$, found in Lemma 3.8 satisfies*

$$u_n(x) \leq u_{n+1}(x), \quad \textit{for almost every } x \in \Omega,$$

and for any compact set $K \subseteq \Omega$, there exists a constant $l = l(K) > 0$ such that for any n, large enough,

$$u_n(x) \geq l \quad \textit{for almost every } x \in K.$$

Proof: Set the notation $w_n(x) = (u_n(x) - u_{n+1}(x))^+$. Then we note that, for any $x \in \Omega$, and $f_n(x) \leq f_{n+1}(x)$,

$$\begin{aligned} &\int_\Omega \frac{f_n(x)}{(u_n(x)+\frac{1}{n})^{q(x)}} w_n(x)\,\mathrm{d}x - \int_\Omega \frac{f_{n+1}(x)}{(u_{n+1}(x)+\frac{1}{n+1})^{q(x)}} w_n(x)\,\mathrm{d}x \\ &= \int_\Omega \left(\frac{f_n(x)}{(u_n(x)+\frac{1}{n})^{q(x)}} - \frac{f_{n+1}(x)}{(u_{n+1}(x)+\frac{1}{n+1})^{q(x)}}\right) w_n(x)\,\mathrm{d}x \\ &= \int_\Omega \left(\frac{f_n(x)}{(u_n(x)+\frac{1}{n})^{q(x)}} - \frac{f_{n+1}(x)}{(u_{n+1}(x)+\frac{1}{n+1})^{q(x)}}\right) (u_n(x)-u_{n+1}(x))^+\,\mathrm{d}x \\ &\leq \int_{\{u_n(x)>u_{n+1}(x)\}} f_{n+1}(x)\left(\frac{(u_{n+1}(x)+\frac{1}{n+1})^{q(x)} - (u_n(x)+\frac{1}{n})^{q(x)}}{(u_n(x)+\frac{1}{n})^{q(x)}(u_{n+1}(x)+\frac{1}{n+1})^{q(x)}}\right)(u_n-u_{n+1})^+\,\mathrm{d}x \\ &\leq 0. \end{aligned}$$

Then the above calculation and Equation (10) implies

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u_n(x)-u_n(y)}{|x-y|^s}\right)\frac{w_n(x)-w_n(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y$$

$$\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u_{n+1}(x)-u_{n+1}(y)}{|x-y|^s}\right)\frac{w_n(x)-w_n(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Now [22, Theorem 1.1] implies that both u_n, u_{n+1} are Hölder continuous up to the boundary. So, we can apply Lemma 3.5 to get $u_n \leq u_{n+1}$ a.e. on $\mathbb{R}^N$. This concludes the proof of the first part.

The second part follows from the continuity of u_n, and Lemma 2.8, which gives $u_n > 0$ on Ω. ■

Proof of Theorem 3.2: By Lemma 3.8, Equation (10) has a weak solution u_n. Let $\varphi \in C_c^\infty(\Omega)$. We have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u_n(x)-u_n(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \frac{f_n(x)\varphi(x)}{\left(u_n(x)+\frac{1}{n}\right)^{q(x)}}\,\mathrm{d}x. \tag{15}$$

First, we claim:

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u_n(x)-u_n(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{16}$$

Proof of the claim: Set $\omega_\delta := \Omega \setminus \Omega_\delta$. Then by Lemma 3.9, there exists a constant $l > 0$ such that $u_n \geq l > 0$ on ω_δ. We get, using Lemma 2.7, and choosing $\varphi = u_n$,

