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Piia Parviainen

Early Mathematical Teaching – From Theoretical Modelling to Teachers' Pedagogical Awareness in Early Childhood Education

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF EDUCATION AND PSYCHOLOGY

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ABSTRACT

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The purpose of this study was to develop a theoretical model of the holistic development of early mathematical skills, as well as to provide knowledge of the teaching of these skills and how teaching could be enhanced to foster children's mathematical learning in early childhood education and care and pre-primary education. The first aim was to summarise research-based knowledge of the holistic development of early mathematical skills through a qualitative systematic review, and to investigate how teaching them corresponds to the development of these skills by using quantitative web survey data from teachers working with 3- to 7-year-old children (N = 206). The second aim was to understand how pedagogical awareness and participation in mathematics professional development (PD) programmes translate to mathematical teaching by utilising the quantitative web survey data (N = 206), qualitative interview data, and quantitative pre-PD and follow-up PD data (N = 7) collected from teachers. In sum, the study yielded a *holistic model of early mathematical skills development*. The results revealed three early mathematical skill categories with bi- and multi-directional interlinkages: (1) numerical skills (NS); (2) spatial thinking skills (STS); and (3) mathematical thinking and reasoning skills (MTRS). Teaching them corresponded partially to research-based understanding of early mathematical skills development, but the study revealed a need for a better comprehension of the development of STS and MTRS. Pedagogical awareness contributed to teaching NS, STS, and MTRS and their appropriate pedagogical practices. Previous participation in mathematics PD programmes was related to a higher frequency of teaching NS, STS, and MTRS. The PD programme carried out in this study revealed new pedagogical practices through enhanced pedagogical awareness of holistic early mathematical teaching. Pre- and in-service teacher education programmes would benefit from applying a holistic understanding of early mathematical skills development, a comprehension of early mathematical learning occurring in different daily situations, and self-reflective practices.

Keywords: development of early mathematical skills, pedagogical awareness, professional development programme, teaching early mathematical skills

TIIVISTELMÄ (ABSTRACT IN FINNISH)

Parviainen, Piia
Varhaisen matematiikan opetus — teoreettisesta mallinnuksesta opettajien pedagogiseen tietoisuuteen varhaiskasvatuksen kontekstissa
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Tämän tutkimuksen tehtävänä oli tuottaa teoreettinen kokonaisvaltainen malli varhaisten matemaattisten taitojen kehittymisestä ja lisätä tietoa näiden taitojen opetuksesta sekä opetuksen laajentamismahdollisuuksista suomalaisessa varhaiskasvatuksessa ja esiopetuksessa. Ensimmäisenä tavoitteena oli koota ajantasainen tieto varhaisten matemaattisten taitojen kokonaisvaltaisesta kehityksestä laadullisen systemaattisen kirjallisuuskatsauksen avulla. Lisäksi tavoitteena oli ymmärtää varhaiskasvatuksen opettajilta kerätyn määrällisen verkkokyselyn (*N* = 206) avulla, miten varhaisten matemaattisten taitojen opettaminen 3–7-vuotiaille vastaa matemaattisten taitojen kehitystä. Toisena tavoitteena oli ymmärtää määrällisen verkkokyselyn (*N* = 206) sekä laadullisen haastattelun ja määrällisen alku- ja seurantakyselyn avulla (*N* = 7), millainen merkitys opettajien pedagogisella tietoisuudella ja heidän osallistumisellaan matematiikan täydennyskoulutuksiin on varhaisen matematiikan opetuksessa. Systemaattisen kirjallisuuskatsauksen perusteella havaittiin kolme varhaisten matemaattisten taitojen kategoriaa, joiden välillä oli kaksi- ja monisuuntaisia yhteyksiä: 1) numeeriset taidot, 2) avaruudellisen ajattelun taidot ja 3) matemaattiset ajattelu- ja päättelytaidot. Näistä muodostui *kokonaisvaltainen malli varhaisten matemaattisten taitojen kehittymisestä*. Varhaisten matemaattisten taitojen opetus vastasi osittain tutkimusperustaista ymmärrystä taitojen tyypillisestä kehityksestä. Opettajien tulisi kuitenkin ymmärtää paremmin avaruudellisen ajattelun taitojen sekä matemaattisten ajattelu- ja päättelytaitojen kehitys. Pedagoginen tietoisuus heijastui matemaattisten taitojen opetukseen ja opetuskäytänteisiin. Matematiikan täydennyskoulutuksiin osallistuminen vahvisti matemaattisten taitojen opetuksen useutta. Tämän tutkimuksen yhteydessä toteutettu täydennyskoulutus sai aikaan uusia opetuskäytänteitä laajentaen pedagogista tietoisuutta kokonaisvaltaisesta varhaisen matematiikan opetuksesta. Johtopäätöksenä voidaan sanoa, että opettajien perustutkinto- ja täydennyskoulutuksissa tulisi huomioida varhaisten matemaattisten taitojen kokonaisvaltainen kehitys, kokonaisvaltainen ymmärrys varhaisten matemaattisten taitojen oppimisesta erilaisissa päivittäisissä tilanteissa ja leikissä sekä itsereflektioon pohjaavat käytänteet.

Avainsanat: pedagoginen tietoisuus, täydennyskoulutus, varhaisten matemaattisten taitojen kehitys, varhaisten matemaattisten taitojen opetus

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I ended this compilation article by borrowing words from an early childhood education and care (ECEC) teacher who said that everyone should need to understand all the things that mathematics encompasses. During this dissertation research journey, I have learnt a lot more about early mathematical teaching than I could have imagined, including what early childhood mathematics education is and can be. I have also come to understand the impact of a mixed methods design on research. The journey has helped me grow not only as a researcher but also as a teacher educator. It has reminded me of who I am, of how I should never give up and be grateful for the steps I have taken related to early childhood (mathematics) education. My work and learning process, however, would not have been successful without the people who have supported me.

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Jyväskylä 4.10.2024
Piia Parviainen

LIST OF ORIGINAL PAPERS

This doctoral study was based on three sub-studies (listed below) published in peer-reviewed academic journals. In this compilation article, the publications are referred to as sub-studies I–III.

SUB-STUDY I	Parviainen, P. (2019). The development of early mathematical skills – A theoretical framework for a holistic model. *Journal of Early Childhood Education Research*, *8*(1), 162–191.
SUB-STUDY II	Parviainen, P., Eklund, K., Koivula, M., Liinamaa, T., & Rutanen, N. (2023). Teaching early mathematical skills to 3- to 7-year-old children – Differences related to mathematical skill category, children's age group and teachers' characteristics. *International Journal of Science and Mathematics Education*, *21*, 1961–1983. https://doi.org/10.1007/s10763-022-10341-y
SUB-STUDY III	Parviainen, P., Eklund, K., Koivula, M., Liinamaa, T., & Rutanen, N. (2024). Enhancing teachers' pedagogical awareness of teaching early mathematical skills – A mixed methods study of a tailored professional development program. *Early Education and Development*, *35*(5), 1103-1125. https://doi.org/10.1080/10409289.2024.2336661

The first author contributed significantly to all three sub-studies by taking a leading role in the overall research design and its implementation as well as the literature review, analyses, writing process, and interpretation of the results of each sub-study. She organised and conducted the data collection of all sub-studies. Sub-study I was conducted independently by the author. The web survey for sub-studies II and III was designed with the co-authors. The construct and psychometric properties of each sub-scale of the survey were examined with Senior Lecturer Kenneth Eklund. He also participated in the statistical analyses and reporting for sub-studies II and III. The sample design of sub-study II was chosen with the other co-authors, who also commented on the manuscript. Sub-study III was designed by the first author, and she, together with a colleague, conducted the early childhood mathematics professional development programme, which was the focus of the study. The data of the semi-structured interviews were collected and analysed by the first author; however, inter-rater coding of two interviews was applied by one co-author and a member-checking technique was undertaken with all co-authors to increase the trustworthiness of the analysis. The co-authors also commented on the manuscript.

FIGURES

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1 INTRODUCTION

Mathematical skills play a pivotal role in our everyday lives (Clements & Sarama, 2014; Clements et al., 2011; Dehaene, 1992). For instance, we calculate how long it might take to reach the supermarket based on its distance from our home and the vehicle we drive. In the supermarket, we estimate the quantity of ingredients we need to prepare our meals. Additionally, we calculate and compare the prices of different ingredients. When we get back home, we use measures and timers to prepare tasty food for ourselves, especially when following a new recipe. This example clearly illustrates that we use mathematics as a tool to reason, describe, and communicate about relationships related to space, time, and number in our daily lives.

In addition to requiring mathematical skills for everyday tasks, mathematical competence is needed for the success of society (Clements & Sarama, 2014; Clements et al., 2011; Dehaene, 1992). As a result, most educational systems, including in the Finnish context, contain a mathematics component intended to ensure basic mathematical understanding and proficiency for all citizens (Clements & Sarama, 2014; van Oers, 2013). To boost science and mathematics competence and understanding in Finnish society, the Finnish Ministry of Education and Culture has recently presented Finnish National Science, Technology, Engineering, and Mathematics (STEM) Strategy and Action Plan 2030 (Ministry of Education and Culture, 2023). STEM Strategy 2030 includes aims and plans for education and research from early childhood education and care (ECEC) to higher education to promote wellbeing and growth that are socially, culturally, ecologically, and economically sustainable. To boost teachers' mathematics teaching skills in different educational settings from ECEC to secondary education, the Finnish Ministry of Education and Culture as well as the Finnish National Agency for Education have funded several professional development (PD) programmes in mathematics (e.g., LUMA2020, LUMATikka, and Joustavaan Matematiikkaan). In addition, specific programmes supporting children's early mathematical development have been created by Finnish universities and research institutes (e.g., Ekapeli-Matikka and Nallematikka).

Acknowledging early childhood mathematics education as an important part of the educational system is fundamental because the foundation for the mathematical skills needed in life is built in early childhood (Clements & Sarama, 2014; Clements et al., 2011). A lot is known about the development of early mathematical skills since the topic has gained a great deal of attention among researchers in Finland and abroad (Aunio & Räsänen, 2016; Lepola & Hannula-Sormunen, 2019; Mulligan & Mitchelmore, 2013; Sarama & Clements, 2009). Several studies have convincingly shown that children's quantitative and numerical understanding, spatial reasoning abilities, and capabilities for mathematical thinking, problem-solving, and reasoning develop gradually from initial to basic and then even more sophisticated skills during early childhood (Aunio & Räsänen, 2016; Clements & Sarama, 2014; Fantozzi et al., 2013; Hribar et al., 2012; Lepola & Hannula-Sormunen, 2019; Vanluydt et al., 2021; Worthington et al., 2019). Despite the vast number of in-depth studies concerning the development of certain mathematical skills, present knowledge is still incomplete. This has resulted in a failure to identify the typical development of early mathematical skills holistically in the extant literature; in other words, existing research has not succeeded in covering versatile mathematical skills comprehensively and acknowledging interlinkages among these skills. Since these previously cited studies have shown that several early mathematical skills together form the foundation for mathematical thinking and promote individuals' abilities to understand the world and manage their daily lives, it is essential to identify and model the development of and connections among these skills holistically.

Gaining a holistic understanding is necessary not only for researchers but also for ECEC teachers, who are responsible for early mathematical teaching within their child groups. This argument can be made from several perspectives. First, on the one hand, an increasing number of studies have attested that good mathematical skills gained in early childhood predict future mathematical learning outcomes (Jordan et al., 2009; Lepola & Hannula-Sormunen, 2019; Simms et al., 2016; Wang et al., 2021), success in reading (Lerkkanen et al., 2005; Romano et al., 2010), and overall achievement in school (Aubrey et al., 2006; Romano et al., 2010). Additionally, mathematical skills have been shown to have a central role in scientific inquiry and STEM/STEAM (n.b. "A" relates to Arts) learning (English, 2023; Morgan et al., 2016), as well as being associated with motor learning (Jylänki et al., 2022; Vanhala et al., 2022). On the other hand, recent studies have indicated that poor mathematical skills in childhood have long-term adverse effects on further education, employment, and even mental health in adulthood (Aro et al., 2019, 2023). Therefore, due to the significant influence of childhood mathematical skills in our lives, promoting the development and learning of early mathematical skills comprehensively and holistically in ECEC and in pre-primary education (6-year-old children) is essential.

Early childhood mathematics education is generally guided by a curriculum, and this is also the case in Finland (cf. Finnish National Agency for

Education, 2020, 2021, 2022). Finnish curriculum guidelines, however, set objectives for teaching not for learning. According to these guidelines, ECEC teachers are required to familiarise children with mathematical content, such as space, time, numeracy, directions, geometrical thinking, measurement, classification, comparison, seriation, and mathematical problem-solving. They are also expected to encourage children to develop their mathematical skills through play, exploration, and the use of different senses in various learning environments. Due to the socio-pedagogical philosophy of Finnish curricular thinking in ECEC and pre-primary education that does not establish firm objectives for mathematical teaching, curricula do not explicitly guide ECEC teachers on how to consider the development of early mathematical skills as a basis for supporting mathematical learning among different aged or individual children. Curricula leave room for teachers to plan and implement early childhood mathematics education that meets the needs and interests of the child group and individual children they are teaching. The idea is to implement curricula that are considered to be the best for the teacher's child group. Consequently, teachers' theoretical understanding of mathematical skills development is critical as it serves as a premise for developmentally appropriate mathematical teaching and learning practices in ECEC and pre-primary education (Clements et al., 2011, Clements et al., 2019; Moss et al., 2016). Based on the essence of Finnish curriculum guidelines (Finnish National Agency for Education, 2020, 2021, 2022), ECEC teachers need such an understanding to implement early childhood mathematics education as described in these documents.

Studies have attested that teachers' theoretical and pedagogical understanding plays a significant role in the implementation of high-quality early childhood mathematics education (Clements et al., 2011; Moss et al., 2016). Furthermore, teachers' pedagogical orientation and intentional practices are critical in promoting children's mathematical learning (Björklund & Palmér, 2024; Björklund et al., 2018; Salomonsen, 2020). However, early mathematical teaching has been shown to be somewhat incidental (Johnston & Bull, 2022; Johnston & Degotardi, 2022) and to lack comprehensive mathematical content (Hindman, 2013; Simpson & Linder, 2014). This may be explained by the fact that teachers themselves seem to feel more competent teaching numerical skills than spatial skills (Björklund & Barendgret, 2016; Gonulates & Gilbert, 2023). Crucially, studies have also indicated that PD programmes in mathematics improve the quality of early mathematical teaching by enhancing content-related teaching confidence and teaching practices (Gonulates & Gilbert, 2023; Knaus, 2017; Lindmeier et al., 2020). However, there is a need for further research concerning teachers' awareness of teaching early mathematical skills in correspondence with children's skills development. The role of mathematics PD programmes in teaching different early mathematical skills and in enhancing practices through conscious reflection have also remained understudied (Barber et al., 2015; Knaus, 2017).

Despite the fact that early mathematical teaching has gained more attention among researchers, especially during recent years, research concerning teaching mathematical skills in ECEC and pre-primary education has, nationally and globally, remained limited, especially compared to that in the context of primary and secondary education. Additionally, the existing body of research has focused on 3- to 5-year-old children (i.e., Simpson & Linder, 2014) and on the teaching of certain mathematical skills, especially numeracy (Linder & Simpson, 2018). Consequently, a study exploring the teaching of early mathematical skills holistically with a broad age range of children is needed.

Additionally, given the previously stated evidence of the fundamental role of early mathematical skills in our lives, the scattered and limited understanding of teaching early mathematical skills in ECEC and pre-primary education, and the role of PD programmes in mathematics, it is critical to explore teaching early mathematical skills comprehensively. This means research should be undertaken concerning the following: (1) our current research-based understanding of the holistic development of early mathematical skills; (2) the teaching of early mathematical skills comprehensively and covering a broad age range of children; (3) teachers' theoretical and pedagogical knowledge and practices in early mathematical teaching and PD; and (4) the role of mathematics PD programmes in early mathematical teaching and its development. Thus, this study developed a theoretical and holistic model of the development of early mathematical skills, investigated how early mathematical teaching is implemented in relation to the different elements that contribute it, and examined how ECEC teachers' knowledge of and practices in early mathematical teaching could be expanded upon to foster 3- to 7-year-old children's mathematical learning in ECEC and pre-primary education.

2 DIFFERENT ELEMENTS OF EARLY MATHEMATICAL TEACHING AS A BASIS FOR TEACHERS´ PEDAGOGICAL AWARENESS

Recently, Clements and colleagues (2023, p. 3) defined early mathematical teaching as ʻthe teaching of math as intentional interactions among children and teachers around mathematics content using deliberately arranged environments, contexts, and tasks, all designed to promote children's learning of increasingly powerful and sophisticated math competencies and positive dispositions ʻ. Their definition crystallises the pivotal role of ECEC teachers' current research-based comprehension of early mathematical skills development and their curricular understanding. It also emphasises teachers' abilities to acknowledge appropriate and beneficial teaching and learning practices to optimally promote children's mathematical skills development and learning. According to this definition, it can also be understood that effective early mathematical teaching requires observation, reflection, and development of one's knowledge and practices. This, based on research, means a combination of child-initiated and adult-initiated practices that take place in different daily situations (e.g., routine events, planned activities, and discussions), during play and teachable moments, as well as in different learning environments (see Anthony & Walshaw, 2009; Cheeseman et al., 2014; Clements et al., 2011; Helenius, 2018; Moss et al., 2016; Salomonsen, 2020). In short, based on Clements and colleagues' (2023) definition, early mathematical teaching requires observation, reflection, and development of one's knowledge and practices related to being able to consciously provide versatile mathematical learning affordances based on up-to-date knowledge and children's individual needs.

2.1 Knowledge of early mathematical skills development and mathematical content as a basis for early mathematical teaching

It has been underscored that teachers need to apply a current research-based understanding of mathematical skills development in their teaching to enable powerful mathematical learning in play, daily situations, teachable moments, and different learning environments (Clements et al., 2011; Helenius, 2018; Moss et al., 2016). This means teachers play a critical role in teaching early mathematical skills to children because their understanding of mathematical skills development as well as their teaching and learning practices are translated into children's possibilities for exploring mathematical phenomena and learning mathematical skills (Clements et al., 2011; MacDonald & Murphy, 2019). Acknowledging current research on mathematical skills development as a basis for early mathematical teaching is important (Clements et al., 2019; Moss et al., 2016) because it has changed our understanding of mathematical skills development drastically, from seeing the child as incapable of abstract and logical thinking (Piaget, 1965) to seeing the child as capable of sophisticated reasoning and mathematical-logical thinking before age 7 (e.g., Clements et al., 2019; Sarama & Clements, 2009). Consequently, teachers should shift their focus from what children cannot do to what they can do (Hachey, 2013).

Contemporary research findings have attested that some mathematical skills are innate, such as number and spatial sense, and that mathematical skills begin in infancy and generally develop gradually as children age (Sarama & Clements, 2009). The concept of 'number sense' in this study refers to understanding quantities and their symbolic representations (number words and numbers) (Jordan et al., 2006; Jordan et al., 2007). However, Dehaene (1997) provides another definition. Studies have shown that infants can discriminate between small quantities (1–3) and use visual information to reach people, objects, and places (Cheries et al., 2006; Clements & Sarama, 2007; Fantozzi et al., 2013; Feigenson et al., 2002; McCrink & Birdsall, 2015). Research has also demonstrated that a primary understanding of the interrelationship between number word and quantity develops during toddlerhood (Clements & Sarama, 2007; Wynn, 1992). Additionally, toddlers learn to build mental images of their surroundings to navigate their way to a target (Fantozzi et al., 2013; Hribar et al., 2012).

Current studies have also shown that between ages 3 and 5, children gradually learn to understand the interrelationships among number words, number symbols, and quantities as well as the principles of commutativity and associativity while counting objects (Hurst et al., 2017; Wynn, 1990, 1992). In addition, at around age 4, children can distinguish between large and small sets of objects, and with age, they learn to understand that large and small quantities are situation-specific (Jordan et al., 2008). Thus, for example, they come to understand that 8 children in one family is a lot but 8 children in an ECEC group is not. Some 3- to 5-year-old children, however, seem to focus on numerosity more spontaneously than others, which has been shown to influence their numerical learning (Hannula, 2005; Hannula & Lehtinen, 2005; Hannula et al., 2010; Hannula-Sormunen et al., 2015; Lepola & Hannula-Sormunen, 2019). Around age 6, children learn counting strategies, such as mental word sequence skills, counting backward, and counting from numbers other than 1, and by age 7, their counting strategies become more sophisticated, including gaining the ability for single sequence or skip counts (Aunio & Räsänen, 2016; Clements & Sarama, 2014; Kullberg et al., 2020). They also learn the inversion of addition and subtraction at this age.

In addition, studies have found that between ages 3 and 6, children learn to use simple maps and spatial coding successfully as well as become more exact and sophisticated in describing spatial qualities, directions, and positions (Clements, 2011; Sarama & Clements, 2009; Fantozzi et al., 2013; Hribar et al., 2012). Research has also revealed that children's understanding of shapes, length, mass, volume, and conservation becomes more precise with age (Clements & Sarama, 2007; Clements & Stephan, 2011). For instance, at age 3, children recognise typical shapes (i.e., triangles and squares); between ages 4 and 5, they recognise different kinds of rectangles; at age 5, they learn typical examples of other shapes (i.e., hexagons and rhombuses); and at age 6, they can distinguish between a rectangle and a parallelogram without right angles (Clements & Sarama, 2014). The same development path can be detected in children's time-related reasoning, which starts from a general comprehension of the past and future and becomes a more accurate understanding of time and use of terminology between ages 3 and 6 (Lyytinen, 2014; Mulligan & Mitchelmore, 2013).

Research has also demonstrated that children learn to compare things and objects based on their attributes or features during toddlerhood; however, their ability to compare different attributes or features expands a lot between ages 3 and 6 (Carraher & Schliemann, 2007; Clements & Sarama, 2014; Sarama & Clements, 2009). Between the same ages, children also learn to discover associations and repeatable sequences, using these discoveries to reach conclusions about features and quantities by categorising things and objects (Vanluydt et al., 2021; Worthington et al., 2019). Learning comparison and repeatable sequences means, for instance, that around age 4, children learn to compare length directly, whereas at age 5, they can arrange length according to serial order (Clements & Sarama, 2014). In addition, with age, children first learn to sort objects according to a single attribute but can reclassify them by changing the attribute. Gradually, they learn to classify and seriate according to two attributes at the same time. Furthermore, children's mathematical problem-solving and reasoning strategies become more sophisticated with age; thus, they learn multiple problem-solving methods as well as come to understand part–whole relations and principles of data modelling (Alsina & Saldago, 2022; Mulligan, 2015; Vanluydt et al., 2021; Warren et al., 2016).

In addition to the separate development of each mathematical skills area, studies have indicated that children use numerical skills and mathematical thinking and reasoning for spatial learning, and vice versa (Cheeseman et al., 2014; Colliver, 2018; Mulligan & Mitchelmore, 2013; Sarama & Clements, 2009). Thus, mathematical skills not only develop separately but also in conjunction with different mathematical skills areas. Therefore, through these connections, one can surmise that mathematical skills become more sophisticated simultaneously. Since studies have revealed that several mathematical skills develop during early childhood both separately and in conjunction, as well as become more sophisticated with age, as described above, it is essential that ECEC teachers have up-to-date knowledge of early mathematical skills development. It is fundamental to foster children's mathematical learning based on their capabilities and competencies instead of disproved knowledge of their mathematical capability (Lee & Ginsburg, 2007, 2009; Lloyd, 2024). Teachers should acknowledge that development and learning are unique processes (cf. Asunta et al., 2016; Carlson et al., 2013; Hannula, 2005; Hannula & Lehtinen, 2005; Hannula et al., 2010; Jordan et al., 2006, 2009; Simms et al., 2016). Children do not necessarily learn specific mathematical skills at a certain age or in a certain order, meaning there is variation among them. Such an understanding is imperative as it allows ECEC teachers to consider individuals' learning paths, thereby enabling flexible and sensitive teaching. It also helps ensure teachers consider individual children's capabilities and learning needs so that they can implement appropriate learning and teaching practices.

In short, it is clear that a current research-based understanding of mathematical skills development promotes effective teaching, whose aim should be to provide children with a strong foundation in the mathematical skills needed in life (Clements & Sarama, 2014; Clements et al., 2011). In the Finnish context, up-to-date knowledge fosters the implementation of early childhood mathematics education based on the needs of children in general as well as those of individual children in relation to the objectives set for early mathematical teaching in the Finnish curriculum guidelines for ECEC and pre-primary education (Finnish National Agency for Education, 2020, 2021, 2022).

Despite the vast body of research concerning the development of early mathematical skills, little is known about comprehensively teaching different early mathematical skills in ECEC and pre-primary education. The focus of existing studies has been on teaching numerical areas, leading to a limited number of studies concerning the teaching of geometry, measurement, algebra, and data analysis (Linder & Simpson, 2018). As the emphasis has also been on 3- to 5-year-old children (i.e., Hindman, 2013; Simpson & Linder, 2014), there is a gap in research concerning other age groups, such as toddlers and children in pre-primary education. Existing studies have, for instance, shown that teaching spatial thinking is not prominent among 3- to 5-year-old children (Hindman, 2013; Simpson & Linder, 2014). It is essential to know how prominent it is among 5- to 7-year-olds for ECEC teachers to consider their early mathematical teaching in relation to children's age-related capabilities. It is also key to learn how prominent the teaching of mathematical thinking and reasoning is among 3- to 7-year-old children. Consequently, there is a need for research to gain a comprehensive understanding of teaching different early mathematical skills (i.e., numeracy, spatial thinking, and mathematical reasoning) to (3- to 7-year-old) children in ECEC and pre-primary education. More precisely, research is needed to understand to what extent the development of early mathematical skills corresponds to their teaching among children of different ages. Additionally, studies should be conducted to ascertain if and why the teaching of different early mathematical skills is emphasised among children from different age groups.

Finally, there are shortcomings in the research concerning a comprehensive and holistic theoretical view of the development of early mathematical skills before age 7. In this study, a current research-based understanding of the development of versatile early mathematical skills as well as interlinkages among these skills was constructed as a complement to a prior model of numerical skills (Aunio & Räsänen, 2016). The theoretical model should benefit (1) ECEC teachers when considering gradual skills development and implementation of age-appropriate mathematical content with children of different ages and individual children in ECEC and pre-primary education as a complement to the curriculum they use (cf. Finnish National Agency for Education, 2020, 2021, 2022); (2) curriculum reforms when taking into account mathematical content in an age-appropriate and comprehensive manner; (3) teacher educators when thinking about content-related mathematical skills development and learning in pre-service and in-service teacher education; and (4) other researchers working on broad explorations of early mathematical learning, such as considering unresearched but important elements in research designs, connecting results to joint theoretical knowledge, and testing the applicability of the model in ECEC contexts. Ideally, the theoretical model can indirectly serve children's mathematical learning via the previously mentioned actions and groups of people.

My study sought to complement the existing body of research concerning the teaching of different early mathematical skills through a broad exploration of the teaching of these skills to 3- to 7-year-old children in correspondence to research-based knowledge of mathematical skills development. It aimed to understand ECEC teachers' emphasis on teaching early mathematical content to children of different ages by examining the frequencies of teaching different early mathematical skills. In this study, the frequencies of teaching different early mathematical skills were considered indications of the importance given to the teaching of these skills in ECEC and pre-primary education in correspondence to children's capabilities. The theoretical model served as a means of comprehensively exploring early mathematical teaching in ECEC and pre-primary education. Additionally, it functioned as a theoretical basis for a PD mathematics programme targeted at ECEC teachers with the aim of enhancing their pedagogical awareness of teaching early mathematical skills in ECEC and pre-primary education.

2.2 Appropriate pedagogical practices in early mathematical teaching and learning

Several studies have indicated that in early mathematical teaching, teachers play an important role in shaping mathematical learning possibilities they introduce and offer for children to explore, become familiar with, and learn about (Lee & Ginsburg, 2007, 2009; Lutovac & Kaasila, 2018; MacDonald & Murphy, 2019). Contemporary research has also revealed that ECEC teachers' pedagogical orientation and pedagogical practices are fundamental in supporting the development of early mathematical skills (Björklund et al., 2018; Clements et al., 2019; Hawes et al., 2017; Johnston & Bull, 2022; Salomonsen, 2020).

Additionally, studies have shown that in addition to understanding the development of early mathematical skills, early mathematical teaching requires appropriate pedagogical practices that promote the learning of these skills (Björklund & Palmér, 2024; Clements et al., 2011; Helenius, 2018; Moss et al., 2016). These include a combination of child- and adult-initiated practices, which take place in different daily situations, such as during play, routine events, teachable moments, and planned activities and discussions, in different learning environments (Anthony & Walshaw, 2009; Cheeseman et al., 2014; Clements et al., 2011; Helenius, 2018; Moss et al., 2016; Salomonsen, 2020). Finnish curriculum guidelines for ECEC and pre-primary education (Finnish National Agency for Education, 2020, 2021, 2022) emphasise such an approach to early mathematical teaching enabling ECEC teachers to implement and apply pedagogical practices comprehensively in different situations and environments. The guidelines also stress that ECEC teachers need to consider children's interests and initiatives as premises for beneficial mathematical learning.

Considering children's interests and initiatives in early mathematical teaching is an important pedagogical practice in early mathematical teaching as it promotes children's mathematical learning (Björklund et al., 2018; Brandt, 2013; Moss et al., 2016; Salomonsen, 2020; Wager, 2014). Studies have shown that ECEC teachers need to respond to children's interests and initiatives with sufficient learning activities, object exploration, and mathematics-related interactions in various learning environments. Additionally, they have revealed that such pedagogical practices are critical because concrete experiences and meaningful activities promote children's mathematical learning in various ways. For example, when children learn measurement through concrete experiences, they come to understand how length and number (Sarama & Clements, 2009) or mass and number (Cheeseman et al., 2014) are connected. Concrete experiences and materials also strengthen children's capacity for mathematical reasoning and assist them in comprehending mathematical concepts (Cheeseman et al., 2014; Lee et al., 2016). Since concrete exploration and experiences have been shown to positively influence mathematical learning outcomes (Trawick-Smith et al., 2016; Vogt et al., 2018), it is essential that ECEC teachers are aware of children's interests and curiosity related to exploration. They should introduce concrete materials in different situations for children's play and exploration (Björklund & Palmér, 2024; Cooke, 2022), although merging play and teaching has been shown to be difficult for teachers (Palmér & Björklund, 2023).

Appropriate pedagogical practices also include sensitive and pedagogically oriented teacher–child interactions, as well as collaborative practices, which are consciously employed in play and different daily situations (Björklund et al., 2018; Björklund & Palmér, 2024; Brandt, 2013; Moss et al., 2016; Palmér & Björklund, 2023; Salomonsen, 2020; Wager, 2014). These are important because consciously employed teacher–child interactions in play and different daily situations have been shown to improve mathematical learning outcomes (Trawick-Smith et al., 2016; Vogt et al., 2018). For instance, these may involve mathematical discussions with children and teachers' use of conceptual mathematical language (Björklund & Palmér, 2024). Discussions with children have been found to be natural ways to promote mathematical learning as joint problem-solving expands children's awareness of mathematical phenomena (Björklund et al., 2018; Brandt, 2013). In addition, such an expansion can happen through ECEC teachers' encouragement of children to explore mathematical phenomena. Studies have also demonstrated that children differ in their spontaneous focusing on numerosity (SFON); Hannula, 2005) and teachers can promote the development of children's mathematical understanding by recognising and supporting children's attention to mathematical aspects of everyday situations (Hannula-Sormunen et al., 2020; Hannula-Sormunen et al., 2021; Mattinen, 2006). Based on such evidence, ECEC teachers should acknowledge that pedagogically oriented collaborative experiences help children develop their mathematical thinking (Björklund & Palmér, 2024; van Oers, 2013) and apply these in their pedagogical practices.

Importantly, studies have also revealed that appropriate pedagogical practices include acknowledging children's participation, which enables ECEC teachers to better consider equitable and meaningful mathematical learning opportunities within their child groups (Helenius, 2018; Polly et al., 2017; Wager, 2014). Children's participation is especially important in promoting mathematical learning among multilingual children (Banse, 2021; Kultti, 2013). Since participation has been shown to be beneficial when applying pedagogical practices, which are meaningful for children but also extend possibilities for learning among different kinds of learners, it has been suggested that early mathematical teaching should focus on pedagogy, which completes the whole day in ECEC and pre-primary education. This means that ECEC teachers catch the teachable moments and learning affordances whenever they occur in play and different daily situations to foster mathematical learning possibilities among all children (Björklund, 2012; Björklund & Palmér, 2024; Cooke, 2022; Johnston & Degotardi, 2022). Such an orientation requires knowledge of pedagogically appropriate mathematical practices, which can be somewhat difficult to fully implement if there is an incomplete understanding of early mathematical teaching (Björklund & Palmér, 2024; Johnston & Bull, 2022; Johnston & Degotardi, 2022; Palmér & Björklund, 2023).

The majority of existing research has focused on early mathematical teaching during planned activities, leading to a lack of studies exploring early mathematical teaching from a holistic perspective, including, for instance, during play, children's initiatives, different daily routines, and teachable moments (see Björklund & Palmér, 2024; Johnston & Bull, 2022; Johnston & Degotardi, 2022). Further research is needed for a comprehensive exploration of whether teachers apply the previously described appropriate teaching and learning practices holistically during play and in different daily situations and teachable moments when they teach early mathematical skills to 3- to 7-year-old children, or if their teaching is focused on planned mathematical activities. Therefore, my study examined early mathematical teaching holistically to gain a comprehensive understanding of the practical implementation of early childhood mathematics education in ECEC and pre-primary education, as well as to discover whether there is a need to develop appropriate pedagogical teaching and learning practices.

2.3 Observation, reflection, evaluation, and development in early mathematical teaching and professional development

Although ECEC teachers need to be aware of the early mathematical skills development and content to be taught, as well as apply appropriate teaching and learning practices, early mathematical teaching should also entail teachers' observation, reflection, evaluation, and development of their knowledge and pedagogical practices because these translate into children's possibilities for mathematical learning (Chen et al., 2014; Gasteiger & Benz, 2018; Gonulates & Gilbert, 2023; Lindmeier et al., 2020). These practices have also been shown to be central to high-quality early mathematical teaching (Björklund et al., 2018; Clements et al., 2019; Lutovac & Kaasila, 2018; MacDonald & Murphy, 2019; Salomonsen, 2020). For example, with such an orientation, teachers can plan and modify the appropriate pedagogical practices based on the recognised learning needs and interests of the child group as well as of the individual children in the group (Callejo et al., 2022; Dunekacke et al., 2015; Hannula-Sormunen et al., 2020; Hannula-Sormunen et al., 2021; Muños-Catalán et al., 2022; Polly et al., 2017). In addition, such an orientation aids teachers in identifying the need to and actually improve knowledge of early mathematical teaching and skills development (Alsina et al., 2021; Dunekacke et al., 2015; Ertle et al., 2008; Gasteiger & Benz, 2018; Gonulates & Gilbert, 2023; Hannula-Sormunen et al., 2020).

Helenius (2018, p. 183), in relation to professional modes of action concerning developing early mathematical teaching, stated that 'visibility is not about what you see but about what you teach your eyes to look for'. According to this idea, early mathematical teaching requires a conscious approach to one's skills, attitudes, knowledge, and practices. Several studies have supported the interpretation by indicating that teachers' awareness of mathematical content

(Callejo et al., 2022; Muños-Catalán et al., 2022; Polly et al., 2017) and their content-related teaching confidence (Alsina et al., 2021; Dunekacke et al., 2015; Galeano et al., 2024) explicitly influence early mathematical teaching. Teachers' teaching confidence has been shown to have both a positive and negative influence on teaching depending on the mathematical content being taught (Çelic, 2017; Ertle et al., 2008; Galeano et al., 2024). For instance, insufficient instruction in spatial content has been attributed to a lack of content awareness (Björklund & Barendgret, 2016). In addition, teachers' biased assumptions about children's mathematical skills development and learning (e.g., mathematics should be taught only to gifted children) and their limited awareness of age-appropriate mathematical content have been revealed to influence teaching negatively (Lee & Ginsburg, 2007, 2009; Lloyd, 2024). Studies have also indicated that teachers do not always recognise the mathematical learning affordances that appear in play, pre-planned situations, and different daily routines (Helenius, 2018; Johnston & Bull, 2022; Johnston & Degotardi, 2022; Palmér et al., 2016) and that their assessment practices are partly incomplete (Chen et al., 2014; Lee & Ginsburg, 2007, 2009). Consequently, opportunities for early mathematical teaching are occasionally missed. Promisingly, studies have also stressed that the more comfortable teachers are in teaching mathematics, the more optimistic are their expectations regarding children's mathematical knowledge and learning (Çelic, 2017; Ertle et al., 2008).

Due to the importance of being aware of one's own knowledge, practices, and thought processes, because of their evident translation to early mathematical teaching, it has been suggested that teachers' understanding of early mathematical teaching in play and different daily situations, as well as their up-to-date understanding of mathematical skills development and appropriate practices, should be promoted through PD and PD programmes (Chen et al., 2014; Clements et al., 2023; Hannula-Sormunen et al., 2020; Johnston & Bull, 2022; Johnston & Degotardi, 2022). To promote teachers' PD, different theories have been applied. One of these is Mezirow's (1991, 1997) transformative learning theory, which has been widely used in the PD of teachers due to its emphasis on a critical approach to teaching and self-reflective cycles of learning. Similarities to this theory can be identified in Mason's (1998, 2002, 2011) theory, which was developed specifically for mathematical teaching. In both theories, the foundation for making sustainable changes in teaching lies in a critical approach to one's professionalism and requires teachers to be cognisant of their thinking structures and practices through self-reflection. Therefore, effective PD should allow teachers to critically examine their practices, question their thought structures, and pursue alternative means of understanding teaching (Cranton, 2016; Cranton & King, 2003; Mason, 1998, 2002, 2011).

Because the PD described above cultivates teachers' habits of mind about teaching, ECEC teachers themselves have underscored the importance of reflective practices and the cyclic nature of learning in the PD of early mathematical teaching (Barber et al., 2014; Hadley et al., 2015). However, due to limited improvement in self-reflection (Knaus, 2017; Lindmeier et al., 2020), commitment to self-reflection in the PD of early mathematical teaching has been stressed (Chen et al., 2014; Knaus, 2017; Mason, 2002, 2011). According to Cranton (2016) and Mason (1998, 2002, 2011), teachers need to commit to undergoing a comprehensive PD process. They have both emphasised that teachers must have ownership in their learning to achieve sustainable changes in their teaching. This can be supported through collaborative practices and participatory methods, such as involving teachers in the process from planning to evaluation, and the development of practices (Cranton, 2016). Teachers themselves have also stressed the importance of ownership and collaborative practices in their PD related to early childhood mathematics (Barber et al., 2014; Hadley et al., 2015). It can be concluded that teachers' empowerment is a prerequisite for transformative learning and at the same time, their empowerment is an outcome of transformative learning (Cranton, 2016).

Based on extant research, the application of transformative learning theory in the design of PD programmes of early childhood mathematics is important to enhance teachers' understanding of early mathematical content and advance their pedagogical practices (Knaus, 2017). These, in turn, improve children's mathematical skills development and engagement in mathematical learning despite the earlier mentioned lack of improvement in self-reflection. Moreover, PD programmes in early childhood mathematics have been shown to support teachers' PD regarding their mathematical content knowledge and pedagogical practices (Gonulates & Gilbert, 2023; Hadley et al., 2015; Knaus, 2017; Palmér & Björklund, 2023). Additionally, mathematics PD programmes enhance content-related teaching confidence (Gonulates & Gilbert, 2023; Lindemeir et al., 2020). Nevertheless, recent studies have suggested that PD programmes in mathematics should emphasise teachers' participation related to the approach and methods used when aiming to enhance pedagogical awareness of teaching early mathematical skills during daily situations (Johnston & Bull, 2022; Johnston & Degotardi, 2022; Palmér & Björklund, 2023).

Although PD programmes in mathematics promote teachers' content knowledge, teaching confidence, and pedagogical practices (Gonulates & Gilbert, 2023; Hadley et al., 2015; Knaus, 2017; Lindemeir et al., 2020), further research is needed about how PD programmes modify early mathematical teaching when teaching different early mathematical skills to 3- to 7-year-old children and teachers' knowledge of pedagogical practices of early mathematical teaching. Additionally, we must improve our understanding of the role of PD mathematics programmes, which apply principles of transformative learning, such as a critical approach to one's professionalism, thinking structures, and practices through self-reflection (Cranton, 2016; Knaus, 2017; Mason, 1998, 2002, 2011; Mezirow, 1991, 1997). My study explores these topics to increase our comprehension of how ECEC teachers' participation in mathematics PD programmes contributes into teaching different early mathematical skills and their PD in early mathematical teaching.

2.4 Teachers’ pedagogical awareness in early mathematical teaching

Based on previous discussions in this chapter, it can be concluded that different issues contribute to early mathematical teaching in different ways, meaning there are broad variations in the implementation of early childhood mathematics education among ECEC teachers regardless of the curriculum, which in Finland, is based on the Finnish curriculum guidelines for ECEC and pre-primary education (Finnish National Agency for Education, 2020, 2021, 2022). Because these variations have been connected to teachers’ knowledge and practices, several theoretical models have been introduced to explain the elements that contribute to teaching mathematics so that teachers are aware of them and can benefit from the understanding in their personal PD to improve their mathematical teaching. For these reasons, several models have been used in mathematics PD programmes and research concerning the teaching of mathematics.

Probably one of the best known and widely used in the natural sciences is Shulman’s (1986) categorisation of the teacher’s subject-specific knowledge into subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Shulman’s model, however, has been criticised for a lack of practice-based elements. Consequently, alternative models have been introduced aiming to extend our understanding of the practice-based elements that also play a critical role in teachers’ knowledge. These include, for instance, emphasising certain practices based on learners’ needs and interests (Ball et al., 2008; Carrillo-Yañez et al., 2018; Mason, 1998, 2002, 2011). In a professional model of knowledge targeted specifically at ECEC teachers, a practice-based element means seizing and designing mathematical learning opportunities, such as asking sufficient questions and choosing adequate materials, that are necessary to implement early childhood mathematics education successfully during play and everyday situations (Gasteiger & Benz, 2018). Also, in this model, conscientious evaluation and analysis of children’s learning processes is considered a critical element in applying practices that benefit children’s learning.

