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THE RANK-ONE THEOREM ON RCD SPACES

GIOACCHINO ANTONELLI, CAMILLO BRENA AND ENRICO PASQUALETTO

We extend Alberti's rank-one theorem to $\mathrm{RCD}(K, N)$ metric measure spaces.

1. Introduction

1A. ***The rank-one theorem in the Euclidean setting.*** Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathrm{BV}(\Omega; \mathbb{R}^k)$, i.e., $u = (u_1, \ldots, u_k) \in (\mathrm{BV}(\Omega))^k$. By using the Lebesgue–Radon–Nikodým theorem, one can write the distributional derivative of u as

$$\mathrm{D}u = \mathrm{D}^a u + \mathrm{D}^s u,$$

where $\mathrm{D}^a u$ is the absolutely continuous part of $\mathrm{D}u$ with respect to the Lebesgue measure $\mathcal{L}^n$ and $\mathrm{D}^s u$ is the singular part of $\mathrm{D}u$. We denote by $\mathrm{D}u/|\mathrm{D}u|$ the matrix-valued Lebesgue–Radon–Nikodým density of $\mathrm{D}u$ with respect to the total variation $|\mathrm{D}u|$. Notice that the total variation of the singular part $|\mathrm{D}^s u|$ is equal to the singular part of the total variation $|\mathrm{D}u|^s$.

De Giorgi and Ambrosio [1988] — motivated by the study of some functionals coming from mathematical physics — conjectured the following:

<u>Rank-one property</u>: For every $u \in \mathrm{BV}(\Omega; \mathbb{R}^k)$, the matrix $\mathrm{D}u/|\mathrm{D}u|$ has rank-1 $|\mathrm{D}u|^s$-almost everywhere.

Alberti [1993] solved in the affirmative the previous conjecture; see also the account in [De Lellis 2008].

It is worth observing that the ideas used in [Alberti 1993] proved to be very robust for further developments of geometric measure theory and the rectifiability theory in Euclidean spaces and even beyond in the metric setting. As a main step of the proof, Alberti [1993] proved that, given an arbitrary Radon measure μ on a k-dimensional plane V in $\mathbb{R}^n$ that is singular with respect to $\mathcal{H}^k \llcorner V$, one can associate to μ a bundle $E(\mu, \cdot)$ whose fibers have dimension at most 1. The fiber $E(\mu, x)$ of this bundle is made by all the vectors $v \in \mathbb{R}^k$ such that $v\mu$ is *tangent* in x, in a precise sense, to the derivative of a BV function on V. What happens, moreover, is that the restriction of μ to the set where $E(\mu, \cdot)$ is one-dimensional can be written as $\int_I \mu_t \, \mathrm{d}t$, where $\mu_t = \mathcal{H}^{k-1} \llcorner S_t$ and S_t is $(k-1)$-rectifiable in V.

In the language of [Alberti et al. 2010], which collects several other fine results for the theory of rectifiability in $\mathbb{R}^n$, the previous result means that, on the set where the fiber is one-dimensional, μ is $(k-1)$-*representable*: namely, it can be written as the integral of measures that are $(k-1)$-rectifiable. At the basis of this possibility of representing a measure as the integral of rectifiable measures is the idea of the Alberti representations.

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Another interesting contribution that originated from this circle of ideas is a result by Alberti and Marchese [2016], where they associate to every Radon measure μ on $\mathbb{R}^n$ the minimal (unique μ-almost everywhere) bundle $V(\mu, \cdot)$ such that every real-valued Lipschitz function on $\mathbb{R}^n$ is differentiable along $V(\mu, x)$ for μ-almost every $x \in \mathbb{R}^n$. Alberti representations were also used by Bate [2015] and Bate and Li [2017] in the study of rectifiability in the general metric setting. For further readings, one can consult the survey by Mattila [2023], in particular, Chapters 8 and 13.

Besides its theoretical interest, the rank-one theorem soon gave important consequences in the calculus of variations. Ambrosio and Dal Maso [1992] exploited it to derive the expression of the relaxation (in BV) of a functional defined on C^1 functions as the integral of a quasiconvex function of linear growth of the gradient. See also [Kristensen and Rindler 2010] for a generalization. Moreover, Fonseca and Müller [1993] generalized the result in [Ambrosio and Dal Maso 1992] for integrands that might not depend solely on the gradient, but also on the space variable and the function itself. For further details we refer the reader to [Ambrosio et al. 2000, Chapter 5].

As an added value to the theoretical interest of the rank-one theorem, De Philippis and Rindler [2016] showed a general structure theorem for $\mathcal{A}$-free vector-valued Radon measures on Euclidean spaces, where $\mathcal{A}$ is a linear constant-coefficient differential operator, from which the rank-one theorem can be derived as a consequence. We also remark that Massaccesi and Vittone [2019] recently gave a very short proof of the rank-one theorem based on the theory of sets of finite perimeter, and with Don they used this simplified strategy to prove the analogue of the rank-one theorem in some Carnot groups [Don et al. 2019].

1B. ***Main result.*** Nowadays a well-established notion of a BV function is available in the metric measure setting. Such a notion was proposed by Miranda [2003] then studied by Ambrosio [2001; 2002] and more recently by Ambrosio and Di Marino [2014].

According to this approach, given a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$, the total variation of the derivative of $f \in L^1_{\mathrm{loc}}(\mathsf{X}, \mathsf{m})$ is the relaxation in $L^1_{\mathrm{loc}}(\mathsf{X}, \mathsf{m})$ of the energy given by the integral of the local Lipschitz constant. Such a definition can be readily extended to define the total variation of a vector-valued function whose components are in $\mathrm{BV}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathsf{m})$; see Definition 2.14 for the precise definition.

In this way one is giving a meaning to the total variation $|\mathrm{D}F|$ of an arbitrary $F \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathsf{m})^k$, while it is in general missing a good notion for the Lebesgue–Radon–Nikodým derivative $\mathrm{D}F/|\mathrm{D}F|$.

In the setting of RCD metric measure spaces, the study of calculus has been blossoming very fast in the last decade. In particular, in [Debin et al. 2021] the authors propose and study the notion of a $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module, and the notion of a capacitary tangent module $L^0_{\mathrm{Cap}}(T\mathsf{X})$, where Cap denotes the usual Capacity (2-3). We refer to Section 2B for the definitions and further details.

A fundamental contribution of [Bruè et al. 2023b], building on [Debin et al. 2021], is the fact that, in the setting of $\mathrm{RCD}(K, N)$ spaces, for an arbitrary set of finite perimeter E with finite mass, one can give a meaning to the unit normal $\nu_E = \mathrm{D}\chi_E/|\mathrm{D}\chi_E|$ as an element of the capacitary tangent module $L^0_{\mathrm{Cap}}(T\mathsf{X})$ such that the Gauss–Green formula holds; see [Bruè et al. 2023b, Theorem 2.4]. The Gauss–Green formula has been successfully employed, together with the former work by Ambrosio, Bruè and Semola [Ambrosio et al. 2019], to obtain the $(n-1)$-rectifiability of the essential boundary of any set of locally finite perimeter in an RCD space of essential dimension n; see [Bruè et al. 2023a; 2023b].

The Gauss–Green formula [Bruè et al. 2023b, Theorem 2.4] has been generalized by the second author together with Gigli for vector-valued BV functions [Brena and Gigli 2024]. We give below the statement of the Gauss–Green formula presented there, where the density $\nu_F = \mathrm{D}F/|\mathrm{D}F|$ is implicitly defined.

Theorem 1.1 [Brena and Gigli 2024, Theorem 3.13]. *Let $k \geq 1$ be a natural number, let $K \in \mathbb{R}$, and let $N \geq 1$. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space, and let $F \in \mathrm{BV}(\mathsf{X}, \mathsf{d}, \mathsf{m})^k$. Then there exists a unique $\nu_F \in L^0_{\mathrm{Cap}}(T\mathsf{X})^k$, up to $|\mathrm{D}F|$-almost everywhere equality, such that $|\nu_F| = 1$ $|\mathrm{D}F|$-almost everywhere and*

$$\sum_{j=1}^{k} \int_{\mathsf{X}} F_j \,\mathrm{div}(v_j)\,\mathrm{d}\mathsf{m} = -\int_{\mathsf{X}} \pi_{|\mathrm{D}F|}(v)\cdot \nu_F \,\mathrm{d}|\mathrm{D}F| \quad \textit{for every } v = (v_1, \dots, v_k) \in \mathrm{TestV}(\mathsf{X})^k.$$

For the notion of divergence of a vector field, the notion of test vector fields $\mathrm{TestV}(\mathsf{X})$, the notion of the projection $\pi_{|\mathrm{D}F|}$ and of the norm $|\cdot|$ in $L^0_{\mathrm{Cap}}(T\mathsf{X})^k$, we refer the reader to Section 2B.

Theorem 1.1 tells us that in the setting of $\mathrm{RCD}(K, N)$ spaces we can give a precise meaning to $\mathrm{D}F/|\mathrm{D}F|$ for an arbitrary vector-valued BV function F. Hence it is meaningful to ask if $\mathrm{D}F/|\mathrm{D}F|$ is a rank-1 matrix $|\mathrm{D}F|^s$-almost everywhere, where $|\mathrm{D}F|^s$ is the singular part of the total variation $|\mathrm{D}F|$. Before giving the main result of this paper we clarify this last sentence by means of a definition. For the definition of the space $L^0(\mathrm{Cap})$, see Section 2B.

Definition 1.2. Let $k \geq 1$ be a natural number, let $K \in \mathbb{R}$, and let $N \geq 1$. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space, let $\nu \in L^0_{\mathrm{Cap}}(T\mathsf{X})^k$, and let $\mu \ll \mathrm{Cap}$ be a Radon measure, where Cap is the usual Capacity (2-3). We say that

$$\mathrm{Rk}(\nu) = 1 \quad \mu\text{-almost everywhere}$$

if there exist $\omega \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ and $\lambda_1, \dots, \lambda_k \in L^0(\mathrm{Cap})$ such that, for every $i = 1, \dots, k$,

$$\nu_i = \lambda_i \omega \quad \mu\text{-almost everywhere.}$$

We remark that this is one of the possible definitions we could have given of having rank 1. For example, one can give an alternative and equivalent definition exploiting the existence of a local basis (with respect to a decomposition of the space in Borel sets) of $L^0_{\mathrm{Cap}}(T\mathsf{X})$ to recover the language of rank of a matrix. It is however clear that in Euclidean spaces, the definition given above coincides with the usual one.

We are now ready to state the main theorem of this paper, which is the generalization of the rank-one theorem in the setting of $\mathrm{RCD}(K, N)$ metric measure spaces $(\mathsf{X}, \mathsf{d}, \mathsf{m})$.

Theorem 1.3 (rank-one theorem for $\mathrm{RCD}(K, N)$ spaces). *Let $k \geq 1$ be a natural number, let $K \in \mathbb{R}$, and let $N \geq 1$. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space, and let $F \in \mathrm{BV}(\mathsf{X}, \mathsf{d}, \mathsf{m})^k$. Then*

$$\mathrm{Rk}(\nu_F) = 1 \quad |\mathrm{D}F|^s\textit{-almost everywhere}$$

in the sense of Definition 1.2, where ν_F is defined in Theorem 1.1 and $|\mathrm{D}F|^s$ is the singular part of the total variation $|\mathrm{D}F|$.

As far as we know, apart from the result of Don, Massaccesi and Vittone [Don et al. 2019] that holds for a special class of Carnot groups, Theorem 1.3 is one of the first instances of the validity of the rank-one theorem in a large class of metric measure spaces.

We stress that, even if the proof of [Don et al. 2019] covers a large class of Carnot groups, some distinguished examples are still not covered. For example, as of today it is not known if the rank-one theorem holds for vector-valued BV functions in the first Heisenberg group $\mathbb{H}^1$. We stress that our strategy for the proof of Theorem 1.3 seems not to apply to the rank-one theorem in $\mathbb{H}^1$. Indeed, we are fundamentally exploiting the fact that we have good bi-Lipschitz charts on the space valued in the tangents. But, even if on $\mathbb{H}^1$ the boundary of a set of locally finite perimeter is intrinsic C^1-rectifiable, see [Franchi et al. 2001], it is nowadays not known whether intrinsic C^1 surfaces can be almost everywhere covered by (bi)-Lipschitz images of their tangents; see [Di Donato et al. 2022] for partial results in this direction.

We stress that our strategy cannot be easily adapted to prove rank-one-type results for BV functions in $\mathrm{RCD}(K,\infty)$ spaces. In fact, our proof works mainly by blow-up. Since $\mathrm{RCD}(K,\infty)$ spaces might not be locally doubling, we do not have a good notion of the Gromov–Hausdorff tangent at their points. In particular, it would even be challenging to understand whether the results in [Ambrosio 2001; 2002; Ambrosio et al. 2019; Bruè et al. 2023a; 2023b], which are the starting point of our analysis, can be adapted to the $\mathrm{RCD}(K,\infty)$ setting.

Moreover, we point out that very recently Lahti proposed an alternative formulation of Alberti's rank-one theorem that could make sense in arbitrary metric measure spaces [Lahti 2022, Section 6].

1C. ***Outline of the proof.*** Our proof is inspired by the one in [Massaccesi and Vittone 2019]. First, given $F \in \mathrm{BV}(\mathsf{X},\mathsf{d},\mathsf{m})^k$, the singular part of the total variation $|\mathrm{D}F|^s$ can be written as the sum of the jump part $|\mathrm{D}F|^j$, which is concentrated on the set where the approximate lower and upper limits of the components of F do not coincide, and the Cantor part $|\mathrm{D}F|^c$; see Definition 2.12. As a consequence of a result by the second author and Gigli, see the forthcoming Lemma 3.13, it is enough to show the rank-one theorem only on the Cantor part.

We stress that, in the proofs of the main results in Section 3, we shall always restrict to sets where the Cantor part of the components of F is concentrated and where we have good density and blow-up properties: we collect all the necessary properties in the technical Proposition 3.7.

The core and the most technically demanding part of the proof is Lemma 3.11, in which we adapt to our setting the main lemma of the short proof of the rank-one theorem in [Massaccesi and Vittone 2019]. In fact, those authors prove that, given two C^1-hypersurfaces Σ_1, Σ_2 in $\mathbb{R}^n \times \mathbb{R}$, the set T of points $p \in \Sigma_1$ such that there exists $q \in \Sigma_2$ for which p and q have the same first n coordinates, $\nu_{\Sigma_1}(p)_{n+1} = \nu_{\Sigma_2}(q)_{n+1} = 0$, and $\nu_{\Sigma_1}(p) \neq \pm\nu_{\Sigma_2}(q)$, is $\mathcal{H}^n$-negligible. Clearly the latter statement makes no sense in our nonsmooth setting, but what one really needs for the proof of the rank-one theorem is Lemma 3.11.

Following the strategy in the proof of the lemma of [Massaccesi and Vittone 2019], one writes T as the projection of a set $\widetilde{T} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ adding one fake coordinate, and proves that $T = \pi(\widetilde{T})$ is $\mathcal{H}^n$-negligible by means of the area formula. In Lemma 3.11 we adapt the same strategy; compare with the definition of the set (3-28). We prove the analogue of the Massaccesi–Vittone lemma substituting the hypersurfaces Σ_i with the (essential) boundaries of sets of the form $\mathcal{G}_f := \{(x,t) : t < f(x)\}$, where $f \in \mathrm{BV}(\mathsf{X},\mathsf{d},\mathsf{m})$. This is enough to implement their strategy in our setting. However, to adapt the proof in [Don et al. 2019; Massaccesi and Vittone 2019] to our framework, one faces nontrivial technical difficulties. Indeed, the key ingredient used by Massaccesi–Vittone was a well-known transversality lemma: given two hypersurfaces in $\mathbb{R}^{n+2}$, their intersection is locally an n-dimensional manifold provided that at every intersection point the given hypersurfaces meet transversally, i.e., have different tangent spaces. This result then extends to the case of the intersection of two $(n+1)$-rectifiable subsets of $\mathbb{R}^{n+2}$: their intersection is σ-finite with respect to $\mathcal{H}^n$ provided that the transversality condition is satisfied and that one discards a set that turns out to be negligible when proving the rank-one property.

It is clear that one needs also information on codimension-2 objects (namely, the intersection of two transverse hypersurfaces) and this kind of information is unavailable on RCD spaces. Therefore, adopting directly this approach is not possible in our framework. Our strategy is then to translate part of the problem from the RCD setting to the Euclidean setting (which allows us to use transversality results as above) via the use of suitable δ-splitting maps that play the role of charts, relying heavily on the results of [Bruè et al. 2023a; 2023b]. The fact that the domains of these charts are not open sets is a source of difficulty and is morally the burden of the proof of Lemma 3.11. In other words, we could not work directly arguing with infinitesimal considerations in the RCD case (i.e., using directly difference of blow-ups) but we had to argue locally and then infinitesimally in a Euclidean space.

As an important part of the proof, to manipulate the vector that is normal to the boundary of the set $\mathcal{G}_f$, we need to introduce a family of charts in which we write those normals in coordinates; see Definition 3.6. We construct these charts in Definition 2.29, and we call them a *good collection of splitting maps*. The latter definition is based on the following fact, which is proved in Lemma 2.28. Given an RCD space of essential dimension n, we prove that, for every $\eta > 0$ small enough, we can find a sequence of n-tuples of harmonic maps $\{u_{k,\eta}\}_{k\in\mathbb{N}}$ defined on balls and a disjointed family of Borel sets $\{D_{k,\eta}\}_{k\in\mathbb{N}}$ such that, for every $x \in D_{k,\eta}$, we have that $u_{k,\eta}$ is an η-splitting map on $B_{r_k}(x)$ and the total variation of every $\mathrm{BV}_{\mathrm{loc}}$ function is concentrated on $\bigsqcup_{k\in\mathbb{N}} D_{k,\eta}$.

The other two ingredients to adapt in our setting the strategy of [Massaccesi and Vittone 2019] are Lemma 3.9 and Theorem 3.8. In the first we prove that, given $f \in \mathrm{BV}$, restricting to the good set on the Cantor part as in Proposition 3.7, we have that (in coordinates) the density $\nu_f(x)$ is equal to the first coordinates of the normal $\nu_{\mathcal{G}_f}(x, f(x))$, where $\mathcal{G}_f := \{(x,t) : t < f(x)\}$. In the second we prove that, restricting to the good set on the Cantor part as in Proposition 3.7, the $(n+1)$-th coordinate of the normal $\nu_{\mathcal{G}_f}$ is almost everywhere zero. This is essentially due to the fact that we are on the singular part of $\mathrm{D}f$.

Again, not having at our disposal a linear structure is a source of difficulty, as the distributional derivative has no more a direction-wise meaning, in the sense that it is not possible to define the distributional derivative of a function of bounded variation with respect to a given direction without giving up the differential meaning of this object. To overcome this difficulty, we employ blow-up arguments and density arguments.

Finally, putting together Lemmas 3.11 and 3.9 and Theorem 3.8, we conclude that, given two BV functions f, g, we have $\nu_f = \pm\nu_g$ $|\mathrm{D}f| \wedge |\mathrm{D}g|$-almost everywhere on the intersection of the good sets $C_f \cap C_g$ defined in Proposition 3.7; see Lemma 3.12. Here $\wedge$ stands for the minimum between the two measures, i.e., the biggest measure ϕ such that $\phi \le \mu$ and $\phi \le \nu$. This, together with the same property on the jump part, see Lemma 3.13, concludes the proof.

We stress that along the way in Section 3A, building on [Deng 2020] (compare with [Colding and Naber 2012] for the Hölder continuity property of tangents along geodesics in the Ricci-limit case), we improve a previous result of [Bruè et al. 2023a] by showing that every BV function on an RCD space of essential dimension n has total variation concentrated on the set $\mathcal{R}_n^*$ of n-regular points with positive and finite n-density, see Theorem 3.3. We exploit the latter result to answer in the affirmative a conjecture proposed in [Semola 2020] about the representation of the perimeter measure, see Theorem 3.4.

In the Appendix we exploit the previously described result proved in Theorem 3.3, together with the recently proved metric variant of the Marstrand–Mattila rectifiability criterion [Bate 2022], to give an alternative and shorter proof of the $(n-1)$-rectifiability of the essential boundaries of sets of locally finite perimeter in RCD spaces with essential dimension n. We believe that this result is of independent interest but we point out that it originated as a side remark due to the fact that we were interested in proving the rank-one property in general $\mathrm{RCD}(K, N)$ spaces without restricting ourselves to *noncollapsed* RCD spaces. Indeed, the information that the perimeter measure and the $\mathcal{H}^{n-1}$ measure restricted to the reduced boundary are mutually absolutely continuous (already known in the *noncollapsed* case) is crucial in the proof of Lemma 3.11. Anyway, we point out that even if the proof presented in the Appendix is much shorter than the original one, it is heavily based on the ideas and techniques exploited in [Bruè et al. 2023a; 2023b], i.e., looking at what happens at the space locally and infinitesimally by using well-behaved charts.

Structure of the paper. In Section 2 we discuss the basic tools and notation that we shall use throughout the paper.

In particular, in Section 2A we discuss the basic toolkit for metric measure spaces. We recall the definition of PI space, the notion of pointed measured Gromov–Hausdorff convergence and tangents, and the basic Sobolev and BV calculus in arbitrary metric measure spaces.

In Section 2B we recall basic structure results of RCD spaces and the main important notions of Sobolev and BV calculus on RCD spaces. We further recall the notion of *good coordinates* introduced in [Bruè et al. 2023a] and the notion of splitting maps, and finally we prove Lemma 2.28 that leads to the notion of *good collection of splitting maps*, see Definition 2.29.

In Section 3 we prove the main results of this paper, and in particular we give the proof of the rank-one theorem in Theorem 1.3.

In particular in Section 3A, building on [Deng 2020], we prove Theorem 3.3 described above.

In Section 3B we prove some auxiliary results toward the proof of the rank-one theorem, namely Proposition 3.7, Lemma 3.9, and Theorem 3.8.

Finally, in Section 3C we exploit the previous results together with the main result in Lemma 3.11, which is the adaptation to our setting of the lemma of [Massaccesi and Vittone 2019], to show the rank-one property on the Cantor part, see Lemma 3.12. This is enough to conclude the proof of the rank-one theorem by exploiting also Lemma 3.13, which is the rank-one property on the jump part.

In the Appendix we give the alternative proof of the rectifiability of the essential boundaries of sets of locally finite perimeter in RCD spaces that we described above.

2. Preliminaries

We often need to bound quantities in terms of constants that depend only on geometric parameters but whose precise value is not important. For this reason, we denote with $C_{a,b,\dots}$ a constant depending only on the parameters $a, b, \dots$, whose value might change from line to line or even within the same line.

