Convergence Rate of the Euler Scheme for Diffusion Processes

Master's Thesis

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Abstract

The aim of this thesis is to consider the strong and weak convergence of the Euler scheme for a one-dimensional diffusion process whereas the results for the strong convergence are more classical, and the more recent results concerning the weak convergence are due to Bally and Talay, [1, 2].

Contents

Chapter 1

Introduction

1.1 Stochastic Differential Equations

Stochastic differential equations play an important role in stochastic modeling. They are very popular in many areas of science and economics because of their ability to imitate the behavior of random phenomena. The difference between a stochastic differential equation and an ordinary differential equation is the random component added to the latter. A stochastic differential equation has the general form of

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x$$

or, equivalently

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,$$

where the function b is called *drift coefficient* and the function σ is called *diffusion coefficient*. The initial value of the process is $X_0 = x$ and $t \in [0, T]$ with $T > 0$ represents the time. The drift describes the steepness of the process whereas the diffusion describes the variation of the process around its mean. There are two main definitions of a solution to an stochastic differential equation, the strong solution and the weak solution. The difference between these two solutions lies in the underlying probability space.

An important example is the equation for the geometric Brownian motion

$$dX_t = bX_t dt + \sigma X_t dW_t, \quad X_0 = x$$

which is the equation used to describe the dynamics of the price of a stock in the famous Black-Scholes options pricing formula in financial mathematics. In this equation one has $b(t, X_t) = bX_t$ and $\sigma(t, X_t) = \sigma X_t$. Figure 1.1 represents a trajectory of the geometric Brownian motion.

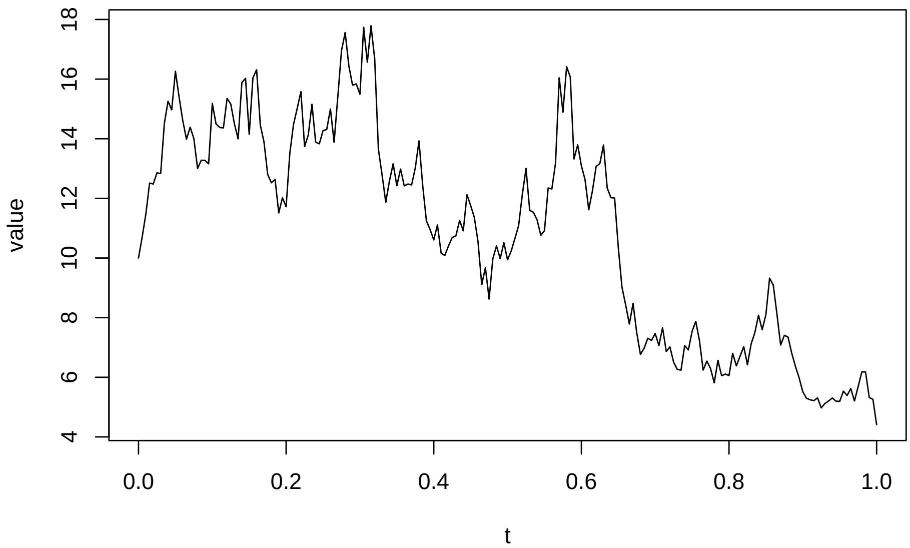

Figure 1.1: *Trajectory of a geometric Brownian motion on* $[0, 1]$*, where* $b = 0$*,* $\sigma = 1$ *and* $X_0 = 10$*.*

1.2 On Simulations of Stochastic Processes

Simulations are an important tool in stochastic modeling. There are often situations where we do not obtain an exact solution of a stochastic differential equation and therefore we need to use a computer simulation method to obtain an approximate solution to the problem. To be able to simulate a stochastic process (X_t) we need to have a discretization scheme (X_t^n). This means that we divide the corresponding time-interval into n sub-intervals and compute the approximated value of the process in each discrete time-point. There are various schemes for this, see for example the article of Talay and Tubaro, [10]. The precision or rate of convergence of an approximation scheme can be measured in various ways whereas there are two principal ones: the strong convergence and the weak convergence. The strong convergence plays a role, for example, in evaluating or estimating the risk of a portfolio in Stochastic Finance. The weak convergence is used in the pricing of bonds and options in Stochastic Finance. In this context the mathematical problem consists of two parts: Firstly, one has to determine the rate of convergence for an appropriate approximation scheme (in this thesis done for the Euler scheme), secondly one has to simulate the discretized process by a Monte

Carlo method. The source of the error depends now on two factors: the number of sub-intervals n, the discretization scheme works with, and the number of simulated trajectories N. We focus on the first one.

1.3 About This Thesis

In this thesis we study the strong convergence and weak convergence of the Euler scheme, an important discrete-time approximation of diffusion processes. The main results are Proposition 3.2 and Proposition 3.3. The first one concerns the strong convergence and is based on the book of Gard, [5]. The latter concerns the weak convergence and is based on articles of Bally and Talay, [1, 2].

Chapter 2

Preliminaries

We start by introducing some basics of probability theory. In this thesis we are considering a one dimensional diffusion process of type

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

In general, we do not know (X_t) or its law, and therefore we need an appropriate approximation for it. We consider the Euler scheme to approximate the process (X_t) and denote it by (X_t^n). There are several other schemes to approximate the process (X_t), for example the Milstein scheme and the second order scheme considered by Talay and Tubaro, [10].

Remark 2.1. The coefficients b and σ has the following properties throughout the thesis:

- b and σ are time homogeneous, that means: $b(t, x) = b(x)$ and $\sigma(t, x) = \sigma(x)$,
- $\inf_{x \in \mathbb{R}} \sigma^2(x) > 0$,
- $b, \sigma \in \mathcal{C}_b^\infty(\mathbb{R})$,

and we implicate these properties on b and σ by writing $b, \sigma \in \mathcal{D}$.

2.1 Probability Theory

In this section we introduce some basics of probability theory like, the concepts of a σ-algebra, a probability measure, a probability space, a filtration, a stochastic basis and a random variable.

Definition 2.2. A system of subsets $\mathcal{F}$ of Ω is called σ-*algebra*, if *(i)* $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$,

(ii) $A \in \mathcal{F}$ implies that the complement of A belongs to $\mathcal{F}$,

(iii) $A_1, A_2, \ldots \in \mathcal{F}$ implies that $\bigcup_{i=i}^{\infty} A_i \in \mathcal{F}$.

An *algebra* is defined by replacing the condition *(iii)* by:

(iv) $A, B \in \mathcal{F}$ implies that $A \cup B \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called *measurable space.*

Definition 2.3. A mapping $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$ is called *probability measure* if the following conditions hold:

(i) $\mathbb{P}(\Omega) = 1$,

(ii) $\mathbb{P}\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$, for $A_1, A_2, \ldots \in \mathcal{F}$ with $A_i \cap A_j = \emptyset$ when $i \neq j$.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called *probability space.*

Definition 2.4. Assume a family of σ-algebras $(\mathcal{F}_t)_{t\in[0,T]}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F},$$

for $0 \le s < t \le T$, then the sequence $(\mathcal{F}_t)_{t\in[0,T]}$ is called *filtration.*

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is called *stochastic basis* and denoted by $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$.

Definition 2.5. Let $(\Omega, \mathcal{F})$ be a measurable space. A map $f : \Omega \longrightarrow \mathbb{R}$ is called a *random variable*, if for every $B \in \mathcal{B}(\mathbb{R})$ one has that

$$\{\omega \in \Omega : f(\omega) \in B\} \in \mathcal{F}.$$

In some context the function f is also called *$\mathcal{F}$-measurable.*

Remark 2.6. For $\Omega := \mathbb{R}$ and $\mathcal{F} := \mathcal{B}(\mathbb{R})$, the fuction $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called *Borel-measurable* provided that for all $B \in \mathcal{B}(\mathbb{R})$ one has

$$\{\omega \in \Omega : f(\omega) \in B\} \in \mathcal{B}(\mathbb{R}).$$

Definition 2.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $f : \Omega \to \mathbb{R}$ be a random variable. The *expected value* or *mean* of f is

$$\mathbb{E}f := \int_{\Omega} f(\omega) d\mathbb{P}(\omega)$$

and the *variance* of f is

$$Var(f) := \mathbb{E}(f - \mathbb{E}f)^2.$$

Definition 2.8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $f : \Omega \longrightarrow \mathbb{R}$ be a random variable and $1 \leq p < \infty$. The L_p-*semi-norm* of f is defined by

$$\|f\|_p := \Big(\int_\Omega |f(\omega)|^p d\mathbb{P}(\omega) \Big)^{\frac{1}{p}}$$

and the L_∞-*semi-norm* of f by

$$\|f\|_\infty := \sup_{t \geq 0} \big\{ \mathbb{P}(\{\omega \in \Omega : |f(\omega)| \geq t\}) > 0 \big\}.$$

Proposition 2.9 (Hölder's inequality)**.** *Let $f, g : \Omega \to \mathbb{R}$ be random variables with $\mathbb{E}|fg| < \infty$, $p > 1$ and $q = \frac{p}{p-1}$. Then*

$$\mathbb{E}fg \leq \|f\|_p \|g\|_q.$$

2.2 Stochastic Processes

Definition 2.10. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and J is an arbitrary set. A family

$$X_t(\omega), \quad \omega \in \Omega, \quad t \in J \tag{2.1}$$

of random variables with $X_t : \Omega \longrightarrow \mathbb{R}$ is called a *stochastic process* with index set J. In our case the index set is the time interval $[0, T]$ with $T > 0$ and we denote the process X by $(X_t)_{t \in [0,T]}$.