Based on the criticism of the emphasis on knowledge in Shulman’s (1986) model, Lindmeier’s (2011) and Lindmeier and colleagues’ (2020) models include a reflective element in addition to an action-related element, as in Gasteiger and Benz (2018), although they called the reflective element ‘evaluation’. However, teacher-related elements in Lindmeier’s (2011) and Lindmeier and colleagues’ (2020) models are conceptualised as professional competence aiming to complement knowledge with action-related competence and reflective competence. In Lindmeier and colleagues’ (2020) model, which they introduced in early childhood mathematics education, reflective competence refers to the ability to plan and implement mathematical activities, as well as the ability to observe mathematical learning and development and to use this diagnostic information to support individuals’ mathematical learning (cf. Gasteiger & Benz, 2018). Yet, in Lindmeier and colleagues' (2020) model, action-related competence refers to the ability to recognise age-appropriate mathematical affordances in play and daily situations. It also includes the ability to react spontaneously to emerging learning moments, as in the model of Gasteiger and Benz (2018).

Similarities concerning the elements in the two models introduced for early childhood education (Gasteiger & Benz, 2018; Lindmeier et al., 2020) can be identified despite their divergent conceptualisations. Both models view high-quality early mathematical teaching as being contingent on curricular mathematical content; teachers' understanding of the needs, interests, and development of learners; and sufficient promotion of mathematical learning through intentional, pedagogically appropriate practices in play and different daily situations. However, as discussed in Chapters 2.1, 2.2, and 2.3, several studies have indicated that in addition to observing and evaluating children's learning and applying appropriate practices based on those recognitions, teachers need to reflect on their up-to-date knowledge of early mathematical skills development and mathematical content as well as pedagogical practices because these also translate into early mathematical teaching (e.g., Callejo et al., 2022; Chen et al., 2014; Dunekacke et al., 2015; Muños-Catalán et al., 2022; Polly et al., 2017). Teachers need to reflect on and evaluate their confidence and assumptions regarding early mathematical teaching and learning as well because they also have an effect on early mathematical teaching (Alsina et al., 2021; Çelic, 2017; Ertle et al., 2008; Gasteiger & Benz, 2018).

Given the evidence of teachers' self-reflection and self-evaluation of their skills, knowledge, assumptions, and pedagogical practices as essential parts of shaping early mathematical teaching, I include these as critical elements for high-quality early mathematical teaching. In my study, I conceptualise these as teachers' *pedagogical awareness*, which includes three elements: (1) knowledge of early mathematical skills development and content to be taught; (2) appropriate pedagogical teaching and learning practices; and (3) observation, reflection, evaluation, and (professional) development; additionally, the conceptualisation considers teachers' skills, knowledge, and attitudes (Figure 1).

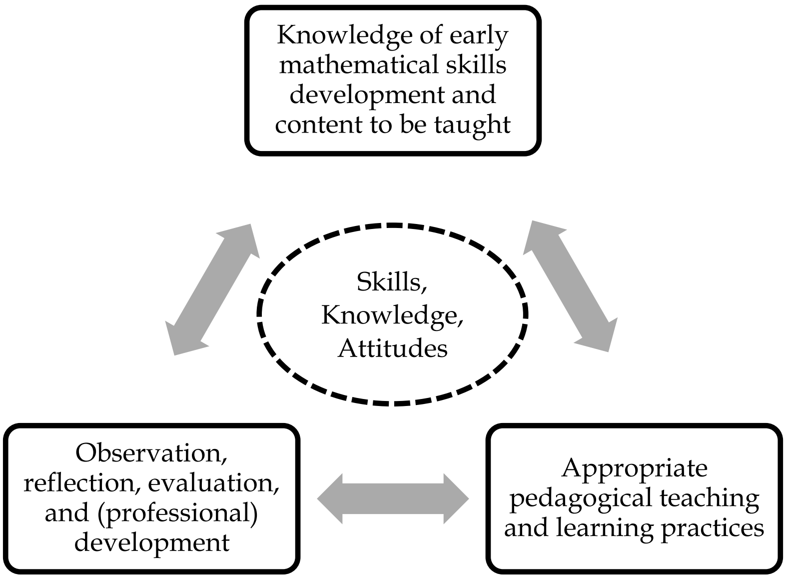

FIGURE 1 Teachers' pedagogical awareness in early mathematical teaching

I have discussed each of these three elements in Chapters 2.1, 2.2, and 2.3 to emphasise their specific contribution to early mathematical teaching, including how separately and together they affect this type of teaching. The concept of 'pedagogical awareness', which brings together all three elements as illustrated in Figure 1, can be connected to Mason's (1998, 2002) conceptualisation of awareness, which relates to the importance of noticing what and how one teaches and why. The word 'pedagogy' stems from the Greek root 'paidagōgos' (παιδαγωγός) (MOT Oxford Dictionary of English, n.d.), which consists of two words, (1) 'pais'=child and (2) agōgos=leader, and is commonly connected to professionals who use their multi-scientific knowledge to plan and implement goal-oriented practices to enhance children's development and learning, whilst emphasising a child-centred approach in ECEC (Farquhar & White, 2014). The conceptualisation of teachers' pedagogical awareness, thus, relates to their ability to use their knowledge of early mathematical skills development and content in applying appropriate pedagogical practices that support children's learning, as well as to their abilities to observe, reflect, evaluate, and develop their skills, knowledge, practices, and attitudes as needed to implement high-quality early childhood mathematics education (see Figure 1).

Although different elements that modify early mathematical teaching have been explored separately and from various perspectives, there is a need for research related to the role of ECEC teachers' pedagogical awareness in early mathematical teaching and the teaching of different early mathematical skills. More studies are also required in relation to teachers' self-reflection and evaluation of their awareness of early mathematical teaching since when ECEC teachers have participated in mathematics PD programmes, improvements have been indicated only in teaching practices and content knowledge (Chen et al., 2014; Gasteiger & Benz, 2018; Knaus, 2017; Lindmeier et al., 2020) not in reflection

(Chen et al., 2014; Knaus, 2017; Lindmeier et al., 2020). Consequently, we do not know how teachers reflect on and develop their pedagogical awareness of early mathematical teaching, such as understanding mathematical skills development, as well as their pedagogically appropriate teaching and learning practices, in relation to their existing knowledge and practices when they participate in mathematics PD programmes. Additionally, our knowledge is lacking about how they describe changes in their knowledge and practices, as well as what practices and elements of PD and PD programmes are critical to the changes. These research gaps relate back to Helenius' (2018) thought about teaching eyes to look for mathematical phenomena and the needed modes of action as a critical part of PD and high-quality early mathematical teaching. My study, therefore, explores these issues to provide a comprehensive understanding of the role of teachers' pedagogical awareness in early mathematical teaching and how teachers expand their awareness when they participate in PD programmes for early childhood mathematics.

3 AIMS OF THE RESEARCH

Based on the addressed shortcomings of research in the field of early childhood mathematics education, the purpose of this mixed methods research (MMR) is to provide a theoretical framework for the holistic development of early mathematical skills based on current research, empirical knowledge of early mathematical teaching, and the pivotal role of teachers' pedagogical awareness and mathematics PD programmes in early mathematical teaching. The first aim is to construct a theoretical model of the holistic development of early mathematical skills based on current research-based evidence and thereafter, to investigate the correspondence with teaching these skills to 3- to 7-year-old children to understand early mathematical teaching in Finnish ECEC and pre-primary education. The second aim is to explore the role of ECEC teachers' pedagogical awareness and mathematics PD programmes in early mathematical teaching in ECEC and pre-primary education to comprehend how these are translated into early mathematical teaching. I formulated the following research questions to achieve these aims:

1) What is the current research-based understanding of the holistic development of early mathematical skills, and how does teaching early mathematical skills to 3- to 7-year-old children correspond to it in ECEC and pre-primary education?
2) To what extent and how does ECEC teachers' pedagogical awareness of early mathematical teaching and their participation in mathematics PD programmes contribute to the early mathematical teaching of 3- to 7-year-old children?

I designed the three sub-studies to assist in building knowledge of the explored phenomena both from theoretical and empirical perspectives and applied MMR design to answer the research questions comprehensively. The relationships among the purpose and aims of the study, the research questions of this compilation article, and the three sub-studies are presented in Table 1.

TABLE 1 Study purpose, research aims and questions, and their connections to related sub-studies I–III

Purpose of the study	Aims of the research	Research questions of the compilation article	Related sub-studies
To provide up-to-date knowledge of the holistic development of early mathematical skills and the teaching of these skills to 3- to 7-year-old children, and the pivotal role of teachers' pedagogical awareness and mathematics PD programmes in early mathematical teaching.	To construct a research-based model of the holistic development of early mathematical skills and to explore the teaching of different early mathematical skills to 3- to 7-year-old children in correspondence of early mathematical skills development in Finnish ECEC and pre-primary education.	(1) What is the current research-based understanding of the holistic development of early mathematical skills, and how does teaching early mathematical skills to 3- to 7-year-old children correspond to it in ECEC and pre-primary education?	**Sub-study I:** Holistic development of early mathematical skills (qualitative) **Sub-study II:** Frequencies of teaching numerical skills, spatial thinking skills, and mathematical thinking and reasoning skills to 3- to 7-year-old children in Finnish ECEC and pre-primary education (quantitative)
	To explore the role of teachers' pedagogical awareness and mathematics PD programmes in teaching early mathematical skills to 3- to 7-year-old children in Finnish ECEC and pre-primary education through a comprehensive understanding of their contribution to teaching.	(2) To what extent and how does ECEC teachers' pedagogical awareness of early mathematical teaching and their participation in mathematics PD programmes contribute to the early mathematical teaching of 3- to 7-year-old children?	**Sub-study II:** Associations between ECEC teachers' pedagogical awareness, participation in PD programmes, and teaching different early mathematical skills to 3- to 7-year-old children (quantitative) **Sub-study III:** ECEC teachers' descriptions of the changes in their pedagogical awareness of teaching early mathematical skills to 3- to 7-year-old children and the sustainability of the change in the context of mathematics PD programme (qualitative and quantitative)

To answer the first research question, sub-study I built a comprehensive understanding of early mathematical skills development related to 0- to 7-year-old children and the connections and interlinkages between the skills, leading to a holistic model of the development of early mathematical skills. Sub-study I served as the theoretical basis for my further investigation into how frequently different early mathematical skills are taught to groups of 3- to 5-year-old, 5- to 6-year-old, and 6- to 7-year-old children in Finnish ECEC and pre-primary education in sub-study II. To answer the first research question, sub-studies I and II together provided a comprehensive picture of the correspondence between teaching different early mathematical skills to children of different ages and current research-based knowledge of the holistic development of early mathematical skills.

To answer to the second research question, sub-study II provided new insights into variations in ECEC teachers' pedagogical awareness of teaching different early mathematical skills. It also provided understanding of how teachers' pedagogical awareness and participation in PD programmes in mathematics are associated with the frequencies of their teaching different early mathematical skills to 3- to 7-year-old children. Sub-study III aimed to comprehend ECEC teachers' awareness of their strengths and weaknesses related to teaching different early mathematical skills, as well as their teaching and learning practices. Sub-study III also provided understanding of the mechanisms of mathematics PD programmes in enhancing pedagogical awareness of early mathematical teaching. Sub-studies II and III endeavoured to ascertain the preconditions for a high-quality early childhood mathematics education and the successful expansion of teachers' pedagogical awareness in teaching early mathematical skills in ECEC and pre-primary education.

4 MIXED METHODS RESEARCH DESIGN

My study is aligned with MMR design as I utilised both qualitative and quantitative approaches and research methods to obtain a complete and in-depth understanding of current research-based theory of the holistic development of early mathematical skills, correspondence of teaching these skills to 3- to 7-year-old children in relation to typical skills development, and the pivotal role of teachers' pedagogical awareness and mathematics PD programmes in teaching early mathematical skills (Johnson et al., 2007; Johnson & Onwuegbuzie, 2004; Schoonenboom & Johnson, 2017). I considered MMR design to be the optimal for constructing the study to explore and clarify the relationships among the studied phenomena and thus, to answer the research questions comprehensively (Creswell & Plano Clark, 2018; Johnson & Turner, 2003; Schoonenboom & Johnson, 2017).

Quantitative and qualitative components, approaches, and thinking were equally valued in my study, and they alternated and interacted throughout the research process (Creswell & Plano Clark, 2018; Greene, 2015). In doing so, I aimed to seek complementary strengths and avoid overlapping weaknesses of quantitative and qualitative research, which is the fundamental principle of MMR, as stated by Johnson and Turner (2003). The MMR design was comprehensive in my study as I integrated MMR at multiple levels, including in the: methods, methodology and paradigm (Creswell & Plano Clark, 2018; Greene, 2015) and present next the integration by following this order.

4.1 Data and methods

My MMR design study consists of three sub-studies, each based on their own data and methods (Table 2).

TABLE 2 Research data and its collection, research questions, and analysis of sub-studies I–III

Sub-studies	Data and its collection	Research questions (RQ)	Analysis
Sub-study I	134 peer-reviewed articles and research studies published from 2003–2018, identified via electronic searches using the Education Resources Information Center	RQ1: In what way are early mathematical skills conceptualised, and can they be categorised based on the existing research literature? RQ2: What kind of holistic framework describing the development of early mathematical skills can be built?	Qualitative content analysis focusing on literature analysis
Sub-study II	*N* = 206 ECEC teachers, web survey (Webropol) carried out from 1–3/2020	RQ1: Are there differences in the frequency of teaching or in teachers' pedagogical awareness of teaching numerical skills, spatial thinking skills, and mathematical thinking and reasoning skills? Additionally, are these potential differences moderated by the children's age group? RQ2: To what extent do teachers' pedagogical awareness, duration of mathematics PD programmes, age and work experience, as well as the children's age group, explain variations in teaching of NS, STS, and MTRS? What is the relative importance of these factors?	Parametrical statistical analysis, IBM SPSS Statistics 27 RQ1: Multivariate analysis for repeated measures RQ2: Correlation analysis + a univariate analysis of variance
Sub-study III	*N* = 7 ECEC teachers, semi-structured individual interviews over Zoom carried out during 6/2020, pre-PD and follow-up questionnaires administered during 1/2020 and 2/2021, respectively	RQ1: How do teachers in ECEC describe the changes occurring in their pedagogical awareness of teaching early mathematical skills after participation in a tailored mathematics PD programme? RQ2: Do the pre-PD and follow-up-questionnaires completed by the teachers validate possible increases in teachers' pedagogical awareness of teaching early mathematical skills (i.e., numerical skills, spatial thinking skills, and mathematical thinking and reasoning skills) and reveal long-lasting changes nine months after the end of the PD program?	RQ1: Qualitative thematic analysis RQ2: Non-parametrical statistical analysis, IBM SPSS Statistics 28: One-tailed Wilcoxon Signed Ranks Test

Although I utilised different data and methods in the sub-studies, as Table 2 illustrates, integration of methods in the compilation, as a basic component of integration in MMR design (Greene, 2015), enabled me to engage in a holistic and comprehensive investigation of early mathematical teaching of 3- to 7-year-old children in relation to their early mathematical skills development as well as the role of teachers' pedagogical awareness and PD programmes in early mathematical teaching.

4.1.1 Data and methods of sub-study I

In sub-study I, I identified English-language peer-reviewed articles for the data of the systematic review through the Education Resources Information Center (ERIC) for electronic searches. I limited the education level to early childhood education. I determined the initial search terms (i.e., 'mathematics education', 'mathematics instruction', 'mathematics', 'mathematics skills', 'numeracy', 'numerical skills', 'geometric', 'measurement', 'spatial ability', 'spatial relationship', 'algebra', 'early algebra', and 'early childhood education') based on Clements and Sarama's (2007, 2009) extensive pioneering work and, with the help of a thesaurus, I completed them with subject headings and frequently used keywords in the articles. A comprehensive and high-quality search (see Cooper, 2019; Reed & Baxter, 2009; White, 2019) yielded 492 peer-reviewed articles.

Since the aim of sub-study I was to obtain data from the latest studies into the learning and development of early mathematical skills and broaden Clements and Sarama's (2007, 2009) earlier findings, I selected relevant articles for detailed investigation based on their titles and dates (limited to the period from 2003 to 2018) (Page et al., 2021; White, 2019). By selecting articles based on the aforementioned criteria, I was left with 273 articles for the next step in the analysis. I continued the identification by refining the search through an examination of the articles' abstracts (Page, 2021; White, 2019). During this phase, I also thoroughly reviewed the content of the articles, which covered age groups from infants to 8-year-old children. In doing so, I narrowed the data down to 134 articles and literature related to the development of early mathematical skills (Cooper et al., 2019; Page et al., 2021; White, 2019), which I analysed using qualitative content analysis focusing on literature analysis (Cooper, 2019; Krippendorff, 2004; Mason, 2009; Patton, 2015; Ryan & Bernard, 2000; Wilson, 2019).

I first categorised the articles according to conceptual codes (e.g., number sense, symmetry, geometry, and patterning), and then, I organised them into 13 broader thematic clusters emerging from the data (e.g., counting skills, spatial reasoning, geometrical awareness, and problem-solving strategies). Thereafter, I further divided the thematic clusters into typologies, and I formed the following three early mathematical skills categories: (1) numerical skills (NS), (2) spatial thinking skills (STS), and (3) mathematical thinking and reasoning skills (MTRS). Of the 134 analysed articles, 62 were related to numerical learning, 54 focused on spatial thinking skills (e.g., spatial and geometrical learning, and measurement), and 18 discussed mathematical thinking and reasoning processes (e.g., algebra, data analysis, and patterning) separately. Most of the articles focused on 7- to 8-year-old children.

I continued the analysis by further examining the connections among the conceptual codes, thematic clusters, and skills categories through crosswise comparison (e.g., articles connecting numerical learning and mathematical reasoning processes; spatial and numerical learning; and numerical learning, spatial learning, and mathematical thinking and reasoning). In doing so, I found interconnections among the skills categories with which to construct my holistic model of early mathematical skills development.

4.1.2 Data and methods of sub-study II

In sub-study II, I used research data from 206 teachers of 3- to 7-year-old children, who had formal teaching qualifications in ECEC and pre-primary education (varying from a university-level master's degree to a college-level degree) and worked in Finnish-language early education centres in the public sector. As Table 2 illustrates, I collected the data via the 'Teaching Early Mathematical Skills' web survey (Webropol) between January and March 2020.

To ensure a representative sample of Finnish ECEC teachers, I followed a thorough sample selection procedure (Johnson & Christensen, 2017; Newby, 2014). Thus, I used stratified sampling and included a variety of cities and towns from different geographical areas of Finland to guarantee geographical representativeness and the inclusion of different-sized municipalities (Table 3). In this phase, I obtained research permissions from the administration of the municipalities' early education services in accordance with their decision-making protocols, and thereafter, I selected early education centres within each municipality using systematic sampling (Johnson & Christensen, 2017; Newby, 2014). I conducted the systematic sampling of early education centres by choosing every fifth centre from an alphabetical or area list I found on the municipal website. After receiving administrative approvals for conducting my research, I sent emails to the heads of early education centres asking them to distribute research invitations to centre teachers who worked with 3- to 7-year-old children and had formal teaching qualifications in ECEC and pre-primary education. I also requested that they report the number of teachers to whom they sent the research invitations to determine the actual sample size and attrition rate.

TABLE 3 Number of responses based on the location and number of inhabitants of the city, town, or municipality of the workplaces

Location and number of inhabitants related to workplaces	Number of early education centres in the sample (*N* = 102)	Number of teachers who received the survey (*N* = 557)	Percentage of answers (*N* = 206)

Location of work-place in Finland			
Lapland	4	12	1%
Northern Finland	19	105	12.5%
Eastern Finland	16	80	16.5%
Western and Central Finland	19	128	32.5%
Southern Finland	34	151	27.5%
South Western Finland	10	81	9%
Unknown			1%
Number of inhabitants of the city, town, or municipality			
~200000	47	244	29%
100000–150000	29	194	45.5%
<500000	26	126	24%
Unknown		7	1.5%

As Table 3 illustrates, 557 teachers from 102 early education centres received the web survey, of whom 206 responded within the four-week deadline. This resulted in a response rate of 37%. Because I distributed the invitations via the heads of early education centres, I did not get information concerning those who declined the survey. If the respondents' gender division (*N* = 206; 196 women, 8 men, 1 gender of 'other', 1 no answer) is compared to teachers' gender distribution in Finnish ECEC and pre-primary education (Finnish National Agency for Education, 2017), there is a clear connection between the respondents' genders and teachers' gender distribution.

Measures

I developed the 'Teaching Early Mathematical Skills' web survey, which I used in sub-study II and partly in sub-study III, based on *the holistic model of early mathematical skills development*, which I constructed in sub-study I. I employed the content of the three skill categories (NS, STS, and MTRS) as the basis for formulating the survey items and calculating the scale scores. In other words, I operationalised the theoretical concepts into quantitative measures for the analyses. Before the final survey, I conducted two pilot web surveys (*N* = 20 and *N* = 18) to test the internal consistency of the scales. Based on the feedback from the teachers who completed the pilot web surveys, I clarified the formulation of the items for the final survey. The final 'Teaching Early Mathematical Skills' web survey included 86 closed-ended questions, which I divided into three parts (Table 4).

TABLE 4 Categories, number of questions, and reliability

Categories of questions	Number of questions	Cronbach's alpha
Respondents' background information	9	
Teaching frequency of		
NS	18	.80
STS	20	.75
MTRS	24	.84
Pedagogical awareness of teaching		
NS	5	.71
STS	5	.73
MTRS	5	.67

Note. NS=Numerical skills, STS=Spatial thinking skills, MTRS=Mathematical thinking and reasoning skills.

The first part included the respondents' background information, which I asked via nine questions. These concerned age; gender; professional qualifications; professional title; work experience in ECEC and pre-primary education; duration of PD programmes in mathematics; workplace location; number of residents of the city, town, or municipality; and the children's age group (3- to 5-year-olds, 5- to 6-year-olds, or 6- to 7-year-olds). The children's age groups were based on their typical age-based groupings in Finnish early education centres, meaning teachers were able to select the children's age group based on these prescribed groupings into which they commonly organise daily activities. Because Multivariate analyses of variance (MANOVA) showed that neither the area of Finland (Lapland, North, East, West and Central, South, South West, $F(15, 541) = 0.51$, $p = .936$) nor the size of the municipality (city, town, municipality, $F(6, 396) = 0.88$, $p = .514$) had an effect on the frequency of teaching NS, STS, and MTRS, I did not consider these in the final analysis.

As Table 4 shows, the second part of the 'Teaching Early Mathematical Skills' web survey included 62 questions focusing on how frequently respondents taught NS (18 questions), STS (20 questions) and MTRS (24 questions). I divided questions related to the frequency of teaching NS into three subscales: number and quantity knowledge, counting skills, and addition and subtraction skills. I also divided questions related to the frequency of teaching STS into three subscales: spatial reasoning, geometrical awareness, and sense of time. Questions related to the frequency of teaching MTRS, I divided into four subscales: mathematical-logical and analytical thinking, problem-solving and reasoning; comparison; classification; and seriation.

In addition to asking respondents to answer questions about the frequency of teaching NS, STS, and MTRS more generally using an interval scale with three questions (one for each skill category), I asked them to answer the rest of the claims concerning the teaching frequencies of NS, STS, and MTRS on a sliding scale, i.e., from 1 to 7 (1='I strongly disagree' and 7='I strongly agree'). The items in each scale included both direct and indirect claims related to teaching certain skills. Sample statements concerning the frequency of teaching NS include "I often teach counting skills (e.g., counting children during a morning circle, play-based counting activities, counting spoons during mealtimes)" and concerning the frequency of teaching MTRS include 'I often teach mathematical-logical thinking (i.e., logic games, construction series, and problem-solving assignments) '. I added one reversed item for each scale to keep the respondents focused and prevent mechanical answering. For example, 'I rarely teach directions and locations (e.g., above, beneath, in front of, behind, far, near) ', which concerned the frequency of teaching STS.

Because I categorised the three skill categories (NS, STS, and MTRS) in sub-study I (*the holistic model of early mathematical skills development*), I formulated and classified the questions based on its results. Each item in the survey concerning the teaching of NS, STS, and MTRS belonged primarily to a pre-determined sub-scale, for instance, counting skills (included in NS), spatial reasoning (included in STS), or comparison (included in MTRS). I also designed the scales to cover their teaching from initial to more sophisticated skills. Since I also explored the interconnections of these skill categories in sub-study I, several of the claims measured, to some extent, the teaching of one or both of the other two skill categories (i.e., NS and STS or NS, STS, and MTRS). I derived scale scores for frequencies of teaching NS, STS, and MTRS by calculating the arithmetic means from their items. The Cronbach's alpha for each score, determined to ensure the internal consistency of each measure (see Table 4), showed that the reliabilities of all three scales were above the preferred ≥.70 (Johnson & Christensen, 2017).

The third part of the 'Teaching Early Mathematical Skills' web survey included 15 questions (Table 4) related to teachers' self-evaluation of their pedagogical awareness of teaching NS, STS, and MTRS (five questions for each). I presented respondents with similar questions related to pedagogical awareness of teaching each skill category (NS, STS, and MTRS). Each category covered the following five topics: (1) content knowledge of the skill category; (2) significance of the skill category in the teaching of early mathematical skills; (3) evaluation of how strongly one's teaching is based on a firm theoretical understanding of the development of the skill category; (4) up-to-date knowledge of the development of each skill in children; and (5) evaluation of the need for new practices for teaching the skill category.

Similarly, as in the second part of the 'Teaching Early Mathematical Skills' web survey, I asked respondents to answer the questions regarding their pedagogical awareness of teaching NS, STS, and MTRS on a sliding scale, i.e., 1 to 7 (1='I strongly disagree' and 7='I strongly agree'). For instance, pedagogical awareness of teaching NS included the following statement: 'My teaching of NS is based on strong content knowledge of the development of NS in children'. Again, I used one reversed item in each scale. This involved asking the respondents to evaluate their pedagogical awareness from an opposite perspective. For instance, pedagogical awareness of teaching NS included the following statement: 'I do not have up-to-date knowledge on how children learn NS'.

I derived scale scores for the frequencies of teaching NS, STS, and MTRS, as well as for pedagogical awareness of teaching NS, STS, and MTRS, by calculating the arithmetic means from their items. The Cronbach's alpha for each score was determined to ensure the internal consistency of each measure. As Table 4 shows, the reliabilities of the scales concerning the frequencies of teaching NS, STS, and MTRS and the scales concerning pedagogical awareness of teaching NS and STS were above the preferred ≥.70 (Johnson & Christensen, 2017). However, the reliability of the scale concerning pedagogical awareness of teaching MTRS was .67.

I continued ensuring that the requirements for the parametric statistical analyses were fulfilled by examining distributions of the mathematical scale scores. Table 5 shows that all distributions were normal or close to normal as, in all measures, skewness/standard error of skewness and kurtosis/standard error of kurtosis were below or close to 2.

TABLE 5 Descriptive statistics of scale scores related to frequency and pedagogical awareness of teaching early mathematical skills

	Age group of children											
	3- to 5-year-olds				5- to 6-year-olds				6- to 7-year-olds			
Scale	Mean	SD	Skew (SE)	Kurt (SE)	Mean	SD	Skew (SE)	Kurt (SE)	Mean	SD	Skew (SE)	Kurt (SE)
Teaching frequency of												
NS	4.38	0.91	-0.32 (0.24)	-0.19 (0.47)	5.01	0.84	-0.51 (0.44)	0.13 (0.86)	5.40	0.74	-0.46 (0.28)	-0.33 (0.56)
STS	4.35	0.80	-0.19 (0.24)	-0.07 (0.47)	4.56	0.78	-0.56 (0.44)	-0.50 (0.86)	4.75	0.86	-0.12 (0.28)	-0.23 (0.56)
MTRS	3.89	0.90	-0.16 (0.24)	-0.02 (0.47)	4.42	0.83	0.51 (0.44)	-0.40 (0.86)	4.68	0.96	-0.12 (0.28)	-0.48 (0.56)
Pedagogical awareness of teaching												
NS	3.95	1.16	0.26 (0.24)	-0.20 (0.47)	4.52	1.18	-0.15 (0.45)	0.39 (0.87)	4.91	1.13	0.04 (0.28)	-0.97 (0.56)
STS	3.67	1.17	0.24 (0.24)	0.01 (0.47)	4.04	1.10	0.22 (0.45)	0.39 (0.87)	4.15	1.18	0.31 (0.29)	-0.10 (0.57)
MTRS	3.85	1.17	0.31 (0.24)	-0.16 (0.47)	4.38	1.06	-0.02 (0.45)	1.24 (0.87)	4.70	1.06	0.25 (0.28)	-0.62 (0.56)

Note. NS=Numerical skills, STS=Spatial thinking skills, MTRS=Mathematical thinking and reasoning skills.

To answer the first research question of sub-study II, whether the frequency of teaching early mathematical skills varied according to skill category and children's age group, I used a MANOVA for repeated measures (Table 2). For this, I employed the scale score of the teaching frequency in each skill category (NS, STS, and MTRS) as the within-subject factor and the children's age group as the between-subject factor. Additionally, I used Post hoc pairwise comparisons of skill categories, using Bonferroni correction for significance to examine possible differences in the mean frequencies of teaching NS, STS, and MTRS separately for each children's age group. To comprehensively answer the first research question, I also employed a MANOVA for repeated measures to investigate whether teachers' pedagogical awareness varied according to skill category (NS, STS, and MTRS). For this, I utilised the scale score of pedagogical awareness in each skill category (NS, STS, and MTRS) as the within-subject factor and the children's age group as the between-subject factor. Again, I employed Post hoc pairwise comparisons, using Bonferroni correction for the significance, to explore the differences in the mean level of teachers' pedagogical awareness of teaching NS, STS, and MTRS separately for each children's age group.

To answer the second question of sub-study II, i.e., to figure out to what extent certain characteristics of teachers (age, work experience, pedagogical awareness, and duration of mathematics PD programmes) and children's age group were related to the teaching frequency of NS, STS, and MTRS, as well as to determine the relative importance of these factors, I first inspected Pearson correlations between the background measures and the scale scores of the frequencies of teaching NS, STS, and MTRS. Thereafter, I employed univariate analysis of variance separately for each scale score to determine the significant factors for the teaching frequency in each skill category. I aimed to determine which factors had a unique effect on the outcome when added simultaneously to the model. In addition, I explored the relative importance of each factor by reporting the percentage of variance explained by each independent factor (i.e., characteristics of teachers and children's age group).

4.1.3 Data and methods of sub-study III

As I applied the MMR design in sub-study III, I used both quantitative and qualitative data collected from seven ECEC teachers who worked with 3- to 7-year-old children and participated in a tailored PD programme in mathematics (part of the national LUMA2020 Development programme). The qualitative data consisted of semi-structured individual interviews and took place in June 2020 after the PD programme had ended, whereas quantitative pre-PD and follow-up questionnaires were completed in January 2020 and February 2021, respectively. The aim of the data collection was to explore changes in ECEC teachers' pedagogical awareness of early mathematical teaching when participating in a tailored PD programme in mathematics (Table 2).

None of the teachers participating in the study had previously taken a mathematics PD programme. They represented five early education centres and worked with typical age groups: three teachers worked with 3- to 5-year-old children and four with 5- to 7-year-old children. The teachers' work experience varied from fewer than five years (n = 1), to more than 10 years (n = 2), and to more than 20 years (n = 4).

The aim of the tailored PD programme was to enhance teachers' pedagogical awareness of teaching early mathematical skills holistically in play and different daily life situations in ECEC and pre-primary education. Based on this aim, I designed and conducted the PD programme with a colleague from the University of Jyväskylä. The holistic model of the development of early mathematical skills, which I had constructed in sub-study I, was the theoretical basis for the PD programme. It was used, for example, for collegial brainstorming around topics such as how mathematical content could be taught to children of different ages. I had also developed a reflective journal based on the model, which teachers used for self-reflection during the PD programme. However, the data from the reflective journals were not included in this doctoral dissertation.

I utilised characteristics of participatory action research to some extent in sub-study III (Kemmis & McTaggart, 2005; McIntyre, 2008) because of their philosophical similarities to transformative learning in PD. I employed self-reflective cycles and collaborative learning, which are common in participatory action research (Kemmis & McTaggart, 2005; McIntyre, 2008) and transformative learning (Cranton, 2016; Mezirow, 1991; see also Mason, 1998, 2002, 2011) in the programme, while the study aimed to understand how teachers constructed and attached meanings to their pedagogical awareness of teaching early mathematical skills.

The qualitative semi-structured interview data, which I collected after the PD programme in June 2020, focused on three areas. The first concerned questions related to elements of the PD programme (e.g., 'How did the LUMA2020 Development programme promote your pedagogical awareness of early mathematical teaching?' and 'During the programme, what prevented or slowed down your PD related to early mathematical teaching?'). The second area focused on questions related to teachers' pedagogical awareness in teaching early mathematical skills (e.g., 'What area of teaching early mathematical skills was your weakest at the beginning of the programme, and what happened to it during the programme?'). The third area concerned reflection (e.g., 'How would you reflect on your professional development during the LUMA2020 Development programme?'). During the interview, the teachers had the printed training materials, namely, the theoretical packages and personal reflective journals. Thus, the teachers could and did use the printed material to reflect on their answers. I video-recorded all the interviews (ranging from 75 to 90 minutes). These were later transcribed, yielding 112 pages (font 10, single spaced) of transcribed text.

To answer the first research question of sub-study III, which was about exploring ECEC teachers' descriptions of the changes in their pedagogical awareness of teaching early mathematical skills after participation in a tailored mathematics PD programme, I used the interview data and analysed them with thematic analysis. According to Braun and Clarke (2006, 2022), thematic analysis includes six phases, which I also applied in the analytical process. In the first phase, I read the data several times to familiarise myself with them. In the second phase, I generated initial codes and collated initial themes through inductive and explorative orientations. In the third phase, I applied deductive orientation, informed by extant research literature, for arranging the codes according to main and sub-themes related to pedagogical awareness and early mathematical skills. In the fourth phase of the thematic analysis, I reviewed the themes and sub-themes to determine whether they worked in relation to the codes and the entire dataset. Therefore, at this point, I used inductive, explorative, and critical orientations for rearranging the coded data extracts to generate themes, which concerned mathematical skills and content, children's perspectives on mathematical learning, and teachers' perspectives on mathematics teaching (cf. Braun & Clarke, 2006, 2022). Again, I re-examined these.

In the re-examination included in the fifth phase of the thematic analysis, I employed both deductive orientation and a critical approach to arrive at a clearer understanding of the changes that occurred in the teachers' pedagogical awareness of teaching early mathematical skills during the PD programme (Braun & Clarke, 2006, 2022). Thereafter, I employed a deductive orientation and critical approach to more precisely comprehend the three main themes and sub-themes as follows: (1) developmentally appropriate mathematical content, including holistic skills development, NS, STS, and MTRS; (2) child-initiated mathematical learning, including children's interests and initiatives and collaborative mathematical learning; and (3) holistic mathematical teaching, including integrating mathematics into play, daily situations, and pre-planned activities. Essentially, I alternated between the dataset and the literature throughout the analytical process to more deeply refine the analytic work (Braun & Clarke, 2006, 2022). By doing so, in the sixth and final phase of the thematic analysis, I generated a categorisation of themes through which changes in teachers' pedagogical awareness of teaching early mathematical skills during the tailored PD program could be discussed (see Chapter 5).

To minimise the incidence of subjective bias when analysing the data, I distributed the coded material in each analytical phase to my supervisors and discussed each phase critically with them. Together, we carefully scrutinised the interpretations. This member-checking technique was an essential part of my study as it permitted the in-depth scrutiny of the analytical results and increased their trustworthiness since I planned and conducted the PD programme and interviews (Newby, 2014). Therefore, the member-checking technique also added a high level of transparency to my analytical process. To further confirm and strengthen the trustworthiness of the analysis results, I calculated the inter-rater reliability from the coding of two interviews according to the sub-themes conducted by one of my supervisors, reaching 93% agreement.

To answer the second research question of sub-study III, whether the pre-PD and follow-up questionnaires completed by the ECEC teachers validate possible increases in teachers' pedagogical awareness of teaching NS, STS, and MTRS and reveal long-lasting changes nine months after the end of the PD programme, I collected quantitative pre-PD data in January 2020 and follow-up data in February 2021 (Table 2). For this data collection, I utilised part of the 'Teaching Early Mathematical Skills' web survey that I created for sub-study II (cf. Chapter 2.1.2). In sub-study III, I used 15 questions from the survey related to teachers' pedagogical awareness of teaching NS, STS, and MTRS. Cronbach's alpha and descriptive statistics related to the categories are presented in Tables 4 and 5, respectively. I used a Non-parametric One-tailed Wilcoxon Signed Ranks Test separately for the scale scores of pedagogical awareness related to NS, STS, and MTRS to test whether the teachers' pedagogical awareness increased during the PD programme and the changes remained nine months after its conclusion. Additionally, to calculate their effect sizes, I applied Cohen's d using the pooled standard deviation of the two assessments (cf. Cohen, 1992 for criteria of different magnitudes of effect size).

4.2 Methodology and paradigm

The second level of integration of MMR involved decisions related to the methodology (Creswell & Plano Clark, 2018; Greene, 2015). One concerned the timing of the quantitative and qualitative components, which in this study were utilised partially concurrently and partially sequentially (Johnson et al., 2007; Johnson & Onwuegbuzie, 2004; Schoonenboom & Johnson, 2017). I conducted sub-studies II and III concurrently, meaning that I collected the data for these studies partly simultaneously, whereas I carried out sub-study I before these two studies (see Figure 2).

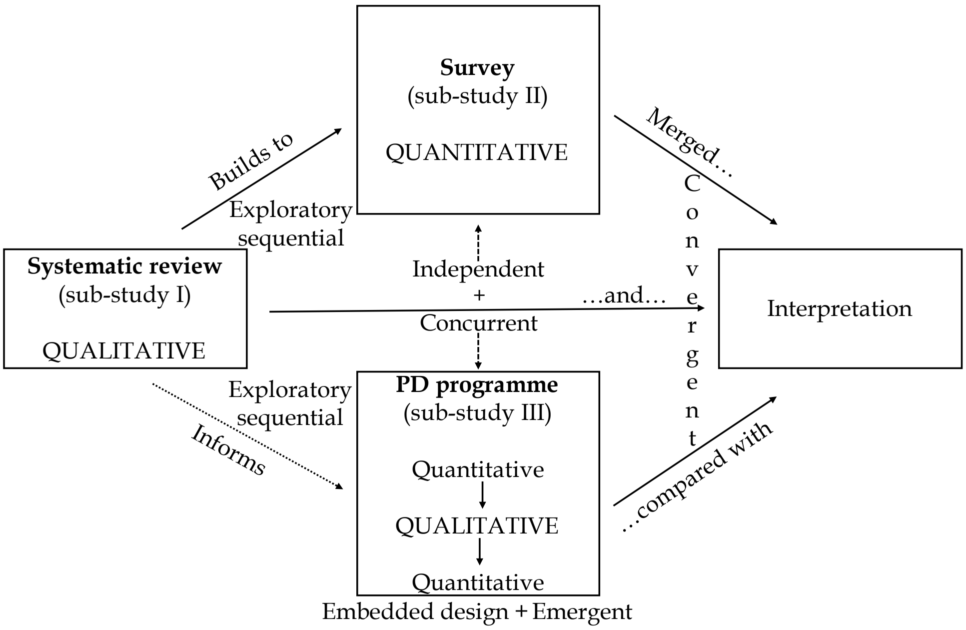

FIGURE 2 Hybrid multiphase mixed methods research design

I chose this method because the results of sub-study I served as a theoretical basis for the other two studies. I used the results of sub-study I in the development of the 'Teaching Early Mathematical Skills' web survey (sub-studies II and III) and in conducting the PD programme and exploring ECEC teachers' pedagogical awareness of teaching early mathematical skills during the PD programme (sub-study III). Therefore, sub-studies II and III were dependent on the results of sub-study I, meaning the MMR design of this part of the study was exploratory sequential (Creswell & Plano Clark, 2018; Schoonenboom & Johnson, 2017). Conversely, sub-studies II and III were independent although I conducted them concurrently (see Figure 2) because neither their implementation nor results were dependent on the results of the data analysis of the other study (Creswell & Plano Clark, 2018; Schoonenboom & Johnson, 2017). Additionally, in sub-study III, I conducted qualitative and quantitative components sequentially while they were also independent.

The other methodological decision for my study concerned the point of integrating the quantitative and qualitative components (Creswell & Plano Clark, 2018; Schoonenboom & Johnson, 2017). In sub-study III, I used integration for triangulation, meaning that I employed the quantitative findings to determine whether those together with the qualitative findings yielded convergent results and to discuss possible divergent results (Creswell & Plano Clark, 2018; Schoonenboom & Johnson, 2017). The emphasis was on the qualitative method, whereas the quantitative method was complementary (see Figure 2). In this compilation report, the point of integration took place in the setting of the two research questions (Chapter 3) and in aiming to answer these by explaining the diverse results of the sub-studies (Chapter 6). I applied principles of convergent design in this part of the study as I integrated the quantitative and qualitative methods to answer the research questions thoroughly (Creswell & Plano Clark, 2018).

Although I planned to use a qualitative approach for sub-study I and a quantitative method for sub-study II in the initial phase of the dissertation study, employing a quantitative component in sub-study III was emergent (Figure 2) (Creswell & Plano Clark, 2018; Schoonenboom & Johnson, 2017). My initial idea in sub-study III was to utilise only qualitative data due to its richness. I, however, added the quantitative results to validate the qualitative findings and to explore the sustainability of the changes investigated in the study (e.g., improved self-reflection) (Barber et al., 2014; Knaus, 2017; Lindmeier et al., 2020). The emergent design was, thus, a critical part of conducting sub-study III. Adding the quantitative results increased the credibility of the qualitative results, as these were not dependent on my possible influence through my dual role of trainer and an interviewer.

In short, the typology of this doctoral dissertation, as proposed by Schoonenboom and Johnson (2017), could be considered a hybrid design (see Figure 2) since different typologies could be used describe the overall design and its parts (sub-studies I–III). Characteristics of multiphase design can be detected in the timing (simultaneity and dependence), and convergent parallel design can be found in sub-studies II and III, as those were performed independently, while their results together with the results of sub-study I were brought together in the overall interpretation (related to the point of integration) (see Figure 2) (Creswell & Plano Clark, 2018). However, the typology of sub-study III was embedded design because the quantitative component was added to strengthen the design and validate the results (Creswell & Plano Clark, 2018). Given this overall complexity, a hybrid design best describes the typology of this doctoral dissertation.