Given $n \in \mathbb{N}$ and nonempty sets $\mathsf{X}_1, \dots, \mathsf{X}_n$, for any $i = 1, \dots, n$, we will always tacitly denote by π^i the projection of the Cartesian product $\mathsf{X}_1 \times \cdots \times \mathsf{X}_n$ onto its i-th factor:

$$\pi^i : \mathsf{X}_1 \times \cdots \times \mathsf{X}_n \to \mathsf{X}_i, \quad (x_1, \dots, x_n) \mapsto x_i.$$

Moreover, we denote by $\pi^{i,j}$ the projection of the Cartesian product $\mathsf{X}_1 \times \cdots \times \mathsf{X}_n$ onto its (i, j) factor, namely

$$\pi^{i,j} : \mathsf{X}_1 \times \cdots \times \mathsf{X}_n \to \mathsf{X}_i \times \mathsf{X}_j, \quad (x_1, \dots, x_n) \mapsto (x_i, x_j).$$

Finally, we denote by τ the inversion map on the last two factors on a product of three factors, namely

$$\tau : \mathsf{X}_1 \times \mathsf{X}_2 \times \mathsf{X}_3 \to \mathsf{X}_1 \times \mathsf{X}_3 \times \mathsf{X}_2, \quad (x_1, x_2, x_3) \mapsto (x_1, x_3, x_2). \tag{2-1}$$

2A. *Metric measure spaces.* For the purposes of this paper, a *metric measure space* is a triple $(\mathsf{X}, \mathsf{d}, \mathsf{m})$, where (X, d) is a complete and separable metric space while $\mathsf{m} \geq 0$ is a boundedly finite Borel measure on X. By a *pointed metric measure space* $(\mathsf{X}, \mathsf{d}, \mathsf{m}, p)$ we mean a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ together with a distinguished point $p \in \operatorname{spt}(\mathsf{m})$, where

$$\operatorname{spt}(\mathsf{m}) := \{x \in \mathsf{X} \mid \mathsf{m}(B_r(x)) > 0 \text{ for every } r > 0\}$$

stands for the *support* of m. Given an open set $\Omega \subseteq \mathsf{X}$, we denote by $\mathrm{LIP}_{\mathrm{loc}}(\Omega)$ and $\mathrm{LIP}(\Omega)$ the spaces of all locally Lipschitz and Lipschitz functions on Ω, respectively, while we set

$$\mathrm{LIP}_{\mathrm{bs}}(\Omega) := \{f \in \mathrm{LIP}(\Omega) \mid \operatorname{spt}(f) \text{ is bounded and } \mathsf{d}(\partial\Omega, \operatorname{spt}(f)) > 0\}.$$

Given any $f \in \mathrm{LIP}_{\mathrm{loc}}(\Omega)$, its *local Lipschitz constant* $\operatorname{lip} f := \Omega \to [0, +\infty)$ is defined as

$$\operatorname{lip} f(x) := \begin{cases} \overline{\lim}_{y \to x} |f(x) - f(y)|/\mathsf{d}(x, y) & \text{if } x \in \Omega \text{ is an accumulation point,} \\ 0 & \text{if } x \in \Omega \text{ is an isolated point.} \end{cases}$$

For any $k \in [0, +\infty)$ and $\delta > 0$, we will denote by $\mathcal{H}^k_\delta$ and $\mathcal{H}^k$ the *k-dimensional Hausdorff δ-premeasure* and the *k-dimensional Hausdorff measure* on (X, d), respectively. Namely,

$$\mathcal{H}^k_\delta(E) := \inf\left\{ \sum_{i=1}^\infty \omega_k \left(\frac{\operatorname{diam}(E_i)}{2} \right)^k \,\middle|\, E \subseteq \bigcup_{i \in \mathbb{N}} E_i \subseteq \mathsf{X},\ \sup_{i \in \mathbb{N}} \operatorname{diam}(E_i) < \delta \right\},$$
$$\mathcal{H}^k(E) := \lim_{\delta \searrow 0} \mathcal{H}^k_\delta(E) = \sup_{\delta > 0} \mathcal{H}^k_\delta(E)$$

for every set $E \subseteq \mathsf{X}$, where

$$\omega_k := \frac{\pi^{k/2}}{\Gamma\left(1 + \frac{k}{2}\right)}$$

and Γ stands for Euler's gamma function. For every $n \in \mathbb{N}$, notice that ω_n is the Euclidean volume of the unit ball in $\mathbb{R}^n$.

PI *spaces.* Throughout the whole paper, we will work in the setting of PI spaces. We say that a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is *uniformly locally doubling* provided that, for every radius $R > 0$, there exists a constant $C_D > 0$ such that

$$\mathsf{m}(B_{2r}(x)) \leq C_D \mathsf{m}(B_r(x)) \quad \text{for every } x \in \mathsf{X} \text{ and } r \in (0, R).$$

Moreover, we say that $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ supports a *weak local* $(1, 1)$*-Poincaré inequality* provided there exists a constant $\lambda \geq 1$ for which the following property holds: given any $R > 0$, there exists a constant $C_P > 0$ such that, for any function $f \in \mathrm{LIP}_{\mathrm{loc}}(\mathsf{X})$,

$$⨍_{B_r(x)} \left| f - ⨍_{B_r(x)} f \, \mathrm{dm} \right| \mathrm{dm} \leq C_P r ⨍_{B_{\lambda r}(x)} \mathrm{lip}\, f \, \mathrm{dm} \quad \text{for every } x \in \mathsf{X} \text{ and } r \in (0, R).$$

Definition 2.1 (PI space). We say that a metric measure space is a *PI space* provided it is uniformly locally doubling and it supports a weak local (1, 1)-Poincaré inequality.

In the context of PI spaces, we will consider the *codimension*-1 *Hausdorff* δ*-premeasure* $\mathcal{H}^h_\delta$ (for any $\delta > 0$) and the *codimension*-1 *Hausdorff measure* $\mathcal{H}^h$, which are given by

$$\mathcal{H}^h_\delta(E) := \inf\left\{ \sum_{i=1}^{\infty} \frac{\mathsf{m}(B_{r_i}(x_i))}{\mathrm{diam}(B_{r_i}(x_i))} \;\middle|\; E \subseteq \bigcup_{i\in\mathbb{N}} B_{r_i}(x_i),\ \sup_{i\in\mathbb{N}} \mathrm{diam}(B_{r_i}(x_i)) < \delta \right\},$$
$$\mathcal{H}^h(E) := \lim_{\delta \searrow 0} \mathcal{H}^h_\delta(E) = \sup_{\delta > 0} \mathcal{H}^h_\delta(E),$$

respectively, for every set $E \subseteq \mathsf{X}$.

Measured Gromov–Hausdorff convergence and tangents. Let us recall the notion of *pointed measured Gromov–Hausdorff* convergence (see, e.g., [Gigli et al. 2015]). We say that a pointed metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m}, p)$ is *normalized* provided $C^1_p(\mathsf{m}) = 1$, where we set

$$C^r_p = C^r_p(\mathsf{m}) := \int_{B_r(p)} \left(1 - \frac{\mathsf{d}(\cdot\,, p)}{r}\right) \mathrm{dm} \quad \text{for every } r > 0.$$

If $(\mathsf{X}, \mathsf{d}, \mathsf{m}, p)$ is any pointed metric measure space, then $(\mathsf{X}, \mathsf{d}, \mathsf{m}^1_p, p)$ is normalized, where

$$\mathsf{m}^r_p := C^r_p(\mathsf{m})^{-1} \mathsf{m} \quad \text{for every } r > 0.$$

Let $C\colon (0, +\infty) \to (0, +\infty)$ be a given nondecreasing function. Then we denote by $\mathbb{X}_{C(\cdot)}$ the family of all the equivalence classes of normalized pointed metric measure spaces that are $C(\cdot)$-doubling, in the sense that

$$\mathsf{m}(B_{2r}(x)) \leq C(R)\mathsf{m}(B_r(x)) \quad \text{for every } x \in \mathsf{X} \text{ and } 0 < r \leq R.$$

The equivalence classes are intended with respect to the following equivalence relation: we identify two pointed metric measure spaces $(\mathsf{X}_1, \mathsf{d}_1, \mathsf{m}_1, p_1)$ and $(\mathsf{X}_2, \mathsf{d}_2, \mathsf{m}_2, p_2)$ provided there exists a bijective isometry $\varphi\colon \mathrm{spt}(\mathsf{m}_1) \to \mathrm{spt}(\mathsf{m}_2)$ such that $\varphi(p_1) = p_2$ and $\varphi_* \mathsf{m}_1 = \mathsf{m}_2$.

Definition 2.2 (pointed measured Gromov–Hausdorff). Let $C\colon (0,+\infty)\to(0,+\infty)$ be nondecreasing. Let $(\mathsf{X},\mathsf{d},\mathsf{m},p)$, $(\mathsf{X}_i,\mathsf{d}_i,\mathsf{m}_i,p_i)\in\mathbb{X}_{C(\cdot)}$ for $i\in\mathbb{N}$ be given. Then we say that $(\mathsf{X}_i,\mathsf{d}_i,\mathsf{m}_i,p_i)\to(\mathsf{X},\mathsf{d},\mathsf{m},p)$ in the *pointed measured Gromov–Hausdorff sense* (briefly, in the *pmGH sense*) provided there exist a proper metric space $(\mathsf{Z},\mathsf{d}_\mathsf{Z})$ and isometric embeddings $\iota\colon\mathsf{X}\to\mathsf{Z}$ and $\iota_i:\mathsf{X}_i\to\mathsf{Z}$ for $i\in\mathbb{N}$ such that $\iota_i(p_i)\to\iota(p)$ and $(\iota_i)_*\mathsf{m}_i\rightharpoonup\iota_*\mathsf{m}$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$, meaning that $\int f\circ\iota_i\,\mathrm{d}\mathsf{m}_i\to\int f\circ\iota\,\mathrm{d}\mathsf{m}$ for every $f\in C_{\mathrm{bs}}(\mathsf{Z})$. The space Z is called a *realization* of the pmGH convergence $(\mathsf{X}_i,\mathsf{d}_i,\mathsf{m}_i,p_i)\to(\mathsf{X},\mathsf{d},\mathsf{m},p)$.

For brevity, we will identify $(\iota_i)_*\mathsf{m}_i$ with m_i itself. It is possible to construct a distance $\mathsf{d}_{\mathrm{pmGH}}$ on $\mathbb{X}_{C(\cdot)}$ whose converging sequences are exactly those converging in the pointed measured Gromov–Hausdorff sense. Moreover, the metric space $(\mathbb{X}_{C(\cdot)},\mathsf{d}_{\mathrm{pmGH}})$ is compact.

Definition 2.3 (pmGH tangent). Let $C\colon (0,+\infty)\to(0,+\infty)$ be nondecreasing. Then

$$\mathrm{Tan}_p(\mathsf{X},\mathsf{d},\mathsf{m}):=\Big\{(\mathsf{Y},\mathsf{d}_\mathsf{Y},\mathsf{m}_\mathsf{Y},q)\in\mathbb{X}_{C(\cdot)}\;\Big|\;\exists r_i\searrow 0:(\mathsf{X},r_i^{-1}\mathsf{d},\mathsf{m}_p^{r_i},p)\xrightarrow{\mathrm{pmGH}}(\mathsf{Y},\mathsf{d}_\mathsf{Y},\mathsf{m}_\mathsf{Y},q)\Big\}.$$

Notice that $(\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_p^r,p)\in\mathbb{X}_{C(\cdot)}$ holds for every $(\mathsf{X},\mathsf{d},\mathsf{m},p)\in\mathbb{X}_{C(\cdot)}$ and $r\in(0,1)$, and thus accordingly the family $\mathrm{Tan}_p(\mathsf{X},\mathsf{d},\mathsf{m})$ is (well defined and) nonempty.

Definition 2.4 (regular set). Let $n\in\mathbb{N}$ be given. Let $C\colon(0,+\infty)\to(0,+\infty)$ be any nondecreasing function such that $(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)\in\mathbb{X}_{C(\cdot)}$, where d_e stands for the Euclidean distance $\mathsf{d}_e(x,y):=|x-y|$ on $\mathbb{R}^n$ while $\underline{\mathcal{L}}^n$ is the normalized measure $(\mathcal{L}^n)_0^1=((n+1)/\omega_n)\mathcal{L}^n$. Then the set of *n-regular points* of a given element $(\mathsf{X},\mathsf{d},\mathsf{m},p)\in\mathbb{X}_{C(\cdot)}$ is defined as

$$\mathcal{R}_n=\mathcal{R}_n(\mathsf{X}):=\{x\in\mathsf{X}\mid\mathrm{Tan}_x(\mathsf{X},\mathsf{d},\mathsf{m})=\{(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)\}\}.$$

Remark 2.5. We point out that the set $\mathcal{R}_n(\mathsf{X})$ of n-regular points is Borel measurable. To check it, define $\phi\colon\mathsf{X}\to[0,+\infty)$ as

$$\phi(x):=\varlimsup_{r\searrow 0}\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)).$$

One can readily verify that $(0,1)\ni r\mapsto(\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x)\in\mathbb{X}_{C(\cdot)}$ is $\mathsf{d}_{\mathrm{pmGH}}$-continuous for any given $x\in\mathsf{X}$, whence

$$\phi(x)=\inf_{k\in\mathbb{N}}\sup_{q\in\mathbb{Q}\cap(0,1/k)}\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X},q^{-1}\mathsf{d},\mathsf{m}_x^q,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0))\quad\text{for every }x\in\mathsf{X}.\tag{2-2}$$

Since $\mathsf{X}\ni x\mapsto(\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x)\in\mathbb{X}_{C(\cdot)}$ is $\mathsf{d}_{\mathrm{pmGH}}$-continuous for any given $r\in(0,1)$, we deduce that $\mathsf{X}\ni x\mapsto\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X},q^{-1}\mathsf{d},\mathsf{m}_x^q,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0))$ is a continuous function for any $q\in\mathbb{Q}\cap(0,1)$. Consequently, (2-2) ensures that $\mathcal{R}_n(\mathsf{X})=\{x\in\mathsf{X}:\phi(x)=0\}$ is a Borel set (in fact, a countable intersection of F_σ sets), as we claimed.

Definition 2.6 (convergences along pmGH converging sequences). Let $(\mathsf{X}_i,\mathsf{d}_i,\mathsf{m}_i,p_i)\in\mathbb{X}_{C(\cdot)}$ for $i\in\mathbb{N}$ and $(\mathsf{X},\mathsf{d},\mathsf{m},p)\in\mathbb{X}_{C(\cdot)}$ satisfy $(\mathsf{X}_i,\mathsf{d}_i,\mathsf{m}_i,p_i)\to(\mathsf{X},\mathsf{d},\mathsf{m},p)$ in the pmGH sense, with realization Z. Then we give the following definitions:

(i) Let $f_i:\mathsf{X}_i\to\mathbb{R}$ for $i\in\mathbb{N}$ and $f:\mathsf{X}\to\mathbb{R}$ be given functions. Then we say that f_i *uniformly converges* to f provided, for any $\varepsilon>0$, there exists $\delta>0$ such that $|f_i(x_i)-f(x)|\le\varepsilon$ for every $i\ge\delta^{-1}$ and $x_i\in\mathsf{X}_i$, $x\in\mathsf{X}$ with $\mathsf{d}_\mathsf{Z}(x_i,x)\le\delta$.

(ii) Let $f_i : \mathsf{X}_i \to \mathbb{R}$ for $i \in \mathbb{N}$ and $f : \mathsf{X} \to \mathbb{R}$ be given functions. Then we say that f_i *locally uniformly converges* to f provided, for any $R > 0$, we have that $f_{i|B_R(p_i)}$ uniformly converges to $f_{|B_R(p)}$.

(iii) Let $E_i \subseteq \mathsf{X}_i$ for $i \in \mathbb{N}$ and $E \subseteq \mathsf{X}$ be given Borel sets. Suppose that $\mathsf{m}_i(E_i) < +\infty$ for every $i \in \mathbb{N}$ and $\mathsf{m}(E) < +\infty$. Then we say that $E_i \to E$ (*strongly*) *in* L^1 provided $\mathsf{m}_i(E_i) \to \mathsf{m}(E)$ and $\mathsf{m}_i \llcorner E_i \rightharpoonup \mathsf{m} \llcorner E$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$.

(iv) Let $E_i \subseteq \mathsf{X}_i$ for $i \in \mathbb{N}$ and $E \subseteq \mathsf{X}$ be given Borel sets. Then we say that $E_i \to E$ (*strongly*) *in* L^1_{loc} provided $E_i \cap B_R(p_i) \to E \cap B_R(p)$ in L^1 for every $R > 0$.

Sobolev calculus. Given a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$, we define the *Sobolev space* $W^{1,2}(\mathsf{X})$ as the set of all functions $f \in L^2(\mathsf{m})$ for which there exists $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ such that $f_n \to f$ in $L^2(\mathsf{m})$ and $(\mathrm{lip}\, f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(\mathsf{m})$. Then $W^{1,2}(\mathsf{X})$ becomes a Banach space if endowed with the norm

$$\|f\|_{W^{1,2}(\mathsf{X})} := \left(\int |f|^2 \,\mathrm{d}\mathsf{m} + \inf_{(f_n)_n} \varliminf_{n\to\infty} \int \mathrm{lip}^2 f_n \,\mathrm{d}\mathsf{m} \right)^{1/2} \quad \text{for every } f \in W^{1,2}(\mathsf{X}),$$

where the infimum is taken among all those sequences $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ such that $f_n \to f$ in $L^2(\mathsf{m})$ and $(\mathrm{lip}\, f_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathsf{m})$. Given any function $f \in W^{1,2}(\mathsf{X})$, there exists a unique element $|\mathrm{D}f| \in L^2(\mathsf{m})$, called the *minimal relaxed slope* of f, such that the Sobolev norm of f can be expressed as $\|f\|^2_{W^{1,2}(\mathsf{X})} = \|f\|^2_{L^2(\mathsf{m})} + \||\mathrm{D}f|\|^2_{L^2(\mathsf{m})}$. Moreover, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ such that $f_n \to f$ and $\mathrm{lip}\, f_n \to |\mathrm{D}f|$ in $L^2(\mathsf{m})$. This notion of Sobolev space, proposed in [Ambrosio et al. 2013], is an equivalent reformulation of the one introduced in [Cheeger 1999]. See [Ambrosio et al. 2013] for the equivalence between these two and other approaches.

The *Sobolev capacity* is the set-function on X defined as

$$\mathrm{Cap}(E) := \inf_f \|f\|^2_{W^{1,2}(\mathsf{X})} \quad \text{for every set } E \subseteq \mathsf{X}, \tag{2-3}$$

where the infimum is taken among all $f \in W^{1,2}(\mathsf{X})$ such that $f \geq 1$ holds m-a.e. on some open neighborhood of E. Here we adopt the convention that $\mathrm{Cap}(E) := +\infty$ whenever no such f exists. It holds that Cap is a submodular outer measure on X, which is finite on bounded sets and satisfies $\mathsf{m}(E) \leq \mathrm{Cap}(E)$ for every $E \subseteq \mathsf{X}$ Borel.

We shall also work with *local Sobolev spaces*, whose definition we are going to recall. Fix an open set $\Omega \subseteq \mathsf{X}$. Then we define $W^{1,2}_{\mathrm{loc}}(\Omega)$ as the space of all functions $f \in L^2_{\mathrm{loc}}(\Omega, \mathsf{m})$ such that $\eta f \in W^{1,2}(\mathsf{X})$ for every $\eta \in \mathrm{LIP}_{\mathrm{bs}}(\Omega)$. Since the minimal relaxed slope is a local object, meaning that, for any choice of $f_1, f_2 \in W^{1,2}(\mathsf{X})$,

$$|\mathrm{D}f_1| = |\mathrm{D}f_2| \quad \text{holds } \mathsf{m}\text{-a.e. on } \{f_1 = f_2\},$$

it makes sense to associate to any $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ the function $|\mathrm{D}f| \in L^2_{\mathrm{loc}}(\Omega, \mathsf{m})$ given by

$$|\mathrm{D}f| := |\mathrm{D}(\eta f)| \quad \mathsf{m}\text{-a.e. on } \{\eta = 1\}$$

for every $\eta \in \mathrm{LIP}_{\mathrm{bs}}(\Omega)$. The local Sobolev space $W^{1,2}(\Omega)$ is defined as

$$W^{1,2}(\Omega) := \{ f \in W^{1,2}_{\mathrm{loc}}(\Omega) \mid f, |\mathrm{D}f| \in L^2(\mathsf{m}) \}.$$

Finally, we define $W^{1,2}_0(\Omega)$ as the closure of $\mathrm{LIP}_{\mathrm{bs}}(\Omega)$ in $W^{1,2}(\Omega)$.

Following the terminology introduced in [Gigli 2015], we say that a given metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is *infinitesimally Hilbertian* provided $W^{1,2}(\mathsf{X})$ (and thus also $W^{1,2}(\Omega)$ for any $\Omega \subseteq \mathsf{X}$ open) is a Hilbert space. Under this assumption, the mapping

$$W^{1,2}(\Omega) \times W^{1,2}(\Omega) \ni (f, g) \mapsto \nabla f \cdot \nabla g := \frac{|\mathrm{D}(f+g)|^2 - |\mathrm{D}f|^2 - |\mathrm{D}g|^2}{2} \in L^1(\Omega, \mathsf{m})$$

is bilinear and continuous. We say that a given function $f \in W^{1,2}(\Omega)$ has a *Laplacian*, briefly $f \in D(\Delta, \Omega)$, provided there exists a function $\Delta f \in L^2(\Omega, \mathsf{m})$ such that

$$\int_\Omega \nabla f \cdot \nabla g \, \mathrm{d}\mathsf{m} = -\int_\Omega g \Delta f \, \mathrm{d}\mathsf{m} \quad \text{for every } g \in W_0^{1,2}(\Omega). \tag{2-4}$$

No ambiguity may arise, since Δf is uniquely determined by (2-4). The set $D(\Delta, \Omega)$ is a linear subspace of $W^{1,2}(\Omega)$, and the resulting operator $\Delta\colon D(\Delta, \Omega) \to L^2(\Omega, \mathsf{m})$ is linear. For the sake of brevity, we shorten $D(\Delta, \mathsf{X})$ to $D(\Delta)$. By a *harmonic* function on Ω we mean an element $f \in D(\Delta, \Omega)$ such that $\Delta f = 0$.

BV *calculus.* We begin by recalling the notions of a function of bounded variation and of a set of finite perimeter in the context of metric measure spaces following [Miranda 2003].

Definition 2.7 (function of bounded variation). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space. Let a function $f \in L^1_{\mathrm{loc}}(\mathsf{X}, \mathsf{m})$ be given. Then we define

$$|Df|(\Omega) := \inf\left\{ \varliminf_{i \to \infty} \int_\Omega \mathrm{lip}\, f_i \, \mathrm{d}\mathsf{m} \;\middle|\; (f_i)_{i \in \mathbb{N}} \subseteq \mathrm{LIP}_{\mathrm{loc}}(\Omega),\ f_i \to f \text{ in } L^1_{\mathrm{loc}}(\Omega, \mathsf{m}) \right\}$$

for any open set $\Omega \subseteq \mathsf{X}$. We declare that a function $f \in L^1_{\mathrm{loc}}(\mathsf{X}, \mathsf{m})$ is of *local bounded variation*, briefly $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$, if $|Df|(\Omega) < +\infty$ for every $\Omega \subseteq \mathsf{X}$ open and bounded. In this case, it is well known that $|\mathrm{D}f|$ extends to a locally finite measure on X. Moreover, a function $f \in L^1(\mathsf{X}, \mathsf{m})$ is said to belong to the space of *functions of bounded variation* $\mathrm{BV}(\mathsf{X}) = \mathrm{BV}(\mathsf{X}, \mathsf{d}, \mathsf{m})$ if $|Df|(\mathsf{X}) < +\infty$.

