

Using Copulas in Finding a Lower Bound for Functions of Dependent Risks

Eija Päivinen

A thesis submitted for the degree of Master of Philosophy in Mathematics

Department of Mathematics and Statistics
University of Jyväskylä
22nd March 2006

Contents

1	Introduction	2
2	Copulas	4
2.1	Fréchet bounds	12
2.2	Comonotonicity of random variables	17
2.3	Some properties of copulas	24
3	Lower bound for functions of dependent risks	27
3.1	A lower bound using partial information about dependence . .	28
3.2	A lower bound in the case of no information on dependence . .	40

1 Introduction

In finance theory risk is defined as the dispersion of unexpected outcomes due to movements in financial variables i.e. risk can be considered as a random variable and it is best measured in terms of probability distribution functions.

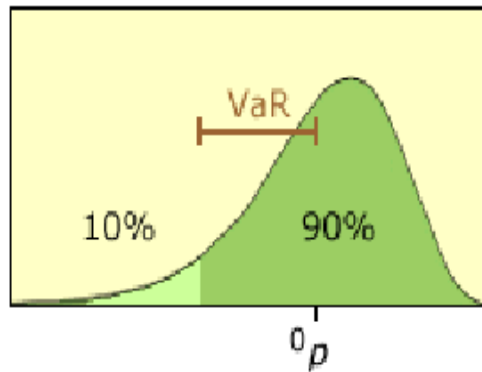
Value-at-Risk (VaR) is a category of risk measures that describe probabilistically the market risk of a trading portfolio. VaR is widely used in banks, securities firms, commodity and energy merchants, and other trading organizations.

We shortly introduce VaR, for more information see [1]. Computing VaR requires first the definition of a period over which to measure unfavorable outcomes. We consider X as the random variable of loss that might occur over a chosen period of time. Let F_X be the distribution function of X , $F_X(x) = \mathbb{P}(X \leq x)$. For $0 \leq \alpha \leq 1$ the Value-at-Risk at probability level α of X is its α -quantile, i.e.

$$\text{VaR}_\alpha(X) := F_X^{-1}(\alpha) := \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\},$$

which is the left-continuous inverse of the distribution function. It means that with probability α the portfolio's value will drop at most $\text{VaR}_\alpha(X)$.

For example, express the time in trading days and let 0 be the current time. We know our portfolio's current market value 0_p , but its market value 1_p after one trading day is unknown. It is a random variable. We might for example report the 90%-quantile of the portfolio's single-period loss $0_p - 1_p$, $\text{VaR}_{0,90}(0_p - 1_p)$.



In the picture it is shown the density function of the portfolio's value 1_p after one trading day. VaR is the 90%-quantile of the portfolio's single-period loss distribution, which is the same as the distribution of $0_p - 1_p$. It means, that with probability 0,9 the portfolio's value after one trading day will be at least $0_p - \text{VaR}$.

The problem is to manage the VaR of a joint position $\psi(X_1, \dots, X_n)$ resulting from the combination of different dependent risks X_1, \dots, X_n . In many

situations only partial or no information at all about the dependence between X_1, \dots, X_n is available i.e. the joint distribution F_{X_1, \dots, X_n} is unknown.

We study the problem of finding the best-possible lower bound on the distribution function of $\psi(X) = \psi(X_1, \dots, X_n)$. It will provide us an upper bound for the VaR. This problem has a long history. The first result was provided by Makarov [2] in response to a question of A.N. Kolmogorov for $n = 2$ and $\psi(X_1, X_2) := X_1 + X_2$. A few years later, in [3], the bounds were proved to be best-possible and to hold in arbitrary dimensions for any continuous non-decreasing ψ . Dependence information was used in [4] in estimating the distribution for $n = 2$ and sharpness of the lower bound was proved for non-decreasing functions $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$. In [3] and [4], the distribution functions are left-continuous.

In this paper we will first study some properties of copulas, that are functions that will later be used in estimating the right-continuous distribution of $\psi(X) = \psi(X_1, \dots, X_n)$. In Section 3 there will be presented a lower bound for the distribution of $\psi(X)$ for an increasing function ψ in arbitrary dimensions. This bound can be tightened in the case that we have some information on dependence. We improve the bound presented in [5], and show that the bound presented in [6] is indeed best-possible in two dimensions when ψ is left-continuous in its second coordinate. For a special case, there will be also presented an improvement to this lower bound in the case nothing is known about the dependence. The theorem was first introduced in [5].

There is also an upper bound. In [6] it is shown that one can provide a best-possible upper bound in arbitrary dimensions when ψ is left-continuous and one can also use information about dependence to tighten the bound.

2 Copulas

The dependence between random variables X_1, \dots, X_n on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is completely determined by their joint distribution function $F(x_1, \dots, x_n) = \mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n)$. We separate F into two parts: a copula, which describes the dependence structure, and the marginal distributions. The word copula originates from the Latin word for connecting two different things. And this is what copulas do; they connect the margins to a joint distribution. To define a copula, we use the definition introduced in [7].

Definition 2.1 A function $C : [0, 1]^n \rightarrow [0, 1]$, is a copula provided that

1. (a) for every $u \in [0, 1]^n$,

$$C(u) = 0 \text{ if at least one coordinate of } u \text{ is } 0,$$

and

- (b)

$$C(u) = u_k \text{ if } u_i = 1 \text{ for all } i \in \{1, \dots, k-1, k+1, \dots, n\}$$

and

2. C is n -increasing, which means that if $a \in [0, 1]^n$ and $b \in [0, 1]^n$ such that $a_i \leq b_i$ for all $i = 1, \dots, n$, then

$$\sum_{j_1=1}^2 \dots \sum_{j_n=1}^2 (-1)^{j_1+\dots+j_n} C(u_{1j_1}, \dots, u_{nj_n}) \geq 0, \quad (1)$$

where $u_{i1} = a_i$ and $u_{i2} = b_i$ for all $i = 1, \dots, n$.

The left-hand side of (1) is called the C -volume of $(a_1, b_1] \times \dots \times (a_n, b_n]$ and we denote it by $V_C((a_1, b_1] \times \dots \times (a_n, b_n])$.

In Section 2.3. we will show that a copula is a distribution function on its domain, and that its margins are standard uniformly distributed. By the following theorem we can use copulas to describe any distribution function F . It is called *Sklar's Theorem* after A. Sklar, who first published the theorem in 1959. The proof in this paper follows the proof in [7] (Theorem 2.3.3).

Theorem 2.2 For any n -dimensional distribution function F with margins F_1, \dots, F_n there exists a copula $C : [0, 1]^n \rightarrow [0, 1]$ such that

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)). \quad (2)$$

Conversely, for any copula $C : [0, 1]^n \rightarrow [0, 1]$ and any margins F_1, \dots, F_n the function $C(F_1, \dots, F_n)$ is an n -dimensional distribution function.

For the convenience of the reader we recall here also the definition of an n -dimensional distribution function from [9]. We will denote by $\overline{\mathbb{R}}$ the extended real line, $\overline{\mathbb{R}} := [-\infty, \infty]$.

Definition 2.3 A function $F, F : \overline{\mathbb{R}}^n \rightarrow [0, 1]$, is an an n -dimensional distribution function provided that

- (i) F is n -increasing i.e. for $a \in \overline{\mathbb{R}}^n$ and $b \in \overline{\mathbb{R}}^n$ such that $a_i \leq b_i$ for all $i = 1, \dots, n$ it holds

$$\sum_{j_1=1}^2 \dots \sum_{j_n=1}^2 (-1)^{j_1+\dots+j_n} F(u_{1j_1}, \dots, u_{nj_n}) \geq 0, \quad (3)$$

where $u_{i1} = a_i$ and $u_{i2} = b_i$ for all $i = 1, \dots, n$,

- (ii) F is right-continuous i.e. if $x^{(k)} \downarrow x$, then $F(x^{(k)}) \downarrow F(x)$ for all $x \in \overline{\mathbb{R}}^n \setminus \{+\infty, \dots, +\infty\}$,
- (iii) if $x^{(k)} \downarrow x$ and $x_i = -\infty$ for some $i = 1, \dots, n$, then $F(x^{(k)}) \downarrow 0$ and
- (iv) $F(+\infty, \dots, +\infty) = 1$.

We only prove the two-dimensional case of Sklar's theorem following the proof in [7]. For the proof we first introduce the concept of *subcopulas*.

Definition 2.4 A two-dimensional subcopula is a function C' with the following properties:

0. The domain of C' is a cartesian product $S_1 \times S_2$, where S_1 and S_2 are subsets of $[0, 1]$ containing the points 0 and 1.
1. For all $u_1 \in S_1$ and $u_2 \in S_2$ it holds
 - (a) $C'(u_1, 0) = 0 = C'(0, u_2)$ and
 - (b) $C'(u_1, 1) = u_1$ and $C'(1, u_2) = u_2$.
2. C' is 2-increasing.

From 0.-2. it follows directly that C' is a copula if $S_1 = S_2 = [0, 1]$ and like for copulas, we define the C' -volume of $(a, c] \times (b, d]$, where $a, c \in S_1$ and $b, d \in S_2$, by

$$V_{C'}((a, c] \times (b, d]) := C'(c, d) - C'(c, b) - C'(a, d) + C'(a, b) \geq 0.$$

We start the proof of Sklar's Theorem with

Lemma 2.5 ([7] (Lemma 2.1.5)) *Let T_1 and T_2 be subsets of $\overline{\mathbb{R}}$ containing the points $-\infty$ and ∞ . Let $H : T_1 \times T_2 \rightarrow [0, \infty)$ be a function that is 2-increasing and grounded i.e. $H(x_1, -\infty) = 0 = H(-\infty, x_2)$ for all $x_1 \in T_1$ and $x_2 \in T_2$. Let F_1 and F_2 be the margins of H , meaning $F_1(x_1) = H(x_1, \infty)$ and $F_2(x_2) = H(\infty, x_2)$ for all $x_1 \in T_1$ and $x_2 \in T_2$. Then for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $T_1 \times T_2$ one has*

$$|H(y_1, y_2) - H(x_1, x_2)| \leq |F_1(y_1) - F_1(x_1)| + |F_2(y_2) - F_2(x_2)|.$$

Proof: From the triangle inequality we have for all x and y in $T_1 \times T_2$, that

$$|H(y_1, y_2) - H(x_1, x_2)| \leq |H(y_1, y_2) - H(x_1, y_2)| + |H(x_1, y_2) - H(x_1, x_2)|.$$

Now assume $x_1 \leq y_1$. Since H is 2-increasing and grounded, for $(x_1, -\infty), (y_1, y_2) \in T_1 \times T_2$ it holds

$$\begin{aligned} V_H((x_1, y_1] \times (-\infty, y_2]) &= H(y_1, y_2) - H(x_1, y_2) - H(y_1, -\infty) + H(x_1, -\infty) \\ &= H(y_1, y_2) - H(x_1, y_2) \geq 0 \end{aligned}$$

and

$$V_H((x_1, y_1] \times (y_2, \infty]) = H(y_1, \infty) - H(x_1, \infty) - H(y_1, y_2) + H(x_1, y_2) \geq 0,$$

where the latter is equivalent to $H(y_1, \infty) - H(x_1, \infty) \geq H(y_1, y_2) - H(x_1, y_2)$. We get by the definition of F_1 and F_2 , that

$$0 \leq H(y_1, y_2) - H(x_1, y_2) \leq H(y_1, \infty) - H(x_1, \infty) = F_1(y_1) - F_1(x_1).$$

An analogous inequality holds when $y_1 \leq x_1$, hence it follows that for any x_1 and y_1 in T_1

$$|H(y_1, y_2) - H(x_1, y_2)| \leq |F_1(y_1) - F_1(x_1)|.$$

Similarly, for any x_2 and y_2 in T_2 we can show that

$$|H(x_1, y_2) - H(x_1, x_2)| \leq |F_2(y_2) - F_2(x_2)|.$$

Now

$$\begin{aligned} |H(y_1, y_2) - H(x_1, x_2)| &\leq |H(y_1, y_2) - H(x_1, y_2)| + |H(x_1, y_2) - H(x_1, x_2)| \\ &\leq |F_1(y_1) - F_1(x_1)| + |F_2(y_2) - F_2(x_2)|. \end{aligned}$$

□

The next lemma will show that the first statement of Sklar's Theorem holds for subcopulas.

Lemma 2.6 ([7] (Lemma 2.3.4)) *Let H be a joint distribution function with margins F_1 and F_2 . Then there exists a unique subcopula C' such that*

1. $\text{Dom } C' = S_1 \times S_2$, where S_1 and S_2 are the ranges of F_1 and F_2 , respectively, and
2. for all x_1 and x_2 in $\overline{\mathbb{R}}$, it holds $H(x_1, x_2) = C'(F_1(x_1), F_2(x_2))$.

Proof: Let us define a relation C' ,

$$C' := \{((F_1(x_1), F_2(x_2)), H(x_1, x_2)) : x_1, x_2 \in \overline{\mathbb{R}}\},$$

from the set $A := \{(F_1(x_1), F_2(x_2)) : x_1, x_2 \in \overline{\mathbb{R}}\}$ to the set $B := \{H(x_1, x_2) : x_1, x_2 \in \overline{\mathbb{R}}\}$. We have to show that C' is a subcopula.

We will first check whether the relation C' is a function. For this let $(x_1, x_2), (y_1, y_2) \in \overline{\mathbb{R}}^2$ such that $b_1 = H(x_1, x_2)$ and $b_2 = H(y_1, y_2)$. Now if $a = (F_1(x_1), F_2(x_2)) = (F_1(y_1), F_2(y_2))$ for some $a \in A$, it implies $F_1(x_1) = F_1(y_1)$ and $F_2(x_2) = F_2(y_2)$. As a distribution function H satisfies the assumptions of Lemma 2.5, so that

$$|H(y_1, y_2) - H(x_1, x_2)| \leq |F_1(y_1) - F_1(x_1)| + |F_2(y_2) - F_2(x_2)| = 0,$$

whence $b_1 = H(x_1, x_2) = H(y_1, y_2) = b_2$. We have shown that for all $a \in A$ there exists $b \in B$ such that $C'(a) = b$ and if a is in relation with $b_1 = H(x_1, x_2)$ and $b_2 = H(y_1, y_2)$, then $b_1 = b_2$. These two assertions give that C' is a function.

Next, let us check the three assertions of Definition 2.4 to show that C' is a subcopula.

0. $\text{Dom } C' = \text{Ran } F_1 \times \text{Ran } F_2 = S_1 \times S_2$ and for $i = 1, 2$ it holds $0 = F_i(-\infty) \in S_i$ and $1 = F_i(\infty) \in S_i$ and $S_i \subseteq [0, 1]$ since F_i is nondecreasing.

1. and 2. For any $(u_1, u_2) \in S_1 \times S_2$ there exists $(x_1, x_2) \in \overline{\mathbb{R}}^2$ such that $u_1 = F_1(x_1)$ and $u_2 = F_2(x_2)$. Now

$$C'(u_1, 0) = C'(F_1(x_1), F_2(-\infty)) = H(x_1, -\infty) = 0$$

and

$$C'(0, u_2) = C'(F_1(-\infty), F_2(x_2)) = H(-\infty, x_2) = 0.$$

The same way we get

$$C'(u_1, 1) = C'(F_1(x_1), F_2(\infty)) = H(x_1, \infty) = F_1(x_1) = u_1$$

and

$$C'(1, u_2) = C'(F_1(\infty), F_2(x_2)) = H(\infty, x_2) = F_2(x_2) = u_2.$$

To show that C' is 2-increasing, take $(u_1, u_2), (v_1, v_2) \in S_1 \times S_2$ such that $u_1 \leq v_1$ and $u_2 \leq v_2$. There exist $(x_1, x_2), (y_1, y_2) \in \overline{\mathbb{R}}^2$ such that $F_i(x_i) = u_i$ and $F_i(y_i) = v_i$ for $i = 1, 2$. Since F_1 and F_2 are nondecreasing, we can choose $x_i \leq y_i$ for $i = 1, 2$. Then

$$\begin{aligned} V_{C'}((u_1, v_1] \times (u_2, v_2]) &= C'(v_1, v_2) - C'(u_1, v_2) - C'(v_1, u_2) + C'(u_1, u_2) \\ &= C'(F_1(y_1), F_2(y_2)) - C'(F_1(x_1), F_2(y_2)) \\ &\quad - C'(F_1(y_1), F_2(x_2)) + C'(F_1(x_1), F_2(x_2)) \\ &= H(y_1, y_2) - H(x_1, y_2) - H(x_2, y_1) + H(x_1, x_2) \\ &\geq 0, \end{aligned}$$

since H is 2-increasing as a distribution function.