Finally, I integrated MMR design into the paradigm, in my study pragmatism (Creswell & Plano Clark, 2018; Greene, 2015). As pragmatism is considered to offer a useful middle position philosophically and methodologically, it allowed me to take a non-purist stance on quantitative and qualitative research (Johnson et al., 2007; Johnson & Onwuegbuzie, 2004). In this study, it enabled me to investigate the research questions from both viewpoints (i.e., utilise both qualitative and quantitative data) and thereby have the best opportunities to answer them (Johnson et al., 2007; Schoonenboom & Johnson, 2017). The aim was to fully understand the studied phenomenon and to build knowledge (theory and practice) by considering different perspectives and standpoints (qualitative and quantitative) (Johnson et al., 2007). Hence, as stated by Johnson and Onwuegbuzie (2004), pragmatism offered me an epistemological justification to construct knowledge based on acknowledging its tentative nature, as well as the fact that knowledge is built on our lived experiences and thus, provisional. During the study, I made pragmatic choices regarding the methodology as explained earlier (e.g., timing, point of integration, and planned versus emergent design). Simultaneously, pragmatism offered a logic for constructing an MMR design to answer the research questions compared to a more basic study design (Creswell & Plano Clark, 2018; Schoonenboom & Johnson, 2017).

4.3 Ethical considerations

A fundamental premise for high-quality research is that it is conducted in an ethically responsible manner. In this study, I meticulously followed Finnish ethical principles of research with human participants (Finnish National Board on Research Integrity, 2019) and other research ethics guidelines on good scientific practice (e.g., respecting the human dignity and autonomy of participants; engaging in systematic data collection; and ensuring confidential and cautious data processing, storage, and analysis) throughout the study (Byrne, 2016; Christians, 2018; Johnson & Christensen, 2017). My main ethical considerations in sub-study I concerned systematic data collection, meticulous analysis, and cautious reporting of results to ensure ethical conduct in the research review (Cooper, 2019; Reed & Baxter, 2009; White, 2019; Wilson, 2019). Since there were human participants in sub-studies II and III, in compliance with ethical standards, I informed the research participants of the voluntary, strictly confidential, and anonymous nature of the study (Byrne, 2016; Christians, 2018; Finnish National Board on Research Integrity, 2019; Johnson & Christensen, 2017). As instructed by the guidelines and the Human Sciences Ethics Committee of the University of Jyväskylä, I followed official informed consent procedure, including research notifications, privacy notices, and consents to participate in both studies.

In sub-study II, I provided information about the official consent procedure for the study in the research invitation email (e.g., confidential use of research participants' background information, careful handling and storing of data, and access to data limited to the research team). Additionally, I included a web link to the research notification and privacy notice in the invitation. Before the actual questions of the web survey opened for participants, I asked for their approval to use their information to ensure they were willing to participate in my study by completing the survey. As per good scientific protocol, I also informed the participants of the estimated survey completion time and that they were able to complete the survey in parts if they chose.

In sub-study III, applying the official consent procedure meant that before the PD programme started, I let the ECEC teachers know that they were able to take part in the PD programme without any commitment to being a participant in my research. Every teacher, however, was willing to participate in my study. At the beginning of the PD programme, I provided the ECEC teachers with a research notification and privacy notice that included consent to participate in the research. This ensured that the teachers were aware of the overall research procedure (e.g., time commitment), that their involvement in the study was completely voluntary, and that I was committed to the protection and confidentiality of their data and privacy throughout the process of handling and storing data as well as reporting the results. After assuring myself that they were cognisant of this information, I asked them to sign two identical contracts (one for them, one for me) concerning their consent to participate in the study.

Before engaging with the participants in sub-studies II and III, I applied for research permissions from the administrations of the municipalities' early education services in accordance with their decision-making protocols. In sub-study II, I had to obtain permissions from 15 municipal administrations, of these, one declined to participate in the study. In sub-study III, I obtained permissions from two administrations.

After I received municipal level administrative approvals for conducting my research in sub-study II, I asked for research permissions from the heads of 113 early education centres via email. Of these, 102 heads answered my email and gave me the approval to conduct my study in their centres. Thereafter, via the heads, I distributed the research invitation, including the link to the web survey, research notifications, privacy notices, and the consents to participate, to ECEC teachers from these centres. Of the 113 early education centres, 11 did not reply to my email, thus I do not know if they declined the study or had other reasons for not replying to my inquiry. In sub-study III, I did not need separate research permissions from the heads of the early education centres as the ECEC teachers already had their heads' approval to participate in the LUMA2020 development programme as part of their PD; therefore, I only needed these teachers' consent to participate in my research. In addition to following the official consent procedure to participate in research, the ethical conduct of sub-studies II and III also included trustworthiness of analysing and reporting the results (Byrne, 2016; Christians, 2018; Creswell & Plano Clark, 2018; Johnson & Christensen, 2017). I naturally also applied these principles in this compilation report.

Finally, one of the fundamental ethical principles is to avoid any harm, risks, or damages to research participants (Finnish National Board on Research Integrity, 2019). Accordingly, I pseudonymised the interviews in sub-study III so that the individual ECEC teachers could not be identified to guarantee their anonymity in presenting the results. In addition, I anonymised the names of the early education centres and staff. In sub-study II, I did not know who the teachers were answering the web survey and used the provided teacher-related background information for searching for variations in early mathematical teaching and generalising the results of the sample. To guarantee confidentiality, I stored all data in protected folders. I reported the results of all the sub-studies in addition to discussing the MMR design applied in this compilation study to maximise the openness and transparency of the design and its phases. I distributed published results of the web survey 'Teaching Early Mathematical Skills' (sub-study II) to the administrations of the municipalities' early education services and to early education centres. The published results of the PD programme in mathematics (sub-study III) were distributed to the ECEC teachers and the administrations of the municipalities' early education services as agreed when I obtained approval for the study.

5 MAIN RESULTS OF THE SUB-STUDIES

I designed this doctoral dissertation based on the indicated research gaps to provide novel research-based knowledge of the holistic development of early mathematical skills as well as of the elements that contribute to early mathematical teaching. I next describe the research focuses, results, and conclusions of sub-studies I–III.

5.1 Sub-study I: Theoretical framework for holistic early mathematical skills development

There is a shared understanding that children develop versatile mathematical skills in early childhood. However, in systematic reviews, categorisations and classifications of early mathematical skills development has been limited to a numerical standpoint. So, there is a gap in systematic reviews addressing the learning and development of early mathematical skills comprehensively. A complete theoretical model of early mathematical skills, which illustrates the connections and development of skills, was needed. Therefore, I explored the ways of conceptualising early mathematical skills development and how to categorise them based on the systematic review in sub-study I. Additionally, I explored what kind of holistic framework describing the development of early mathematical skills could be built based on a literature review. I presented corresponding figures in relation to the results in the original article. Therefore, I only describe them in what follows.

The results of the systematic review yielded three early mathematical skill categories: (1) *Numerical skills*, including innate number sense, which serves as the basis for the gradual development of counting skills and basic skills in arithmetic; (2) *Spatial thinking skills*, including innate spatial sense, which serves as the basis for spatial reasoning, geometrical awareness, and sense of time; and (3) *Mathematical thinking and reasoning skills*, which are not innate but develop gradually and include different reasoning, logical thinking, and problem-solving strategies as well as the understanding of patterns and functions and their relations (e.g., comparison, classification, and seriation). As previous studies have shown, several mathematical skills develop gradually and simultaneously, and the development of different mathematical skills is connected (i.e., numerical skills are needed in spatial learning and vice versa). Thus, I constructed a theoretical *holistic model of early mathematical skills development* showing bi- and multi-directional relationships among the elements of numerical skills, spatial thinking skills, and mathematical thinking and reasoning skills.

In the conclusion, I suggested that the importance of strengthening versatile mathematical skills in parallel and simultaneously through conscious teaching and learning practices should be stressed in early childhood mathematics education as well as in pre- and in-service teacher education.

5.2 Sub-study II: Teaching early mathematical skills to 3- to 7-year-old children in relation to mathematical skill category, children's age groups, and teachers' characteristics

Research on mathematical skills development has been limited to a numerical standpoint. Similarly, a recent research review showed that most studies on the teaching of early mathematics have focused on numerical areas, with a limited number also covering geometry, measurement, algebra, and data analysis. The need for research on mathematics teaching from a broad perspective of mathematical content, including, for instance, time, spatial reasoning, and mathematical-logical thinking, has been indicated, especially teaching these skills to broader age groups of children, including those in pre-primary education. To respond to this research gap through the second sub-study, I applied the results of sub-study I to investigate variations in the frequency with which different early mathematical skills, namely NS, STS, and MTRS, are taught to 3- to 7-year-old children, aiming to gain a broad perspective on mathematics. I also explored to what extent teacher-related characteristics and the age group of children explained variations in teaching frequencies. I employed the 'Teaching Early Mathematical Skills' web survey, which I designed based on *the holistic model of early mathematical skills development* (sub-study I), to collect answers from teachers in ECEC (N = 206).

The results from repeated MANOVAs showed that the frequency of teaching early mathematical skills depended on the skill category and that children's age groups moderated these differences. In 3- to 5-year-old children's groups, NS and STS were taught with the same frequency, whereas in 5- to 6-year-old children's groups and 6- to 7-year-old children's groups, NS was taught more often than STS. MTRS was taught least in all age groups. NS was taught more frequently to children in older age groups than to 3- to 5-year-olds, and MTRS was taught more frequently to 6- to 7-year-olds than to 3- to 5-year-olds. Also, pedagogical awareness of teaching early mathematical skills depended on the skill category, and children's age groups moderated these differences. The results revealed that in all age groups, pedagogical awareness was lowest in teaching STS. However, only in 6- to 7-year-old children's groups was teachers' pedagogical awareness of teaching NS higher than for STS and MTRS. Furthermore, the results from univariate analysis of variance showed that of the teacher-related characteristics, pedagogical awareness and mathematics PD programmes were strongly associated with teaching frequency in all skill categories, whereas children's age group was associated with the frequency of teaching NS and MTRS.

Because children's opportunities to learn early mathematical skills depend on teachers' pedagogical awareness of teaching NS, STS, and MTRS as well as teachers' participation in PD programmes in early childhood mathematics, I proposed that their critical role in promoting high-quality early childhood mathematics education should be acknowledged.

5.3 Sub-study III: Teachers' pedagogical awareness of teaching early mathematical skills to 3- to 7-year-old children framed through a professional development programme

Both teachers' pedagogical awareness of early mathematical teaching and their participation in PD programmes in early childhood mathematics promote high-quality early childhood mathematics education. The two elements influence children's possibilities for exploring mathematical phenomena and learning mathematical skills as shown by several studies, including sub-study II in this dissertation. However, it remained unclear what kind of tailored PD programme in early childhood mathematics would support teachers' commitment to self-reflection as well as to the development of their pedagogical awareness and teaching practices. There was also little data on how teachers change their thinking during PD programmes. To respond to these gaps in the research, in sub-study III, I investigated how a tailored PD programme in early childhood mathematics, which was designed based on self-identified needs for learning, can enhance pedagogical awareness of teaching early mathematical skills holistically to 3- to 7-year-old children, in any situation or at any teachable moment in play or daily life (i.e., discussions, routine events, pre-planned activities).

I designed the PD programme around principles of transformative learning to prompt sustainable changes in teaching. Additionally, I utilised *the holistic model of early mathematical skills development* (sub-study I) as a theoretical basis for ECEC teachers to explore and examine their own teaching practices. I made these choices because the need for applying our current research-based understanding of mathematical skills development to early childhood mathematics education has been underscored by prior research.

I carried out semi-structured individual interviews to explore ECEC teachers' descriptions of the changes in their pedagogical awareness of teaching early mathematical skills after participating in a tailored PD programme. Additionally, I used a pre-PD and follow-up questionnaire to validate possible increases in pedagogical awareness of teaching early mathematical skills, namely, NS, STS, and MTRS. I also employed the questionnaires to explore permanence in the possible changes, as I collected the follow-up questionnaire nine months after the end of the PD programme.

The thematic analysis showed that ECEC teachers enhanced their pedagogical awareness of teaching early mathematical skills: (1) in developmentally appropriate mathematical content, (2) in child-initiated mathematical learning, and (3) in holistic mathematical teaching. Importantly, reflecting on and examining one's own practices individually and collaboratively, recognising children's interests and initiatives, and taking actions to develop teaching and learning practices in daily life and play aligned with different elements of the PD programme (i.e., theory, collaborative meetings, self-reflective journal, and instructional package) and comprised the foundation of the transformative process.

The results of the Non-parametric Wilcoxon Signed Ranks Test showed that teachers' pedagogical awareness increased significantly in NS and STS but not in MTRS nine months after the end of the PD programme. Effect sizes (Cohen's d using the pooled standard deviation of the two assessments) were large in NS (1.04), moderate in STS (0.68), and small in MTRS (0.30) (see Cohen, 1992 for the criteria of different magnitudes of effect size).

Finally, I suggested that PD programmes in early childhood mathematics enhance ECEC teachers' pedagogical awareness of teaching early mathematical skills holistically when they are tailored to their needs. Importantly, I also recommended that programmes need to include reflective elements and follow the principles of transformative learning.

6 DISCUSSION

The purpose of my study was to provide an up-to-date research-based theoretical framework of the holistic development of early mathematical skills, the teaching of early mathematical skills holistically to 3- to 7-year-old children, and the role of teachers' pedagogical awareness and mathematics PD programmes in early mathematical teaching in Finnish ECEC and pre-primary education. To achieve these objectives, I first explored the current research-based understanding of early mathematical skills development and interlinkages among different mathematical skills to construct a *holistic model of early mathematical skills development*. I thereafter explored the correspondence between the current research-based understanding of early mathematical skills development and the teaching of different early mathematical skills to 3- to 7-year-old children. Additionally, I investigated to what extent and how ECEC teachers' pedagogical awareness of teaching early mathematical skills as well as their participation in PD programmes in mathematics contribute to early mathematical teaching. In these explorations, I utilised a theoretical understanding of early mathematical skills development based on extant research (with a focus on current studies) as well as web survey data, pre-PD and follow-up questionnaire data, and interview data. I applied MMR design to clarify the relationships among the studied phenomena to answer the research questions thoroughly (Creswell & Plano Clark, 2018; Johnson & Turner, 2003; Schoonenboom & Johnson, 2017).

6.1 Research-based understanding of the holistic development of early mathematical skills and the correspondence with teaching them to 3- to 7-year-old children

The first aim of my study was to explore the current research-based understanding of the holistic development of early mathematical skills to construct a theoretical framework of skills development. Subsequently, my aim was to investigate how teaching early mathematical skills to 3- to 7-year-old children in Finnish ECEC and pre-primary education corresponds to the development of these skills.

6.1.1 Holistic model of early mathematical skills development

The results from the systematic review (sub-study I) revealed three mathematical skill categories: (1) numerical skills, (2) spatial thinking skills, and (3) mathematical thinking and reasoning skills; additionally, the results showed bi- and multi-directional relationships among them.

A closer exploration of the systematic review concerning current studies of the development of NS, STS, and MTRS demonstrated on the one hand, that NS and STS are based on innate number sense and spatial sense, and on the other hand, that NS, STS, and MTRS develop from initial skills in toddlerhood to basic skills and even further into sophisticated skills during early childhood and the pre-primary year (e.g., Carraher & Schliemann, 2007; Clements, 2011; Clements & Sarama, 2014; Fantozzi et al., 2013; Hannula-Sormunen et al., 2015; Hribar et al., 2012; Hurst et al., 2017; Jordan et al., 2008; Mulligan & Mitchelmore, 2013). The review also showed that some of children's mathematical skills progress along with motor, language, and cognitive development, for instance, understanding time, conservation, and numerical and spatial relationships, as well as the capability for mathematical reasoning (e.g., Asunta et al., 2016; Clements & Sarama, 2007; Donnelly et al., 2017; Lee et al., 2016; Lyytinen, 2014; Warren et al., 2016). The findings indicated that children learn versatile mathematical skills progressively and that learning mathematical skills is a holistic process, which is also closely connected to the development of other skills. Consequently, although it is essential to pay attention to the teaching of NS, STS, and MTRS in a developmentally appropriate manner, it is also key to focus on how motor, language, and cognitive development can be utilised to promote the development of mathematical skills. In other words, it is important to consider how to integrate mathematical learning into other learning areas taught in ECEC and pre-primary education.

The systematic review of the current studies also showed interlinkages among NS, STS, and MTRS. More precisely, bi- and multi-directional relationships among NS, STS, and MTRS were found by showing that children need STS to learn NS and vice versa (e.g., Cheeseman et al., 2014; Laski & Siegler, 2014; Sarama & Clements, 2009; Zhang et al., 2017); additionally, they need MTRS to learn NS and STS, and they may use NS and STS to learn MTRS (e.g., Colliver, 2018; English, 2013; Mulligan & Mitchelmore, 2013; Sarama & Clements, 2009). The systematic review demonstrated that the three skill areas are interlinked (e.g., Laski et al., 2013). These findings resulted in the theoretical *holistic model of early mathematical skills development* (Figure 3).

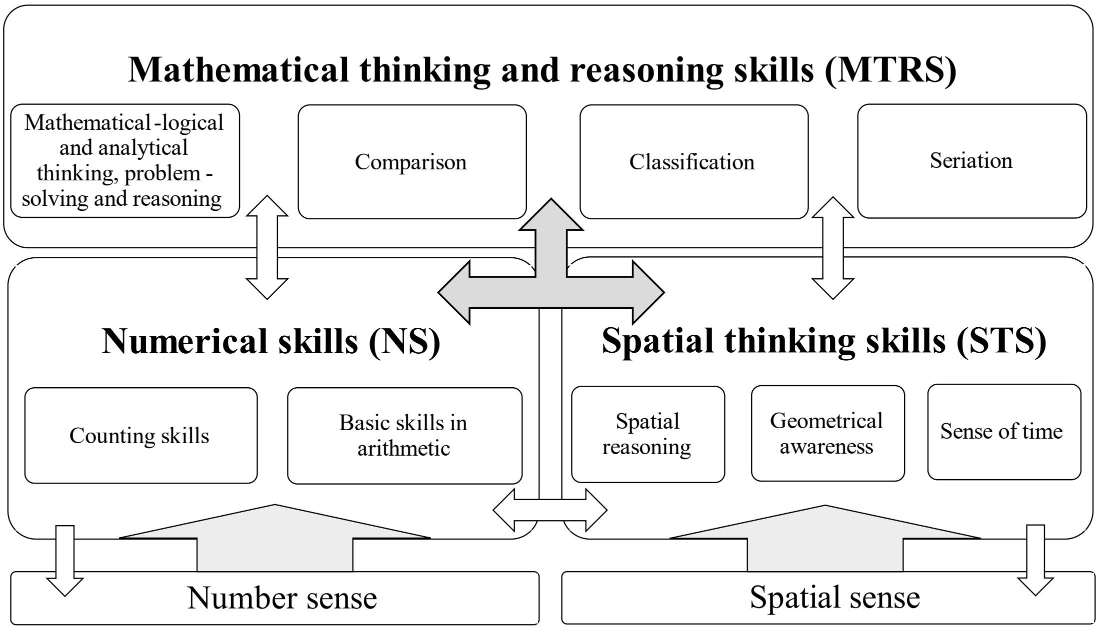

FIGURE 3 Holistic model of early mathematical skills development (Parviainen, 2019; Parviainen et al., 2023)

As Figure 3 illustrates, constructing the theoretical model based on the systematic review (sub-study I) broadened previous knowledge concerning numerical categorisations and compilations of early mathematical skills development (Aunio & Räsänen, 2016; Clements & Sarama, 2007; Sarama & Clements, 2009) by bringing together versatile skills that develop before age seven, and by showing interlinkages and relationships among these skills. It amalgamated the results of independent studies, which have indicated relationships among the skill categories (e.g., Clements, 2011; Hawes et al., 2017; Kyttälä et al., 2014; Laski & Siegler, 2014; Mulligan & Mitchelmore, 2013). Based on the findings, in addition to paying attention to the developmentally appropriate teaching of NS, STS, and MTRS and the connections between mathematical learning and learning other skills, it is essential to focus on teaching that considers interlinkages and connections among NS, STS, and MTRS in learning early mathematical skills. In doing so, the development of early mathematical skills can be promoted comprehensively by consciously connecting and integrating mathematical learning affordances into different pedagogical practices in ECEC and pre-primary education.

6.1.2 Teaching early mathematical skills to different age groups and within an age group in correspondence with early mathematical skills development

The examination of the variations of teaching NS, STS, and MTRS to 3- to 7-year-old children in Finland using a web survey (sub-study II) showed that teaching NS and MTRS depended on the children’s age group; NS was taught less often to 3- to 5-year-old children than to 5- to 6-year-old children and 6- to 7-year-old children, and MTRS less often to 3- to 5-year-old children than to 6- to 7-year-old children. In contrast, I did not find age-related variations in the frequencies of teaching STS. I also discovered variations in the frequencies of teaching NS, STS, and MTRS within the different age groups. NS and STS were taught with the same frequency; however, MTRS was taught with the least frequency to 3- to 5-year-old children, but NS was taught most often to 5- to 6-year-old and 6- to 7-year-old children. In these age groups, STS and MTRS were taught equally frequently.

Since several studies in the systematic review (sub-study I) demonstrated that older children are capable of more sophisticated learning concerning NS, STS, and MTRS compared to younger children (e.g., Aunio & Räsänen, 2016; Battista, 2007; Carraher & Schliemann, 2007; Clements & Sarama, 2014; Hannula-Sormunen et al., 2015; Jones & Tzekaki, 2016; Jordan et al., 2006; Merkley & Ansari, 2016; Mulligan & Mitchelmore, 2013; Vasilyeva & Bowers, 2006), I assumed that these skills are taught more often to older than to younger children. I based my assumption on the view that teachers would emphasise NS, STS, and MTRS with older children more often in recognition of their more developed skills in these areas compared to the skills of younger children. The findings from the web survey (sub-study II) showing that NS was taught more often to 6- to 7-year-old children and to 5- to 6-year-old children than to 3- to 5-year-old children indicate that variations in teaching NS correspond to the development of NS (Kreilinger et al., 2022; Kullberg et al., 2020; Lepola & Hannula-Sormunen, 2019; Sutherland et al., 2021).

Based on the systematic review of studies attesting that older children are more advanced in STS than younger children (sub-study I), I had expected that the results of the web survey (sub-study II) would have shown STS being taught more often to children in older age groups than to children in the youngest age groups. However, I was somewhat surprised not to find any differences in the frequencies of teaching STS among the age groups. Consequently, the findings indicate a need for enhancing our current research-based understanding of the development of STS among ECEC teachers because the frequencies of teaching STS to 3- to 7-year-old children did not correspond to the development of STS (e.g., Clements & Stephan, 2011; Hawes et al., 2017; Jones & Tzekaki, 2016; Liu & Zhan, 2022). Attention should be paid in particular to the capabilities and development of STS among 5- to 6-year-old and 6- to 7-year-old children.

The results from the web survey (sub-study II) showing that MTRS was taught more often to 6- to 7-year-old children than to 3- to 5-year-old children indicate that variations in teaching MTRS to these age groups correspond to the development of MTRS (Alsina & Saldago, 2022; Carraher & Schliemann, 2007; Clements & Sarama, 2014; Clements et al., 2019; Lee et al., 2016; Vanluydt et al., 2021; Vanluydt et al., 2022; Worthington et al., 2019). The frequency of teaching MTRS to 5- to 6-year-old children fell in between these two age groups; thus, it also corresponds to the development of MTRS. Based on the findings of the development of MTRS shown by the systematic review (sub-study I), I thought MTRS would also be taught more often to 5- to 6-year-old children than to 3- to 5-year-old children. Inspecting the 95% confidence intervals of the mean frequency of teaching MTRS revealed that the difference was close to being statistically significant. Not finding differences between the teaching of MTRS to 5- to 6-year-old children and 6- to 7-year-old children indicates that ECEC teachers are aware of children's rather sophisticated capabilities and strategies concerning MTRS between ages 5 and 7, and therefore, they emphasise these with this age group (Alsina & Saldago, 2022; Clements & Sarama, 2014; Lee et al., 2016; Mulligan, 2015; Vanluydt et al., 2022). It, however, seems that there is room for a current research-based understanding of the development of MTRS among ECEC teachers, as not finding differences between the frequencies of teaching MTRS to 3- to 5-year-old children and 5- to 6-year-old children may indicate that ECEC teachers are not fully aware of the development of MTRS (e.g., Alsina & Saldago, 2022; Carraher & Schliemann, 2007; Clements & Sarama, 2014; Lee et al., 2016; Mulligan & Mitchelmore, 2013; Vanluydt et al., 2021; Vanluydt et al., 2022).

After taking a closer look at the frequencies of teaching NS, STS, and MTRS within different age groups, I found that in the 3- to 5-year-old children's groups, NS and STS were taught as frequently, whereas MTRS was clearly taught less frequently than NS and STS (sub-study II). These results correspond to the development of NS, STS, and MTRS among children in these age groups, as several studies in the systematic review (sub-study I), as well as a large number of current studies, have indicated clearly that NS and STS are based on innate senses and basic skills that develop a lot between ages 3 and 6, whereas MTRS is not innate but develops between ages 3 and 6 along with cognitive and language development (cf. Alsina & Saldago, 2022; Clements et al., 2019; Kreilinger et al., 2022; Kullberg et al., 2020; Lepola & Hannula-Sormunen, 2019; Liu & Zhan, 2022; Sutherland et al., 2021; Vanluydt et al., 2021; Worthington et al., 2019). It is, therefore, very important to consider that MTRS is taught less often than NS and STS to 3- to 5-year-old children, but learning basic skills in NS and STS is supported with the same frequency. The findings reveal that among 3- to 5-year-old children, the balance between teaching NS, STS, and MTRS corresponds to the development of these skills among children in this age range. These findings provided novel insights into teaching STS to 3- to 5-year-old children, as previous studies have indicated such teaching is not prominent among children in this age group (Hindman, 2013; Simpson & Linder, 2014).

The results from the web survey (sub-study II) showed that in 5- to 6-year-old children's groups and 6- to 7-year-old children's groups, NS was taught more often than STS, whereas STS and MTRS were taught with the same frequency. When these results are compared to studies concerning the development of NS, STS, and MTRS in the systematic review (sub-study I) and current studies in the field, the frequencies of teaching NS and MTRS correspond to the development of these skills, but more effort could be put into teaching STS to 5- to 7-year-old children (cf., Alsina & Saldago, 2022; Hawes et al., 2017; Keisar & Peled, 2018; Kullberg et al., 2020; Liu & Zhan, 2022; Sutherland et al., 2021; Vanluydt et al., 2021; Worthington et al., 2019). The argument is supported particularly by the studies concerning the development of STS indicating that children's understanding of directions, locations, and conservation becomes more precise between ages 5 and 7 (Clements, 2011; Clements & Stephan, 2011; Hawes et al., 2017; Hribar at al., 2013; Jones & Tzekaki, 2016; Lyytinen, 2014). Therefore, it is critical to teach STS to 5- to 7-year-old children by emphasising these developing skills more consciously and more often than is done at present.

Taken together, teaching NS to different aged children and within different age groups corresponded the best to the development of NS, whereas there exists an evident need for ECEC teachers to gain up-to-date knowledge about the development of STS to ensure the developmentally appropriate teaching of STS. Although the teaching of MTRS corresponded well to different age groups of children, the comparison among age groups also indicated room for up-to-date knowledge of the development of MTRS. The need for current knowledge of the development of STS and MTRS relates specifically to teaching 5- to 7-year-old children.

Paying attention to teaching STS and MTRS is important because the studies in the systematic review (sub-study I) and several current studies have indicated bi- and multi-directional relationships among NS, STS, and MTRS (e.g., Cheeseman et al., 2014; Clements & Stephan, 2011; Colliver 2018; English, 2013; Laski & Siegler, 2014; Liu & Zhang, 2022; Mulligan, 2015; Vanluydt et al., 2021, 2022). In other words, STS supports learning NS and vice versa, and the same applies among MTRS, STS, and NS. Paying attention to teaching STS and MTRS is also important because several studies have shown that not only NS but also STS and MTRS influence later mathematical learning and overall school achievement (English, 2023; Jordan et al., 2009; Lepola & Hannula-Sormunen, 2019; Lerkkanen et al., 2005; Romano et al., 2010; Simms et al., 2016). In short, conscious and developmentally aware teaching of NS, STS, and MTRS to children of different ages and within different age groups according to the development of these skills among children need to be emphasised in ECEC and pre-primary education to promote the holistic learning of versatile mathematical skills.

6.2 The pivotal role of teachers’ pedagogical awareness and professional development programmes in promoting the quality of early mathematical teaching

The second aim of my study was to explore the role of ECEC teachers’ pedagogical awareness and PD programmes in mathematics in early mathematical teaching in ECEC and pre-primary education. I aimed to gain an understanding of to what extent and how ECEC teachers’ pedagogical awareness explains the teaching of NS, STS, and MTRS as well as the implementation of pedagogical teaching and learning practices. Additionally, through a comprehensive exploration, I aimed to reveal to what extent and how PD programmes in early childhood mathematics explain the teaching of NS, STS, and MTRS as well as the role of mathematics PD programmes in changing and enhancing teachers’ pedagogical awareness of early mathematical teaching.

6.2.1 Connection of teachers' pedagogical awareness to variations in early mathematical teaching

The results from the web survey (sub-study II) revealed that the stronger the teachers' pedagogical awareness, the more often they taught NS, STS, and MTRS to 3- to 7-year-old children. The results also showed that teachers' pedagogical awareness of teaching NS, STS, and MTRS varied depending on the mathematical skill category; STS was the weakest regardless of the age group being taught. Furthermore, the findings from the interviews with the ECEC teachers who participated in the mathematics PD programme (sub-study III) showed that all ECEC teachers evaluated their pedagogical awareness of teaching NS as the strongest; most of them evaluated their pedagogical awareness of teaching STS and MTRS, as well as their appropriate learning and teaching practices, as limited or somewhat limited before participating in the programme. Comprehensive elaboration of the role of teachers' pedagogical awareness in teaching early mathematical skills demonstrated that the effect of teachers' pedagogical awareness on early mathematical teaching was connected to (1) the frequency of teaching early mathematical skills to children of different ages; (2) the scope of teaching early mathematical skills to children from different age groups; and (3) appropriate pedagogical teaching and learning practices in the implementation of early childhood mathematics education (sub-studies II and III).

In terms of the frequencies of teaching early mathematical skills, the results from the web survey (sub-study II) revealed that teachers' pedagogical awareness was strongly associated with the frequencies of teaching NS, STS, and MTRS; its association was largest for MTRS (17.1%), moderate for NS (8.7%), and for STS (9.4%). Pedagogical awareness was the factor explaining the biggest portion of the variation in the frequency of teaching STS and MTRS compared to other factors (teacher's age, mathematics PD programmes, and age group of children). Furthermore, the results revealed variation in pedagogical awareness concerning the teaching of NS, STS, and MTRS by showing that it was lowest for STS among all ECEC teachers. Only among teachers of 6- to 7-year-old children was teachers' pedagogical awareness of teaching NS higher than that for STS and MTRS. Interestingly, compared to the indicated need for up-to-date knowledge concerning the development of early mathematical skills, STS and MTRS were the areas needing improvement (see Chapter 6.1.2). Consequently, it can be understood that teachers' pedagogical awareness has a significant role in teaching early mathematical skills. These findings show that teachers' pedagogical awareness of teaching different early mathematical skills is associated with the frequencies of teaching them and varies depending on the skills category, which adds to previous research knowledge, as teachers' awareness of early mathematical content (Callejo et al., 2022; Dunekacke et al., 2015; Muños-Catalán et al., 2022) and their content-related teaching confidence (Alsina et al., 2021; Dunekacke et al., 2015) have been shown to influence early mathematical teaching.

Second, in terms of the scope of teaching early mathematical skills, the findings of the ECEC teachers' interviews (sub-study III) concerning teachers' pedagogical awareness of teaching NS, STS, and MTRS were partly confirmative and partly contradictory compared to the results of the web survey (sub-study II). All interviewed teachers working with 3- to 7-year-old children described having the strongest pedagogical awareness of teaching NS, which was different compared to the results from the web survey, showing no difference between the teachers' pedagogical awareness of teaching NS and MTRS to 3- to 5-year-old and 5- to 6-year-old children's groups. The interviewed ECEC teachers evaluated their pedagogical awareness after the PD programme in mathematics in relation to the theoretical *holistic model of early mathematical skills development* (Figure 3), as opposed to the ECEC teachers who answered the web survey, which may explain these disparities. Since previous studies have shown an emphasis on NS in mathematics education in initial teacher education programmes (Clements & Sarama, 2011; Simpson & Linder, 2014), it is natural that of the three areas, all teachers indicated their awareness of NS to be the strongest. Additionally, when considering that the age groups of the children explained variation in the frequencies of teaching NS, and the results indicated that the variation corresponded to the development of NS (sub-studies I and II, see Chapter 6.1.2), the role of teacher education in teachers' awareness of teaching NS is supported.

As in the results from the web survey (sub-study II), most of the interviewed teachers described their pedagogical awareness of teaching STS as being weakest (sub-study III). These findings confirm teachers' pedagogical awareness of teaching STS, as both the web survey and the interviewed teachers' self-evaluation of their awareness concerning the three skill areas in relation to the theoretical *holistic model of early mathematical skills development* (Figure 3) yielded similar results. These confirmatory findings are important as previous studies have shown teachers' limited awareness of age-appropriate mathematical content has a negative influence on early mathematical teaching (Lee & Ginsburg, 2007, 2009; Lloyd, 2024). In particular, insufficient teaching of spatial content has been connected to a lack of content awareness (Björklund & Barendgret, 2016). A similar influence can also be identified in my study, because the frequencies of teaching STS did not vary among different age groups (as discussed in Chapter 6.1.2), whilst pedagogical awareness of teaching STS was lowest among ECEC teachers.

Most of the interviewed teachers also described having limited awareness of teaching MTRS (sub-study III), which was also different compared to the results of the web survey (sub-study II) showing no difference between teachers' pedagogical awareness of teaching NS and MTRS to 3- to 5-year-old and 5- to 6-year-old children's groups. However, the web survey also revealed a strong association between teachers' pedagogical awareness and the frequency of teaching MTRS, which indicated large and systematic variations in teaching MTRS to 3- to 7-year-old children. One explanation could be that the interviewed ECEC teachers belonged to the group of teachers whose awareness of MTRS was limited before they participated in the PD mathematics programme, since they, again, evaluated their prior awareness in relation to the *holistic model of early mathematical skills development* (see Figure 3). As the results regarding the frequencies of teaching MTRS indicated a need for a research-based understanding of the development of MTRS (sub-study II, see Chapter 6.1.2), the interviewed teachers may have evaluated their awareness as limited concerning the development of MTRS among older children. Although current research has broadened our knowledge of the development of MTRS considerably (Alsina & Saldago, 2022; Vanluydt et al., 2021; Worthington et al., 2019), the findings may not have reached ECEC teachers comprehensively, which may explain variations in pedagogical awareness of teaching MTRS to some extent.

Third, by elaborating on the appropriate pedagogical teaching and learning practices in the implementation of early childhood mathematics education, the results from the ECEC teachers' interviews showed that their awareness of applying pedagogical teaching and learning practices in play, teachable moments, and different daily situations was critically influenced by their pedagogical awareness (sub-study III). The results demonstrated that most interviewed teachers described having had a teacher-initiated approach to mathematical teaching and teaching mathematics during planned mathematical activities only, whereas only two teachers detailed having a child-initiated approach to mathematical learning and teaching mathematics in daily situations in addition to planned mathematical activities before participating in the mathematics PD programme. The findings indicate that in most of these child groups, mathematical learning affordances had been somewhat limited and the implementation of early childhood mathematics education varied a lot among the child groups. These results supplement recent studies with similar findings (Björklund et al., 2018; Helenius, 2018; Johnston & Bull, 2022; Johnston & Degotardi, 2022; Palmér et al., 2016). These results are also important from a curriculum perspective. They indicate that objectives set for early mathematical teaching in the Finnish curriculum guidelines for ECEC and pre-primary education (Finnish National Agency for Education, 2020, 2021, 2022), emphasising that teaching early mathematical skills and mathematical content should take place in play and different daily situations and in different environments as well as through considering children's interests and needs, were not fully met in most of these teachers' child groups.

In conclusion, the results from the web survey (sub-study II) and teachers' interviews (sub-study III) are corroborative concerning teachers' pedagogical awareness of early mathematical teaching. Together, they indicate that teachers' pedagogical awareness influences 'what' and 'how' teachers teach when it comes to early mathematical skills in Finnish ECEC and pre-primary education and that variation exists among ECEC teachers. The findings also reveal that teachers' pedagogical awareness concerning teaching different mathematical skills to children of different ages is not steady across age groups; instead, children's possibilities for learning versatile mathematical skills are broadly and completely vulnerable to teachers' capabilities (cf. Björklund & Barendgret, 2016; Dunekacke et al., 2015; Galeano et al., 2024; Gasteiger & Benz, 2018; Johnston & Bull, 2022). The pivotal role of teachers' pedagogical awareness in early mathematical teaching is unquestionable, as revealed in other studies (Björklund et al., 2018; Callejo et al., 2022; Çelic, 2017; Dunekacke et al., 2015; Ertle et al., 2008; Muños-Catalán et al., 2022; Polly et al., 2017; Salomonsen, 2020). The results indicate a need for a current research-based understanding concerning the development of STS and

MTRS as well as a broader comprehension of pedagogical teaching and learning practices in pre-service and in-service ECEC teacher education. This will help teachers and teacher students implement Finnish curricula related to early mathematical teaching as intended (Finnish National Agency for Education, 2020, 2021, 2022).

6.2.2 Importance of mathematics professional development programmes in enhancing early mathematical teaching

The results from the web survey (sub-study II) showed that the more experience in PD programmes in mathematics ECEC teachers had, the more often they taught NS, STS, and MTRS to 3- to 7-year-old children. The exploration of teachers' pedagogical awareness in the context of a tailored mathematics PD programme (sub-study III) enhanced my understanding of the connection between ECEC teachers' pedagogical awareness and their participation in a mathematics PD programme in relation to early mathematical teaching. The results showed that through participation in the mathematics PD programme and by self-evaluating their practices and developing those individually and collaboratively in relation to the theoretical content of the programme, the teachers enhanced their pedagogical awareness of early mathematical teaching concerning developmentally appropriate mathematical content, child-initiated learning, and holistic mathematical teaching. The results from the pre-PD and follow-up questionnaires (sub-study III) revealed sustainable changes in the teachers' pedagogical awareness of teaching NS and STS.

In terms of the role of mathematics PD programmes in teaching early mathematical skills, the results from the web survey (sub-study II) showed that the more experience the ECEC teachers had with mathematics PD programmes the more often they taught NS, STS, and MTRS in ECEC and pre-primary education. The association was the largest on STS (5.8%) and mildest in MTRS (2.3%), NS (3.9%) felling in between these. The results from the ECEC teachers' interviews complemented the understanding of the role of mathematics PD programmes in teaching NS, STS, and MTRS by showing that teachers' awareness of early mathematical skills development as well as age-appropriate teaching of NS, STS, and MTRS were enhanced during the PD programme (sub-study III). The teachers' descriptions of recognising the essence of practicing number sense and counting skills with 3- to 5-year-old children, paying more attention to teaching basic skills in arithmetic, and strengthening skills in counting and understanding quantity with 5- to 7-year-olds indicate that participation in a mathematics PD programme can also strengthen the teaching of these skills areas (i.e., NS) that teachers themselves have evaluated as being strong already (cf. Callejo et al., 2022; Çelic, 2017; Ertle et al., 2008; Gonulates & Gilbert, 2023; Muños-Catalán et al., 2022; Polly et al., 2017).

The findings also demonstrated that participation in a mathematics PD programme can strengthen teachers' capacity for teaching their self-evaluated weaker areas (cf. Alsina et al., 2021; Çelic, 2017; Dunekacke et al., 2015; Callejo et al., 2022; Gonulates & Gilbert, 2023; Hadley et al., 2015; Knaus, 2017; Muños-

Catalán et al., 2022). The interviewed ECEC teachers described how they broadened their age-specific teaching of STS as teachers of 3- to 5-year-olds began to emphasise the teaching of time, spatial relations, and basic shapes; teachers of 5- to 7-year-olds became more aware of stressing the teaching of more sophisticated shapes and figures, as well as the measurement of length, mass, volume, and time. Concerning the awareness of teaching MTRS, the teachers of 3- to 5-year-olds learnt to promote children's reasoning skills, whereas the teachers of 5- to 7-year-olds promoted children's capacity for problem-solving and reasoning.

The PD programme in mathematics also provided ECEC teachers with a new awareness of the bi- and multi-directional relationships among developing early mathematical skills (see Figure 3; cf. Cheeseman et al., 2014; English, 2013; Laski et al., 2013; Laski & Siegler, 2014; Mulligan & Mitchelmore, 2013) as none of them was aware of the interlinkages and relationships before the programme. Through this awareness, they described gaining a more conscious and holistic understanding of mathematical skills development, leading to practical ways of strengthening children's learning of NS, STS, and MTRS. These findings together indicate that if teachers become aware of the development of NS, STS, and MTRS as well as their interlinkages through their participation in mathematics PD programmes, they can become more capable of teaching NS, STS, and MTRS to children of different ages and foster broader learning opportunities related to mathematical phenomena for children (cf. Alsina et al., 2021; Callejo et al., 2022; Çelic, 2017; Dunekacke et al., 2015; Muños-Catalán et al., 2022).

The results of the pre-PD and follow-up questionnaire (sub-study III) completed by the ECEC teachers were a testament to the pivotal role of mathematics PD programmes in enhancing teachers' pedagogical awareness of teaching early mathematical skills. They revealed that teachers' increased pedagogical awareness of teaching NS and STS showed up nine months after the PD programme. Even though an increase in MTRS was not detected later, despite teachers describing changes in this area in the interviews, the small effect size could be understood as an indicator of a small and sustained influence of the programme on MTRS as well. Based on these findings, it can be concluded that PD programmes in mathematics influence the frequencies of teaching different early mathematical skills but also developmental and age-appropriate awareness of teaching different early mathematical skills (cf. Alsina et al., 2021; Callejo et al., 2022; Çelic, 2017; Dunekacke et al., 2015; Hadley et al., 2015; Knaus, 2017; Muños-Catalán et al., 2022). Nevertheless, more attention needs to be paid in future PD programmes to teachers' awareness of teaching MTRS. Interestingly, Gonulates and Gilbert (2023) have recently detected similar findings related to teachers' awareness of teaching NS, STS, and MTRS; however, teachers, according to their results, still struggled with the implementation of developmentally appropriate mathematical content.