Definition 2.8 (set of finite perimeter). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space. Let $E \subseteq \mathsf{X}$ be a Borel set and $\Omega \subseteq \mathsf{X}$ an open set. Then we define the *perimeter* of E in Ω as

$$P(E, \Omega) := \inf\left\{ \varliminf_{i \to \infty} \int_\Omega \mathrm{lip}\, f_i \, \mathrm{d}\mathsf{m} \;\middle|\; (f_i)_{i \in \mathbb{N}} \subseteq \mathrm{LIP}_{\mathrm{loc}}(\Omega),\ f_i \to \chi_E \text{ in } L^1_{\mathrm{loc}}(\Omega, \mathsf{m}) \right\},$$

in other words $P(E, \Omega) := |D\chi_E|(\Omega)$. We say that E has *locally finite perimeter* if $P(E, \Omega) < +\infty$ for every $\Omega \subseteq \mathsf{X}$ open and bounded. Moreover, we say that E has *finite perimeter* if $P(E, \mathsf{X}) < +\infty$, and we write $P(E) := P(E, \mathsf{X})$.

Given a uniformly locally doubling space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ and a Borel set $E \subseteq \mathsf{X}$, we define the *essential boundary* of E as

$$\partial^* E := \left\{ x \in \mathsf{X} \;\middle|\; \varlimsup_{r \searrow 0} \frac{\mathsf{m}(E \cap B_r(x))}{\mathsf{m}(B_r(x))} > 0,\ \varlimsup_{r \searrow 0} \frac{\mathsf{m}(E^c \cap B_r(x))}{\mathsf{m}(B_r(x))} > 0 \right\}.$$

Then $\partial^* E$ is a Borel subset of the topological boundary ∂E of E. Moreover, if $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is a PI space, then $P(E, \cdot)$ is concentrated on $\partial^* E$; see [Ambrosio 2002, Theorem 5.3].

Definition 2.9 (precise representative). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space, and let $f : \mathsf{X} \to \mathbb{R}$ be a Borel function. Then we define the *approximate lower* and *upper limits* as

$$f^\wedge(x) := \operatorname{ap}\varliminf_{y\to x} f(y) := \sup\left\{ t \in \overline{\mathbb{R}} : \lim_{r\searrow 0} \frac{\mathsf{m}(B_r(x)\cap\{f<t\})}{\mathsf{m}(B_r(x))} = 0 \right\},$$
$$f^\vee(x) := \operatorname{ap}\varlimsup_{y\to x} f(y) := \inf\left\{ t \in \overline{\mathbb{R}} : \lim_{r\searrow 0} \frac{\mathsf{m}(B_r(x)\cap\{f>t\})}{\mathsf{m}(B_r(x))} = 0 \right\}$$

for every $x \in \mathsf{X}$. Here we adopt the convention that

$$\inf\varnothing = +\infty \quad \text{and} \quad \sup\varnothing = -\infty.$$

Moreover, we define the *precise representative* $\bar{f} : \mathsf{X} \to \overline{\mathbb{R}}$ of f as

$$\bar{f}(x) := \tfrac{1}{2}(f^\wedge(x) + f^\vee(x)) \quad \text{for every } x \in \mathsf{X},$$

where we adopt the convention that $+\infty - \infty = 0$.

We define the *jump set* $J_f \subseteq \mathsf{X}$ of the function f as the Borel set

$$J_f := \{x \in \mathsf{X} : f^\wedge(x) < f^\vee(x)\}.$$

It is well known that if $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is a PI space and $f \in \mathrm{BV}(\mathsf{X})$, then J_f is a countable union of essential boundaries of sets of finite perimeter, so that in particular $\mathsf{m}(J_f) = 0$. See [Ambrosio et al. 2004, Proposition 5.2]. Moreover, as proved in [Kinnunen et al. 2014, Lemma 3.2], we have that

$$|\mathrm{D}f|(\mathsf{X}\setminus\mathsf{X}_f) = 0, \quad \text{where } \mathsf{X}_f := \{x \in \mathsf{X} \mid -\infty < f^\wedge(x) \le f^\vee(x) < +\infty\}, \tag{2-5}$$

and thus in particular $-\infty < \bar{f}(x) < +\infty$ holds for $|\mathrm{D}f|$-a.e. $x \in \mathsf{X}$.

Definition 2.10. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space, and let $f : \mathsf{X} \to \mathbb{R}$ be Borel. Then we define the *subgraph* of f, denoted by $\mathcal{G}_f \subseteq \mathsf{X}\times\mathbb{R}$, as the Borel set

$$\mathcal{G}_f := \{(x,t) \in \mathsf{X}\times\mathbb{R} : t < f(x)\}.$$

Lemma 2.11. *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a locally uniformly doubling metric measure space, and let $f : \mathsf{X} \to \mathbb{R}$ be a Borel function. Then*

$$(x,t) \in \partial^*\mathcal{G}_f \implies t \in [f^\wedge(x), f^\vee(x)] \qquad \text{and} \qquad t \in (f^\wedge(x), f^\vee(x)) \implies (x,t) \in \partial^*\mathcal{G}_f.$$

In particular, if $x \in \mathsf{X}_f \setminus J_f$, then $\partial^*\mathcal{G}_f \cap (\{x\}\times\mathbb{R}) \subseteq \{(x, \bar{f}(x))\}$.

Proof. In the proof, the constant C_D may change from line to line and it only depends on the doubling constant at scale $R = 1$. We can compute, for $r \in (0, \varepsilon)$, using Fubini's theorem,

$$\begin{aligned}
\frac{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t)\cap\mathcal{G}_f)}{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t))} &\le \frac{(\mathsf{m}\otimes\mathcal{L}^1)((B_r(x)\times B_r(t))\cap\mathcal{G}_f)}{(\mathsf{m}\otimes\mathcal{L}^1)(B_{r/2}(x)\times B_{r/2}(t))} \\
&\le C_D \frac{(\mathsf{m}\otimes\mathcal{L}^1)(\{(y,t) \in B_r(x)\times B_r(t) : t < f(y)\})}{r\,\mathsf{m}(B_r(x))} \\
&\le C_D \frac{\fint_{t-r}^{t+r} \mathsf{m}(\{y \in B_r(x) : s < f(y)\})\,\mathrm{d}s}{\mathsf{m}(B_r(x))} \le C_D \frac{\mathsf{m}(B_r(x)\cap\{f > t-\varepsilon\})}{\mathsf{m}(B_r(x))}.
\end{aligned}$$

Therefore, if $(x,t) \in \partial^*\mathcal{G}_f$, then $t \le f^\vee(x)$. Similarly, we can show that if $r \in (0,\varepsilon)$,

$$\frac{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t)\setminus\mathcal{G}_f)}{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t))} \le C_D \frac{\mathsf{m}(B_r(x)\cap\{f<t+\varepsilon\})}{\mathsf{m}(B_r(x))},$$

which in turn shows that if $(x,t) \in \partial^*\mathcal{G}_f$, then $t \ge f^\wedge(x)$. Conversely, arguing as above, we can show that if $r \in (0,\varepsilon)$,

$$\frac{(\mathsf{m}\otimes\mathcal{L}^1)(B_{2r}(x,t)\cap\mathcal{G}_f)}{(\mathsf{m}\otimes\mathcal{L}^1)(B_{2r}(x,t))} \ge C_D \frac{\mathsf{m}(B_r(x)\cap\{f>t+\varepsilon\})}{\mathsf{m}(B_r(x))}$$

and

$$\frac{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t)\setminus\mathcal{G}_f)}{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t))} \ge C_D \frac{\mathsf{m}(B_r(x)\cap\{f<t-\varepsilon\})}{\mathsf{m}(B_r(x))},$$

which yield the second claim. □

Definition 2.12 (decomposition of the total variation measure). Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be a PI space and $f \in \mathrm{BV}(\mathsf{X})$. Then we write $|\mathrm{D}f|$ as $|\mathrm{D}f|^a + |\mathrm{D}f|^s$, where $|\mathrm{D}f|^a \ll \mathsf{m}$ and $|\mathrm{D}f|^s \perp \mathsf{m}$. We can decompose the singular part $|\mathrm{D}f|^s$ as $|\mathrm{D}f|^j + |\mathrm{D}f|^c$, where the *jump part* is given by $|\mathrm{D}f|^j := |\mathrm{D}f| \llcorner J_f$ while the *Cantor part* is given by $|\mathrm{D}f|^c := |\mathrm{D}f|^s \llcorner (\mathsf{X}\setminus J_f)$.

By [Ambrosio et al. 2015, Theorem 5.1] and its proof, taking into account the elementary inequality

$$a \le \sqrt{1+a^2} \le 1+a \quad \text{for every } a>0,$$

(or see [Ambrosio et al. 2004, Proposition 4.2]) we obtain the following proposition.

Proposition 2.13. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be a* PI *space and* $f \in \mathrm{BV}(\mathsf{X})$. *Then* $\mathcal{G}_f$ *is a set of locally finite perimeter in* $\mathsf{X}\times\mathbb{R}$ *and, denoting with* π *the projection map* $\mathsf{X}\times\mathbb{R}\to\mathsf{X}$,

$$|\mathrm{D}f| \le \pi_*|\mathrm{D}\chi_{\mathcal{G}_f}| \le |\mathrm{D}f| + \mathsf{m}.$$

In particular, if $C \subseteq \mathsf{X}$ *is a Borel set satisfying* $|\mathrm{D}f|^c = |\mathrm{D}f|\llcorner C$, *then*

$$\pi_*(|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner C\times\mathbb{R}) = |\mathrm{D}f|\llcorner C.$$

Definition 2.14. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be a metric measure space and $F \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})^k$. We define

$$|\mathrm{D}F|(\Omega) := \inf\left\{ \varliminf_{i\to\infty} \int_\Omega \left(\sum_{j=1}^k (\mathrm{lip}\, F_i^j)^2 \right)^{1/2} \mathrm{d}\mathsf{m} \;\middle|\; (F_i)_i \subseteq \mathrm{LIP}_{\mathrm{loc}}(\Omega)^k,\ F_i \to F \text{ in } L^1_{\mathrm{loc}}(\Omega)^k \right\}$$

for any open set $\Omega \subseteq \mathsf{X}$. Then we extend this definition to Borel subsets of X, as done in the scalar case; see [Brena and Gigli 2024, Section 2.3]. We also define

$$J_F := \bigcup_{i=1}^k J_{F_i}$$

It is clear that Definition 2.12 extends immediately to the vector-valued case.

2B. RCD *spaces.* We assume the reader is familiar with the language of $\mathrm{RCD}(K,N)$ spaces. Recall that an $\mathrm{RCD}(K,N)$ space is an infinitesimally Hilbertian metric measure space verifying the curvature-dimension condition $\mathrm{CD}(K,N)$, in the sense of Lott–Villani–Sturm, for some $K\in\mathbb{R}$ and $N\in[1,\infty)$. Here we only consider finite-dimensional $\mathrm{RCD}(K,N)$ spaces, namely we assume $N<\infty$. Finite-dimensional RCD spaces are PI. As proven in [Bruè and Semola 2020; De Philippis et al. 2017; Gigli and Pasqualetto 2021; Kell and Mondino 2018; Mondino and Naber 2019], the following structure theorem holds.

Theorem 2.15. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space. Then there exists a number* $n\in\mathbb{N}$ *with* $1\le n\le N$, *called the* ***essential dimension*** *of* $(\mathsf{X},\mathsf{d},\mathsf{m})$, *such that* $\mathsf{m}(\mathsf{X}\setminus\mathcal{R}_n)=0$. *Moreover, the regular set* $\mathcal{R}_n$ *is* (m,n)*-rectifiable and* $\mathsf{m}\ll\mathcal{H}^n\llcorner\mathcal{R}_n$.

Recall that $\mathcal{R}_n$ is said to be (m,n)*-rectifiable* provided there exist Borel subsets $(A_i)_{i\in\mathbb{N}}$ of $\mathcal{R}_n$ such that each A_i is bi-Lipschitz equivalent to a subset of $\mathbb{R}^n$ and $\mathsf{m}\big(\mathcal{R}_n\setminus\bigcup_i A_i\big)=0$.

Sobolev calculus on RCD *spaces.* We assume the reader is familiar with the language of $L^p(\mathsf{m})$*-normed* $L^\infty(\mathsf{m})$*-modules* [Gigli 2018b] and $L^0(\mathrm{Cap})$*-normed* $L^0(\mathrm{Cap})$*-modules* [Debin et al. 2021]. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be a given $\mathrm{RCD}(K,N)$ space. We denote by $L^2(T^*\mathsf{X})$ and $L^2(T\mathsf{X})$ the *cotangent module* and the *tangent module* of $(\mathsf{X},\mathsf{d},\mathsf{m})$, respectively. Moreover, $L^0(T\mathsf{X})$ stands for the $L^0(\mathsf{m})$-completion of $L^2(T\mathsf{X})$, in the sense of [Gigli 2018a, Theorem/Definition 2.7]. A fundamental class of Sobolev functions on X is the algebra of *test functions* [Savaré 2014; Gigli 2018b]:

$$\mathrm{Test}^\infty(\mathsf{X}):=\{f\in D(\Delta)\cap L^\infty(\mathsf{m})\mid |\mathrm{D}f|\in L^\infty(\mathsf{m}),\ \Delta f\in W^{1,2}(\mathsf{X})\cap L^\infty(\mathsf{m})\}.$$

Since RCD spaces enjoy the *Sobolev-to-Lipschitz property*, each function in $\mathrm{Test}^\infty(\mathsf{X})$ has a Lipschitz representative. Moreover, $\mathrm{Test}^\infty(\mathsf{X})$ is dense in $W^{1,2}(\mathsf{X})$ and $\nabla f\cdot\nabla g\in W^{1,2}(\mathsf{X})$ for every $f,g\in\mathrm{Test}^\infty(\mathsf{X})$. The class of *test vector fields* is then defined as

$$\mathrm{TestV}(\mathsf{X}):=\left\{\sum_{i=1}^k f_i\nabla g_i\;\middle|\;k\in\mathbb{N},\ (f_i)_{i=1}^k,\ (g_i)_{i=1}^k\subseteq\mathrm{Test}^\infty(\mathsf{X})\right\}\subseteq L^2(T\mathsf{X}).$$

We denote by $L^0_{\mathrm{Cap}}(T\mathsf{X})$ the *capacitary tangent module* on $(\mathsf{X},\mathsf{d},\mathsf{m})$ introduced in [Debin et al. 2021, Theorem 3.6] and by $\bar\nabla\colon\mathrm{Test}^\infty(\mathsf{X})\to L^0_{\mathrm{Cap}}(T\mathsf{X})$ the capacitary gradient operator. Given any Borel measure μ on X such that $\mu\ll\mathrm{Cap}$ (meaning that $\mu(N)=0$ for every $N\subseteq\mathsf{X}$ Borel with $\mathrm{Cap}(N)=0$), we denote by $\pi_\mu\colon L^0(\mathrm{Cap})\to L^0(\mu)$ the canonical projection.

Letting $L^0_\mu(T\mathsf{X})$ be the quotient of $L^0_{\mathrm{Cap}}(T\mathsf{X})$ up to μ-a.e. equality (where we identify two elements $v,w\in L^0_{\mathrm{Cap}}(T\mathsf{X})$ if $\pi_\mu(|v-w|)=0$ holds μ-a.e.), we have a natural projection map $\pi_\mu\colon L^0_{\mathrm{Cap}}(T\mathsf{X})\to L^0_\mu(T\mathsf{X})$, which satisfies $|\pi_\mu(v)|=\pi_\mu(|v|)$ μ-a.e. for all $v\in L^0_{\mathrm{Cap}}(T\mathsf{X})$. The space $L^0_\mu(T\mathsf{X})$ is an $L^0(\mu)$-normed $L^0(\mu)$-module. As pointed out in [Debin et al. 2021, Proposition 3.9], the quotient $L^0_{\mathsf{m}}(T\mathsf{X})$ can be identified with the tangent module $L^0(T\mathsf{X})$ and the projection $\pi_{\mathsf{m}}\colon L^0_{\mathrm{Cap}}(T\mathsf{X})\to L^0(T\mathsf{X})$ satisfies $\nabla f=\pi_\mu(\bar\nabla f)$ for every $f\in\mathrm{Test}^\infty(\mathsf{X})$. Due to this consistency, to ease the notation we will indicate the capacitary gradient of a test function f with ∇f instead of $\bar\nabla f$.

The *Hessian* of $f\in\mathrm{Test}^\infty(\mathsf{X})$ is the unique tensor $\mathrm{Hess}(f)\in L^2(T^*\mathsf{X})\otimes L^2(T^*\mathsf{X})$ with

$$2\int h\,\mathrm{Hess}(f)(\nabla g_1\otimes\nabla g_2)\,\mathrm{d}\mathsf{m}=-\int\nabla f\cdot\nabla g_1\,\mathrm{div}(h\nabla g_2)+\nabla f\cdot\nabla g_2\,\mathrm{div}(h\nabla g_1)+h\nabla f\cdot\nabla(\nabla g_1\cdot\nabla g_2)\,\mathrm{d}\mathsf{m}$$

for every $g_1, g_2, h \in \mathrm{Test}^\infty(\mathsf{X})$. Recall that a vector field $v \in L^2(T\mathsf{X})$ is said to have a *divergence*, briefly $v \in D(\mathrm{div})$, provided there exists a function $\mathrm{div}(v) \in L^2(\mathsf{m})$ such that

$$\int \nabla f \cdot v \, \mathrm{d}\mathsf{m} = -\int f \, \mathrm{div}(v) \, \mathrm{d}\mathsf{m} \quad \text{for every } f \in W^{1,2}(\mathsf{X}); \tag{2-6}$$

note that $\mathrm{div}(v)$ is uniquely determined by (2-6). The Hessian above is a local object:

$$\chi_{\{f_1=f_2\}} \cdot \mathrm{Hess}(f_1) = \chi_{\{f_1=f_2\}} \cdot \mathrm{Hess}(f_2) \quad \text{for every } f_1, f_2 \in \mathrm{Test}^\infty(\mathsf{X}). \tag{2-7}$$

The validity of this property allows us to define the Hessian of a harmonic function f defined on an open set $\Omega \subseteq \mathsf{X}$, as we are going to discuss. As proven in [Jiang 2014], the harmonic function $f : \Omega \to \mathbb{R}$ is locally Lipschitz. In particular, $\eta f \in \mathrm{Test}^\infty(\mathsf{X})$ for every cut-off function $\eta \in \mathrm{Test}^\infty(\mathsf{X})$ such that $\mathrm{spt}(\eta) \Subset \Omega$. As shown in [Ambrosio et al. 2016; Mondino and Naber 2019], there are plenty of cut-off test functions: given any $x \in \mathsf{X}$ and $0 < r < R$, there exists $\eta \in \mathrm{Test}^\infty(\mathsf{X})$ with $0 \le \eta \le 1$ such that $\eta = 1$ on $B_r(x)$ and $\mathrm{spt}(\eta) \Subset B_R(x)$. Thanks to this fact and to (2-7), it makes sense to m-a.e. define the measurable function $|\mathrm{Hess}(f)| : \Omega \to [0, +\infty)$ as

$$|\mathrm{Hess}(f)| := |\mathrm{Hess}(\eta f)| \quad \mathsf{m}\text{-a.e. on } \{\eta = 1\}$$

for every $\eta \in \mathrm{Test}^\infty(\mathsf{X})$ such that $\mathrm{spt}(\eta) \Subset \Omega$.

BV *calculus on* RCD *spaces.* Now we focus on BV functions and sets of finite perimeter on $\mathrm{RCD}(K, N)$ spaces. The following notion was introduced in [Ambrosio et al. 2019, Definition 4.1].

Definition 2.16 (tangents to a set of finite perimeter). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m}, p)$ be a pointed $\mathrm{RCD}(K, N)$ space and $E \subseteq \mathsf{X}$ a set of locally finite perimeter. Then we define $\mathrm{Tan}_p(\mathsf{X}, \mathsf{d}, \mathsf{m}, E)$ as the family of all quintuplets $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathsf{m}_\mathsf{Y}, q, F)$ that verify the following two conditions:

(1) $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathsf{m}_\mathsf{Y}, q) \in \mathrm{Tan}_p(\mathsf{X}, \mathsf{d}, \mathsf{m})$.

(2) $F \subseteq \mathsf{Y}$ is a set of locally finite perimeter with $\mathsf{m}_\mathsf{Y}(F) > 0$ for which the following property holds: along a sequence $r_i \searrow 0$ such that $(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_p^{r_i}, p) \to (\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathsf{m}_\mathsf{Y}, q)$ in the pmGH sense, with realization Z, it holds that $\chi_E^i \to \chi_F$ in L^1_{loc}, where by χ_E^i we mean the characteristic function of E intended in the rescaled space $(\mathsf{X}, r_i^{-1}\mathsf{d})$. If this is the case, we write

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_p^{r_i}, p, E) \to (\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathsf{m}_\mathsf{Y}, q, F).$$

The following theorem is extracted from [Brena and Gigli 2024, Theorem 3.13], see also [Bruè et al. 2023b, Theorem 2.4].

Theorem 2.17. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $F \in \mathrm{BV}(\mathsf{X})^k$. *Then there exists a unique, up to* $|\mathrm{D}F|$*-a.e. equality,* $\nu_F \in L^0_{\mathrm{Cap}}(T\mathsf{X})^k$ *such that* $|\nu_F| = 1$ $|\mathrm{D}F|$*-a.e. and*

$$\sum_{j=1}^{k} \int_\mathsf{X} F_j \, \mathrm{div}(v_j) \, \mathrm{d}\mathsf{m} = -\int_\mathsf{X} \pi_{|\mathrm{D}F|}(v) \cdot \nu_F \, \mathrm{d}|\mathrm{D}F| \quad \textit{for every } v = (v_1, \dots, v_k) \in \mathrm{TestV}(\mathsf{X})^k.$$

Notice that if $F \in \mathrm{BV}(\mathsf{X})^k$, we consider ν_F as an element of $L^0_{\mathrm{Cap}}(T\mathsf{X})^k$ that is defined $|\mathrm{D}F|$-a.e.. This allows us, via a standard localization procedure, to define ν_F even if F is a vector-valued function of locally bounded variation, or, in other words, if F is a k-tuple of functions of locally bounded variation. In particular, if E is a set of locally finite perimeter, we naturally have a unique, up to $|\mathrm{D}\chi_E|$-a.e. equality, $\nu_E \in L^0_{\mathrm{Cap}}(T\mathsf{X})$, where we understand $\nu_E = \nu_{\chi_E}$.