$$\begin{aligned}
&C(G)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u_n(x)-u_n(y)}{|x-y|^s}\right)\frac{u_n(x)-u_n(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_\Omega \frac{f_n(x)u_n(x)}{\left(u_n(x)+\frac{1}{n}\right)^{q(x)}}\,\mathrm{d}x\\
&= \int_{\Omega_\delta} \frac{f_n(x)u_n(x)}{\left(u_n(x)+\frac{1}{n}\right)^{q(x)}}\,\mathrm{d}x + \int_{\omega_\delta} \frac{f_n(x)u_n(x)}{\left(u_n(x)+\frac{1}{n}\right)^{q(x)}}\,\mathrm{d}x\\
&\leq \int_{\Omega_\delta\cap\{u_n\leq 1\}} f_n(x)\,\mathrm{d}x + \int_{\Omega_\delta\cap\{u_n>1\}} f_n(x)u_n(x)\,\mathrm{d}x + \int_{\omega_\delta}\frac{f_n(x)u_n(x)}{l^{q(x)}}\,\mathrm{d}x\\
&\leq \|f\|_{L^1(\Omega)} + (1+\|l^{-q(\cdot)}\|_{L^\infty(\omega_\delta)})\|f\|_{L^{\overline{G_*}}(\Omega)}\|u_n\|_{L^{G_*}(\Omega)}\\
&\leq \|f\|_{L^1(\Omega)} + C_1\|u_n\|_{W_0^{s,G}(\Omega)}.
\end{aligned}$$

Assuming $\alpha := \|u_n\|_{W_0^{s,G}(\Omega)} > 1$, we get, using Lemma 2.1,

$$\begin{aligned}
1 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|u_n(x)-u_n(y)|}{\alpha|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\\
&\leq \frac{1}{\alpha^{p^-}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \leq \frac{\|f\|_{L^1(\Omega)}}{\alpha^{p^-}} + C_1\frac{1}{\alpha^{p^--1}}
\end{aligned}$$

This shows that $\|u_n\|_{W_0^{s,G}(\Omega)}$ must be bounded. So $u_n \rightharpoonup u$ in $W_0^{s,G}$ weakly. By Lemma 2.7, $u_n \to u$ strongly in $L^1(\Omega)$, and hence $u_n \to u$ pointwise a.e. up to a subsequence.

Now applying Lemma 2.1

$$\overline{G}(g(t)) = \int_0^{g(t)} \overline{g}(\tau)\,\mathrm{d}\tau = \int_0^t \overline{g}(g(\tau))g'(\tau)\,\mathrm{d}\tau = \int_0^t \tau g'(\tau)\,\mathrm{d}\tau.$$

This implies

$$(p^- - 1)G(t) \le \overline{G}(g(t)) \le (P^+ - 1)G(t). \tag{17}$$

This, along with Lemma 2.2, shows that the sequence of functions $(x,y) \mapsto g\left(\frac{u_n(x)-u_n(y)}{|x-y|^s}\right)$ is bounded in $L^{\overline{G}}\left(\mathbb{R}^N \times \mathbb{R}^N, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)$. So it has a weakly convergent subsequence; without loss of generality, we assume it to be itself. It is easy to check that the the function $(x,y) \mapsto \frac{\varphi(x)-\varphi(y)}{|x-y|^s}$ is in $L^G\left(\mathbb{R}^N \times \mathbb{R}^N, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)$. Hence Equation (16) follows and the claim is true.

Now, in order to complete the proof, taking into account Equation (15), we only need to show the convergence of the right-hand side of Equation (15). Note that

$$\left|\frac{f_n(x)\varphi(x)}{\left(u_n(x)+\frac{1}{n}\right)^{q(x)}}\right| \le |l^{-q(x)}f(x)\varphi(x)| \in L^1(\Omega),$$

where we get l from applying Lemma 3.9 on supp (φ). Therefore, we can apply DCT to get

$$\lim_{n\to\infty}\int_\Omega \frac{f_n(x)\varphi(x)}{\left(u_n(x)+\frac{1}{n}\right)^{q(x)}}\,\mathrm{d}x = \int_\Omega \frac{f(x)\varphi(x)}{u(x)^{q(x)}}\,\mathrm{d}x.$$