A particularly gratifying finding was that of teachers' pedagogical awareness of teaching STS; it was detected as the weakest skillset both in the web survey (sub-study II) and ECEC teachers' interviews (sub-study III), but the results from the teachers' interviews and from the pre-PD and follow-up questionnaires (sub-study III) showed that with the aid of a PD programme, their understanding of the development of STS as well as age-appropriate awareness of teaching STS were enhanced, and sustainable changes could be detected. These findings are promising as it is likely that broader awareness increased teachers' ability to capture mathematical learning affordances more frequently but also that weaker areas can be developed (cf. Björklund & Barendgret, 2016; Björklund et al., 2018; Gonulates & Gilbert, 2023; Hadley et al., 2015; Johnston & Bull, 2022; Johnston & Degotardi, 2022; Knaus, 2017). When ECEC teachers participate in PD programmes in mathematics, children should benefit from their improved ability to recognise opportunities to engage with versatile mathematical phenomena related to numerical and spatial learning as well as mathematical thinking and reasoning in play, planned activities, and different daily routines. These, in turn, have been shown to be essential parts of early mathematical teaching (Clements et al., 2011; Clements et al., 2023; MacDonald & Murphy, 2019; Palmér et al., 2016) and high-quality early childhood mathematics education (Björklund et al., 2018; Helenius, 2018; Lutovac & Kaasila, 2018; Moss et al., 2016; Salomonsen, 2020).

The results from the interviews and pre-PD and follow-up questionnaire (sub-study III) emphasise that an up-to-date understanding of early mathematical skills development can be supported by PD programmes in mathematics. The results also demonstrate that teachers are able to translate current knowledge into early mathematical teaching practices by strengthening both stronger and weaker areas as well as by taking actions with a holistic understanding of skills development. The findings, together with the evidence from the web survey (sub-study II) regarding the role of PD programmes and teachers' pedagogical awareness in teaching early mathematical skills (Chapter 6.1.1), indicate that PD programmes in mathematics have an important role in early mathematical teaching by helping teachers learn how to support children's mathematical skills development. Based on these indications, one can conclude that a holistic understanding of early mathematical skills development complements teachers' comprehension of the intended implementation of early childhood mathematics education found in the curricula (cf. Finnish National Agency for Education, 2020, 2021, 2022).

The results from the ECEC teachers' interviews also showed that participation in a mathematics PD programme enhanced teachers' pedagogical awareness in early mathematical teaching concerning pedagogically appropriate teaching and learning practices (sub-study III). The findings demonstrated a broader awareness of connecting mathematical learning affordances in play, planned activities, teachable moments, and different daily routines through consideration of children's interests, needs and capabilities, as well as conscious teacher–child interactions, which are key to high-quality early mathematical teaching (Björklund et al., 2018; Clements et al., 2023; Moss et al., 2016; Salomonsen, 2020; Trawick-Smith et al., 2016; Vogt et al., 2018; Wager, 2014). ECEC teachers, for instance, connected mathematical learning to tidy up routines; one teacher supported numerical learning through toy collecting and other classification through toy sorting. Teachers also described more conscious practices in using teachable moments, as they detailed how they connected mathematical learning to clothing, gardening, meal-time discussions, and so on, choosing a suitable moment to teach one child or the whole group of children. Integrating mathematical phenomena into different planned activities strengthened pedagogical practices as the teachers connected possibilities for mathematical learning while their focus was, for instance, on music, crafts, or visual arts. The teachers' conscious actions also covered possibilities for mathematical learning during play as they used mathematical language when they played with children and also offered mathematical materials for children's play.

As these advances in appropriate pedagogical teaching and learning practices were described by those teachers who had not applied such practices consciously before the PD programme, as well as by those who had, it can be concluded that the quality of early mathematical teaching improved within the child groups the teachers were teaching as a result of the PD programme in mathematics. These findings are important as previous studies have shown that teachers have found it difficult to connect early mathematical teaching to play (Björklund & Palmér, 2024; Palmér & Björklund, 2023) and daily routines (Johnston & Bull, 2022; Johnston & Degotardi, 2022) instead of instructional mathematical teaching. My findings also supplement the results of other studies, which have indicated that mathematics PD programmes provide an important platform for PD, such as learning to enhance pedagogical awareness of early mathematical teaching in versatile ways (Gasteiger & Benz, 2018; Gonulates & Gilbert, 2023; Hadley et al., 2015; Knaus, 2017; Lindmeier et al., 2020; Simpson & Linder, 2014). My findings also indicate that a PD programme in mathematics can promote the implementation of a curriculum, as the expanded practices among all ECEC teachers could better meet the goals for early mathematical teaching in the Finnish curriculum for ECEC and pre-primary education (cf. Finnish National Agency for Education, 2020, 2021, 2022).

Several studies have clearly stated that the most influential PD programmes in mathematics promote both action-based and reflective learning (Alsina et al., 2021; Çelic, 2017; Dunekacke et al., 2015; Gasteiger & Benz, 2018; Lindmeier et al., 2020; Mason, 1998, 2002, 2011). The results from the ECEC teachers' interviews (sub-study III) showed that the input of a tailored PD programme in mathematics applying principles of transformative learning, such as reflecting on and critically examining one's own practices individually and collaboratively in relation to the theory of early mathematical skills development, and taking actions to develop appropriate learning and teaching practices, comprised the foundation for teachers' transformative learning process (cf. Cranton, 2016; Mezirow, 1991, 1997; see also Mason, 1998, 2002, 2011). These led to changes both in reflection and action-related teaching, as teachers became aware of their strengths and limitations related to developmentally appropriate mathematical content as well as their pedagogically appropriate learning and teaching practices. The findings indicated that the elements of teachers' pedagogical awareness in early mathematical teaching (see Figure 1, Chapter 2) as well as principles of transformative learning, which emphasise teachers' commitment to and their ownership of the PD process, need to be comprehensively applied in the designs of PD programmes to enhance ECEC teachers' PD and achieve sustainable changes in early mathematical teaching (cf. Barber et al., 2014; Hadley et al., 2015; Knaus, 2017; Nurmi et al., 2021).

Taken together, the results from the PD programme (sub-study III) provided new insights into the applicable elements of the PD programme in mathematics and demonstrated that ECEC teachers' pedagogical awareness of early mathematical teaching can be changed and enhanced via a mathematics PD programme (sub-study III). These findings included the results from the web survey (sub-study II) showing that the more experience ECEC teachers have with PD programmes in mathematics, the more often they teach NS, STS, and MTRS to children in ECEC and pre-primary education. The PD programme design applying the principles of transformative learning (Cranton, 2016; Mezirow, 1991, 1997; see also Mason 1998, 2002, 2011) revealed sustainable changes in teachers' pedagogical awareness in early mathematical teaching and indicated that reflection and self-evaluation of one's own practices were critical parts of this change. Therefore, I suggest utilising these elements in future PD programme designs, as earlier studies have shown improvement only in action-related teaching, not in reflection (Knaus, 2017; Lindmeier et al., 2020).

Finally, the mathematical categorisations, in my study *the holistic model of early mathematical skills development,* which have been suggested for use in the development of teaching practices (Purpura & Lonigan, 2015), supported ECEC teachers, who participated in a mathematics PD programme, in examining and developing their knowledge of early mathematical skills development, as well as their learning and teaching practices. These categorisations also aided them in differentiating mathematical instructions as necessary. I, therefore, further suggest that holistic mathematical categorisations, such as *the holistic model of early mathematical skills development*, should be included in PD programme designs to improve teachers' pedagogical awareness of early mathematical teaching (see Figure 1, Chapter 2). In doing so, the implementation of early childhood mathematics education as found in Finnish curricula for ECEC and pre-primary education can be promoted (cf. Finnish National Agency for Education, 2020, 2021, 2022).

6.3 Practical, pedagogical, and policy implications

The purpose of my study was to explore understudied areas of early mathematical teaching from novel perspectives. Personally, I wanted to develop my understanding of the topic as a researcher but also as a teacher educator to be able to improve my theoretical understanding and teaching practices as well as mathematics course content both for pre-service and in-service ECEC teacher education programmes based on my research. I intended for my study to benefit children's mathematical learning through readers of the published articles and this compilation article who would ensure its contributions through academic, practical, pedagogical, and policy implications.

Further, the *holistic model of early mathematical skills development*, which proved to be successful in a mathematics PD programme design, should also be used in pre-service ECEC teacher education. This will promote a shift in ECEC teacher education, removing numerical learning from its central role and instead emphasising the importance of learning versatile mathematical skills, namely NS, STS, and MTRS. It will also increase our understanding of the interlinkages and relationships among these skills to improve mathematics education courses (cf. Clements & Sarama, 2011; Lloyd, 2024; Simpson & Linder, 2014). In both pre-service and in-service ECEC teacher education, pedagogical practices, which support early mathematical learning and teaching in different daily situations, such as play, daily routines, discussions, planned activities, and teachable moments, should be discussed and experienced by teachers and teacher students to promote high-quality early mathematical teaching. Thus, a transition from planned mathematical activities and teacher-initiated practices, which tend to be somewhat problematic in early mathematical teaching as my findings also illustrated, towards balanced teaching between teacher-initiated and child-initiated practices taking place holistically throughout the day in ECEC and pre-primary education can be developed consciously in pre-service and in-service ECEC teacher education (cf. Björklund & Palmér, 2024; Cheeseman et al., 2014; Palmér & Björklund, 2023; Salomonsen, 2020).

Based on my findings, developing pre- and in-service ECEC teacher education, as well as considering one's own PD while working with children, requires consciousness of one's own skills, knowledge, and attitudes. I suggest that different elements of teachers' pedagogical awareness should be considered in PD programme designs to promote ECEC teachers' and teacher students' action-related and reflective learning (see Figure 1, cf. Gasteiger & Benz, 2018; Lindmeier et al., 2020; Mason, 1998, 2002, 2011). Pedagogical awareness aids teachers and teacher students in self-evaluating what they do in relation to research-based knowledge of skills development and content as well as appropriate pedagogical practices related to observations of children's needs, skills, and interests, as my findings also demonstrated. Self-evaluative practices may also bring to light faulty assumptions about early mathematical learning and teaching (Anthony & Walshaw, 2009; Lee & Ginsburg, 2007, 2009; Lloyd, 2024; Piaget, 1965; Salomonsen, 2020) as well as personal attitudes and considerations of oneself as a mathematics teacher (Çelic, 2017; Ertle et al., 2008). Self-evaluative practices can aid teachers and teacher students in overcoming obstacles they encounter related to early mathematical teaching. Such practices promote PD aiming for high-quality early mathematical teaching, which considers children's participation and their interests, and takes place in daily practices, teachable moments, and play; hence, they enable mathematical learning affordances for all children (cf. Banse, 2021; Björklund & Palmér, 2024; Helenius, 2018; Johnston & Degotardi, 2022; Kultti, 2013; Polly et al., 2017; Trawick-Smith et al., 2016).

Taken together, developing pre- and in-service ECEC teacher education should focus on understanding that mathematical phenomena are everywhere in our everyday lives (Clements & Sarama, 2014; Clements et al., 2011; van Oers, 2013); therefore, mathematics should be taught holistically in different daily situations, including play, which promotes the implementation of high-quality child-centred early childhood mathematics education (Clements et al., 2011; Helenius, 2018; Moss et al., 2016; Wager, 2014; see also Finnish National Agency for Education, 2020, 2021, 2022). Such an approach to early childhood mathematics education promotes STEM strategy 2030, especially from the perspective of life-long learning (Ministry of Education and Culture, 2023). Children need to be encouraged to become familiar with the mathematical phenomena around them, including when they put their clothes on, get in line, tidy up their toys, play with sand, build huts, eat lunch, bake buns, prepare instruments, paint butterflies, and so on, in ECEC and pre-primary education, an objective that can be achieved through ECEC teachers' conscious pedagogical practices. This kind of early mathematical teaching embodies the idea of teaching our eyes to look for mathematical phenomena around us (Helenius, 2018).

Through these conscious steps, those who already work as ECEC teachers and those studying to become teachers will be able to apply a research-based understanding of the development of versatile early mathematical skills as well as the pedagogically appropriate implementation of early childhood mathematics education in ECEC and pre-primary education. Consequently, they will be capable of implementing mathematical teaching as intended in the Finnish curricula for ECEC and pre-primary education (Finnish National Agency for Education, 2020, 2021, 2022). Nevertheless, I suggest that in the next curriculum reforms for ECEC and pre-primary education, attention should be paid to holistic early mathematical skills development in descriptions of early childhood mathematics education, similar to what has already been done in the curriculum for two-year pre-primary education trial (Finnish National Agency for Education, 2021).

6.4 Evaluation of the reliability and trustworthiness of the study

Evaluation criteria of the quality of MMR design vary from detailed (e.g., O'Cathain, 2010) to broad (e.g., Bryman et al., 2008; Creswell & Plano Clark, 2018). However, they share the idea of evaluating quality by considering standards for quantitative research, qualitative research, and MMR. I evaluated the quality of this study by following the core set of evaluation criteria for MMR design provided by Creswell and Plano Clark (2018), which includes evaluation of (1) the collection and analyses of both quantitative and qualitative data in responding to research questions, (2) the intentionality of integrating the two forms of data and their results, (3) the logical organisation of conducting the MRR design, and (4) the frame of the MMR design within theory and philosophy.

A careful electronic search for current English-language peer-reviewed articles concerning the development of early mathematical skills with the help of ERIC, Clements and Sarama's (2007, 2009) pioneering work, and a thesaurus led me to identify high-quality and comprehensive research on my area of interest (sub-study I) (Cooper, 2019; Reed & Baxter, 2009). Conducting the systematic review by meticulously following the procedure of content analysis focusing on literature analysis (e.g., familiarising myself deeply with the content of the articles and checking the stability of the category definitions) increased the trustworthiness of the analysis (Cooper, 2019; Krippendorff, 2004; Mason, 2009; Patton, 2015; Ryan & Bernard, 2000; Wilson, 2019). Nevertheless, I could have also searched articles written in languages other than English and employed other search tools in addition to ERIC to identify an even greater number of high-quality studies. Although 134 articles can be considered a sufficient number of studies for analysis, using other search tools and considering other languages might have revealed more studies concerning MTRS in particular, for which I was only able to find 18 articles.

I built the quantitative measure 'Teaching Early Mathematical Skills' web survey for further exploration based on research-based knowledge concerning the theoretical framework for the *holistic model of early mathematical skills development*. After validating the measure by testing the internal consistency of the scales and following a cautious and systematic sample selection procedure (Johnson & Christensen, 2017; Newby, 2014), I collected data from ECEC teachers (sub-studies II and III). In terms of assessing the validity of the sample, most of the respondents who answered the web survey (sub-study II) were women, and the sample represented the average teacher's gender distribution in Finnish ECEC and pre-primary education (cf. Finnish National Agency for Education, 2017). Moreover, the respondents represented different locations and different-sized municipalities in Finland (see Table 3). In this sense, the results could be considered generalisable (Creswell & Plano Clark, 2018; Johnson & Christensen, 2017). However, due to the sampling technique, I did not get any background information from those who did not answer the survey, which limits the validity because it produced a knowledge gap related to their heterogeneity in terms of age, gender, educational qualifications, work experience, and duration of mathematics PD. I confirmed the reliability of the statistical analysis of the data collected with the web survey (sub-studies II and III) by conducting the analysis process under the supervision of one of my supervisors. Parametrical statistical analysis, on the one hand, enabled me to examine the frequencies of teaching early mathematical skills to children of different ages as well as to examine how different teacher-related characteristics (e.g., teacher's age, work experience, duration of PD programmes in mathematics, and pedagogical awareness of teaching early mathematical skills) were associated with early mathematical teaching (sub-study II). On the other hand, non-parametrical statistical analysis allowed me to determine the sustainability of the changes in teachers' pedagogical awareness of teaching early mathematical skills when they participated in the mathematics PD programme (sub-study III).

The seven ECEC teachers who participated in a PD programme in mathematics for the first time in their educational careers held various ECEC teacher qualifications, had both short and long work experience, worked as teachers in small or big municipalities, and had experience teaching children of different ages in early education centres (sub-study III). Although the sample covered only seven teachers, the group of teachers was rather heterogenous in terms of their backgrounds, thus they represented the teacher population rather well, which improves the trustworthiness of the results (Creswell, 2012; Johnson & Christensen, 2017). I applied Braun and Clarke's (2006, 2022) six phases of thematic analysis cautiously in the analysis of the interview data of these teachers to ensure a high level of trustworthiness in analysis results while investigating the changes teachers described in their pedagogical awareness as a result of the PD programme. My critical discussions of each analytical phase with my supervisors also increased the trustworthiness of the process and guaranteed in-depth scrutiny of the interpretations and analytical results (Creswell, 2012; Creswell & Plano Clark, 2018; Newby, 2014). This collaboration process was important as I planned and conducted the PD programme and interviews on my own. Additionally, a key part of confirming and strengthening the trustworthiness of analysis results was calculating the inter-rater reliability from the coding of two interviews, which one of my supervisors conducted (Creswell, 2012; Creswell & Plano Clark, 2018). Achieving 93% agreement for the inter-rater reliability indicated the process was highly trustworthy.

Integration of the results in this compilation report demonstrates that the qualitative components provided knowledge of the researched-based theory of the development of early mathematical skills (sub-study I) and increased our understanding of the transformation of teachers' pedagogical awareness resulting from the PD programme in mathematics (sub-study III), while the quantitative components provided knowledge of teaching NS, STS, and MTRS to children of different ages as well as variations in teachers' pedagogical awareness of teaching NS, STS, and MTRS (sub-study II). The quantitative components also furnished information about the role of teachers' pedagogical awareness and mathematics PD programmes in early mathematical teaching (sub-studies II and III).

The combination of qualitative and quantitative research components expanded and strengthened this compilation article's interpretations and conclusions related to the connection between the development of early mathematical skills and teaching them to 3- to 7-year-old children, as well as the pivotal role of teachers' pedagogical awareness and mathematics PD programmes in early mathematical teaching (cf. Creswell & Plano Clark, 2018; Schoonenboom & Johnson, 2017). Answering the research questions in a comprehensive manner was facilitated by being able to discuss the findings of the three sub-studies in an integrative and comparative manner because of the ideology of MMR (see Figure 2; cf. Creswell & Plano Clark, 2018; Johnson & Turner, 2003; Schoonenboom & Johnson, 2017). Additionally, the combination of qualitative and quantitative components resulted in a better understanding of the changes in teachers' pedagogical awareness, especially through input from the PD programme (sub-study III). Consequently, the combination of qualitative and quantitative research components provided new in-depth insights into the identified research gaps in early childhood mathematics education (Creswell & Plano Clark, 2018; Schoonenboom & Johnson, 2017). The qualitative and quantitative research components complemented each other in this compilation article, including the exploration of the PD programme in mathematics (Creswell & Plano Clark, 2018). However, the use of self-reported teacher data only (sub-studies II and III) limits the gained insights to teachers' self-reported experiences and practices. Potential response bias could therefore not be assessed (Rosenman et al., 2011). The use of observational data, for instance, in addition to self-reported data could have complemented and extended the interpretations and conclusions by providing an external perspective on the phenomenon under study.

Regarding the logic of organising and conducting the MMR design as well as framing the design within theory and philosophy (Creswell & Plano Clark, 2018), my study followed the principle of integrating MMR at multiple levels, including in the: methods, methodology, and paradigm (Figure 2) (see Chapter 4.3; cf. Creswell & Plano Clark, 2018; Greene, 2015). I valued quantitative and qualitative components, approaches, and thinking equally in my study. In practice, these two components alternated and interacted throughout the research process (Creswell & Plano Clark, 2018). However, I could have used MMR design in the PD programme (sub-study III) even more comprehensively by collecting questionnaire data at the end of the programme in addition to collecting it before the programme and nine months after the programme ended. Pragmatic choices that I made concerning the methods and methodology (e.g., timing, point of integration, and planned versus emergent design) still offered me a logic for constructing an MMR design that would thoroughly address the research questions (Creswell & Plano Clark, 2018; Schoonenboom & Johnson, 2017). Pragmatism also fostered an epistemological justification for knowledge construction (Johnson & Onwuegbuzie, 2004). By acknowledging the tentative nature of knowledge included in pragmatism, I could integrate the qualitative and quantitative components while answering the research questions by considering the knowledge they offered from dual perspectives (Johnson & Turner, 2003).

Finally, pragmatism recognises the fact that knowledge is based on our lived experiences (Johnson & Onwuegbuzie, 2004). In my study, this meant that in the integrative discussion of the results, I could search interpretations for similarities and differences resulting from the sub-studies to understand the explored phenomena comprehensively. In addition, having a background as a teacher in ECEC and working as a teacher educator in an ECEC teacher education programme, my own lived experiences and theoretical understanding of educational issues helped me conduct this study. For instance, while designing appropriate research (web-survey and reflective journal, N.B. the latter was not used in this doctoral dissertation), I aimed to create tools that ECEC teachers could use for their PD in terms of pedagogical self-evaluation in the PD programme. The feedback from the ECEC teachers who answered the pilot web survey and ECEC teachers using the reflective journal during the PD programme served the purpose from both perspectives, namely, research and means for PD. My background also aided me in designing and conducting the PD programme as I could use my understanding of the combination of theoretical and practical elements needed in teaching and teacher education, while I applied principles of transformative learning (Cranton, 2016; Mezirow, 1991, 1997), as well as, to some extent, the characteristics of action research (Kemmis & McTaggart, 2005; McIntyre, 2008) in the programme design. However, I took steps to guarantee the trustworthiness and transparency of the study, as explained earlier, to reduce the possible unwanted influence of personal bias in conducting and exploring the PD programme. In sum, I followed the criteria of MMR design meticulously during the whole research process and while writing this compilation article (Creswell & Plano Clark, 2018; Greene, 2015; Johnson et al., 2007; Johnson & Turner, 2003; Johnson & Onwuegbuzie, 2004; Schoonenboom & Johnson, 2017).

6.5 Concluding remarks and future directions

This study provided a new *holistic model of early mathematical skills development* that proved to be beneficial in the teaching of early mathematical skills and the design of mathematics PD programmes for ECEC teachers. The study broadened our understanding of the early mathematical teaching of 3- to 7-year-old children by showing that teaching these skills, namely NS, STS, and MTRS, corresponds partially to our current research-based understanding of the development of early mathematical skills. The study revealed the need for PD among ECEC teachers, especially in relation to STS but also MTRS, particularly for 5- to 7-year-old children. The study also provided novel insights into the pivotal role of teachers' pedagogical awareness and mathematics PD programmes in early mathematical teaching by revealing that both separately and together these contribute to the improvement of early mathematical teaching. Based on these findings, I suggest acknowledging elements of teachers' pedagogical awareness in the designs of mathematics PD programmes and initial ECEC teacher education.

A hybrid multiphase MMR design (see Figure 2) was appropriate to my study as it offered useful methods, including methodological and paradigmatic choices, to follow from the first steps of the research process to the final steps of integrating and discussing the results in relation to the research questions in this compilation study (Creswell & Plano Clark, 2018; Johnson & Turner, 2003; Schoonenboom & Johnson, 2017). Because of the design and how I used quantitative and qualitative components in this study, the summary of the results is rich and deep. Despite improving our knowledge of early mathematical teaching through MMR design in this study, more research is needed to understand issues related to early childhood mathematics education more broadly, both in Finland and around the world. Therefore, I suggest MMR designs be used in future studies to gain in-depth knowledge of issues that have not received sufficient scholarly attention. Concurrently, it is important to ensure the use of versatile data sources (e.g., use of observational data in addition to self-reported data), in addition to considering quantitative and qualitative components of MMR designs.

Since my focus was on teaching early mathematical skills to 3- to 7-year-old children, more research about teaching NS, STS, and MTRS to children under 3 years is needed. To this end, the web survey has already been adapted for data collection from ECEC teachers working with this age group. Research is also needed on ECEC teachers' pedagogical awareness of the role of PD programmes in early mathematical teaching for children who are under 3 years old. Such endeavours would fill a gap in the literature since research focusing on early mathematical teaching of very young children in ECEC is limited (MacDonald & Murphy, 2019; Simpson & Linder, 2014). Research concerning this age group would broaden our understanding of early mathematical teaching to cover the whole age range of children participating in ECEC and pre-primary education.

Research is also needed concerning the teaching of sub-skills included in NS, STS, and MTRS to gain insights into how teachers working with children of different ages differentiate their teaching to correspond to their developmental understanding of these skills. Teachers' individual PD processes during their participation in PD programmes in mathematics also require more attention among researchers. PD programmes with more participants than were in my study would allow for such explorations. The effectiveness of PD programmes should also be examined with a greater number of participants than were in my study. The web survey could be used for these purposes. It would also be important to explore the role of reflective journals in PD processes as this data was not utilised in my study. In doing so, collaborative studies, with the aim of simultaneously investigating and developing early mathematical teaching, would bring value to both research and practice (Cai et al., 2019).

As a teacher educator, I call for studies exploring the content and practices of mathematics education courses in ECEC teacher education programmes, as a recent study has shown that teacher educators' beliefs about early mathematical learning are translated into their teaching (Lloyd, 2024). Moreover, it would be important to know how the addressed topics, that is to say (1) the development of early mathematical skills, (2) appropriate practices in early mathematical teaching, and (3) teachers' pedagogical awareness, are taught in in-service ECEC teacher education to determine potential areas for improvement.

Through such research, we could gain new insights in unresearched areas. By applying the results of my study as well as those of existing and future studies concerning early mathematical teaching, pre- and in-service ECEC teacher education can be developed to help people teach their eyes to look for the mathematical phenomena all around us (cf. Helenius, 2018). As one of the ECEC teachers participating in the PD programme for early childhood mathematics answered when I asked about the need for future training: 'Everyone needs to understand all the things that mathematics encompasses. We must twig and train'.

SUMMARY IN FINNISH

Tässä väitöskirjatutkimuksessa tarkastellaan nykytutkimukseen perustuvaa tietämystä varhaisten matemaattisten taitojen kokonaisvaltaisesta kehityksestä. Lisäksi tarkastellaan varhaisten matemaattisten taitojen opetusta 3–7-vuotiaille suhteessa taitojen kehitykseen suomalaisessa varhaiskasvatus- ja esiopetuskontekstissa. Näiden lisäksi tarkastelun kohteena on opettajan pedagogisen tietoisuuden ja matematiikan täydennyskoulutuksen rooli varhaisten matemaattisten taitojen opetuksessa 3–7-vuotaiden opetusta koskien.

Suomessa varhaiskasvatuksen ja esiopetuksen matematiikan opetusta ohjaavat valtakunnallisten opetussuunnitelmien perusteiden pohjalta tehtävät paikalliset opetussuunnitelmat (Finnish National Agency for Education, 2020, 2021, 2022). Opetussuunnitelmat eivät kuitenkaan määritä tavoitteita matemaattisten taitojen oppimiselle. Sen sijaan ne määrittävät pedagogisia tavoitteita taitojen opettamiseksi. Tätä vasten varhaiskasvatuksessa ja esiopetuksessa toimivan opettajan on tunnettava matemaattisten taitojen kehityspolut, jotta hän voi tukea eri ikäisten lasten matemaattisten taitojen oppimista. Lisäksi hänen tulee tietää, millaiset pedagogiset menetelmät tukevat matemaattisten taitojen oppimista parhaimmin.

Erityisesti 2000-luvun tutkimukset ovat avartaneet ymmärrystä varhaisten matemaattisten taitojen oppimisesta kumoten aiempaa ymmärrystä siitä, ettei lapsi kykene abstraktiin ja loogiseen matemaattiseen ajatteluun ennen kouluikää (Piaget, 1965). Uusi tutkimustieto on tuottanut aiemman ymmärryksen tilalle kuvan lapsesta, joka oppii varhaislapsuuden aikana monia matemaattisia perustaitoja sekä kykenee kehittyneeseen matemaattiseen ajatteluun jo ennen kouluikää (esim. Alsina & Saldago, 2021; Vanluydt ym., 2021; Worthington ym., 2019). Tutkimukset ovat myös avanneet käsitystä siitä, että osa matemaattisista taidoista on synnynnäisiä (lukumääräisyyden taju ja avaruudellinen taju), kun taas osa taidoista kehittyy iän myötä (Clements & Sarama, 2014). Matemaattisten taitojen kehitys on myös yhteydessä kognitiiviseen, kielen ja motoriikan oppimiseen (Clements & Sarama, 2014; Donnelly ym., 2017; Lyytinen, 2014). Tutkimusten myötä on vahvistunut myös käsitys siitä, että lapsi kykenee jo varhaiskasvatusiässä käsittelemään isoja numeroita sekä ymmärtää mittaamisen ja symmetrian periaatteet (Clements & Stephan, 2011; Kullberg ym., 2020). Kehittyvien ajattelu- ja päättelystrategioiden myötä hän kykenee myös matemaattisloogiseen ongelmanratkaisuun ja tilastointiin (ks. Mulligan, 2015; Mulligan & Mitchelmore, 2013). Laajasta tutkimustiedosta huolimatta kokonaisvaltainen teoreettinen malli taitojen kehityksestä puuttuu, sillä vain numeerisia taitoja koskeva malli on rakennettu (Aunio & Räsänen, 2016).

Kehittyvien matemaattisten taitojen lisäksi viimeaikaiset varhaista matematiikkaa koskevat tutkimukset ovat vahvistaneet käsitystä siitä, että lapset oppivat matemaattisia taitoja parhaimmin, kun erilaiset matemaattisen oppimisen mahdollisuudet ovat läsnä varhaiskasvatuksen ja esiopetuksen arjessa kokopäiväpedagogisesti (Helenius, 2018; Salomonsen ym., 2020). Tämä tarkoittaa matemaattisen oppimisen yhdistymistä erilaisiin arjen tilanteisiin, kuten arkirutiineihin, keskusteluihin, ohjattuun toimintaan ja leikkiin, sekä spontaanisti vastaan tuleviin pedagogisiin hetkiin (Moss ym., 2016; Salomonsen ym., 2020). Tutkimusten mukaan matemaattisten taitojen oppimista edistävät myös sensitiivinen vuorovaikutus ja lasten osallisuutta korostava pedagogiikka (Björklund ym., 2018; Salomonsen ym., 2020) ja tasapaino lapsilähtöisten ja opettajajohtoisten opetuskäytänteiden välillä (Celements ym., 2011).

Varhaiskasvatuksen opettajat eivät tutkimusten mukaan kuitenkaan tunne matemaattisten taitojen kehitystä (ks. Lee & Ginsburg, 2007, 2009) eivätkä varhaisia matemaattisia opetussisisältöjä (Callejo ym., 2022; Muños-Catalan ym., 2022), ja nämä osaamisen puutteet heijastuvat matematiikan opetukseen. Muun muassa heikon ainesisällöllisen ja opetusmenetelmällisen osaamisen on todettu vaikuttavan opetusvarmuuteen (Alsina ym., 2021; Galeano ym., 2024), ja johtavan siihen, että esimerkiksi tilallisuuteen ja muotoihin liittyvien sisältöjen opetus on suppeaa (Hindman, 2013; Simpson & Linder, 2014). Tuoreen tutkimusnäytön perusteella opettajat myös toisinaan sivuuttavat matemaattisen oppimisen mahdollisuudet, vaikka tulokset korostavat leikin ja kokopäiväpedagogisten käytänteiden sekä lasten tarpeet ja intressit kohtaavan matematiikan opetuksen merkitystä osana matemaattisten taitojen oppimista (Björklund & Palmér, 2024; Helenius, 2018). Edellä kuvattujen löydösten perusteella onkin pääteltävissä, että varhaisten matemaattisten taitojen opetus on osin kapeaa, painottuu tiettyihin matemaattisiin sisältöihin eikä toteudu niin laajasti eri oppimisen mahdollisuuksia hyödyntäen kuin voisi. Lisäksi voidaan päätellä, että opettajan toimet ja tietoisuus matematiikan opetuksesta kytkeytyvät olennaisesti lasten mahdollisuuksiin olla matemaattisten ilmiöiden ja matemaattisen oppimisen äärellä osana varhaiskasvatusta ja esiopetusta (Clements ym., 2011; MacDonald & Murphy, 2019). Tutkimustietoa ei ole siitä, missä määrin varhaisten matemaattisten taitojen opetus vastaa käsitystä taitojen kehityksestä, eikä miten opettajat itse arvioivat kykyään opettaa taitoja arjen eri tilanteissa.

Koska opettajan tiedot, taidot ja asenteet vaikuttavat olennaisella tavalla matematiikan opetukseen, on näistä opetusta selittävistä tekijöistä koottu erilaisia malleja, jotka havainnollistavat matematiikan opetukseen vaikuttavia asioita (ks. Carrillo-Yañez ym., 2018; Lindmeier, 2011; Shulman, 1986). Varhaisen matematiikan opetukseen kehitettyjä malleja on hyödynnetty myös opetuksen kehittämisessä (ks. Gasteiger & Benz, 2018; Lindmeier ym., 2020). Huomionarvioista useisiin varhaisen matematiikan opetusta koskeviin tutkimuslöydöksiin liittyen onkin, että sekä opettajien tietämystä matematiikan opetussisällöistä että heidän opetuskäytänteitään on mahdollista kehittää täydennyskoulutusten avulla (Hadley ym., 2015; Knaus, 2017). Erityisesti opettajien osallisuutta ja omia ammatillisen kehittymisen tarpeita korostavien reflektiivisten menetelmien (ks. Cranton, 2016; Mason, 1998, 2011; Mezirow, 1991, 1997) on havaittu tukevan opetuskäytänteiden kehittämistä (Barber ym., 2014; Hadley ym., 2015; Knaus, 2017). Näiden tutkimusten lisäksi tarvitaan tietoa siitä, millainen rooli pedagogisella tietoisuudella sekä täydennyskoulutuksilla on matemaattisten taitojen opetuksessa, opetuskäytänteissä ja niiden kehittämisessä.

Lisätutkimustarve indikoitiin aiempien tutkimusten avulla seuraaviin alueisiin: 1) ajantasainen tutkimusperustainen teoriamalli varhaisten matemaattisten taitojen kokonaisvaltaisesta kehitystä, 2) varhaisten matemaattisten taitojen opetus 3–7-vuotiaiden ryhmässä suhteessa varhaisten matemaattisten taitojen kehitykseen, 3) opettajien tietoisuus varhaisten matemaattisten taitojen opetuksesta ja 4) täydennyskoulutuksen rooli varhaisten matemaattisten taitojen opetuksessa sekä pedagogisesti tietoisessa matematiikan opetuksessa. Näiden tarpeiden pohjalta tutkimukselle muodostettiin seuraavat tutkimuskysymykset:

1) Mikä on tämänhetkinen tutkimusperustainen ymmärrys varhaisten matemaattisten taitojen kokonaisvaltaisesta kehityksestä, ja miten varhaisten matemaattisten taitojen opetus 3–7-vuotiaille vastaa tätä?
2) Millä tavoin varhaiskasvatuksen opettajien pedagoginen tietoisuus matematiikan opetuksesta sekä heidän osallistumisensa matematiikan täydennyskoulutuksiin muovaavat varhaisten matemaattisten taitojen opetusta 3–7-vuotiaille?

Väitöskirjatutkimus toteutettiin monimenetelmätutkimuksena, jossa asetettuihin tutkimuskysymyksiin pyrittiin vastaamaan kattavasti ja syvällisesti sekä laadullisia että määrällisiä tekniikoita ja asetelmallisia menetelmiä hyödyntäen (Creswell & Plano Clark, 2018; Schoonenboom & Johnson, 2017). Ensimmäiseen tutkimuskysymykseen liittyen laadullisena aineistona käytettiin 134:ää vertaisarvioitua englanninkielistä tutkimusartikkelia varhaisten matemaattisten taitojen kehityksestä. Tuloksia peilattiin määrällistä verkkokyselyä (*N* = 206) vasten, jossa oli selvitetty varhaiskasvatuksen opettajien varhaisten matemaattisten taitojen opetusta 3–7-vuotiaiden opetusryhmissä eri puolilla Suomea. Toiseen tutkimuskysymykseen vastattiin hyödyntämällä määrällisestä verkkokyselystä osuuksia, joiden avulla selvitettiin opettajien pedagogisen tietoisuuden ja matematiikan täydennyskoulutusten yhteyttä varhaisten matemaattisten taitojen opetukseen. Ymmärrystä pedagogisen tietoisuuden ja matematiikan täydennyskoulutuksen merkityksestä varhaisten matemaattisten taitojen opetukseen syvennettiin varhaiskasvatuksen opettajilta (*N* = 7) matematiikan täydennyskoulutuksen yhteydessä kerätyillä laadullisilla yksilöhaastatteluilla sekä määrällisillä alku- ja seurantakyselyillä.

Laadullinen systemaattinen kirjallisuuskatsaus tuotti kolme varhaisten matemaattisten taitojen kategoriaa, joiden väliltä löytyi kaksi- ja monisuuntaisia yhteyksiä: 1) numeeriset taidot, 2) avaruudellisen ajattelun taidot sekä 3) matemaattiset ajattelu- ja päättelytaidot (esim. Cheeseman ym., 2014; Clements, 2011; English, 2013; Mulligan & Mitchelmore, 2013; Sarama & Clements, 2009). Tuloksena muodostettiin *kokonaisvaltainen malli varhaisten matemaattisten taitojen kehittymisestä* (Parviainen, 2019). Opetuksen nykytilan selvitys eri puolilta Suomea kerätyllä verkkoaineistolla osoitti, että varhaisten matemaattisten taitojen opetuksen useus 3–7-vuotiaiden ryhmissä vastasi osin nykytietämystä taitojen kehityksestä. Ikäryhmien vertailun perusteella numeeristen taitojen opetuksen useus vastasi tietämystä numeeristen taitojen kehityksestä kaikissa tutkimuksen kohteena olleissa ikäryhmissä (3–5-vuotiaat, 5–6-vuotiaat ja 6–7-vuotiaat). Myös matemaattisten ajattelu- ja päättelytaitojen opetuksen useus eri ryhmien opetusta vertailtaessa vastasi nykytietämystä suhteellisen hyvin. Eri ikäryhmien sisäisen tarkastelun perusteella 3–5-vuotiaiden ryhmissä kaikkien kolmen eri taitokategorian opetus oli tasapainossa suhteessa taitojen kehitykseen toisin kuin aiemmat tutkimukset ovat osoittaneet (Hindman, 2013; Simpson & Linder, 2014). Tulokset osoittivat, että opettajat tarvitsisivat lisää tietoa erityisesti 5–7-vuotiaiden avaruudellisen ajattelun taitojen sekä matemaattisten ajattelu- ja päättelytaitojen kehityksestä.

Pedagogisen tietoisuuden osalta verkkokyselyaineisto osoitti, että mitä vahvempi varhaiskasvatuksen opettajien pedagoginen tietoisuus on kussakin taitokategoriassa, sitä useammin he opettivat näitä taitoja 3–7-vuotiaille. Opettajien arvioiden mukaan avaruudellisen ajattelun taitojen opetukseen liittyvä pedagoginen tietoisuus oli heikointa kuten aiemminkin on osoitettu (Björklund & Barendgret, 2016). Täydennyskoulutuksen yhteydessä kerätty haastatteluaineisto täydensi ymmärrystä pedagogisesta tietoisuudesta eri taitojen opetukseen liittyen. Haastatellut opettajat arvioivat numeeristen taitojen opetukseen liittyvän tietoisuuden olleen heillä vahvinta ennen matematiikan täydennyskoulutukseen osallistumista. Lisäksi suurin osa opettajista arvioi tietoisuuden olevan heikointa avaruudellisen ajattelun taitoihin liittyen, mutta rajoittunutta myös matemaattisten ajattelu- ja päättelytaitojen osalta kuten myös hiljattain julkaistu tutkimus osoittaa (Gonulates & Gilbert, 2023). Näiden lisäksi opettajien pedagogisessa tietoisuudessa oli eroja heidän opetuskäytänteitään koskien. Ennen matematiikan täydennyskoulutusta vain kaksi opettajaa seitsemästä oli hyödyntänyt lapsilähtöisiä kokopäiväpedagogisia matemaattisen opetuksen mahdollisuuksia viiden muun kertoessa opetuksen olleen opettajajohtoista ja keskittyneen etukäteen suunniteltuun matemaattiseen toimintaan, mikä ei vastaa suositusta matematiikan opetuksesta (Anthony & Walshaw, 2009; Moss ym., 2016). Löydösten perusteella voitiin päätellä opettajien pedagogisen tietoisuuden vaikuttavan sekä matemaattisten taitojen opetuksen useuteen että sisältöihin mutta myös toteutuksen tapoihin. Tämä vahvistaa aiempaa ymmärrystä siitä, että varhainen matematiikan opetus on laadultaan kirjavaa (Callejo ym., 2022; Dunekacke ym., 2015).

Verkkokyselyn tulosten perusteella täydennyskoulutuksen yhteys varhaisten matemaattisten taitojen opetuksen useuteen oli samankaltainen kuin pedagogisella tietoisuudella; mitä enemmän opettajilla oli kokemusta matematiikan täydennyskoulutuksista, sitä useammin he opettivat 3–7-vuotiaille numeerisia taitoja, avaruudellisen ajattelun taitoja ja matemaattisia ajattelu- ja päättelytaitoja. Täydennyskoulutuksen merkitys oli suurin avaruudellisen ajattelun taitojen opetukseen liittyen. Matematiikan täydennyskoulutuksen yhteydessä kerätty haastatteluaineisto syvensi ymmärrystä täydennyskoulutuksen merkityksestä. Tulosten perusteella varhaiskasvatuksen opettajat laajensivat pedagogista tietoisuuttaan kaikkien kolmen taitokategorian opetukseen liittyen, mikä on erilainen löydös kuin Gonulatesin ja Gilbertin (2023) tutkimuksessa. Opettajilta kerättyjen alku- ja seurantakyselyjen tulokset vahvistivat täydennyskoulutuksen merkitystä, sillä ne osoittivat, että täydennyskoulutuksen aikana laajentuneessa pedagogisessa tietoisuudessa tapahtui pysyvää muutosta sekä numeeristen taitojen että avaruudellisen ajattelun taitojen opetuksen osalta. Vaikka matemaattisten ajattelu- ja päättelytaitojen osalta ei havaittu tilastollisesti merkitsevää muutosta, määrällinen tarkastelu osoitti pientä kasvua myös tällä alueella yhdeksän kuukauden jälkeen täydennyskoulutuksen päättymisestä.