Next we recall that, as proven in [Bruè et al. 2023b], each set of locally finite perimeter E in an $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ satisfies $|\mathrm{D}\chi_E| \ll \mathrm{Cap}$. Notice however that the same result holds in every metric measure space; see [Brena and Gigli 2024, Theorem 2.5]. By the coarea formula, this absolute continuity extends immediately to total variations, so that

$$|\mathrm{D}F| \ll \mathrm{Cap} \quad \text{for every } F \in \underset{\mathrm{loc}}{\mathrm{BV}}(\mathsf{X})^n.$$

The following proposition summarizes results about sets of finite perimeter that are now well known in the context of PI spaces and are proved in [Ambrosio 2002; Eriksson-Bique et al. 2021]; see also [Ambrosio 2001].

Proposition 2.18. *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a PI space and let $E \subseteq \mathsf{X}$ be a set of locally finite perimeter. Then, for $|\mathrm{D}\chi_E|$-a.e. $x \in \mathsf{X}$ the following hold*:

(i) *E is **asymptotically minimal** at x, in the sense that there exist $r_x > 0$ and a function $\omega_x : (0, r_x) \to (0, \infty)$ with $\lim_{r\searrow 0} \omega_x(r) = 0$ satisfying*

$$|\mathrm{D}\chi_E|(B_r(x)) \le (1 + \omega_x(r))|\mathrm{D}\chi_{E'}|(B_r(x)) \quad \text{if } r \in (0, r_x) \text{ and } E' \Delta E \Subset B_r(x).$$

(ii) *$|\mathrm{D}\chi_E|$ is **asymptotically doubling** at x*:

$$\varlimsup_{r\searrow 0} \frac{|\mathrm{D}\chi_E|(B_{2r}(x))}{|\mathrm{D}\chi_E|(B_r(x))} < \infty.$$

(iii) *We have the estimates*

$$0 < \varliminf_{r\searrow 0} \frac{r|\mathrm{D}\chi_E|(B_r(x))}{\mathsf{m}(B_r(x))} \le \varlimsup_{r\searrow 0} \frac{r|\mathrm{D}\chi_E|(B_r(x))}{\mathsf{m}(B_r(x))} < \infty.$$

(iv) *The following density estimate holds*:

$$\varliminf_{r\searrow 0} \min\left\{\frac{\mathsf{m}(B_r(x) \cap E)}{\mathsf{m}(B_r(x))}, \frac{\mathsf{m}(B_r(x) \setminus E)}{\mathsf{m}(B_r(x))}\right\} > 0.$$

Remark 2.19. It is well known (see [Heinonen et al. 2015, Theorem 3.4.3 and p. 77]) that for an asymptotically doubling measure the Lebesgue differentiation theorem holds. In particular, if E is a set of locally finite perimeter in a PI space and $f \in L^1(|\mathrm{D}\chi_E|)$, then, for $|\mathrm{D}\chi_E|$-a.e. x,

$$\lim_{r\searrow 0} \fint_{B_r(x)} |f(y) - f(x)| \,\mathrm{d}|\mathrm{D}\chi_E|(y) = 0.$$

Let us now introduce the notion of reduced boundary of a set of locally finite perimeter. First, we introduce the set $\mathcal{R}_n^*$. Following [Ambrosio and Tilli 2004], given a metric measure space $(\mathsf{X}, \mathsf{d}, \mu)$ and a real number $k \geq 0$, we define the *upper* and *lower k-dimensional densities* of μ as

$$\overline{\Theta}_k(\mu, x) := \varlimsup_{r \searrow 0} \frac{\mu(B_r(x))}{\omega_k r^k} \quad \text{and} \quad \underline{\Theta}_k(\mu, x) := \varliminf_{r \searrow 0} \frac{\mu(B_r(x))}{\omega_k r^k} \qquad \text{for every } x \in \mathsf{X},$$

respectively. In the case where $\overline{\Theta}_k(\mu, x)$ and $\underline{\Theta}_k(\mu, x)$ coincide, we denote their common value by $\Theta_k(\mu, x) \in [0, +\infty]$, and we call it the *k-dimensional density* of μ at x.

Definition 2.20. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space having essential dimension n. Then we define the set $\mathcal{R}_n^* = \mathcal{R}_n^*(\mathsf{X}) \subseteq \mathcal{R}_n$ as

$$\mathcal{R}_n^* := \{x \in \mathcal{R}_n \mid \exists \Theta_n(\mathsf{m}, x) \in (0, +\infty)\}.$$

In the case in which $\mathsf{m} = \mathcal{H}^N$, by the Bishop–Gromov comparison, one has that $\Theta_N(\mathcal{H}^N, x)$ exists and is positive for every $x \in \mathsf{X}$. Moreover, the volume convergence results in [De Philippis and Gigli 2018] and the lower semicontinuity of the density imply that $\Theta_N(\mathcal{H}^N, x) \leq 1$ for every $x \in \mathsf{X}$. Notice that the set $\mathcal{R}_n^*$ is Borel, see Remark 2.5. As shown in [Ambrosio et al. 2018, Theorem 4.1], $\mathsf{m}(\mathsf{X} \setminus \mathcal{R}_n^*) = 0$.

Definition 2.21 (reduced boundary). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $E \subseteq \mathsf{X}$ be a set of locally finite perimeter. Then we define the *reduced boundary* $\mathcal{F}E \subseteq \partial^* E$ of E as the set of all points $x \in \mathcal{R}_n^*$ satisfying all four conclusions of Proposition 2.18 and such that

$$\mathrm{Tan}_x(\mathsf{X}, \mathsf{d}, \mathsf{m}, E) = \{(\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, \{x_n > 0\})\}, \tag{2-8}$$

where $n \in \mathbb{N}$, $n \leq N$ stands for the essential dimension of $(\mathsf{X}, \mathsf{d}, \mathsf{m})$. We recall that the set of points $x \in X$ that satisfy (2-8) is denoted by $\mathcal{F}_n E$.

As proven in [Bruè et al. 2023a] after [Ambrosio et al. 2019; Bruè et al. 2023b], taking into account the forthcoming Theorem 3.3, the perimeter measure $|\mathrm{D}\chi_E|$ is concentrated on the reduced boundary $\mathcal{F}E$.

Remark 2.22. By the proof of [Ambrosio et al. 2019, Corollary 4.10], by [Ambrosio et al. 2019, Corollary 3.4], and by the membership to $\mathcal{R}_n^*$, we see that the following hold for any $x \in \mathcal{F}E$:

(i) If $r_i \searrow 0$ is such that
$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0) \tag{2-9}$$
in a realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$, then, up to not relabeled subsequences and a change of coordinates in $\mathbb{R}^n$,
$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x, E) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, \{x_n > 0\})$$
in the same realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$. Notice that, given a sequence $r_i \searrow 0$, it is always possible to find a subsequence satisfying (2-9).

(ii) If $r_i \searrow 0$ is such that
$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x, E) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, \{x_n > 0\})$$
in a realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$, then $|\mathrm{D}\chi_E|$ weakly converges to $|\mathrm{D}\chi_{\{x_n > 0\}}|$ in duality with $C_{\mathrm{bs}}(Z)$.

(iii) We have

$$
\lim_{r\searrow 0}\frac{\mathsf{m}(B_r(x))}{r^n}=\omega_n\Theta_n(\mathsf{m},x)\in(0,+\infty),
$$
$$
\lim_{r\searrow 0}\frac{C_x^r}{r^n}=\frac{\omega_n}{n+1}\Theta_n(\mathsf{m},x), \tag{2-10}
$$
$$
\lim_{r\searrow 0}\frac{|\mathrm{D}\chi_E|(B_r(x))}{r^{n-1}}=\omega_{n-1}\Theta_n(\mathsf{m},x).
$$

Definition 2.23 (good coordinates). Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n. Let $E\subseteq\mathsf{X}$ be a set of locally finite perimeter and $x\in\mathcal{F}E$ be given. Then we say that an n-tuple $u=(u^1,\dots,u^n)$ of harmonic functions $u^\ell\colon B_{r_x}(x)\to\mathbb{R}$ is a *system of good coordinates* for E at x provided the following properties are satisfied:

(1) For any $\ell,j=1,\dots,n$,

$$
\lim_{r\searrow 0}\fint_{B_r(x)}|\nabla u^\ell\cdot\nabla u^j-\delta_{\ell j}|\,\mathrm{d}\mathsf{m}=\lim_{r\searrow 0}\fint_{B_r(x)}|\nabla u^\ell\cdot\nabla u^j-\delta_{\ell j}|\,\mathrm{d}|\mathrm{D}\chi_E|=0.
$$

(2) For any $\ell=1,\dots,n$, there exists $\nu_\ell(x)$ defined as follows:

$$
\nu_\ell(x):=\lim_{r\searrow 0}\fint_{B_r(x)}\nu_E\cdot\nabla u^\ell\,\mathrm{d}|\mathrm{D}\chi_E|,\qquad \lim_{r\searrow 0}\fint_{B_r(x)}|\nu_\ell(x)-\nu_E\cdot\nabla u^\ell|\,\mathrm{d}|\mathrm{D}\chi_E|=0. \tag{2-11}
$$

(3) The resulting vector $\nu(x):=(\nu_1(x),\dots,\nu_n(x))\in\mathbb{R}^n$ satisfies $|\nu(x)|=1$.

The following theorem is proved in [Bruè et al. 2023a, Theorem 3.6].

Theorem 2.24. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n. Let $E\subseteq\mathsf{X}$ be a set of locally finite perimeter and $x\in\mathcal{F}E$ be given. Then, good coordinates exist at $|\mathrm{D}\chi_E|$-a.e. point $x\in\mathcal{F}E$.*

Remark 2.25. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n, let $x\in\mathsf{X}$ and let $u=(u^1,\dots,u^n)$ be an n-tuple of harmonic functions satisfying

$$
\lim_{r\searrow 0}\fint_{B_r(x)}|\nabla u^\ell\cdot\nabla u^j-\delta_{\ell j}|\,\mathrm{d}\mathsf{m}=0.
$$

Given a sequence of radii $r_i\searrow 0$ such that

$$
(\mathsf{X},r_i^{-1}\mathsf{d},\mathsf{m}_x^{r_i},x)\to(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)
$$

and a fixed realization of such convergence, it follows from the results recalled in [Bruè et al. 2023b, Section 1.2.3] (see also [Bruè et al. 2023b, (1.22)], a consequence of the improved Bochner inequality in [Han 2018]) that, up to extracting a not relabeled subsequence, the functions in

$$
\{r_i^{-1}u^j\}_i\quad\text{for } j=1,\dots,n
$$

converge locally uniformly to orthogonal coordinate functions of $\mathbb{R}^n$.

The ensuing result is taken from [Bruè et al. 2023a, Proposition 4.8].

Proposition 2.26. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n. *Let* $E \subseteq \mathsf{X}$ *be a set of locally finite perimeter. Then, for* $|D\chi_E|$*-a.e.* $x \in \mathsf{X}$, *the following property holds. Suppose that* $u = (u^1, \dots, u^n)\colon B_r(x) \to \mathbb{R}^n$ *is a system of good coordinates for* E *at* x. *Let* $\nu(x) \in \mathbb{R}^n$ *be as in Definition 2.23. If the coordinates* (x_ℓ) *on the* (*Euclidean*) *tangent space to* X *at* x *are chosen so that the maps* (u^ℓ) *converge to* $(x_\ell)\colon \mathbb{R}^n \to \mathbb{R}^n$ *when properly rescaled, then the blow-up* H *of* E *at* x (*in the sense of finite perimeter sets*) *is*

$$H = \{y \in \mathbb{R}^n \mid y \cdot \nu(x) \geq 0\}.$$

Splitting maps. Let us now present the notion of a δ-splitting map. We follow closely the presentation in [Bruè et al. 2023b], compare with [Bruè et al. 2023b, Definition 3.4].

Definition 2.27 (splitting map). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $x \in \mathsf{X}$, $k \in \mathbb{N}$, and $r, \delta > 0$ be given. Then a map $u = (u_1, \dots, u_k)\colon B_r(x) \to \mathbb{R}^k$ is said to be a δ*-splitting map* provided the following properties hold:

(i) u_ℓ is harmonic, meaning that, for every $\ell = 1, \dots, k$, we have $u_\ell \in D(\Delta, B_r(x))$ and $\Delta u_\ell = 0$, and u_ℓ is C_N-Lipschitz for every $\ell = 1, \dots, k$.

(ii) $r^2 \fint_{B_r(x)} |\mathrm{Hess}(u_\ell)|^2 \, \mathrm{dm} \leq \delta$ for every $\ell = 1, \dots, k$.

(iii) $\fint_{B_r(x)} |\nabla u_\ell \cdot \nabla u_j - \delta_{\ell j}| \, \mathrm{dm} \leq \delta$ for every $\ell, j = 1, \dots, k$.

As already noticed in [Bruè et al. 2023b, Remark 3.6], in the classical definition of δ-splitting maps in the smooth setting, in item (i) above the stronger condition $|\nabla u| \leq 1 + \delta$ is required. Anyway we stress that when $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is an $\mathrm{RCD}(-\delta, N)$ space and u is a δ-splitting map as above, we have that $\sup_{y \in B_{r/2}(x)} |\nabla u|(y) \leq 1 + C_N \delta^{1/2}$, see [Bruè et al. 2022, Remark 3.3], and compare with [Cheeger and Naber 2015, Equations (3.42)–(3.46)]. This means that, for δ small enough, if u is a δ-splitting map on $B_r(x)$ on an $\mathrm{RCD}(-\delta, N)$ space as above, then it is a $C_N \delta^{1/2}$-splitting map on $B_{r/2}(x)$ in the classical smooth sense.

In the following lemma we slightly improve previous results obtained in [Bruè et al. 2023a; 2023b], and we show that we can find good coordinates with respect to *every* $\mathrm{BV}_{\mathrm{loc}}$ function.

Lemma 2.28. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n *and* $\eta \in (0, 1)$. *Then there exists a sequence of* n*-tuples of harmonic* $C_{K,N}$*-Lipschitz maps* $\{u_k\}_k$,

$$u_k = (u_k^1, \dots, u_k^n) : B_{2r_k}(x_k) \to \mathbb{R}^n,$$

and a sequence of pairwise disjoint Borel sets $\{D_k\}_k$ *with* $D_k \subseteq B_{r_k}(x_k)$ *such that*

(i) *for every* $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$,

$$|Df|\Big(\mathsf{X} \setminus \bigcup_k D_k\Big) = 0,$$

(ii) *for every* $x \in D_k$, u_k *is an* η*-splitting map on* $B_r(x)$ *for any* $r \in (0, r_k)$,

(iii) *there exists a Borel matrix-valued map* $M = (M_{\ell,j}) : D_k \to \mathbb{R}^{n\times n}$ *satisfying*

$$\lim_{r\searrow 0} \fint_{B_r(x)} |\nabla u_k^\ell \cdot \nabla u_k^j - M(x)_{\ell,j}| \,\mathrm{dm} = 0. \tag{2-12}$$

To any such collection of η-splitting maps, we can therefore associate a natural map

$$\bigcup_k D_k \to \mathbb{N}, \quad x \mapsto k(x).$$

Proof. The proof follows the arguments given in the proof of [Bruè et al. 2023b, Theorem 3.2]. However, as we need a slightly stronger statement, we include the details of the proof.

Fix a countable dense set $S \subseteq \mathcal{R}_n$. Let $y \in S$ be given. If $\varepsilon > 0$ is small enough and $r \in (0, \sqrt{\varepsilon/|K|}) \cap \mathbb{Q}$ is such that

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X}, r^{-1}\mathsf{d}, \mathsf{m}_y^r, y), (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0)) < \varepsilon,$$

then, by [Bruè et al. 2023b, Corollary 3.10], we obtain a δ-splitting map $u_{y,r} : B_{5r}(y) \to \mathbb{R}^n$ for some δ (which can be made arbitrarily small, taking ε small enough). Let

$$D_{y,r} := \left\{x \in B_{(5/4)r}(y) \;\middle|\; u_{y,r} \text{ is an } \eta\text{-splitting map on } B_s(x) \text{ for every } s \in \left(0, \tfrac{5}{4}r\right)\right\}.$$

The claim of the lemma will be proved with the sequence of sets $\{D_{y,r}\}_{y,r}$ and maps $\{u_{y,r}\}_{y,r}$ after making the sets disjoint and restricting the maps.

Assume now, by contradiction, that the claim is false. Then, using a locality argument and the coarea formula, we find a set of finite perimeter $E \subseteq \mathsf{X}$ such that

$$|\mathrm{D}\chi_E|\left(\mathsf{X} \setminus \bigcup_{y,r} D_{y,r}\right) > 0. \tag{2-13}$$

Fix $\varepsilon > 0$ to be determined later. If $x \in \mathcal{F}E$, then there exists $r = r(x) \in \mathbb{Q} \cap (0, 1)$ such that $|K| r^2 < \varepsilon < 4$ and

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X}, r^{-1}\mathsf{d}, \mathsf{m}_x^r, x), (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0)) < \varepsilon \quad \text{and} \quad \frac{r|\mathrm{D}\chi_E|(B_{r/4}(x))}{\mathsf{m}(B_{r/4}(x))} > 2\frac{\omega_{n-1}}{\omega_n}.$$

By density of S and thanks to an easy continuity argument, we deduce that, for some point $y = y(x) \in S \cap B_{r/2}(x)$,

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X}, r^{-1}\mathsf{d}, \mathsf{m}_y^r, y), (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0)) < \varepsilon \quad \text{and} \quad \frac{r|\mathrm{D}\chi_E|(B_{r/4}(y))}{\mathsf{m}(B_{r/4}(y))} > 2\frac{\omega_{n-1}}{\omega_n}. \tag{2-14}$$

By the discussion above (that is, [Bruè et al. 2023b, Corollary 3.10]), we obtain a δ-splitting map $u_{y,r} : B_{5r}(y) \to \mathbb{R}^n$ for some $\delta = \delta(\varepsilon)$ (which can be made arbitrarily small, taking ε small enough). By [Bruè et al. 2023b, Corollary 3.12], $u_{y,r}$ is a $C_N\delta^{1/4}$-splitting map on $B_s(x)$ for any $x \in D^\varepsilon_{y,r} \subseteq B_{(5/4)r}(y)$ and $s \in \left(0, \frac{5}{4}r\right)$, where

$$\mathcal{H}^h_5(B_{(5/4)r}(y) \setminus D^\varepsilon_{y,r}) \le C_N \delta^{1/2} \frac{\mathsf{m}(B_{(5/2)r}(x))}{\frac{5}{2}r}.$$

Therefore, $D^\varepsilon_{y,r} \subseteq D_{y,r}$ if $C_N\delta^{1/4} < \eta$.

We apply the Vitali covering lemma to the family $\{B_{r(x)/4}(y(x))\}_{x\in\mathcal{F}E}$ constructed as above, and we obtain a sequence of disjoint balls $\{B_{r(x_i)/4}(y(x_i))\}_i$ such that

$$\mathcal{F}E \subseteq \bigcup_i B_{(5/4)r(x_i)}(y(x_i)).$$

Set

$$D^\varepsilon := \bigcup_i D^\varepsilon_{y(x_i),r(x_i)}.$$

Following the computations in the proof of [Bruè et al. 2023b, Theorem 3.2], we obtain

$$\begin{aligned}\mathcal{H}^h_5(\mathcal{F}E\setminus D^\varepsilon) &\le \sum_{i\in\mathbb{N}} \mathcal{H}^h_5(B_{(5/4)r(x_i)}(y(x_i))\setminus D^\varepsilon_{y(x_i),r(x_i)}) \le C_N\delta^{1/2}\sum_{i\in\mathbb{N}}\frac{\mathsf{m}(B_{(5/2)r(x_i)}(y(x_i)))}{\frac{5}{2}r(x_i)}\\ &\le C_N\delta^{1/2}\sum_{i\in\mathbb{N}}\frac{\mathsf{m}(B_{r(x_i)/4}(y(x_i)))}{\frac{1}{4}r(x_i)} \le C_N\delta^{1/2}|\mathrm{D}\chi_E|(\mathsf{X}),\end{aligned} \tag{2-15}$$

where the constants C_N may change from line to line, in the third inequality we are using the doubling property together with the fact that $r(x_i)$ is sufficiently small, and in the last inequality we are using (2-14) together with the fact that the $\{B_{r(x_i)/4}(y(x_i))\}$ are disjoint. Let now $\{\varepsilon_i\}_i$ with $\varepsilon_i \searrow 0$ be such that the corresponding $\{\delta_i\}_i$ satisfy both $\delta_i^{1/2} \le 2^{-i}$ and $C_N\delta_i^{1/4} < \eta$, and set

$$G := \bigcup_i D^{\varepsilon_i} \subseteq D_{y,r}.$$

Then $\mathcal{H}^h_5(\mathcal{F}E\setminus G) = 0$, which contradicts (2-13).

Finally, item (iii) is a direct consequence of the fact that, since u^ℓ_k is harmonic for every $\ell = 1,\dots,n$ and $k\in\mathbb{N}$, one can give a pointwise meaning to $\nabla u^\ell_k(x)\cdot\nabla u^j_k(x)$, compare with [Bruè et al. 2023a, Remark 2.10]. □

Definition 2.29. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space having essential dimension n. Then by a *good collection of splitting maps* on X we mean a family $\{\boldsymbol{u}_\eta : \eta\in(0,n^{-1})\cap\mathbb{Q}\}$ of sequences $\boldsymbol{u}_\eta = (u_{\eta,k})_{k\in\mathbb{N}}$ of maps

$$u_{\eta,k} = (u^1_{\eta,k},\dots,u^n_{\eta,k})\colon B_{r_{\eta,k}}(x_{\eta,k}) \to \mathbb{R}^n$$

as in Lemma 2.28. We will denote by $D_{\eta,k}\subseteq B_{r_{\eta,k}}(x_{\eta,k})$ the sets associated to $\boldsymbol{u}_\eta$ as in Lemma 2.28. We define

$$D_\eta := \bigcup_{k=1}^\infty D_{\eta,k},$$

and by $k_\eta(x) : D_\eta \to \mathbb{N}$ we denote the unique index satisfying $x\in D_{\eta,k_\eta(x)}$. For every $x\in D_{\eta,k}$ we define a matrix $A_\eta(x)\in\mathbb{R}^{n\times n}$ such that, with the same notation of Lemma 2.28, $A_\eta(x)M_\eta(x)A_\eta(x)^T = \mathrm{Id}_{n\times n}$. The existence of such a matrix follows from the choice of $\bar\eta_n$. Indeed, from the construction of the symmetric matrix $B_\eta(x)$, it follows that $\|\mathrm{Id} - M_\eta(x)\|_{L^\infty} < n^{-1}$, thus $\|\mathrm{Id} - M_\eta(x)\|_{\mathrm{op}} < 1$, so that the conclusion follows from the spectral theorem.