Remark 2.11. There are two different views of the stochastic process (2.1). For each fixed $t \in J$,

$$X_t = X_t(\cdot)$$

denotes a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and for each fixed $\omega \in \Omega$,

$$X_\cdot(\omega)$$

corresponds to a real-valued function defined on J, which is called a *trajectory* or *sample path* of the process, [5].

Remark 2.12. For now on, if $X_0 = x$ ($X_0^n = x$ resp.), we write $(X_t(x))_{t \in [0,T]}$ ($(X_t^n(x))_{t \in [0,T]}$ resp.) to emphasize the starting value of the process.

Definition 2.13. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$ be a stochastic basis and $X = (X_t)_{t \in [0,T]}$ a stochastic process. Then,

(*i*) X is called *measurable*, if the function $(\omega, t) \to X_t(\omega)$ considered as map from $\Omega \times [0, T]$ into $\mathbb{R}$ is measurable with respect to $\mathcal{F} \times \mathcal{B}([0, T])$ and $\mathcal{B}(\mathbb{R})$.

(*ii*) X is called *progressively measurable* with respect to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, if for all $T \geq T' \geq 0$ the function $(\omega, t) \rightarrow X_t(\omega)$ considered as map from $\Omega \times [0, T']$ into $\mathbb{R}$ is measurable with respect to $\mathcal{F}_{T'} \times \mathcal{B}([0, T'])$ and $\mathcal{B}(\mathbb{R})$.

(*iii*) X is called *adapted* with respect to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, if for all $t \in [0, T]$ one has that X_t is $\mathcal{F}_t$-measurable.

Next we introduce the concept of the Brownian motion, which describes a certain type of random movement. The range of applications of the Brownian motion goes far beyond a study of microscopic particles in suspension and includes modeling of stock prices, of thermal noise in electrical circuits, of certain limiting behavior in queueing and inventory systems, and of random perturbations in a variety of other physical, biological, economic, and management systems (Karatzas, Shreve, [7]).

Definition 2.14. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which is complete, and let $T > 0$ be fixed. A stochastic process $W = (W_t)_{t\in[0,T]}$ is called *Brownian motion* provided that the following conditions are satisfied:

(*i*) the map $t \longrightarrow W_t(\omega)$ in continuous for all $\omega \in \Omega$,

(*ii*) $W_0 \equiv 0$,

(*iii*) for all $0 \leq s < t \leq T$ the increment $W_t - W_s$ is independent from $(W_u)_{u\in[0,s]}$, that means for all $0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq s$ and $A, A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}&\mathbb{P}(W_t - W_s \in A, W_{s_1} \in A_1, \ldots, W_{s_n} \in A_n)\\ &\quad = \mathbb{P}(W_t - W_s \in A)\mathbb{P}(W_{s_1} \in A_1, \ldots, W_{s_n} \in A_n),\end{aligned}$$

(*iv*) and each of these increments has a Gaussian distribution with

$$\begin{aligned}\mathbb{E}(W_t - W_s) &= 0,\\ Var(W_t - W_s) &= t - s.\end{aligned}$$

Now we take a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and Brownian motion $(W_t)_{t\in[0,T]}$. Without loss of generality we can choose the filtration $\mathcal{F}$ to be the completion of the σ-algebra $\sigma(W_s : s \in [0, T])$ and $\mathcal{F}_t = \sigma(W_s : s \leq t) \vee \mathcal{N}$ where

$$\mathcal{N} := \{A \subseteq \Omega : \exists B \in \mathcal{F} \text{ with } A \subseteq B \text{ and } \mathbb{P}(B) = 0\}.$$

This leads to the following Lemma.

Lemma 2.15 (Karatzas, Shreve, [7]). *Let $W = (W_t)_{t\in[0,T]}$ be a Brownian motion in the sense of Definition 2.14 and $\mathcal{F}$ and $(\mathcal{F}_t)_{t\in[0,T]}$ as defined above. Then $(W_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, that means that $(W_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted and satisfies the conditions (i), (ii) and (iv). The condition (iii) is replaced by the following:*

$(iii)'$ for all $0 \leq s < t \leq T$ the random variable $W_t - W_s$ is independent of $\mathcal{F}_s$.

As mentioned before, there are two main definitions of a solution to an stochastic differential equation. In this thesis we consider only strong solutions, that meas we fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$, a Brownian motion $W = (W_t)_{t\in[0,T]}$ and then we define a solution of the stochastic differential equation as:

Definition 2.16. Let $x \in \mathbb{R}$ and $b, \sigma \in \mathcal{D}$. A continuous, adapted and square integrable process $(X_t)_{t\in[0,T]}$ with $X_0 = x$ is a solution of the *stochastic differential equation*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \tag{2.2}$$

provided that the following conditions hold:

(i) $X_0 \equiv x$.

(ii) $X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$ for $t \in [0,T]$ almost surely.

Remark 2.17. The principle of weak solutions is that we do not start with a stochastic basis but we construct a particular basis to our problem.

Now we obtain the strong uniqueness and existence of solutions of stochastic differential equations.

Proposition 2.18 (Strong uniqueness, [7], Theorem on page 287). *Let $b, \sigma \in \mathcal{D}$ and assume that $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ are solutions of (2.2). Then*

$$\mathbb{P}(X_t = Y_t, t \geq 0) = 1.$$

Proposition 2.19 (Existence of solutions, [7], Theorem on page 289). *Let $b, \sigma \in \mathcal{D}$. There exists a solution of the stochastic differential equation (2.2).*

Now we introduce the famous formula of Itô[1] which is widely used in stochastics and especially in financial mathematics.

[1]Kiyoshi Itô (born 1915) is a Japanese mathematician whose work is now called Itô calculus. He was awarded the Gauss prize in 2006 for his lifetime achievements.

Definition 2.20. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis and $(W_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion. Let $b(s)$ and $\sigma(s)$ be progressively measurable stochastic processes such that

$$\int_0^T \sigma^2(s)ds < \infty \quad \mathbb{P}\text{-a.s.}$$

and

$$\int_0^T |b(s)|ds < \infty \quad \mathbb{P}\text{-a.s.}$$

An *Itô process* is a stochastic process $X = (X_t)_{t\in[0,T]}$ of the form

$$X_t = x + \int_0^t b(s)ds + \int_0^t \sigma(s)dW_s, \quad t \in [0,T], \quad a.s.,$$

where $X_0 = x \in \mathbb{R}$.

Proposition 2.21 (Itô's formula, [5])**.** *Let $(X_t)_{t\in[0,T]}$ be an Itô-process with representation*

$$X_t = x + \int_0^t b(u)du + \int_0^t \sigma(u)dW_u,$$

for all $t \in [0,T]$, almost surely, and let $f \in \mathcal{C}_b^{1,2}$. Then one has that

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial u}(u, X_u)du + \int_0^t \frac{\partial f}{\partial x}(u, X_u)\sigma(u)dW_u \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(u, X_u)b(u)du + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(u, X_u)\sigma^2(u)du \end{aligned}$$

for all $t \in [0,T]$, almost surely.

The following Proposition establishes a link between partial differential equations and stochastic processes.

Proposition 2.22 (Feynman-Kac)**.** *Let $dX_t = b(X_t)dt + \sigma(X_t)dW_t$, with $b, \sigma \in \mathcal{D}$, be a stochastic differential equation and f be a Borel-measurable function which satisfies*

$$|f(x)| \leq c(1 + |x|^q)$$

for some $c > 0$ and $q \geq 1$. Define

$$u(t,x) := \mathbb{E}f(X_{T-t}(x)) \text{ for } (t,x) \in [0,T] \times \mathbb{R}.$$

Then $u(t,x)$ solves the partial differential equation

$$\frac{\partial u}{\partial t}(t,x) + b(x)\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}[\sigma(x)]^2\frac{\partial^2 u}{\partial x^2} = 0$$

on $[0, T) \times \mathbb{R}$ *with the terminal condition*

$$u(T, x) = f(x).$$

For the proof we need the following Proposition presented in [4] on page 263, and Lemma 2.24.