Clearly C' satisfies the two assertions of this lemma, since $A = S_1 \times S_2$ and $B = \text{Ran } H \subseteq [0, 1]$ and $H(x_1, x_2) = C'(F_1(x_1), F_2(x_2))$ for all $x_1, x_2 \in \overline{\mathbb{R}}$. \square

The following theorem gives an interesting property of copulas and subcopulas, which is that they are continuous.

Theorem 2.7 ([7](Theorem 2.2.4)) *Let C' be a subcopula with domain $S_1 \times S_2$. Then for every $(u_1, u_2), (v_1, v_2) \in S_1 \times S_2$,*

$$|C'(u_1, u_2) - C'(v_1, v_2)| \leq |u_1 - v_1| + |u_2 - v_2|.$$

Hence C' is uniformly continuous.

Proof: First note that C' is increasing in each coordinate when the other coordinate is fixed, since for $v_1, u_1 \in S_1$, $v_1 \geq u_1$ and $v_2 \in S_2$ we have by 2-increasingness that

$$\begin{aligned} 0 \leq V_{C'}((u_1, v_1] \times (0, v_2]) &= C'(v_1, v_2) - C'(u_1, v_2) - C'(v_1, 0) + C'(u_1, 0) \\ &= C'(v_1, v_2) - C'(u_1, v_2) \end{aligned}$$

and for $v_2, u_2 \in S_2$, $v_2 \geq u_2$ and $v_1 \in S_1$

$$\begin{aligned} 0 \leq V_{C'}((0, v_1] \times (u_2, v_2]) &= C'(v_1, v_2) - C'(0, v_2) - C'(v_1, u_2) + C'(0, u_2) \\ &= C'(v_1, v_2) - C'(v_1, u_2). \end{aligned}$$

Let $(u_1, u_2), (v_1, v_2) \in S_1 \times S_2$. We can assume that $v_1 \geq u_1$ and we check the cases $v_2 \geq u_2$ and $v_2 \leq u_2$ separately.

It holds by 2-increasingness that

$$\begin{aligned} V_{C'}((u_1, v_1] \times (u_2, 1]) &= C'(v_1, 1) - C'(u_1, 1) - C'(v_1, u_2) + C'(u_1, u_2) \\ &= v_1 - u_1 - C'(v_1, u_2) + C'(u_1, u_2) \geq 0, \end{aligned}$$

which implies $0 \leq C'(v_1, u_2) - C'(u_1, u_2) \leq v_1 - u_1$, hence

$$|C'(v_1, u_2) - C'(u_1, u_2)| \leq |v_1 - u_1|. \quad (4)$$

In the same way we get for $u_2 \leq v_2$

$$\begin{aligned} V_{C'}((v_1, 1] \times (u_2, v_2]) &= C'(1, v_2) - C'(1, u_2) - C'(v_1, v_2) + C'(v_1, u_2) \\ &= v_2 - u_2 - C'(v_1, v_2) + C'(v_1, u_2) \geq 0, \end{aligned}$$

which gives

$$|C'(v_1, v_2) - C'(v_1, u_2)| \leq |v_2 - u_2|. \quad (5)$$

Now by the triangle inequality and the inequalities (4) and (5) we have

$$\begin{aligned} |C'(v_1, v_2) - C'(u_1, u_2)| &= |C'(v_1, v_2) - C'(v_1, u_2) + C'(v_1, u_2) - C'(u_1, u_2)| \\ &\leq |C'(v_1, v_2) - C'(v_1, u_2)| + |C'(v_1, u_2) - C'(u_1, u_2)| \\ &\leq |v_2 - u_2| + |v_1 - u_1|. \end{aligned} \quad (6)$$

In the same way (6) can be shown for $v_2 \leq u_2$. □

Since any copula is also a subcopula, we get the following result:

Corollary 2.8 *Any copula is continuous.*

The following lemma states that subcopulas can be extended to copulas.

Lemma 2.9 ([7] (Lemma 2.3.5)) *Let C' be a subcopula. Then there exists a copula C such that $C(u_1, u_2) = C'(u_1, u_2)$ for all $(u_1, u_2) \in \text{Dom } C'$.*

Proof: We will extend the domain of C' first to its closure and then to $[0, 1]^2$. In both steps we will define a function that is a subcopula (or a copula) and equal to C' on its domain.

First, let $\text{Dom } C' = S_1 \times S_2$ and define a function C'' on the closure $\bar{S}_1 \times \bar{S}_2$ such that $C''(u) = C'(u)$ for all $u \in S_1 \times S_2$. For $u \in (\bar{S}_1 \times \bar{S}_2) \setminus (S_1 \times S_2)$ define $C''(u) := \lim_{v \rightarrow u} C'(v)$. The limits exist because C' is uniformly continuous. Let us check the three assertions of Definition 2.4 to show that C'' is a subcopula as well.

0. $\text{Dom } C'' = \bar{S}_1 \times \bar{S}_2 \subseteq [0, 1]^2$ and $0, 1 \in S_i \subseteq \bar{S}_i$ for $i = 1, 2$.

1. For all $u_1 \in \bar{S}_1$ and $u_2 \in \bar{S}_2$ it holds

$$C''(u_1, 0) = \lim_{v_1 \rightarrow u_1} C'(v_1, 0) = 0 \quad \text{and} \quad C''(0, u_2) = \lim_{v_2 \rightarrow u_2} C'(0, v_2) = 0,$$

$$\begin{aligned}
C''(u_1, 1) &= \lim_{v_1 \rightarrow u_1} C'(v_1, 1) = \lim_{v_1 \rightarrow u_1} v_1 = u_1 \quad \text{and} \\
C''(1, u_2) &= \lim_{v_2 \rightarrow u_2} C'(1, v_2) = \lim_{v_2 \rightarrow u_2} v_2 = u_2.
\end{aligned}$$

2. To show that C'' is 2-increasing, take $u, v \in \bar{S}_1 \times \bar{S}_2$ such that $u_1 \leq v_1$ and $u_2 \leq v_2$. If $u_1 = v_1$ or $u_2 = v_2$, then

$$\begin{aligned}
&V_{C''}((u_1, v_1] \times (u_2, v_2]) \\
&= C''(v_1, v_2) - C''(v_1, u_2) - C''(u_1, v_2) + C''(u_1, u_2) \\
&= \begin{cases} C'''(v_1, v_2) - C'''(v_1, u_2) - C'''(v_1, v_2) + C'''(v_1, u_2) = 0 \\ C'''(v_1, v_2) - C'''(v_1, v_2) - C'''(u_1, v_2) + C'''(u_1, v_2) = 0. \end{cases}
\end{aligned}$$

Assume $u_1 < v_1$ and $u_2 < v_2$. There exists sequences $(u_1^k)_{k=1}^\infty$ and $(v_1^k)_{k=1}^\infty$ in S_1 such that $u_1^k \rightarrow u_1$ and $v_1^k \rightarrow v_1$. Since $u_1 < v_2$, there exists $k_1 \in \{1, 2, \dots\}$ such that $u_1^k \leq v_1^k$ for all $k \geq k_1$. We will also consider sequences $(u_2^k)_{k=1}^\infty$ and $(v_2^k)_{k=1}^\infty$ in S_2 such that $u_2^k \rightarrow u_2$ and $v_2^k \rightarrow v_2$. Like above, there exists k_2 such that $u_2^k \leq v_2^k$ for all $k \geq k_2$. Defining $k^* := \max\{k_1, k_2\}$ it holds

$$\begin{aligned}
&V_{C''}((u_1, v_1] \times (u_2, v_2]) \\
&= C''(v_1, v_2) - C''(v_1, u_2) - C''(u_1, v_2) + C''(u_1, u_2) \\
&= \lim_{k \rightarrow \infty} [C'(v_1^k, v_2^k) - C''(v_1^k, u_2^k) - C''(u_1^k, v_2^k) + C''(u_1^k, u_2^k)] \\
&= \lim_{k^* \leq k, k \rightarrow \infty} [C'(v_1^k, v_2^k) - C''(v_1^k, u_2^k) - C''(u_1^k, v_2^k) + C''(u_1^k, u_2^k)] \\
&\geq 0.
\end{aligned}$$

Now define a function C on $[0, 1]^2$. Let $(a, b) \in [0, 1]^2$ and define

$$\begin{aligned}
a_1 &:= \sup\{x \in \bar{S}_1 : x \leq a\}, \quad a_2 := \inf\{x \in \bar{S}_1 : x \geq a\} \\
b_1 &:= \sup\{x \in \bar{S}_2 : x \leq b\} \quad \text{and} \quad b_2 := \inf\{x \in \bar{S}_2 : x \geq b\}.
\end{aligned}$$

It holds $a_1 \leq a \leq a_2$ and $b_1 \leq b \leq b_2$. Note that $a \in \bar{S}_1$ if and only if $a_1 = a_2$ and $b \in \bar{S}_2$ if and only if $b_1 = b_2$. Now let

$$\lambda_1 := \begin{cases} \frac{a-a_1}{a_2-a_1}, & \text{if } a_1 < a_2, \\ 1, & \text{if } a_1 = a_2 \end{cases} \quad \text{and} \quad \mu_1 := \begin{cases} \frac{b-b_1}{b_2-b_1}, & \text{if } b_1 < b_2, \\ 1, & \text{if } b_1 = b_2 \end{cases}$$

and define

$$\begin{aligned}
C(a, b) &:= (1 - \lambda_1)(1 - \mu_1)C''(a_1, b_1) + (1 - \lambda_1)\mu_1C''(a_1, b_2) \\
&\quad + \lambda_1(1 - \mu_1)C''(a_2, b_1) + \lambda_1\mu_1C''(a_2, b_2).
\end{aligned}$$

The function C is defined on $[0, 1]^2$ and we will prove that it satisfies the two assertions of Definition 2.1, hence it is a copula.

1. $C(a, 0) = (1 - \lambda_1)C''(a_1, 0) + \lambda_1C''(a_2, 0) = 0$ for all $a \in [0, 1]$ and $C(0, b) = (1 - \mu_1)C''(0, b_1) + \mu_1C''(0, b_2) = 0$ for all $b \in [0, 1]$. For $b = 1$ it holds $\mu_1 = 1$, so that for all $a \in [0, 1]$ it holds

$$\begin{aligned} & C(a, 1) \\ &= [(1 - \lambda_1)(1 - \mu_1) + (1 - \lambda_1)\mu_1]C''(a_1, 1) + [\lambda_1(1 - \mu_1) + \lambda_1\mu_1]C''(a_2, 1) \\ &= (1 - \lambda_1)a_1 + \lambda_1a_2 \\ &= a_1 + \lambda_1(a_2 - a_1) \\ &= \begin{cases} a_1 + \frac{a-a_1}{a_2-a_1}(a_2 - a_1) = a, & \text{if } a_1 < a_2, \\ a, & \text{if } a_1 = a_2 = a. \end{cases} \end{aligned}$$

The same way one can show that $C(1, b) = \dots = b$ for all $b \in [0, 1]$.

2. Let us show that C is 2-increasing. Let $(a, b) \in [0, 1]^2$ and $(c, d) \in [0, 1]^2$ such that $a \leq c$ and $b \leq d$ and define $a_1, a_2, b_1, b_2, \lambda_1, \mu_1$ like before. Let $c_1, c_2, d_1, d_2, \lambda_2, \mu_2$ be related to c and d the same way. We want to show that $V_C((a, c] \times (b, d]) \geq 0$.

By definition of V_C and C we have

$$\begin{aligned} V_C((a, c] \times (b, d]) &= C(c, d) - C(c, b) - C(a, d) + C(a, b) \\ &= (1 - \lambda_2)(1 - \mu_2)C''(c_1, d_1) + (1 - \lambda_2)\mu_2C''(c_1, d_2) \\ &\quad + \lambda_2(1 - \mu_2)C''(c_2, d_1) + \lambda_2\mu_2C''(c_2, d_2) \\ &\quad - (1 - \lambda_2)(1 - \mu_1)C''(c_1, b_1) - (1 - \lambda_2)\mu_1C''(c_1, b_2) \\ &\quad - \lambda_2(1 - \mu_1)C''(c_2, b_1) - \lambda_2\mu_1C''(c_2, b_2) \tag{7} \\ &\quad - (1 - \lambda_1)(1 - \mu_2)C''(a_1, d_1) - (1 - \lambda_1)\mu_2C''(a_1, d_2) \\ &\quad - \lambda_1(1 - \mu_2)C''(a_2, d_1) - \lambda_1\mu_2C''(a_2, d_2) \\ &\quad + (1 - \lambda_1)(1 - \mu_1)C''(a_1, b_1) + (1 - \lambda_1)\mu_1C''(a_1, b_2) \\ &\quad + \lambda_1(1 - \mu_1)C''(a_2, b_1) + \lambda_1\mu_1C''(a_2, b_2). \end{aligned}$$

If there is no element of \bar{S}_1 between a and c such that it is not equal to a or not equal to c , then $a_1 = c_1$ and $a_2 = c_2$. It follows $\lambda_1 = \lambda_2$ and the sum (7) becomes zero. Similarly, if there is no $x \in \bar{S}_2$ such that $b \leq x \leq d$ and it holds $x \neq b$ or $x \neq d$, then $b_1 = d_1$, $b_2 = d_2$ and $\mu_1 = \mu_2$. Hence $V_C((a, c] \times (b, d]) = 0$.

Assume $a < c$ and $b < d$ and that there is an element of \bar{S}_1 between a and c and an element of \bar{S}_2 between b and d . Rearranging the terms in (7) we get

$$V_C((a, c] \times (b, d]) = (1 - \lambda_1)(1 - \mu_1)V_{C''}((a_1, a_2] \times (b_1, b_2])$$

$$\begin{aligned}
& + (1 - \lambda_1)V_{C''}((a_1, a_2] \times (b_2, d_1]) \\
& + (1 - \lambda_1)\mu_2V_{C''}((a_1, a_2] \times (d_1, d_2]) \\
& + (1 - \mu_1)V_{C''}((a_2, c_1] \times (b_1, b_2]) \\
& + V_{C''}((a_2, c_1] \times (b_2, d_1]) \\
& + \mu_2V_{C''}((a_2, c_1] \times (d_1, d_2]) \\
& + (1 - \mu_1)\lambda_2V_{C''}((c_1, c_2] \times (b_1, b_2]) \\
& + \lambda_2V_{C''}((c_1, c_2] \times (b_2, d_1]) \\
& + \lambda_2\mu_2V_{C''}((c_1, c_2] \times (d_1, d_2]),
\end{aligned}$$

which is a sum of nine nonnegative quantities with nonnegative coefficients and therefore nonnegative. \square

Proof of Theorem 2.2: The first statement of the theorem follows directly from Lemma 2.6 and Lemma 2.9. The second statement follows almost trivially from the properties of marginal distributions and copulas. \square

2.1 Fréchet bounds

From now on, we will consider the componentwise order for real-valued functions on \mathbb{R}^n defined as: $f \leq g$, if $f(x_1, \dots, x_n) \leq g(x_1, \dots, x_n)$ for all $(x_1, \dots, x_n) \in \mathbb{R}^n$. Now we can present the lower and upper Fréchet bounds C_l and C_u for a copula C . Fréchet bounds and their properties can be found in many papers that concern copulas or multivariate distributions.