Hence the proof is complete. ■

Lemma 3.10: *For any $a, b \in \mathbb{R}$, we have*

$$|g(a) - g(b)| \le C\frac{|a-b|g(|a|+|b|)}{|a|+|b|} \le Cg(|a|+|b|).$$

Proof:

$$g(b) - g(a) = \int_0^1 g'(a+(b-a)t)(b-a)\,\mathrm{d}t.$$

Now since g' is increasing one has for $t \in (0,1)$, $|a+(b-a)t| \le ||a|+|b||$. So we get

$$|g(a) - g(b)| \le |a-b|g'(|a|+|b|).$$

The results now follow trivially using the hypothesis on g. ■

Lemma 3.11: *Let $\Phi : (0,\infty) \to (0,\infty)$ be a strictly convex, C^1-function such that Φ' is increasing and there exists $\theta_1, \theta_2 \ge 0$ such that $\theta_1\frac{\Phi(x)}{x} \le \Phi'(x) \le \theta_2\frac{\Phi(x)}{x}$. For $x, y \in \mathbb{R}$ and $\varepsilon > 0$, define $S_\varepsilon^x := \{x \ge \varepsilon\} \cap \{y \ge 0\}$, and $S_\varepsilon^y := \{x \ge 0\} \cap \{y \ge \varepsilon\}$. Then for $(x,y) \in S_\varepsilon^x \cup S_\varepsilon^y$,*

$$|\Phi(x) - \Phi(y)| \ge C\Phi'(\epsilon)|x-y| \quad \textit{with} \quad C := \max(\theta_1, 1).$$

Proof: By symmetry, without loss of generality, we can assume $x > y$. Now for some $\lambda \in (y, x)$, we have $\Phi(x) - \Phi(y) = \Phi'(\lambda)(x - y)$. If we assume $x \geq y \geq \varepsilon > 0$, then we have

$$|\Phi(x) - \Phi(y)| \geq \Phi'(\lambda)|x - y| \geq \Phi'(\varepsilon)|x - y|.$$

For, $0 \leq y < \varepsilon \leq x$, then by strict convexity of Φ, we get

$$\frac{\Phi(x) - \Phi(y)}{x - y} \geq \frac{\Phi(x)}{x} \geq \theta_1 \Phi'(x) \geq \theta_1 \Phi'(\varepsilon)$$

thus concluding the assertion. ∎

Lemma 3.12: *Let Φ, H, f, q be as in Theorem* 3.3, *u_n be as in Lemma* 3.8. *Then there is a constant $C > 0$, independent of n such that $\|\Phi(u_n)\|_{W_0^{s,G}(\Omega)}, \|\Phi(u)\|_{W_0^{s,G}(\Omega)} \leq C$, where u is the pointwise limit of u_n.*

Proof: We have, for $t > 0$,

$$\Phi(t) := \int_0^t G^{-1}\left(G(1)\tau^{q^*-1}\right) \mathrm{d}\tau,$$

that is

$$\Phi'(t) := G^{-1}\left(G(1)t^{q^*-1}\right),$$

which gives, applying the fact that $\Phi'(t)$ is increasing and hence $\Phi(t) \leq t\Phi'(t)$,

$$g(\Phi'(t))\Phi(t) = \frac{\Phi'(t)g(\Phi'(t))}{G(\Phi'(t))}\frac{G(\Phi'(t))\Phi(t)}{\Phi'(t)} \leq p^+G(1)t^{q^*-1}\frac{\Phi(t)}{\Phi'(t)} \leq p^+G(1)t^{q^*}. \quad (18)$$

Using Equation (18), Lemma 3.6, and the fact $q^* > 1$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|\Phi(u_n(x)) - \Phi(u_n(y))|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \\
&\quad \leq C\int_\Omega \frac{f_n(x)}{\left(u_n(x) + \frac{1}{n}\right)^{q(x)}} g(\Phi'(u_n(x)))\Phi(u_n(x))\,\mathrm{d}x \\
&\quad = C\left(\int_{\Omega_\delta, u_n<1} + \int_{\Omega_\delta, u_n \geq 1} + \int_{\omega_\delta, u_n<1} + \int_{\omega_\delta,\, u_n\geq 1}\right)\frac{f_n(x)}{\left(u_n(x) + \frac{1}{n}\right)^{q(x)}} g(\Phi'(u_n(x)))\Phi(u_n(x)) \\
&\quad \leq C\int_{\Omega\cap\{u_n<1\}} f_n(x) + C\int_{\Omega\cap\{u_n\geq 1\}} f_n(x)u_n(x)^{q^*}. \qquad (19)
\end{aligned}$$