Proposition 2.23. *For* $b, \sigma \in \mathcal{D}$ *there exists a transition density* $\Gamma : (0, T] \times \mathbb{R} \times \mathbb{R} \to [0, \infty) \in \mathcal{C}^\infty$ *such that* $\mathbb{P}(X_t(x) \in B) = \int_{\mathbb{R}} \Gamma(t, x, \xi) d\xi$ *for* $t \in (0, T]$ *and* $B \in \mathcal{B}(R)$*, where* $(X_t(x))_{t \in [0,T]}$ *is the strong solution of the stochastic differential equation (2.2) starting from* x*, such that the following is satisfied:*

(i) For $(s, x, \xi) \in (0, T] \times \mathbb{R} \times \mathbb{R}$ *one has*

$$\frac{\partial \Gamma}{\partial s}(s, x, \xi) = \frac{\sigma(x)^2}{2} \frac{\partial^2 \Gamma}{\partial x^2}(s, x, \xi) + b(x) \frac{\partial \Gamma}{\partial x}(s, x, \xi).$$

(ii) For $k, l \in \{0, 1, 2, \ldots\}$ *there exists a constant* $c = c(k, l) > 0$ *such that for* $(s, x, \xi) \in (0, T] \times \mathbb{R} \times \mathbb{R}$ *one has that*

$$\left| \frac{\partial^{k+l} \Gamma}{\partial s^k \partial x^l}(s, x, \xi) \right| \leq c s^{-k-l/2} \gamma_{cs}(x - \xi) \quad \text{where } \gamma_t(\eta) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{\eta^2}{2t}}.$$

Consequently, for $f \in \mathcal{C}_\gamma$ *one has*

$$\frac{\partial^{k+l}}{\partial s^k \partial x^l} \int_{\mathbb{R}} \Gamma(s, x, \xi) f(\xi) d\xi = \int_{\mathbb{R}} \frac{\partial^{k+l} \Gamma}{\partial s^k \partial x^l}(s, x, \xi) f(\xi) d\xi$$

on $(0, T] \times \mathbb{R}$ *for* $k, l \in \{0, 1, 2, \ldots\}$*, where the differentiation can be taken in any order and where* $\mathcal{C}_\gamma$ *is defined (Geiss, [6]) as the linear space of all Borel measurable functions* $f : \mathbb{R} \to \mathbb{R}$ *such that there is some* $m > 0$ *with*

$$\sup_{x \in \mathbb{R}} e^{-m|x|} \mathbb{E} f^2(x + tg) < \infty$$

for all $t > 0$*, where* $g \sim \mathcal{N}(0, 1)$*.*

Lemma 2.24. *Let* $f : \mathbb{R} \times \Omega \to \mathbb{R}$ *such that*

(i) $\frac{\partial}{\partial x} f(\cdot, \omega)$ *is continuous for all* $\omega \in \Omega$*,*

(ii) $\frac{\partial}{\partial x} f(x, \cdot)$ *and* $f(x, \cdot)$ *are random variables,*

(iii) there exists a random variable $g : \Omega \to [0, \infty)$ *such that*

$$\left| \frac{\partial}{\partial x}(x, \omega) \right| \leq g(\omega), \text{ for all } x \in \mathbb{R}, \quad \omega \in \Omega$$

and $\mathbb{E} g(\omega) < \infty$*,*

(iv)

$$\int_\Omega |f(x,\omega)| d\mathbb{P}(\omega) < \infty, \text{ for all } x \in \mathbb{R}.$$

Then

$$\frac{\partial}{\partial x}\int_\Omega f(x,\omega) d\mathbb{P}(\omega) = \int_\Omega \frac{\partial}{\partial x} f(x,\omega) d\mathbb{P}(\omega).$$

Proof of Proposition 2.22. The function u can be written as follows

$$u(t,x) = \mathbb{E}f(X_{T-t}(x)) = \int_{\mathbb{R}} f(\xi)\Gamma(T-t,x,\xi)d\xi$$

and by taking derivatives with respect to t one has

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{d}{dt}\int_{\mathbb{R}} f(\xi)\Gamma(T-t,x,\xi)d\xi \\ &= \int_{\mathbb{R}} f(\xi)\frac{d}{dt}\Gamma(T-t,x,\xi)d\xi \\ &= -\int_{\mathbb{R}} f(\xi)\frac{\partial \Gamma}{\partial s}(T-t,x,\xi)d\xi, \end{aligned}$$

where we have used the Lemma 2.24 to interchange integration and differentiation. Taking derivatives with respect to x one has

$$\frac{\partial u}{\partial x}(t,x) = \int_{\mathbb{R}} f(\xi)\frac{\partial \Gamma}{\partial x}(T-t,x,\xi)d\xi.$$

Now, by using the Proposition 2.23, one has that

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) + b(x)\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\sigma(x)^2\frac{\partial^2 u}{\partial x^2}(t,x) \\ &= \int_{\mathbb{R}} f(\xi)\Big[-\frac{\partial \Gamma}{\partial s}(T-t,x,\xi) + b(x)\frac{\partial \Gamma}{\partial x}(T-t,x,\xi) \\ &\qquad + \frac{\sigma(x)^2}{2}\frac{\partial^2 \Gamma}{\partial x^2}(T-t,x,\xi)\Big]d\xi \\ &= 0. \end{aligned}$$

□

Lemma 2.25 (Gronwall, [5]). *Let A, B, $T \geq 0$ and $f : [0,T] \to \mathbb{R}$ be a continuous function such that*

$$f(t) \leq A + B\int_0^t f(s)ds$$

for all $t \in [0,T]$. Then one has that $f(T) \leq Ae^{BT}$.

Let $b, \sigma \in \mathcal{D}$ and assume $(X_t)_{t\in[0,T]}$ is the strong solution of the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x.$$

This leads to the following two properties of the solution (Bouleau and Lépingle, [3]).

Proposition 2.26. *For* $1 \le p < \infty$ *and* $t \le T$

$$\|X_t\|_p \le C(p,T)(1+|x|).$$

Proof. We have that

$$\begin{aligned}
\|X_t\|_p &\le |x| + \Big\| \int_0^t \sigma(X_u)dW_u \Big\|_p + \Big\| \int_0^t b(X_u)du \Big\|_p \\
&\le |x| + c_p \Big\| \Big(\int_0^t \sigma^2(X_u)du \Big)^{1/2} \Big\|_p + t\|b\|_\infty \\
&\le |x| + c_p\|\sigma\|_\infty \sqrt{t} + \|b\|_\infty t,
\end{aligned}$$

where have used the Burkholder-Davis-Gundy inequality, [7] (Theorem 3.28 on page 166). □

Proposition 2.27. *For* $1 \le p < \infty$ *and* $t \le T$

$$\|X_t - X_s\|_p \le c(p,T)(1+|x|)(t-s)^{1/2}.$$

Proof. Here we have that

$$\begin{aligned}
\|X_t - X_s\|_p &\le \Big\| \int_s^t \sigma(X_u)dW_u \Big\|_p + \Big\| \int_s^t b(X_u)du \Big\|_p \\
&\le c_p \Big\| \Big(\int_s^t |\sigma(X_u)|^2 du \Big)^{1/2} \Big\|_p + \|b\|_\infty (t-s) \\
&\le c_p \|\sigma\|_\infty \sqrt{t-s} + \|b\|_\infty (t-s) \\
&\le (c_p\|\sigma\|_\infty + \sqrt{T}\|b\|_\infty)\sqrt{t-s},
\end{aligned}$$

where have used again the Burkholder-Davis-Gundy inequality. □

Chapter 3

Convergence Rate of the Euler Scheme

In this chapter the strong and weak convergence of the Euler scheme for a one-dimensional diffusion processes is studied. The results for the weak convergence go back to articles of Bally and Talay [1, 2]. Let the process $(X_t(x))_{t\in[0,T]}$, which takes values in $\mathbb{R}$, be the unique strong solution of

$$X_t(x) = x + \int_0^t b(X_s(x))ds + \int_0^t \sigma(X_s(x))dW_s, \tag{3.1}$$

where $(W_t)_{t\in[0,T]}$ is a 1-dimensional Brownian motion.

We start by defining the Euler scheme of a diffusion process. Then we study the strong and weak convergence of the Euler scheme.

Definition 3.1. Let $(X_t(x))_{t\in[0,T]}$ be the solution of (3.1). The *Euler scheme* for $(X_t(x))_{t\in[0,T]}$ with step-size $\frac{T}{n}$ and $T > 0$ is defined by

$$\begin{aligned} X^n_{(p+1)T/n}(x) \quad &:= \quad X^n_{pT/n}(x) + b\big(X^n_{pT/n}(x)\big)\frac{T}{n} \\ &\qquad + \sigma\big(X^n_{pT/n}(x)\big)\big(W_{(p+1)T/n} - W_{pT/n}\big) \end{aligned}$$

and, for $\frac{pT}{n} \leq t < \frac{(p+1)T}{n}$,

$$\begin{aligned} X^n_t(x) \quad &:= \quad X^n_{pT/n}(x) + b\big(X^n_{pT/n}(x)\big)(t - \frac{pT}{n}) \\ &\qquad + \sigma\big(X^n_{pT/n}(x)\big)\big(W_t - W_{pT/n}\big), \end{aligned}$$

where $p = 0, \ldots, n-1$.

3.1 Strong Convergence

Here we represent the strong convergence of the Euler scheme based on the book of Gard, [5]. Because we are considering time homogeneous processes, the following Proposition will slightly differ from [5].

Proposition 3.2. *Let $b, \sigma \in \mathcal{D}$. Then we have*

$$\mathbb{E}\big|X_{kT/n}(x) - X^n_{kT/n}(x)\big|^2 \leq \frac{c^2}{n}(1 + |x|^2)$$

for some constant $c > 0$ and $k = 0, \ldots, n$.