Proposition 2.10 *Let $C : [0, 1]^n \rightarrow [0, 1]$ be a copula and let the functions $C_l : [0, 1]^n \rightarrow [0, 1]$ and $C_u : [0, 1]^n \rightarrow [0, 1]$ be defined by*

$$C_l(u_1, \dots, u_n) := \left(\sum_{i=1}^n u_i - n + 1 \right)^+$$

and

$$C_u(u_1, \dots, u_n) := \min \{u_1, \dots, u_n\}.$$

Then $C_l \leq C \leq C_u$.

Proof: Let us prove the upper Fréchet bound C_u first. Let $(u_1, \dots, u_n) \in [0, 1]^n$ and u_{i^*} be the minimum of u_1, \dots, u_n . In the proof of Theorem 2.7 we saw that copulas are increasing in each coordinate, hence

$$C(u_1, \dots, u_n) \leq C(1, \dots, 1, u_{i^*}, 1, \dots, 1) = u_{i^*} = \min\{u_1, \dots, u_n\}.$$

For the lower Fréchet bound C_l we know that for all $(u_1, \dots, u_n) \in [0, 1]^n$ it holds $0 \leq C(u_1, \dots, u_n)$. Now we have to show

that $\sum_{i=1}^n u_i - n + 1 \leq C(u_1, \dots, u_n)$. We get for standard uniformly distributed random variables U_i that

$$\begin{aligned}
\sum_{i=1}^n u_i - n + 1 &= 1 + \sum_{i=1}^n (u_i - 1) \\
&= 1 - \sum_{i=1}^n \mathbb{P}(U_i(\omega) > u_i) \\
&\leq 1 - \mathbb{P}\left(\bigcup_{i=1}^n \{U_i(\omega) > u_i\}\right) \\
&= \mathbb{P}\left(\left(\bigcup_{i=1}^n \{U_i(\omega) > u_i\}\right)^c\right) \\
&= \mathbb{P}\left(\bigcap_{i=1}^n \{U_i(\omega) \leq u_i\}\right) \\
&= \mathbb{P}(U_1 \leq u_1, \dots, U_n \leq u_n) \\
&= C(u_1, \dots, u_n).
\end{aligned}$$

□

The functions C_u and C_l have some interesting properties, which will be introduced in the following propositions.

Proposition 2.11 C_u is a copula on $[0, 1]^n$ for all $n = 2, 3, \dots$ and C_l is a copula on $[0, 1]^2$.

Proof: Let us first show that the upper Fréchet bound C_u , $C_u(u_1, \dots, u_n) = \min\{u_1, \dots, u_n\}$, is a copula.

1. If $u_i = 0$ for some $i = 1, \dots, n$, then $C_u(u) = \min\{u_1, \dots, u_n\} = 0$ and if $u_i = 1$ for all $i = 1, \dots, k-1, k+1, \dots, n$, then $C_u(u) = \min\{u_1, \dots, u_n\} = \min\{u_k, 1\} = u_k$.

2. Take $a = (a_1, \dots, a_n) \in [0, 1]^n$ and $b = (b_1, \dots, b_n) \in [0, 1]^n$ such that $a_i \leq b_i$ for all $i = 1, \dots, n$ and define $x_{i1} = a_i$ and $x_{i2} = b_i$ for all $i = 1, \dots, n$.

We will show that

$$\begin{aligned}
&V_{C_u}((a_1, b_1] \times \dots \times (a_n, b_n]) \\
&= \sum_{j_1=1}^n \dots \sum_{j_n=1}^n (-1)^{j_1 + \dots + j_n} \min\{x_{1j_1}, \dots, x_{nj_n}\} \\
&= \max\{(\min\{b_1, \dots, b_n\} - \max\{a_1, \dots, a_n\}), 0\}.
\end{aligned}$$

We consider the two cases

- 1° $\max\{a_1, \dots, a_n\} \leq \min\{b_1, \dots, b_n\}$ and
- 2° $\max\{a_1, \dots, a_n\} > \min\{b_1, \dots, b_n\}$.

For any permutation $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$

$$\min\{u_1, \dots, u_n\} = \min\{u_{\pi(1)}, \dots, u_{\pi(n)}\}$$

and there exists a permutation $\pi' : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that

$$a_{\pi'(1)} \leq a_{\pi'(2)} \leq \dots \leq a_{\pi'(n)},$$

so that by the symmetry of V_{C_u} we can assume that $a_1 \leq a_2 \leq \dots \leq a_n$.

First note that if $a_i \leq \min\{b_1, \dots, b_n\}$, then

$$\begin{aligned} & \sum_{j_{i+1}=1}^2 \dots \sum_{j_n=1}^2 (-1)^{j_{i+1}+\dots+j_n} \min\{b_1, \dots, b_{i-1}, a_i, x_{(i+1)j_{i+1}}, \dots, x_{nj_n}\} \\ = & \sum_{j_{i+2}=1}^2 \dots \sum_{j_n=1}^2 (-1)^{j_{i+2}+\dots+j_n} [\min\{b_1, \dots, b_{i-1}, a_i, b_{i+1}, x_{(i+2)j_{i+1}}, \dots, x_{nj_n}\} \\ & \quad - \min\{b_1, \dots, b_{i-1}, a_i, a_{i+1}, x_{(i+2)j_{i+1}}, \dots, x_{nj_n}\}] \\ = & \sum_{j_{i+2}=1}^2 \dots \sum_{j_n=1}^2 (-1)^{j_{i+2}+\dots+j_n} [a_i - a_i] \\ = & 0. \end{aligned}$$

1°. Since $a_i \leq \min\{b_1, \dots, b_n\}$ for all $i = 1, \dots, n$, we have

$$\sum_{j_{i+1}=1}^2 \dots \sum_{j_n=1}^2 (-1)^{j_{i+1}+\dots+j_n} \min\{b_1, \dots, b_{i-1}, a_i, x_{(i+1)j_{i+1}}, \dots, x_{nj_n}\} = 0 \quad (8)$$

for all $i = 1, \dots, n$. Thus

$$\begin{aligned} & \sum_{j_1=1}^n \dots \sum_{j_n=1}^n (-1)^{j_1+\dots+j_n} \min\{x_{1j_1}, \dots, x_{nj_n}\} \\ = & \sum_{j_2=1}^2 \dots \sum_{j_n=1}^2 (-1)^{j_2+\dots+j_n} [\min\{b_1, x_{2j_2}, \dots, x_{nj_n}\} - \min\{a_1, x_{2j_2}, \dots, x_{nj_n}\}] \\ = & \sum_{j_2=1}^2 \dots \sum_{j_n=1}^2 (-1)^{j_2+\dots+j_n} \min\{b_1, x_{2j_2}, \dots, x_{nj_n}\} \end{aligned}$$

$$\begin{aligned}
& - \sum_{j_2=1}^2 \cdots \sum_{j_n=1}^2 (-1)^{j_2+\cdots+j_n} \min\{a_1, x_{2j_2}, \dots, x_{nj_n}\} \\
&= \sum_{j_2=1}^2 \cdots \sum_{j_n=1}^2 (-1)^{j_2+\cdots+j_n} \min\{b_1, x_{2j_2}, \dots, x_{nj_n}\} + 0 \\
&= \sum_{j_3=1}^2 \cdots \sum_{j_n=1}^2 (-1)^{j_3+\cdots+j_n} [\min\{b_1, b_2, x_{3j_3}, \dots, x_{nj_n}\} \\
&\quad - \min\{b_1, a_2, x_{3j_3}, \dots, x_{nj_n}\}] \\
&= \cdots = \min\{b_1, \dots, b_n\} - \min\{b_1, \dots, b_{n-1}, a_n\} \\
&= \min\{b_1, \dots, b_n\} - a_n \\
&= \min\{b_1, \dots, b_n\} - \max\{a_1, \dots, a_n\}.
\end{aligned}$$

2°. Now we assume that $a_n > \min\{b_1, \dots, b_n\}$. There exists a number $m \in \{2, \dots, n\}$ such that

$$a_1, \dots, a_{m-1} \leq \min\{b_1, \dots, b_n\} < a_m \leq a_{m+1} \leq \cdots \leq a_n.$$

For all $i = m, \dots, n$ it follows $\min\{b_1, \dots, b_n\} \leq b_i$ since $a_i \leq b_i$. We get that $\min\{b_1, \dots, b_n\} = \min\{b_1, \dots, b_{m-1}\} = \min\{b_1, \dots, b_{m-1}, x_{mj_m}, \dots, x_{nj_n}\}$ and by (8)

$$\begin{aligned}
& \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n (-1)^{j_1+\cdots+j_n} \min\{x_{1j_1}, \dots, x_{nj_n}\} \\
&= \sum_{j_2=1}^2 \cdots \sum_{j_n=1}^2 (-1)^{j_2+\cdots+j_n} [\min\{b_1, x_{2j_2}, \dots, x_{nj_n}\} - \min\{a_1, x_{2j_2}, \dots, x_{nj_n}\}] \\
&= \cdots = \sum_{j_m=1}^2 \cdots \sum_{j_n=1}^2 (-1)^{j_m+\cdots+j_n} [\min\{b_1, \dots, b_{m-2}, b_{m-1}, x_{mj_m}, \dots, x_{nj_n}\} \\
&\quad - \min\{b_1, \dots, b_{m-2}, a_{m-1}, x_{mj_m}, \dots, x_{nj_n}\}] \\
&= \sum_{j_{m+1}=1}^2 \cdots \sum_{j_n=1}^2 (-1)^{j_{m+1}+\cdots+j_n} [\min\{b_1, \dots, b_m, x_{(m+1)j_{m+1}}, \dots, x_{nj_n}\} \\
&\quad - \min\{b_1, \dots, b_{m-1}, a_m, x_{(m+1)j_{m+1}}, \dots, x_{nj_n}\}] \\
&= \sum_{j_{m+1}=1}^2 \cdots \sum_{j_n=1}^2 (-1)^{j_{m+1}+\cdots+j_n} [\min\{b_1, \dots, b_{m-1}\} - \min\{b_1, \dots, b_{m-1}\}] \\
&= 0.
\end{aligned}$$

Now let us show that $C_l : [0, 1]^2 \rightarrow [0, 1]$, $C_l(u_1, u_2) = (u_1 + u_2 - 1)^+$, is a copula. Obviously, $C_l(0, u_2) = 0 = C_l(u_1, 0)$ and $C_l(u_1, 1) = u_1$, $C_l(1, u_2) = u_2$ for all $u_1, u_2 \in [0, 1]$.

Take (a_1, a_2) and $(b_1, b_2) \in [0, 1]^2$ such that $a_1 \leq b_1$ and $a_2 \leq b_2$. Then we get

$$\begin{aligned}
& V_C((a_1, b_1] \times (a_2, b_2]) \\
&= C_l(b_1, b_2) - C_l(a_1, b_2) - C_l(b_1, a_2) + C_l(a_1, a_2) \\
&= (b_1 + b_2 - 1)^+ - (a_1 + b_2 - 1)^+ - (b_1 + a_2 - 1)^+ + (a_1 + a_2 - 1)^+ \\
&= \begin{cases} 0 - 0 - 0 + 0 = 0, & \text{if } b_1 + b_2 \leq 1 \\ b_1 + b_2 - 1 \geq 0, & \text{if } b_1 + b_2 \geq 1, b_1 + a_2 \leq 1 \text{ and } a_1 + b_2 \leq 1 \\ b_1 + b_2 - 1 - (a_1 + b_2 - 1) \\ = b_1 - a_1 \geq 0, & \text{if } a_1 + b_2 \geq 1 \text{ and } b_1 + a_2 \leq 1 \\ b_1 + b_2 - 1 - (b_1 + a_2 - 1) \\ = b_2 - a_2 \geq 0, & \text{if } a_1 + b_2 \leq 1 \text{ and } b_1 + a_2 \geq 1 \\ b_1 + b_2 - 1 - (a_1 + b_2 - 1) \\ -(b_1 + a_2 - 1) + (a_1 + a_2 - 1) = 0, & \text{if } a_1 + a_2 \geq 1 \text{ and} \\ b_1 + b_2 - 1 - (a_1 + b_2 - 1) \\ -(b_1 + a_2 - 1) + 0 \geq 0, & \text{if } a_1 + b_2 \geq 1, b_1 + a_2 \geq 1 \text{ and } a_1 + a_2 \leq 1. \end{cases}
\end{aligned}$$

So that C_l is 2-increasing, hence it is a copula. \square

Remark 2.12 C_l is not a copula for $n > 2$.

Proof: Choose $a = (a_1, \dots, a_n) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \dots, 0)$ and $b = (1, \dots, 1)$. Denote $x_{i1} = a_i$ and $x_{i2} = 1$ for all $i = 1, \dots, n$. Since $C_l(x_{1j_1}, \dots, x_{nj_n}) = 0$ if $j_i = 1$ for some $i = 4, \dots, n$, we have

$$\begin{aligned}
& V_C((a_1, 1] \times \dots \times (a_n, 1]) \\
&= \sum_{j_1=1}^2 \dots \sum_{j_n=1}^2 (-1)^{j_1 + \dots + j_n} C_l(x_{1j_1}, \dots, x_{nj_n}) \\
&= \sum_{j_1=1}^2 \sum_{j_2=1}^2 \sum_{j_3=1}^2 (-1)^{j_1 + j_2 + j_3} C_l(x_{1j_1}, x_{2j_2}, x_{3j_3}, 1, \dots, 1) \\
&= \sum_{j_1=1}^2 \sum_{j_2=1}^2 \sum_{j_3=1}^2 (-1)^{j_1 + j_2 + j_3} (x_{1j_1} + x_{2j_2} + x_{3j_3} - 2)^+ \\
&= (1 + 1 + 1 - 2)^+ - (1 + 1 + \frac{1}{2} - 2)^+ + (1 + \frac{1}{2} + \frac{1}{2} - 2)^+
\end{aligned}$$

$$\begin{aligned}
& - (1 + \frac{1}{2} + 1 - 2)^+ + (\frac{1}{2} + 1 + \frac{1}{2} - 2)^+ - (\frac{1}{2} + 1 + 1 - 2)^+ \\
& + (\frac{1}{2} + \frac{1}{2} + 1 - 2)^+ - (\frac{1}{2} + \frac{1}{2} + \frac{1}{2} - 2)^+ \\
& = 1 - \frac{1}{2} + 0 - \frac{1}{2} + 0 - \frac{1}{2} + 0 - 0 \\
& = -\frac{1}{2},
\end{aligned}$$

so that C_l is not n -increasing, hence it is not a copula. \square

2.2 Comonotonicity of random variables

Random variables X_1, \dots, X_n with a C_u -dependence structure are called comonotonic. A necessary and sufficient condition for comonotonicity is that there exist increasing functions $f_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, \dots, n$, and a random variable $Z : \Omega \rightarrow \mathbb{R}$ such that (X_1, \dots, X_n) and $(f_1(Z), \dots, f_n(Z))$ have the same distribution. It is obvious that this condition is sufficient, and that it is necessary is shown in the proof of the proposition below. In [6] (Proposition 3.1) it is shown the following result.

Proposition 2.13 *Assume $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is increasing and left-continuous in each coordinate, $0 \leq \alpha \leq 1$ and X_1, \dots, X_n are comonotonic random variables. Then it holds*

$$\text{VaR}_\alpha(\psi(X_1, \dots, X_n)) = \psi(\text{VaR}_\alpha(X_1), \dots, \text{VaR}_\alpha(X_n)), \quad (9)$$

provided that the both sides are finite.