Notice that, for every $f \in \mathrm{BV}(\mathsf{X})$, we have $|\mathrm{D}f|(\mathsf{X} \setminus D_\eta) = 0$. Let us fix $\eta \in (0, n^{-1}) \cap \mathbb{Q}$. Since for every $x \in D_\eta$ there exists a unique $k_\eta(x)$ such that $x \in D_{\eta,k_\eta(x)}$, and since there exists also a splitting map $u_{\eta,k_\eta(x)}$ on some ball around x, one has that the limit

$$\lim_{r\to 0} \fint_{B_r(x)} \nabla u^\ell_{\eta,k_{\eta(x)}} \cdot \nabla u^j_{\eta,k_{\eta(x)}} \, \mathrm{dm}$$

exists for every $\ell, j \in \{1, \dots, n\}$, compare the end of the proof of Lemma 2.28 and [Bruè et al. 2023a, Remark 2.10]. Hence, for every $\eta \in (0, n^{-1}) \cap \mathbb{Q}$, one can give a pointwise meaning to the $\mathbb{R}^{n\times n}$-valued map

$$M_\eta : x \in D_\eta \mapsto (\nabla u^\ell_{\eta,k_{\eta(x)}} \cdot \nabla u^j_{\eta,k_{\eta(x)}})_{\ell,j\in\{1,\dots,n\}}(x) \tag{2-16}$$

such that (2-12) holds.

3. Main results

3A. *Representation formula for the perimeter.* In this section we prove, by exploiting [Deng 2020] and the same argument of [Bruè et al. 2023a], that the total variation of every BV function is concentrated on $\mathcal{R}^*_n$. We use the latter information to deduce that the perimeter measure of every set of locally finite perimeter is mutually absolutely continuous with respect to $\mathcal{H}^{n-1}$. We will be using the following theorem, which is proved in [Deng 2020, Theorem 1.3].

Theorem 3.1. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space, with* $K \in \mathbb{R}$ *and* $N \geq 1$, *and* $\mathrm{spt}(\mathsf{m}) = \mathsf{X}$. *Then* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *is nonbranching, i.e., if* $\gamma, \sigma : [0, L] \to \mathsf{X}$ *are two unit speed geodesics satisfying* $\gamma(0) = \sigma(0)$ *and* $\gamma(t_0) = \sigma(t_0)$ *for some* $t_0 \in (0, L)$, *then* $\gamma = \sigma$.

Proposition 3.2. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space having essential dimension n. Suppose that* $\gamma : [0, 1] \to \mathsf{X}$ *is a geodesic satisfying* $\gamma_t \in \mathcal{R}^*_n$ *for a dense family of* $t \in (0, 1)$. *Then* $\gamma_t \in \mathcal{R}^*_n$ *for every* $t \in (0, 1)$.

Proof. Let $\delta \in \left(0, \frac{1}{20}\right)$ be fixed. Theorem 3.1 ensures that the constant-speed reparametrization of $\gamma|_{[\delta/2,1-\delta/2]}$ on $[0, 1]$ is the unique geodesic between its endpoints. Then [Deng 2020, (166)] gives $\varepsilon = \varepsilon(N, \delta) > 0$, $\bar r = \bar r(N, \delta) > 0$, and $C = C(N, \delta) > 0$ such that

$$\left|\frac{\mathsf{m}(B_r(\gamma_s))}{\mathsf{m}(B_r(\gamma_{s'}))} - 1\right| \leq C|s - s'|^{\frac{1}{2(1+2N)}} \quad \text{for every } r \in (0, \bar r) \text{ and } s, s' \in [\delta, 1 - \delta] \text{ with } |s - s'| < \varepsilon.$$

In particular, for any $s, s' \in [\delta, 1 - \delta]$ with $|s - s'| < \varepsilon$, we have

$$\left|\frac{\mathsf{m}(B_r(\gamma_s))}{\omega_n r^n}\left(\frac{\mathsf{m}(B_r(\gamma_{s'}))}{\omega_n r^n}\right)^{-1} - 1\right| \leq C|s - s'|^{\frac{1}{2(1+2N)}} \quad \text{for every } r \in (0, \bar r). \tag{3-1}$$

Now let $t \in [\delta, 1 - \delta]$ be fixed, and choose a sequence $(t_i)_{i\in\mathbb{N}} \subseteq \gamma^{-1}(\mathcal{R}^*_n) \cap [\delta, 1 - \delta] \cap (t - \varepsilon, t + \varepsilon)$ such that $t_i \to t$. Up to a not relabeled subsequence, we can assume that $\Theta_n(\mathsf{m}, \gamma_{t_i}) \to \lambda$ for some $\lambda \in [0, +\infty]$. Pick sequences $(r_j)_{j\in\mathbb{N}}, (\tilde r_j)_{j\in\mathbb{N}} \subseteq (0, \bar r)$ such that

$$\frac{\mathsf{m}(B_{r_j}(\gamma_t))}{\omega_n r_j^n} \to \overline{\Theta}_n(\mathsf{m}, \gamma_t) \quad \text{and} \quad \frac{\mathsf{m}(B_{\tilde r_j}(\gamma_t))}{\omega_n \tilde r_j^n} \to \underline{\Theta}_n(\mathsf{m}, \gamma_t).$$

Plugging $(s, s', r) = (t, t_i, r_j)$ or $(s, s', r) = (t, t_i, \tilde{r}_j)$ into (3-1) and letting $j \to \infty$, we deduce that $\overline{\Theta}_n(\mathsf{m}, \gamma_t) < +\infty$ and

$$\left|\frac{\overline{\Theta}_n(\mathsf{m}, \gamma_t)}{\Theta_n(\mathsf{m}, \gamma_{t_i})} - 1\right|, \left|\frac{\underline{\Theta}_n(\mathsf{m}, \gamma_t)}{\Theta_n(\mathsf{m}, \gamma_{t_i})} - 1\right| \le C|t - t_i|^{\frac{1}{2(1+2N)}} \quad \text{for every } i \in \mathbb{N}. \tag{3-2}$$

Similarly, plugging $(s, s', r) = (t_i, t, r_j)$ or $(s, s', r) = (t_i, t, \tilde{r}_j)$ into (3-1) and letting $j \to \infty$, we deduce that $\underline{\Theta}_n(\mathsf{m}, \gamma_t) > 0$ and

$$\left|\frac{\Theta_n(\mathsf{m}, \gamma_{t_i})}{\overline{\Theta}_n(\mathsf{m}, \gamma_t)} - 1\right|, \left|\frac{\Theta_n(\mathsf{m}, \gamma_{t_i})}{\underline{\Theta}_n(\mathsf{m}, \gamma_t)} - 1\right| \le C|t - t_i|^{\frac{1}{2(1+2N)}} \quad \text{for every } i \in \mathbb{N}. \tag{3-3}$$

Observe that (3-2) and (3-3) imply, respectively, that, for every $i \in \mathbb{N}$,

$$|\overline{\Theta}_n(\mathsf{m}, \gamma_t) - \underline{\Theta}_n(\mathsf{m}, \gamma_t)| \le 2C|t - t_i|^{\frac{1}{2(1+2N)}}\,\Theta_n(\mathsf{m}, \gamma_{t_i}), \tag{3-4a}$$

$$|\overline{\Theta}_n(\mathsf{m}, \gamma_t) - \underline{\Theta}_n(\mathsf{m}, \gamma_t)| \le 2C|t - t_i|^{\frac{1}{2(1+2N)}}\,\frac{\overline{\Theta}_n(\mathsf{m}, \gamma_t)\underline{\Theta}_n(\mathsf{m}, \gamma_t)}{\Theta_n(\mathsf{m}, \gamma_{t_i})}. \tag{3-4b}$$

Hence, we can conclude that $\overline{\Theta}_n(\mathsf{m}, \gamma_t) = \underline{\Theta}_n(\mathsf{m}, \gamma_t)$ by letting $i \to \infty$ in (3-4a) if $\lambda < +\infty$, or in (3-4b) if $\lambda = +\infty$. This shows that $\gamma_t \in \mathcal{R}_n^*$ for every $t \in [\delta, 1 - \delta]$. Thanks to the arbitrariness of δ, we proved that $\gamma_t \in \mathcal{R}_n^*$ for every $t \in (0, 1)$, as desired. □

Theorem 3.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space having essential dimension n. Then*

$$|\mathrm{D}f|(\mathsf{X} \setminus \mathcal{R}_n^*) = 0 \quad \textit{for every } f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X}).$$

Proof. The statement can be achieved by repeating verbatim the proof of [Bruè et al. 2023a, Theorem 3.1], using $\mathcal{R}_n^*$ instead of $\mathcal{R}_n$ and Proposition 3.2 instead of [Bruè et al. 2023a, Proposition 2.14]. □

The following theorem answers [Semola 2020, Conjecture 5.32] in the affirmative.

Theorem 3.4 (representation formula for the perimeter). *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space having essential dimension n. Let* $E \subseteq \mathsf{X}$ *be a set of locally finite perimeter. Then*

$$|\mathrm{D}\chi_E| = \Theta_n(\mathsf{m}, \cdot)\mathcal{H}^{n-1} \llcorner \mathcal{F}E. \tag{3-5}$$

In particular, $\Theta_{n-1}(|\mathrm{D}\chi_E|, x) = \Theta_n(\mathsf{m}, x)$ *for* $\mathcal{H}^{n-1}$*-a.e.* $x \in FE$.

Proof. Up to a standard localization argument, we can suppose that E is of finite perimeter. Define $R_j := \{x \in \mathcal{R}_n^* : 2^j \le \Theta_n(\mathsf{m}, x) < 2^{j+1}\}$ for all $j \in \mathbb{Z}$. Notice that $\{R_j\}_{j \in \mathbb{Z}}$ is a measurable partition of $\mathcal{R}_n^*$. Given $j \in \mathbb{Z}$ and $B \subseteq \mathsf{X}$ Borel, for any $x \in B \cap R_j \cap \mathcal{F}E$, there exists

$$\Theta_{n-1}(|\mathrm{D}\chi_E|, x) = \frac{\omega_n}{\omega_{n-1}} \lim_{r \searrow 0} \frac{r|\mathrm{D}\chi_E|(B_r(x))}{\mathsf{m}(B_r(x))} \frac{\mathsf{m}(B_r(x))}{\omega_n r^n} = \Theta_n(\mathsf{m}, x) \in [2^j, 2^{j+1}). \tag{3-6}$$

Therefore, an application of [Ambrosio and Tilli 2004, Theorem 2.4.3] yields, for any $B \subseteq \mathsf{X}$ Borel,

$$2^j \mathcal{H}^{n-1}(B \cap R_j \cap \mathcal{F}E) \le |\mathrm{D}\chi_E|(B \cap R_j) \le 2^{j+n}\mathcal{H}^{n-1}(B \cap R_j \cap \mathcal{F}E),$$

whence $2^j\mathcal{H}^{n-1}\llcorner(R_j\cap\mathcal{F}E)\le|\mathrm{D}\chi_E|\llcorner R_j\le 2^{j+n}\mathcal{H}^{n-1}\llcorner(R_j\cap\mathcal{F}E)$. Thanks to Theorem 3.3, we deduce that $\mu_E:=\mathcal{H}^{n-1}\llcorner\mathcal{F}E$ is a σ-finite Borel measure on X satisfying $|\mathrm{D}\chi_E|\ll\mu_E\ll|\mathrm{D}\chi_E|$. In particular, we know from [Bruè et al. 2023b, Theorem 4.1] that the set $\mathcal{F}E$ is countably $\mathcal{H}^{n-1}$-rectifiable, so that [Ambrosio and Kirchheim 2000, Theorem 5.4] and the computation in (3-6) ensure that

$$\frac{\mathrm{d}|\mathrm{D}\chi_E|}{\mathrm{d}\mu_E}(x)=\lim_{r\searrow 0}\frac{|\mathrm{D}\chi_E|(B_r(x))}{\omega_{n-1}r^{n-1}}=\Theta_n(\mathsf{m},x)$$

is satisfied for $\mathcal{H}^{n-1}$-a.e. $x\in\mathcal{F}E$. Therefore, the identity stated in (3-5) is achieved. □

Remark 3.5. Notice that, as a consequence of [Bruè et al. 2023a, Corollary 3.2], for any set E of locally finite perimeter in an $\mathrm{RCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathsf{m})$ of essential dimension n, we have

$$|D\chi_E|=\frac{\omega_{n-1}}{\omega_n}\mathcal{H}^h\llcorner\mathcal{F}E.$$

Hence, taking also (3-5) into account, we conclude that the measure $\mathcal{H}^h$ and $\mathcal{H}^{n-1}$ are mutually absolutely continuous on the reduced boundary $\mathcal{F}E$.

3B. ***Auxiliary results.*** Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n. Notice that if a given function $u:B_r(\bar{x})\to\mathbb{R}$ is harmonic, then ∇u admits a quasicontinuous representative in a localization of $L^0_{\mathrm{Cap}}(T\mathsf{X})$. Also, by tensorization of the energy, if $k\in\mathbb{N}$, then the function

$$\mathsf{X}\times\mathbb{R}^k\supseteq B_r(\bar{x})\times\mathbb{R}^k\ni(x,y)\mapsto u(x)$$

is harmonic, and hence it admits a quasicontinuous representative in a localization of $L^0_{\mathrm{Cap}}(T(\mathsf{X}\times\mathbb{R}^k))$ with respect to the relevant capacity. Therefore, the following definition is meaningful.

Definition 3.6. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space having essential dimension n. Let $f\in\mathrm{BV}(\mathsf{X})$ be given. Fix a good collection $\{\boldsymbol{u}_\eta\}_\eta$ of splitting maps on X. Then, given any $\eta\in(0,n^{-1})\cap\mathbb{Q}$, the $|\mathrm{D}f|$-measurable map $\nu_f^{\boldsymbol{u}_\eta}:\mathsf{X}\to\mathbb{R}^n$ is defined at $|\mathrm{D}f|$-a.e. $x\in\mathsf{X}$ as

$$\nu_f^{\boldsymbol{u}_\eta}(x):=((\nu_f\cdot\nabla u^1_{\eta,k_\eta(x)})(x),\dots,(\nu_f\cdot\nabla u^n_{\eta,k_\eta(x)})(x)).$$

The $|\mathrm{D}\chi_{\mathcal{G}_f}|$-measurable map $\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}:\mathsf{X}\times\mathbb{R}\to\mathbb{R}^{n+1}$ is defined at $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $p=(x,t)\in\mathsf{X}\times\mathbb{R}$ as

$$\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(p):=((\nu_{\mathcal{G}_f}\cdot\nabla u^1_{\eta,k_\eta(x)})(p),\dots,(\nu_{\mathcal{G}_f}\cdot\nabla u^n_{\eta,k_\eta(x)})(p),(\nu_{\mathcal{G}_f}\cdot\nabla\pi^2)(p));$$

notice that $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $p=(x,t)$ satisfies $x\in D_\eta$ as a consequence of Lemma 2.28(i), Proposition 2.13, and the existence of functions of locally bounded variation whose total variation equals m.

In view of the following proposition, recall the definition of the reduced boundary in use in this note, Definition 2.21. In particular, notice that, by definition, $\mathcal{F}\mathcal{G}_f\subseteq\mathcal{R}^*_{n+1}(\mathsf{X}\times\mathbb{R})$, and we will use this inclusion throughout (in particular, recall the properties stated in Remark 2.22). Notice finally that the matrix valued maps $C_f\ni x\mapsto A_\eta(x)$ in the proposition below are independent of f (up to the choice of their domain).

Proposition 3.7. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space having essential dimension n. Let* $f\in\mathrm{BV}(\mathsf{X})$ *be given. Let* $\{\boldsymbol{u}_\eta\}_\eta$ *be a good collection of splitting maps on* X*. Then there exists a Borel set* $C_f\subseteq\mathsf{X}$ *satisfying the following properties*:

(i) $|\mathrm{D}f|^c=|\mathrm{D}f|\llcorner C_f$ *and* $\mathsf{m}(C_f)=0$.

(ii) $C_f\subseteq\mathcal{R}_n^*(\mathsf{X})\setminus J_f$ *and* $\mathcal{F}\mathcal{G}_f\cap(C_f\times\mathbb{R})=(\mathrm{id}_{\mathsf{X}},\bar f)(C_f)$.

(iii) *Given any* $\eta\in(0,n^{-1})\cap\mathbb{Q}$ *and* $x\in C_f$, *for* $A_\eta(x)\in\mathbb{R}^{n\times n}$ *as in Definition 2.29,* $(A_\eta(x)u_{\eta,k(x)},\pi^2)$ *is a set of good coordinates for* $\mathcal{G}_f$ *at* $(x,\bar f(x))$.

(iv) *If* $u=(u^1,\dots,u^{n+1})\colon B_{r_x}(x,\bar f(x))\to\mathbb{R}^{n+1}$ *is a system of good coordinates for* $\mathcal{G}_f$ *at* $(x,\bar f(x))$ *for some* $x\in C_f$, *and the coordinates* (x_ℓ) *on the* (*Euclidean*) *tangent space to* $\mathsf{X}\times\mathbb{R}$ *at* $(x,\bar f(x))$ *are chosen so that the maps* (u^ℓ) *converge to* $(x_\ell)\colon\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ (*when properly rescaled, see Remark 2.25*), *then the blow-up of* $\mathcal{G}_f$ *at* $(x,\bar f(x))$ *can be written as*

$$H:=\{y\in\mathbb{R}^{n+1}\mid y\cdot\nu(x,\bar f(x))\geq 0\},$$

where the unit vector $\nu(x,\bar f(x)):=(\nu^1(x,\bar f(x)),\dots,\nu^{n+1}(x,\bar f(x)))$ *is given by* (2-11).

(v) *If* $p=(x,\bar f(x))\in C_f\times\mathbb{R}$, *then, for every* $\eta\in(0,n^{-1})\cap\mathbb{Q}$, *we have* $x\in D_{\eta,k_\eta(x)}$ *for some* $k_\eta(x)$ *and p is a point of density* 1 *of* $D_{\eta,k_\eta(x)}\times\mathbb{R}$ *for* $|\mathrm{D}\chi_{\mathcal{G}_f}|$.

Proof. Let us start this proof by defining several sets whose intersection will define C_f. Hence we will define C_f in (3-11), and we will verify each item separately.

For every $\eta\in(0,n^{-1})\cap\mathbb{Q}$ and every $k\in\mathbb{N}$, take $\mathcal{D}_{\eta,k}$ to be the set of points of density 1 in $(D_{\eta,k}\times\mathbb{R})\cap\mathcal{F}\mathcal{G}_f$ with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$. We thus have that $\bigcup_{k\in\mathbb{N}}\mathcal{D}_{\eta,k}$ covers $|\mathrm{D}\chi_{\mathcal{G}_f}|$-almost all $D_\eta\times\mathbb{R}$. Hence, by Proposition 2.13 and Lemma 2.28, the set $\pi^1\big(\bigcup_{k\in\mathbb{N}}\mathcal{D}_{\eta,k}\big)$ covers $|\mathrm{D}f|$-almost all X for every $\eta\in(0,n^{-1})\cap\mathbb{Q}$. As a consequence, if we denote $\mathcal{D}:=\bigcap_{\eta\in(0,n^{-1})\cap\mathbb{Q}}\pi^1\big(\bigcup_{k\in\mathbb{N}}\mathcal{D}_{\eta,k}\big)$, then

$$|\mathrm{D}f|(\mathsf{X}\setminus\mathcal{D})=0. \tag{3-7}$$

Let $\mathcal{A}\subseteq\mathsf{X}\times\mathbb{R}$ be the set of points $(x,t)\in\mathsf{X}\times\mathbb{R}$ such that, if $u=(u^1,\dots,u^{n+1}):B_{r_{(x,t)}}(x,t)\to\mathbb{R}^{n+1}$ is a system of good coordinates for $\mathcal{G}_f$ at (x,t), and the coordinates (x_ℓ) on the (Euclidean) tangent space to $\mathsf{X}\times\mathbb{R}$ at (x,t) are chosen so that the maps (u^ℓ) converge to $(x_\ell)\colon\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ (when properly rescaled), then the blow-up of $\mathcal{G}_f$ at (x,t) can be written as

$$\{y\in\mathbb{R}^{n+1}\mid y\cdot\nu(x,t)\geq 0\},$$

where the unit vector $\nu(x,t):=(\nu^1(x,t),\dots,\nu^{n+1}(x,t))$ is given by (2-11). Then, by Proposition 2.26, we have also that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|((\mathsf{X}\times\mathbb{R})\setminus\mathcal{A})=0. \tag{3-8}$$

Let $\eta\in(0,n^{-1})\cap\mathbb{Q}$, and let $\mathcal{T}_\eta$ be the Lebesgue points of $\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}$ (defined in Definition 3.6) with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$. Let $\mathcal{T}:=\bigcap_{\eta\in(0,n^{-1})\cap\mathbb{Q}}\mathcal{T}_\eta$, and notice that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|((\mathsf{X}\times\mathbb{R})\setminus\mathcal{T})=0. \tag{3-9}$$

Let us fix $\eta \in (0, n^{-1}) \cap \mathbb{Q}$ and $k \in \mathbb{N}$. Let $\widetilde{M}_\eta := (M_\eta, \pi^2)$ be defined on $\mathsf{X} \times \mathbb{R}$, where M_η is defined in (2-16). Notice that $\widetilde{M}_\eta$ is $|\mathrm{D}\chi_{\mathcal{G}_f}|$-measurable. Let $\mathcal{S}_\eta$ be the Lebesgue points of $\widetilde{M}_\eta$ with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$, and let $\mathcal{S} := \bigcap_{\eta\in(0,n^{-1})} \mathcal{S}_\eta$. Notice that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|((\mathsf{X} \times \mathbb{R}) \setminus \mathcal{S}) = 0. \tag{3-10}$$

Let $S \subseteq \mathsf{X}_f$ with $\mathsf{m}(S) = 0$ be such that $|\mathrm{D}f|^s$ is concentrated on S (recall (2-5)). Let us now define

$$C_f := S \cap (\mathcal{R}_n^*(\mathsf{X}) \setminus J_f) \cap \Bigg(\bigcap_{\eta\in(0,n^{-1})\cap\mathbb{Q}} D_\eta \Bigg) \cap \pi^1(\mathcal{A} \cap \mathcal{T} \cap \mathcal{S} \cap \mathcal{F}\mathcal{G}_f) \cap \mathcal{D}, \tag{3-11}$$

where D_η is defined in Definition 2.29, J_f is the jump set of f, $\mathcal{F}\mathcal{G}_f$ is the reduced boundary of $\mathcal{G}_f$, and $\mathcal{A}$, $\mathcal{T}$, $\mathcal{S}$ are defined above. Let us verify each item separately.

Item (i). Notice that $|\mathrm{D}f|^c$ is concentrated on S. Moreover, $|\mathrm{D}f|^c$ is concentrated on $\mathsf{X} \setminus J_f$, and, due to Lemma 2.28, $|\mathrm{D}f|^c$ is concentrated on $\bigcap_{\eta\in(0,n^{-1})\cap\mathbb{Q}} D_\eta$ as well. Due to (3-7), $|\mathrm{D}f|$ is concentrated on $\mathcal{D}$. Furthermore, $|\mathrm{D}\chi_{\mathcal{G}_f}|$ is concentrated on $\mathcal{A} \cap \mathcal{T} \cap \mathcal{S} \cap \mathcal{F}\mathcal{G}_f$ due to (3-8)–(3-10) and to the definition of reduced boundary, see Definition 2.21. Thus, due to Proposition 2.13, $|\mathrm{D}f|$ is concentrated on $\pi^1(\mathcal{A} \cap \mathcal{T} \cap \mathcal{S} \cap \mathcal{F}\mathcal{G}_f)$. Putting this all together, we get that $|\mathrm{D}f|^c$ is concentrated on C_f.