Matematiikan täydennyskoulutus laajensi haastatteluaineiston tulosten mukaan myös opettajien tietoisuutta lapsilähtöisistä käytänteistä sekä matemaattisten taitojen oppimisen ja opetuksen mahdollisuuksista arjen eri tilanteissa, pedagogisissa hetkissä ja leikeissä. Tämä tapahtui sekä niiden opettajien kohdalla, joille nämä käytänteet olivat olleet aiemmin vieraita mutta myös heidän, jotka olivat jo ennen koulutusta toteuttaneet kyseisenlaista matematiikan opetusta. Opettajien omien arvioiden mukaan täydennyskoulutuksen sisällöt, vuorovaikutukselliset ja osallisuutta tukevat käytänteet sekä itsearviointiin ja reflektointiin perustuvat menetelmät tukivat heidän matematiikan opetuksensa kehittämistä mutta auttoivat myös tunnistamaan alueita, jotka edelleen tarvitsevat kehittämistä. Tulokset vahvistavat aiempia päätelmiä matematiikan täydennyskoulutusten merkityksestä eri matemaattisten taitojen opetuksessa (Gasteiger & Benz, 2018; Gonulates & Gilbert, 2023). Täydennyskoulutusten avulla on mahdollista laajentaa pedagogisesti tietoista matematiikan opetusta niin sisällöllisesti kuin käytänteiden osalta, kun koulutuksen suunnittelussa huomioidaan reflektiota ja sitoutumista painottavia muutokseen tähtääviä menetelmiä (ks. Mason, 2002; Nurmi ym., 2021). Tämä löydös oli uusi aiempiin tuloksiin verrattuna (Hadley ym., 2015; Knaus, 2017).

Tulokset suosittavat, että varhaiskasvatuksen opettajien perustutkinto- ja täydennyskoulutusten kehittämisessä huomioidaan nykytietämys varhaisten matemaattisten taitojen kehityksestä erilaisten mallien, kuten *kokonaisvaltainen malli varhaisten matemaattisten taitojen kehittymisestä* (Parviainen, 2019), avulla. Näin on suositettu jo aiemmassa tutkimuksessa (Purpura & Lonigan, 2015). Lisäksi tulokset suosittavat, että koulutusten kehittämisessä painotetaan sellaisia varhaisen matematiikan opetuksen käytänteitä, jotka tähtäävät kokonaisvaltaiseen ymmärrykseen matemaattisten taitojen oppimisesta erilaisissa päivittäisissä tilanteissa ja leikissä lapsilähtöisiä ja aikuisjohtoisia käytänteitä yhdistäen (vrt. Anthony & Walshaw, 2009; Helenius, 2018). Tulokset myös suosittavat koulutuksellisia elementtejä, jotka huomioivat oppijoiden itsereflektion merkityksen osana laadukasta matemaattisten taitojen opetusta (vrt. Barber ym., 2014; Lindmeier ym., 2020; Mason, 2002). Löydösten perusteella perustutkinto- ja täydennyskoulutuksen tavoitteena tulisi varhaisen matematiikan osalta olla sellainen opetus, joka tähtää kaikkien lasten saatavilla olevien matemaattisten ilmiöiden tarkasteluun erilaisissa ympäristöissä, koko varhaiskasvatus- ja esiopetuspäivän ajan (vrt. Banse, 2021; Helenius, 2018; Johnston & Degotardi, 2022; Kultti, 2013; Polly ym., 2017).

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ORIGINAL PAPERS

I

THE DEVELOPMENT OF EARLY MATHEMATICAL SKILLS – A THEORETICAL FRAMEWORK FOR A HOLISTIC MODEL

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The Development of Early Mathematical Skills – A Theoretical Framework for a Holistic Model

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ABSTRACT: This article presents a theoretical framework for a holistic model of the development of early mathematical skills in early childhood education. The first aim of this study was to conduct a comprehensive international review of the literature to explore early mathematical skills categories. The literature review yielded three early mathematical skills categories, namely (1) numerical skills, (2) spatial thinking skills and (3) mathematical thinking and reasoning skills. Previous studies have shown that several mathematical skills develop gradually and simultaneously in early ages and that these skills areas are interconnected in mathematical skills learning. Accordingly, the second aim of this study was to build a holistic theoretical model illustrating the interconnections between the three developing skills categories. The mathematical skills gained in early childhood influence later mathematical achievement. Thus, the importance of simultaneous strengthening of versatile early mathematical skills through conscious early learning practices should be addressed in early mathematical learning.

Keywords: *early mathematical skills, numerical skills, spatial thinking skills, mathematical thinking and reasoning skills, holistic model of early mathematical skills*

Introduction

Mathematics is part of everyday life and necessary for being an autonomous citizen; it helps people organise their insights and discoveries about the world in systematic ways, such as when they need to represent quantities and spatial relationships (Cross, Woods,

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Schweingruber & National Research Council, 2009). Cross and colleagues (2009) hold the view that mathematics provides means for reasoning, describing and understanding the world and its phenomena. Because of the importance of mathematics in human life, most governments require that education systems provide basic mathematical understanding and proficiency (van Oers, 2013). The importance of strengthening mathematical skills and stressing mathematical learning in early childhood education is thus clear (Sarama & Clements, 2009).

Contemporary research findings attest to the importance of early mathematical learning. They show that the understanding of complex mathematics and abstract reasoning develops much earlier than was once believed (Clements, Fuson & Sarama, 2019; Cross et al., 2009; Mulligan & Mitchelmore, 2013). Studies have broadened our understanding by showing that humans are born with a number sense (meaning quantity and number knowledge) and spatial sense (Clements & Sarama, 2007) and that the development of mathematical skills begins at a very early age. Studies also show that an early understanding of mathematics correlates positively with both children's future success in mathematics (Aunio & Niemivirta, 2010; Jordan, Kaplan, Ramineni & Locuniak, 2009; Jordan, Mulhern & Wylie, 2009) and overall achievements (Aubrey, Dahl, Godfrey, 2006; Romano, Kohen, Babchishin & Pagini, 2010). Besides, the quality of mathematics education is central to the development of mathematical skills (Aunio, Korhonen, Bashash & Khoshbakht, 2014; MacDonald & Carmichael, 2018).

Broadly speaking, mathematics education is often divided into two parts: numeracy and geometry (e.g. Tsamir, Tirosh & Levenson, 2011). Less attention is given to mathematical reasoning processes, even though children also learn these skills (e.g. Mulligan & Mitchelmore, 2013; Sarama & Clements, 2009). In 2007, Clements and Sarama were the first researchers who compiled the development of early mathematical skills through an extensive inclusion of the studies on mathematics learning and teaching in early childhood. Their comprehensive research review included six areas: (1) number and quantitative thinking, (2) geometry and spatial thinking, (3) geometric measurement, (4) patterns and algebraic thinking, (5) data analysis and (6) mathematical processes. Later, Sarama and Clements (2009) combined the latter three areas under the umbrella of 'other content domains and processes'.

Although Clements and Sarama (2007, 2009) presented their comprehensive compilations some time ago, the literature reviews on early childhood mathematics education remains limited. Only two studies (see Fox & Diezmann, 2007; Linder, Ramey & Zambak, 2013) have focused on children's perspectives whereas three other studies (see Antohony & Walshaw, 2009; Linder & Simpson, 2018, MacDonald & Murphy, 2019) have concentrated on teachers' perspectives. Early mathematical skills have thus far been categorised and classified from a numerical standpoint (see Aunio & Räsänen, 2016). A literature review that would address the learning and development of early mathematical skills comprehensively does not yet exist. The new research findings concerning children's early mathematical skills suggest that a comprehensive theoretical model of early mathematical skills illustrating the interconnections of simultaneous development of skills is needed.

Aims

The purpose of this study is to propose a theoretical holistic model of the development of early mathematical skills through the conceptual analysis of the research literature. The first aim is to form early mathematical skills categories and find possible gaps in the existing studies by answering the following research question:

1. In what way are early mathematical skills conceptualised, and can they be categorised based on the existing research literature?

Drawing on the answers to the first research question, the second aim is to build a comprehensive, holistic theoretical model of the development of early mathematical skills through the following question:

2. What kind of holistic framework describing the development of early mathematical skills can be built?

Method

The research literature on early mathematical skills served as data in this investigation. The literature review was conducted, and English-language peer-reviewed articles were identified via electronic searches using the Education Resources Information Center (ERIC). Clements and Sarama's (2007, 2009) extensive pioneering work was used to decide on the initial search terms, which were completed with subject headings and frequently used keywords of the articles with the help of a thesaurus. The search terms included 'mathematics education', 'mathematics instruction', 'mathematics', 'mathematics skills', 'numeracy', 'numerical skills', 'geometric', 'measurement', 'spatial ability', 'spatial relationship', 'algebra', 'early algebra' and 'early childhood education'. Search terms were carefully selected based on earlier research knowledge about early mathematical skills and with the help of frequently used search terms. Education level was limited to early childhood education. The ERIC identified 492 peer-reviewed articles.

The aim was to attain a comprehensive and high-quality search, which would correspond with and broaden the existing knowledge (see Cooper, 2009; Reed & Baxter, 2009).

The initial analysis of the titles helped in selecting relevant articles written between 2003 and 2018 for detailed examination, yielding 273 articles. The rationale behind the above time frame was to enable accessing contemporary data from the latest studies into the learning and development of early mathematical skills and broadening Clements and Sarama's (2007, 2009) earlier findings. This choice was supported by the changes that have taken place in early mathematics education and pedagogy during the present century (Newton & Alexander, 2013). The identified 273 articles were collected using the RefWorks programme for further processing. The next step was to refine the search by exploring the articles' abstracts. During this analysis, age groups (from infants to 8-year-olds) and article contents were carefully reviewed, and the articles related to early mathematical skills in early childhood education were selected for qualitative content analysis. Also, other relevant materials and sources were searched to avoid gaps in data collection.

Altogether, 134 articles were analysed. The process of qualitative content analysis (see Krippendorff, 2004; Mason, 2009; Patton, 2015; Ryan & Bernard, 2000) focusing on literature analysis (see Cooper, 2009; Wilson, 2009) was followed. Articles were first categorised according to conceptual codes (e.g. number sense, symmetry, geometry and patterning). Second, the concepts were carefully organised into broader thematic clusters. Thirteen thematic clusters emerged from the data (e.g. counting skills, spatial reasoning, geometrical awareness and problem-solving strategies). Third, the thematic clusters were further divided into typologies, and the following three early mathematical skills categories were formed: (1) numerical skills, (2) spatial thinking skills and (3) mathematical thinking and reasoning skills. Figure 1 describes the process of the content analysis evolving from conceptual codes into thematic clusters and further into the broader three themes, answering to the first research question.

Sixty-two of the analysed articles were related to numerical learning, whereas 49 focused on spatial and geometrical learning; only five articles were concerned with measurement. Of the remaining articles (of 134), 18 discussed mathematical thinking and reasoning processes (e.g. algebra, data analysis and patterning) separately, and most of them focused on seven- to eight-year-old children.

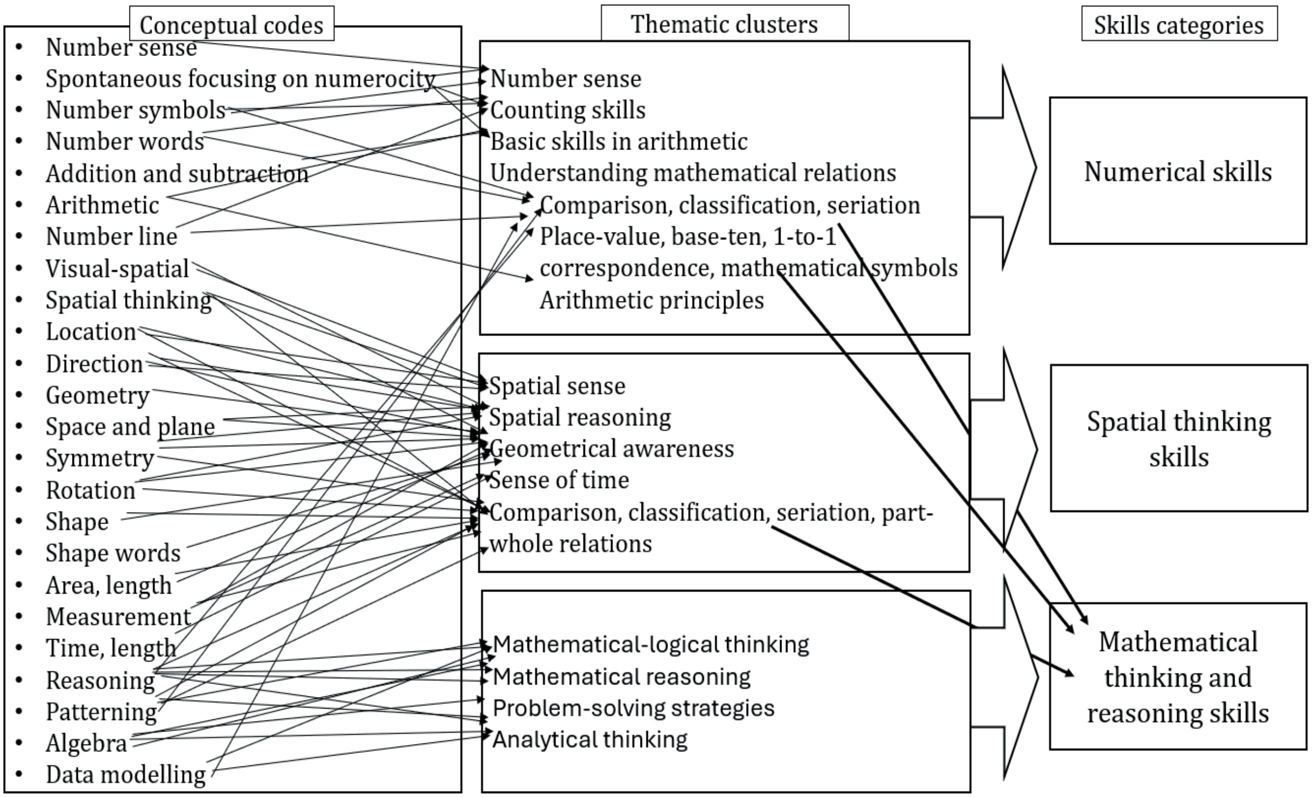

FIGURE 1 The process of the thematic content analysis.

After indicating the conceptual codes and grouping them into skills categories (research question 1), the next step of the analysis was to further explore the connections between the conceptual codes, thematic clusters and skills categories. This was carried out through crosswise comparison. The final phase of the analysis was to build up the holistic model of the early mathematical skills development (research question 2), in which the found interconnections of the skills categories are represented. Throughout the whole research process, standard ethical standards were followed. These included systematic data collection and cautious analysis (e.g. Byrne, 2016; Cooper, 2009; Christians, 2011; Reed & Baxter, 2009; Wilson, 2009).

Results

Three early mathematical skills categories were found in the research literature on mathematics in early childhood education, namely (1) numerical skills, (2) spatial thinking skills and (3) mathematical thinking and reasoning skills. The elements and sub-skills of these categories and the existing gaps in the research literature will be discussed first. Thereafter, the focus will turn to the second research question: The framework for the holistic model of the development of early mathematical skills, which combines early mathematical skills comprehensively.

Numerical skills

The articles related to numerical skills revealed that numerical skills, in general, refer to the number and quantitative thinking and are founded on an intrinsic number sense. As categorised by Clements and Sarama (2007, 2009), the number and quantitative thinking also includes verbal and object counting, early arithmetic strategies, comparing and ordering skills and learning to compose and decompose mathematical symbols. One recent article by Aunio and Räsänen (2016) proposed a model for teaching core numerical skills. This model is based on a series of longitudinal studies and includes similar elements to earlier categorisations by Clements and Sarama (2007, 2009). The model is built around four elements, which include several sub-skills. These are number sense (symbolic and non-symbolic), counting skills (knowledge of quantities and number symbols, number word sequence and enumeration), basic skills in arithmetic (addition, subtraction and arithmetic combinations) and 'understanding mathematical relations' (mathematical-logical principles, arithmetic principles, mathematical symbols, place-value and base-ten system). Next, the elements of numerical skills will be presented.

Number sense

The literature showed that *number sense* requires understanding numbers and quantities in symbolic and non-symbolic number sense (Aunio & Räsänen, 2016; Jordan, Glutting & Ramineni, 2008; Toll, Kroesbergen & Van Luit, 2016) and infants' progress in recognising small quantities (Clements & Sarama, 2007; Kaufmann, 2008; Sarama & Clements, 2009). Although number sense is an innate skill, the ability to differentiate quantities develops gradually in young children and varies from person to person (Hannula-Sormunen, Lehtinen & Räsänen, 2015). Therefore, several articles suggested number sense as the basis for later mathematics skills development (Aunio & Niemivirta, 2010; Kolkman, Kroesbergen & Leseman, 2013). The reviewed articles showed that a strong number sense has a positive influence on children's subsequent mathematical skills, whereas weak number sense explains mathematical learning difficulties (Jordan et al., 2008; Jordan, Kaplan, Nabors & Locuniak, 2006; Merkley & Ansari, 2016; Price & Ansari, 2013; Toll et al., 2016). In addition, recent longitudinal studies indicated that children's spontaneous focusing on numerosity in early childhood predicts their mathematical performance in school (Bojorque, Torbeyns, Hannula-Sormunen, Van Nijlen & Verschaffel, 2017; Hannula, Lepola & Lehtinen, 2010; Hannula-Sormunen et al., 2015; Nanu, McMullen, Munck, Pipari Study Group & Hannula-Sormunen, 2018). These findings support Aunio and Räsänen's (2016) model showing that number sense is a vital part of numerical skills and form a base for mathematical learning.

Counting skills, basic skills in arithmetic and understanding mathematical relations

In the literature, *counting skills* were held to include skills and knowledge of quantities, number words and number symbols, number word sequence and enumeration (e.g. Aunio & Räsänen, 2016; Purpura & Lonigan, 2015). Besides, the literature showed that children need mathematical reasoning skills for fluent counting. As conceptualised by Aunio and Räsänen (2016), 'understanding mathematical relations' include early mathematical-logical principles (comparison, classification, seriation and one-to-one correspondence) as well as place-value logic, including ordinality and cardinality.

The literature review revealed that children learn number word sequence skills, meaning naming numbers and counting those forwards or backwards from a given number during early childhood (Aunio & Räsänen, 2016). Furthermore, understanding ordinality of numbers is part of numerical symbol knowledge and helps children learn the place dependency of a number (Merkley & Ansari, 2016; Slusser & Sarnecka, 2011), requiring awareness that numbers themselves have magnitudes (Jordan et al., 2006). Additionally, Aunio and Räsänen (2016) showed that by comparing and seriating numbers, children become aware of the place-value system, which is a requirement for fluent number word sequence skills. Hence, mastering mental number word sequence skills has been recommended as a foundation for successful arithmetic skills because when adding amounts, children need to be able to count the number word sequence forwards or backwards for subtraction (Aunio & Räsänen, 2016; Jordan, Kaplan et al., 2009; Jordan, Mulhern et al., 2009).

Articles related to counting skills state that in counting numerosities, children use their number word sequence skills to enumerate (Aunio & Räsänen, 2016; Jordan et al., 2006). Besides, successful enumeration requires understanding the logic of one-to-one correspondence, cardinality and ordinality (Sarama & Clements, 2009; Slusser & Sarnecka, 2011). Gaining insights into the interrelationships between number word, number symbol and quantity has been shown to help the child comprehend the principle of ordinality (Sarama & Clements, 2009; Merkley & Ansari, 2016). In learning cardinality, instead, the literature points out that children need to understand that when one starts listing the number word sequence forwards from the first object, the last counted object and its referring number reveal the total amount of the counted objects (Aunio & Räsänen, 2016; Muldoon, Lewis & Francis, 2007; Slusser & Sarnecka, 2011). Thus, based on the reviewed literature, it can be argued that children need the skills above to gain fluent counting skills. Additionally, they need to understand that all kinds of objects can be counted; they also need to know that every countable object is counted only once and that each number word is connected towards one particular object (Clements & Sarama, 2007; Sarama & Clements, 2009). Thus, the literature indicates that to count successfully, children must understand that objects can be counted in any order as long as other counting principles are followed, such as a number word sequence listed in the right order (Aunio & Räsänen, 2016; Muldoon et al., 2007).

Learning *basic skills in arithmetic* requires children to learn that numbers form a coherent system whereby they can be composed, decomposed and commuted (Aunio & Räsänen, 2016; Clements & Sarama, 2007). They, therefore, need to develop an understanding of mathematical relationships (e.g. part-whole) and arithmetic principles to solve arithmetic tasks successfully (Aunio & Räsänen, 2016; Cross et al., 2009). Thus, literature asserted that children need to know that entities or parts need to be counted into sums or wholes and that larger sets are made up of smaller sets. Additionally, children must understand that by adding entities in a different order, they will reach the same answer (commutativity) and that sets decomposed and recombined in different orders will still yield the same answer (associativity) (Aunio & Räsänen, 2016; Cross et al., 2009). Several studies stressed that developing numerical skills widely during early childhood education is crucial to understanding arithmetic principles as they together form the foundation for subsequent mathematical development (Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; Merkley & Ansari, 2016; Purpura & Lonigan, 2013).

As conceptualised above, numerical learning is founded on number sense and, alongside counting skills, forms a basis for developing basic skills in arithmetic (Figure 2). Besides, 'understanding mathematical relations' (mathematical-logical principles, arithmetic principles, mathematical symbols, place-value and the base-ten system) is essential to counting and arithmetic processes (Aunio & Räsänen, 2016). For the congruence of conceptualisation of this study, these skills will later be referred to as *mathematical thinking and reasoning skills.* Figure 2 shows the skills, sub-skills and the interconnections of the numerical skills.

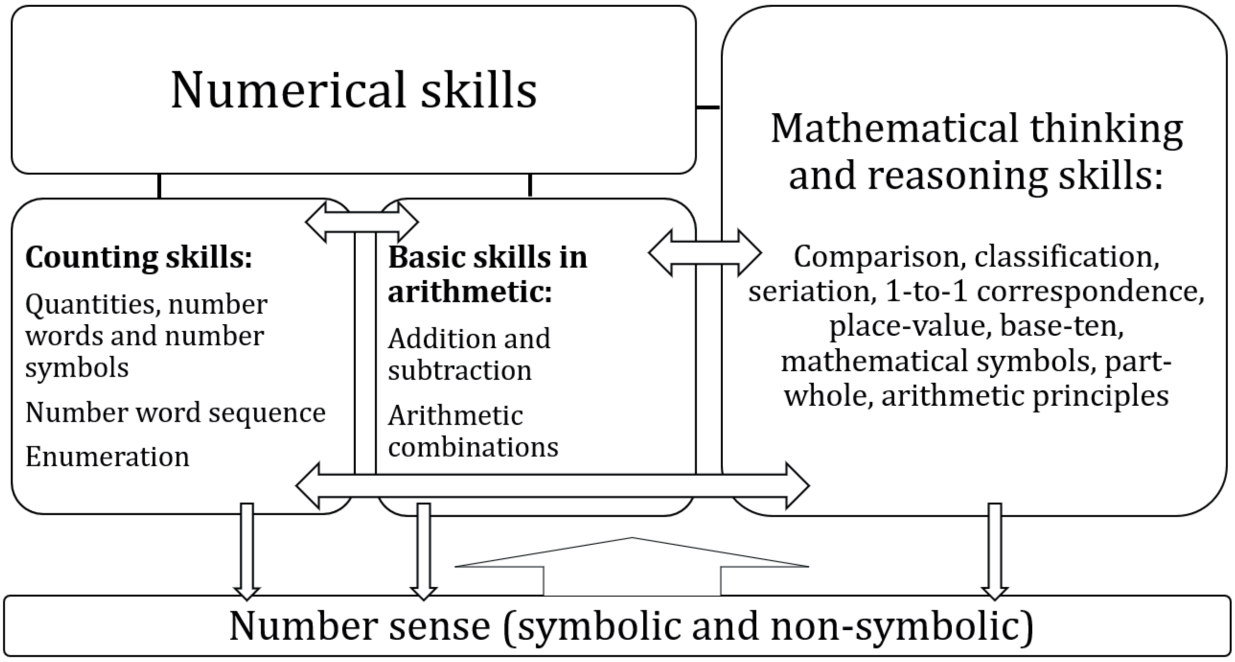

FIGURE 2 Numerical skills (adapted from the original model of Aunio and Räsänen, 2016).

Aunio and Räsänen's (2016) model for teaching core numerical skills has been an important part of categorising early mathematical skills. Such conceptualisation can aid in assessing children's numerical learning reliably and differentiating instructions as required (Purpura & Lonigan, 2015). A common categorisation may also boost global endeavours aimed at increasing teaching competencies in early mathematics education. The need for categorisation can be advocated by contemporary studies, which show that teaching systems and educational backgrounds of early childhood education teachers in different countries can heavily influence children's early numerical learning, although the learning objectives set by the curriculum have been similar (e.g. Aunio et al., 2014). The importance of learning numerical skills during early childhood has also been noted in the literature as the majority of the studies (62 articles) reviewed in this study dealt with this skill area, centred on counting skills.

Spatial thinking skills

The literature review illustrated that geometry and spatial thinking skills base on an innate spatial sense (e.g. Clements & Sarama, 2007) and generally speaking include spatial thinking, geometry and geometric measurement (Battista, 2007; Clements & Sarama, 2007; Jones & Tzekaki, 2016; Sarama & Clements, 2009). Most articles indicated that learning these skills requires mathematical reasoning skills, such as comparison, classification, seriation and part-whole understanding (Battista, 2007; Clements &

Sarama, 2007; Jones & Tzekaki, 2016; Sarama & Clements, 2009). However, inconsistencies in categorising geometry and spatial thinking skills were detected in the data. While some categorisations refer to the skill area as 'geometry and measurement' (see Cross et al., 2009), others that follow the two-part mathematics education division (numbers and geometry) call the area 'geometry' (Tsamir et al., 2011). Some articles on measurement discussed geometric measurement, although two of them also discussed time measurement (Meaney, 2011; Russell, Alexis & Clayton, 2010). Such findings indicate that a new classification of this broad skill area is necessary. Because of the discrepancies in categorisations, this study conceptualises these spatial thinking skills, covering the development of the skills aimed at comprehending spatial environment. Spatial thinking skills are founded on spatial sense and include spatial reasoning, geometrical awareness and sense of time and require *mathematical thinking and reasoning skills*. Learning measurement is categorised as a sub-skill that enhances understanding in the main skills area. The next section presents the conceptual arguments of this new classification.

Spatial sense

The reviewed articles showed that babies and toddlers have the requisite abilities to locate both themselves and objects in space by using either landmarks or geometric cues (Clements & Sarama, 2007; Cross et al., 2009). Spatial sense evolves, and early spatial thinking skills develop through motor learning, in which perceptual-motor skills, especially the development of depth perception, play an essential role (Donnelly, Mueller & Gallahue, 2017). A few articles hold the view that perceptual-motor skills help in understanding spatial relationships (location and direction, depth perception and motion perception) and becoming aware of one's own body (body actions and positions) (Clements & Sarama, 2007; Donnelly et al., 2017). Additionally, one recent article indicated that some connections to the spatial thinking development are similar to cognitive processes development of motor areas (Asunta, Viholainen, Ahonen, Rintala & Cantell, 2016). The literature review, in general, revealed that *spatial sense* forms an essential basis for spatial and geometric learning (Cross et al., 2009; Howse & Howse, 2014).

Spatial reasoning, geometrical awareness and sense of time

Spatial reasoning was conceptualised in the literature as the ability to visualise, explore and reflect on spatial objects and images as well as their relationships and transformations (Battista, 2007). It connects the web of skills, such as the ability to structure, think visually and form mental images (Mulligan, Woolcott, Mitchelmore & Davis, 2018). According to Clements and Sarama (2007), spatial reasoning includes location, navigation and spatial orientation. The literature shows that the development of spatial reasoning begins at an early age when toddlers learn to use visual information to function, for example, for reaching places, people and objects (Cross et al., 2009). It also suggests that this development continues throughout early childhood. Several studies indicated that children gradually learn to build mental images of the surrounding environment and create simple maps to navigate their way to their target (Clements & Sarama, 2007; Krüger, Kaiser, Mahler, Bartels & Krist, 2014; Tsubota & Chen, 2012). Many studies also indicated that young children can use maps and spatial coding successfully (Chen, 2007; Fantozzi, Cottino & Gennarelli, 2013; Hribar, Haun & Call, 2012; Vasilyeva & Bowers, 2006, 2010). Additionally, the literature showed that through spatial reasoning, children come to understand symmetry, asymmetry, mirror images, rotation and patterns (Hawes, Moss, Caswell, Naqvi & MacKinnon, 2017; Krüger et al., 2014; Ramful, Ho & Lorie, 2015; Sarama & Celements, 2009). In these reasoning processes, children also use mathematical thinking and reasoning skills, such as comparison and part-whole relationships.

Contemporary articles addressed that enhancing children's spatial reasoning is important, mainly because acquired skills increase conceptual understanding, visuospatial reasoning and use of spatial language (Hawes et al., 2017; Jones & Tzekaki, 2016). The literature also showed that the development of spatial reasoning skills is influenced by language: Children first learn to express directions, then positions and finally abstract attributes (Lyytinen, 2014; Owens, 2014). However, part of the literature also indicated that the relationship between spatial thinking and mathematics is not straightforward (Clements & Sarama, 2009; Kyttälä, Aunio, Lepola & Hautamäki, 2014). Several articles, nevertheless, stressed that spatial reasoning is proved to be a fundamental part of early mathematical learning because children´s spatial abilities (e.g. visuospatial skills) predict their mathematics achievements (Carlson, Rowe & Curby, 2013; Simms, Clayton, Cragg, Gilmore & Johnson, 2016). Additionally, some studies indicated that spatial reasoning is critical to science, technology, engineering, arts and mathematics (STEAM) education (e.g. Mulligan et al., 2018). Importantly, studies emphasised that spatial reasoning is spatial and not visual, although visual input is essential for the full development of spatial reasoning (Clements & Sarama, 2007; Hribar et al., 2012; Sarama & Clements, 2009). These conceptual arguments require acknowledging spatial reasoning as a component of spatial thinking skills rather than underestimating the value of this complex content area by conceptualising it as being part of geometry.

Regardless of the need for a separate classification, the data also revealed that spatial reasoning is needed in *geometrical awareness*, as the literature stresses the relationship between spatial sense and geometric reasoning (Howse & Howse, 2014). Geometrical awareness connects concepts and ways of reasoning and presenting spatial environments through physical attributes, such as shape and space (Battista, 2007; Hawes et al., 2017; Howse & Howse, 2014). During early childhood, children first learn to name the common shapes verbally, then gradually gain skills to use shape-related attributes and finally express geometrical features by using complex terminology (Cross et al., 2009; Skoumpourdi, 2016). Additionally, the literature reviewed in this study asserted that the same progress applies to the depiction of shapes, starting from drawing a circle at the age of three to gaining skills to draw a star at the age of six, and the progress is influenced by the maturation of fine motor skills (Payne & Isaacs, 2017; Sarama & Clements, 2009; Villarroel & Sanz, 2017).

This literature review indicated that the aforementioned skills tend to become somewhat sophisticated by the age of eight (Kaur, 2015); this is the time when children know several geometric concepts and can depict a variety of shapes and figures in both two and three dimensions (Dagli & Halat, 2016; Skoumpourdi, 2016; Villarroel & Sanz, 2017). The studies indicated that understanding the two- and three-dimensionality of shapes and figures requires that children can compare and classify features but also understand part-whole relationships (Clements & Sarama, 2007; Hallowell, Okamoto, Romo & La Joy, 2015). Some studies suggested that to learn the principles of rotation, pattern and seriation related to two- and three-dimensionality, rigid ways of presenting shapes should be avoided (Clements & Sarama, 2007; Sinclair & Moss, 2012). Additionally, recent studies have suggested that the diversity of two- and three-dimensional learning fosters spatial-knowledge development and spatial reasoning (Hawes et al., 2017; Jones & Tzekaki, 2016). These conceptual arguments illustrate that versatile skills related to geometrical awareness are a fundamental part of spatial thinking skills as they promote understanding of certain aspects of the spatial environment.

The data also revealed that learning all aspects of spatial environment requires *a sense of time*. In environmental phenomena, the space-time relationship is conceptualised as part of the spatial environment (see Golledge, Marsh & Battersby, 2008), hence part of spatial thinking skills. However, the reviewed articles showed that sense of time was widely ignored in the literature on early mathematical skills since it was addressed by three studies only. These articles emphasised that children use spatial thinking skills while learning time-related reasoning (Meaney, 2011; Russell et al., 2010) and structural reasoning while familiarising themselves with clock faces (Mulligan & Mitchelmore, 2013). The literature review revealed that global categorisations mainly focus on geometry, measurement and spatial thinking skills. This attention may explain why learning about the concept of time was ignored in early mathematical skills. However, the Finnish curricula (Finnish National Agency for Education, 2017; Finnish National Board of Education, 2016) suggest that time should be explored through seasons and other time-related concepts and children should be familiarised with chronological order. Russell and colleagues´ (2010) study supports this view by showing that young children are capable of future-oriented thinking. Additionally, only a few articles stressed that language learning can influence understanding the concept of time, as children first learn to speak about the past and the future then gradually learn to express time in more specific ways (Lyytinen, 2014). The findings of the review illustrate that the sense of time has been neglected in the studies while it should have been acknowledged as part of spatial thinking skills.

The literature indicated that spatial environment reasoning requires the aforementioned skills (spatial reasoning, geometrical awareness and sense of time) as well as understanding basics of measurement (Battista, 2007; Solomon, Vasilyeva, Huttenlocher & Levine, 2015). In several mathematical theories, measurement relates to geometry and includes length, area, volume and mass (Battista, 2007; Clements & Sarama, 2007; Jones & Tzekaki, 2016). The articles argued that measurement enables children to learn about the size of the objects but also to determine locations in space (Clements & Sarama, 2007; Fantozzi et al., 2013). Additionally, a few studies highlighted that spatial reasoning required understanding conservation, although it is somewhat demanding at an early age and very often leads to wrong judgements between the measured items (Cross et al., 2009). It was also shown that children learn space-time reasoning through measurement (Meaney, 2011). The findings above support earlier assertions that understanding versatile spatial relationships supports human endeavours, such as architecture, art, engineering and science (Cross et al., 2009; Mulligan et al., 2018). Thus, the findings of this study indicate that measurement needs to be classified as a sub-skill, which is a requirement for learning several spatial relationships.

The conceptual arguments about *the elements of spatial thinking skills*, showing the discrepancies and gaps in earlier categorisations, indicate the necessity of a new classification among these skills. This need is supported by the knowledge that teaching geometry and spatial thinking skills are usually minimised in early childhood education (Clements & Sarama, 2011; Sarama & Clements, 2009; Tsamir et al., 2011). Therefore, this study suggests that this skill cluster be called ‘spatial thinking skills’, according to the model in Figure 3.

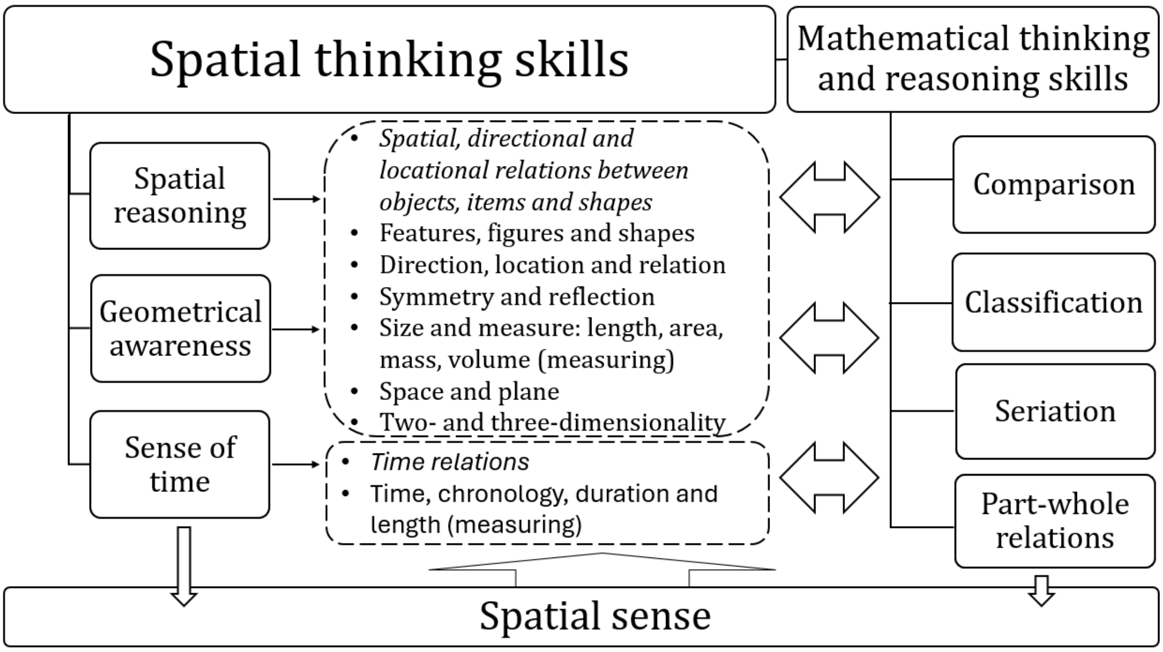

FIGURE 3 Spatial thinking skills.

The model illustrates that spatial thinking skills are based on intrinsic spatial sense and includes spatial reasoning, geometrical awareness and sense of time. These skills are connected and include several sub-skills (e.g. measurement), which develop during early childhood. Additionally, learning spatial thinking skills requires *mathematical thinking and reasoning skills*. Together these skills support a child´s learning about versatile aspects of the spatial environment. Therefore, it is proposed that this suggested model be used as a guiding model for teaching spatial thinking skills in early childhood education.

Mathematical thinking and reasoning skills

A significant part of early mathematical skills involves developing the reasoning process used in learning about the phenomena surrounding us and the world. Mathematical reasoning processes generally require an understanding of the associations and interrelationships between objects, items, number symbols, quantities and their features. The articles reviewed in this study revealed that alongside analogical and statistical reasoning, children learn quantitative, numerical and spatial reasoning (Mulligan, 2015; Mulligan & Mitchelmore, 2013; Mulligan, Mitchelmore, English & Crevensten, 2013; Mulligan et al., 2018). In some articles, these skills were conceptualised as early algebraic thinking and reasoning skills (Carraher & Schliemann, 2007), covering the beginning of primary education (Warren, Trigueros & Ursini, 2016), and as algebraic thinking later at school. Clements and Sarama (2007, 2009) categorise these skills using number and quantity thinking, geometry and spatial thinking, patterns and structures (algebraic thinking), data analysis and mathematical processes. Mulligan and Mitchelmore (2013), on the other hand, define these processes as spatial and quantitative reasoning, deduction and induction, analogical reasoning, and statistical reasoning. Mathematical reasoning processes require several parallel logical thinking and reasoning skills and given that the literature review revealed an absence of a common classification in early mathematics education, it is suggested here that the cluster of these skills be conceptualised as *mathematical thinking and reasoning skills*. The conceptual arguments will be presented next.

Cluster of logical thinking, reasoning, analysing and problem-solving skills

According to Carraher and Schliemann (2007), children come to understand that certain patterns, functions and their relationships need to be considered in mathematical thinking and reasoning processes. Several articles shared this conceptualisation, addressing that children learn to search for certain structures and regularities to order, predict and create cohesion (McGarvey, 2013; Sarama & Clements, 2009; Schultz, Gopnik & Glymour, 2007). Children practise these skills, for instance, by comparing, sorting, classifying, ordering and seriating objects according to specific features (Carraher & Schliemann, 2007; Warren et al., 2016). Hence, children learn to discover associations and repeatable sequences and reach conclusions about quantities and features by categorising objects. Clements and Sarama (2007) suggested that young children can usually sort things by using a single attribute, although they can reclassify them by using different attributes. Additionally, children can collect and find patterns and functions in graphs or diagrams, which assist in learning the principles of data analysis (Sarama & Clements, 2009). A few studies supported this view by showing that young children are capable of data modelling (English, 2013; Mulligan, 2015).

The reviewed articles in this study revealed that mathematical thinking and reasoning processes guide children on how to use problem-solving and reasoning strategies that are not innate but emerge early in their lives (Clements & Sarama, 2007). Children are also capable of learning structuring process through word problems and numerical and geometrical processes (Mulligan & Mitchelmore, 2013); they are also capable of learning multiple problem-solving methods (Tsamir, Tirosh, Tabach & Levenson, 2010). Additionally, recent studies indicated that children learn somewhat sophisticated thinking during early childhood, such as functional relationships reasoning (Blanton, Brizuela, Gardiner, Sawrey & Newman-Owens, 2015; Lee, Collins & Melton, 2016). These studies also showed that children are capable of processing reasoning and using the required concepts (Lee et al., 2016). Current studies also advanced prior understandings by showing that the development of mathematical thinking and reasoning skills positively correlates with general mathematical achievement (see Lee et al., 2016; Warren et al., 2016). However, the studies also emphasised that children who do not achieve basic mathematical thinking and reasoning competencies during early childhood will often face problems in mathematics later at school (Lee et al., 2016; Warren et al., 2016).

Based on the conceptualisation presented in this study, a cluster of mathematical thinking and reasoning skills (Figure 4) is an essential component of early mathematical skills. These skills enable children to understand and consider patterns, functions and their relationships in mathematical processes while reasoning, solving problems and learning analytical thinking and mathematical-logical thinking.