For the proof we introduce the following lemma:

Lemma 2.14 *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function and define the generalized left- and right-continuous inverses of φ by $\varphi^{-1}, \varphi^\wedge : \mathbb{R} \rightarrow \overline{\mathbb{R}}$,*

$$\varphi^{-1}(y) := \inf\{x \in \mathbb{R} : \varphi(x) \geq y\} \quad \text{and} \quad \varphi^\wedge(y) := \sup\{x \in \mathbb{R} : \varphi(x) \leq y\}.$$

Then one has the following assertions:

- (i) φ^{-1} and φ^\wedge are increasing.
- (ii) φ^{-1} is left-continuous and φ^\wedge is right-continuous.
- (iii) If φ is right-continuous and $\varphi^{-1}(y) > -\infty$, then $\varphi(x) \geq y$ if and only if $x \geq \varphi^{-1}(y)$.

(iv) If φ is left-continuous and $\varphi^\wedge(y) > -\infty$, then $\varphi(x) \leq y$ if and only if $x \leq \varphi^\wedge(y)$.

Proof: (i) We take real numbers $y_1, y_2 \in \mathbb{R}$ such that $y_1 \leq y_2$ and show that $\varphi^{-1}(y_1) \leq \varphi^{-1}(y_2)$ and $\varphi^\wedge(y_1) \leq \varphi^\wedge(y_2)$. For all $x \in \{z \in \mathbb{R} : \varphi(z) \geq y_2\}$ it holds $\varphi(x) \geq y_2 \geq y_1$, so that $x \in \{z \in \mathbb{R} : \varphi(z) \geq y_1\}$. It follows $\{z \in \mathbb{R} : \varphi(z) \geq y_2\} \subseteq \{z \in \mathbb{R} : \varphi(z) \geq y_1\}$, which implies

$$\varphi^{-1}(y_1) = \inf\{z \in \mathbb{R} : \varphi(z) \geq y_1\} \leq \inf\{z \in \mathbb{R} : \varphi(z) \geq y_2\} = \varphi^{-1}(y_2).$$

For all $x \in \{z \in \mathbb{R} : \varphi(z) \leq y_1\}$ it holds $\varphi(x) \leq y_1 \leq y_2$. Hence $\{x \in \mathbb{R} : \varphi(x) \leq y_1\} \subseteq \{x \in \mathbb{R} : \varphi(x) \leq y_2\}$. Now

$$\varphi^\wedge(y_1) = \sup\{x \in \mathbb{R} : \varphi(x) \leq y_1\} \leq \sup\{x \in \mathbb{R} : \varphi(x) \leq y_2\} = \varphi^\wedge(y_2).$$

(ii) We show that φ^{-1} is left-continuous. Take $y \in \mathbb{R}$ and a sequence $(y_k)_{k=1}^\infty \subseteq \mathbb{R}$ such that $y_k \uparrow y$. Since

$$\{x \in \mathbb{R} : \varphi(x) \geq y_{k+1}\} \subseteq \{x \in \mathbb{R} : \varphi(x) \geq y_k\}$$

we have $\inf\{x \in \mathbb{R} : \varphi(x) \geq y_{k+1}\} \geq \inf\{x \in \mathbb{R} : \varphi(x) \geq y_k\}$ for all $k = 1, 2, \dots$, hence $\lim_{k \rightarrow \infty} \inf\{x \in \mathbb{R} : \varphi(x) \geq y_k\}$ exists. Define

$$z := \lim_{k \rightarrow \infty} \varphi^{-1}(y_k) = \lim_{k \rightarrow \infty} \inf\{x \in \mathbb{R} : \varphi(x) \geq y_k\}$$

and show that $z \leq \varphi^{-1}(y)$ and $z \geq \varphi^{-1}(y)$.

Since $y_k \leq y$ for all $k = 1, 2, \dots$ we have $\inf\{x \in \mathbb{R} : \varphi(x) \geq y_k\} \leq \inf\{x \in \mathbb{R} : \varphi(x) \geq y\} = \varphi^{-1}(y)$, hence

$$z = \lim_{k \rightarrow \infty} \inf\{x \in \mathbb{R} : \varphi(x) \geq y_k\} \leq \inf\{x \in \mathbb{R} : \varphi(x) \geq y\} = \varphi^{-1}(y).$$

Because φ^{-1} is increasing by (i) and $z = \lim_{k \rightarrow \infty} \varphi^{-1}(y_k)$ for $y_k \leq y_{k+1}$ for all $k = 1, 2, \dots$, we have

$$z \geq \inf\{x \in \mathbb{R} : \varphi(x) \geq y_k\} \text{ for all } k = 1, 2, \dots$$

This implies that for all for all $k = 1, 2, \dots$ and for all $\varepsilon > 0$ the number $z + \varepsilon$ belongs to the set $\{x \in \mathbb{R} : \varphi(x) \geq y_k\}$, so that $\varphi(z + \varepsilon) \geq y_k$. Taking the supremum over k one gets

$$\varphi(z + \varepsilon) \geq \sup_{k=1,2,\dots} y_k = y.$$

This implies that for all $\varepsilon > 0$ it holds $z + \varepsilon \in \{x \in \mathbb{R} : \varphi(x) \geq y\}$ and therefore $z + \varepsilon \geq \inf\{x \in \mathbb{R} : \varphi(x) \geq y\} = \varphi^{-1}(y)$. Now we have that

$\lim_{k \rightarrow \infty} \varphi^{-1}(y_k) + \varepsilon \geq \varphi^{-1}(y)$ for all $\varepsilon > 0$, hence $\lim_{k \rightarrow \infty} \varphi^{-1}(y_k) \geq \varphi^{-1}(y)$. Finally, $\lim_{k \rightarrow \infty} \varphi^{-1}(y_k) \leq \varphi^{-1}(y)$ and $\lim_{k \rightarrow \infty} \varphi^{-1}(y_k) \geq \varphi^{-1}(y)$ imply

$$\lim_{k \rightarrow \infty} \varphi^{-1}(y_k) = \varphi^{-1}(y).$$

Let us show now that φ^\wedge is right-continuous. Take $y \in \mathbb{R}$ and a sequence $(y_k)_{k=1}^\infty \subseteq \mathbb{R}$ such that $y_k \downarrow y$. Consider the function $\phi(x) := -\varphi(-x)$. For x_1 and $x_2 \in \mathbb{R}$ such that $x_1 \leq x_2$ it holds $\varphi(-x_2) \leq \varphi(-x_1)$, so that $\phi(x_1) = -\varphi(-x_1) \leq -\varphi(-x_2) = \phi(x_2)$, hence ϕ is increasing. It follows by the proof above that ϕ^{-1} is left-continuous. Note that for any $z \in \mathbb{R}$ we have

$$\begin{aligned} \phi^{-1}(z) &= \inf\{x \in \mathbb{R} : -\varphi(-x) \geq z\} = \inf\{x \in \mathbb{R} : \varphi(-x) \leq -z\} \\ &= \inf\{-x \in \mathbb{R} : \varphi(x) \leq -z\} = \sup\{x \in \mathbb{R} : \varphi(x) \leq -z\} \\ &= \varphi^\wedge(-z). \end{aligned}$$

Now since $-y_k \uparrow -y$ and ϕ^{-1} is left-continuous, it holds

$$\lim_{k \rightarrow \infty} \varphi^\wedge(y_k) = \lim_{k \rightarrow \infty} \phi^{-1}(-y_k) = \phi^{-1}(-y) = \varphi^\wedge(y).$$

(iii) Assume that φ is right-continuous and $y \in \mathbb{R}$ is such that $\varphi^{-1}(y) > -\infty$. If $\varphi(x) \geq y$, then

$$\varphi^{-1}(y) = \inf\{z \in \mathbb{R} : \varphi(z) \geq y\} \leq x.$$

Now let us consider the other direction. For all $x \in \mathbb{R}$ with $x > \varphi^{-1}(y) = \inf\{z \in \mathbb{R} : \varphi(z) \geq y\}$ it holds $\varphi(x) \geq y$. Hence

$$y \leq \inf_{x > \varphi^{-1}(y)} \varphi(x)$$

and, by the right-continuity of φ , it holds

$$\inf_{x > \varphi^{-1}(y)} \varphi(x) = \varphi(\varphi^{-1}(y)).$$

Hence $y \leq \varphi(\varphi^{-1}(y))$, so that for all $x \geq \varphi^{-1}(y)$ it holds $\varphi(x) \geq y$.

(iv) Assume that φ is left-continuous and $y \in \mathbb{R}$ is such that $\varphi^\wedge(y) > -\infty$. If $\varphi(x) \leq y$, then

$$x \leq \sup\{z \in \mathbb{R} : \varphi(z) \leq y\} = \varphi^\wedge(y).$$

On the other hand, assume first that $x < \varphi^\wedge(y) = \sup\{z \in \mathbb{R} : \varphi(z) \leq y\}$. It means $x \in \{z \in \mathbb{R} : \varphi(z) \leq y\}$, hence $\varphi(x) \leq y$. If $x = \varphi^\wedge(y)$, then we get by the left-continuity of φ that $\varphi(\varphi^\wedge(y)) \leq y$, so that $x \leq \varphi^\wedge(y)$ implies $\varphi(x) \leq y$. \square

Proof of Proposition 2.13: Let Z be a real valued random variable and suppose that $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and left-continuous. Suppose that $\text{VaR}_\alpha(Z) = F_Z^{-1}(\alpha)$ is finite for a given $\alpha \in [0, 1]$. The distribution function of $\varphi(Z)$ is $F_{\varphi(Z)}(z) = \mathbb{P}(\varphi(Z) \leq z)$. Let $t \in \mathbb{R}$. If $\varphi^\wedge(t) = -\infty$, then $\varphi(x) > t$ for all $x \in \mathbb{R}$ and

$$F_{\varphi(Z)}(t) = \mathbb{P}(\varphi(Z) \leq t) = 0 = F_Z(-\infty) = F_Z(\varphi^\wedge(t)).$$

And if $\varphi^\wedge(t) > -\infty$, then by Lemma 2.14 (iv)

$$F_{\varphi(Z)}(t) = \mathbb{P}(\varphi(Z) \leq t) = \mathbb{P}(Z \leq \varphi^\wedge(t)) = F_Z(\varphi^\wedge(t)).$$

Since F_Z is right-continuous and $F_Z^{-1}(\alpha) > -\infty$, we have by Lemma 2.14 (iii), that $F_Z(\varphi^\wedge(t)) \geq \alpha$ if and only if $\varphi^\wedge(t) \geq F_Z^{-1}(\alpha)$. Now, since we can assume that there exists a number $t \in \mathbb{R}$ such that $\varphi^\wedge(t) > -\infty$,

$$\begin{aligned} \text{VaR}_\alpha(\varphi(Z)) &= \inf \{t \in \mathbb{R} : F_{\varphi(Z)}(t) \geq \alpha\} = \inf \{t \in \mathbb{R} : F_Z(\varphi^\wedge(t)) \geq \alpha\} \\ &= \inf \{t \in \mathbb{R} : \varphi^\wedge(t) \geq F_Z^{-1}(\alpha)\} \\ &= \inf \{t \in \mathbb{R} : t \geq \varphi(F_Z^{-1}(\alpha))\} \\ &= \varphi(F_Z^{-1}(\alpha)) = \varphi(\text{VaR}_\alpha(Z)). \end{aligned} \tag{10}$$

Let F_1, \dots, F_n be the distribution functions of X_1, \dots, X_n and define

$$\varphi(\alpha) := \psi(F_1^{-1}(\alpha), \dots, F_n^{-1}(\alpha)). \tag{11}$$

The function φ is increasing since ψ and $F_1^{-1}, \dots, F_n^{-1}$ are increasing. We get that φ is left-continuous, since ψ is left-continuous and the increasingness of F_i 's imply by Lemma 2.14 (ii) that F_i^{-1} 's are left-continuous for all $i = 1, \dots, n$.

Let U be standard uniformly distributed random variable. Then it holds

$$\begin{aligned} \mathbb{P}\left(\bigcup_{i=1}^n \{F_i^{-1}(U) = -\infty\}\right) &\leq \sum_{i=1}^n \mathbb{P}(F_i^{-1}(U) = -\infty) \\ &= \sum_{i=1}^n \mathbb{P}(\inf\{x \in \mathbb{R} : F_i(x) \geq U\} = -\infty) \\ &= \sum_{i=1}^n \mathbb{P}(F_i(x) < U \forall x \in \mathbb{R}) \\ &\leq \sum_{i=1}^n \mathbb{P}(1 \leq U) = 0, \end{aligned}$$

so that by Lemma 2.14 (iii)

$$\begin{aligned}
\mathbb{P}(F_1^{-1}(U) \leq x_1, \dots, F_n^{-1}(U) \leq x_n) &= \mathbb{P}\left(\bigcap_{i=1}^n \{-\infty < F_i^{-1}(U) \leq x_i\}\right) \\
&= \mathbb{P}\left(\bigcap_{i=1}^n \{U \leq F_i(x_i)\}\right) \\
&= \mathbb{P}(U \leq \min\{F_1(x_1), \dots, F_n(x_n)\}) \\
&= \min\{F_1(x_1), \dots, F_n(x_n)\} \\
&= C_u(F_1(x_1), \dots, F_n(x_n)) \\
&= \mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n),
\end{aligned}$$

which means that the random vectors (X_1, \dots, X_n) and $(F_1^{-1}(U), \dots, F_n^{-1}(U))$ have the same distribution. Now since $\text{VaR}_\alpha(U) = F_U^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_U(x) \geq \alpha\} = \alpha$, we have by (10) and (11)

$$\begin{aligned}
\text{VaR}_\alpha(\psi(X_1, \dots, X_n)) &= \text{VaR}_\alpha(\psi(F_1^{-1}(U), \dots, F_n^{-1}(U))) = \text{VaR}_\alpha(\varphi(U)) \\
&= \varphi(\text{VaR}_\alpha(U)) = \varphi(\alpha) \\
&= \psi(F_1^{-1}(\alpha), \dots, F_n^{-1}(\alpha)) \\
&= \psi(\text{VaR}_\alpha(X_1), \dots, \text{VaR}_\alpha(X_n)).
\end{aligned}$$

□

The equation (9) is often used in evaluating Value-at-Risk when the dependence structure is not known. But this approximation is not good, because it can be

$$\text{VaR}_\alpha(\psi(X)) > \psi(\text{VaR}_\alpha(X_1), \dots, \text{VaR}_\alpha(X_n))$$

or

$$\text{VaR}_\alpha(\psi(X)) < \psi(\text{VaR}_\alpha(X_1), \dots, \text{VaR}_\alpha(X_n)).$$

This was observed in [8] and we follow their example in showing the first inequality.

Example 2.15 Let X and Y be independent random variables with identical distribution $F(x) = 1 - x^{-1/2} \mathbb{1}_{\{x \geq 1\}}(x)$. Let us approximate $\mathbb{P}(X+Y \leq s)$ and calculate $\mathbb{P}(X+X \leq s)$ to show that for $\alpha \in (0, 1)$ it holds that

$$\text{VaR}_\alpha(X+Y) > \text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y).$$

The density function of the random variables X and Y is $f(x) = \frac{1}{2}x^{-3/2}\mathbb{1}_{\{x \geq 1\}}(x)$. For $s > 2$ we obtain by convolution

$$\begin{aligned}
\mathbb{P}(X + Y \leq s) &= \int_{-\infty}^{\infty} F(s - y)dF(y) \\
&= \int_{-\infty}^{\infty} \left(1 - (s - y)^{-\frac{1}{2}}\right) \mathbb{1}_{\{s - y \geq 1\}}(y) \frac{1}{2}y^{-3/2} \mathbb{1}_{\{y \geq 1\}}(y)dy \\
&= \int_1^{s-1} \left(1 - (s - y)^{-\frac{1}{2}}\right) \frac{1}{2}y^{-3/2}dy \\
&= \int_1^{s-1} \frac{1}{2}y^{-3/2}dy - \frac{1}{2} \int_1^{s-1} (s - y)^{-\frac{1}{2}}y^{-3/2}dy \\
&= 1 - (s - 1)^{-\frac{1}{2}} - \frac{1}{2s} \int_{\frac{1}{s}}^{\frac{s-1}{s}} (1 - x)^{-\frac{1}{2}}x^{-3/2}dx,
\end{aligned}$$

where we used a change of variables $x = y/s$.