Set $r := \frac{p^-}{p^- + q^* - 1}$. We have, for large enough t_0, and for any $t > t_0$,

$$\begin{aligned}
t^{\frac{1}{r}} &= \frac{1}{r}\int_0^1 \tau^{\frac{1}{r}-1}\,\mathrm{d}\tau + \frac{1}{r}\int_1^t \tau^{\frac{1}{r}-1}\,\mathrm{d}\tau \leq \frac{2}{r}\int_1^t \tau^{\frac{1}{r}-1}G^{-1}(G(1))\,\mathrm{d}\tau \\
&\leq \frac{2}{r}\int_1^t G^{-1}\left(G(1)\tau^{\frac{p^-(1-r)}{r}}\right)\mathrm{d}\tau \leq \frac{2}{r}\int_0^t G^{-1}\left(G(1)\tau^{\frac{p^-(1-r)}{r}}\right)\mathrm{d}\tau = \frac{2}{r}\Phi(t). \qquad (20)
\end{aligned}$$

Applying Equation (20) on Equation (19) and then using Hölder's inequality, and finally the fact that $|f_n| \le |f|$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|\Phi(u_n(x)) - \Phi(u_n(y))|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \\
&\le C\int_{\Omega\cap\{u_n<1\}} f_n(x) + C\int_{\Omega\cap\{u_n\ge 1\}} f_n(x)\Phi(u_n(x))^{rq^*} \\
&\le C\|f_n\|_{L^1(\Omega)} + C\|f_n\|_{L^{\overline{H}}(\Omega)}\|\Phi(u_n)^{rq^*}\|_{L^H(\Omega)} \\
&\le C\|f\|_{L^1(\Omega)} + C\|f\|_{L^{\overline{H}}(\Omega)}\|\Phi(u_n)^{rq^*}\|_{L^H(\Omega)}. \qquad (21)
\end{aligned}$$

Observe that

$$\begin{aligned}
\left\|\Phi^{rq^*}(u_n)\right\|_{L^H(\Omega)} &= \inf\left\{\lambda>0 \,\middle|\, \int_\Omega H\left(\frac{\Phi(u_n)^{rq^*}}{\lambda}\right)\le 1\right\} \\
&= \inf\left\{\lambda^{rq^*}>0 \,\middle|\, \int_\Omega H\left(\frac{\Phi(u_n)^{rq^*}}{\lambda^{rq^*}}\right)\le 1\right\} \\
&= \left(\inf\left\{\lambda>0 \,\middle|\, \int_\Omega H\left(\frac{\Phi(u_n)^{rq^*}}{\lambda^{rq^*}}\right)\right\}\right)^{rq^*} \\
&= \left(\inf\left\{\lambda>0 \,\middle|\, \int_\Omega G_*\left(\frac{\Phi(u_n)}{\lambda}\right)\right\}\right)^{rq^*} = \|\Phi(u_n)\|^{rq^*}_{L^{G_*}(\Omega)},
\end{aligned}$$

to see the last line recall that $G_*(t) := H(t^{rq^*})$. Combining this with Equation (21) gives

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|\Phi(u_n(x)) - \Phi(u_n(y))|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \le C\|f\|_{L^1(\Omega)} + C\|f\|_{L^{\overline{H}}(\Omega)}\|\Phi(u_n)\|^{rq^*}_{L^{G_*}(\Omega)}.$$

From Lemma 2.7, we can write

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|\Phi(u_n(x)) - \Phi(u_n(y))|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \le C\|f\|_{L^1(\Omega)} + C\|f\|_{L^{\overline{H}}(\Omega)}\|\Phi(u_n)\|^{rq^*}_{W_0^{s,G}(\Omega)}.$$