Proof. From our assumptions on b and σ we have that there is a constant K such that for all $s, t \in [0, T]$, $x, y \in \mathbb{R}$,

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y| \tag{3.2}$$

and

$$|b(x)|^2 + |\sigma(x)|^2 \leq K^2(1 + |x|^2).$$

Let us denote $\frac{kT}{n}$ by t_k, $X_{t_k} - X^n_{t_k}$ by D_{t_k} and $\mathbb{E}D^2_{t_k}$ by γ_k. We obtain

$$\begin{aligned}
D_{t_k} &= X_{t_k}(x) - X^n_{t_k}(x) \\
&= X_0 + \int_0^{t_k} b(X_s(x))ds + \int_0^{t_k} \sigma(X_s(x))dW_s \\
&\quad -X^n_{t_{k-1}}(x) - \int_{t_{k-1}}^{t_k} b\big(X^n_{t_{k-1}}(x)\big)ds - \int_{t_{k-1}}^{t_k} \sigma\big(X^n_{t_{k-1}}(x)\big)dW_s \\
&= X_0 + \int_0^{t_{k-1}} b(X_s(x))ds + \int_0^{t_{k-1}} \sigma(X_s(x))dW_s \\
&\quad + \int_{t_{k-1}}^{t_k} b(X_s(x))ds + \int_{t_{k-1}}^{t_k} \sigma(X_s(x))dW_s \\
&\quad -X^n_{t_{k-1}}(x) - \int_{t_{k-1}}^{t_k} b\big(X^n_{t_{k-1}}(x)\big)ds - \int_{t_{k-1}}^{t_k} \sigma\big(X^n_{t_{k-1}}(x)\big)dW_s \\
&= D_{t_{k-1}} + \int_{t_{k-1}}^{t_k} b(X_s(x)) - b\big(X^n_{t_{k-1}}(x)\big)ds \\
&\quad + \int_{t_{k-1}}^{t_k} \sigma(X_s(x)) - \sigma\big(X^n_{t_{k-1}}(x)\big)dW_s \qquad \text{a.s.}
\end{aligned}$$

Now we apply Itô's formula to $f(t, x) = x^2$, $b_s := b(X_s(x)) - b(X^n_{t_{k-1}}(x))$ and $\sigma_s := \sigma(X_s(x)) - \sigma(X^n_{t_{k-1}}(x))$ and get that

$$
\begin{aligned}
D^2_{t_k} &= f(t_k, D_{t_k}) \\
&= f(t_{k-1}, D_{t_{k-1}}) + \int_{t_{k-1}}^{t_k} 2D_s\sigma_s dW_s + \int_{t_{k-1}}^{t_k} (2D_s b_s + \sigma_s^2) ds \\
&= D^2_{t_{k-1}} + \int_{t_{k-1}}^{t_k} 2D_s\sigma_s dW_s + \int_{t_{k-1}}^{t_k} (2D_s b_s + \sigma_s^2) ds \qquad \text{a.s.}
\end{aligned}
$$

and, by taking expected value on both sides,

$$
\begin{aligned}
\gamma_k &= \gamma_{k-1} + \int_{t_{k-1}}^{t_k} \mathbb{E}\Big[2\big(X_s(x) - X_s^n(x)\big)\big[b(X_s(x)) - b(X^n_{t_{k-1}}(x))\big] \\
&\quad + \big[\sigma(X_s(x)) - \sigma(X^n_{t_{k-1}}(x))\big]^2\Big] ds \\
&\leq \gamma_{k-1} + \int_{t_{k-1}}^{t_k} \mathbb{E}\Big[\big|X_s(x) - X_s^n(x)\big|^2 + \big|b(X_s(x)) - b(X^n_{t_{k-1}}(x))\big|^2 \\
&\quad + \big|\sigma(X_s(x)) - \sigma(X^n_{t_{k-1}}(x))\big|^2\Big] ds, \qquad (3.3)
\end{aligned}
$$

where the stochastic integral disappears because we have that

$$
\int_{t_{k-1}}^{t_k} \mathbb{E} D_s^2 \sigma_s^2 ds < \infty.
$$

By (3.2) we get that

$$
\begin{aligned}
&\big|b(X_s(x)) - b(X^n_{t_{k-1}}(x))\big|^2 \\
&\quad \leq 2\Big[\big|b(X_s(x)) - b(X_{t_{k-1}}(x))\big|^2 + \big|b(X_{t_{k-1}}(x)) - b(X^n_{t_{k-1}}(x))\big|^2\Big] \\
&\quad \leq 2K^2\Big(\big|X_s(x) - X_{t_{k-1}}(x)\big|^2 + \big|X_{t_{k-1}}(x) - X^n_{t_{k-1}}(x)\big|^2\Big), \qquad (3.4)
\end{aligned}
$$

and the same for σ. From Proposition 2.27 we get that there is a constant K_1 depending on b, σ and T only, such that

$$
\mathbb{E}\big|X_s(x) - X_{t_{k-1}}(x)\big|^2 \leq K_1(s - t_{k-1})(1 + |x|^2). \qquad (3.5)
$$

To shorten the notation we let $K_2 := K_1(1 + |x|^2)$. Applying (3.4) and (3.5) to (3.3) we get,

$$\begin{aligned}
\gamma_k &\leq \gamma_{k-1} + \int_{t_{k-1}}^{t_k} \mathbb{E}\Big[\big|X_s(x) - X_s^n(x)\big|^2 \\
&\quad +4K^2\Big(\big|X_s(x) - X_{t_{k-1}}(x)\big|^2 + \big|X_{t_{k-1}}(x) - X_{t_{k-1}}^n(x)\big|^2\Big)\Big]ds \\
&\leq \gamma_{k-1} + \int_{t_{k-1}}^{t_k} \mathbb{E}\big|X_s(x) - X_s^n(x)\big|^2 ds \\
&\quad + \int_{t_{k-1}}^{t_k} 4K^2\big(K_2(s - t_{k-1}) + \gamma_{k-1}\big)ds \\
&= \gamma_{k-1} + \int_{t_{k-1}}^{t_k} \mathbb{E}\big|X_s(x) - X_s^n(x)\big|^2 ds + 2K^2K_2h^2 + 4K^2\gamma_{k-1}h \\
&= \gamma_{k-1}(1 + 4K^2h) + 2K^2K_2h^2 + \int_{t_{k-1}}^{t_k} \mathbb{E}\big|X_s(x) - X_s^n(x)\big|^2 ds,
\end{aligned}$$

where h is the time increment with constant length of T/n. From the previous computation we have that

$$\mathbb{E}\big|X_{t_k}(x) - X_{t_k}^n(x)\big|^2 \leq \alpha + \int_{t_{k-1}}^{t_k} \mathbb{E}\big|X_s(x) - X_s^n(x)\big|^2 ds$$

for $\alpha := \gamma_{n-1}(1 + 4K^2h) + 2K^2K_2h^2$. Following the same proof we also have

$$\mathbb{E}|X_t(x) - X_t^n(x)|^2 \leq \alpha + \int_{t_{k-1}}^{t} \mathbb{E}|X_s(x) - X_s^n(x)|^2 ds$$

for $t \in [t_{k-1}, t_k]$. Letting

$$\psi(t) := \mathbb{E}|X_t(x) - X_t^n(x)|^2,$$

we obtain a continuous function $\psi : [t_{k-1}, t_k] \to [0, \infty)$ with

$$\psi(t) \leq \alpha + \int_{t_{k-1}}^{t} \psi(s)ds$$

for all $t \in [t_{k-1}, t_k]$. We obtain the continuity of ψ by

$$\begin{aligned}
|\psi(t)^{1/2} - \psi(s)^{1/2}| &= \big|\|X_t(x) - X_t^n(x)\|_{L_2} - \|X_s(x) - X_s^n(x)\|_{L_2}\big| \\
&\leq \|(X_t(x) - X_s(x)) - (X_t^n(x) - X_s^n(x))\|_{L_2} \\
&\leq \|X_t(x) - X_s(x)\|_{L_2} + \|X_t^n(x) - X_s^n(x)\|_{L_2} \\
&= \|A + B\|_{L_2} + \|A^n + B^n\|_{L_2} \\
&\leq \|A\|_{L_2} + \|B\|_{L_2} + \|A^n\|_{L_2} + \|B^n\|_{L_2}
\end{aligned}$$

where $A := \int_s^t b(X_u(x))du$, $B := \int_s^t \sigma(X_u(x))dW_u$, $A^n := \int_s^t b^n(u)du$ and $B^n := \int_s^t \sigma^n(u)dW_u$ with

$$b^n(u) = \sum_{k=1}^{n} b(X_{(k-1)T/n^n(x)})\mathbb{I}_{((k-1)T/n,kT/n]}(u)$$

and

$$\sigma^n(u) = \sum_{k=1}^{n} \sigma(X_{(k-1)T/n^n(x)})\mathbb{I}_{((k-1)T/n,kT/n]}(u).$$