Moreover, $\mathbb{P}(X + X \leq s) = \mathbb{P}(X \leq s/2) = 1 - (s/2)^{-1/2}$ and for $s = 2$ it holds $\mathbb{P}(X + X \leq 2) = 0$ and $\mathbb{P}(X + Y \leq 2) = 0$. For $s > 2$ define $G(s) := \mathbb{P}(X + Y \leq s)$ and $H(s) := \mathbb{P}(X \leq s/2)$ and calculate

$$\begin{aligned}
\frac{d}{ds}(sG(s)) &= \frac{d}{ds} \left[s - s(s - 1)^{-\frac{1}{2}} - \frac{1}{2} \int_{\frac{1}{s}}^{\frac{s-1}{s}} (1 - x)^{-\frac{1}{2}}x^{-3/2}dx \right] \\
&= 1 - (s - 1)^{-\frac{1}{2}} + \frac{s}{2}(s - 1)^{-\frac{3}{2}} \\
&\quad - \frac{1}{2} \left[\left(1 - \frac{s-1}{s}\right)^{-\frac{1}{2}} \left(\frac{s-1}{s}\right)^{-\frac{3}{2}} \frac{1}{s^2} + \left(1 - \frac{1}{s}\right)^{-\frac{1}{2}} \left(\frac{1}{s}\right)^{-\frac{3}{2}} \frac{1}{s^2} \right] \\
&= 1 - (s - 1)^{-\frac{1}{2}} + \frac{s}{2}(s - 1)^{-\frac{3}{2}} - \frac{1}{2}(s - 1)^{-\frac{3}{2}} - \frac{1}{2}(s - 1)^{-\frac{1}{2}} \\
&= 1 - (s - 1)^{-\frac{1}{2}}
\end{aligned}$$

and

$$\frac{d}{ds}(sH(s)) = \frac{d}{ds} \left(s - (2s)^{\frac{1}{2}} \right) = 1 - (2s)^{-\frac{1}{2}}.$$

Now since for $s > 2$ it holds $s - 1 < 2s$, we have

$$\frac{d}{ds}(sG(s)) = 1 - \frac{1}{\sqrt{s-1}} < 1 - \frac{1}{\sqrt{2s}} = \frac{d}{ds}(sH(s)).$$

From $2G(2) = 2H(2) = 0$ it follows $sG(s) < sH(s)$ for $s > 2$, hence

$$\mathbb{P}(X + Y \leq s) = G(s) < H(s) = \mathbb{P}(X + X \leq s) \text{ for } s > 2.$$

Since for $x \leq 2$ it holds $F_{X+Y}(x) = \mathbb{P}(X+Y \leq x) = \int_1^{x-1} F(x-y)f(y)dy = 0$, we have, for $\alpha \in (0, 1)$, that

$$\begin{aligned}\text{VaR}_\alpha(X + Y) &= F_{X+Y}^{-1}(\alpha) = \inf \{x \in \mathbb{R} : F_{X+Y}(x) \geq \alpha\} \\ &= \inf \{x > 2 : F_{X+Y}(x) \geq \alpha\}\end{aligned}$$

and, knowing that for $x > 2$ it holds $F_{X+Y}(x) < F_{X+X}(x)$, we have

$$\begin{aligned}\text{VaR}_\alpha(X + Y) &= \inf \{x > 2 : F_{X+Y}(x) \geq \alpha\} \\ &> \inf \{x > 2 : F_{X+X}(x) \geq \alpha\} \\ &\geq \inf \{x \in \mathbb{R} : F_{X+X}(x) \geq \alpha\} \\ &= F_{X+X}^{-1}(\alpha) = \text{VaR}_\alpha(X + X).\end{aligned}$$

The joint distribution of X and X is $F_{X,X}(x_1, x_2) = \mathbb{P}(X \leq x_1, X \leq x_2) = \min\{F(x_1), F(x_2)\}$, which means that X is comonotonic with itself. Hence by Proposition 2.13 it holds $\text{VaR}_\alpha(X+X) = \text{VaR}_\alpha(X)+\text{VaR}_\alpha(X) = \text{VaR}_\alpha(X)+\text{VaR}_\alpha(Y)$. It follows

$$\text{VaR}_\alpha(X + Y) > \text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y) \quad \text{for } \alpha \in (0, 1).$$

We can use this example of Embrechts et al. to show that the converse inequality can also happen. Consider the random variables $-X$ and $-Y$ and let $s > 2$. Since X and Y have continuous distribution functions, it holds

$$\begin{aligned}F_{-X-Y}(-s) &= \mathbb{P}(-X - Y \leq -s) = \mathbb{P}(X + Y \geq s) \\ &= 1 - \mathbb{P}(X + Y \leq s) = 1 - G(s)\end{aligned}$$

and

$$\begin{aligned}F_{-2X}(-s) &= \mathbb{P}(-2X \leq -s) = \mathbb{P}(X \geq s/2) \\ &= 1 - \mathbb{P}(X \leq s/2) = 1 - H(s).\end{aligned}$$

Because $G(s) < H(s)$ for $s > 2$, we have

$$F_{-X-Y}(x) = 1 - G(-x) > 1 - H(-x) = F_{-2X}(x) \quad \text{for } x < -2.$$

For $\alpha \in (0, 1)$ it follows by the fact that $F_{-2X}(x) \geq 1$ and $F_{-X-Y}(x) \geq 1$ for $x \geq -2$ that

$$\begin{aligned}\text{VaR}_\alpha(-X) + \text{VaR}_\alpha(-Y) &= \text{VaR}_\alpha(-2X) \\ &= \inf \{x < -2 : 1 - H(-x) \geq \alpha\} \\ &> \inf \{x < -2 : 1 - G(-x) \geq \alpha\} \\ &= \text{VaR}_\alpha(-X - Y).\end{aligned}$$

2.3 Some properties of copulas

In the previous section we found out in Corollary 2.8 that copulas are continuous. In this section we introduce more of the interesting properties of copulas. In many papers copulas are introduced as distribution functions from $[0, 1]^n$ to $[0, 1]$ with standard uniform margins. We show that it is indeed a necessary and sufficient condition for a function to be a copula.

Proposition 2.16 *Let C be a function from $[0, 1]^n$ to $[0, 1]$. It holds that C is copula if and only if it is a distribution function which has standard uniform margins. The latter means that*

- (i) C is n -increasing i.e. for $a \in [0, 1]^n$ and $b \in [0, 1]^n$ such that $a_i \leq b_i$ for all $i = 1, \dots, n$ it holds

$$\sum_{j_1=1}^2 \dots \sum_{j_n=1}^2 (-1)^{j_1+\dots+j_n} C(u_{1j_1}, \dots, u_{nj_n}) \geq 0, \quad (12)$$

where $u_{i1} = a_i$ and $u_{i2} = b_i$ for all $i = 1, \dots, n$,

- (ii) C is right-continuous i.e. if $x^{(k)} \downarrow x$, then $C(x^{(k)}) \downarrow C(x)$ for all $x^{(k)}, x \in [0, 1]^n \setminus \{1, \dots, 1\}$,
- (iii) if $x^{(k)} \downarrow x$ and $x_i = 0$ for some $i = 1, \dots, n$, then $C(x^{(k)}) \downarrow 0$,
- (iv) $C(1, \dots, 1) = 1$ and
- (v) the margins of C are standard uniformly distributed i.e. for all $k = 1, \dots, n$ and $u_k \in [0, 1]$ it holds $C(1, \dots, 1, u_k, 1, \dots, 1) = \mathbb{P}(U \leq u_k)$, where $U \sim U[0, 1]$.

Proof: Assertion (i) is equivalent to 2. in the definition of a copula, Definition 2.1. And since for a random variable $U \sim U[0, 1]$ it holds $\mathbb{P}(U \leq u_k) = u_k$ for $u_k \in [0, 1]$, it follows that assertions (v) and 1.(b) are equivalent. 1.(a) follows from (iii), hence it is shown that (i), (iii) and (v) imply that C is a copula.

We have to show that (ii)-(iv) hold for a copula. By Corollary 2.8 a copula is continuous and therefore right-continuous as is demanded in assertion (ii). Right-continuity and 1.(a) imply (iii) and 1.(b) implies (iv). \square

What is the copula of independent random variables? We introduce the product copula $\Pi : [0, 1]^n \rightarrow [0, 1]$, $\Pi(u) := \prod_{i=1}^n u_i$ for all $u \in [0, 1]^n$. Π is a copula, since it is a distribution function as a product of standard uniform distribution functions, $\Pi(u) = \prod_{i=1}^n u_i = \prod_{i=1}^n F_{U_i}(u_i)$ and $\Pi(1, \dots, 1, u_i, 1, \dots, 1) = u_i = F_{U_i}(u_i)$ for $U_i \sim U[0, 1]$ for all $i = 1, \dots, n$, which means that the margins of Π are uniformly distributed.

Proposition 2.17 *The random variables X_1, \dots, X_n are independent if and only if their joint distribution function can be presented by the product copula Π .*

Proof: Let X_1, \dots, X_n be independent with distribution functions F_1, \dots, F_n and a joint distribution function $F = F_1 \cdots F_n$. Then

$$\Pi(F_1(x_1) \dots F_n(x_n)) = F_1(x_1) \cdots F_n(x_n) = F(x_1, \dots, x_n),$$

i.e. Π is a copula of X_1, \dots, X_n .

Now let us prove the other direction. Let X_1, \dots, X_n be random variables with the copula Π . Then

$$\begin{aligned} & \mathbb{P}(X_1 \in (-\infty, x_1], \dots, X_n \in (-\infty, x_n]) \\ &= \mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n) = \Pi(F_1(x_1), \dots, F_n(x_n)) \\ &= \prod_{i=1}^n F_i(x_i) = \prod_{i=1}^n \mathbb{P}(X_i \leq x_i) = \prod_{i=1}^n \mathbb{P}(X_i \in (-\infty, x_i]). \end{aligned}$$

Since $\mathcal{B}(\mathbb{R}) = \sigma\{(-\infty, x] : x \in \mathbb{R}\}$, we get by the Uniqueness Theorem, which uses Π -systems, that

$$\mathbb{P}(X_1 \in B_1, \dots, X_n \in B_n) = \mathbb{P}(X_1 \in B_1) \cdots \mathbb{P}(X_n \in B_n)$$

for all $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$, which means that the random variables X_1, \dots, X_n are independent. \square

Proposition 2.18 *If the distribution functions F_1, \dots, F_n of X_1, \dots, X_n are continuous, then the copula C of X_1, \dots, X_n is unique.*

Proof: Let F be the joint distribution function of X_1, \dots, X_n and assume that C and C' are two copulas that satisfy equation (2). Then for all $(u_1, \dots, u_n) \in [0, 1]^n$ one has that

$$\begin{aligned} C(u_1, \dots, u_n) &= C(F_1(F_1^{-1}(u_1)), \dots, F_n(F_n^{-1}(u_n))) \\ &= F(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)) \\ &= C'(F_1(F_1^{-1}(u_1)), \dots, F_n(F_n^{-1}(u_n))) \\ &= C'(u_1, \dots, u_n). \end{aligned}$$

\square

One can find examples of random variables with non-unique copulas. For instance, let X and Y be random variables with distribution functions

$F_X(x) = \mathbb{1}_{\{x \geq a\}}(x)$ and $F_Y(y) = \mathbb{1}_{\{y \geq b\}}(y)$, where $a, b \in \mathbb{R}$. We get that

$$C_l(F_X(x), F_Y(y)) = (F_X(x) + F_Y(y) - 1, 0)^+ = \begin{cases} 0, & \text{if } x < a \text{ or } y < b, \\ 1, & \text{if } x \geq a \text{ and } y \geq b \end{cases}$$

and

$$C_u(F_X(x), F_Y(y)) = \min\{F_X(x), F_Y(y)\} = \begin{cases} 0, & \text{if } x < a \text{ or } y < b, \\ 1, & \text{if } x \geq a \text{ and } y \geq b, \end{cases}$$

which implies that $F_{X,Y}(x, y) = C(F_X(x), F_Y(y))$ for any copula C .

3 Lower bound for functions of dependent risks

Let X_1, \dots, X_n be n real-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with given distribution functions $F_i(x) := \mathbb{P}(\omega : X_i(\omega) \leq x)$ for $i = 1, \dots, n$. The random vector $X := (X_1, \dots, X_n)$ can be seen as a vector of one-period financial (or insurance) risks. Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Borel-function of these risks, for instance the sum. We consider the problem of bounding from below the distribution function of the random variable $\psi(X)$, over the class of possible distribution functions for X having fixed margins. In fact, we search for

$$m_\psi^+(s) := \inf \{ \mathbb{P}(\{\omega : \psi(X(\omega)) \leq s\}) : X_i \sim F_i, i = 1, \dots, n \}.$$

Let C be a copula and μ_C be the image-measure of X_1, \dots, X_n with C -dependence structure and define

$$\begin{aligned} \sigma_{C,\psi}^+(F_1, \dots, F_n)(s) &:= \mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) \\ &= \int_{\{y \in \mathbb{R}^n : \psi(y) \leq s\}} C(F_1(x_1), \dots, F_n(x_n)). \end{aligned}$$

Now we can denote $m_\psi^+(s) = \inf \{ \mu_C(\psi(y) \leq s) : C \text{ is a copula} \}$. For a function $C_L : [0, 1]^n \rightarrow [0, 1]$ define

$$\tau_{C_L,\psi}(F_1, \dots, F_n)(s) := \sup_{x_1, \dots, x_{n-1} \in \mathbb{R}} C_L(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n^-(\psi_{x_{-n}}^\wedge(s)))$$

and

$$\tau_{C_L,\psi}^+(F_1, \dots, F_n)(s) := \sup_{x_1, \dots, x_{n-1} \in \mathbb{R}} C_L(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n(\psi_{x_{-n}}^\wedge(s))),$$

where $F_n^-(x_n)$ denotes the left limit of F_n at x_n , $\psi_{x_{-n}}^\wedge(s)$ is the right-continuous inverse of the function $\psi_{x_{-n}}(y) := \psi(x_{-n}, y)$, which means

$$\psi_{x_{-n}}^\wedge(s) := \sup\{x_n \in \mathbb{R} : \psi(x_{-n}, x_n) \leq s\}$$

and

$$x_{-n} := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

For $n = 2$ we will also use the notations $\psi_{x_1}^\wedge(s) := \sup\{x_2 : \psi(x_1, x_2) \leq s\}$ and $\psi_{x_2}^\wedge(s) := \sup\{x_1 : \psi(x_1, x_2) \leq s\}$.

3.1 A lower bound using partial information about dependence

Assume that we have some information about the dependence structure of X_1, \dots, X_n in form that we have a lower bound C_L for the copula C of the portfolio. In this case we can reduce our search to

$$\begin{aligned} m_\psi^+(s) &= \inf \{ \sigma_{C,\psi}^+(F_1, \dots, F_n)(s) : C \geq C_L \} \\ &= \inf \{ \mathbb{P}(\psi(X) \leq s) : X_i \sim F_i, i = 1, \dots, n, F_X \geq C_L(F_1, \dots, F_n) \}. \end{aligned}$$

This kind of dependence information can be used in evaluating $m_\psi^+(s)$. Unlike in [3] and [4], we consider right-continuous distributions and provide a lower bound for $m_\psi^+(s)$ that was first introduced in [6]. Also a slightly more general lower bound will be introduced.

In [5] the distribution functions are considered to be right-continuous, and in Theorem 3.1 of [5], it is shown that for all $s \in \mathbb{R}$

$$\mu_C(\psi(X) < s) \geq \sup_{x_1, \dots, x_{n-1} \in \mathbb{R}} C_L(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n^-(\psi_{x_{-n}}^{-1}(s))),$$

where $\psi_{x_{-n}}^{-1}(s) = \sup\{x_n \in \mathbb{R} : \psi(x_{-n}, x_n) < s\}$ is the left-continuous inverse of the function $\psi_{x_{-n}}(y) = \psi(x_{-n}, y)$. It provides a lower bound for the distribution of $\psi(X)$.