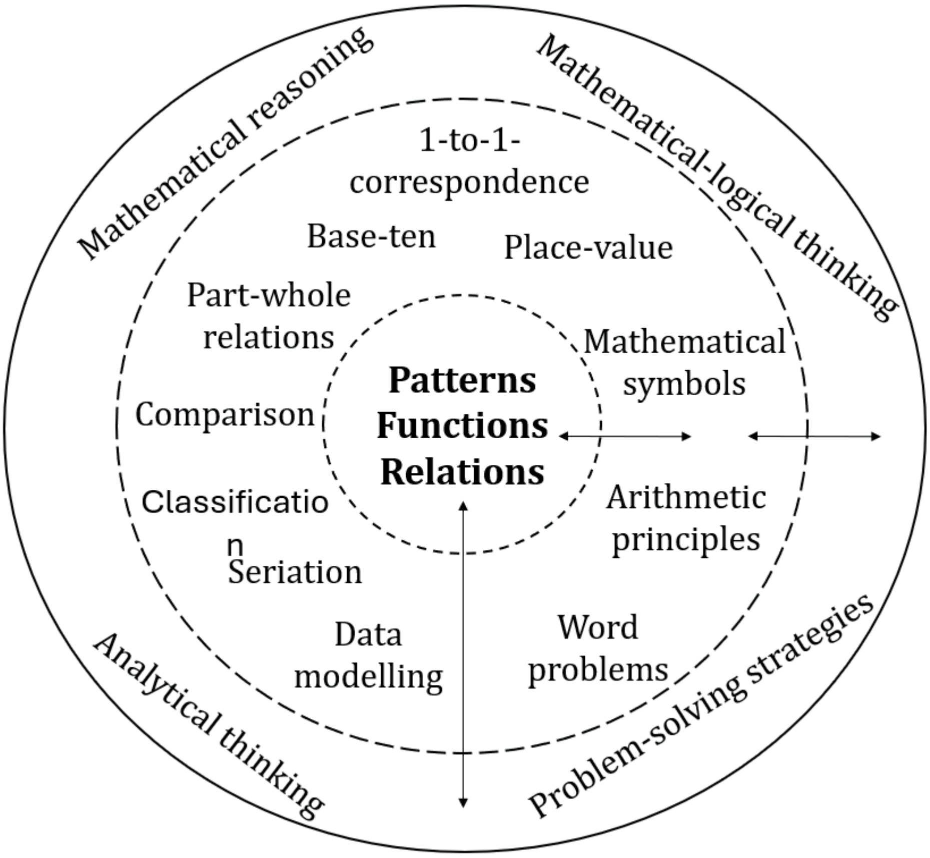

FIGURE 4 Mathematical thinking and reasoning skills.

Figure 4 illustrates how children use versatile mathematical thinking and reasoning skills while learning different aspects of mathematics. Skills are, therefore, presented in a cluster displaying their interconnections. Mathematical thinking and reasoning skills should not be acknowledged as part of numerical skills (Figure 2) and spatial thinking skills (Figure 3) only. It is also crucial to support the development of these skills as a separate skill set.

Holistic model of early mathematical skills

The literature reviewed in this study was utilised in forming early mathematical skills categories, namely (1) numerical skills, (2) spatial thinking skills and (3) mathematical thinking and reasoning skills. The review revealed that these categories include several skills and sub-skills, which are essential to gaining strong mathematical skills necessary for life on the one hand (Clements & Sarama, 2007; Cross et al., 2009; van Oers, 2013) and school mathematics on the other (Aunio & Niemivirta, 2010; Kolkman et al., 2013; Lee et al., 2016; Nanu et al., 2018; Romano et al., 2010; Toll et al., 2016; Warren et al., 2016). Several articles stressed how each skills category influences gaining strong mathematical skills during early childhood (e.g. Hannula & Lehtinen, 2005; Hawes et al., 2017; Howse & Howse, 2014; Kolkman et al., 2013; Mulligan & Mitchelmore, 2013; Simms et al., 2016). Therefore, for children to gain strong mathematical skills, it is vital to ensure the simultaneous development of all early mathematical skills.

Interconnections of skills categories

The reviewed literature revealed that not only do early mathematical skills develop simultaneously but learning new skills is also connected between the skills categories. Some studies stressed that learning about spatial environment requires numerical learning and vice versa. While learning measurement, children connect length and number (Clements & Sarama, 2007; Fantozzi et al., 2013; Solomon et al., 2015) and mass and number (Cheeseman, McDonough & Ferguson, 2014). Several studies showed that children use spatial reasoning while learning counting strategies and basics of arithmetic (Kyttälä et al., 2014; Laski et al., 2013; Laski & Siegler, 2014; Toll et al., 2016). The studies also stressed that children use numerical skills while learning spatial reasoning (e.g. rotation) (Hawes et al., 2017) or spatial contents (e.g. understanding area or time) (Chigeza & Sorin, 2016; Fantozzi et al., 2013; Mulligan & Mitchelmore, 2013; Mulligan et al., 2013). The literature showed that similar to number sense, strong spatial sense positively correlates with a child´s mathematical learning, illustrating that visuospatial system is needed to strengthen number sense (Bobis, 2008; Toll et al., 2016), represent magnitudes (Sarama & Celements, 2009) and understand arithmetic principles (e.g. composing and decomposing) (Laski et al., 2013).

A few earlier studies indicated that in terms of evolving spatial reasoning, neurological maturation influences learning relative magnitude between two numerals, leading to misjudgements between the counted objects at an early age (Ansari, Garcia, Lucas, Hamon & Dhital, 2005). Additionally, the literature showed that to understand the connections between numerical and spatial topics fully, children need conceptualisation skills, which are very often related to learning and development of language (Kyttälä et al., 2014;

Lyytinen, 2014; Owens, 2014). The findings above suggest that children use numerical skills for learning spatial thinking skills and vice versa. Due to the strong relationship between numerical and spatial learning, several studies suggested that spatial-numerical congruence should be better addressed in numerical learning (Laski & Siegler, 2014; Patro & Haman, 2012; Zhang et al., 2017). A finding of Toll's research group (2016) supported this view, showing that concerning number sense, visuospatial working memory strongly influences subsequent mathematical achievement.

In addition to discovering the interconnections between numerical skills and spatial thinking skills, mathematical thinking and reasoning skills were found to be connected to both skills areas. In their categorisation of numerical skills, Aunio and Räsänen (2016) already acknowledged this connection, although they conceptualised it in their model as 'understanding mathematical relations'. According to the literature, children learn the general aspects of variables and entities, such as understanding arithmetic principles, one-to-one correspondence, mathematical symbols (e.g. +, -, <, >, =) and part-whole relationships through mathematical thinking, reasoning skills (i.e. composing and decomposing), place-value and base-ten system (Carraher & Schliemann, 2007; Sarama & Clements, 2009; Slusser & Sarnecka, 2011; Warren et al., 2016). Articles also asserted that children learn the functional relationships between the quantities (Blanton et al., 2015; Brizuela, Blanton, Sawrey, Newman-Owens & Gardiner, 2015; Purpura & Lonigan, 2013; Sarama & Clements, 2009). Additionally, some articles indicated that learning mathematical thinking and reasoning skills (e.g. data modelling) fosters numerical learning (Mulligan, 2015). The review illustrated that children use several mathematical thinking and reasoning processes while learning numerical skills (see Figure 2) but learn mathematical thinking and reasoning skills through numerical learning (see Figure 4).

The literature review also revealed a connection between spatial thinking skills and mathematical thinking and reasoning skills (see Figure 3). The literature addressed that children use mathematical thinking and reasoning skills while learning spatial thinking skills. Several studies showed that children compare, classify, seriate and apply their understanding of part-whole relationships while learning to understand various aspects of spatial environment (Clements & Sarama, 2007; Hallowell et al., 2015; Hawes et al., 2017; Jones & Tzekaki, 2016; Ramful et al., 2015; Zacharos, Antonopoulos & Ravanis, 2011). Additionally, the literature showed that children apply logical thinking skills while reasoning spatial environment (Chen, 2007; Tsubota & Chen, 2012; Vasilyeva & Bowers, 2006, 2010). Again, the same connection would be applied the other way round, as the articles indicated that children use spatial reasoning while learning to understand patterns, functions and their relationships (Mulligan & Mitchelmore, 2013; Mulligan et al., 2013) or arithmetic principles (e.g. decomposition) (Laski et al., 2013). Importantly, one article emphasised that it is crucial to separate visual learning from spatial reasoning, mainly because poorly visualised mathematics software can hardly promote mathematical thinking and reasoning skills; this would mean that children are then not using spatial reasoning skills but instead limited to visual thinking (Ginsburg, Jamalian & Creighan, 2013).

The literature reviewed in this study indicated the interconnections between the three skills categories and asserted that the connection between the categories was bidirectional. Further, it indicated that because of bidirectional connections, children draw on the skills of all categories while learning mathematics. For example, using visuospatial skills while learning arithmetic principles (Kyttälä et al., 2014; Laski et al., 2013) connects numerical skills, spatial thinking skills and mathematical thinking and reasoning skills. Thus, these findings illustrate that early mathematical skills learning and development are interconnected across the three categories.

Simultaneous development of interconnected early mathematical skills

Versatile interconnected mathematical skills begin to develop simultaneously at an early age in a child's life. The literature addresses that children use several different skills when learning certain aspects of mathematics (English, 2013; Hawes et al., 2017; Laski et al., 2013; Mulligan et al., 2018; Sarama & Clements, 2009; Warren et al., 2016). In this study, the skills areas were first categorised separately to build a framework for the holistic learning and development of early mathematical skills. Subsequently, it was necessary to show the interconnections of these skills categories to be able to illustrate the complex learning and development of early mathematical skills fully. The holistic theoretical model of early mathematical skills (Figure 5) was constructed by analysing the literature, noting the early mathematical skills that children learn and develop in early childhood, including the connections and foundations of these skills. The elements of early mathematical skills, namely (1) numerical skills (including number sense), (2) spatial thinking skills (including spatial sense) and (3) mathematical thinking and reasoning skills and their interrelationships are presented in Figure 5.

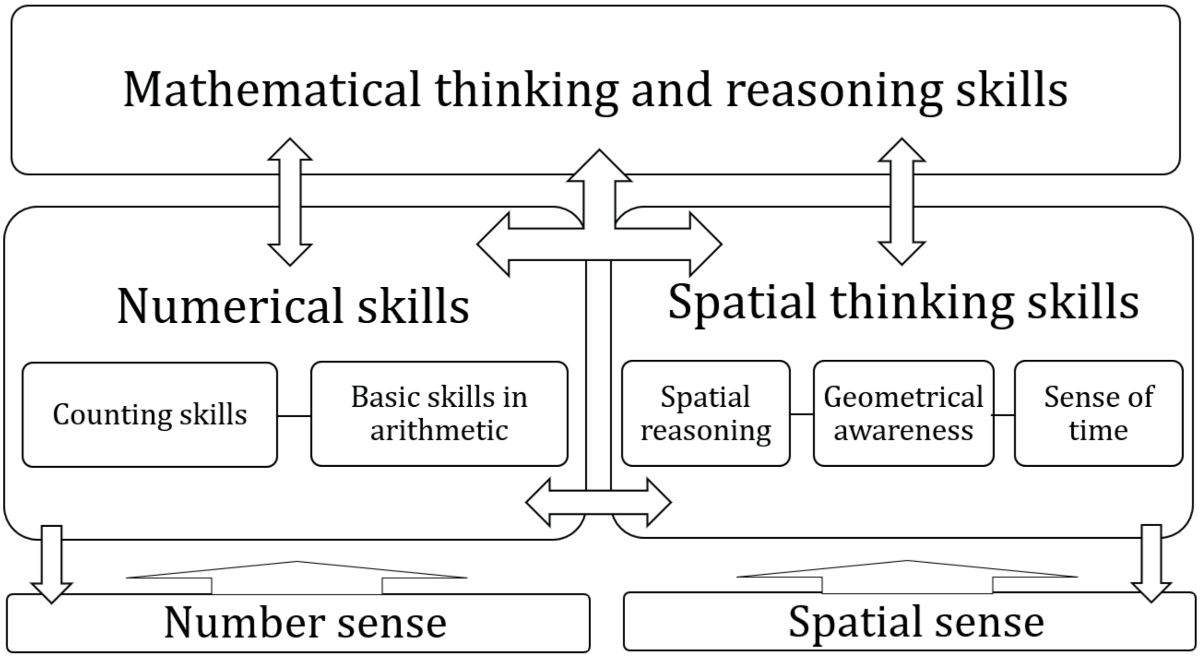

FIGURE 5 Holistic model of the development of early mathematical skills.

As Figure 5 illustrates, different elements of early mathematical skills are completely interconnected, and there exist bi- and multi-directional relationships among these elements. The reviewed articles in this study showed that number sense (symbolic and non-symbolic) and spatial sense develop from an early age. They also showed that they form the foundation for learning numerical skills (Aunio & Niemivirta, 2010; Bobis, 2008; Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; Kolkman et al., 2013) and spatial thinking skills (Cross et al., 2009; Howse & Howse, 2014). The literature review also revealed that learning numerical skills requires spatial thinking skills and vice versa (Bobis, 2008; Laski & Siegler, 2014; Sarama & Clements, 2009, Zhang et al., 2017). Additionally, it showed that during early childhood, children learn mathematical thinking and reasoning skills and that they use these to learn numerical skills and spatial thinking skills and vice versa (Carraher & Schliemann, 2007; Clements & Sarama, 2007; Mulligan & Mitchelmore, 2013; Sarama & Clements, 2009). Furthermore, the review showed that the three skills areas are interlinked while children learn mathematics (e.g. Laski et al., 2013).

The articles reviewed in this study showed that understanding the simultaneous development of mathematical skills and the interconnections of the skills categories is essential for promoting the holistic development of mathematical skills in early childhood. Reaching a balanced approach to teaching early mathematical skills to children requires that teachers need to be aware of what and how they teach. Several articles showed that when children become able to simultaneously develop their mathematical skills in versatile ways (e.g. learning numeracy with visuospatial aid), their mathematical learning improves holistically (Cheeseman et al., 2014; Cheun & McBridge, 2017; Colliver, 2018; Laski & Siegler, 2014; Lee et al., 2016; MacDonald & Lowric, 2011; McNeil, Fyfe & Dunwiddie, 2015; Zur & Gelman, 2004). The contemporary studies also indicated that children learn somewhat sophisticated mathematical thinking during early childhood when they are taught appropriate mathematical curricular contents (Blanton et al., 2015; Lee et al., 2016). The aims and results of this present study helped in forming a theoretical framework for a holistic model of the development of early mathematical skills in early childhood. The model is suggested to be considered in curriculum reforms and used as a guide on teaching early mathematical skills, as it is grounded on the understanding that early mathematical skills are interconnected and develop simultaneously.

Discussion

This literature review draws a holistic theoretical model of the learning and development of early mathematical skills. Besides, the categorisation of the core mathematical skills showed that children learn these skills simultaneously, and these apects should, therefore, be acknowledged in early childhood education. This understanding illustrates that there exist comprehensive compilations of early mathematical skills development (Clements & Sarama, 2007; Sarama & Clements, 2009) and many independent studies that focus on certain mathematical skills areas (i.e. Cheeseman et al., 2014; Hannula & Lehtinen, 2005; Krüger et al., 2014; Mulligan, 2015). This literature-based holistic model study into early mathematical skills aims to promote discussions of versatile mathematical skills which children learn and develop during early childhood.

The literature review shows that from birth to the age of eight, children learn and develop several mathematical skills, which form the bedrock of the skills that they will learn later (e.g. Aunio & Räsänen, 2016; Hannula-Sormunen et al., 2015; Sarama & Clements, 2009; Simms et al., 2016; Toll et al., 2016). By the age of eight, for example, children have fully developed counting strategies, learned the basics of arithmetic (McNeil et al., 2015) and acquired spatial and geometrical principles (Clements & Sarama, 2007; Kaur, 2015; Sarama & Clements, 2009). They know how to use visuospatial skills for mapping (Cross et al., 2009; Vasleyeva & Bowers, 2006, 2010). Furthermore, they can describe the qualities and features that can help them readily distinguish directions and locations (Clements & Sarama, 2007; Owens, 2014). They also understand certain measures, such as time, length, mass, volume and area (Battista, 2007; Jones & Tzekaki, 2016; Meaney, 2011). Besides, they can draw on mathematical thinking and reasoning skills to solve mathematical problems (Carraher & Schlieman, 2007; Mulligan & Mitchelmore, 2013).

The literature review also shows that versatile mathematical skills develop simultaneously at an early age (e.g. Aunio & Räsänen, 2016; Howse & Howse, 2014; Jones & Tzekaki, 2016; Merkley & Ansari, 2016; Mulligan et al. 2018). Moreover, newer studies clearly illustrate strong interconnections between different early mathematical skills (e.g. Chigeza & Sorin, 2016; Hawes et al., 2017; Kyttälä et al., 2014; Laski & Siegler, 2014; Mulligan et al., 2013; Zhang et al., 2017). Early mathematical skills have thus far been categorised and classified only from a numerical perspective (Aunio & Räsänen, 2016). Hence, in addition to numerical skills, it is essential to categorise all early mathematical skills and build a holistic theoretical framework, illustrating the interconnections of these developing skills areas. The practical aim of the model, therefore, is to raise teachers' awareness that mathematical skills in early childhood education are interconnected and develop simultaneously and encourage them to seek a holistic approach to their teaching. Building an understanding of the interconnections of early mathematical skills can also illustrate how significant it is that children's mathematical skills develop systematically, simultaneously and holistically. Acting accordingly will assure effective planning for balanced mathematical programmes for early childhood education settings.

The literature review illustrates that the number of studies on the development of mathematical thinking and reasoning skills is limited and that more studies are needed to understand this skills category better. Additionally, researchers are advised to extend their investigations into younger age groups, as most of the studies are focused on children aged seven or eight years. More extensive research evidence would illustrate the importance of these skills and perhaps advance our knowledge of individual variations within them. Given the current limited number of studies, more explorations into the learning and development of the sense of time are also necessary to deepen our understanding of this skills area. Additionally, studies exploring the connections between the skills categories would broaden our understanding of the holistic learning and development of early mathematical skills.

One perceived limitation of this study is that the literature search was confined to specific search terms and the ERIC database. Using other databases (e.g. PsycINFO) and more diverse search terms could have yielded a more comprehensive number of relevant articles for review and analysis.

Conclusion

To support children´s mathematical skills, the findings in this study illustrate the need for more research into the practical implications of early mathematics teaching practices, teacher education and professional development (see Piasta, Logan, Pelatti, Capps &

Petrill, 2015). Additionally, the findings suggest that a comprehensive analysis investigating how early mathematical skills are conceptualised and categorised in curricula could provide valuable insights. Collectively, these efforts can broaden our understanding of teaching mathematical skills in early childhood education.

The arguments presented above were validated in the 2009 Conference of European Research in Mathematics Education, which reiterated the importance of mathematics teaching and research for children aged between three and eight years (Tsamir et al., 2011). This literature review suggests that to recognise the importance of early mathematical learning and mathematics teaching, future researchers should focus on (1) specific skills areas that need to be better understood (sense of time, mathematical thinking and reasoning skills); (2) connections between skills development aimed at broadening current understanding and (3) practical implementations to promote the learning of versatile early mathematical skills.

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II

TEACHING EARLY MATHEMATICAL SKILLS TO 3- TO 7-YEAR-OLD CHILDREN – DIFFERENCES RELATED TO MATHEMATICAL SKILL CATEGORY, CHILDREN'S AGE GROUP AND TEACHERS' CHARACTERISTICS

by

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Teaching Early Mathematical Skills to 3- to 7-Year-Old Children — Differences Related to Mathematical Skill Category, Children's Age Group and Teachers' Characteristics

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Abstract
This study explored teaching early mathematical skills to 3- to 7-year-old children in early childhood education and care (ECEC) and pre-primary education. Teachers in ECEC ($N=206$) answered a web survey. The first aim was to determine whether teaching frequency or pedagogical awareness of teaching early mathematical skills varied according to the category of skills (numerical skills, spatial thinking skills and mathematical thinking and reasoning skills) and whether children's age group moderated these differences. The second aim was to explore to what extent teacher-related characteristics and children's age group explained variations in teaching frequency concerning early mathematical skills. Results from repeated MANOVAs demonstrated that the frequency and pedagogical awareness of teaching early mathematical skills depended on the skill category and that children's age group moderated these differences. In 5- to 6-year-olds and 6- to 7-year-olds, numerical skills were taught more often than spatial thinking skills, whereas in 3- to 5-year-olds, they were taught as frequently. In all age groups, mathematical thinking and reasoning skills were taught the least. Pedagogical awareness was lowest in teaching spatial thinking skills in all age groups, but only in 6- to 7-year-olds was teachers' pedagogical awareness in teaching numerical skills higher than in the two other categories. According to a univariate analysis of variance, pedagogical awareness and mathematics professional development programmes were strongly associated with teaching frequency in all skill categories. The results emphasise that children's opportunities to learn early mathematical skills depend on teachers' characteristics.

Keywords Mathematical thinking and reasoning skills · Numerical skills · Pedagogical awareness · Spatial thinking skills · Teaching early mathematical skills

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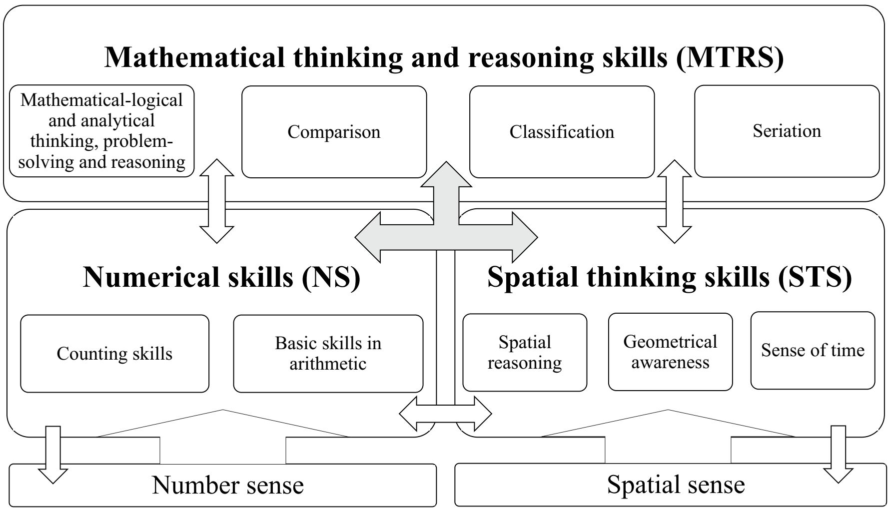

Fig. 1 Holistic model of early mathematical skills development (Parviainen, 2019)

Introduction

As mathematics is a demonstrably important part of human life, poor mathematical skills in childhood have been shown to have long-term adverse effects on further education, employment and even mental health in adulthood (Aro et al., 2019). Therefore, mathematics is included in most educational systems with the aim of ensuring basic mathematical proficiency and understanding for all citizens (van Oers, 2013). While contemporary research findings clearly show that children develop and learn versatile mathematical skills before school age, skills which are the basis for those learnt at school (Aunio & Räsänen, 2016; Lepola & Hannula-Sormunen, 2019; Mulligan & Mitchelmore, 2013; Sarama & Clements, 2009), research on mathematics education in early childhood education and care (ECEC) and pre-primary education is still limited when compared to that in primary and secondary education. Since the 2009 Conference of European Research in Mathematics Education, there has been a call for studies exploring mathematics education in early childhood from various perspectives (Linder & Simpson, 2018; Tsamir et al., 2011).

Linder and Simpson's (2018) recent research review revealed that most studies on the teaching of early mathematics have focused on numerical areas with comparatively limited coverage of algebra, geometry, measurement and data analysis. There is also a lack of research on mathematics teaching from a broad perspective of mathematical contents, including, e.g. spatial and mathematical reasoning skills. By applying this broad perspective on mathematics, we investigated variations in the frequency with which different early mathematical skills are taught in ECEC and pre-primary education. Our research was based on Parviainen's (2019) holistic model of early mathematical skills development (see Fig. 1) and was aimed at better understanding the teaching of different skills to 3- to 7-year-old children. In this study, teaching was broadly understood as covering all teaching situations and spontaneously emerging teachable moments in daily life (e.g. discussions and routine events) and play that enhance children's learning.

Teachers in ECEC play a critical role in shaping children's mathematical learning opportunities. Studies have revealed that the more comfortable teachers are with teaching mathematics, the more optimistic they are regarding children's learning (Çelic, 2017; Lutovac & Kaasila, 2018; Sumpter, 2020). However, we lack knowledge concerning potential differences in teachers' pedagogical awareness of teaching different early mathematical skills to 3- to 7-year-olds. Additionally, it remains unclear how certain teacher characteristics (teachers' pedagogical awareness of teaching mathematical skills, duration of professional development (PD) programmes in mathematics, age and work experience and children's age group) are linked to the frequency of teaching different early mathematical skills to 3- to 7-year-old children.

Theoretical and Conceptual Framework

The purpose of early childhood mathematics education is to promote children's development of mathematical skills, strengthen their capacity for mathematical learning and cultivate positive attitudes towards mathematics (Sarama & Clements, 2009). It also aims to enhance children's numerical and spatial learning as well as to bolster their memorisation, problem-solving and reasoning skills (Clements et al., 2011; Keisar & Peled, 2018). Therefore, the elements of early mathematical skills and their teaching are discussed first, followed by a conceptualisation of existing knowledge of teacher-related variations in teaching frequency.

Early Mathematical Skills and Variations in Their Teaching Frequency

Early mathematical skills and their teaching are typically categorised into numeracy and geometry (Tsamir et al., 2011), although broader perspectives covering spatial thinking and mathematical reasoning processes have also been presented (Clements & Sarama, 2007; Sarama & Clements, 2009). Based on a systematic literature review, Parviainen (2019) introduced a broad theoretical framework for a holistic model of early mathematical skills development (Fig. 1). The present study was grounded on this model, as it permitted the operationalisation of different mathematical skills by offering a logical basis for their division.

The *holistic model of early mathematical skills development* (Parviainen, 2019) categorises early mathematical skills into three skill categories: (1) *numerical skills (NS)*, which include innate number sense, gradual development of counting skills and basic skills in arithmetic; (2) *spatial thinking skills (STS)*, including innate spatial sense, which serves as the basis for spatial reasoning, geometrical awareness and sense of time; and (3) *mathematical thinking and reasoning skills (MTRS)*, which are not innate but develop gradually and include the understanding of patterns, functions and their relations as well as different reasoning, logical thinking and problem-solving strategies. Despite this categorisation, these skills overlap and are mutually interactive, e.g. MTRS are needed in NS and STS, and vice versa.

The conceptualisation of NS is based on the knowledge that they develop gradually from birth, such as the sense of numbers and quantities, and strengthen as children age (Baroody, 2011; Clements & Sarama, 2007; Lepola & Hannula-Sormunen, 2019). In ECEC and pre-primary education, primary counting skills (e.g. interrelationships between number word, number symbol and quantity) develop first, followed by counting strategies (e.g. mental number word sequence skills develop during pre-primary education). Counting skills are essential for learning basic arithmetic skills covering principles of commutativity and associativity generally by age 5 and the inversion of addition and subtraction by age 7 (Aunio & Räsänen, 2016; Baroody, 2011; Kullberg et al., 2020).

STS develop alongside NS (Fig. 1). Spatial sense develops from birth, with children progressively learning versatile spatial and geometrical principles (Clements, 2011; Sarama & Clements, 2009), such as mapping (Clements, 2011; Vasilyeva & Bowers, 2006, 2010) and discriminating directions and locations, which are subskills of spatial reasoning. Additionally, children become aware of the principles of measuring while learning spatial relations, geometry and time (Baroody, 2011; Battista, 2007; Clements & Stephan, 2011; Jones & Tzekaki, 2016). Geometrical awareness skills become more precise with age, such as the understanding of shapes (Clements, 2011; Hawes et al., 2017), conservation, mass and volume (Clements & Sarama, 2007; Clements & Stephan, 2011). Furthermore, as children age and their language develops, they gain skills to describe spatial qualities in a more sophisticated way (Clements & Sarama, 2007). Moreover, time-related reasoning develops and becomes more accurate alongside language development (Lyytinen, 2014; Mulligan & Mitchelmore, 2013).

In contrast, MTRS do not constitute an innate skill (Fig. 1) but instead develop as children gradually learn to consider patterns, functions and their relationships in mathematical thinking and reasoning processes (Carraher & Schliemann, 2007; Vandlyndt et al., 2021; Worthington et al., 2019). MTRS develop when children learn mathematical-logical and analytical thinking, problem-solving and reasoning strategies and principles of comparison, classification and seriation (Baroody, 2011; Keisar & Peled, 2018; Mulligan & Mitchelmore, 2013). Such learning includes understanding part–whole relations, place-value logic and data modelling (Aunio & Räsänen, 2016; Mulligan, 2015). MTRS develop alongside cognitive development as older children learn to solve mathematical problems by using their logic and reasoning strategies (Alsina & Salgado, 2021; Vandlyndt et al., 2021; Warren et al., 2016).

Parviainen's (2019) model connects the three skill categories, demonstrating bi- and multi-directional relationships between their skills (Fig. 1). For instance, seriation and place-value logic (in MTRS) are applied in NS (e.g. number word sequencing), and part–whole relations and comparison (in MTRS) are applied in STS (e.g. understanding the two- and three-dimensionality of shapes) (Baroody, 2011; Clements, 2011; Sarama & Clements, 2009). In addition, learning to measure area or time (in STS) or data modelling (in MTRS) requires NS (Baroody, 2011; Clements & Stephan, 2011; Mulligan, 2015; Mulligan & Mitchelmore, 2013). Furthermore, learning to understand magnitudes (in NS) requires spatial reasoning (in STS) (Laski & Siegler, 2014; Sarama & Clements, 2009) and MTRS (Baroody, 2011). Because of these relationships, paying more or less attention to one skill category over another does not support the holistic development of early mathematical skills (see Parviainen, 2019).

Existing research on teaching early mathematical skills focuses on 3- to 5-year-olds and reveals several types of variations, with some studies showing counting and learning about shapes to be taught the most (Gonzales & Paik, 2011; Hindman, 2013). Moreover, calendar-related activities appear to be a frequent part of teaching, whereas learning about measurement and telling time is taught less to 3- to 5-year-olds (Hindman, 2013; Sarama & DiBiase, 2004). Although NS, STS and MTRS develop gradually, and despite existing relationships between these categories, extant research on teaching early mathematical skills does not comprehensively investigate such teaching to 3- to 7-year-olds in different age groups. We thus found it necessary to comprehensively examine the frequency of teaching early mathematical skills, namely NS, STS and MTRS, in different age groups in the current study. In this study, Parviainen's (2019) holistic model was applied to investigate these variations.

Teacher-Related Variations in Teaching Early Mathematical Skills

Different theoretical models, including those incorporating teachers' cognition and action competence, have been developed to describe teacher-related factors in early mathematical teaching (Lindmeier, 2011; Lindmeier et al., 2020). According to Gasteiger and Benz (2018), cognition, conceptualised as teachers' knowledge, is crucial to coherently teaching mathematics. Cognition includes mathematical content knowledge, age-appropriate conceptual and developmental understanding of mathematical skills, a variety of learning activities, and observations of mathematical skills' development. Gasteiger and Benz (2018) conceptualised action competence through pedagogical and didactical actions, including situational observing, perceiving and evaluation. In this study, cognition and action competence were integrated into one concept: pedagogical awareness, including mathematical content knowledge, theoretical understanding of early mathematical skills' development, current knowledge about learning these skills, the significance of specific mathematical skills in teaching, coherent practical implications and evaluations of the aforementioned elements. High pedagogical awareness in teachers can be regarded as a prerequisite for teaching and supporting children's early mathematical development in versatile ways.

Earlier studies have indicated teacher-related variations in mathematics teaching (see Lutovac & Kaasila, 2018). Teachers' attitudes towards mathematics (Çelic, 2017) and their pedagogically aware practices are positively linked to teaching early mathematical skills (MacDonald & Murphy, 2019). Components of pedagogical awareness, such as teachers' content knowledge (Callejo et al., 2022; Dunekacke et al., 2015; Muños-Catalán et al., 2022) and content-related teaching confidence, explicitly influence mathematics teaching (Alsina et al., 2021; Gasteiger & Benz, 2018). Although teachers in general are rather confident about their ability to teach mathematical content (Björklund & Barendregt, 2016; Chen et al., 2014; Sumpter, 2020), Chen et al.'s (2014), study revealed that teachers are more confident in teaching rotation, distance estimation, problem-solving and data analysis than arithmetic. Furthermore, some studies have indicated that teachers are less aware of teaching geometry than numbers (Björklund & Barendregt, 2016; Tsamir et al., 2011). Besides, Björklund and Barendregt (2016) discerned that teachers' awareness of mathematical problem-solving is limited. To expand knowledge related to pedagogical awareness of teaching early mathematical skills to 3- to 7-year-olds, this study explored possible variations in pedagogical awareness of teaching NS, STS and MTRS and the potential moderating effect of children's age group on differences between the three skill categories.

In addition to pedagogical awareness, participation in mathematics PD programmes explains teacher-related variations in teaching mathematics, as these programmes have been shown to increase the quality of early childhood mathematics education (Bruns et al., 2017; Tirosh et al., 2011; Tsamir et al., 2014). PD programmes especially improve reflective and action-related mathematics teaching (Gasteiger & Benz, 2018; Lindmeier et al., 2020). For example, sorting and patterning were previously more commonly taught than shapes (Sarama & DiBiase, 2004), which are now more frequently taught (Hindman, 2013). A potential explanation for this shift in early childhood mathematics education is the development of PD and teacher education programmes aimed at enhancing understanding of the importance of geometry and spatial reasoning (Clements & Sarama, 2011). Besides, teachers have heterogeneous educational backgrounds, ages and work experience, which may be reflected in their teaching (see Gasteiger et al., 2021; Sumpter, 2020). To expand knowledge on the influence of teacher-related factors in early childhood mathematics education, this study explored their prospective effects on teaching NS, STS and MTRS to 3- to 7-year-old children.

Aims

The objective of the present study was to explore early childhood mathematics education from the perspective of teaching frequency of different early mathematical skills to 3- to 7-year-old children. We were interested in whether teaching early mathematical skills reflects the pace of development of different skills as theoretically described in Fig. 1. In other words, does the relative teaching frequency of NS, STS and MTRS vary according to the children's age group, e.g. are MTRS taught less frequently than NS and STS to 3- to 5-year-olds, and can clear differences in teaching frequency be detected between the three skill categories among 5- to 6-year-olds and 6- to 7-year-olds. A related aim was to assess whether teachers' pedagogical awareness of teaching these skills varies between the skill categories or the age groups. Another aim was to determine how teacher-related characteristics (pedagogical awareness, duration of PD in mathematics, teachers' age and work experience) and children's age group affect the teaching of the aforementioned skills. These aims were achieved by answering the following research questions:

(1) Are there differences in the frequency of teaching or in teachers' pedagogical awareness of teaching NS, STS and MTRS? Additionally, are these potential differences moderated by the children's age group?
(2) To what extent do teachers' pedagogical awareness, duration of mathematics PD programmes, age and work experience, as well as children's age group, explain variations in the teaching of NS, STS and MTRS? What is the relative importance of these factors?

Study Design

The research data ($N=206$) were collected in Finland between January and March 2020 using a web survey (Webropol) targeted at teachers of 3- to 7-year-old children working in Finnish-language early education centres in the public sector who had formal teaching qualifications in ECEC and pre-primary education (varying from university-level master's degree to former college-level degree).

Method

A cautious sample selection procedure (Johnson & Christensen, 2017; Newby, 2014) was followed to ensure a representative sample of Finnish teachers in ECEC. Geographical representativeness as well as the inclusion of different-sized municipalities was ensured by using stratified sampling and including a variety of cities and towns from different geographical areas of Finland in the sample. Research permissions were obtained from the administration of early education services in accordance with their decision-making protocols.

Next, early education centres within each municipality were selected using systematic sampling (Johnson & Christensen, 2017; Newby, 2014): every fifth centre from an alphabetical or areal list found on the municipal website was chosen. After receiving administrative approval, research invitations were distributed to teachers by the heads of early education centres. Four weeks were allowed for submitting the survey, and reminder messages were sent three times to improve the response rate.

To determine the actual sample size and the size of attrition, the heads of the early education centres were asked to report the number of teachers to whom they sent the research invitation. Altogether, 557 teachers from 102 early education centres received the web survey, of whom 206 responded, resulting in a response rate of 37%. No information was available concerning those who declined the survey. The majority of respondents (196) were women. Eight men responded, one respondent indicated a gender of 'other' and one did not answer the question. The gender division of the respondents represented the teachers' gender distribution in Finnish ECEC and pre-primary education (Finnish National Agency for Education, 2017).

In compliance with ethical standards, the teachers were informed of the voluntary, confidential and anonymous nature of the web survey, including the official informed consent procedure, and their approval for the use of their information (Byrne, 2016; Johnson & Christensen, 2017). Finnish ethical principles of research with human participants (Finnish National Board on Research Integrity, 2019) and other research ethics guidelines (Byrne, 2016) related to, e.g. data storage and handling, were followed throughout the study.

Measures

The web survey was developed based on Parviainen's (2019) *holistic model of early mathematical skills development*. The content of the three skill categories (NS, STS and MTRS) served as the basis for formulating the survey items and calculating the scale scores, i.e. in operationalising the theoretical concepts into quantitative measures used in the analyses. Two pilot web surveys ($N=20$ and $N=18$) were conducted to test the internal consistency and reliability of the scales and to sharpen the formulation of the items. The final survey included 86 closed-ended questions and was divided into three parts.

The first part of the survey included nine questions concerning the respondents' background information (gender, age, qualification, professional title, work experience in ECEC and pre-primary education, location of the workplace, number of residents of the municipality, town or city, children's age group and the duration of PD programmes in mathematics). The respondents were asked to select which of the following age groups of children they taught: 3- to 5-year-olds, 5- to 6-year-olds and 6- to 7-year-olds (i.e. pre-primary education). These represent the typical age-based groupings of children at Finnish early education centres. Daily activities and the broad learning objectives of the socio-pedagogical curricula are organised and prescribed based on these groupings. Multivariate analyses of variance (MANOVA) showed that neither the area of Finland (Lapland, North, East, West and Central, South, South West, $F(15, 541)=0.51$, $p=0.936$) nor the size of the municipality (city, town, municipality, $F(6, 396)=0.88$, $p=0.514$) had an effect on the teaching of different early mathematical skills (NS, STS and MTRS), and thus, they were not considered in the final analysis.

The second part of the survey included 59 questions focusing on how frequently respondents taught NS, STS and MTRS. NS included 17 questions, divided into three subscales: number and quantity knowledge, counting skills and skills in addition and subtraction. STS included 19 questions, divided into three subscales: spatial reasoning, geometrical awareness and sense of time. MTRS included 23 questions, divided into four subscales: mathematical-logical and analytical thinking, problem-solving and reasoning, comparison, classification and seriation. Claims concerning the frequency of teaching NS, STS and MTRS were answered by positioning a sliding clutch according to one's opinion between the extremes of the scale, i.e. 1 and 7 (1 = 'I strongly disagree' and 7 = 'I strongly agree'). The items included both direct and indirect claims related to teaching certain skills, e.g. in NS 'I often teach counting skills (e.g. counting children during a morning circle, play-based counting activities, counting spoons during mealtimes)' or in MTRS 'I often teach mathematical-logical thinking (i.e. logic games, construction series and problem-solving assignments)'. Each scale included one reversed item to maintain the respondents' attention and to prevent mechanical answers, e.g. in STS 'I rarely teach directions and locations (e.g. above, beneath, in front of, behind, far, near)'. Based on its content, each claim was classified as belonging primarily to one of the three skill categories. However, several of these claims measured, to some extent, the teaching of one or both of the other two skill categories. Three questions regarding the frequency of teaching NS, STS and MTRS more generally (one for each skill category) were answered using an interval scale, resulting in the following final numbers of items: NS 18 items, STS 20 items and MTRS 24 items.

The third part of the survey included 15 questions regarding teachers' self-evaluation of their pedagogical awareness of teaching NS, STS and MTRS (five questions for each). Similar questions related to each skill category (NS, STS and MTRS) were presented separately, covering the following five topics: (1) content knowledge of the skill category, (2) significance of the skill category in the teaching of early mathematical skills, (3) evaluation of how strongly one's teaching is based on a firm theoretical understanding of the development of the skill category, (4) up-to-date knowledge of the development of each skill in children and (5) evaluation of the need for new practices for teaching the skill category. The questions were answered by a sliding clutch between the extremes of the scale, i.e. 1 and 7 (1 = 'I strongly disagree' and 7 = 'I strongly agree'). One reversed item was used in each scale that asked the respondents to evaluate their pedagogical awareness from an opposite perspective. For instance, pedagogical awareness of teaching NS included the following statements: 'My teaching of NS is based on strong content knowledge of the development of NS in children', and 'I do not have up-to-date knowledge on how children learn NS'.

Scale scores were derived by calculating the arithmetic means from their items. The Cronbach's alpha for each score, determined to ensure the internal consistency of each measure, is reported in Table 1. The reliabilities of all scales were above the preferred ≥ 0.70 (Johnson & Christensen, 2017) — except for one subscale, the reliability of which was 0.67.

Table 1 Internal consistencies of the scales on the 'teaching early mathematical skills' questionnaire

Scale	Number of items	Cronbach's alpha
Teaching frequency of NS	18	0.80
STS	20	0.75
MTRS	24	0.84
Pedagogical awareness of teaching		
NS	5	0.71
STS	5	0.73
MTRS	5	0.67

NS, numerical skills; *STS*, spatial thinking skills; *MTRS*, mathematical thinking and reasoning skills

Results

Distributions of the mathematical scale scores were examined to ensure that the requirements for the parametric statistical analyses were fulfilled. All distributions were normal or close to normal as, in all measures, skewness/standard error of skewness and kurtosis/standard error of kurtosis were below or close to 2 (see Table 2).

Frequency and Pedagogical Awareness of Teaching Early Mathematical Skills

To examine whether the frequency of teaching early mathematical skills varied according to skill category and children's age group, a MANOVA for repeated measures was used, in which the scale score of teaching frequency in each skill category (NS, STS and MTRS) was used as the within-subject factor and the children's age group was used as the between-subject factor. The analysis showed that the skill category × children's age group interaction was significant ($F(4, 400) = 11.76$, $p < 0.001$, $\eta_p^2 = 0.10$) (see Fig. 2).