Now we have that $|b^n| \leq \|b\|_\infty$, $|\sigma^n| \leq \|\sigma\|_\infty$ and

$$\begin{aligned}
\|A\|_{L_2} &\leq (t-s)\|b\|_\infty, \\
\|A^n\|_{L_2} &\leq (t-s)\|b\|_\infty, \\
\|B\|_{L_2} &\leq \Big(\mathbb{E}\int_s^t |\sigma(X_u(x))|^2 du\Big)^{1/2} \leq (t-s)^{1/2}\|\sigma\|_\infty, \\
\|B^n\|_{L_2} &\leq \Big(\mathbb{E}\int_s^t |\sigma^n(u)|^2 du\Big)^{1/2} \leq (t-s)^{1/2}\|\sigma\|_\infty.
\end{aligned}$$

Finally,

$$|\psi(t)^{1/2} - \psi(s)^{1/2}| \leq 2[(t-s)\|b\|_\infty + \sqrt{t-s}\|\sigma\|_\infty].$$

Applying Lemma 2.25 gives that

$$\psi(t) \leq \alpha e^h. \tag{3.6}$$

From the initial condition $\gamma_0 = 0$ and iterating (3.6) one gets that,

$$\gamma_n \leq 2K^2 K_2 h^2 e^h \Big(\frac{1-r^n}{1-r}\Big), \tag{3.7}$$

where $r = (1+4K^2h)e^h$ and $r > 1$. In detail, let us denote $1+4K^2h$ by Γ and $2K^2K_2h^2$ by Υ so that one gets from (3.6) that

$$\gamma_n \leq (\gamma_{n-1}\Gamma + \Upsilon)e^h.$$

Using the same inequality for $\gamma_{n-1}, \ldots, \gamma_1$ one gets that

$$\begin{aligned}
\gamma_n &\le \big[(\gamma_{n-2}\Gamma + \Upsilon)e^h\Gamma + \Upsilon\big]e^h \\
&= \gamma_{n-2}\Gamma^2 e^{2h} + \Upsilon\Gamma e^{2h} + \Upsilon e^h \\
&\le (\gamma_{n-3}\Gamma + \Upsilon)e^h\Gamma^2 e^{2h} + \Upsilon\Gamma e^{2h} + \Upsilon e^h \\
&= \gamma_{n-3}\Gamma^3 e^{3h} + \Upsilon\Gamma^2 e^{3h} + \Upsilon\Gamma e^{2h} + \Upsilon e^h \\
&\;\;\vdots \\
&\le \gamma_0\Gamma^n e^{nh} + \Upsilon\Gamma^{n-1}e^{nh} + \ldots + \Upsilon\Gamma e^{2h} + \Upsilon e^h \\
&= \Upsilon e^h\big[(\Gamma e^h)^{n-1} + \ldots + (\Gamma e^h)^2 + \Gamma e^h + 1\big] \\
&= \Upsilon e^h \frac{1-(\Gamma e^h)^n}{1-\Gamma e^h},
\end{aligned}$$

which proves (3.7). Now the right-hand side of (3.7) can be arranged as

$$2K^2K_2(1 - r^{\frac{T}{h}})\Big(\frac{he^h}{1-r}\Big)h, \tag{3.8}$$

where $h = T/n$. When $h \to 0$, $r^{\frac{1}{h}} = (1 + 4K^2h)^{\frac{1}{h}}e \to e^{4K^2+1}$, and by L'Hôspital's rule,

$$\frac{he^h}{1-r} = \frac{h}{e^{-h} - (1 + 4K^2h)} \longrightarrow \frac{-1}{1+4K^2}.$$

This implies that (3.8) divided by h converges to

$$2K^2K_2\Big[\frac{e^{(4K^2+1)T} - 1}{4K^2+1}\Big],$$

as $h \to 0$. Now we can conclude that,

$$\gamma_n = \mathcal{O}(h).$$

□

Recent results about convergence properties of approximation schemes for stochastic differential equations based on the notion of complexity can be found in Müller-Gronbach, [8, 9].

3.2 Weak Convergence

Before we formulate the main result of this chapter in Proposition 3.3 we introduce some notation. Let us denote by $\mathcal{L}$ the second-order differential operator

$$\mathcal{L} := b(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}.$$

Let f be a bounded and measurable function and $u(t,x) := \mathbb{E}f(X_{T-t}(x))$ for $(t,x) \in [0,T] \times \mathbb{R}$. By Proposition 2.22 we have

$$\frac{\partial u}{\partial t} + \mathcal{L}u = 0 \text{ on } [0,T) \times \mathbb{R} \text{ and } u(T,\cdot) = f(\cdot). \tag{3.9}$$

Let $\Psi(t,x)$ be defined by

$$\begin{aligned}\Psi(t,x) \quad := \quad & \frac{1}{2}b^2(x)\frac{\partial^2 u}{\partial x^2}(t,x) + \frac{1}{2}b(x)\sigma^2(x)\frac{\partial^3 u}{\partial x^3}(t,x) \\ & + \frac{1}{8}\sigma^4(x)\frac{\partial^4 u}{\partial x^4}(t,x) + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}(t,x) \\ & + b(x)\frac{\partial^2 u}{\partial t \partial x}(t,x) + \frac{1}{2}\sigma^2(x)\frac{\partial^3 u}{\partial t \partial x^2}(t,x),\end{aligned}$$

where $(t,x) \in [0,T) \times \mathbb{R}$.

Proposition 3.3. *Let $b, \sigma \in \mathcal{D}$. If f is a measurable and bounded function, then the error of the Euler scheme satisfies*

$$|\mathbb{E}f(X_T^n(x)) - \mathbb{E}f(X_T(x))| \leq c\|f\|_\infty(1 + |x|^Q)\frac{1}{n}.$$

where $c, Q > 0$ depend at most on b, σ and T.

The following lemmas are needed to prove Proposition 3.3. Lemma 3.5 is not proven in this thesis, but the proof can be found in [1].

Lemma 3.4. *Let u be a function as in (3.9). Then, for any smooth function g such that each derivative is of polynomial growth and $\alpha \in \{0,1,2,\ldots\}$ there exist $K, Q > 0$ depending on g, b, σ, α and T such that*

$$\left|\mathbb{E}\Big(g(X_t(x))\partial_\alpha^x u(t, X_t(x))\Big)\right| \leq K\|f\|_\infty(1 + |x|^Q) \text{ for all } t \in [0,T), \tag{3.10}$$

and

$$\left|\mathbb{E}\Big(g(X_t^n(x))\partial_\alpha^x u(t, X_t^n(x))\Big)\right| \leq K\|f\|_\infty(1 + |x|^Q) \text{ for all } t \in \big[0, T - \frac{T}{n}\big]. \tag{3.11}$$

Proof of Lemma 3.4. We prove (3.10) and (3.11) only for $0 \leq t \leq T/2$. First, remind that

$$u(t,x) = \int_{\mathbb{R}} p_{T-t}(x,y)f(y)dy$$

with $p_s(x,y) = \Gamma(s,x,y)$ where Γ is the transition density as in Proposition 2.23. By taking partial derivative with respect to x, we get

$$\partial_\alpha^x u(t,x) = \int_{\mathbb{R}} \partial_\alpha^x p_{T-t}(x,y) f(y) dy.$$

Using the estimate from Proposition 2.23 we get

$$\begin{aligned} |\partial_\alpha^x u(t,x)| &\le \int_{\mathbb{R}} |\partial_\alpha^x p_{T-t}(x,y)| |f(y)| dy \\ &\le \frac{c}{(T-t)^{\alpha/2}} \|f\|_\infty \int_{\mathbb{R}} \gamma_{cs}(x-y) dy \\ &= \frac{c}{(T-t)^{\alpha/2}} \|f\|_\infty. \end{aligned}$$

Consequently, for some $Q \ge 1$,

$$\begin{aligned} |\mathbb{E} g(X_t(x)) \partial_\alpha^x u(t, X_t(x))| &\le \mathbb{E}|g(X_t(x))| \frac{c}{(T-t)^{\alpha/2}} \|f\|_\infty \\ &\le c' \mathbb{E}(1 + |X_t(x)|^Q) \frac{c}{(T-t)^{\alpha/2}} \|f\|_\infty \\ &= (c' + c' \|X_t(x)\|_Q^Q) \frac{c}{(T-t)^{\alpha/2}} \|f\|_\infty \\ &\le c''(1 + |x|^Q) \frac{1}{(T-t)^{\alpha/2}} \|f\|_\infty, \end{aligned}$$

where we used Proposition 2.26. Because of

$$\frac{1}{T-t} \le \frac{2}{T} \quad \text{for } 0 \le t \le T/2$$

we obtain (3.10) for those t. Let us turn to (3.11). Here we get in the same way

$$\begin{aligned} |\mathbb{E} g(X_t(x)) \partial_\alpha^x u(t, X_t(x))| &\le (c' + c' \|X_t^n(x)\|_Q^Q) \frac{c}{(T-t)^{\alpha/2}} \|f\|_\infty \\ &\le \left[c' + c'(\|X_t(x)\|_Q + \|X_t^n(x) - X_t(x)\|_Q)^Q\right] \\ &\quad \times \frac{c}{(T-t)^{\alpha/2}} \|f\|_\infty. \end{aligned}$$

Now we can finish the proof by Proposition 3.2 and, again, Proposition 2.26. □

Lemma 3.5 (Bally and Talay, [1]). *Under the assumptions of the Proposition 3.3, for some integer Q and some non decreasing function $K(T)$, one has that*

$$\left| \mathbb{E} f(X_T^n(x)) - \mathbb{E}\left[P_{T/n} f(X_{T-T/n}^n(x))\right] \right| \le \frac{K(T)}{n^2} \|f\|_\infty (1 + |x|^Q)$$

with $u(t,\cdot) = P_{T-t} f(\cdot) = \mathbb{E} f(X_{T-t}(\cdot))$.