In the following theorem we improve this bound by taking the right-continuous inverse $\psi_{x_{-n}}^\wedge(s)$ instead of $\psi_{x_{-n}}^{-1}(s)$. We will also show as was done in [6], Theorem 3.1, that if ψ is left-continuous in its last coordinate, it is not necessary to take the left limit of F_n . The proof follows the proof of Theorem 3.1 in [5].

Theorem 3.1 *Let $X = (X_1, \dots, X_n) : \Omega \rightarrow \mathbb{R}^n$ be a random vector on $(\Omega, \mathcal{F}, \mathbb{P})$ and let F_1, \dots, F_n be the distribution functions of X_1, \dots, X_n , respectively. Let C be their copula. Assume there exists a copula (or a grounded function) $C_L : [0, 1]^n \rightarrow [0, 1]$ such that $C(u) \geq C_L(u)$ for all $u \in [0, 1]^n$. If $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function, which is non-decreasing in each coordinate, then for every $s \in \mathbb{R}$ we have*

$$\sigma_{C,\psi}^+(F_1, \dots, F_n)(s) \geq \tau_{C_L,\psi}(F_1, \dots, F_n)(s). \quad (13)$$

If, in addition, ψ is left-continuous in its last coordinate, we have for all $s \in \mathbb{R}$ that

$$\sigma_{C,\psi}^+(F_1, \dots, F_n)(s) \geq \tau_{C_L,\psi}^+(F_1, \dots, F_n)(s). \quad (14)$$

Proof: By Corollary 2.8, for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$\begin{aligned}
& \mu_C(\{y \in \mathbb{R}^n : y_1 \leq x_1, \dots, y_{n-1} \leq x_{n-1}, y_n < x_n\}) \\
&= \mu_C\left(\bigcup_{k=1}^{\infty} \left\{y \in \mathbb{R}^n : y_1 \leq x_1, \dots, y_{n-1} \leq x_{n-1}, y_n \leq x_n - \frac{1}{k}\right\}\right) \\
&= \lim_{k \rightarrow \infty} \mu_C\left(\left\{y \in \mathbb{R}^n : y_1 \leq x_1, \dots, y_{n-1} \leq x_{n-1}, y_n \leq x_n - \frac{1}{k}\right\}\right) \\
&= \lim_{k \rightarrow \infty} C\left(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n\left(x_n - \frac{1}{k}\right)\right) \\
&= C\left(F_1(x_1), \dots, F_{n-1}(x_{n-1}), \lim_{k \rightarrow \infty} F_n\left(x_n - \frac{1}{k}\right)\right) \\
&= C\left(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n^-(x_n)\right).
\end{aligned}$$

Let $s \in \mathbb{R}$. We want to show the inequality

$$\begin{aligned}
\sigma_{C,\psi}^+(F_1, \dots, F_n)(s) &= \mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) \\
&\geq \sup_{x_{-n} \in \mathbb{R}^{n-1}} C_L(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n^-(\psi_{x_{-n}}^\wedge(s))) \\
&= \tau_{C_L,\psi}(F_1, \dots, F_n)(s).
\end{aligned}$$

For this, let $(a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$ and define

$$a_n := \psi_{a_{-n}}^\wedge(s) = \sup\{x_n \in \mathbb{R} : \psi(a_{-n}, x_n) \leq s\}.$$

We will consider the cases $a_n \in \mathbb{R}$, $a_n = \infty$ and $a_n = -\infty$ and show that for $a_n = \infty$ and $a_n = -\infty$ it holds

$$\begin{aligned}
\mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) &\geq C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n(a_n)) \\
&\geq C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n^-(a_n)).
\end{aligned}$$

In the case a_n is finite, we will show that

$$\mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) \geq C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n^-(a_n))$$

and if ψ is left-continuous in its last coordinate, then

$$\mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) \geq C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n(a_n)).$$

If $a_n = \infty$, then $\psi(a_1, \dots, a_{n-1}, x_n) \leq s$ for all $x_n \in \mathbb{R}$ and

$$\begin{aligned}
\mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) &\geq \mu_C(\{y \in \mathbb{R}^n : y_1 \leq a_1, \dots, y_{n-1} \leq a_{n-1}, y_n \in \mathbb{R}\}) \\
&= C(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n(\infty)) \\
&= C(F_1(a_1), \dots, F_n(a_n)).
\end{aligned}$$

If $a_n = -\infty$, then $\psi(a_1, \dots, a_{n-1}, x_n) > s$ for all $x_n \in \mathbb{R}$ and

$$\begin{aligned} C_L(F_1(a_1), \dots, F_n(a_n)) &= C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n(-\infty)) \\ &= C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), 0) \\ &= 0 \\ &\leq \mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}). \end{aligned}$$

Assume now that a_n is finite. Then $\psi(y) \leq s$ for all $y \in \mathbb{R}^n$ such that $y_1 \leq a_1, \dots, y_{n-1} \leq a_{n-1}$ and $y_n < a_n$ so that

$$\{y \in \mathbb{R}^n : y_1 \leq a_1, \dots, y_{n-1} \leq a_{n-1}, y_n < a_n\} \subseteq \{y \in \mathbb{R}^n : \psi(y) \leq s\}.$$

Hence

$$\begin{aligned} \mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) &\geq \mu_C(\{y \in \mathbb{R}^n : y_1 \leq a_1, \dots, y_{n-1} \leq a_{n-1}, y_n < a_n\}) \\ &= C(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n^-(a_n)) \\ &\geq C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n^-(a_n)). \end{aligned}$$

If ψ is left-continuous in its last coordinate, then $\psi(y) \leq s$ for all $y \in \mathbb{R}^n$ such that $y_i \leq a_i$ for all $i = 1, \dots, n$. Hence

$$\{y \in \mathbb{R}^n : y_1 \leq a_1, \dots, y_n \leq a_n\} \subseteq \{y \in \mathbb{R}^n : \psi(y) \leq s\}$$

and

$$\begin{aligned} \mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) &\geq \mu_C(\{y \in \mathbb{R}^n : y_1 \leq a_1, \dots, y_n \leq a_n\}) \\ &= C(F_1(a_1), \dots, F_n(a_n)) \\ &\geq C_L(F_1(a_1), \dots, F_n(a_n)). \end{aligned}$$

Since $C_L(F_1(a_1), \dots, F_n(a_n)) \geq C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n^-(a_n))$ and $a_n = \psi_{a_n}^\wedge(s)$, we have that

$$\mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) \geq C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n^-(\psi_{a_n}^\wedge(s)))$$

for all $a_{-n} \in \mathbb{R}^{n-1}$. Hence

$$\begin{aligned} \sigma_{C,\psi}^+(F_1, \dots, F_n)(s) &= \mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) \\ &\geq \sup_{a_{-n} \in \mathbb{R}^{n-1}} C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n^-(\psi_{a_{-n}}^\wedge(s))) \\ &= \tau_{C_L,\psi}(F_1, \dots, F_n)(s). \end{aligned}$$

If ψ is left-continuous in its last coordinate, then we have

$$\mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) \geq C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n(\psi_{a_n}^\wedge(s)))$$

for all $a_{-n} \in \mathbb{R}^{n-1}$, which implies

$$\begin{aligned}\sigma_{C,\psi}^+(F_1, \dots, F_n)(s) &= \mu_C(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) \\ &\geq \sup_{a_{-n} \in \mathbb{R}^{n-1}} C_L(F_1(a_1), \dots, F_{n-1}(a_{n-1}), F_n(\psi_{a_{-n}}^\wedge(s))) \\ &= \tau_{C_L,\psi}^+(F_1, \dots, F_n)(s).\end{aligned}$$

□

Remark 3.2 Taking the right-continuous inverse $\psi_{x_{-n}}^\wedge(s)$ instead of left-continuous $\psi_{x_{-n}}^{-1}(s)$ actually improves the bound, i.e. it can be

$$\tau_{C_L,\psi}(F_1, \dots, F_n)(s) > \sup_{x_1, \dots, x_{n-1} \in \mathbb{R}} C_L(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n^-(\psi_{x_{-n}}^{-1}(s))).$$

For instance, take $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\psi(x_1, x_2) := \mathbb{I}_{(a,\infty)^2}(x_1, x_2)$ for some positive real number a . Then for $s = 0$ it holds

$$\psi_{x_{-n}}^{-1}(0) = \inf\{x_2 : \psi(x_1, x_2) \geq 0\} = \inf \mathbb{R} = -\infty$$

and

$$\psi_{x_{-n}}^\wedge(0) = \sup\{x_2 : \psi(x_1, x_2) \leq 0\} = \begin{cases} a, & \text{if } x_1 > a \\ \infty, & \text{if } x_1 \leq a. \end{cases}$$

Now

$$\begin{aligned}\tau_{C_L,\psi}(F_1, F_2) &= \sup_{x_1 \in \mathbb{R}} C_L(F_1(x_1), F_2^-(\psi_{x_1}^\wedge(0))) \\ &\geq C_L(F_1(a), F_2(\infty)) = F_1(a)\end{aligned}$$

and

$$\sup_{x_1 \in \mathbb{R}} C_L(F_1(x_1), F_2^-(\psi_{x_1}^{-1}(0))) \leq C_L(F_1(\infty), F_2(-\infty)) = 0.$$

Remark 3.3 Since for the lower Fréchet bound C_l it holds $C \geq C_l$ for every copula C and C_l is grounded, this bound will hold also for C_l . Thus we get a lower bound also in the case that there is no information on dependence.

Now one might ask how sharp are the bounds (13) and (14). We will show that if ψ is left-continuous in its last coordinate, then for any copula C_L and any fixed $s \in \mathbb{R}$ there exists a copula $C \geq C_L$ such that equality holds in (14) for $n = 2$. First we introduce the copula in a lemma, then we will show that it attains the bound.

Lemma 3.4 Let $C_L : [0, 1]^2 \rightarrow [0, 1]$ be a copula and define for $t \in [0, 1]$ the function $C_t : [0, 1]^2 \rightarrow [0, 1]$ as follows:

$$C_t(u_1, u_2) = \begin{cases} \max\{t, C_L(u_1, u_2)\}, & \text{when } (u_1, u_2) \in [t, 1]^2, \\ \min\{u_1, u_2\} & \text{otherwise.} \end{cases}$$

Then the function C_t is a copula.

Proof: We have to show that C_t fulfills the two conditions of Definition 2.1.

1. Let $(u_1, u_2) \in [0, 1]^2$ have at least one coordinate equal to 0. If $t > 0$, then $C_t(u_1, u_2) = \min\{u_1, u_2\} = 0$. If $t = 0$, then $C_t(u_1, u_2) = \max\{t, C_L(u_1, u_2)\} = 0$, because $t = 0$ and $C_L(u_1, u_2) = 0$ since C_L is a copula.

$C_t(u_1, 1) = u_1$ and $C_t(1, u_2) = u_2$, because for instance in the first case

$$\begin{aligned} C_t(u_1, 1) &= \begin{cases} \max\{t, C_L(u_1, 1)\}, & \text{when } u_1 \geq t, \\ \min\{u_1, 1\} & \text{otherwise,} \end{cases} \\ &= \begin{cases} \max\{t, u_1\}, & \text{when } u_1 \geq t, \\ \min\{u_1, 1\} & \text{otherwise,} \end{cases} \\ &= u_1. \end{aligned}$$

2. Let us show that C_t is 2-increasing. Take $(a_1, a_2) \in [0, 1]^2$ and $(b_1, b_2) \in [0, 1]^2$ such that $a_1 \leq b_1$ and $a_2 \leq b_2$. We have to show that

$$V_{C_t}((a_1, b_1] \times (a_2, b_2]) = C_t(b_1, b_2) - C_t(b_1, a_2) - C_t(a_1, b_2) + C_t(a_1, a_2) \geq 0.$$

Assume first that $a_1, a_2 \geq t$. If $C_L(a_1, a_2) \geq t$, then

$$V_{C_t}((a_1, b_1] \times (a_2, b_2]) = V_{C_L}((a_1, b_1] \times (a_2, b_2]) \geq 0,$$

since C_L is a copula. If $C_L(a_1, a_2) \leq t$, $C_L(a_1, b_2) \geq t$ and $C_L(a_2, b_1) \geq t$, then

$$\begin{aligned} V_{C_t}((a_1, b_1] \times (a_2, b_2]) &= C_L(b_1, b_2) - C_L(b_1, a_2) - C_L(a_1, b_2) + t \\ &\geq C_L(b_1, b_2) - C_L(b_1, a_2) - C_L(a_1, b_2) + C_L(a_1, a_2) \geq 0. \end{aligned}$$

If $C_L(a_1, b_2) \geq t$ and $C_L(a_2, b_1) \leq t$, then

$$V_{C_t}((a_1, b_1] \times (a_2, b_2]) = C_L(b_1, b_2) - C_L(a_1, b_2) - t + t,$$

which is non-negative since C_L is increasing. The same is true for $C_L(a_1, b_2) \leq t$ and $C_L(a_2, b_1) \geq t$. If $C_L(b_1, b_2) \geq t$, $C_L(a_1, b_2) \leq t$ and $C_L(a_2, b_1) \leq t$, then

$$V_{C_t}((a_1, b_1] \times (a_2, b_2]) = C_L(b_1, b_2) - t - t + t \geq 0.$$

In the case that $C_L(b_1, b_2) \leq t$ the volume $V_{C_t}((a_1, b_1] \times (a_2, b_2]) = t - t - t + t = 0$.

Now assume that $a_1 < t$ or $a_2 < t$. If $a_1 \leq a_2$, then

$$\begin{aligned} & V_{C_t}((a_1, b_1] \times (a_2, b_2]) \\ &= C_t(b_1, b_2) - C_t(b_1, a_2) - \min\{a_1, b_2\} + \min\{a_1, a_2\} \\ &= C_t(b_1, b_2) - C_t(b_1, a_2) - a_1 + a_1 \\ &\geq 0, \end{aligned}$$

since C_t is increasing. The same holds when $a_2 \leq a_1$. \square

In [5] (Theorem 3.2) it was shown that for the copula C_t from the previous lemma it holds

$$\mu_{C_t}(\psi(X) < s) = \sup_{x_1 \in \mathbb{R}} C_L(F_1(x_1), F_2^-(\psi_{x_1}^{-1}(s))) \quad (15)$$

for an arbitrary but fixed number s , which means that the right-hand side of (15) provides a sharp lower bound for $m_\psi^+(s)$ if $\mu_{C_t}(\psi(X) \leq s) = \mu_{C_t}(\psi(X) < s)$. However in case that $\mu_{C_t}(\psi(X) \leq s) > \mu_{C_t}(\psi(X) < s)$, this bound fails to be sharp.

In [4] (Theorem 3.3.3/1) the same result, i.e. equality (15), was proved for $n = 2$ and left-continuous distribution functions. Furthermore, it was assumed that ψ is continuous and at least one of F_1 and F_2 should be continuous in a certain set of \mathbb{R}^2 .

We provide a stronger result for right-continuous distribution functions with less assumptions concerning ψ and the marginal distributions. The following theorem shows that for an increasing function ψ which is continuous in its last coordinate, the copula C_t reaches the bound (14) stated in Theorem 3.1. To show this we have combined the proofs in [4] (Theorem 3.3.3/1) and a former version of [5] (Theorem 3.2) which no longer exists. In [5] the theorem was formulated for an arbitrary n , but unfortunately, C_t is no longer a copula for $n > 2$ as will be shown later.