When $\|\Phi(u_n)\|_{W_0^{s,G}(\Omega)} > t_0$, using Lemma 2.2, we get

$$\|\Phi(u_n)\|^{p^-}_{W_0^{s,G}(\Omega)} \le C\|f\|_{L^1(\Omega)} + C\|f\|_{L^{\overline{H}}(\Omega)}\|\Phi(u_n)\|^{rq^*}_{W_0^{s,G}(\Omega)}.$$

From the hypothesis, we have, $rq^* < p^-$. This implies that the norm $\|\Phi(u_n)\|_{W_0^{s,G}(\Omega)}$ cannot increase arbitrarily. So, there exists a constant $C>0$, independent of n, such that $\|\Phi(u_n)\|_{W_0^{s,G}(\Omega)} \le C$.

By Lemma 3.9, u_n is a monotone increasing sequence. So, we can define u as the pointwise limit of u_n. Direct application of Fatou's lemma and Lemma 2.2 implies that $\|\Phi(u)\|_{W_0^{s,G}(\Omega)} \le C$. ■

Proof of Theorem 3.3: By Lemma 3.9, u_n is a monotone increasing sequence. So, we can define u as the pointwise limit of u_n. Next, we show that this u is the required solution.

We know from Lemma 3.8 that there are u_n which satisfy

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u_n(x)-u_n(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,dx\,dy = \int_\Omega \frac{f_n(x)\phi(x)}{\left(u_n(x)+\frac{1}{n}\right)^{q(x)}}\,dx.$$

Note that, on $\operatorname{supp}(\phi)$, as $f \in L^1(\Omega)$,

$$\left|\frac{f_n(x)\phi(x)}{\left(u_n(x)+\frac{1}{n}\right)^{q(x)}}\right| \le \|l^{-q(\cdot)}\|_{L^\infty}|f||\phi| \in L^1.$$

Hence, by dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_\Omega \frac{f_n(x)\phi(x)}{\left(u_n(x)+\frac{1}{n}\right)^{q(x)}} = \int_\Omega \frac{f(x)\phi(x)}{u(x)^{q(x)}}.$$

So, we need to show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u_n(x)-u_n(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,dx\,dy$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,dx\,dy.$$

We have, $\Phi(u) \in W_0^{s,G}(\Omega)$ and by Lemma 2.6, it follows that $\Phi(u) \in L^G(\Omega)$. Comparing integrals, where $u > 1$, it follows that $u \in L^G(\Omega)$. We see, using Lemma 3.10,

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u_n(x)-u_n(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,dx\,dy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,dx\,dy\right|$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|g\left(\frac{u_n(x)-u_n(y)}{|x-y|^s}\right)-g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\right|\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{N+s}}\,dx\,dy$$
$$\le C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{|u_n(x)-u_n(y)|+|u(x)-u(y)|}{|x-y|^s}\right)\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{N+s}}\,dx\,dy$$
$$= C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_n \quad \text{(assume)}$$

The proof will be complete if we can show that $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_n \to 0$. To do this, first, set

$$\mathcal{S}_\phi := \operatorname{supp}\phi, \quad \text{and} \quad \mathcal{Q}_\phi := (\mathbb{R}^N \times \mathbb{R}^N) \setminus ({\mathcal{S}_\phi}^c \times {\mathcal{S}_\phi}^c).$$

Now using Hölder's inequality with respect to the measure $\frac{dx\,dy}{|x-y|^N}$, we get for any compact set $K \subseteq \mathbb{R}^N \times \mathbb{R}^N$,

$$\iint_{\mathbb{R}^{2N}\setminus K} I_n = \iint_{\mathcal{Q}_\phi\setminus K} I_n$$
$$\le C\left\|g\left(\frac{|u_n(x)-u_n(y)|+|u(x)-u(y)|}{|x-y|^s}\right)\right\|_{L^{\overline{G}}\left(\mathcal{Q}_\phi\setminus K,\frac{dx\,dy}{|x-y|^N}\right)}$$

$$\times \left\| \frac{|\varphi(x)-\varphi(y)|}{|x-y|^s} \right\|_{L^G\left(\mathcal{Q}_\phi \setminus K, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)}.$$