Item (ii). By Lemma 2.11, one has that if $x \in C_f \setminus J_f$, then $\mathcal{F}\mathcal{G}_f \cap (\{x\} \times \mathbb{R}) = \{(x, \bar{f}(x))\}$. Indeed, $x \in C_f \subseteq \pi^1(\mathcal{F}\mathcal{G}_f)$, and then $\mathcal{F}\mathcal{G}_f \cap (\{x\} \times \mathbb{R})$ is nonempty. Hence $\mathcal{F}\mathcal{G}_f \cap (C_f \times \mathbb{R}) = (\mathrm{id}_{\mathsf{X}}, \bar{f})(C_f)$.

Item (iii). Let $x \in C_f$. Hence, by item (ii) and by definition of C_f, we have that $x = \pi^1(x, \bar{f}(x))$ and $(x, \bar{f}(x)) \in \mathcal{T} \cap \mathcal{S}$.

Let $\eta \in (0, n^{-1}) \cap \mathbb{Q}$. We have that there exists $k_\eta(x)$ such that $x \in D_{\eta,k_\eta(x)}$. By Lemma 2.28(iii), compare with (2-16), we get the existence of a matrix $M(x) \in \mathbb{R}^{n\times n}$ such that, for every $\ell, j = 1, \dots, n$,

$$\lim_{r\searrow 0} ⨍_{B_r(x)} |\nabla u^\ell_{\eta,k_\eta(x)} \cdot \nabla u^j_{\eta,k_\eta(x)} - M(x)_{\ell,j}| \,\mathrm{dm} = 0.$$

Hence, taking the matrix $A_\eta(x)$ from Definition 2.29, we conclude that, calling $v_{\eta,k_\eta(x)} := A_\eta(x) u_{\eta,k_\eta(x)}$, we have, for every $\ell, j = 1, \dots, n$,

$$\lim_{r\searrow 0} ⨍_{B_r(x)} |\nabla v^\ell_{\eta,k_\eta(x)} \cdot \nabla v^j_{\eta,k_\eta(x)} - \delta_{\ell j}| \,\mathrm{dm} = 0.$$

Hence, as a consequence of the previous equality, the independence of the coordinates in $\mathsf{X}\times\mathbb{R}$, and Fubini's theorem, calling $\tilde{v}_{\eta,k_\eta(x)} := (v_{\eta,k_\eta(x)}, \pi^2)$, we get that the following holds for every $\ell, j = 1, \dots, n+1$:

$$\lim_{r\searrow 0} ⨍_{B_r(x,\bar{f}(x))} |\nabla \tilde{v}^\ell_{\eta,k_\eta(x)} \cdot \nabla \tilde{v}^j_{\eta,k_\eta(x)} - \delta_{\ell j}| \,\mathrm{d}(\mathsf{m} \otimes \mathcal{H}^1) = 0. \tag{3-12}$$

Now, since $(x, \bar{f}(x)) \in \mathcal{S}_\eta$ and $\mathcal{S}_\eta$ is the set of the Lebesgue points of (M_η, π^2) (see (2-16)) with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$, we also get, for every $\ell, j = 1, \dots, n+1$,

$$\lim_{r\searrow 0} ⨍_{B_r(x,\bar{f}(x))} |\nabla \tilde{v}^\ell_{\eta,k_\eta(x)} \cdot \nabla \tilde{v}^j_{\eta,k_\eta(x)} - \delta_{\ell j}| \,\mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| = 0. \tag{3-13}$$

Finally, notice that $(x, \bar f(x)) \in \mathcal{T}_\eta$ and $\mathcal{T}_\eta$ are the Lebesgue points of $\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}$ with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$. Hence, $(x, \bar f(x))$ is also a Lebesgue point of the $|\mathrm{D}\chi_{\mathcal{G}_f}|$-measurable map defined for $p = (y, t)$ as

$$\tilde\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(p) := ((\nu_{\mathcal{G}_f} \cdot \nabla A_\eta(x) u^1_{\eta,k_\eta(x)})(p), \dots, (\nu_{\mathcal{G}_f} \cdot \nabla A_\eta(x) u^n_{\eta,k_\eta(x)})(p), (\nu_{\mathcal{G}_f} \cdot \nabla \pi^2)(p)). \tag{3-14}$$

Arguing as in the last part of [Bruè et al. 2023a, Proposition 3.6], we get that the norm of the $|\mathrm{D}\chi_{\mathcal{G}_f}|$-Lebesgue representative of $\tilde\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}$ at $(x, \bar f(x))$ is 1. Hence the last information, together with (3-12) and (3-13), give that $\tilde v_{\eta,k_\eta(x)}$ is a set of good coordinates for $\mathcal{G}_f$ at $(x, \bar f(x))$.

Item (iv). It follows from item (ii) and the definition of $\mathcal{A}$.

Item (v). It follows from item (ii) and the definition of $\mathcal{D}$. □

Theorem 3.8. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space having essential dimension* n. *Fix a function* $f \in \mathrm{BV}(\mathsf{X})$ *and a good collection* $\{\boldsymbol{u}_\eta\}_\eta$ *of splitting maps on* X. *Let* $C_f \subseteq \mathsf{X}$ *be as in Proposition 3.7. Then, for any given* $\eta \in (0, n^{-1}) \cap \mathbb{Q}$,

$$(\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta})_{n+1}(p) = 0 \quad \textit{for } \mathcal{H}^n\textit{-a.e. } p \in \mathcal{F}\mathcal{G}_f \cap (C_f \times \mathbb{R}).$$

Proof. We recall from Proposition 2.13 that $\pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner (\mathcal{F}\mathcal{G}_f \cap (C_f \times \mathbb{R}))) = |\mathrm{D}f|^c$. Moreover, Lemma 2.11 ensures that the mapping $\pi^1 \colon \mathcal{F}\mathcal{G}_f \cap (C_f \times \mathbb{R}) \to C_f$ is the inverse of $(\mathrm{id}_{\mathsf{X}}, \bar f) \colon C_f \to \mathcal{F}\mathcal{G}_f \cap (C_f \times \mathbb{R})$. Given any $k, j \in \mathbb{N}$ and $\alpha \in (0,1) \cap \mathbb{Q}$, we define

$$C_f^{k,\alpha,j} := \left\{ x \in C_f \cap D_{\eta,k} \;\middle|\; |(\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta})_{n+1}(x, \bar f(x))| \geq \alpha,\ j^{-1} \leq \Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, (x, \bar f(x))) \leq j \right\}.$$

Notice that the sets $C_f^{k,\alpha,j}$ obviously depend on η, but, as we are working with a fixed $\eta \in (0, n^{-1}) \cap \mathbb{Q}$, we do not make this dependence explicit. Recalling Theorem 3.3, we see that

$$\{x \in C_f \mid (\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta})_{n+1}(x, \bar f(x)) \neq 0\} = \bigcup_{k,\alpha,j} C_f^{k,\alpha,j} \quad \text{up to } |\mathrm{D}f|\text{-null sets.}$$

Hence, proving the statement amounts to showing that each set $\mathcal{F}\mathcal{G}_f \cap (C_f^{k,\alpha,j} \times \mathbb{R})$ is $\mathcal{H}^n$-negligible. Given any $\varepsilon > 0$, by Lusin's theorem we can find $\Sigma \subseteq \mathcal{F}\mathcal{G}_f \cap (C_f^{k,\alpha,j} \times \mathbb{R})$ Borel such that $\bar f$ is continuous on $\pi^1(\Sigma)$ and $\mathcal{H}^n((\mathcal{F}\mathcal{G}_f \cap (C_f^{k,\alpha,j} \times \mathbb{R})) \setminus \Sigma) < \varepsilon$.

Our aim is to show that

$$\mathcal{H}^n(\Sigma) = 0 \tag{3-15}$$

since this would imply $\mathcal{H}^n(\mathcal{F}\mathcal{G}_f \cap (C_f^{k,\alpha,j} \times \mathbb{R})) = 0$ by the arbitrariness of $\varepsilon > 0$. Up to discarding an $\mathcal{H}^n$-null set from Σ, we can also assume (thanks to Remark 2.19 and Theorem 3.4) that $\Theta_n(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner \Sigma, p) = \Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)$ for every $p \in \Sigma$. Now we claim that

$$\lim_{r \searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma \cap B_r(p)) \setminus (\mathsf{X} \times B_{\beta r}(t)))}{r^n} = 0 \quad \text{for every } p = (x, t) \in \Sigma, \tag{3-16}$$

where we set $\beta = \beta(\alpha) := \sqrt{1 - \alpha^2} \in (0,1)$. The role played by α will be made clear in what follows. To show the claim, fix $p = (x, t) \in \Sigma$ and take any sequence $\{r_i\}_i \subseteq (0, +\infty)$ with $r_i \searrow 0$. Since $x \in \mathcal{R}_n(\mathsf{X})$, one has that

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0) \quad \text{in the pmGH topology.}$$

Let $(\mathsf{Z}, \mathsf{d}_\mathsf{X})$ be a realization of such convergence. Then $(\mathsf{Z}\times\mathbb{R}, \mathsf{d}_\mathsf{Z}\times\mathsf{d}_e)$ is a realization of

$$(\mathsf{X}\times\mathbb{R}, r_i^{-1}(\mathsf{d}\times\mathsf{d}_e), (\mathsf{m}\otimes\mathcal{L}^1)_p^{r_i}, p, \mathcal{G}_f) \to (\mathbb{R}^{n+1}, \mathsf{d}_e, \underline{\mathcal{L}}^{n+1}, 0, H),$$

where $H \subseteq \mathbb{R}^{n+1}$ is a halfspace. We also know from Proposition 3.7(v) that, up to passing to a not relabeled subsequence, the rescaled perimeters $|\mathrm{D}\chi_{\mathcal{G}_f}|$ weakly converge to $\mathcal{H}^n \llcorner \partial H$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$. Moreover, by Proposition 3.7(iv), ∂H is normal to $\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(p)$. Thus, since $(\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta})_{n+1}(p) \geq \alpha$, we have $\partial H \cap B_1(0) \subseteq B_1(0)\times B_\beta(0)$ by our choice of β. From the latter the claim (3-16) follows, taking into account also (2-10). For $\gamma \in (0,+\infty)$ and $(x,t)\in \mathsf{X}\times\mathbb{R}$, we define the cone

$$C_\gamma(x,t) := \{(y,s)\in\mathsf{X}\times\mathbb{R} \mid \gamma\mathsf{d}(y,x) \geq |s-t|\}.$$

Now take $\gamma=\gamma(\beta)=\sqrt{(1+\beta)/(1-\beta)} \in (1,+\infty)$. Notice that $\gamma^2 > \beta/(1-\beta)$. Next we claim that

$$\lim_{r\searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma\cap B_r(p))\setminus C_\gamma(p))}{r^n} = 0 \quad \text{for every } p=(x,t)\in\Sigma. \tag{3-17}$$

In order to prove it, fix $\delta > 0$. By virtue of (3-16), we can take $r_0 > 0$ small enough that

$$\sup_{r\in(0,r_0)} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma\cap B_r(p))\setminus(\mathsf{X}\times B_{\beta r}(t)))}{r^n} \leq \delta. \tag{3-18}$$

Notice that

$$B_{r_0}(p)\setminus C_\gamma(p) \subseteq \bigcup_i B_{r_i}(p)\setminus(\mathsf{X}\times B_{\beta r_i}(t)), \tag{3-19}$$

where, for any $i\in\mathbb{N}$ with $i\geq 1$, we define

$$r_i := \beta\sqrt{\frac{\gamma^2+1}{\gamma^2}}\, r_{i-1} = \biggl(\beta\sqrt{\frac{\gamma^2+1}{\gamma^2}}\biggr)^i r_0.$$

Given that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma\cap B_{r_i}(p))\setminus(\mathsf{X}\times B_{\beta r_i}(p))) \overset{(3\text{-}18)}{\leq} \delta r_i^n = \delta\biggl(\beta\sqrt{\frac{\gamma^2+1}{\gamma^2}}\biggr)^{ni} r_0^n,$$

it follows from the inclusion in (3-19) that

$$\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma\cap B_{r_0}(p))\setminus C_\gamma(p))}{r_0^n} \leq \delta\sum_i \biggl(\beta\sqrt{\frac{\gamma^2+1}{\gamma^2}}\biggr)^{ni}.$$

Thanks to the arbitrariness of $\delta>0$ and the finiteness of $\sum_i(\beta\sqrt{(\gamma^2+1)/\gamma^2})^{ni}$, (3-17) is proved.

Let now $\varepsilon' > 0$. We wish to show that there exists a set $\Sigma'\subseteq\Sigma$ with $\mathcal{H}^n(\Sigma\setminus\Sigma') < \varepsilon'$ such that there exists $r_0\in(0,1)$ satisfying

$$(\Sigma'\cap B_{r_0}(p))\setminus C_{2\gamma}(p) = \varnothing \quad \text{for every } p\in\Sigma'. \tag{3-20}$$

We do it using a standard argument, see, e.g., the proof of [Simon 1983, Theorem 1.6]. By Egorov's theorem, we can choose $\Sigma'\subseteq\Sigma$ Borel with $\mathcal{H}^n(\Sigma\setminus\Sigma')<\varepsilon'$ such that, for any given $\delta'>0$, there exists

$r_0 \in (0,1)$ such that, for every $r \in (0, 2r_0)$ and $p \in \Sigma'$,

$$\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma \cap B_r(p))}{\Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)\omega_n r^n} \geq 1 - \delta', \tag{3-21a}$$

$$\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma \cap B_r(p)) \setminus C_\gamma(p))}{\Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)\omega_n r^n} \leq \delta'; \tag{3-21b}$$

the former follows from the fact that $\Theta_n(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner \Sigma, p) = \Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)$, the latter from (3-17). We aim to show that if $\delta' > 0$ is small enough, then this choice of Σ' and r_0 satisfies (3-20). Assume now that there exists $q \in (\Sigma' \cap B_{r_0}(p)) \setminus C_{2\gamma}(p)$ for some $p \in \Sigma'$. Then

$$B_\rho(q) \subseteq B_{\tilde{\mathsf{d}}(p,q)+\rho}(p) \setminus C_\gamma(p), \quad \text{where } \rho := \tilde{\mathsf{d}}(p,q)\sin(\arctan(2\gamma) - \arctan(\gamma)), \tag{3-22}$$

where we write $\tilde{\mathsf{d}} := \mathsf{d} \times \mathsf{d}_e$ for brevity. Therefore, we can estimate

$$\begin{aligned} \delta' &\overset{\text{(3-21b)}}{\geq} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma \cap B_{\tilde{\mathsf{d}}(p,q)+\rho}(p)) \setminus C_\gamma(p))}{\Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)\omega_n (\tilde{\mathsf{d}}(p,q)+\rho)^n} \\ &\overset{\text{(3-22)}}{\geq} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma \cap B_\rho(q))}{\Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)\omega_n (\tilde{\mathsf{d}}(p,q)+\rho)^n} \\ &\overset{\text{(3-21b)}}{\geq} (1-\delta')\frac{\rho^n}{(\tilde{\mathsf{d}}(p,q)+\rho)^n} = (1-\delta')\frac{(\sin(\arctan(2\gamma) - \arctan(\gamma)))^n}{(1+\sin(\arctan(2\gamma) - \arctan(\gamma)))^n}, \end{aligned}$$

which leads to a contradiction provided $\delta' > 0$ was chosen small enough, proving (3-20).

Finally, our aim is to show that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma') = 0 \tag{3-23}$$

since this, by the arbitrariness of $\varepsilon' > 0$, would imply (3-16) and accordingly the statement. Take $p = (x,t) \in \Sigma'$. Since $\bar{f}$ is continuous on $\pi^1(\Sigma')$, there exists $r_1 \in (0, r_0/\sqrt{2})$ such that $|\bar{f}(y) - \bar{f}(x)| < r_0/\sqrt{2}$ for all $y \in B_{r_1}(x) \cap \pi^1(\Sigma')$. As $\Sigma' \subseteq \{(x,t) \in \mathsf{X} \times \mathbb{R} : t = \bar{f}(x)\}$, we see that

$$\Sigma' \cap (B_{r_1}(x) \times \mathbb{R}) \subseteq \Sigma' \cap B_{r_0}(p) \subseteq C_{2\gamma}(p)$$

by (3-20), so that, setting $\lambda := \sqrt{1+4\gamma^2}$,

$$\Sigma' \cap (B_r(x) \times \mathbb{R}) \subseteq \Sigma' \cap B_{\lambda r}(p) \quad \text{for every } r \in (0, r_1). \tag{3-24}$$

It follows that, for every $p = (x,t) \in \Sigma'$, we have

$$\begin{aligned} \overline{\Theta}_n(\pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner \Sigma'), x) &= \varlimsup_{r \searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma' \cap (B_r(x) \times \mathbb{R}))}{\omega_n r^n} \overset{\text{(3-24)}}{\leq} \varlimsup_{r \searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma \cap B_{\lambda r}(p))}{\omega_n r^n} \\ &= \lambda^n \Theta_n(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner \Sigma, p) = \lambda^n \Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p) \leq \lambda^n j, \end{aligned}$$

where the last inequality stems from the inclusion $\Sigma' \subseteq \mathcal{F}\mathcal{G}_f \cap (C_f^{k,\alpha,j} \times \mathbb{R})$. Therefore, by applying [Ambrosio and Tilli 2004, Theorem 2.4.3] and using the fact that $\pi^1(\Sigma') \subseteq C_f$, we can conclude that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma') = \pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner \Sigma')(\pi^1(\Sigma')) \leq (2\lambda)^n j \mathcal{H}^n(\pi^1(\Sigma')) \leq (2\lambda)^n j \mathcal{H}^n(C_f) = 0,$$

thus obtaining (3-23). Consequently, the statement is achieved. □

Lemma 3.9. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension n. Fix a function* $f \in \mathrm{BV}(\mathsf{X})$ *and a good collection* $\{\boldsymbol{u}_\eta\}_\eta$ *of splitting maps on* X. *Let* C_f *be as in the statement of Proposition 3.7. Then, for any* $\eta \in (0, n^{-1}) \cap \mathbb{Q}$,

$$\nu_f^{\boldsymbol{u}_\eta}(x) = (\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(x, \bar{f}(x)))_{1,\dots,n} \quad \textit{for } |\mathrm{D}f| \llcorner C_f\textit{-a.e. } x \in \mathsf{X}.$$

Proof. Recall that $|\mathrm{D}f| \llcorner C_f = \pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner (C_f \times \mathbb{R}))$, so that the statement makes sense. By the coarea formula, it is enough to show that, for a.e. t, we have $\nu_f^{\boldsymbol{u}_\eta}(x) = (\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(x, \bar{f}(x)))_{1,\dots,n}$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E_t \cap C_f$, where we define $E_t := \{f > t\}$. Taking [Brena and Gigli 2024, Lemma 3.27] into account, we see that it is sufficient to prove that, for a.e. t and for every $k \in \mathbb{N}$,

$$\nu_{\chi_{E_t}}^{\boldsymbol{u}_\eta}(x) = (\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(x, \bar{f}(x)))_{1,\dots,n} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{F}E_t \cap C_f \cap D_{\eta,k}. \tag{3-25}$$

Let $x \in \mathcal{F}E_t \cap C_f \cap D_{\eta,k}$ be a given point where the conclusions of Proposition 2.26 hold with $E = E_t$; notice that $\mathcal{H}^{n-1}$-a.e. point of $\mathcal{F}E_t \cap C_f \cap D_{\eta,k}$ has this property. We aim to show that the identity in (3-25) is verified at x. Write $p := (x, \bar{f}(x))$ for brevity. Thanks to Remark 2.22(i) and Proposition 3.7(v), we can find a sequence $r_i \searrow 0$, halfspaces $H \subseteq \mathbb{R}^{n+1}$ and $H' \subseteq \mathbb{R}^n$, and a proper metric space $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ such that

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x, E_t) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, H'), \tag{3-26a}$$

$$(\mathsf{X} \times \mathbb{R}, r_i^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}}, (\mathsf{m} \otimes \mathcal{H}^1)_p^{r_i}, p, \mathcal{G}_f) \to (\mathbb{R}^{n+1}, \mathsf{d}_e, \underline{\mathcal{L}}^{n+1}, 0, H) \tag{3-26b}$$

in the realizations Z and $\mathsf{Z} \times \mathbb{R}$, respectively. Notice also that

$$\{(y, s) \in \mathsf{X} \times \mathbb{R} \mid s < t\} \to \{(y, s) \in \mathbb{R}^n \times \mathbb{R} \mid s < 0\} \quad \text{in } L^1_{\mathrm{loc}} \tag{3-27}$$

in the realization $\mathsf{Z} \times \mathbb{R}$. Therefore, by stability, we deduce from (3-26b) and (3-27) that

$$\{(y, s) \in \mathsf{X} \times \mathbb{R} \mid s < f(y),\ s < t\} \to H \cap \{(y, s) \in \mathbb{R}^n \times \mathbb{R} \mid s < 0\} \quad \text{in } L^1_{\mathrm{loc}}.$$

Recalling (3-26a) and using Fubini's theorem and dominated convergence, we see that

$$E_t \times (-\infty, t) \to H' \times (-\infty, 0) \quad \text{in } L^1_{\mathrm{loc}}.$$

Given that $E_t \times (-\infty, t) \subseteq \{(y, s) \in \mathsf{X} \times \mathbb{R} : s < f(y),\ s < t\}$, we obtain that

$$H' \times (-\infty, 0) \subseteq H \cap \{(y, s) \in \mathbb{R}^n \times \mathbb{R} \mid s < 0\}.$$

Thanks to our choice of x and to items (iv) and (v) of Proposition 3.7, we can see that $\nu_{\chi_{E_t}}^{\boldsymbol{u}_\eta}(x)$ and $(\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(p))_{1,\dots,n}$ have the same direction, namely there exists $\lambda(x) \in [0, 1]$ such that

$$\nu_{\chi_{E_t}}^{\boldsymbol{u}_\eta}(x) = \lambda(x)(\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(p))_{1,\dots,n}.$$

Now notice that the conclusion of Theorem 3.8 forces $\lambda(x)$ to equal 1, up to discarding a $|\mathrm{D}f| \llcorner C_f$-negligible set. ☐

3C. *Rank-one theorem.* In this final subsection we prove Theorem 1.3. We first start with an auxiliary definition and a technical result taken from [Bruè et al. 2023b].

Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n and $E\subseteq\mathsf{X}$ a set of locally finite perimeter. Let $\varepsilon>0$ and $r>0$ be given. Then, following [Bruè et al. 2023b, Definition 4.6], we define $(\mathcal{F}_nE)_{r,\varepsilon}$ as the set of all points $x\in\mathcal{F}_nE$ such that

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X},s^{-1}\mathsf{d},\mathsf{m}_x^s,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0))<\varepsilon,$$
$$\left|\frac{\mathsf{m}(E\cap B_s(x))}{\mathsf{m}(B_s(x)}-\frac{1}{2}\right|+\left|\frac{s|\mathrm{D}\chi_E|(B_s(x))}{\mathsf{m}(B_s(x))}-\frac{\omega_{n-1}}{\omega_n}\right|<\varepsilon$$

for every $s\in(0,r)$. We remark that, for every $x\in\mathcal{F}_nE$ and for every $\varepsilon>0$, there exists $r>0$ such that $x\in(\mathcal{F}_nE)_{r,\varepsilon}$. We now recall the following result, which was proved in [Bruè et al. 2023b, Proposition 4.7].

Proposition 3.10. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n. Let $E\subseteq\mathsf{X}$ be a set of locally finite perimeter. Then, for any $\eta>0$, there exists $\varepsilon=\varepsilon(N,\eta)>0$ such that the following property is satisfied: if $p\in(\mathcal{F}_nE)_{2r,\varepsilon}$ for some $0<r<|K|^{-1/2}$ and there exists an ε-splitting map $u\colon B_{2r}(p)\to\mathbb{R}^{n-1}$ such that*

$$\frac{r}{\mathsf{m}(B_{2r}(p))}\int_{B_{2r}(p)}|\nu_E\cdot\nabla u^\ell|\,\mathrm{d}|\mathrm{D}\chi_E|<\varepsilon\quad\text{for every }\ell=1,\dots,n-1,$$

then there exists a Borel set $G\subseteq B_r(p)$ with $\mathcal{H}^h_5(B_r(p)\setminus G)\le C_N\eta\mathsf{m}(B_r(p))/r$ such that

$$u\colon G\cap(\mathcal{F}_nE)_{2r,\varepsilon}\to\mathbb{R}^{n-1}\quad\text{is bi-Lipschitz onto its image.}$$

We pass to the following lemma, which is the technical core of the proof of Theorem 1.3.

Lemma 3.11. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n. Fix any two functions $f,g\in\mathrm{BV}(\mathsf{X})$. Let $\{\boldsymbol{u}_\eta\}_\eta$ be a good collection of splitting maps on X. Let us consider the sets $C_f,C_g\subseteq\mathsf{X}$ given by Proposition 3.7. Let τ be the inversion map defined in (2-1), and let*

$$\Sigma_f:=\mathcal{FG}_f\cap(C_f\times\mathbb{R}),\quad\widetilde{\Sigma}_f:=\Sigma_f\times\mathbb{R},$$
$$\Sigma_g:=\mathcal{FG}_g\cap(C_g\times\mathbb{R}),\quad\widetilde{\Sigma}_g:=\tau(\Sigma_g\times\mathbb{R}).$$

Moreover, let us set $R:=\pi^1(\widetilde{R})\subseteq\mathsf{X}$, where the set $\widetilde{R}\subseteq\mathsf{X}\times\mathbb{R}^2$ is defined as

$$\bigcap_{\substack{\eta\in\mathbb{Q},\\ 0<\eta<n^{-1}}}\{(x,t,s)\in\widetilde{\Sigma}_f\cap\widetilde{\Sigma}_g\mid\nu^{\boldsymbol{u}_\eta}_{\mathcal{G}_f}(x,t)\neq\pm\nu^{\boldsymbol{u}_\eta}_{\mathcal{G}_g}(x,s),\ (\nu^{\boldsymbol{u}_\eta}_{\mathcal{G}_f}(x,t))_{n+1}=(\nu^{\boldsymbol{u}_\eta}_{\mathcal{G}_g}(x,s))_{n+1}=0\}.\tag{3-28}$$

Then

$$(|\mathrm{D}f|\wedge|\mathrm{D}g|)(R)=0.$$

Proof. Let us fix a ball $\bar{B}$ in X, set

$$\Omega_f:=(C_f\times\mathbb{R})\cap(\bar{B}\times\mathbb{R})\cap\mathcal{FG}_f,$$

and define similarly Ω_g.

For $i \in \mathbb{N}$, set $\eta_i := 2^{-i}\eta_0$. Here $\eta_0 \in (0, n^{-1}) \cap \mathbb{Q}$ satisfies $\eta_0 C_N < 1$, where C_N is given in Proposition 3.10. We claim that, for every i, there exists a decomposition of the kind

$$\Omega_f = G_i(f) \cup M_i(f) \cup R_i(f),$$

and similarly for g, for which the following hold:

- We have the inequality
$$\mathcal{H}^h_5(M_i(f)) + |\mathrm{D}\chi_{\mathcal{G}_f}|(R_i(f)) \le C_{K,N}\eta_i(|\mathrm{D}\chi_{\mathcal{G}_f}|(\bar{B} \times \mathbb{R}) + 1), \tag{3-29}$$
and similarly for g, where $C_{K,N}$ is, in particular, independent of i.
- Set $\widehat{G}_i(f) := \pi^1(G_i(f))$ and $\widehat{G}_i(g) := \pi^1(G_i(g))$. Define similarly $\widehat{M}_i(f)$, $\widehat{M}_i(g)$, $\widehat{R}_i(f)$, and $\widehat{R}_i(g)$. Then
$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)(R \cap \widehat{G}_i(f) \cap \widehat{G}_i(g)) = 0. \tag{3-30}$$

We show now how this decomposition allows us to conclude the proof of the lemma. We set

$$\widehat{G} := \bigcup_{i\in\mathbb{N}} \widehat{G}_i(f) \cap \widehat{G}_i(g).$$

As (3-30) implies that

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)(R \cap \widehat{G}) = 0,$$

it suffices to show (recall that $R \subseteq C_f \cap C_g$)

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)((C_f \cap C_g \cap \bar{B}) \setminus \widehat{G}) = 0,$$

as the ball $\bar{B}$ was arbitrary.

Let us go through the proof of the last equality. Notice that, for every i,

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)((C_f \cap C_g \cap \bar{B}) \setminus \widehat{G}) \le |\mathrm{D}f|(\widehat{M}_i(f) \cup \widehat{R}_i(f)) + |\mathrm{D}g|(\widehat{M}_i(g) \cup \widehat{R}_i(g)).$$

Therefore, it is enough to show that (as a similar statement will hold for g),

$$\lim_{i\to\infty} |\mathrm{D}f|(\widehat{M}_i(f) \cup \widehat{R}_i(f)) = 0,$$

so that, recalling Proposition 2.13 and that $\pi^1|_{\mathcal{F}\mathcal{G}_f}$ is injective on $C_f \times \mathbb{R}$, we can just show

$$\lim_{i\to\infty} |\mathrm{D}\chi_{\mathcal{G}_f}|\biggl(\bigcup_{j\ge i} M_j(f)\biggr) + |\mathrm{D}\chi_{\mathcal{G}_f}|(R_i(f)) = 0,$$

which follows from (3-29), since (3-29) again and the definition of η_i imply that

$$\mathcal{H}^h_5\biggl(\bigcap_{i\in\mathbb{N}}\bigcup_{j\ge i} M^j(f)\biggr) = 0.$$

For the sake of clarity, we subdivide the rest of the proof into five steps. In Step 1 we construct a candidate decomposition as above in such a way that (3-29) is satisfied. The remaining steps are to prove (3-30) for the decomposition obtained in Step 1. Step 2 and Step 4 are used to obtain technical estimates, whereas Step 3 is the most important and proves a σ-finiteness property via transverse intersection. With these results in mind, we conclude the proof in Step 5. In the rest of the proof, we are going to use heavily all the conditions ensured by the membership to C_f and C_g without pointing it out every time. In other words, we are morally partitioning X into good sets, up to an almost negligible set. These sets are good in the sense that $\widetilde{\Sigma}_f$ and $\widetilde{\Sigma}_g$, restricted to the preimage of these sets with respect to the projection onto X, are bi-Lipschitz equivalent to $(n+1)$-rectifiable subsets of $\mathbb{R}^{n+2}$, via the same chart maps. Then, as explained in the introduction, the task is to prove transversality of these two subsets of $\mathbb{R}^{n+2}$, and this is done via a blow-up argument, taking advantage of the fact that we are using the same chart maps.

Step 1: Construction of the decomposition. Let $\varepsilon_i \in (0, n^{-1}) \cap (0, \omega_n/(2\omega_{n+1})) \cap \mathbb{Q}$ be given by Proposition 3.10 applied to $E = \mathcal{G}_f$, with η_i in place of η. Using the good collection of splitting maps, consider

$$\boldsymbol{u}_i = \{u_{i,k}\}_k := \boldsymbol{u}_{\varepsilon_i/(n+1)}, \quad \{D_{i,k}\}_k := \{D_{\varepsilon_i/(n+1),k}\}_k, \quad k_i := k_{\varepsilon_i/(n+1)}, \quad A_i := A_{\varepsilon_i/(n+1)},$$

where we recall that k and A have been defined in Definition 2.29.

We only consider the case of the function f, the construction for g being the same, and we concentrate on a fixed i. Therefore, we do not indicate the dependence on f for what remains of Step 1.

We refer to the discussion at the beginning of Section 3C for the definition (and the basic properties) of the auxiliary set $(\mathcal{F}_{n+1}\mathcal{G}_f)_{r,\varepsilon}$. Let

$$r_i \in (0, |K|^{-1})$$

be small enough that, setting

$$R_i^1 := \Omega_f \setminus (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i},$$

we have

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(R_i^1) < \eta_i.$$

Let also $c = c_i \in (0, 1)$ be small enough that, setting

$$R_i^2 := \Omega_f \setminus \{p \in \mathcal{F}\mathcal{G}_f \mid c < \Theta_n(|\mathrm{D}\chi_{\mathcal{G}_f}|, p) < c^{-1}\},$$

we have

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(R_i^2) < \eta_i.$$

Take now $p = (x, \bar{f}(x)) \in \Omega_f \setminus R_i^1$, so that $x \in D_{i,k}$ for $k = k_i(x)$, see item (v) of Proposition 3.7, and we have an associated invertible matrix $A = A_i(x)$, compare with item (iii) of Proposition 3.7, and the discussion in Definition 2.29. Set $v := (u_{i,k}, \pi^2)$ and $z := (Au_{i,k}, \pi^2)$. Notice, by the fact that $x \in D_{i,k}$, we have that $u_{i,k}$ is ε_i-splitting on a small ball around x. Hence, by tensorization, v is ε_i-splitting on a small ball around p. Recall, moreover, that, by item (iii) of Proposition 3.7, we have that z is a set of good coordinates at $(x, \bar{f}(x))$, see Definition 2.23. Hence, we have that, for some $\nu \in \mathbb{S}^n$,

$$\lim_{r \searrow 0} \fint_{B_r(p)} |\nu^j - \nu_{\mathcal{G}_f} \cdot \nabla z^j| \, \mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| = 0 \quad \text{for every } j = 1, \dots, n+1,$$

so that, for some $\mu \in \mathbb{R}^{n+1} \setminus \{0\}$,

$$\lim_{r \searrow 0} \fint_{B_r(p)} |\mu^j - \nu_{\mathcal{G}_f} \cdot \nabla v^j| \, \mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| = 0 \quad \text{for every } j = 1, \dots, n+1.$$

It follows that, for some $B \in \mathrm{SO}(n+1)$, setting $w = Bv$, we have

$$\lim_{r \searrow 0} \fint_{B_r(p)} |\nu_{\mathcal{G}_f} \cdot \nabla w^j| \, \mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| = 0 \quad \text{for every } j = 1, \dots, n.$$

Indeed, it suffices to take $B \in \mathrm{SO}(n+1)$ such that $B\mu = (0^n, \|\mu\|_{\mathbb{R}^{n+1}})$. The equation above and the membership $p \in \mathcal{F}\mathcal{G}_f$ imply that

$$\lim_{r \searrow 0} \frac{r}{\mathsf{m} \otimes \mathcal{H}^1(B_{2r}(p))} \int_{B_{2r}(p)} |\nu_{\mathcal{G}_f} \cdot \nabla w^j| \, \mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| = 0 \quad \text{for every } j = 1, \dots, n.$$

Take then $\tilde{r} = \tilde{r}_{i,p} \in (0, r_i)$ small enough that w is an ε_i-splitting map on $B_{2\tilde{r}}(p)$ (this is possible thanks to our choice of $\boldsymbol{u}_i$, the fact that v is ε_i-splitting on a small ball around p, and that $B \in \mathrm{SO}(n+1)$[1]), moreover

$$\frac{\tilde{r}}{\mathsf{m} \otimes \mathcal{H}^1(B_{2\tilde{r}}(p))} \int_{B_{2\tilde{r}}(p)} |\nu_{\mathcal{G}_f} \cdot \nabla w^j| \, \mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| < \varepsilon_i \quad \text{for every } j = 1, \dots, n,$$

and finally, using also that $|\mathrm{D}\chi_{\mathcal{G}_f}|$ is asymptotically doubling at p,

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\tilde{r}}(p) \setminus (D_{i,k} \times \mathbb{R})) < \eta_i |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\tilde{r}/5}(p)),$$

where we recall that for deducing the last information we are using item (v) of Proposition 3.7. We can also assume that $B_{\tilde{r}}(x) \subseteq \bar{B}$, which will be useful below. Note that $p \in (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i} \subseteq (\mathcal{F}_{n+1}\mathcal{G}_f)_{2\tilde{r},\varepsilon_i}$. We can thus apply Proposition 3.10 and obtain a set $G = G_{i,p} \subseteq B_{\tilde{r}}(p)$ such that

$$\mathcal{H}^h_5(B_{\tilde{r}}(p) \setminus G) \le C_N \eta_i \frac{\mathsf{m} \otimes \mathcal{H}^1(B_{\tilde{r}}(p))}{\tilde{r}}$$

and $(w^1, \dots, w^n) : G \cap (\mathcal{F}_{n+1}\mathcal{G}_f)_{2\tilde{r},\varepsilon_i} \to \mathbb{R}^n$ is bi-Lipschitz onto its image. Here C_N depends only on N. Clearly, also $v : G \cap (\mathcal{F}_{n+1}\mathcal{G}_f)_{2\tilde{r},\varepsilon_i} \to \mathbb{R}^{n+1}$ is bi-Lipschitz onto its image, so that the image of v is n-rectifiable, due to the fact that $\mathcal{F}_{n+1}\mathcal{G}_f$ is n-rectifiable.

To sum up, for i fixed, for every $p = (x, t) \in \Omega_f \setminus R^1_i$, we have shown that

$$v_{i,p} := (u_{i,k_i(x)}, \pi^2) : G_{i,p} \cap (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i} \to \mathbb{R}^{n+1}$$

is bi-Lipschitz onto its image for some set $G_{i,p} \subseteq B_{\tilde{r}_{i,p}}(p)$, that

$$\mathcal{H}^h_5(B_{\tilde{r}_{i,p}}(p) \setminus G_{i,p}) \le C_N \eta_i \frac{\mathsf{m} \otimes \mathcal{H}^1(B_{\tilde{r}_{i,p}}(p))}{\tilde{r}_{i,p}}, \tag{3-31}$$

and finally that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\tilde{r}_{i,p}}(p) \setminus (D_{i,k_i(x)} \times \mathbb{R})) < \eta_i |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\tilde{r}_{i,p}/5}(p)). \tag{3-32}$$

We apply Vitali's covering lemma to find a sequence of balls $\{B^j_i\}_j$ where, for every j, we have that $B^j_i = B_{r^j_i}(p^j_i) = B_{\tilde{r}_{i,p}}(p)$ for some $p = p^j_i \in \Omega_f \setminus R^1_i$ such that

$$\bigcup_{j \in \mathbb{N}} B^j_i \supseteq \Omega_f \setminus R^i_1$$

and $\{5^{-1} B^j_i\}_j$ are pairwise disjoint; here $5^{-1} B^j_i$ stands for the ball $B_{r^j_i/5}(p^j_i)$. Clearly, to each B^j_i are associated in a natural way the sets G^j_i and D^j_i and maps $v^j_i : G^j_i \cap (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i} \to \mathbb{R}^{n+1}$. We set then

$$M_i := \Omega_f \cap \bigcup_{j \in \mathbb{N}} (B^j_i \setminus G^j_i) \quad \text{and} \quad R^3_i := \Omega_f \cap \bigcup_{j \in \mathbb{N}} (B^j_i \setminus (D^j_i \times \mathbb{R})).$$

[1] Notice that the operator norm of B is bounded above by a function of n, hence the Lipschitz constant of w might increase by at most such a factor, but this is clearly not a problem.

Using (3-31) for the first chain of inequalities and (3-32) for the second chain of inequalities, we have

$$\mathcal{H}^h_5(M_i) \le \sum_{j\in\mathbb{N}} \mathcal{H}^h_5(B_i^j \setminus G_i^j) \le C_N \eta_i \sum_{j\in\mathbb{N}} \frac{\mathsf{m}\otimes\mathcal{H}^1(B_i^j)}{r_i^j} \le C_{K,N}\eta_i \sum_{j\in\mathbb{N}} \frac{\mathsf{m}\otimes\mathcal{H}^1(5^{-1}B_i^j)}{\frac{1}{5}r_i^j}$$
$$\le C_{K,N}\eta_i \sum_{j\in\mathbb{N}} |\mathrm{D}\chi_{\mathcal{G}_f}|(5^{-1}B_i^j) \le C_{K,N}\eta_i |\mathrm{D}\chi_{\mathcal{G}_f}|(\bar{B}\times\mathbb{R}).$$

We stress that in the fourth inequality above we are using that $p_i^j \in (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i}$ and

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(R_i^3) \le \sum_{j\in\mathbb{N}} |\mathrm{D}\chi_{\mathcal{G}_f}|(B_i^j \setminus (D_i^j\times\mathbb{R})) \le \eta_i \sum_{j\in\mathbb{N}} |\mathrm{D}\chi_{\mathcal{G}_f}|(5^{-1}B_i^j) \le \eta_i |\mathrm{D}\chi_{\mathcal{G}_f}(\bar{B}\times\mathbb{R})|.$$

Now set

$$S_i^j := v_i^j((\Omega_f \cap G_i^j \cap (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i}) \setminus (R_i^1 \cup R_i^2 \cup R_i^3)) \subseteq \mathbb{R}^{n+1},$$

and recall that S_i^j is n-rectifiable. For every $j\in\mathbb{N}$, there exists a countable family $\{S_i^{j,\ell}\}_{\ell\in\mathbb{N}}$ of C^1-hypersurfaces in $\mathbb{R}^{n+1}$ such that

$$\mathcal{H}^n\Big(S_i^j \setminus \bigcup_{\ell\in\mathbb{N}} S_i^{j,\ell}\Big) = 0.$$

Define

$$\widehat{S}_i^{j,\ell} := \left\{ y \in S_i^{j,\ell}\cap S_i^j \;\middle|\; \lim_{r\searrow 0} \frac{\mathcal{H}^n(B_r(y)\cap S_i^{j,\ell}\cap S_i^j)}{\omega_n r^n} = 1 \right\}$$

and

$$R_i^4 := \bigcup_{j\in\mathbb{N}} \bigcap_{\ell\in\mathbb{N}} (S_i^j \setminus (v_i^j)^{-1}(\widehat{S}_i^{j,\ell})) \subseteq \Omega_f,$$

and notice that $\mathcal{H}^n(R_i^4) = 0$, so that $|\mathrm{D}\chi_{\mathcal{G}_f}|(R_i^4) = 0$. We set also

$$Q_i^{j,\ell} := (v_i^j)^{-1}(\widehat{S}_i^{j,\ell}) \subseteq \Omega_f,$$

and notice that

$$\text{if } v_i^j = (u_{i,k}, \pi^2),\ \text{ then } Q_i^{j,\ell} \subseteq D_{i,k}\times\mathbb{R} \text{ for every } \ell\in\mathbb{N}. \tag{3-33}$$

Now define

$$R_i^5 := \bigcup_{j,\ell\in\mathbb{N}} \left(Q_i^{j,\ell} \setminus \left\{ p \in Q_i^{j,\ell} \;\middle|\; \lim_{r\searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(p)\cap Q_i^{j,\ell})}{|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(p))} = 1 \right\} \right).$$

We then set

$$R_i := R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4 \cup R_i^5,$$

and finally

$$G_i := \Omega_f \setminus (M_i \cup R_i) \subseteq \bigcup_{j,\ell\in\mathbb{N}} Q_i^{j,\ell}.$$

It is immediate to check that the sets we constructed satisfy (3-29). The rest of the proof shows that they also satisfy (3-30).

Step 2: Almost one-sided Kuratowski convergence. For any i, let

$$p \in \Omega_f \setminus R_i^1(f),$$

and let $\rho_k \searrow 0$ be such that

$$(\mathsf{X}\times\mathbb{R}, \rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}}, (\mathsf{m}\otimes\mathcal{H}^1)_p^{\rho_k}, p, \mathcal{G}_f) \to (\mathbb{R}^{n+1}, \mathsf{d}_e, \underline{\mathcal{L}}^{n+1}, 0, H),$$

where $H \subseteq \mathbb{R}^{n+1}$ is a halfspace. Fix also $\rho > 0$. Assume the convergence is realized in a proper metric space $(\mathsf{Z}, \mathsf{d}_{\mathsf{Z}})$. We show that, for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$B^{\mathsf{Z}}_\rho(p^k) \cap (\Omega_f \setminus R^1_i(f))^k \subseteq B^{\mathsf{Z}}_\varepsilon(\partial H) \quad \text{if } k \geq k_0.$$

Here the superscript k denotes the isometric image in Z through the embedding of the ρ_k-rescaled space.

We argue by contradiction. Up to taking a not relabeled subsequence, by the contradiction assumption, there exist $\{q^k\}_k$ such that, for every k,

$$q^k \in (B^{\mathsf{Z}}_\rho(p^k) \cap (\Omega_f \setminus R^1_i(f))^k) \setminus B^{\mathsf{Z}}_\varepsilon(\partial H).$$

Up to a not relabeled subsequence, $q^k \to q \in \mathsf{Z}$, with $\mathsf{d}_{\mathsf{Z}}(q, \partial H) \geq \frac{1}{2}\varepsilon$. It is easy to see that $q \in \mathbb{R}^{n+1}$. By weak convergence of measures,

$$\lim_{k\to\infty} \frac{\rho_k |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\varepsilon\rho_k/2}(q^k))}{C_p^{\rho_k}} = 0.$$

On the other hand, recalling that $\{q^k\}_k \subseteq (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i}$ and using again the weak convergence of measures,

$$\varliminf_{k\to\infty} \frac{\rho_k |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\varepsilon\rho_k/2}(q^k))}{C_p^{\rho_k}} = \varliminf_{k\to\infty} \frac{\rho_k |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\varepsilon\rho_k/2}(q^k))}{\mathsf{m}(B_{\varepsilon\rho_k/2}(q^k))} \frac{\mathsf{m}(B_{\varepsilon\rho_k/2}(q^k))}{C_p^{\rho_k}} \geq \frac{\omega_n}{2\omega_{n+1}} \mathcal{L}^{n+1}(B_{\varepsilon/2}(q)) > 0,$$

which is a contradiction.