Theorem 3.5 *Let $X = (X_1, X_2)$ be a random vector on \mathbb{R}^2 with marginal distribution functions F_1 and F_2 . Let $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function, that is non-decreasing in each coordinate and left-continuous in the second coordinate for each fixed first coordinate. Fix $s \in \mathbb{R}$ and for a copula C_L , set $t := \tau_{C_L, \psi}^+(F_1, F_2)(s)$. If the corresponding copula of X_1 and X_2 is the copula C_t from Lemma 3.4 defined for C_L and t , then*

$$\sigma_{C_t, \psi}^+(F_1, F_2)(s) = \tau_{C_L, \psi}^+(F_1, F_2)(s).$$

Proof: We already have that

$$\sigma_{C_t, \psi}^+(F_1, F_2)(s) \geq \tau_{C_L, \psi}^+(F_1, F_2)(s), \quad (16)$$

since

$$C_L(u) \leq \max\{t, C_L(u)\} = C_t(u) \quad \text{for } u \in [t, 1]^2$$

and

$$C_L(u) \leq \min\{u_1, u_2\} = C_t(u) \quad \text{for } u \in [0, 1]^2 \setminus [t, 1]^2$$

i.e. $C_t \geq C_L$, which implies by Theorem 3.1 that

$$\sigma_{C_t, \psi}^+(F_1, F_2)(s) \geq \tau_{C_L, \psi}^+(F_1, F_2)(s).$$

Hence we have to prove the inequality

$$\sigma_{C_t, \psi}^+(F_1, F_2)(s) \leq \tau_{C_L, \psi}^+(F_1, F_2)(s) = t.$$

We consider the cases $t = 1$, and $t \in [0, 1)$ separately. For $t = 1$ it is clear, since

$$\sigma_{C_t, \psi}^+(F_1, F_2)(s) = \mu_{C_t}(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) \leq 1.$$

Let $t \in [0, 1)$ and consider the set

$$B_s := \{x \in \mathbb{R}^2 : \psi(x) \leq s\}.$$

We want to show that $\mu_{C_t}(B_s) = \mu_{C_t}(\{y \in \mathbb{R}^n : \psi(y) \leq s\}) \leq t$. If $B_s = \emptyset$, then for all $t \in [0, 1)$

$$\sigma_{C_t, \psi}^+(F_1, F_2)(s) = \mu_{C_t}(B_s) = \mu_{C_t}(\emptyset) = 0 \leq t.$$

Assume that B_s is non-empty. We show that $C_L(F_1(a_1), F_2(a_2)) \leq t$ for all $(a_1, a_2) \in B_s$, and that this implies $C_t(F_1(a_1), F_2(a_2)) = \min\{t, F_1(x_1), F_2(x_2)\}$. Then we will show for $t = 0$ and $t \in (0, 1)$ that $\sigma_{C_t, \psi}^+(F_1, F_2)(s) \leq t$.

Let $a = (a_1, a_2) \in B_s$. Then $\psi(a) \leq s$ and

$$\psi_{a_1}^\wedge(s) = \sup\{x_2 \in \mathbb{R} : \psi(a_1, x_2) \leq s\} \geq a_2.$$

Now

$$\begin{aligned} C_L(F_1(a_1), F_2(a_2)) &\leq C_L(F_1(a_1), F_2(\psi_{a_1}^\wedge(s))) \\ &\leq \sup_{x_1 \in \mathbb{R}} C_L(F_1(x_1), F_2(\psi_{x_1}^\wedge(s))) \end{aligned}$$

$$\begin{aligned}
&= \tau_{C_L, \psi}^+(F_1, F_2)(s) \\
&= t.
\end{aligned}$$

Hence for all $x \in B_s$ it holds

$$C_L(F_1(x_1), F_2(x_2)) \leq t. \quad (17)$$

Now recall that

$$\begin{aligned}
&C_t(F_1(x_1), F_2(x_2)) \\
&= \begin{cases} \max\{t, C_L(F_1(x_1), F_2(x_2))\}, & \text{if } F_1(x_1), F_2(x_2) \in [t, 1]^2 \\ \min\{F_1(x_1), F_2(x_2)\}, & \text{otherwise.} \end{cases}
\end{aligned}$$

By (17) this implies for all $x \in B_s$

$$C_t(F_1(x_1), F_2(x_2)) = \min\{t, F_1(x_1), F_2(x_2)\}. \quad (18)$$

If $t = 0$, then

$$\begin{aligned}
C_t(F_1(x_1), F_2(x_2)) &= C_0(F_1(x_1), F_2(x_2)) \\
&= \min\{0, F_1(x_1), F_2(x_2)\} = 0
\end{aligned}$$

for all $x \in B_s$. Define $Z : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \rightarrow \mathbb{R}$, $Z(x) := 0$. Then for all $x \in B_s$ we have $C_0(F_1(x_1), F_2(x_2)) = Z(x_1, x_2)$ and

$$\begin{aligned}
\sigma_{C_0, \psi}^+(F_1, F_2) &= \int_{B_s} dC_0(F_1(x_1), F_2(x_2)) \\
&= \int_{B_s} dZ(x_1, x_2) \\
&\leq \int_{\mathbb{R}^2} dZ(x_1, x_2) \\
&= Z(\infty, \infty) - Z(\infty, -\infty) - Z(-\infty, \infty) + Z(-\infty, -\infty) \\
&= 0.
\end{aligned}$$

Now let $t \in (0, 1)$ and define $b = (b_1, b_2)$,

$$\begin{aligned}
b_1 &:= \sup\{x_1 : F_1(x_1) < t\}, \text{ and} \\
b_2 &:= \psi_{b_1}^\wedge(s) = \sup\{x_2 \in \mathbb{R} : \psi(b_1, x_2) \leq s\}.
\end{aligned}$$

We will show that

$$\mu_{C_t}(B_s) \leq \mu_{C_t}((-\infty, b_1] \times (-\infty, b_2]) \leq t.$$

Since F_1 is a distribution function, $b_1 = \sup \{x_1 : F_1(x_1) < t\} = \infty$ implies $t = 1$, and $b_1 = \sup \{x_1 : F_1(x_1) < t\} = -\infty$ means $t = 0$, when we define $\sup \emptyset := -\infty$. Hence b_1 is finite. Since F_1 is increasing the definition of b_1 implies that

$$\text{if } x_1 < b_1, \text{ then } F_1(x_1) < t. \quad (19)$$

Since F_1 is right-continuous and $F_1(x_1) \geq t$ for all $x_1 > b_1$ we have $F_1(b_1) \geq t$, so that

$$F_1(x_1) \geq t \text{ for all } x_1 \geq b_1. \quad (20)$$

Next we show that $F_2(x_2) \geq t$ whenever $x_2 \geq b_2$. To show this, assume that there is a number $x'_2 > b_2$ such that $F_2(x'_2) < t$. Define

$$x'_1 := \psi_{x'_2}^\wedge(s) = \sup \{x_1 : \psi(x_1, x'_2) \leq s\}.$$

In the case $x'_1 \geq b_1$ it would hold

$$b_2 = \sup \{x_2 : \psi(b_1, x_2) \leq s\} \geq \sup \{x_2 : \psi(x'_1, x_2) \leq s\} \geq x'_2,$$

which contradicts to the assumption that $x'_2 > b_2$. Hence $x'_1 < b_1$ and by (19) we have $F_1(x'_1) < t$. Now for all $x_1 \leq x'_1$

$$C_L(F_1(x_1), F_2(\psi_{x'_1}^\wedge(s))) \leq F_1(x_1) \leq F_1(x'_1) < t.$$

For $x_1 > x'_1$ it holds $x'_2 \geq \psi_{x_1}^\wedge(s) = \sup \{y_2 \in \mathbb{R} : \psi(x_1, y_2) \leq s\}$, because if $x'_2 < \psi_{x_1}^\wedge(s)$, then

$$x'_1 = \sup \{y_1 \in \mathbb{R} : \psi(y_1, x'_2) \leq s\} \geq \sup \{y_1 \in \mathbb{R} : \psi(y_1, \psi_{x_1}^\wedge(s)) \leq s\} \geq x_1,$$

which contradicts to $x_1 > x'_1$. Hence $x'_2 \geq \psi_{x_1}^\wedge(s)$, which implies

$$C_L(F_1(x_1), F_2(\psi_{x_1}^\wedge(s))) \leq F_2(\psi_{x_1}^\wedge(s)) \leq F_2(x'_2) < t.$$

Hence for all $x_1 > x'_1$ it holds

$$C_L(F_1(x_1), F_2(\psi_{x_1}^\wedge(s))) \leq C_L(1, F_2(x'_2)) = F_2(x'_2) < t.$$

Now we have that $C_L(F_1(x_1), F_2(\psi_{x_1}^\wedge(s))) \leq \max\{F_1(x'_1), F_2(x'_2)\} < t$ for all $x_1 \in \mathbb{R}$ which implies

$$\begin{aligned} t &> \max\{F_1(x'_1), F_2(x'_2)\} \\ &\geq \sup_{x_1 \in \mathbb{R}} C_L(F_1(x_1), F_2(\psi_{x_1}^\wedge(s))) \\ &= \tau_{C_L, \psi}^+(F_1, F_2)(s) \\ &= t, \end{aligned}$$

which is a contradiction and we have that $F_2(x_2) \geq t$ for all $x_2 > b_2$. Because F_2 is right-continuous, it follows that $F_2(b_2) \geq t$ so that

$$F_2(x_2) \geq t \text{ for all } x_2 \geq b_2. \quad (21)$$

From (19), (20) and (21) we get for all $x \in B_s$,

$$\begin{aligned} & C_t(F_1(x_1), F_2(x_2)) \\ &= \min\{t, F_1(x_1), F_2(x_2)\} \\ &= \begin{cases} \min\{F_1(x_1), F_2(x_2)\}, & \text{if } x_1 < b_1 \text{ and } x_2 < b_2, \\ \min\{t, F_1(x_1)\}, & \text{if } x_2 \geq b_2, \\ \min\{t, F_2(x_2)\}, & \text{if } x_1 \geq b_1. \end{cases} \end{aligned} \quad (22)$$

We want to show that

$$\sigma_{C_t, \psi}^+(F_1, F_2) = \mu_{C_t}(B_s) \leq t.$$

Let us consider the following covering of B_s .

$$B_s \subseteq I_1 \cup I_2 \cup T,$$

where

$$I_i := \{x \in \mathbb{R}^2 : x_i > b_i\} \cap B_s, \text{ for } i = 1, 2$$

and

$$T := (-\infty, b_1] \times (-\infty, b_2].$$

Let us show that $\mu_{C_t}(I_i) = 0$ for $i = 1, 2$. Consider the functions $g_j : \overline{\mathbb{R}}^2 \rightarrow \mathbb{R}$, $g_j(x_1, x_2) := \min\{t, F_j(x_j)\}$ for $j = 1, 2$. Now

$$\begin{aligned} \mu_{C_t}(I_1) &= \int_{I_1} dC_t(F_1(x_1), F_2(x_2)) = \int_{\{x \in \mathbb{R}^2 : x_1 > b_1, \psi(x) \leq s\}} d \min\{t, F_2(x_2)\} \\ &= \int_{\{x \in \mathbb{R}^2 : x_1 > b_1, \psi(x) \leq s\}} dg_2(x_1, x_2) \\ &\leq \int_{(b_1, \infty] \times \mathbb{R}} dg_2(x_1, x_2) \\ &= g_2(\infty, \infty) - g_2(b_1, \infty) - g_2(\infty, -\infty) + g_2(b_1, -\infty) \\ &= \min\{t, F_2(\infty)\} - \min\{t, F_2(\infty)\} \\ &\quad - \min\{t, F_2(-\infty)\} + \min\{t, F_2(-\infty)\} \\ &= 0. \end{aligned}$$

In the same way we get

$$\mu_{C_t}(I_2) = \int_{I_2} d \min\{t, F_1(x_1)\} \leq \int_{\mathbb{R} \times (b_2, \infty]} dg_1(x_1, x_2) = 0.$$

Since ψ is left-continuous in the second coordinate and $b_2 = \sup\{x_2 \in \mathbb{R} : \psi(b_1, x_2) \leq s\}$, we have $b = (b_1, b_2) \in B_s$. Hence by (22)

$$\mu_{C_t}(T) = \mu_{C_t}((-\infty, b_1] \times (-\infty, b_2]) = C_t(F_1(b_1), F_2(b_2)) = t.$$

We get

$$\mu_{C_t}(B_s) \leq \mu_{C_t}(I_1 \cup I_2 \cup T) \leq \mu_{C_t}(I_1) + \mu_{C_t}(I_2) + \mu_{C_t}(T) = \mu_{C_t}(T) = t.$$

Since $t \leq \sigma_{C_t, \psi}^+(F_1, F_2)(s) = \mu_{C_t}(B_s) \leq t$, it follows that

$$\sigma_{C_t, \psi}^+(F_1, F_2)(s) = t.$$

□

Remark 3.6 This result was formulated in [6] for arbitrary n , but the proof contains a gap. See [5] (page 11) for more details. In [5] (page 10) it is also noted that the result in [6] is not correct since $\mu_C(\psi(X) \leq s)$ may have no minimum over the set of copulas. This is closer studied in [7] (page 187).

However, for $n = 2$ the minimum exists: Since $C_t \geq C_L$, we have $\mu_{C_t}(\psi(X) \leq s) \geq \inf\{\mu_C(\psi(X) \leq s) : C \geq C_L\}$. By Theorem 3.1 for all $C \geq C_L$ it holds $\mu_C(\psi(X) \leq s) \geq \tau_{C_L, \psi}^+(F_1, F_2)(s)$, hence by Theorem 3.5 $\inf\{\mu_C(\psi(X) \leq s) : C \geq C_L\} \geq \tau_{C_L, \psi}^+(F_1, F_2)(s) = \mu_{C_t}(\psi(X) \leq s)$. It follows that

$$\mu_{C_t}(\psi(X) \leq s) = \min\{\mu_C(\psi(X) \leq s) : C \geq C_L\}$$

and choosing $C_L = C_l$, where C_l is the lower Fréchet bound, we have

$$\mu_{C_t}(\psi(X) \leq s) = \min\{\mu_C(\psi(X) \leq s) : C \text{ is a copula}\}.$$

By the following proposition, which has already been published in [5] (Example 3.1), we obtain that the previous result holds only for $n = 2$.

Proposition 3.7 *There exist numbers s such that the function C_t , defined for $t = \tau_{C_L, \psi}^+(F_1, \dots, F_n)(s)$, by*

$$C_t(u) := \begin{cases} \max\{t, C_L(u)\}, & \text{for } u \in [t, 1]^n \\ \min\{u_1, \dots, u_n\}, & \text{otherwise} \end{cases}$$

is not a copula for $n > 2$.