Now, if $\left\| g\left(\frac{|u_n(x)-u_n(y)|+|u(x)-u(y)|}{|x-y|^s}\right) \right\|_{L^{\overline{G}}\left(\mathcal{Q}_\phi \setminus K, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)} \leq 1$, we get

$$\iint_{\mathbb{R}^{2N}\setminus K} I_n \leq C \left\| \frac{|\varphi(x)-\varphi(y)|}{|x-y|^s} \right\|_{L^G\left(\mathcal{Q}_\phi \setminus K, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)}.$$

Otherwise, we apply Lemma 2.2 and Equation (17) to get

$$\begin{aligned}
\iint_{\mathbb{R}^{2N}\setminus K} I_n &\leq C \left(\iint_{\mathcal{Q}_\phi \setminus K} \overline{G}\left(g\left(\frac{|u_n(x)-u_n(y)|+|u(x)-u(y)|}{|x-y|^s} \right) \right) \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \right)^{\frac{1}{p^-}} \\
&\quad \times \left\| \frac{|\varphi(x)-\varphi(y)|}{|x-y|^s} \right\|_{L^G\left(\mathcal{Q}_\phi \setminus K, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)} \\
&\leq C \left(\iint_{\mathcal{Q}_\phi \setminus K} G\left(\frac{|u_n(x)-u_n(y)|+|u(x)-u(y)|}{|x-y|^s} \right) \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \right)^{\frac{1}{p^-}} \\
&\quad \times \left\| \frac{|\varphi(x)-\varphi(y)|}{|x-y|^s} \right\|_{L^G\left(\mathcal{Q}_\phi \setminus K, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)} \\
&\leq C \left[\iint_{\mathcal{Q}_\phi \setminus K} G\left(\frac{|u_n(x)-u_n(y)|}{|x-y|^s} \right) \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \right. \\
&\quad \left. + \iint_{\mathcal{Q}_\phi \setminus K} G\left(\frac{|u(x)-u(y)|}{|x-y|^s} \right) \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \right]^{\frac{1}{p^-}} \\
&\quad \times \left\| \frac{|\varphi(x)-\varphi(y)|}{|x-y|^s} \right\|_{L^G\left(\mathcal{Q}_\phi \setminus K, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)}.
\end{aligned}$$

By Lemma 3.9, there exists $l = l(\mathcal{S}_\phi) > 0$ such that for n large enough, $u_n(x) > l$. We now apply Lemma 3.11 on the two integrands of the last line to get

$$\begin{aligned}
\iint_{\mathbb{R}^{2N}\setminus K} I_n \leq C &\left[\iint_{\mathcal{Q}_\phi \setminus K} G\left(\frac{|\Phi(u_n)(x)-\Phi(u_n)(y)|}{|x-y|^s} \right) \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \right. \\
&\left. + \iint_{\mathcal{Q}_\phi \setminus K} G\left(\frac{|\Phi(u(x))-\Phi(u(y))|}{|x-y|^s} \right) \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \right]^{\frac{1}{p^+}} \\
&\times \left\| \frac{|\varphi(x)-\varphi(y)|}{|x-y|^s} \right\|_{L^G\left(\mathcal{Q}_\phi \setminus K, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)}.
\end{aligned}$$

By Lemmas 2.2 and 3.12, it is clear that

$$\iint_{\mathbb{R}^{2N}\setminus K} I_n \leq C \left\| \frac{|\varphi(x)-\varphi(y)|}{|x-y|^s} \right\|_{L^G\left(\mathcal{Q}_\phi\setminus K, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)}.$$