The following Lemma is the key point in the proof of Proposition 3.3. For the proof of Proposition 3.3 we need only to find an upper bound for the terms of the expansion given below.

Lemma 3.6. *It holds that*

$$\begin{aligned}\mathbb{E}f(X_T^n(x)) - \mathbb{E}f(X_T(x)) &= \frac{T^2}{n^2}\sum_{k=0}^{n-2}\mathbb{E}\Psi\Big(\frac{kT}{n}, X_{kT/n}^n(x)\Big) \\ &\quad + \mathbb{E}f(X_T^n(x)) - \mathbb{E}\Big[P_{T/n}f(X_{T-T/n}^n(x))\Big] \\ &\quad + \sum_{k=0}^{n-2} I_k^n, \end{aligned} \tag{3.12}$$

where

$$I_k^n = \mathbb{E}\int_{\frac{kT}{n}}^{\frac{(k+1)T}{n}}\int_{\frac{kT}{n}}^{t}\int_{\frac{kT}{n}}^{v}\vartheta_k^n(s)dsdvdt$$

and $\vartheta_k^n(s)$ *is a sum of differentials* $\frac{\partial^\alpha}{\partial x^\alpha}u(s, X_s^n)$ *up to order* 6 *multiplied by products of* $b(X_{kT/n}^n)$ *and* $\sigma(X_{kT/n}^n)$.

Proof. One defines the differential operator $\mathcal{L}_z$ by

$$\mathcal{L}_z g(\cdot) := b(z)\frac{\partial g}{\partial x}(\cdot) + \frac{1}{2}\sigma^2(z)\frac{\partial^2 g}{\partial x^2}(\cdot)$$

where $z \in \mathbb{R}$ is being fixed. Let u be defined by $\mathbb{E}f(X_{T-t}(x))$. This implies that

$$\mathbb{E}f(X_T^n(x) - \mathbb{E}f(X_T(x)) = \mathbb{E}u(T, X_T^n(x)) - u(0,x) = \sum_{k=0}^{n-1}\delta_k^n,$$

with

$$\delta_k^n := \mathbb{E}\Big[u\Big(\frac{(k+1)T}{n}, X_{(k+1)T/n}^n(x)\Big) - u\Big(\frac{kT}{n}, X_{kT/n}^n(x)\Big)\Big].$$

For $u(s, X_s^n(x))$ with $s \in (t_k^n, t_{k+1}^n]$ and $k = 0, \ldots, n-1$ Itô's formula gives, $\mathbb{P}$-a.s.,

$$\begin{aligned} u(s, X_s^n(x)) &= u(t_k^n, X_{t_k^n}^n(x)) + \int_{t_k^n}^s \frac{\partial u}{\partial t}(t, X_t^n(x))dt \\ &\quad + \int_{t_k^n}^s \frac{\partial u}{\partial x}(t, X_t^n(x))\sigma(X_{t_k^n}^n)dW_t + \int_{t_k^n}^s \frac{\partial u}{\partial x}(t, X_t^n(x))b(X_{t_k^n}^n)dt \\ &\quad + \frac{1}{2}\int_{t_k^n}^s \frac{\partial^2 u}{\partial x^2}(t, X_t^n(x))\sigma^2(X_{t_k^n}^n)dt. \end{aligned}$$

Then, for $z = X^n_{kT/n}$, $t^n_k = kT/n$ and $s = (k+1)T/n$, one has that

$$\begin{aligned}\delta^n_k &= \mathbb{E}\int_{kT/n}^{(k+1)T/n}\Big(\frac{\partial u}{\partial t}(t, X^n_t(x)) + \frac{\partial u}{\partial x}(t, X^n_t(x))b(z)\\ &\quad +\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t, X^n_t(x))\sigma^2(z)\Big)dt\\ &= \mathbb{E}\int_{kT/n}^{(k+1)T/n}\Big(\frac{\partial u}{\partial t}(t, X^n_t(x)) + \mathcal{L}_z u(t, X^n_t(x))\Big)dt.\end{aligned}$$

Let us define

$$\widehat{u}(t, X^n_t(x)) := \frac{\partial u}{\partial t}(t, X^n_t(x)) + \mathcal{L}_z u(t, X^n_t(x)).$$

Now we apply Itô's formula to $\widehat{u}$ which gives that, almost surely,

$$\begin{aligned}\widehat{u}(t, X^n_t(x)) &= \widehat{u}\Big(\frac{kT}{n}, X^n_{kT/n}(x)\Big) + \int_{kT/n}^{t}\bigg(\frac{\partial \widehat{u}}{\partial v}(v, X^n_v(x))\\ &\quad +\frac{\partial \widehat{u}}{\partial x}(v, X^n_v(x))b(z) + \frac{1}{2}\frac{\partial^2 \widehat{u}}{\partial x^2}(v, X^n_v(x))\sigma^2(z)\bigg)dv\\ &\quad +\int_{kT/n}^{t}\frac{\partial \widehat{u}}{\partial x}(v, X^n_v(x))\sigma(z)dW_v\\ &= \frac{\partial u}{\partial t}\Big(\frac{kT}{n}, X^n_{kT/n}(x)\Big) + \mathcal{L}_z u\Big(\frac{kT}{n}, X^n_{kT/n}(x)\Big)\\ &\quad +\int_{kT/n}^{t}\Big(\frac{\partial}{\partial v} + \mathcal{L}_z\Big)\frac{\partial u}{\partial v}\big(v, X^n_v(x)\big)\\ &\quad +\Big(\frac{\partial}{\partial v} + \mathcal{L}_z\Big)\mathcal{L}_z u\big(v, X^n_v(x)\big)dv\\ &\quad +\int_{kT/n}^{t}\frac{\partial \widehat{u}}{\partial x}(v, X^n_v(x))\sigma(z)dW_v,\end{aligned}$$

where $t \in (\frac{kT}{n}, \frac{(k+1)T}{n}]$ and, by taking the expected value one gets that

$$\begin{aligned}\delta^n_k &= \mathbb{E}\int_{kT/n}^{(k+1)T/n}\int_{kT/n}^{t}\Big(\frac{\partial}{\partial v} + \mathcal{L}_z\Big)\frac{\partial u}{\partial v}\big(v, X^n_v(x)\big)\\ &\quad +\Big(\frac{\partial}{\partial v} + \mathcal{L}_z\Big)\mathcal{L}_z u\big(v, X^n_v(x)\big)dvdt\end{aligned}$$

because

$$\frac{\partial u}{\partial t}\Big(\frac{kT}{n}, X^n_{kT/n}(x)\Big) + \mathcal{L}_z u\Big(\frac{kT}{n}, X^n_{kT/n}(x)\Big) = 0.$$

The integrands gets the following form:

$$\begin{aligned}&\Big(\frac{\partial}{\partial v}+\mathcal{L}_z\Big)\frac{\partial u}{\partial v}\big(v,X_v^n(x)\big)\\&=\Big(\frac{\partial}{\partial v}+b(z)\frac{\partial}{\partial x}+\frac{1}{2}\sigma^2(z)\frac{\partial^2}{\partial x^2}\Big)\frac{\partial u}{\partial v}\big(v,X_v^n(x)\big)\\&=\Big(\frac{\partial^2 u}{\partial v^2}+b(z)\frac{\partial^2 u}{\partial v\partial x}+\frac{1}{2}\sigma^2(z)\frac{\partial^3 u}{\partial v\partial x^2}\Big)\big(v,X_v^n(x)\big)\end{aligned}$$

and

$$\begin{aligned}&\Big(\frac{\partial}{\partial v}+\mathcal{L}_z\Big)\mathcal{L}_z u\big(v,X_v^n(x)\big)\\&=\Big[b(z)\frac{\partial^2}{\partial v\partial x}+\frac{1}{2}\sigma^2(z)\frac{\partial^3}{\partial v\partial x^2}+\mathcal{L}_z\mathcal{L}_z\Big]u\big(v,X_v^n(x)\big),\end{aligned}$$

where

$$\begin{aligned}&\mathcal{L}_z\mathcal{L}_z u\big(v,X_v^n(x)\big)\\&=\mathcal{L}_z\Big(b(z)\frac{\partial}{\partial x}+\frac{1}{2}\sigma^2(z)\frac{\partial^2}{\partial x^2}\Big)u\big(v,X_v^n(x)\big)\\&=\Big[b(z)\Big(b(z)\frac{\partial^2}{\partial x^2}+\frac{1}{2}\sigma^2(z)\frac{\partial^3}{\partial x^3}\Big)\\&\quad+\frac{1}{2}\sigma^2(z)\Big(b(z)\frac{\partial^3}{\partial^3 x}+\frac{1}{2}\sigma^2(z)\frac{\partial^4}{\partial x^4}\Big)\Big]u\big(v,X_v^n(x)\big)\\&=\Big(b^2(z)\frac{\partial^2}{\partial x^2}+b(z)\sigma^2(z)\frac{\partial^3}{\partial x^3}+\frac{1}{4}\sigma^4(z)\frac{\partial^4}{\partial x^4}\Big)u\big(v,X_v^n(x)\big).\end{aligned}$$