Proof: Let $n > 2$ and choose

$$\psi(x_1, \dots, x_n) = x_1 \cdots x_n \mathbb{1}_{\{x_1 \geq 0, \dots, x_n \geq 0\}}(x_1, \dots, x_n).$$

Let $C_L(u) := \prod_{i=1}^n u_i$ and let F_1, \dots, F_n be the distribution functions of n standard uniformly distributed random variables. For $s > 0$ it holds

$$\begin{aligned} \psi_{-n}^\wedge(s) &= \inf\{x_n \in \mathbb{R} : x_1 \cdots x_n \mathbb{1}_{\{x_1 \geq 0, \dots, x_n \geq 0\}}(x_1, \dots, x_n) \geq s\} \\ &= \begin{cases} \infty, & \text{if } x_i \leq 0 \text{ for some } i = 1, \dots, n-1, \\ \frac{s}{x_1 \cdots x_{n-1}}, & \text{if } x_i > 0 \text{ for all } i = 1, \dots, n-1, \end{cases} \end{aligned}$$

hence for $s \in (0, 1)$ we get

$$\begin{aligned} t &= \tau_{C_L, \psi}^+(F_1, \dots, F_n)(s) \\ &= \sup_{x_1, \dots, x_{n-1} \in \mathbb{R}} C_L(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n(\psi_{-n}^\wedge(s))) \\ &= \sup_{x_1, \dots, x_{n-1} \in (0, 1]} \left(\prod_{i=1}^{n-1} F_i(x_i) \right) F_n\left(\frac{s}{x_1 \cdots x_{n-1}}\right) \\ &= s. \end{aligned}$$

Set $s = \left(\frac{n}{2(n-1)}\right)^2 = t \in (0, 1)$ and let $\alpha = \frac{n}{2(n-1)}$. Choose $a, b \in \mathbb{R}^n$ such that

$$a = (\alpha, \dots, \alpha) \text{ and } b = (1, \dots, 1).$$

We will show that C_t is not n -increasing i.e. the sum

$$\sum_{j_1=1}^2 \cdots \sum_{j_n=1}^2 (-1)^{j_1 + \dots + j_n} C_t(u_{1j_1}, \dots, u_{nj_n}) \quad (23)$$

is negative. We have $a, b \in [\alpha^2, 1]^n = [t, 1]^n$ and using the notation $u_{i1} = a_i$ and $u_{i2} = b_i$ for $i = 1, \dots, n$ it follows that the sum (23) is equal to

$$\sum_{j_1=1}^2 \cdots \sum_{j_n=1}^2 (-1)^{j_1 + \dots + j_n} \max\left\{t, \prod_{i=1}^n u_{ij_i}\right\}. \quad (24)$$

Since $u_{i1} = \alpha$ and $u_{i2} = 1$ for all $i = 1, \dots, n$, the product $\prod_{i=1}^n u_{ij_i}$ is always of form α^k , where k is the number of u_{i1} 's occuring in the product, having the values $k = 0, 1, \dots, n$. In the sum (24) the case $k = 0$ occurs only once, that is when $j_i = 2$ for all $i = 1, \dots, n$. The case $k = 1$ occurs when $j_i = 1$ for one and only one $i = 1, \dots, n$ which means that it occurs n times. There

are $\binom{n}{2}$ cases where $j_i = 1$ exactly two times and so on. Finally, there is one case when $k = n$. Considering the sum like this, we get that the sum (24), and therefore also the sum (23), is equal to

$$\begin{aligned} & \max\{t, \alpha^0\} - n \max\{t, \alpha^1\} + \binom{n}{2} \max\{t, \alpha^2\} - \binom{n}{3} \max\{t, \alpha^3\} + \\ & \cdots + (-1)^n \binom{n}{n} \max\{t, \alpha^n\} \\ & = \sum_{k=0}^n (-1)^k \binom{n}{k} \max\{t, \alpha^k\}. \end{aligned}$$

From $0 < \alpha < 1$ and $t = \alpha^2$ it follows for $k = 2, \dots, n$ that $t \geq \alpha^k$ and the maximum of t and α^k is equal to t . For the values $k = 0$ and $k = 1$ we have $\max\{t, \alpha^0\} = \max\{t, 1\} = 1$ and $\max\{t, \alpha^1\} = \max\{t, \alpha\} = \alpha$. Hence

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \max\{t, \alpha^k\} = 1 - n\alpha + t \sum_{k=2}^n (-1)^k \binom{n}{k}.$$

Recall that $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$ for all $n = 1, 2, \dots$ which implies that

$$\sum_{k=2}^n (-1)^k \binom{n}{k} = - \sum_{k=0}^1 (-1)^k \binom{n}{k} = n - 1.$$

Now the sum (23) reduces to

$$\begin{aligned} & 1 - n\alpha + t(n - 1) \\ & = 1 - n\alpha + (n - 1)\alpha^2 \\ & = 1 - n \frac{n}{2(n - 1)} + (n - 1) \left(\frac{n}{2(n - 1)} \right)^2 \\ & = 1 - \frac{n^2}{4(n - 1)}, \end{aligned}$$

which is negative for $n > 2$. It means that C_t is not n -increasing, hence it is not a copula. \square

3.2 A lower bound in the case of no information on dependence

The bounds (13) and (14) stated in Theorem 3.1 hold especially for the lower Fréchet bound C_l and (14) is sharp for $n = 2$, but in a special case it can be improved.

Assume for the rest of this chapter that the margins are identically distributed having a continuous distribution F and let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be the sum. Because of continuity we have

$$m_\psi^+(s) = m_\psi(s) := \inf\{\mathbb{P}(\psi(X) < s) : X_i \sim F, i = 1, \dots, n\}$$

and a duality result proved in [10] gives

$$\begin{aligned} m_\psi(s) &= \inf \left\{ \mathbb{P} \left(\left\{ \omega : \sum_{i=1}^n X_i(\omega) < s \right\} \right) : X_i \sim F, i = 1, \dots, n \right\} \\ &= 1 - \inf \left\{ n \int f dF : f \text{ is a bounded measurable function on } \mathbb{R} \text{ s.t.} \right. \\ &\quad \left. \sum_{i=1}^n f(x_i) \geq \mathbb{1}_{[s, \infty)} \left(\sum_{i=1}^n x_i \right) \text{ for all } x \in [0, \infty)^n \right\}. \end{aligned} \quad (25)$$

Using this expression it is shown in [5] that for every $s \geq 0$ it holds

$$m_\psi(s) \geq 1 - \inf_{r \in (0, s/n)} \frac{n}{s - nr} \int_r^{s-(n-1)r} (1 - F(x)) dx.$$

However, if F is very small on the interval $(r, s - (n-1)r)$, the right-hand side of the inequality above will be negative. We specify this bound by excluding the negative cases and we show that this bound will be better (i.e. greater or equal) than the bound (14) for some s . The proof follows the one in [5] (Theorem 4.2).

Theorem 3.8 *For every $s \geq 0$*

$$m_\psi(s) \geq \left(1 - \inf_{r \in (0, s/n)} \frac{n}{s - nr} \int_r^{s-(n-1)r} (1 - F(x)) dx \right)^+.$$

Proof: For $r \in (0, \frac{s}{n})$ define $\hat{f}_r : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$\hat{f}_r(x) := \begin{cases} 0, & x < r \\ \frac{x-r}{s-nr}, & r \leq x < s - (n-1)r \\ 1, & \text{otherwise.} \end{cases}$$

We will first show that \hat{f}_r is an admissible function in (25). \hat{f}_r is bounded and measurable, hence we have to show that $\sum_{i=1}^n \hat{f}_r(x_i) \geq \mathbb{1}_{[s, \infty)}(\sum_{i=1}^n x_i)$ for all $x \in [0, \infty)^n$.

Let $x \in [0, \infty)^n$. Since \hat{f}_r is non-negative, it is sufficient to show that $\sum_{i=1}^n \hat{f}_r(x_i) \geq 1$, when $\sum_{i=1}^n x_i \geq s$. If $x_i \geq s - (n-1)r$ for some $i_0 \in \{1, \dots, n\}$, then $\sum_{i=1}^n \hat{f}_r(x_i) \geq \hat{f}_r(x_{i_0}) = 1$. Assume $x_1, \dots, x_n < s - (n-1)r$ with $\sum_{i=1}^n x_i \geq s$ and define

$$I := \{i \in \{1, \dots, n\} : x_i \geq r\} \quad \text{and} \quad J := \{i \in \{1, \dots, n\} : x_i < r\}.$$

Now since $\sum_{i \in I} x_i + \sum_{j \in J} x_j \geq s$ we have $\sum_{i \in I} x_i \geq s - \sum_{j \in J} x_j > s - \#J \cdot r$. It follows

$$\begin{aligned} \sum_{i=1}^n \hat{f}_r(x_i) &= \sum_{i \in I} \hat{f}_r(x_i) = \sum_{i \in I} \frac{x_i - r}{s - nr} = \frac{\sum_{i \in I} x_i - \#I \cdot r}{s - nr} \\ &> \frac{s - \#J \cdot r - \#I \cdot r}{s - nr} = \frac{s - nr}{s - nr} = 1. \end{aligned}$$

By (25) we have that for all $r \in (0, s/n)$ it holds

$$m_\psi(s) \geq 1 - \inf_{r \in (0, s/n)} n \int_{\mathbb{R}} \hat{f}_r(x) dF(x)$$

and using partial integration we get

$$\begin{aligned} &\int_{\mathbb{R}} \hat{f}_r(x) dF(x) \\ &= \int_r^{s-(n-1)r} \frac{x-r}{s-nr} dF(x) + \int_{s-(n-1)r}^{\infty} dF(x) \\ &= \left|_r^{s-(n-1)r} \frac{x-r}{s-nr} F(x) - \int_r^{s-(n-1)r} \frac{F(x)}{s-nr} dx + F(\infty) - F(s-(n-1)r) \right. \\ &= F(s-(n-1)r) - \frac{1}{s-nr} \int_r^{s-(n-1)r} F(x) dx + 1 - F(s-(n-1)r) \\ &= \frac{1}{s-nr} \int_r^{s-(n-1)r} (1-F(x)) dx, \end{aligned}$$

which completes the proof. \square

Now we will show that for large s this bound is greater than or equal to the bound presented in (14). First note that

$$\inf_{r \in (0, s/n)} \int_r^{s-(n-1)r} \frac{1-F(x)}{s-nr} dx \leq \lim_{r > 0, r \uparrow s/n} \left[1 - \int_r^{s-(n-1)r} \frac{F(x)}{s-nr} dx \right].$$

For some $t \in (r, s-(n-1)r)$ it holds

$$\int_r^{s-(n-1)r} \frac{F(x)}{s-nr} dx = \frac{F(t)}{s-nr} (s-(n-1)r - r) = F(t),$$

so that

$$\inf_{r \in (0, s/n)} \int_r^{s-(n-1)r} \frac{1-F(x)}{s-nr} dx \leq 1 - \lim_{t > 0, t \uparrow s/n} F(t) = 1 - F(s/n),$$

since $r \uparrow s/n$ implies $t \uparrow s/n$. We get

$$1 - \inf_{r \in (0, s/n)} \frac{n}{s-nr} \int_r^{s-(n-1)r} (1-F(x)) dx \geq 1 - n + nF\left(\frac{s}{n}\right).$$

Since F is a continuous distribution function, there exists

$$x_F := \inf\{x \geq 0 : F \text{ is concave on } [x, \infty)\}.$$

Let $s \geq nF^{-1}\left(\frac{F(x_F)+n-1}{n}\right)$. We will show that

$$\tau_{C_l, \psi}^+(F, \dots, F)(s) = \left(nF\left(\frac{s}{n}\right) - n + 1\right)^+.$$

Since ψ was the sum, we have by the definition of C_l , that

$$\begin{aligned} \tau_{C_l, \psi}^+(F, \dots, F)(s) &= \sup_{x_1, \dots, x_{n-1} \in \mathbb{R}} \left(\sum_{i=1}^{n-1} F(x_i) + F(\psi_{x-n}^\wedge(s)) - n + 1 \right)^+ \\ &= \sup_{x_1, \dots, x_{n-1} \in \mathbb{R}} \left(\sum_{i=1}^{n-1} F(x_i) + F\left(s - \sum_{i=1}^{n-1} x_i\right) - n + 1 \right)^+ \\ &= \sup_{x \in \mathbb{R}^n, \sum x_i = s} \left(\sum_{i=1}^n F(x_i) - n + 1 \right)^+. \end{aligned}$$

This is always greater than or equal to $(nF(s/n) - n + 1)^+$, hence we have to show that it is also less than or equal for $s \geq nF^{-1}\left(\frac{F(x_F)+n-1}{n}\right)$.

Since F is concave on $[x_F, \infty)$, for $x \in [x_F, \infty)^n$ such that $\sum_{i=1}^n x_i = s$, it holds $\sum_{i=1}^n F(x_i)/n \leq F(\sum_{i=1}^n x_i/n) = F(s/n)$. This implies

$$\sup_{x \in [x_F, \infty)^n, \sum x_i = s} \left(\sum_{i=1}^n F(x_i) - n + 1 \right)^+ \leq (nF(s/n) - n + 1)^+.$$

Since F is right-continuous and non-decreasing, it holds for $s \geq nF^{-1}\left(\frac{F(x_F)+n-1}{n}\right)$ by Lemma 2.14 (iii), that $F(s/n) \geq \frac{F(x_F)+n-1}{n}$, which implies $F(x_F) \leq nF(s/n) - n + 1$. For $x \in \mathbb{R}^n \setminus [x_F, \infty)^n$ there is $i_0 \in \{1, \dots, n\}$ such that $x_{i_0} < x_F$, hence

$$\sum_{i=1}^n F(x_i) - n + 1 \leq F(x_{i_0}) + (n-1)F(\infty) - n + 1 \leq F(x_F)$$

$$\leq nF(s/n) - n + 1.$$

It follows

$$\sup_{x \in \mathbb{R}^n \setminus [x_F, \infty)^n, \sum x_i = s} \left(\sum_{i=1}^n F(x_i) - n + 1 \right)^+ \leq (nF(s/n) - n + 1)^+.$$

We have shown that when $s \geq \left(nF^{-1} \left(\frac{F(x_F) + n - 1}{n} \right) \right)^+$, it holds

$$\begin{aligned} m_\psi(s) &\geq \left(1 - \inf_{r \in (0, s/n)} \frac{n}{s - nr} \int_r^{s - (n-1)r} (1 - F(x)) \, dx \right)^+ \\ &\geq \tau_{C_l, \psi}^+(F, \dots, F)(s). \end{aligned}$$

In [5] (pages 15-17) it is shown that for $n > 2$ it can be

$$\begin{aligned} &\left(1 - \inf_{r \in (0, s/n)} \frac{n}{s - nr} \int_r^{s - (n-1)r} (1 - F(x)) \, dx \right)^+ \\ &> \sup_{x_1, \dots, x_{n-1} \in \mathbb{R}} C_l(F(x_1), \dots, F(x_{n-1}), F^-(\psi_{x_{-n}}^{-1}(s))). \end{aligned} \quad (26)$$

Since $\psi_{x_{-n}}^{-1}(s) = s - \sum_{i=1}^{n-1} x_i = \psi_{x_{-n}}^\wedge(s)$ and F is continuous, the right-hand side of (26) is equal to $\tau_{C_l, \psi}^+(F, \dots, F)(s)$. Hence this bound actually improves estimation.

References

- [1] P. Jorion. Value at Risk: The New Benchmark for Managing Financial Risk. McGraw-Hill, New York. (2001)
- [2] G. D. Makarov. (1981) Estimates for the distribution function of a sum of two random variables when the marginal distributions are fixed. *Theor. Probab. Appl.* 26. 803–806
- [3] M. J. Frank, R.B. Nelsen and B. Schweizer. (1987) Best-possible bounds for the distribution of a sum -a problem of Kolmogorov. *Probability Theory Related Fields.* Springer-Verlag, 74(2). 199–211
- [4] R. C. Williamson. (1989) Probabilistic Arithmetic. A Thesis submitted for the degree of Doctor of Philosophy. Department of Electrical Engineering, University of Queensland. <URL: <http://users.rsise.anu.edu.au/~williams/papers/thesis300dpi.pdf>>, November 2005. 70–80
- [5] P. Embrechts, G. Puccetti. (2004) Bounds for functions of dependent risks. Paul Embrechts Publications. <URL: <http://www.math.ethz.ch/~baltes/ftp/EP04web.pdf>>, June 2005.
- [6] P. Embrechts, A. Höing, A. Juri. (1991) Using Copulae to bound Value-at-Risk for functions of dependent risks. Paul Embrechts Publications. <URL: <http://www.math.ethz.ch/~baltes/ftp/EHJ.pdf>>, June 2005.
- [7] R. B. Nelsen. (1999) An Introduction to Copulas. Volume 139 of Lecture Notes in Statistics. Springer-Verlag, New York. 1–18
- [8] P. Embrechts, A. McNeil, D. Straumann. (2002) Correlation and dependence in risk management: properties and pitfalls. Paul Embrechts Publications. <URL: <http://www.math.ethz.ch/~strauman/preprints/pitfalls.pdf>>, October 2005. 28–29
- [9] A. N. Shiryaev. (1984) Probability. Second edition. Springer-Verlag, New York.
- [10] L. Rüschendorf. (1982) Random variables with maximum sums. *Adv. in Appl. Probab.* 14(3). 623–632