Post hoc pairwise comparisons of skill categories, using Bonferroni correction for significance, revealed that the differences in the mean frequency of teaching the three early mathematical skill categories varied according to the children's age group. In 3- to 5-year-olds, NS and STS were more frequently taught than MTRS at the $p < 0.001$ level but did not differ from each other ($p = 1.00$). The differences between NS and MTRS, on the one hand, and between STS and MTRS, on the other, were of medium size (*Cohen's* $d = 0.54$ in both cases) (see cut-off scores for small, medium and large effect sizes, Cohen, 1992). Among 5- to 6-year-olds and 6- to 7-year-olds, NS were taught more often than STS and MTRS ($p = 0.001$ for 5- to 6-year-olds and $p < 0.001$ for 6- to 7-year-olds). However, STS and MTRS did not differ from each other ($p = 0.79$ for 5- to 6-year-olds and $p = 0.96$ for 6- to 7-year-olds). The difference between NS and STS was of medium size for 5- to 6-year-olds ($d = 0.56$) and of large size for 6- to 7-year-olds ($d = 0.81$). Likewise, the difference between NS and MTRS was of medium size for 5- to 6-year-olds ($d = 0.71$) and of large size for 6- to 7-year-olds ($d = 0.84$).

We also investigated whether teachers' pedagogical awareness varied according to skill category (NS, STS and MTRS) or children's age group. A MANOVA for repeated measures was used. The scale score of pedagogical awareness in each skill category (NS, STS and MTRS) was used as the within-subject factor and the children's age group as the between-subject factor. The analysis revealed that the skill category × children's age group interaction was significant ($F(4, 394) = 4.87$, $p = 0.001$, $\eta_p^2 = 0.05$) (see Fig. 3).

Post hoc pairwise comparisons, using Bonferroni correction for the significance, revealed that the differences in the mean level of teachers' pedagogical awareness of the three skill categories varied according to the children's age group. Among 3- to 5-year-olds, teachers' pedagogical awareness in teaching STS was lower than in teaching NS ($p < 0.001$) and MTRS ($p = 0.02$). Differences were of small size ($d = 0.24$ between STS and NS, and $d = 0.15$ between STS and MTRS). Teachers' pedagogical awareness concerning NS and MTRS did not differ from each other

Table 2 Descriptive statistics of scale scores related to the frequency and pedagogical awareness of teaching early mathematical skills

	Age group of children											
	3- to 5-year-olds				5- to 6-year-olds				6- to 7-year-olds			
Scale	Mean	SD	Skew (SE)	Kurt (SE)	Mean	SD	Skew (SE)	Kurt (SE)	Mean	SD	Skew (SE)	Kurt (SE)
Teaching frequency of												
NS	4.38	0.91	−0.32 (0.24)	−0.19 (0.47)	5.01	0.84	−0.51 (0.44)	0.13 (0.86)	5.40	0.74	-0.46 (0.28)	-0.33 (0.56)
STS	4.35	0.80	−0.19 (0.24)	−0.07 (0.47)	4.56	0.78	−0.56 (0.44)	−0.50 (0.86)	4.75	0.86	-0.12 (0.28)	-0.23 (0.56)
MTRS	3.89	0.90	−0.16 (0.24)	−0.02 (0.47)	4.42	0.83	0.51 (0.44)	−0.40 (0.86)	4.68	0.96	-0.12 (0.28)	-0.48 (0.56)
Pedagogical awareness of teaching												
NS	3.95	1.16	0.26 (0.24)	−0.20 (0.47)	4.52	1.18	−0.15 (0.45)	0.39 (0.87)	4.91	1.13	0.04 (0.28)	-0.97 (0.56)
STS	3.67	1.17	0.24 (0.24)	0.01 (0.47)	4.04	1.10	0.22 (0.45)	0.39 (0.87)	4.15	1.18	0.31 (0.29)	-0.10 (0.57)
MTRS	3.85	1.17	0.31 (0.24)	−0.16 (0.47)	4.38	1.06	−0.02 (0.45)	1.24 (0.87)	4.70	1.06	0.25 (0.28)	-0.62 (0.56)

NS, numerical skills; *STS*, spatial thinking skills; *MTRS*, mathematical thinking and reasoning skills

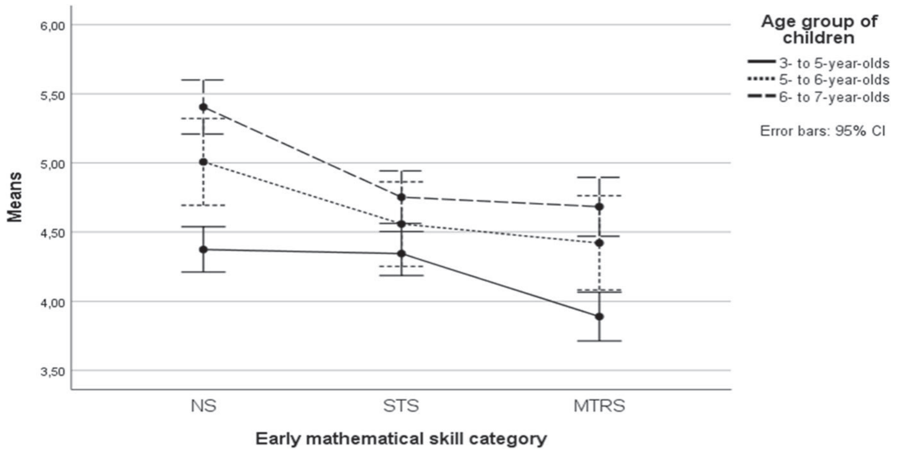

Fig. 2 Mean frequency of teaching early mathematical skills according to skill category and children's age group. Note. NS, numerical skills; STS, spatial thinking skills; MTRS, mathematical thinking and reasoning skills

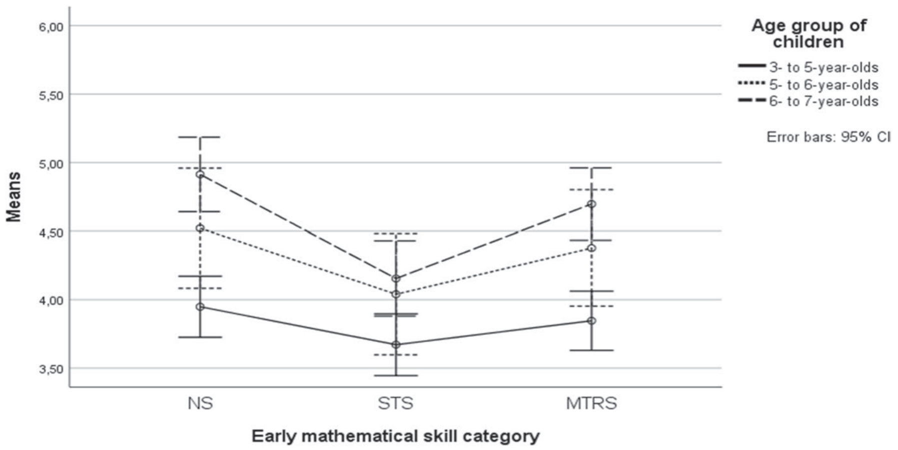

Fig. 3 Means of teachers' pedagogical awareness according to skill category and children's age group. Note. NS, numerical skills; STS, spatial thinking skills; MTRS, mathematical thinking and reasoning skills

($p = 0.45$). Likewise, among 5- to 6-year-olds, teachers' pedagogical awareness was lower in STS compared to NS ($p = 0.01$) and MTRS ($p = 0.03$). The differences between STS and NS, on the one hand, and between STS and MTRS, on the other, were of small size ($d = 0.42$ and $d = 0.32$, respectively). Teachers' pedagogical awareness regarding NS and MTRS did not differ from each other ($p = 0.70$). In contrast, among 6- to 7-year-olds, teachers' pedagogical awareness in all skill categories differed from each other at the $p < 0.05$ level. Pedagogical awareness was highest in teaching NS and lowest in teaching STS, with MTRS falling between these two. The difference between NS and STS, on the one hand, and between MTRS and STS, on the other, were of medium size ($d = 0.66$ and $d = 0.49$, respectively), whereas the difference between NS and MTRS was small ($d = 0.19$).

Associations between Teachers' Characteristics, Children's Age Group and Frequency of Teaching Early Mathematical Skills

We first inspected Pearson correlations between the background measures and the scale scores of the three early mathematical skill categories. Thereafter, a univariate analysis of variance was used separately for each scale score to determine the significant factors for the teaching frequency in each skill category. In other words, we continued the analysis by examining how certain characteristics of teachers (age, work experience, pedagogical awareness and duration of mathematics PD programmes) and children's age group were related to teaching frequency of NS, STS and MTRS.

Correlation analysis revealed, first, modest associations between teachers' age, work experience and duration of mathematics PD programmes in relation to the teaching frequency of all skill categories (see Table 3). In addition, a moderate association was found between teachers' pedagogical awareness of teaching and the frequency of teaching each skill. The correlation between teachers' age and work experience was strong, suggesting potential multicollinearity. However, between other independent measures, associations were weak. Teachers' pedagogical awareness of teaching NS, STS and MTRS were strongly associated with each other, similarly to the frequencies of teaching NS, STS and MTRS. However, these latter internecine

Table 3 Correlations between teachers' characteristics and teaching frequency of different early mathematical skills

	2	3	4	5	6	7	8	9
1. Age	0.82***	0.38***	0.29***	0.19**	0.25***	0.19**	0.10**	0.12**
2. Work experience		0.39***	0.29***	0.20**	0.29***	0.27***	0.17**	0.20**
3. Duration of mathematics PD programmes			0.41***	0.34***	0.36***	0.36***	0.26***	0.32***
Teaching frequency of								
4. NS				0.73***	0.78***	0.48***	0.43***	0.54***
5. STS					0.78***	0.43***	0.49***	0.44***
6. MTRS						0.54***	0.51***	0.52***
Pedagogical awareness of teaching								
7. NS							0.73***	0.78***
8. STS								0.51***
9. MTRS								

NS, numerical skills; *STS*, spatial thinking skills; *MTRS*, mathematical thinking and reasoning skills

associations caused no problem, as separate skill categories were analysed separately in different models.

Next, a univariate analysis of variance was performed separately for each skill category (NS, STS and MTRS) to determine which factors had a unique effect on the outcome when added simultaneously to the model. Moreover, the relative importance of each factor was inspected by reporting the percentage of variance explained by each independent factor. All variables with a significant association with the dependent measure were included in the model first, after which non-significant measures were removed one by one until the final model with only significant measures remained. Only the results related to the final model are presented.

At first, the univariate analysis of variance for NS showed that pedagogical awareness of teaching NS ($F(1, 197) = 18.72$, $p < 0.001$, $\eta_p^2 = 0.087$), duration of PD in mathematics ($F(1, 197) = 7.97$, $p = 0.005$, $\eta_p^2 = 0.039$) and teachers' age ($F(1, 197) = 4.80$, $p = 0.030$, $\eta_p^2 = 0.024$), as well as children's age group ($F(2, 197) = 15.66$, $p < 0.001$, $\eta_p^2 = 0.137$), were significantly associated with the frequency of teaching NS. The stronger the pedagogical awareness, the more PD in mathematics; and the higher the teacher's age, the more frequently NS were taught. Because our data included three different age groups, we ran post hoc pairwise comparisons to determine which age group differed significantly from the others. Using Bonferroni correction for the significance, the analysis showed that NS were taught less frequently to 3- to 5-year-olds compared to 5- to 6-year-olds ($p = 0.014$) and 6- to 7-year-olds ($p < 0.001$). No difference was found between 5- to 6-year-olds and 6- to 7-year-olds ($p = 0.628$). The children's age group had the largest unique effect, explaining 13.7% of the variance in the frequency of teaching NS not explained by other factors in the model. Teachers' pedagogical awareness of teaching NS had a moderate unique effect (8.7%), whereas effect sizes were small for PD in mathematics (3.9%) and teachers' age (2.4%) (see Fig. 4).

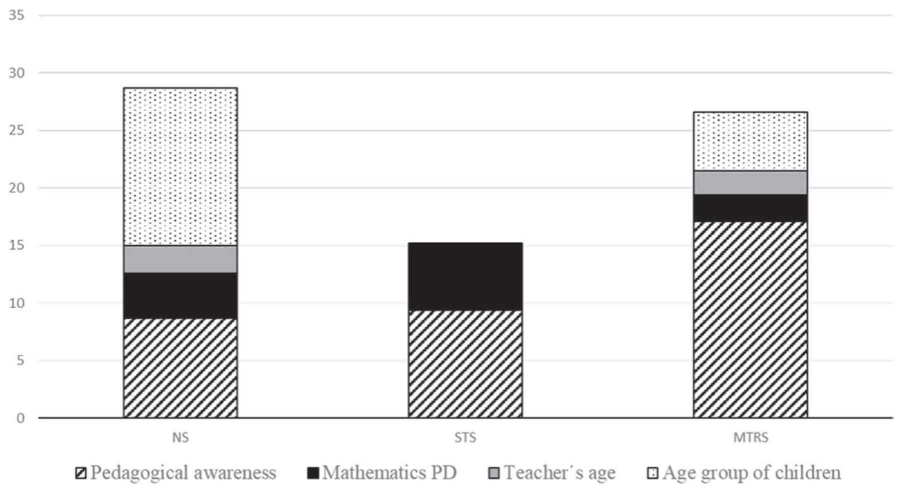

Fig. 4 Portions of variance explained by different factors in numerical skills (NS), spatial thinking skills (STS) and mathematical thinking and reasoning skills (MTRS)

The univariate analysis for STS showed that pedagogical awareness of teaching STS ($F(1, 199) = 47.81$, $p < 0.001$, $\eta_p^2 = 0.094$) and duration of mathematics PD programmes ($F(1, 199) = 12.20$, $p = 0.001$, $\eta_p^2 = 0.058$) were significantly associated with teaching STS. Again, the higher the pedagogical awareness of teaching STS, and the more PD in mathematics, the more often STS were taught. Pedagogical awareness of teaching STS explained 9.4% of the variance in the frequency of teaching STS, and its effect size was larger than that of the duration of mathematics PD programmes (5.8%). Unlike in NS, neither teachers' age nor children's age group had any significance for the frequency of teaching STS (see Fig. 4).

Finally, the univariate analysis of variance for MTRS revealed that pedagogical awareness of teaching MTRS ($F(1, 197) = 40.62$, $p < 0.001$, $\eta_p^2 = 0.171$), duration of mathematics PD programmes ($F(1, 197) = 4.68$, $p = 0.032$, $\eta_p^2 = 0.023$), teachers' age ($F(1, 197) = 4.19$, $p = 0.042$, $\eta_p^2 = 0.021$) and children's age group ($F(2, 197) = 5.25$, $p = 0.006$, $\eta_p^2 = 0.051$) were significantly associated with the frequency of teaching MTRS. The stronger the pedagogical awareness of teaching MTRS, the more PD programmes in mathematics; and the higher the teacher's age, the more frequently MTRS were taught. Concerning children's age group, post hoc pairwise comparisons, using Bonferroni correction for the significance, showed that teaching MTRS was less frequent among 3- to 5-year-olds compared to 6- to 7-year-olds ($p = .008$). No difference was found between 5- to 6-year-olds and 6- to 7-year-olds ($p = 1.000$) or between 3- to 5-year-olds and 5- to 6-year-olds ($p = 0.131$). The effect size of pedagogical awareness of teaching MTRS was large, explaining 17.1% of the variance in the frequency of teaching MTRS not explained by other factors. Children's age group uniquely explained 5.1% of the variance in the frequency of teaching MTRS, whereas the effect sizes were small for the duration of mathematics PD programmes (2.3%) and teachers' age (2.1%) (see Fig. 4).

Discussion

This study investigated early childhood mathematics education by exploring differences in the frequency of teaching NS, STS and MTRS, and in the pedagogical awareness of teaching these skills to 3- to 7-year-old children. The associations between teacher-related characteristics and children's age and the teaching frequency of NS, STS and MTRS were also examined. The study showed that the frequency of teaching early mathematical skills and the pedagogical awareness of teaching these skills varied from one skill category to another. Furthermore, the age group of the children moderated not only the frequency at which NS and MTRS are taught but also the differences between skill categories. NS were taught more frequently to children in older age groups than to 3-to-5-year-olds, and more often than STS and MTRS. However, for 3-to-5-year-olds, NS and STS were taught equally seldom but were taught more often than MTRS, which were taught less to children in this age group than to 6-to-7-year-olds. Teachers' pedagogical awareness was lowest in teaching STS regardless of the children's age group. In contrast, only among 6- to 7-year-olds was teachers' pedagogical awareness higher in teaching NS compared to MTRS and STS. Pedagogical awareness, overall, had a significant effect on teaching frequency in all early mathematical skill categories. That said, according to effect sizes, the strength of this association varied depending on the category, from modest (NS and STS) to high (MTRS). Besides, the duration of PD in mathematics had a rather small but systematic and positive influence on the frequency of teaching NS, STS and MTRS. Teachers' age (work experience) had a small but significant effect on the frequency of teaching NS and MTRS, but not on that of STS.

Studies have shown that not all NS develop at the same time — for instance, understanding the interrelationships between number word, number symbol and quantity develops after the age of 3, whereas understanding the relationship between addition and subtraction usually develops during pre-primary education (Aunio & Räsänen, 2016; Baroody, 2011; Kullberg et al., 2020). Hence, it is unsurprising that teachers teach NS less often to 3- to 5-year-olds compared to older age groups and that children's age group had the largest unique effect in explaining variations in teaching frequency. Besides, the effect size in the teaching frequency of NS and MTRS increased according to age differences among the groups; the effect size was moderate between 3- to 5-year-olds and 5- to 6-year-olds, and large between 3- to 5-year-olds and 6- to 7-year-olds. These findings indicate that teachers emphasise teaching NS as a transition to primary education.

Furthermore, possibly due to the central role of NS in mathematics education in the teacher education (Clements & Sarama, 2011; Simpson & Linder, 2014), teachers in ECEC seem to have knowledge of which specific NS are suitable for children in pre-primary education. This observation is supported by our finding that, only among 6- to 7-year-olds, teachers' pedagogical awareness was higher in NS compared to STS and MTRS. Such interplay between the duration of PD in mathematics, pedagogical awareness of NS and the age-group of children might also explain variation in the frequency of teaching NS to 3- to 7-year-old children. At the same time, however, the moderate association between pedagogical awareness and frequency of teaching NS also demonstrates that some teachers' pedagogical awareness of NS is low and that they do not teach NS very often. Young and inexperienced teachers seem to teach NS less often, as both of these factors were related to teaching frequency and pedagogical awareness of NS. More research is needed to understand how 3- to 7-year-olds are taught numbers and quantity knowledge, counting skills and basic arithmetic skills.

The current study both supported and expanded knowledge concerning teaching STS, as previous studies have shown that STS is not prominent in early childhood mathematics education (Clements & Sarama, 2011; Simpson & Linder, 2014). Overall, the teaching frequency of STS did not vary according to the children's age group. However, we showed that the frequency of teaching STS compared to NS depended on the children's age group. In line with earlier studies, we showed that STS and NS were as frequently taught to 3- to 5-year-olds. In contrast, among 5- to 6-year-olds and 6- to 7-year-olds, STS were less frequently taught than NS, and the effect sizes representing the difference between the two skill categories were moderate and large, respectively. Furthermore, pedagogical awareness of teaching STS was evaluated as being lowest by teachers in all age groups, but it explained a larger portion of the variance in the frequency of teaching STS than the duration of PD in mathematics.

Somewhat surprisingly, the age group of the children was not associated with the teaching frequency for STS. However, children acquire a more complex understanding of time (Lyytinen, 2014), spatial relations and shapes (Clements, 2011; Hawes et al., 2017) and measurement, and mass and volume (Baroody, 2011; Clements & Stephan, 2011) between the ages of three and seven. We expected this to impact the association between the age group of the children and the frequency at which STS are taught, irrespective of the broad learning objectives set in the Finnish curricula. Thus, it appears that the teachers were not fully aware of these developmental changes in children in relation to STS. This might also be significant with regard to not finding a significant effect on the age group. That the teachers' pedagogical awareness of STS was the lowest and they taught STS less frequently than NS to 5- to 6-year-olds and 6- to 7-year-olds supports this observation. Moreover, such low pedagogical awareness might translate to unawareness of different teaching practices that could be used with older children while teaching STS. We already know that measurement is less frequently taught than other content areas to 3- to 5-year-olds (Hindman, 2013; Sarama & DiBiase, 2004), and regular calendar-related activities and discussions about seasons and daily activities take place routinely in this age group (see Gonzales & Paik, 2011; Hindman, 2013). Examining the frequency of teaching specific STS skills linked with understanding the development of mathematical skills is essential to strengthening pedagogical awareness of teaching STS in different age groups and developing mathematics education in teacher training programmes. Thus, training could be used to strengthen awareness of age-appropriate STS content (see Callejo et al., 2022).

As in the teaching frequency of NS, that of MTRS was significantly influenced by the children's age group. The study revealed, first, that MTRS were taught less frequently to 3- to 5-year-olds than to 6- to 7-year-olds. Furthermore, children's age group moderated differences in teaching frequency between different skills. To 3- to 5-year-olds, MTRS were taught less frequently than NS and STS, whereas to 5- to 6-year-old and 6- to 7-year-old children, MTRS were taught just as often as STS. Besides the children's age group, pedagogical awareness of teaching MTRS, duration of PD in mathematics and teachers' age (and work experience) influenced the teaching frequency of MTRS. Furthermore, the effect of pedagogical awareness of teaching MTRS on the teaching frequency of MTRS was double the size compared to NS and STS. The strong association between pedagogical awareness and the frequency of teaching MTRS demonstrates large and systematic variations in teaching MTRS to 3- to 7-year-olds in relation to pedagogical awareness.

The results indicate that teachers in ECEC understand that MTRS develop gradually between age groups (see Mulligan & Mitchelmore, 2013) along with cognitive and language development (Keisar & Peled, 2018; Worthington et al., 2019), and, therefore, teaching certain MTRS, which require sophisticated cognitive thinking, is not yet reasonable with the youngest children. Recent studies that have investigated the development of MTRS have clearly shown that 4- to 5-year-olds become aware of structuring and reasoning processes (Vandlyndt et al., 2021; Warren et al., 2016), 5- to 6-year-olds learn sophisticated mathematical thinking (Alsina & Salgado, 2021), and 6-year-olds are capable of learning functional relationships and data modelling (Keisar & Peled, 2018; Mulligan, 2015). As recent studies have enhanced understanding of the development of MTRS, and MTRS are linked to learning NS and STS, it is essential to explore how frequently specific MTRS are taught to 3- to-7-year-olds in order to promote well-balanced development and learning of mathematical skills in early childhood.

Limitations

Despite learning much about variations in the teaching of early mathematical skills to 3- to 7-year-olds, the present study had some limitations, which must be considered when interpreting the results. First, by guaranteeing respondent anonymity, we excluded information about why some teachers declined to answer the survey. Collecting personal information (names and emails) would have allowed us to remind these teachers to complete the survey, which most likely would have improved the response rate. Despite this limitation, the sample was representative of Finnish ECEC teachers as it did not reveal any differences in location (area of Finland or the size of the municipality). Second, the employed measures had high reliabilities, excluding pedagogical awareness of teaching MTRS (Cronbach's alpha = 0.67), which did not meet the preferred ≥ 0.70. However, despite the reduced reliability, the association between pedagogical awareness and teaching frequency of MTRS was ultimately high. Third, as the study was cross-sectional, it limited us from drawing causal conclusions. Fourth, the role of a particular theoretical approach and the *holistic model of early mathematical skills development* (Fig. 1), framing this research, should be acknowledged (see Parviainen, 2019). This approach, albeit holistic in nature, built on certain assumptions concerning the three skill categories that guided the research design and, ultimately, the survey questions. Although it was beyond the scope of the study to critically reflect on the assumptions proposed in the model, the results, measured by teaching frequencies, supported the pace of development of different skills and the relationships between the skill categories. Fifth, as the study sought to obtain a comprehensive view of teaching early mathematical skills — namely NS, STS and MTRS — there remains a need to further explore variations in the frequency of teaching specific NS, STS and MTRS to 3- to 7-year-old children.

Conclusions

The current study showed that the frequency and pedagogical awareness of teaching early mathematical skills to 3- to 7-year-old children depend on the skill category (NS, STS and MTRS) of mathematics and that children's age group moderate differences between the skill categories. These findings suggest that teachers are capable, to some extent, of effectively considering the children's age and readiness when planning teaching practices related to mathematic skills, as MTRS were taught less frequently to younger children and NS more frequently to older children. In addition, the study revealed that those 3- to 7-year-olds whose ECEC teachers were pedagogically aware of teaching NS, STS and MTRS and had undergone PD in mathematics had opportunities to practise and learn early mathematical skills more frequently than other children. These findings, however, indicate room for further development of pre- and in-service education of ECEC teachers in mathematics education, as pedagogical awareness of teaching STS was low, and teachers seemed to be unable to consider children's age when determining the frequency of teaching STS.

As previous studies have shown, pedagogical awareness and mathematics PD programmes increase the quality of ECEC and pre-primary mathematics education (Bruns et al., 2017; Dunekacke et al., 2015; Gasteiger & Benz, 2018; Lindmeier et al., 2020). Thus, paying attention to the contents of mathematics education during both initial and in-service teacher training could potentially increase awareness of teaching different early mathematical skills to children of different ages (see Callejo et al., 2022; Muños-Catalán et al., 2022). Yet, more research on the frequency of teaching different NS, STS and MTRS to 3- to 7-year-old children might provide deeper insights into possible variations in teaching these skills to different age groups. Such knowledge would benefit the development of early childhood mathematics education and pre- and in-service teacher education programmes while considering the relationships between the development of skill categories in order to promote the holistic development of early mathematical skills within different age groups and age-appropriate teaching practices.

Author Contribution Piia Parviainen processed the original idea. The experiment was designed and the survey was performed by Piia Parviainen with contributions from Kenneth Eklund, Merja Koivula, Tarja Liinamaa and Niina Rutanen. Piia Parviainen and Kenneth Eklund analysed the data and reported the results. Piia Parviainen took the lead in writing the manuscript, with contributions from Kenneth Eklund and Merja Koivula. All authors discussed the results, commented on previous versions of the manuscript and approved the final manuscript.

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Data Availability No extra material is included in the article.

Declarations

Ethics Approval According to the local guidelines, ethical review applies only to precisely defined research configurations. As our project had no such configurations, ethical review was not required. However, research notification, privacy notice and consent to participate were followed as mandated by the Human Sciences Ethics Committee of the university.

Consent to Participate Participation in this research was voluntary. Consent to participate and permission to use participant data were given by every participant. No participant withdrew from the research.

Consent for Publication This work can be published by the *International Journal of Science and Mathematics Education*.

Conflict of Interest The authors declare no competing interests.

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III

ENHANCING TEACHERS' PEDAGOGICAL AWARENESS OF TEACHING EARLY MATHEMATICAL SKILLS – A MIXED METHODS STUDY OF A TAILORED PROFESSIONAL DEVELOPMENT PROGRAM

by

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Enhancing Teachers' Pedagogical Awareness of Teaching Early Mathematical Skills – A Mixed Methods Study of Tailored Professional Development Program

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ABSTRACT

The purpose of this mixed methods study was to explore changes in pedagogical awareness of teaching early mathematical skills among teachers in early childhood education (N =7) when participating in a tailored professional development (PD) program in mathematics. The program, which was designed around principles of transformative learning, was aimed at strengthening the conscious and holistic teaching of early mathematical skills to 3- to 7-year-old children. *Research Findings:* Thematic analysis of semi-structured interviews revealed that teachers enhanced their pedagogical awareness of teaching early mathematical skills concerning developmentally appropriate mathematical content, child-initiated mathematical learning and holistic mathematical teaching in different daily situations and play. Data obtained from pre-PD and follow-up questionnaires completed by teachers confirmed a sustainable increase in their pedagogical awareness of numerical and spatial reasoning. Recognizing children's interests, reflecting on and examining one's own practices individually and collaboratively, and taking actions to develop teaching and learning practices aligned with those of the PD program comprised the foundation for the transformative process. *Practice or Policy:* It was concluded that PD programs in mathematics enhance teachers' pedagogical awareness of teaching early mathematical skills holistically when they are tailored to the needs of teachers, include reflective elements, and follow principles of transformative learning.

Introduction

Successful early childhood mathematics education requires conscious teaching (Clements et al.; 2011, Moss et al., 2016). Such teaching entails the holistic development of children's mathematical skills through versatile, age-appropriate learning experiences in different daily situations (e.g. routine events, planned activities and discussions) and play, together with other people in early childhood education and care (ECEC) and pre-primary education. It has been acknowledged that teachers' underlying pedagogical orientation and practices are critical in supporting the development of children's mathematical skills (Björklund et al., 2018, Brandt, 2013; Salomonsen, 2020). Teachers' pedagogical awareness has been shown to influence children's opportunities to explore mathematical phenomena and learn mathematical skills (Parviainen et al., 2023; Trawick-Smith et al., 2016; Vogt et al., 2018). Through conscious practices, teachers can promote the learning of versatile mathematical skills, such as numerical learning (Laski & Siegler, 2014; McNeil et al., 2015), spatial reasoning (Hawes et al., 2017; Jones & Tzekaki, 2016), and mathematical thinking and reasoning (Clements et al., 2019; Lee et al., 2016).

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Mathematics professional development (PD) (Clements et al., 2011; Hadley et al., 2015) and PD programs play an important role in promoting successful early childhood mathematics education (Parviainen et al., 2023; Simpson & Linder, 2014). PD programs can broaden teachers' mathematical knowledge (Gasteiger & Benz, 2018), increase the quality of teaching (Bruns et al., 2017; Simpson & Linder, 2014) and improve children's mathematical learning outcomes (Knaus, 2017). Nevertheless, studies are needed that will explore PD programs in early childhood mathematics from different perspectives (Bruns et al., 2017). Recent studies have called for investigations of PD programs that focus on early mathematics teachers' pedagogical awareness and actions (Gasteiger & Benz, 2018; Lindmeier et al., 2020). Research is also needed to examine tailored PD programs, which are designed to meet the needs of teachers (Barber et al., 2014; Knaus, 2017).

To respond to these research calls, the present mixed methods study explored how a tailored PD program in early childhood mathematics changed teachers' pedagogical awareness of teaching early mathematical skills to 3- to 7-year-old children. The aim of this program was to enhance pedagogical awareness of teaching early mathematical skills holistically, in any situation or at any teachable moment in play or daily life (i.e. discussions, routine events, preplanned activities). The program was based on the self-identified needs of participating teachers. We utilized Parviainen's (2019) *holistic model of the development of early mathematical skills* (Appendix A) as the theoretical basis for evaluating the program because prior research has underscored the necessity of applying current research-based understandings of mathematical skills development to early childhood mathematics education. Doing so, it has been argued, will enable powerful mathematical learning in different daily situations and during play (Clements et al., 2011; Moss et al., 2016). We also incorporated transformative learning principles into the design of the PD program because they can prompt sustainable changes in teaching (cf. Cranton, 2016; Mezirow 1997).

Pedagogical Awareness of Teaching Early Mathematical Skills

Several practice-based models have been introduced to explain the knowledge, competences and actions needed by teachers to implement high-quality mathematics education (Ball et al., 2008; Carrillo-Yañez et al., 2018; Lindmeier, 2011). Gasteiger and Benz (2018) and Lindmeier et al. (2020) developed specific models to apply in early childhood mathematics education. Both models have been applied to mathematics PD and have been found to enhance teaching practices as well as the analysis of different aspects of teaching. These models view proper mathematics instruction as being contingent not only on curricular mathematical content but also on the extent to which teachers understand the needs and interests of learners and promote mathematical learning through intentional, pedagogically appropriate teaching. Based on these models, in this study, the knowledge, competences and actions needed by teachers to effectively implement mathematics education are conceptualized as pedagogical awareness. Pedagogical awareness includes the following three dimensions: (1) content and skills development, (2) appropriate teaching and learning practices, and (3) reflection and evaluation.

Pedagogical Awareness Concerning Early Mathematical Content and Skills Development

Teachers' knowledge of early mathematical content and the associated curriculum as well as their understanding of current theory regarding the development of early mathematical skills translate to teaching practices, thus serving as a premise for successful early childhood mathematics education (Clements et al., 2011; Gasteiger & Benz, 2018; Lindmeier et al., 2020; Moss et al., 2016). Recent studies of mathematical skills development and learning have changed our understanding of children, who are now seen as capable of sophisticated reasoning and mathematical-logical thinking before the age of 7 (e.g. Clements et al., 2019; Hawes et al., 2017). It is currently understood that some mathematical skills (e.g. number sense and spatial sense) are innate and develop gradually from birth, while other skills (i.e. understanding spatial relationships, conservation and time, and the capacity for mathematical reasoning) progress along with cognitive, language and motor development in interaction with the environment (van Oers, 2013; Parviainen, 2019; Sarama & Clements, 2009). In addition, the realization that development of mathematical skills requires connections between different skills (i.e. numerical skills are needed in spatial learning and vice versa) has enhanced our understanding of the holistic development of early mathematical skills (Parviainen, 2019).

Studies have revealed that teachers' pedagogical awareness of different mathematical content is not comprehensive (Björklund & Barendregt, 2016; Parviainen et al., 2023) – insufficient instruction in spatial thinking, for example, has been attributed to lack of awareness (Björklund & Barendregt, 2016). Limited awareness of age-appropriate mathematical content has also been shown to influence the frequency with which different mathematical skills are taught (Parviainen et al., 2023). Nevertheless, research has also revealed that the more comfortable teachers are in teaching mathematical content, the more optimistic their expectations are regarding children's mathematical knowledge and learning (Çelik, 2017; Ertle et al., 2008).

Pedagogical Awareness Concerning Appropriate Teaching and Learning Practices

Teaching early mathematical skills requires the ability to (1) plan and implement developmentally appropriate mathematical activities, (2) recognize and capture age-appropriate mathematical affordances in play and daily situations, and (3) respond to spontaneously emerging learning moments (cf. Gasteiger & Benz, 2018; Lindmeier et al., 2020). It is important for teachers to consider children's interests and initiatives as premises for mathematical learning and to respond to them with sufficient learning activities, object exploration, and mathematics-related interactions in various learning environments (Björklund et al., 2018; Brandt, 2013; Moss et al., 2016; Salomonsen, 2020). The capacity to respond to children's initiatives, needs and interests, however, requires pedagogically oriented and sensitive teacher – child interaction, as well as collaborative practices. These include, for instance, mathematical discussions with children and teachers' use of conceptual mathematical language in different daily situations and play. Such discussions are essential because joint problem-solving allows teachers to encourage children to explore mathematical phenomena, in turn expanding their awareness (Björklund et al., 2018; Brandt, 2013). It should also be noted that collaborative experiences help children develop their mathematical thinking (van Oers, 2013).

Concrete experiences and meaningful activities promote children's mathematical learning in versatile ways. For instance, measuring concrete objects can enhance understanding of spatial concepts (Cheeseman et al., 2014), and the connection between length and number (Sarama & Clements, 2009) as well as between mass and number (Cheeseman et al., 2014). Because concrete materials strengthen the capacity for mathematical reasoning and the comprehension of mathematical concepts (Lee et al., 2016), in addition to positively influencing mathematical learning outcomes, it is crucial that teachers introduce these materials into different situations and play (Trawick-Smith et al., 2016; Vogt et al., 2018). It is also vital that appropriate practices, such as teacher – child interaction, are consciously employed in these situations because they can also improve mathematical learning outcomes. Overall, by acknowledging children's participation, teachers can better consider meaningful and equitable mathematical learning opportunities within their child groups (Helenius, 2018; Polly et al., 2017).

Pedagogical Awareness Concerning Reflection and Evaluation

The implementation of early childhood mathematics education through age-appropriate content and practices, which promote learning and skills development, entails reflection and evaluation by teachers (cf. Gasteiger & Benz, 2018; Lindmeier et al., 2020). In practice, constant observation and assessment of children's mathematical learning and development are required to appropriately support children's learning in daily situations and play. Observation and assessment are also needed to avoid teaching based on teachers' assumptions about children's learning and skills development (cf. Lee & Ginsburg;

2009). Additionally, teachers must reflect on and evaluate, for instance, the appropriateness of implemented practices and their own attitudes, beliefs and motivations (Gasteiger & Benz, 2018). This is because teachers' awareness of mathematical content (Dunekacke et al., 2015; Polly et al., 2017), confidence in teaching such content (Çelik, 2017; Ertle et al., 2008), and pedagogically informed practices explicitly influence mathematics teaching and thus children's mathematical learning (Parviainen et al., 2023; Trawick-Smith et al., 2016; Vogt et al., 2018). These are also important considerations in PD (Chen et al., 2014) because critically approaching one's own professionalism as well as being cognizant of one's own thinking structures and practices through self-reflection lay the foundation for making real changes in teaching (Mezirow, 1991, 1997).

Professional Development (PD) and PD Programs in Early Childhood Mathematics Education

Mezirow's (1991, 1997) transformative learning theory is widely used in the PD of teachers. Recently, it has been further developed to account for one's own actions and cognition and the group's role in PD (cf., e.g. Cranton, 2016). Effective PD cultivates habits of mind about teaching, allowing teachers to critically examine their practices, question their thought structures and pursue alternative means to understand teaching (Cranton, 2016; Cranton & King, 2003) via self-reflective learning cycles (Cranton, 2016; Mezirow, 1991). Teachers themselves have underscored the importance of reflective and collaborative practices as well as the cyclic nature of learning in the PD of early childhood mathematics (Barber et al., 2014; Hadley et al., 2015). Nevertheless, to foster sustainable changes in teaching, it is essential that teachers commit to completing the PD process, which can be supported by participatory methods (Cranton, 2016) – for instance, by involving teachers in decision making at different stages of the PD program, from planning to evaluation to the development of practices.

The principles of transformative learning in the design of the mathematics PD program have helped teachers enhance their mathematical content knowledge and develop their pedagogical practices, generating more engaged mathematical learning and improved developmental outcomes among children (Knaus, 2017). However, in some programs, improvements have been limited to action-related teaching, with no improvements in self-reflection (Knaus, 2017; Lindmeier et al., 2020). It has been suggested that PD programs should pursue a more strategic approach, one which accounts for and is consequently tailored to the self-reported learning needs of participants (Barber et al., 2014). Additionally, concepts derived from action research (Knaus, 2017) and commitment to self-reflection (Chen et al., 2014; Knaus, 2017) should be incorporated. Such adjustments to PD programs could improve self-reflection, which is critical to the implementation of early childhood mathematics education.

Present Study

This mixed methods study focused on changes in ECEC teachers' pedagogical awareness of teaching early mathematical skills to 3- to 7-year-old children during a tailored PD program in mathematics. The program was designed to familiarize teachers with a holistic approach to teaching mathematical skills in different daily situations and play in ECEC and pre-primary education. It was tailored according to the needs of the teachers and incorporated principles of transformative learning. In response to calls for research on teachers' pedagogical awareness and actions in PD programs (Gasteiger & Benz, 2018; Lindmeier et al., 2020), on the ways in which PD programs are tailored (Barber et al., 2014; Knaus, 2017), and on how teachers change their thinking during PD programs (Bruns et al., 2017), we addressed the following research questions: "How do ECEC teachers describe changes in their pedagogical awareness of teaching early mathematical skills after participating in a tailored PD program in mathematics?" and "Do the pre-PD and follow-up-questionnaires completed by the teachers validate possible increases in teachers" pedagogical awareness of teaching early mathematical skills (i.e. numerical skills, spatial thinking skills and mathematical thinking and reasoning skills) and reveal long-lasting changes nine months after the end of the PD program?'

Research Design

This mixed methods study focused on a tailored PD program in mathematics for ECEC teachers in Finland. The first author conducted the program together with a colleague. The program lasted for one academic year (Appendix B) and was part of the national *LUMA2020 Development Program* (funded by the Ministry of Education and Culture, Finland), established through LUMA Center Finland. The *LUMA2020 Development Program* aimed at supporting the development of STEAM (Science, Technology, Engineering, Arts and Mathematics) teaching and project-based learning from ECEC to secondary school in Finland (Nurmi et al., 2021).

Participants and Ethical Considerations

The study participants were teachers of 3- to 7-year-old children ($N = 7$) who had taken part in the *LUMA2020 Development Program*. The teachers represented five early education centers. Three teachers worked with 3- to 5-year-old children, and four with 5- to 7-year-old children. Four teachers had more than 20 years of experience, two had more than 10 years of experience, and one had fewer than five years of experience. None of the teachers had previously participated in a mathematics PD program.

Ethical guidelines on good scientific practice, including careful and confidential data processing, storage, and analysis, were meticulously followed throughout the study (Byrne, 2016). In addition, the participants were informed that their involvement in the study was completely voluntary. Research notifications, privacy notices and the consents to participate were documented based on the requirements and instructions of the Human Sciences Ethics Committee of the University of Jyväskylä.

Tailored PD Program in Mathematics as a Platform for PD

The mathematics PD program was tailored to meet the needs of the participants concerning the development of early childhood mathematics education. ECEC teachers self-identified their needs and personal learning aims for PD in the beginning of the program through an individual open-format questionnaire and collaborative discussion. The goals and needs outlined by most participants involved gaining theoretical knowledge of early childhood mathematics education, recognizing opportunities to teach mathematics in different daily situations, and acquiring new teaching ideas and materials. Based on these, an overarching plan for the PD program was formulated to enhance pedagogical awareness of teaching early mathematical skills holistically in different situations in daily life and play in ECEC and pre-primary education.

The PD program included meetings designed by two trainers according to the teachers' needs and learning aims (Appendix B). In the meetings, the trainers facilitated the teachers' PD through collegial learning, including collective discussions about teaching experiences and hands-on activities for teaching mathematics. *The holistic model of the development of early mathematical skills* (Appendix A) served as the basis for collegial brainstorming around topics such as how mathematical content could be taught to children of different ages.

As part of the *LUMA2020 Development Program*, the teachers conducted project-based mathematical learning in their child groups. The teachers had planned and implemented these projects together with the children, based on the children's needs and interests. Therefore, in the PD program, the trainers conducted specified trainings separately in each center to support these endeavors. To bolster mathematical teaching and provide new teaching ideas, instructional packages were given to the teachers, which the teachers could utilize upon their choice. Each package included concrete learning materials and teaching ideas for preplanned activities, transition situations and play.

In addition to material support and meetings, strong emphasis was placed on the implementation of early childhood mathematics education in child groups, and on the reflection and evaluation of one's own teaching (see Appendix A). The teachers had the freedom to plan and implement mathematical teaching according to the needs of their own child group. Commitment to personal learning objectives and the development of practices were supported by a personal reflective journal, which enabled the teachers to reflect on, evaluate and develop their mathematical teaching from different perspectives (e.g. teaching early mathematical skills in different daily situations and play, noting how children practiced skills, estimating the intentionality of teaching, and planning further steps) (N.B. reflective journals were not used as data in this study).