Since $\phi \in C_c^\infty(\Omega)$, for a fixed $\varepsilon > 0$, there exists $K = K(\varepsilon)$ such that

$$\iint_{\mathbb{R}^{2N}\setminus K} I_n < \frac{\varepsilon}{2}.$$

We, now have to estimate $\iint_K I_n$. For this, we use Vitali's convergence theorem. Let $E \subseteq K$. Arguing as above, we can get

$$\iint_E I_n \leq C \left\| \frac{|\varphi(x)-\varphi(y)|}{|x-y|^s} \right\|_{L^G\left(E, \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}\right)}.$$

This shows that the integrand in LHS is uniformly integrable, that is $\iint_E I_n \to 0$ as $\mathcal{L}^N(E) \to 0$. Applying Vitali's convergence theorem, we get for large enough n, $\iint_E I_n < \frac{\varepsilon}{2}$. So, from Equation (21), we get $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_n \to 0$ as $n \to \infty$, hence the proof follows. ■

Proof of Theorem 3.4: Let u be a solution of Equation (1) obtained through Theorems 3.2 and 3.3. Then u is pointwise limit of a sequence of solutions, u_n, of Equation (10). Also, by Lemma 3.9, there exists $l(K) > 0$ for any compact set $K \subseteq \Omega$ such that

$$u(x) \geq l(K) > 0 \quad \text{for almost all } x \in K.$$

This implies that there exists some $C_K > 0$ such that $u^{-q(x)}(x) \leq C_K$ for all $x \in K$. Fix $x_0 \in \Omega$ and $r > 0$ such that $B := B(x_0, r) \subset \overline{B(x_0, r)} \subset \Omega$. Again, since u is a weak solution of Equation (1), this implies that for any $\varphi \in C_c^\infty(B(x_0, r))$, where, with $\varphi \geq 0$,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_B f(x)u(x)^{-q(x)}\phi(x)\,\mathrm{d}x\\ \leq C_B\int_B f(x)\phi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{22}$$

where $v \in W^{s,G}(B) \cap L^\infty(B)$, is a solution to the problem

$$\begin{cases}(-\Delta_g)^s v = C_B f, & \text{in } B,\\ v > 0, & \text{in } B,\\ v = 0, & \text{in } \mathbb{R}^N \setminus B\end{cases}$$

obtained through Lemma 3.7. By using Lemma 3.5, we can conclude that $u \leq v$ in B if u is continuous on $\mathbb{R}^N$. That is $u \in L^\infty_{\mathrm{loc}}(\Omega)$ provided u is continuous on $\mathbb{R}^N$.

Again, since we have, from Equation (22),

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,dx\,dy \le C\int_B \phi(x)\,dx,$$

defining the sets

$$U_0 := \left\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \;\middle|\; \frac{|u(x)-u(y)|}{|x-y|^s}\ge 1\right\},$$
$$U_j := \left\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \;\middle|\; \frac{1}{j+1}\le\frac{|u(x)-u(y)|}{|x-y|^s}<\frac{1}{j}\right\} \quad \text{for} \quad j\ge 1,$$

we get from Lemma 2.1 that

$$\begin{aligned}
C\int_B \phi(x)\,dx &\ge \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+s}}\,dx\,dy\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^s}\,\frac{(\varphi(x)-\varphi(y))}{(u(x)-u(y))|x-y|^N}\,dx\,dy\\
&\ge p^-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{(\varphi(x)-\varphi(y))}{(u(x)-u(y))|x-y|^N}\,dx\,dy\\
&= p^-\sum_{j=0}^{\infty} j^{p^+}G\left(\frac{1}{j+1}\right)\iint_{U_j}\frac{|u(x)-u(y)|^{p^+-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp^+}}\,dx\,dy\\
&\ge p^-\sum_{j=0}^{\infty}\frac{j^{p^+}}{(j+1)^{p^+}}G(1)\iint_{U_j}\frac{|u(x)-u(y)|^{p^+-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp^+}}\,dx\,dy\\
&\ge C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p^+-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp^+}}\,dx\,dy.
\end{aligned}$$

We can now apply Corollary 5.5 of [23] to conclude that there is some $\alpha\in(0,1)$ such that $u\in C^\alpha(B)$. This completes the proof. ■

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