Combining the above equations one gets that

$$\begin{aligned}&\Big(\frac{\partial}{\partial v}+\mathcal{L}_z\Big)\frac{\partial u}{\partial v}\big(v,X_v^n(x)\big)+\Big(\frac{\partial}{\partial v}+\mathcal{L}_z\Big)\mathcal{L}_z u\big(v,X_v^n(x)\big)\\&=\Big(b^2(z)\frac{\partial^2 u}{\partial x^2}+b(z)\sigma^2(z)\frac{\partial^3 u}{\partial x^3}+\frac{1}{4}\sigma^4(z)\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial v^2}+2b(z)\frac{\partial^2 u}{\partial v\partial x}\\&\quad+\sigma^2(z)\frac{\partial^3 u}{\partial v\partial x^2}\Big)(v,X_v^n(x)).\end{aligned}$$

Then apply Itô's formula to all partial derivatives of u. For $\frac{\partial^2 u}{\partial v^2}$ one has

$$\begin{aligned}\frac{\partial^2 u}{\partial v^2}(v,X_v^n(x)) &= \frac{\partial^2 u}{\partial v^2}\Big(\frac{kT}{n},X_{kT/n}^n(x)\Big)+\int_{kT/n}^{v}\Big[\frac{\partial^3 u}{\partial v^2\partial s}(s,X_s^n(x))\\&\quad+\frac{\partial^3 u}{\partial v^2\partial x}(s,X_s^n(x))b(z)+\frac{1}{2}\frac{\partial^4 u}{\partial v^2\partial x^2}(s,X_s^n(x))\sigma^2(z)\Big]ds\\&\quad+\int_{kT/n}^{v}\frac{\partial^3 u}{\partial v^2\partial x}(s,X_s^n(x))\sigma(z)dW_s.\end{aligned}$$

The calculations of other partial derivatives are similar and they are not presented here. Finally we collect the first terms of Itô's formula and get that

$$\begin{aligned}
\delta_k^n &= \mathbb{E}\int_{kT/n}^{(k+1)T/n}\int_{kT/n}^{t}\Big(b^2(z)\frac{\partial^2 u}{\partial x^2}+b(z)\sigma^2(z)\frac{\partial^3 u}{\partial x^3}+\frac{1}{4}\sigma^4(z)\frac{\partial^4 u}{\partial x^4}\\
&\quad+\frac{\partial^2 u}{\partial v^2}+2b(z)\frac{\partial^2 u}{\partial v\partial x}+\sigma^2(z)\frac{\partial^3 u}{\partial v\partial x^2}\Big)\Big(\frac{kT}{n},X_{kT/n}^n(x)\Big)dvdt\\
&\quad+\mathbb{E}\int_{kT/n}^{(k+1)T/n}\int_{kT/n}^{t}\int_{kT/n}^{v}\vartheta_k^n(s)dsdvdt\\
&= \mathbb{E}\int_{kT/n}^{(k+1)T/n}\int_{kT/n}^{t}2\Psi\Big(\frac{kT}{n},X_{kT/n}^n(x)\Big)dvdt+I_k^n\\
&= \frac{T^2}{n^2}\mathbb{E}\Psi\Big(\frac{kT}{n},X_{kT/n}^n(x)\Big)+I_k^n.
\end{aligned}$$

Since $u(t,\cdot)=P_{T-t}f(\cdot)=\mathbb{E}f(X_{T-t}(\cdot))$, one has that

$$\begin{aligned}
\delta_{n-1}^n &= \mathbb{E}\Big[u\big(T,X_T^n(x)\big)-u\big(T-T/n,X_{T-T/n}^n(x)\big)\Big]\\
&= \mathbb{E}f(X_T^n(x))-\mathbb{E}\Big[P_{T/n}f(X_{T-T/n}^n(x))\Big]
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathbb{E}f(X_T^n(x))-\mathbb{E}f(X_T(x)) &= \delta_{n-1}^n+\sum_{k=0}^{n-2}\delta_k^n\\
&= \mathbb{E}f(X_T^n(x))-\mathbb{E}\Big[P_{T/n}f(X_{T-T/n}^n(x))\Big]\\
&\quad+\frac{T^2}{n^2}\sum_{k=0}^{n-2}\mathbb{E}\Psi\Big(\frac{kT}{n},X_{kT/n}^n(x)\Big)+\sum_{k=0}^{n-2}I_k^n.
\end{aligned}$$

□

Proof of Proposition 3.3. The expansion given in Lemma 3.6 gives that

$$\begin{aligned}\left|\mathbb{E}f(X_T^n(x)) - \mathbb{E}f(X_T(x))\right| &\leq \frac{T^2}{n^2}\sum_{k=0}^{n-2}\left|\mathbb{E}\Psi\Big(\frac{kT}{n}, X_{kT/n}^n(x)\Big)\right| \\ &\quad + \left|\mathbb{E}f(X_T^n(x)) - \mathbb{E}\Big[P_{T/n}f(X_{T-T/n}^n(x))\Big]\right| \\ &\quad + \sum_{k=0}^{n-2}|I_k^n| \\ &\leq \frac{1}{n}\Big(T^2 \sup_{k=0,\ldots,n-2}\left|\mathbb{E}\Psi\Big(\frac{kT}{n}, X_{kT/n}^n(x)\Big)\right| \\ &\quad + \sup_{k=0,\ldots,n-2} n^2\mathbb{E}|I_k^n|\Big) \\ &\quad + \left|\mathbb{E}f(X_T^n(x)) - \mathbb{E}\Big[P_{T/n}f(X_{T-T/n}^n(x))\Big]\right|.\end{aligned}$$

By Lemma 3.5 the last term is bounded by

$$\frac{K(T)}{n^2}\|f\|_\infty(1+|x|^Q).$$

It remains to show that

$$\sup_{n=1,2,\ldots}\sup_{k=0,\ldots,n-2}\left[\left|\mathbb{E}\Psi\Big(\frac{kT}{n}, X_{kT/n}^n(x)\Big)\right| + n^2\mathbb{E}|I_k^n|\right] < \infty.$$

But this follows from Lemma 3.4 and Lemma 3.6. □

The next proposition is a variant of Proposition 3.3 which provides for $n \to \infty$ an explicit constant for Proposition 3.3.

Proposition 3.7. *Let $b, \sigma \in \mathcal{D}$. If f is a measurable and bounded function, then the error of the Euler scheme satisfies*

$$\mathbb{E}f(X_T(x)) - \mathbb{E}f(X_T^n(x)) = -\frac{C_f(T,x)}{n} + \frac{Q_n(f,T,x)}{n^2},$$

where $C_f(T,x) := \int_0^T \mathbb{E}\Psi(s, X_s(x))ds \in \mathbb{R}$ and $Q_n(f,T,x)$ satisfy the following property: there exists $K > 0$ depending on b, σ, and T, and a positive real number Q such that

$$|C_f(T,x)| + \sup_n |Q_n(f,T,x)| \leq K\|f\|_\infty(1+|x|^Q).$$

Proof. The expansion (3.12) can be written as

$$\begin{aligned}
&\mathbb{E}f(X^n_T(x)) - \mathbb{E}f(X_T(x)) \\
&= \frac{T}{n}\int_0^T \mathbb{E}\Psi(s, X_s(x))ds + \frac{T^2}{n^2}\sum_{k=0}^{n-2}\mathbb{E}\Psi\Big(\frac{kT}{n}, X_{kT/n}(x)\Big) \\
&\quad -\frac{T}{n}\int_0^T \mathbb{E}\Psi(s, X_s(x))ds \\
&\quad +\frac{T^2}{n^2}\sum_{k=0}^{n-2}\mathbb{E}\Big[\Psi\Big(\frac{kT}{n}, X^n_{kT/n}(x)\Big) - \Psi\Big(\frac{kT}{n}, X_{kT/n}(x)\Big)\Big] \\
&\quad +\sum_{k=0}^{n-2} I^n_k + \mathbb{E}f(X^n_T(x)) - \mathbb{E}\Big[P_{T/n}f(X^n_{T-T/n}(x))\Big] \\
&= \frac{T}{n}\int_0^T \mathbb{E}\Psi(s, X_s(x))ds - \frac{T}{n^2}Q_n(f, T, x),
\end{aligned}$$

where

$$\begin{aligned}
-Q_n(f,T,x) \;:=\; & T\sum_{k=0}^{n-2}\mathbb{E}\Psi\Big(\frac{kT}{n}, X_{kT/n}(x)\Big) - n\int_0^T \mathbb{E}\Psi(s, X_s(x))ds \\
& +T\sum_{k=0}^{n-2}\mathbb{E}\Big[\Psi\Big(\frac{kT}{n}, X^n_{kT/n}(x)\Big) - \Psi\Big(\frac{kT}{n}, X_{kT/n}(x)\Big)\Big] \\
& +\frac{n^2}{T}\sum_{k=0}^{n-2} I^n_k + \frac{n^2}{T}\Big[\mathbb{E}f(X^n_T(x)) - \mathbb{E}\big[P_{T/n}f(X^n_{T-T/n}(x))\big]\Big].
\end{aligned}$$