Characteristics of participatory action research were utilized to some extent (McIntyre, 2008) because of their philosophical similarities to transformative learning in PD. Self-reflective cycles and collaborative learning, which are common in participatory action research (McIntyre, 2008) and transformative learning (Cranton, 2016; Mezirow, 1991), were employed in the program, while the aim of the study design was to understand how teachers constructed and attached meanings to their pedagogical awareness of teaching early mathematical skills.

Collection and Analysis of Data

Semi-structured interviews were conducted with each participant to more fully comprehend changes that occurred in their pedagogical awareness of teaching early mathematical skills during the tailored PD program. The interviews were carried out by the first author after the PD program had ended. The questions asked in each interview focused on three areas: (1) elements of the PD program (i.e. "How did the *LUMA2020 Development Program* promote your pedagogical awareness of early mathematical teaching?" and "During the program, what prevented or slowed down your PD on early mathematical teaching?"), (2) pedagogical awareness in teaching early mathematical skills (i.e. "What was your weakest area in the teaching of early mathematical skills at the beginning of the program, and what happened to it during the program?") and (3) reflection (i.e. "How would you reflect your professional development during *LUMA2020 Development Program*?"). The teachers were in possession of the printed training materials (theoretical package and personal reflective journals) during the interview, which they could use to reflect on their answers. All interviews were video-recorded (ranging from 75 to 90 minutes) and later transcribed, yielding 112 pages (font 10, spacing 1) of transcribed text for thematic analysis (Braun & Clarke, 2022).

In the first step of the analysis, the first author became immersed in the data through several readings, and generated initial codes. Through inductive and explorative orientations initial themes were collated. Thereafter, deductive orientation, informed by research literature was applied, as the codes were arranged according to main and sub-themes, which were related to pedagogical awareness and early mathematical skills. The themes and sub-themes were then reviewed to determine whether they worked in relation to the codes and the entire dataset. At this point, the researcher rearranged the coded data extracts to generate in-depth themes using inductive, explorative and critical orientations (Braun & Clarke, 2022). This process yielded themes concerning mathematical skills and content, children's perspectives on mathematical learning, and teachers' perspectives on mathematics teaching. These were reexamined by employing deductive orientation and critical approach to arrive at a clearer understanding of the changes that occurred in the teachers' pedagogical awareness of teaching early mathematical skills during the PD program, and to more precisely comprehend the three main themes and sub-themes (Figure 1). Moreover, the researcher alternated between the dataset and the literature throughout the analytical process to more deeply refine the analytic work (Braun & Clarke, 2022). This generated a categorization of themes through which changes in pedagogical awareness of teaching early mathematical skills during the tailored PD program could be discussed (see the Results section).

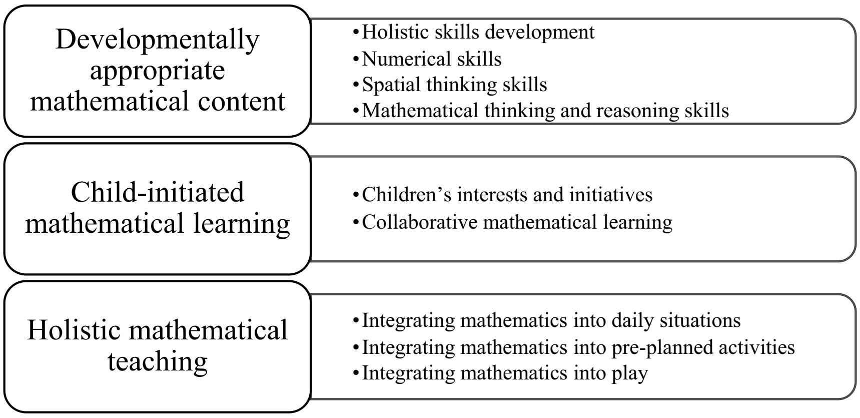

Figure 1. Main themes and sub-themes from the thematic analysis.

To minimize the incidence of subjective bias when analyzing the data, the coded material in each analytical phase was distributed to the whole research team, and the interpretations were carefully and jointly scrutinized. Thus, the analytical phases were critically discussed with the other members of the research team, who did not take part in the planning or implementation of the PD program. This member-checking technique permitted the in-depth scrutiny of the analytical results and increased their trustworthiness (cf. Newby, 2014). It also gave high transparency to the analytical process. To further confirm and strengthen the trustworthiness of analysis results, inter-rater reliability was calculated from the coding of two interviews according to the sub-themes, and 93% agreement was reached.

In addition to the semi-structured interviews, the completed copies of the "Teaching Early Mathematical Skills" questionnaire (Parviainen et al., 2023) were collected at the beginning of the tailored PD program and nine months after the program. The questionnaire was used to assess teachers' pedagogical awareness of teaching early mathematical skills (i.e. numerical skills, spatial thinking skills, and mathematical thinking and reasoning skills), with the intention to validate possible qualitative findings and explore the retention of possible increases. A One-tailed Wilcoxon Signed Ranks Test was used separately for the scale scores of numerical skills, spatial thinking skills and mathematical thinking and reasoning skills to test whether the teachers' pedagogical awareness increased during the PD program and whether the changes remained nine months after the end of the PD program.

Results

Changes in pedagogical awareness of teaching early mathematical skills are described here in terms of the ways in which ECEC teachers enhanced their awareness within the three main themes: (1) from limited to developmentally appropriate mathematical content awareness, (2) from adult-initiated practices to child-initiated mathematical learning, and (3) from preplanned mathematical activities to holistic mathematical teaching. After the presentation of the qualitative findings, results related to the pre-PD and follow-up-questionnaires on the teachers' pedagogical awareness of teaching numerical skills, spatial thinking skills and mathematical thinking and reasoning skills are presented.

Developmentally Appropriate Mathematical Content

All seven interviewed ECEC teachers claimed that the theoretical orientation of the PD program concretized and enhanced their understanding of early mathematical content and skills development, thus helping them to turn their focus toward developmentally appropriate early childhood mathematics education. The teachers reported that they were not aware of the interconnectedness of early mathematical skills areas (cf. Parviainen, 2019) before they took the PD program, but all of them agreed that they had become aware of it and of its importance learning during the program.

Holistic Skills Development

All teachers explained that *the holistic model of the development of early mathematical skills* (see Appendix A) clarified the content and developmental areas included in mathematical learning in ECEC and pre-primary education, as the following examples illustrate:

> Training material of early mathematical skills: I´ve been able to return to it and check how it is with 3- to 4-year-olds. It has increased my confidence about what to do with these children. (T1)

> I wasn´t aware of all the things included in mathematics, but the PD program outlined most of what it includes and what can be done with 5- to 7-year-old children. (T4)

As the examples show, the teachers were not well versed in early mathematical content and skills development, but the training material served as a tool for the age-appropriate instruction of this content and skills. The teachers explained that the model helped them structure their teaching: *"Now that I have this model, I´m able to structure my teaching better" (T6)*. It also helped them to think mathematically: *"I understood that this is what it [certain mathematical content] means and learnt to think and recognize things that are mathematics" (T2)*. The teachers' statements indicate that by bringing different skills areas to their attention through *the holistic model of the development of early mathematical skills* (see Appendix A), the PD program helped them understand mathematical content and skills development more deeply than before.

Every teacher believed that by elaborating and examining their teaching in relation to skills areas, they came to realize that their awareness of teaching numerical skills was broad, but it was more limited regarding spatial thinking skills and mathematical thinking and reasoning skills. Furthermore, they recognized areas they had – and had not – taught as consciously before the program. The teachers explained that by understanding the importance of teaching versatile mathematical skills, they had learnt to teach these skills better:

> Earlier, I stressed numerical skills and paid less attention to spatial thinking skills and mathematical thinking and reasoning skills, and what even belongs to them. Because all these skills where concretely collected there [speaks about the model and the reflective journal], it opened the perspective for pedagogical planning. (T3)

> It is clear to me now that spatial thinking skills and numerical skills are not separate areas but rather connected. This has enhanced my thinking a lot. I now consider more consciously that it is not only about counting but also spatial relations, which I taught less before. (T5)

These examples illustrate how teachers gained a better balance between the skills areas in their content teaching as they began using their new awareness of pedagogical planning and implementation more consciously than before. By examining their practices reflectively in relation to the theory of mathematical skills development, the teachers recognized the importance of paying attention to certain skills areas depending on the children's age.

Yet, six of the teachers also admitted that by critically examining and reflecting on their practices in relation to *the holistic model of the development of early mathematical skills* (see Appendix A), they had recognized their limitations concerning the teaching of different mathematical content after the program had finished:

> Teaching numerical skills for children, that is a big part of my teaching. I wonder if my teaching is limited as I cannot separate other areas, although I teach skills areas a lot. (T2)

> This division of mathematical areas was really good, this holistic mathematical skills development. It was good because I recognized limitations I still have. (T7)

These examples indicate that through the recognition of limitations, the teachers were able to identify areas for future improvement with regard to their awareness of developmentally appropriate mathematical content.

Numerical Skills

The teachers explained that with the aid of the theory of skills development, they better understood different aspects of numerical skills development and the importance of strengthening certain skills among children of a certain age. This, in turn, was reflected in their practices such that the teachers of 3- to 5-year-old children more strongly emphasized practices that strengthened number sense (e.g. interrelationships between number word, number symbol and quantity) and counting skills (e.g. enumeration) in children's learning, whereas teachers of 5- to 7-year-old children paid more attention to teaching basic skills in arithmetic (e.g. base-10 system, addition and subtraction) in addition to strengthening number sense (e.g. understanding quantity) and counting skills (e.g. number word sequencing) during the PD program. The following example illustrates how enhanced awareness of numerical learning and skills development changed a teacher's practices:

> Connecting number symbol and quantity. We hadn't practiced it as often with younger ones as we've practiced now during the program. I hadn't considered it important to connect quantity and number symbol. We've now counted more often because I thought earlier that counting should be practiced with 5- to 6-year-olds. In addition, I hadn't understood that 3- to 4-year-olds can practice adding and subtracting. (T1)

The teacher's description indicates that better understanding of numerical skills development and greater awareness of the importance of teaching certain skills at a certain age helped her orientate numerical content accordingly, which was different compared to her earlier practices. Almost every teacher also noted a sharpened recognition of children's abilities and developmental differences in numerical learning, which helped them to orientate their teaching.

Spatial Thinking Skills

All the teachers reported that their awareness of teaching spatial thinking skills had been limited when the PD program had started. Indeed, five of the teachers reported that this was their weakest area before the PD program. The teachers noted that by examining their practices in relation to the theory of skills development, they both broadened and sharpened their age-specific teaching during the program:

> In the development of language and thinking with children at this age, it is critical for them to learn basic concepts: in front, behind, above, beneath. We practiced these a lot, which was good. I realized that these are essential – therefore, I put a lot of effort into these. (T1)

This example illustrates how the teachers of 3- to 5-year-old children stressed the teaching of spatial relations. Two of these teachers also acknowledged the appropriateness of teaching the basics of length, mass and volume measurement to children in their groups. In addition, the teachers of 3- to 5-year-old children emphasized the teaching of the basics of shapes. In contrast, the teachers of 5- to 7-year-old children observed the enhancement of age-specific awareness concerning the teaching of more sophisticated shapes and figures (e.g. 2- and 3-dimensionality), as well as length, volume and mass measurement, in their child groups. Furthermore, all teachers claimed that greater awareness of teaching time yielded a broader teaching repertoire with children of different ages.

Mathematical Thinking and Reasoning Skills

Most of the teachers mentioned having limited content awareness of mathematical thinking and reasoning skills at the beginning of the PD program. These teachers explained that they had broadened their awareness and perceptions of age-appropriate content by reflecting on their teaching in relation to the theory of skills development and learning and by taking actions based on the knowledge they gained in the program. According to the teachers, enhanced awareness manifested as more conscious age-specific teaching of mathematical-logical thinking, comparison, classification and seriation in their child groups. Teachers of 3- to 5-year-old children described how they focused on strengthening reasoning skills in their child groups, whereas the teachers of 5- to 7-year-old children emphasized that they went a little further by challenging the children's capacity for problem solving and reasoning in relation to their cognitive and language development:

> I have learnt what this area includes and what can be done with these aged children, like practicing data modelling. . . . We collected statistics of weather for many weeks. (T4)

> We Have Practiced Problem Solving Systematically. (T6)

The teachers' explanations also indicate that theoretical knowledge encouraged them to expand and sharpen their practices to cover areas they had not taught as consciously before.

Child-Initiated Mathematical Learning

All seven teachers described how the theoretical orientation of the PD program increased their awareness of the importance of child-initiated mathematical learning. They explained that by accounting for children's interests and needs and by consciously and critically examining their own practices in relation to program aims, collaborative discussions and the theoretical orientation of the program, they better understood how to bring mathematics to the children's attention concretely and naturally. About this, one teacher stated:

> This PD program and examining my own practices have made me aware of the importance of speaking about and introducing mathematics to children. (T7)

Although five of the teachers mentioned that, before the PD program, their mathematical teaching had been quite adult-initiated, it can be concluded that shifting the focus from adult-initiated to child-initiated learning constituted an essential change in their pedagogical awareness of teaching early mathematical skills:

> I have taught mathematics through a teacher-oriented approach, but I have progressed in this area. In the beginning, my teaching was more teacher-oriented, but during the Spring I learnt how to connect children's interests and participation to that. For instance, the children were interested in bugs, so we took loupes and counted spider's legs or ladybug's dots. (T3)

The findings indicate that by realizing the value of the child-initiated approach, the teachers were motivated to consider and develop practices to support children's mathematical learning in ECEC and pre-primary education.

The findings also suggest that shifting the focus from an adult-initiated to a child-initiated approach made more sense for the teachers themselves: *"I now observe my surrounding with a heighted awareness of mathematics and how I can bring it to the children's attention" (T5).* Nevertheless, the enthusiasm expressed for the child-initiated approach varied between the teachers. This indicates that, although this change was essential for all seven teachers, it was more remarkable for some than for others. No one, however, expressed the need to further increase child-initiated mathematical learning in the future.

Children's Interests and Initiatives

Every teacher argued that by recognizing the children's perspectives, and by accounting for their interests and initiatives as premises for learning in different daily situations and play, they developed practices involving child-initiated mathematical learning during the PD program. The teachers explained that they considered the children's interests and listened to their ideas, after which they connected them to preplanned activities:

> The children were interested in the space, so we expanded the topic... The children were curious about the distance of a planet in light years, so the project produced mathematical questions and enriched mathematical thinking. It was easy to build the project around it. (T4)

The teachers also described how they recognized spontaneously emerging teachable moments and integrated the children's interests into mathematical learning:

> The children were interested in their heights and wanted to measure how tall they were, so we drew the contours of a human body and displayed the drawing on the door. We then measured the heights of all the children one by one together with them. . . . The children's heights were measured because the children showed interest in the matter. (T3)

These examples demonstrate how teachers paid attention to how children's interests can bolster child-initiated mathematical learning by supplementing and providing insights into mathematical topics and related phenomena.

Collaborative Mathematical Learning

Furthermore, the teachers described how they came to realize the essence of collaborative learning with children during the PD program. According to them, collaborative learning appeared in their practices through an emphasis on children's participation in planning and brainstorming learning projects – for instance, by measuring and determining how to design a lobby where the children's clothes and shoes could be stored. The teachers also explained how collaborative learning manifested as pondering various issues and problems together in daily life – for instance, during clothing:

> We compared shoes while dressing because several children had similar black winter shoes. A child picked up a pair of shoes and we determined that they could not be his because his feet were so big, and the shoes were so small. Then we searched for another pair and compared whether those were bigger. We came to see that, yes, these were bigger and here also was his name. These fit his feet. (T2)

This teacher's description indicates how she registered the essence of collaborative learning but also how she consciously used mathematical concepts in the situation. Such collaborative learning situations were perceived as affordances for the conscious use of mathematical language by the other six teachers as well.

Holistic Mathematical Teaching

All seven teachers claimed that the PD program had helped them to notice the presence of mathematical phenomena in daily life and play in a different way than before, which allowed them to pursue a more holistic approach to mathematical teaching compared to their earlier practices. According to the teachers, awareness of holistic opportunities emerged through different elements of the program (theoretical components, hands-on explorations, instructional packages, learning projects and collaborative discussions), which assisted them in determining how to teach mathematics more concretely in different situations in ECEC and pre-primary education. The teachers explained that they now recognized mathematical affordances more consciously by observing children and their surroundings as well as by actively reflecting on and examining their practices:

> I actively think, and I am aware. So, I bring mathematics into available moments by catching them like a hawk. (T6)

> My eyes have been opened to these numerous possibilities to use nature, playground, and everything . . . wherever I face the situation. It's like a treasure box. (T7)

Although all seven teachers described these changes in recognizing mathematical phenomena differently, five of them also remarked that their mathematical teaching had focused on preplanned activities before the PD program. It can thus be argued that the teachers' implementation of mathematics education changed in terms of their pedagogical approach because they had learnt to recognize and capture mathematical affordances in daily situations and play. As an example, every teacher stressed that self-reflection had motivated them to use mathematical language more consciously than before in different situations. Yet, acknowledging versatile mathematical affordances permitted all seven teachers to apply a more holistic and integrative approach to mathematical teaching compared to their earlier practices.

Despite the teachers shifting their focus from preplanned mathematical sessions to holistic mathematical teaching in daily life and play during the PD program, they also recognized that integrating numerical learning into different situations was easier compared to spatial learning or mathematical thinking and reasoning. According to the teachers, this realization happened by reflecting on their practices with a reflective journal:

> I recognized that numerical skills were easier to implement. I recognized that I needed to keep in mind what I'm doing and teaching. Although these come automatically to daily events, teaching them is not automatic in everything yet. (T5)

Nevertheless, the recognition that spatial learning and mathematical thinking and reasoning could not yet be automatically integrated into teaching motivated the teachers to examine their practices and pay closer attention to the implementation of these areas, as one teacher noted: *"I recognized that I did not take these that much, I'll take them next week"* (T6). Thus, the findings indicate that recognition helped the teachers become more aware of their limitations in pedagogical awareness of holistic mathematical teaching.

Integrating Mathematics into Daily Situations

The teachers described how they integrated mathematics into daily situations in their practices by, for example, realizing how to connect mathematical learning to transition situations (i.e. from one activity to another), meals and outdoor activities. They, for instance, connected counting utensils, measuring the length of drinking glass stacks, or practicing chronology through problem solving to mealtimes. Additionally, they discussed how they connected practicing number symbols and number word sequencing into transition situations, and comparison and problem solving into clothing situations. The teachers both invented new practices and developed existing ones to more closely link mathematical learning to different daily situations but also to concretize mathematics for children as they recognized how meaningful it was in their child groups:

> To measure time and how long it took to wash your hands. So, we made our own song for that, and the children had to wash their hands until the end of singing. (T1)

> Nowadays, it comes naturally, like during tidy-up-time after outdoor play I ask the children to collect two more toys than is his/her age. The children were excited and explained: "I have a private detective assignment and I have to count it first in my mind, or we count together and then I will do the assignment." I earlier instructed the children to collect three toys, which was not that meaningful for them. (T4)

These examples indicate that the teachers actively searched for ways to implement mathematics education in such a way that the children's mathematical learning would be connected to daily life and would also become more concrete for them.

Integrating Mathematics into Pre-Planned Activities

Although teaching mathematics during preplanned mathematical sessions was a common practice for every teacher before the PD program, the teachers discovered new ways to integrate mathematics into different kinds of preplanned activities (circle times, arts education, physical education, celebrations, field trips, etc.) more consciously than before. One teacher explained how she incorporated measuring into a field trip to a forest with children:

> We measured the passage of time with a watch and measured the number of steps by pedometer. We wondered if coming back took as long, and if not, why was that? (T5)

Another teacher described how she invented a new circle time activity that utilized the learning material provided by the program:

> I put number-symbol cards on the floor and every child chose one to stand on. Then I said a number word and the children had to search it to leave the circle time. I could say that it´s less than fifty-five but greater than forty, then everyone would search for whose number that was. The one whose it was took the card and left. I hadn´t done this earlier, but now I did because I had this material. (T4)

Yet another teacher explained how she discovered a way to enrich the preparation of a rhythmic instrument by connecting mathematical learning to it:

> I have prepared rhythmic instruments as long as those certain plastic bottles have existed. Now I took a few steps further with this activity. We had scales and we explained that you could fill it with a few macaroni only, but with a lot more rice. And we checked how the scale swayed. So, we not only scooped the amount randomly like before. . . . In addition, we put into words that craft balls are in a measure and their volume is one liter, instead saying that those are in a jar. (T6)

The findings indicate that connecting mathematics to different kinds of preplanned activities in various ways helped the teachers take steps toward more holistic teaching of mathematics, whilst it also broadened their ability to teach mathematics through the integration of different subjects.

Integrating Mathematics into Play

The teachers mentioned that by understanding that mathematical learning must be concrete and connected to children's daily life, they registered affordances of play more so than before. Among six teachers, this resulted in organizing learning environments so that tangible and concrete mathematical learning materials (e.g. hourglasses, measures and shapes) were available for children during free play:

> It was a new practice for me to arrange the learning environment so that there were mathematical options for children to choose from during free play, also for the youngest children. (T1)

> I´ve had shapes in a box in a cabinet, waiting to be used during pre-planned mathematical activity. Now I placed them into a hanging storage pouch. The children took them to play with and talked together: "This one lives in the blue circle." They used shapes in play, which made me wonder why I kept them in the cabinet, as they should be available for the children. (T5)

These examples illustrate that enriching play with mathematical learning materials was a new practice for the teachers, many of whom wondered why they had not figured it out earlier. The findings suggest that the teachers were pleased with the new arrangements because they recognized the positive influence they had on the children's play and mathematical learning. One teacher encapsulated the essence of learning mathematics through play: *"It grows and comes alive in children's play, so I need to offer them a chance" (T6)*. Three teachers also explained how they came to realize play as an opportunity for children to explore mathematical materials for a longer period of time compared to guided activities. They thus understood play as an optimal means to strengthen mathematical learning.

In addition to providing materials for play, the same six teachers also expressed that the realization of versatile affordances for mathematical learning in play resulted in more active observation and participation in the children's play. This, according to them, yielded recognizing play situations that could be employed for mathematical teaching. The teachers described how they supported mathematical learning by, for instance, encouraging the children to organize a supermarket according to the principles of classification, or by discussing directions and shapes while preparing a car track. Some teachers also stressed how they used mathematical language and concepts intentionally through their guidance or while playing with the children:

> I pondered how to integrate the children's interests by enriching play and by using mathematical language during play. For instance, we baked with sand, so we pretended that a bucket was one liter, and a cup was one deciliter, and at the same time, we spoke about milk cartons et cetera, so that the children would understand them. (T3)

This example shows that integrating mathematics into play in various ways broadened the teachers' repertoire of mathematical teaching but also provided them with important insights into teaching mathematics through play.

Validating the Increases in Pedagogical Awareness of Teaching Different Early Mathematical Skills

The descriptive statistics of the scale scores for pedagogical awareness of teaching different early mathematical skills in the pre-PD and follow-up-tests are presented in Table 1. The Non-parametric Wilcoxon Signed Ranks Test was used to test whether the PD program resulted in long-lasting increases in the teachers' pedagogical awareness of teaching numerical skills, spatial thinking skills, and mathematical thinking and reasoning skills. The results showed a significant increase in the teachers' pedagogical awareness of teaching numerical skills and spatial thinking skills, which remained nine months after the end of the PD program, but no increase in mathematical thinking and reasoning skills was shown. The effect sizes (Cohen's d using the pooled standard deviation of the two assessments) were large for numerical skills (1.04), moderate for spatial thinking skills (0.68), and small for mathematical thinking and reasoning skills (0.30) (cf. Cohen, 1992 for criteria of different magnitudes of effect size).

Discussion

This mixed methods study sought to explore changes in the pedagogical awareness of ECEC teachers about teaching early mathematical skills to 3- to 7-year-old children when participating in a tailored PD program in early childhood mathematics. The teachers were interviewed after the program to examine how they exemplified changes in their pedagogical awareness of teaching early mathematical skills in ECEC and pre-primary education. Furthermore, the teachers' responses to "Teaching Early Mathematical Skills" pre-PD and follow-up-questionnaires (Parviainen et al., 2023) were used to validate the qualitative findings and to explore the long-lasting changes regarding the teachers' pedagogical awareness of teaching different early mathematical skills.

The results demonstrated that by examining and reflecting on teaching practices, by perceiving the children's interests, and by sharing thoughts with other participating teachers in relation to different elements of the program (e.g. theory and instructional packages), the teachers broadened their pedagogical awareness concerning age-specific mathematical content, captured the essence of child-initiated learning, and widened their teaching repertoire toward more holistic mathematical teaching in different daily situations and play. Six of the teachers also recognized their limitations concerning the teaching of different mathematical content after the program had finished.

The findings indicate, first, that it is possible to enhance ECEC teachers' pedagogical awareness of teaching early mathematical skills through a tailored PD program. The data obtained from the completed pre-PD and follow-up-questionnaires validated this finding by showing an increase in

Table 1. Scale scores in pedagogical awareness of teaching different early mathematical skills in the beginning and nine months after the PD program.

	Pre-test					Follow-up-test					
	Min	Max	Mean	Median	SD	Min	Max	Mean	Median	SD	Z
Pedagogical awareness											
NS	3.42	4.82	4.48	4.66	.49	3.68	6.20	5.18	5.32	.88	−1.69*
STS	2.58	4.66	3.81	3.81	.74	3.32	6.28	4.82	4.86	1.10	−2.03*
MTRS	2.86	5.35	4.28	4.14	.79	3.10	6.06	4.90	5.36	1.05	−1.52

*$p < .05$. one-tailed test.
NS = Numerical skills, STS = Spatial thinking skills, MTRS = Mathematical thinking and reasoning skills.

the teachers' pedagogical awareness of teaching numerical skills and spatial thinking skills, which remained nine months after the end of the PD program, suggesting that the increases were long-lasting. Second, the findings indicate that teachers recognized a need to continue PD in daily life, which the program helped them to acknowledge. This is important, as pivotal principles of transformative learning emphasize the role of self-reflective cycles and critical thinking in PD processes (Cranton, 2016; Mezirow, 1991).

The results revealed that all seven ECEC teachers gained greater awareness of the connections between various facets of early mathematical skills development (cf. Parviainen, 2019), which allowed them to critically examine their practices and to apply their newfound awareness to their teaching. Through self-reflection, they recognized that it was more natural for them to apply numerical skills in different situations, whereas teaching spatial thinking skills and mathematical thinking and reasoning skills required more consideration. This in turn directed them to pay more attention to the implementation of these skills areas, leading to an enhanced teaching repertoire.

The teachers also acquired alternative ways to teach mathematics by critically examining their thinking structures and practices, both independently and together with others. Such a finding indicates that the teachers were committed to completing the PD process, which is an essential part of effective PD as it can culminate in sustainable changes in teaching (cf. Cranton, 2016; Cranton & King, 2003). This point is crucial, as successful early childhood mathematics education requires conscious teaching (Clements et al., 2011; Moss et al., 2016) and an awareness of pedagogy (Björklund et al., 2018; Brandt, 2013) in which reflection and evaluation are the core practices (Gasteiger & Benz, 2018; Lindmeier et al., 2020).

Enhanced age-specific content awareness manifested in the teaching of foundational skills to 3- to 5-year-old children (i.e. number sense, counting, spatial relations, shapes and classification) and more sophisticated skills to 5- to 7-year-old children (i.e. addition and subtraction, 2- and 3-dimensionality of figures, mass, volume and length measurement, and mathematical problem solving). The findings concerning broadened understanding of early mathematical content and gradual skills development are significant because these are the premises for successful early childhood education (Clements et al., 2011; Sarama & Clements, 2009). Yet, the ECEC teachers' age-specific content awareness of spatial thinking skills, which most of them had evaluated as their weakest area at the beginning of the program, was broadened. This broader awareness, gained through practices and reflection during the PD program, is a promising finding as earlier studies showed no differences in the teaching frequency of spatial thinking skills to children of different ages (Parviainen et al., 2023) and found that these skills were not taught to children because of teachers' limited awareness (Björklund & Barendregt, 2016).

The results also showed that with the aid of the theoretical orientation of the program and by observing the children, the ECEC teachers came to understand the essence of the children's interests and initiatives and learned to appreciate collaborative learning as a premise for mathematical teaching. These insights culminated in the stronger implementation of the child-initiated approach in mathematics education. The findings also indicate a better balance between adult-initiated and child-initiated approaches to considering appropriate content in these groups, which has been shown to optimally support mathematical learning (cf. Anthony & Walshaw, 2009; Cheeseman et al., 2014; Salomonsen, 2020). However, some teachers did not discuss these topics as intensively as the others in the interviews. Thus, this suggests that utilizing the children's interests and collaborative learning in early mathematical teaching should be emphasized more consciously in PD programs. Therefore, PD programs should focus on strengthening the child-initiated approach in mathematics education because collaborative learning and acknowledging children's perspectives can promote mathematical learning (cf. Björklund et al., 2018; Trawick-Smith et al., 2016; Vogt et al., 2018; van Oers, 2013).

Furthermore, the results demonstrated that by critically examining their practices and observing children in relation to different elements of the PD program, the ECEC teachers grasped mathematical learning and teaching affordances in different daily situations, teachable moments and play. This helped the teachers to invent new practices and discover new ways to bolster existing ones, like connecting mathematical learning to routine events, integrating such learning with other subjects or celebrations, and considering teaching mathematics through play. The teachers thus expanded their repertoire from preplanned activities to more holistic mathematical teaching. Again, this indicates a stronger connection between the child-initiated approach and adult-initiated practices (e.Anthony & Walshaw, 2009), but it also suggests a greater understanding of mathematical learning as taking place in various learning environments (Helenius, 2018; Moss et al., 2016).

Essentially, through self-reflection, six teachers recognized that they could implement numerical skills more automatically in different situations and play due to strong content awareness at the beginning of the program. Although teaching spatial thinking skills and mathematical thinking and reasoning skills required more deliberation, the results indicated that self-reflection brought these areas in particular to the teachers' attention. This permitted them to search for ways to broaden their teaching repertoire because of their self-identified learning aims and personal commitment to PD. This development is critical, as awareness of content-related confidence and practices can substantially influence teaching and children's mathematical learning (Çelik, 2017; Ertle et al., 2008; Parviainen et al., 2023).

Results regarding the "Teaching Early Mathematical Skills" pre- and follow-up-questionnaire (Parviainen et al., 2023) showed that the teachers' pedagogical awareness of numerical skills increased the most, based on its large effect size. Our earlier study, which focused on comparisons of 206 ECEC teachers' teaching of different mathematical skills, showed that the teachers' pedagogical awareness of numerical skills is strongest and that the teachers promote age-appropriate learning of numerical skills (Parviainen et al., 2023). The findings of the present study indicate that the area that teachers are most familiar with is also the easiest to strengthen and sustain. Interestingly, the teachers in the present study still found need for PD in teaching numerical skills despite this being their strongest area.

In addition, the moderate changes in the teachers' pedagogical awareness of teaching spatial thinking skills evidenced by pre-PD and follow-up questionnaires support the teachers' descriptions of such changes occurred in interviews and indicate their retention of their increased pedagogical awareness of such skills. The finding is promising particularly because pedagogical awareness of teaching spatial thinking skills has previously been shown to be the weakest among ECEC teachers (Parviainen et al., 2023). The teachers in the present study also mentioned the need for deliberate thinking when teaching spatial thinking skills, and five of seven teachers indicated this as their weakest area at the beginning of the PD program. However, based on the findings of the present study, teachers' pedagogical awareness of teaching spatial thinking skills can be increased through PD.

Mathematical thinking and reasoning skills was the only area, in which the pre-PD and follow-up-questionnaires did not show sustain changes in the teachers' pedagogical awareness, although the teachers described changes in this area in the interviews, right after the PD program. Our earlier study revealed large variations in pedagogical awareness of teaching mathematical thinking and reasoning skills (Parviainen et al., 2023), which together with the present study's findings, may reflect a need for teachers' more systematic reflection of teaching mathematical thinking and reasoning skills in daily life, as well as need for the development of pre- and in-service teacher education concerning the teaching of mathematical thinking and reasoning skills.

In sum, the results of this mixed methods study indicated that the design of the tailored PD program helped to enhance the ECEC teachers' pedagogical awareness from several dimensions. For instance, it assisted them in taking conscious steps to support developmentally appropriate mathematical learning, in striking a balance between the child-initiated approach and teacher-initiated practices, and in responding to mathematical affordances in daily situations and play. The results support Knaus's (2017) earlier finding, which showed that designing a PD program based on principles of transformative learning (Cranton, 2016; Mezirow, 1991) helped teachers develop their mathematics teaching practices. Our results, however, revealed that self-reflection and critical examination of one's own teaching practices played a key role in PD during the program. When the ECEC teachers were prompted to self-reflect on their teaching from different perspectives, they became aware of what they did and why. This allowed them to make concerted efforts to develop their practices and recognize their need for further PD. This finding is critical as it provides a new understanding of the role of self-reflection and examination of one's own practices in enhancing pedagogical awareness of teaching early mathematical skills (cf. Gasteiger & Benz, 2018, Lindmeier et al., 2020).

Limitations

While this study yielded important insights into how to enhance pedagogical awareness of teaching early mathematical skills through a tailored PD program in mathematics, it also had some limitations. One of the limitations is the restricted validation of results. As only pre-PD and follow-up questionnaires were presented to the teachers for completion and we did not ask the teachers to complete a questionnaire right after the PD program had ended, we do not know whether there were also changes in the teachers' pedagogical awareness of teaching mathematical thinking and reasoning skills right after the PD program, but that these had faded away by the time the teachers completed the follow-up questionnaires, or whether we would not have been able to validate the changes the teachers described in it even then.

Additionally, the teachers received input from the program, which supported their ability to evaluate their learning processes using selective terminology and viewpoints regarding, for instance, Parviainen's (2019) *holistic model of the development of early mathematical skills* (Appendix A). However, through the versatile and diverse ways in which the teachers described their implementation of early childhood mathematics education in their everyday practices we were able to interpret the integration of the theoretical and practical content of the PD program into their pedagogies.

Pertaining to the study design, one methodological limitation was the subjectivity of the main researcher. It is acknowledged that qualitative data collection, thematic analysis and data interpretation are always subjective (Braun & Clarke, 2022; Newby, 2014). Therefore, qualitative research is susceptible to researcher bias since it is nearly impossible to separate one's own values from the research. In addition, the main researcher in the present study was also susceptible to bias, being the designer and conductor of the PD program. However, there were measures taken to minimize this bias by inviting a member outside the team to collaborate in conducting the program. This arrangement enabled a critical reflection on the researcher's position throughout the process. The questions asked in the semi-structured interviews were also formulated together with another member of the research team, and the data analysis process was subjected to the member-checking technique. In addition, the inter-rater reliability related to the semi-structure interviews was calculated, and results related to pre-PD and follow-up questionnaires were reported to validate the findings of teachers' interviews.

Only seven ECEC teachers participated in the PD program, and they were all oriented toward developing their mathematical teaching, which may have caused some homogeneity in the results. Larger sample sizes are thus warranted in future studies. Nevertheless, as the teachers represented different early education centers and were committed to PD through self-identified learning aims, the processes they employed were somewhat dissimilar. The teachers described their experiences and deliberations in detail and from their own perspectives in each interview, and thus the analysis generated a collective, multivocal synthesis of the varied ways in which the teachers pursued PD.

It is also widely known that teaching in ECEC and pre-primary education is vulnerable to sudden changes in daily life. Absences of teaching staff can greatly impact teaching. COVID-19 significantly challenged teachers during the PD program, which may have adversely impacted their ability to concentrate on their learning aims. However, COVID-19 also inspired the teachers to rely on exceptional circumstances to invent novel ways to teach mathematical skills, like counting the number of persons per room. Notwithstanding the fact that the PD program was not part of regular teaching and transpired during the COVID-19 pandemic, it was beneficial insofar as it was based on self-identified learning aims and the implementation of mathematics education in one's own child group. In other words, the teachers would have taught mathematics to the children anyway, but now they had versatile means with which to do so.

Conclusions

The results of the present study showed that teachers' pedagogical awareness of teaching early mathematical skills can be enhanced through a tailored PD program as sustainable changes were detected in the teaching of numerical skills and spatial thinking skills, although no such changes were detected in the teaching of mathematical thinking and reasoning skills. To support ECEC teachers' efforts to develop their mathematical teaching practices in different daily situations, teachable moments and play, and thus promote substantial mathematical learning among children (cf. Björklund et al., 2018; Clements et al., 2011; Helenius, 2018; Salomonsen, 2020), we suggest that tailored PD programs in early childhood mathematics should apply four principles. Programs should (1) be based on teachers' self-identified learning needs and current research-based theory of early mathematical learning (cf. Knaus, 2017), (2) apply practice-based models that explain the pedagogical awareness of teaching early mathematical skills (cf. Gasteiger & Benz, 2018; Lindmeier et al., 2020), (3) promote commitment to PD through individual and collaborative practices (cf. Barber et al., 2014; Hadley et al., 2015), and (4) incorporate principles of transformative learning (cf. Cranton, 2016; Mezirow, 1997).

We consider these important because our earlier study showed that both teachers' increased pedagogical awareness and participation in PD programs positively influence children by giving them opportunities to learn different mathematical skills (Parviainen et al., 2023). It can thus be argued that participation in a mathematics PD program that encourages a critical examination of the teaching early mathematical skills through reflective practices and principles of transformative learning can greatly increase children's opportunities to learn mathematical skills and lead to sustainable changes in mathematical teaching while simultaneously indicating the need for further PD.

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No potential conflict of interest was reported by the author(s).

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Authors' Contributions

Piia Parviainen processed the original idea. The PD program, including materials, were designed by Piia Parviainen. Interviews were conducted by Piia Parviainen with contribution to questions from Niina Rutanen. Piia Parviainen took the lead in thematic data analyses, however inter-rater coding was conducted by Merja Koivula, member-checking technique and analytical cycles were utilized with Niina Rutanen, Merja Koivula, Tarja Liinamaa and Kenneth Eklund. Statistical data analyses were conducted with Kenneth Eklund. Piia Parviainen took the lead in writing the manuscript, with contributions from all the other authors. All authors discussed the results, commented on previous versions of the manuscript, and approved the final manuscript.

Data Availability Statement

No extra material is included in the article.

Consent to Participate

Participation in this research was voluntary. Consent to participate and permission to use participant data were given by every participant. No participant withdrew from the research.

Consent for Publication

This work can be published by *Early Education and Development*.

Ethics Approval

According to the local guidelines, ethical review applies only to precisely defined research configurations. As our project had no such configurations, ethical pre-review was not required nor sought. However, research notification, privacy notice and consent to participate were followed as mandated by the Human Sciences Ethics Committee of the university.

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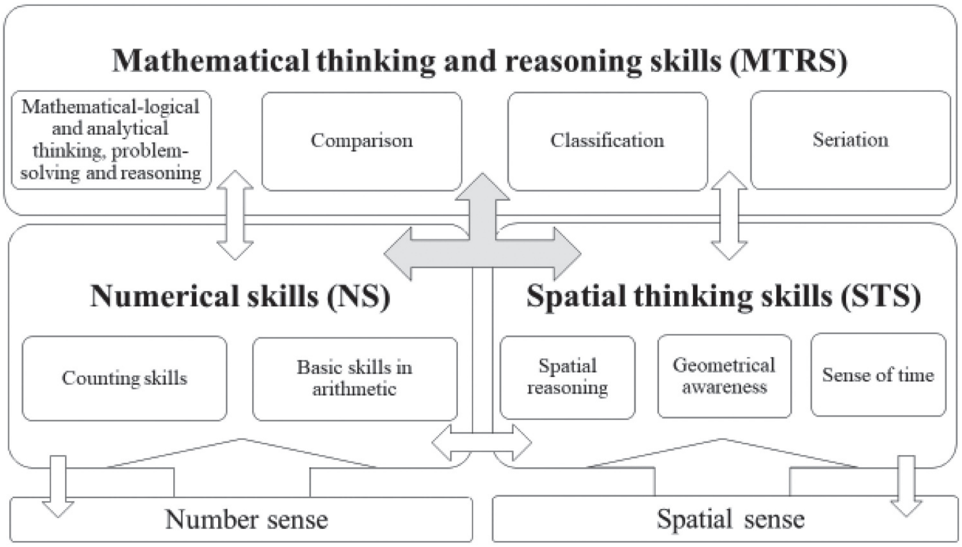
Appendices

Appendix A

Holistic model of early mathematical skills development (Parviainen, 2019).



Appendix B

Structure of the tailored PD program regarding meetings and the implementation phase in early education centers.

Timeline for monthly meetings and specified trainings in early education center	Timeline for implementation in early education centers and self-reflection
Joint meeting 1 (10/2019) • Introducing *LUMA2020 Development Program* and its aims by the trainers • Getting to know each other as trainers and participants Joint meeting 2 (11/2019) • Introducing the idea of project-based learning in the *LUMA2020 Development Program* by the trainers • Theory-based discussion of early mathematical skills development facilitated by the trainers • Exploring hands-on mathematical learning activities designed by the trainers • Setting personal learning aims and training needs for PD by each teacher	Teaching early mathematical skills in different daily situations (11/2019) Brainstorming project-based mathematical learning in child groups according to needs and interests of children (11/2019)

(*Continued*)

(Continued).

Specified training and support for each early education center (12/2019–1/2020) • Targeted training based on indicated needs and support of mathematical learning projects by participants, facilitated by the trainers • Supplying instructional packages for teachers Joint meeting 3 (2/2020) • Collective sharing of experiences and ideas regarding early mathematical teaching and projects • Collective sharing of good practices and experiences used in child groups • Theory-based discussion of the development of different early mathematical skills and ways to support the development facilitated by the trainers Joint meeting 4 (3/2020) • Collective discussion about teaching early mathematical skills based on personal learning aims and information of the first four reflective journals, facilitated by the trainers • Brainstorming how certain content could be taught to children of different ages • Exploring hands-on mathematical learning activities, designed based on specific needs indicated by participants Joint meeting 5 (4/2020) • Collective discussion of PD in teaching mathematics (e.g. achievements, ponderings and new practices) facilitated by the trainers • Collective feedback on the program	Teaching early mathematical skills in different daily situations (12/2019–5/2020) Planning project-based mathematical learning in child groups together with children and teaching team (12/2019) Implementing project-based mathematical learning in child groups together with children and teaching team (1–5/2020) Reflecting and evaluating mathematical teaching through individual reflective journals, filled every other week, 10 times in total (1–5/2020)