Equation (3.10) implies that $\int_0^T \mathbb{E}\Psi(s, X_s(x))ds$ is finite. In order to prove the proposition one can show that

(*i*)

$$\Big|\frac{T}{n}\sum_{k=0}^{n-2}\mathbb{E}\Psi\Big(\frac{kT}{n}, X_{kT/n}(x)\Big) - \int_0^T \mathbb{E}\Psi(s, X_s(x))ds\Big| \leq K\|f\|_\infty(1+|x|^Q)\frac{1}{n}, \tag{3.13}$$

(*ii*)

$$\Big|\mathbb{E}\Psi\Big(\frac{kT}{n}, X_{kT/n}(x)\Big) - \mathbb{E}\Psi\Big(\frac{kT}{n}, X^n_{kT/n}(x)\Big)\Big| \leq K\|f\|_\infty(1+|x|^Q)\frac{1}{n},$$

for $k = 0, \ldots, n-2$,

(*iii*)

$$\Big|\sum_{k=0}^{n-2} I_k^n\Big| \le K\|f\|_\infty(1+|x|^Q)\frac{1}{n^2},$$

(*iv*)

$$\Big|\mathbb{E}f(X_T^n(x)) - \mathbb{E}\big[P_{T/n}f(X_{T-T/n}^n(x))\big]\Big| \le K\|f\|_\infty(1+|x|^Q)\frac{1}{n^2},$$

where $K > 0$ is a constant depending at most on b, σ and T. Inequality (iv) we mentioned in Lemma 3.5. From the remaining inequalities we only show (i). We use Itô's formula and the estimate (3.10). For simplicity we denote $\frac{kT}{n}$ by t_k^n. To estimate (3.13) we start with

$$\begin{aligned}
&\Big|\frac{T}{n}\sum_{k=0}^{n-2}\mathbb{E}\Psi(t_k^n, X_{t_k^n}(x)) - \int_0^T \mathbb{E}\Psi(s, X_s(x))ds\Big| \\
&= \Big|\sum_{k=0}^{n-2}\mathbb{E}\int_{t_k^n}^{t_{k+1}^n}[\Psi(t_k^n, X_{t_k^n}(x)) - \Psi(s, X_s(x))]ds - \int_{t_{n-1}^n}^T \mathbb{E}\Psi(s, X_s(x))ds\Big| \\
&\le \sum_{k=0}^{n-2}\int_{t_k^n}^{t_{k+1}^n}\Big|\mathbb{E}[\Psi(t_k^n, X_{t_k^n}(x)) - \Psi(s, X_s(x))]\Big|ds \\
&\quad + \int_{t_{n-1}^n}^T \Big|\mathbb{E}\Psi(s, X_s(x))\Big|ds. \qquad (3.14)
\end{aligned}$$

Itô's formula gives, almost surely, that

$$\begin{aligned}
\Psi(s, X_s(x)) &= \Psi(0, X_0(x)) + \int_0^s \frac{\partial\Psi}{\partial u}(u, X_u(x))du \\
&\quad + \int_0^s \frac{\partial\Psi}{\partial x}(u, X_u(x))\sigma(X_u(x))dW_u \\
&\quad + \int_0^s \frac{\partial\Psi}{\partial x}(u, X_u(x))b(X_u(x))du \\
&\quad + \frac{1}{2}\int_0^s \frac{\partial^2\Psi}{\partial x^2}(u, X_u(x))\sigma^2(X_u(x))du
\end{aligned}$$

and by taking the expected value we have

$$\mathbb{E}\Psi(s, X_s(x)) = \mathbb{E}\Psi(0, X_0(x)) + \mathbb{E}\int_0^s \hat{\Psi}(u, X_u(x))du,$$

where

$$\begin{aligned}
\hat{\Psi}(u, X_u(x)) &:= \frac{\partial\Psi}{\partial u}(u, X_u(x)) + \frac{\partial\Psi}{\partial x}(u, X_u(x))b(X_u(x)) \\
&\quad + \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}(u, X_u(x))\sigma^2(X_u(x)).
\end{aligned}$$

For $s = t_k^n$ we have

$$\mathbb{E}\Psi(t_k^n, X_{t_k^n}(x)) = \mathbb{E}\Psi(0, X_0(x)) + \mathbb{E}\int_0^{t_k^n} \hat{\Psi}(u, X_u(x))du.$$

Now the difference between above expectations can be written as

$$\mathbb{E}[\Psi(X_s(x), s) - \Psi(t_k^n, X_{t_k^n}(x))] = \int_{t_k^n}^{s} \mathbb{E}\hat{\Psi}(u, X_u(x))du.$$

Using the above difference in (3.14) we have

$$\begin{aligned}
&\Big|\frac{T}{n}\sum_{k=0}^{n-2} \mathbb{E}\Psi(t_k^n, X_{t_k^n}(x)) - \int_0^T \mathbb{E}\Psi(s, X_s(x))ds\Big| \\
&\leq \sum_{k=0}^{n-2}\int_{t_k^n}^{t_{k+1}^n}\int_{t_k^n}^{s} \Big|\mathbb{E}\hat{\Psi}(u, X_u(x))\Big|duds + \int_{t_{n-1}^n}^{T}\Big|\mathbb{E}\Psi(s, X_s(x))\Big|ds.
\end{aligned}$$

By replacing all partial derivatives of the form $\frac{\partial u}{\partial t}$ by $\mathcal{L}u$ in $\hat{\Psi}(t, X_t(x))$, the expansions $\mathbb{E}\hat{\Psi}(t, X_t(x))$ and $\mathbb{E}\Psi(t, X_t(x))$ are of the form treated in Lemma 3.4. Hence

$$\Big|\frac{T}{n}\sum_{k=0}^{n-2} \mathbb{E}\Psi\Big(\frac{kT}{n}, X_{kT/n}\Big) - \int_0^T \mathbb{E}\Psi(s, X_s)ds\Big| \leq C\|f\|_\infty(1 + |x|^Q)\frac{1}{n}.$$

□

3.3 Conclusions

With $b, \sigma \in \mathcal{D}$ and $(X_t^n(x))_{t\in[0,T]}$ being the Euler scheme related to the process $(X_t(x))_{t\in[0,T]}$, we have proved that,

- the strong L_2-error $\sup_{k=0,\ldots,n} \|X_{kT/n}(x) - X_{kT/n}^n(x)\|_{L_2}$ is of order $1/\sqrt{n}$,
- and in contrast to this, the weak error $|\mathbb{E}f(X_T(x)) - \mathbb{E}f(X_T^n(x))|$ is of order $1/n$ even in the case no smoothness assumptions on f are imposed. This better rate of convergence is useful, for example, in Value at Risk (VaR) computations in Stochastic Finance.

Bibliography

[1] Bally, V., Talay, D., *"The Law of the Euler Scheme for Stochastic Differential Equations: I. Convergence Rate of the Distribution Function"*, (INRIA report 2244, 1994).

[2] Bally, V., Talay, D., *"The Law of the Euler Scheme for Stochastic Differential Equations: II. Convergence Rate of the Density"*, (INRIA report 2675, 1995).

[3] Bouleau, N., Lépingle, D., *"Numerical Methods for Stochastic Processes"*, (John Wiley & Sons, Inc., 1994).

[4] Friedman, A., *"Partial Differential Equations of Parabolic Type"*, (Prentice-Hall, 1964).

[5] Gard, Thomas C., *"Introduction to Stochastic Differential Equations"*, (Marcel Dekker, Inc., 1988).

[6] Geiss, C., Geiss, S., *"On Approximation of a Class of Stochastic Integrals and Interpolation"*, (Stochastic and Stochastic Reports, Vol. 76, No. 4, 2004): 339-362.

[7] Karatzas, I., Shreve, S.E., *"Brownian Motion and Stochastic Calculus"*, (Springer-Verlag, 1991).

[8] Müller-Gronbach, T., *"Strong Approximation of Systems of Stochastic Differential Equations"*, (Habilitation thesis, Technische Universität Darmstadt, 2002).

[9] Müller-Gronbach, T., *"The Optimal Uniform Approximation of Systems of Stochastic Differential Equations"*, (Ann. Appl. Probab., Vol. 12, No. 2, 2002): 664-690.

[10] Talay, D., Tubaro, L., *"Expansion of the Global Error for Numerical Schemes Solving Stochastic Differential Equations"*, (INRIA report 1069, 1989).