Step 3: Proof of the σ-finiteness claim. We use the same notation as in Step 1. We claim that, for every i,

$$\mathcal{H}^n \llcorner \{(x,t,s) \in \widetilde{R} \mid x \in \widehat{G}_i(f) \cap \widehat{G}_i(g)\}$$

is σ-finite. To show this, it is enough to prove that, for every $i, j, k, \ell, m, \xi \in \mathbb{N}$,

$$\mathcal{H}^n \llcorner \widetilde{T}_{i,j,k,\ell,m,\xi}$$

is σ-finite, where we set

$$\widetilde{T}_{i,j,k,\ell,m,\xi} := \{(x,t,s) \in \widetilde{R} \mid x \in \widehat{G}_i(f) \cap \widehat{G}_i(g) \cap D_{i,k},\ (x,t) \in Q_i^{j,m}(f),\ (x,s) \in Q_i^{\ell,\xi}(g)\}.$$

Fix then $i, j, k, \ell, m, \xi \in \mathbb{N}$, and set for simplicity $\widetilde{T} = \widetilde{T}_{i,j,k,\ell,m,\xi}$. Now define

$$v := (u_{i,k}, \pi^2, \pi^3) : (Q_i^{j,m}(f) \times \mathbb{R}) \cup \tau(Q_i^{\ell,\xi}(g) \times \mathbb{R}) \to \mathbb{R}^{n+2}.$$

By the construction in Step 1,

$$v|_{Q_i^{j,m}(f)\times\mathbb{R}} \quad \text{and} \quad v|_{\tau(Q_i^{\ell,\xi}(g)\times\mathbb{R})} \tag{3-34}$$

are bi-Lipschitz onto their image. Therefore, as $\widetilde{T} \subseteq (Q_i^{j,m}(f) \times \mathbb{R}) \cap \tau(Q_i^{\ell,\xi}(g) \times \mathbb{R})$, it is enough to show that

$$\mathcal{H}^n \llcorner v(\widetilde{T})$$

is σ-finite. Here a central point is that $\widetilde{T} \subseteq D_{i,k} \times \mathbb{R} \times \mathbb{R}$, so that, by the construction in Step 1, the map v as above will be suitable both for the part concerning f and the part concerning g (see (3-33)). Now notice that

$$v(\widetilde{T}) \subseteq (\widehat{S}_i^{j,m}(f) \times \mathbb{R}) \cap \tau(\widehat{S}_i^{\ell,\xi}(g) \times \mathbb{R}),$$

so that, by a standard result of geometric measure theory on Euclidean spaces, we can simply show that, at every $p = (x,t,s) \in \widetilde{T}$, we have that $\widehat{S}_i^{j,m}(f) \times \mathbb{R}$ and $\tau(\widehat{S}_i^{\ell,\xi}(g) \times \mathbb{R})$ intersect transversally at $v(p)$, or, equivalently, that $\widehat{S}_i^{j,m}(f)\times\mathbb{R}$ and $\tau(\widehat{S}_i^{\ell,\xi}(g)\times\mathbb{R})$ have different tangent spaces at $v(p)$. We can, and will, assume that $v(p)=0$.

By our assumptions, compare with items (iii) and (iv) of Proposition 3.7, we know that there exists a sequence $\rho_k\searrow 0$ and a proper metric space $(\mathsf{Z},\mathsf{d}_\mathsf{Z})$ such that $(\mathsf{Z}\times\mathbb{R}\times\mathbb{R},\mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}})$ realizes both the convergence

$$(\mathsf{X}\times\mathbb{R}\times\mathbb{R},\rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}\times\mathbb{R}},(\mathsf{m}\otimes\mathcal{H}^1\otimes\mathcal{H}^1)_p^{\rho_k},p,\mathcal{G}_f\times\mathbb{R})\to(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},\mathsf{d}_e,\underline{\mathcal{L}}^{n+2},0,H\times\mathbb{R}\times\mathbb{R}) \tag{3-35}$$

and the convergence

$$(\mathsf{X}\times\mathbb{R}\times\mathbb{R},\rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}\times\mathbb{R}},(\mathsf{m}\otimes\mathcal{H}^1\otimes\mathcal{H}^1)_p^{\rho_k},p,\tau(\mathcal{G}_g\times\mathbb{R}))\to(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},\mathsf{d}_e,\underline{\mathcal{L}}^{n+2},0,H'\times\mathbb{R}\times\mathbb{R}), \tag{3-36}$$

where H and H' are halfspaces in $\mathbb{R}^n$. Notice that this can be done since the $(n+1)$-coordinate of the v's are zero, see the definition of $\widetilde{R}$. We have endowed $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ with the coordinates given by the (locally uniform) limits of appropriate rescalings of the components of z, where

$$z:=(A_i(x)u_{i,k},\pi^2,\pi^3)\colon B_\rho(p)\to\mathbb{R}^{n+2}$$

for some $\rho>0$ (see Remark 2.25). To do so, we needed to take a not relabeled subsequence of $\{\rho_k\}_k$, but this will make no difference. Hence, recalling also the definition of $\widetilde{R}$, it follows that $H\neq H'$.

Fix $D\geq 5$ greater than the bi-Lipschitz constants of the maps in (3-34) and such that

$$|(A_i(x),\pi^1,\pi^2)c|\leq(D-4)|c|\quad\text{for every } c\in\mathbb{R}^{n+2}. \tag{3-37}$$

Let $\delta\in(0,D^{-1})$ be small enough that we can find $a\in(\partial H\times\mathbb{R}\times\mathbb{R})\cap B_1(0)\subseteq\mathbb{R}^{n+2}$ such that $B_{D\delta}(a)\cap(\partial H'\times\mathbb{R}\times\mathbb{R})=\varnothing$.

As a consequence of the density assumption made by removing R_i^5, we can find a sequence $\{a^k\}_k\subseteq\mathsf{X}\times\mathbb{R}\times\mathbb{R}$ with

$$a^k\in(Q_i^{j,m}(f)\times\mathbb{R})\cap B_{\rho_k}(p)\quad\text{for every } k\in\mathbb{N}$$

and $a^k\to a$ in $\mathsf{Z}\times\mathbb{R}\times\mathbb{R}$, where here and below the superscript k denotes the isometric image in $\mathsf{Z}\times\mathbb{R}\times\mathbb{R}$ through the embedding of the ρ_k-rescaled space.

By weak convergence of measures,

$$\varliminf_{k\to\infty}\frac{\rho_k|\mathrm{D}\chi_{\mathcal{G}_f\times\mathbb{R}}|(B_{D^{-1}\delta\rho_k}(a^k))}{C_p^{\rho_k}}>0,$$
$$\varlimsup_{k\to\infty}\frac{\rho_k|\mathrm{D}\chi_{\tau(\mathcal{G}_g\times\mathbb{R})}|(B_{D\delta\rho_k}(a^k))}{C_p^{\rho_k}}=0.$$

Recalling again the density assumption made by removing R_i^5 together with the bounds on $\Theta_n(|\mathrm{D}\chi_{\mathcal{G}_f}|,\cdot)$ by removing R_i^2, and finally the weak convergence of measures, this reads as

$$\varliminf_{k\to\infty}\rho_k^{-n-1}\mathcal{H}^{n+1}(B_{D^{-1}\delta\rho_k}(a^k)\cap(Q_i^{j,m}\times\mathbb{R}))>0, \tag{3-38}$$
$$\varlimsup_{k\to\infty}\rho_k^{-n-1}\mathcal{H}^{n+1}(B_{D\delta\rho_k}(a^k)\cap\tau(Q_i^{\ell,\xi}\times\mathbb{R}))=0. \tag{3-39}$$

It is easy to verify by contradiction that (3-38) implies, by our choice of D, that

$$\varliminf_{k\to\infty} \rho_k^{-n-1}\mathcal{H}^{n+1}(B_{\delta\rho_k}(v(a^k))\cap(\widehat{S}_i^{j,m}(f)\times\mathbb{R})\cap B_{2D\rho_k}(0))>0. \tag{3-40}$$

Now we show

$$\varliminf_{k\to\infty} \rho_k^{-n-1}\mathcal{H}^{n+1}(B_{\delta\rho_k}(v(a^k))\cap\tau(\widehat{S}_i^{\ell,\xi}(g)\times\mathbb{R})\cap B_{2D\rho_k}(0))=0. \tag{3-41}$$

By Step 2, we get that, for $\varepsilon\in(0,\delta)$, there exists k_0 such that if $k\geq k_0$, then, for every $b\in(B_{2D^2\rho_k}(p)\setminus B_{D\delta\rho_k}(a^k)\cap\tau(Q_i^{\ell,\xi}\times\mathbb{R}))^k$ there exists $b'\in\partial H'\times\mathbb{R}\times\mathbb{R}$ such that

$$\mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(b,b')<\varepsilon.$$

Up to increasing k_0, we may assume that, for every $k\geq k_0$,

$$\mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(a,a^k)<\varepsilon.$$

Notice that if b is as above, then

$$|b'-a|\geq D\delta-2\varepsilon$$

and, by local uniform convergence, up to enlarging k_0 and provided $\varepsilon>0$ is small enough,

$$|\rho_k^{-1}z(b)-\rho_k^{-1}z(a^k)|\geq|b'-a|-2\delta,$$

so that

$$|z(b)-z(a^k)|\geq((D-2)\delta-2\varepsilon)\rho_k\geq(D-4)\delta\rho_k,$$

which implies, recalling (3-37),

$$|v(b)-v(a^k)|\geq\delta\rho_k.$$

Notice that the above inequality does *not* follow from the fact that the maps in (3-34) are D-bi-Lipschitz, but implies that (3-41) follows from (3-39) by the choice of D.

We can now conclude the proof of Step 3, as by (3-40) and (3-41) it follows easily that $\widehat{S}_i^{j,m}(f)\times\mathbb{R}$ and $\tau(\widehat{S}_i^{\ell,\xi}(g)\times\mathbb{R})$ have different tangent spaces at 0.

Step 4: A technical estimate. For some $i\in\mathbb{N}$, let us assume $\widetilde{R}'$ is such that

$$\widetilde{R}'\subseteq\widetilde{R}\cap(\widehat{G}_i(f)\times\mathbb{R}\times\mathbb{R})\cap(\widehat{G}_i(g)\times\mathbb{R}\times\mathbb{R})$$

and that $\widetilde{R}'$ has finite $\mathcal{H}^n$-measure. Let $p\in\widetilde{R}'$ be fixed. We claim that

$$\lim_{r\searrow 0}\frac{\mathcal{H}_5^n(\pi^{1,2}(\widetilde{R}'\cap B_r(p)))}{r^n}=0.$$

Let us prove the claim. Take a sequence $\rho_k\searrow 0$. We recall that, with the same notation as above, up to a not relabeled subsequence, (3-35) and (3-36) hold. Let

$$I:=I((\partial H\cap\partial H')\times\mathbb{R}\times\mathbb{R})$$

be a neighborhood (in $\mathsf{Z}\times\mathbb{R}\times\mathbb{R}$) of $((\partial H\cap\partial H')\times\mathbb{R}\times\mathbb{R})\cap B_2(0)$ that satisfies

$$\mathcal{H}_5^n(\pi^{1,2}(I))<\varepsilon.$$

As a consequence of Step 2, there exists $k_0 \in \mathbb{N}$ such that

$$B_1^{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(p^k) \cap \widetilde{R}' \subseteq I \quad \text{for every } k \ge k_0,$$

from which, taking the projection $\pi^{1,2}$, the claim follows.

Step 5: Conclusion. Let us finally prove (3-30). By Step 3, it is enough to show that

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)(\pi^1(\widetilde{R}')) = 0,$$

where $\widetilde{R}'$ is as in Step 4. Fix $\varepsilon > 0$. For every $j \in \mathbb{N}$, $j \ge 1$ we consider the sets

$$\widetilde{R}'_j := \left\{ p \in \widetilde{R}' \;\middle|\; \frac{\mathcal{H}^n_5(\pi^{1,2}(\widetilde{R}' \cap B_r(p)))}{r^n} < \varepsilon \text{ for every } r \in (0, j^{-1}) \right\}$$

and

$$\widetilde{R}''_j := \widetilde{R}'_j \setminus \bigcup_{i<j} \widetilde{R}'_i.$$

Notice that, by Step 4,

$$\widetilde{R}' = \bigcup_{j\ge 1} \widetilde{R}''_j$$

and, by construction, this union is disjoint. For every $j \ge 1$, we take a countable family of balls $\{B_{r_i^j}(p_i^j)\}_i$ such that, for every $i \in \mathbb{N}$, we have $r_i^j < j^{-1}$ and $p_i^j \in \widetilde{R}''_j$, as well as

$$\widetilde{R}''_j \subseteq \bigcup_{i\in\mathbb{N}} B_{r_i^j}(p_i^j) \quad \text{and} \quad \sum_{i\in\mathbb{N}} (r_i^j)^n \le 2^n \mathcal{H}^n(\widetilde{R}''_j) + 2^{-j}. \tag{3-42}$$

We can compute, recalling the definition of $\widetilde{R}''_j$ and (3-42),

$$\mathcal{H}^n_5(\pi^{1,2}(\widetilde{R}''_j)) \le \mathcal{H}^n_5\left(\pi^{1,2}\left(\widetilde{R}'' \cap \bigcup_{i\in\mathbb{N}} B_{r_i^j}(p_i^j)\right)\right) \le \sum_{i\in\mathbb{N}} \varepsilon (r_i^j)^n \le \varepsilon(2^n\mathcal{H}^n(\widetilde{R}''_j) + 2^{-j}).$$

Therefore,

$$\mathcal{H}^n_5(\pi^{1,2}(\widetilde{R}')) \le \varepsilon(2^n \mathcal{H}^n(\widetilde{R}') + 1)$$

and, $\varepsilon > 0$ being arbitrary, $|\mathrm{D}\chi_{\mathcal{G}_f}|(\pi^{1,2}(\widetilde{R}')) = 0$, whence the result follows due to Proposition 2.13. □

Lemma 3.12. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n, *and let* $f, g \in \mathrm{BV}(\mathsf{X})$. *Choose two* Cap-*vector field representatives for* ν_f *and* ν_g. *Then*

$$\nu_f = \pm\nu_g \quad (|\mathrm{D}f| \wedge |\mathrm{D}g|)\text{-a.e. on } C_f \cap C_g.$$

Proof. From Lemmas 3.9 and 3.11 together with Theorem 3.8 we have that, for $(|\mathrm{D}f| \wedge |\mathrm{D}g|)$-a.e. $x \in C_f \cap C_g$, there exists $\eta = \eta(x) \in (0, n^{-1}) \cap \mathbb{Q}$ such that

$$\nu_f^{\boldsymbol{u}_\eta}(x) = \pm\nu_g^{\boldsymbol{u}_\eta}(x).$$

It remains to show that if, for some $\eta \in (0, n^{-1}) \cap \mathbb{Q}$, it holds that $\nu_f^{\boldsymbol{u}_\eta} = \pm\nu_g^{\boldsymbol{u}_\eta}$ Cap-a.e. on a Borel set A, then $\nu_f = \pm\nu_g$ Cap-a.e. on A. This follows since the gradients of the functions in $\boldsymbol{u}_{\eta,k}$ are a generating subspace of $L^0_{\mathrm{Cap}}(T\mathsf{X})$ on $D_{\eta,k}$ since the $L^0_{\mathrm{Cap}}(T\mathsf{X})$ module has local dimension at most n. Indeed, if $h_1, \dots, h_{n+1} \in \mathrm{TestF}(\mathsf{X})$ then $\det(\nabla h_i \cdot \nabla h_j)_{i,j} = 0$ m-a.e. hence Cap-a.e., so that it is now easy to bound the local dimension of $L^0_{\mathrm{Cap}}(T\mathsf{X})$. □

The following lemma is extracted from [Brena and Gigli 2024, Proposition 3.30].

Lemma 3.13. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n, *and let* $f, g \in \mathrm{BV}(\mathsf{X})$. *Choose two* Cap-*vector field representatives for* ν_f *and* ν_g. *Then*

$$\nu_f = \pm \nu_g \quad (|\mathrm{D}f| \wedge |\mathrm{D}g|)\text{-a.e. on } J_f \cap J_g.$$

Proof of Theorem 1.3. We first notice that, for every $i = 1, \dots, k$,

$$(\nu_F)_i = \frac{\mathrm{d}|\mathrm{D}F_i|}{\mathrm{d}|\mathrm{D}F|} \nu_{F_i} \quad |\mathrm{D}F|\text{-a.e.}$$

The conclusion on the jump part is given by Lemma 3.13 applied to every pair of components of F together with the well-known fact that, for every $i = 1, \dots, k$, we have $|\mathrm{D}F_i|(J_F \setminus J_{F_i}) = 0$. On the Cantor part, the result follows from Lemma 3.12 applied to every pair of components of F. □

Appendix: Rectifiability of the reduced boundary

In this appendix, we give an alternative proof of the known fact that reduced boundaries of sets of finite perimeter in finite-dimensional RCD spaces are rectifiable. Roughly speaking, this is a consequence of the rectifiability result of [Bate 2022] and the uniqueness of tangents to sets of finite perimeter proved in [Bruè et al. 2023b], once one takes into account the regularity result Theorem 3.3.

Let us recall part of the statement of [Bate 2022, Theorem 1.2].

Theorem A.1. *Let* (X, d) *be a complete metric space*, $k \in \mathbb{N}$, *and* $S \subseteq \mathsf{X}$ *such that* $\mathcal{H}^k(S) < \infty$. *Hence the following are equivalent*:

(1) *S is k-rectifiable.*

(2) *For $\mathcal{H}^k$-almost every $x \in S$, we have $\underline{\Theta}_k(S, x) > 0$ and the existence of a k-dimensional Banach space $(\mathbb{R}^k, \|\cdot\|_k)$ such that*

$$\mathrm{Tan}_x(\mathsf{X}, \mathsf{d}, \mathcal{H}^k \llcorner S) = \{(\mathbb{R}^k, \|\cdot\|_x, \mathcal{H}^k, 0)\}. \tag{A-1}$$

Let us fix $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ an $\mathrm{RCD}(K, N)$ space of essential dimension n. Let $E \subseteq \mathsf{X}$ be a set of locally finite perimeter. Now by Theorem 3.3 and the first part of the argument of Theorem 3.4, we have:

(1) $|\mathrm{D}\chi_E|(\mathsf{X} \setminus \mathcal{R}_n^*) = 0$, and hence $|\mathrm{D}\chi_E|$ is concentrated on $\mathcal{F}E$.

(2) $\mathcal{H}^{n-1} \llcorner \mathcal{F}E$ is a σ-finite Borel measure that is mutually absolutely continuous with respect to $|\mathrm{D}\chi_E|$. Notice that, for the precise computation of the density of $|\mathrm{D}\chi_E|$ with respect to $\mathcal{H}^{n-1} \llcorner \mathcal{F}E$ in Theorem 3.4, we needed the rectifiability of $\mathcal{F}E$, which we will not use in the following argument.

Hence let us call $f \in L^1_{\mathrm{loc}}(|\mathrm{D}\chi_E|)$ the function such that $\mathcal{H}^{n-1} \llcorner \mathcal{F}E = f|\mathrm{D}\chi_E|$, and let $\mathcal{D} \subseteq \mathcal{F}E$ be the set of the Lebesgue points of f with respect to the asymptotically doubling measure $|\mathrm{D}\chi_E|$ that are also differentiability points of $\mathcal{H}^{n-1} \llcorner \mathcal{F}E$ with respect to $|\mathrm{D}\chi_E|$, i.e., for every $x \in \mathcal{D}$,

$$\lim_{r \to 0} \fint_{B_r(x)} |f - f(x)| \,\mathrm{d}|\mathrm{D}\chi_E| = 0 \tag{A-2}$$

and

$$f(x) = \lim_{r \to 0} \frac{\mathcal{H}^{n-1} \llcorner \mathcal{F}E(B_r(x))}{|\mathrm{D}\chi_E|(B_r(x))}. \tag{A-3}$$

Notice that $|D\chi_E|(X\setminus\mathcal{D})=\mathcal{H}^{n-1}(\mathcal{F}E\setminus\mathcal{D})=0$ due to the Lebesgue differentiation theorem [Heinonen et al. 2015, p. 77], and the Lebesgue–Radon–Nikodým theorem [Heinonen et al. 2015, p. 81 and Remark 3.4.29]. Notice, moreover, that since $|D\chi_E|$ is mutually absolutely continuous with respect to $\mathcal{H}^{n-1}\llcorner\mathcal{F}E$, we have $f(x)>0$ for $|D\chi_E|$-almost every $x\in\mathsf{X}$, or equivalently for $\mathcal{H}^{n-1}\llcorner\mathcal{F}E$-almost every $x\in\mathsf{X}$.

Let us now prove that $\mathcal{F}E$ is $(n-1)$-rectifiable by exploiting Theorem A.1. Let us verify item (2) there. By the third line in (2-10) together with the fact that $x\in\mathcal{R}_n^*$ and (A-3), we get that $\underline{\Theta}_{n-1}(\mathcal{F}E,x)>0$ for every $x\in\mathcal{D}$, and hence for $\mathcal{H}^{n-1}$-almost every $x\in\mathcal{F}E$. Let us now verify the second part of item (2). Let us fix $x\in\mathcal{D}$, and let us take an arbitrary sequence $r_i\to 0$. We have that, up to subsequences,

$$\mathsf{X}_i:=(\mathsf{X},r_i^{-1}\mathsf{d},\mathsf{m}_x^{r_i},x,E)\to(\mathbb{R},\mathsf{d}_e,\underline{\mathcal{L}}^n,0,\{x_n>0\})$$

and, in a realization of the previous convergence, we have that the $|D\chi_E|_{\mathsf{X}_i}$ weakly converge to $|D\chi_{\{x_n>0\}}|$. For the sake of clarity, we denoted by $|D\chi_E|_{\mathsf{X}_i}$ the perimeter measure of E in the rescaled space X_i. Notice that $|D\chi_E|_{\mathsf{X}_i}=(r_i/C_x^{r_i})|D\chi_E|$, where $|D\chi_E|$ is the perimeter measure on X. Let $g\in C_{\mathrm{bs}}(\mathsf{Z})$, where Z is a realization of the previous convergence. Hence we have

$$\int_{\mathsf{X}_i} g\,\mathrm{d}\frac{r_i\mathcal{H}^{n-1}\llcorner FE}{C_x^{r_i}}=\int_{\mathsf{X}_i} gf\,\mathrm{d}|D\chi_E|_{\mathsf{X}_i}=\int_{\mathsf{X}_i} g(y)f(x)\,\mathrm{d}|D\chi_E|_{\mathsf{X}_i}(y)+\int_{\mathsf{X}_i} g(y)(f(y)-f(x))\,\mathrm{d}|D\chi_E|_{\mathsf{X}_i}(y),$$

and hence, by using (A-2) and the fact that

$$|D\chi_E|(B_{r_i}(x))\sim\frac{(n+1)\omega_{n-1}}{\omega_n}\frac{C_x^{r_i}}{r_i}$$

as a consequence of the second and third line of (2-10), we conclude that[2]

$$\frac{r_i\mathcal{H}^{n-1}\llcorner\mathcal{F}E}{C_x^{r_i}}\rightharpoonup f(x)|D\chi_{\{x_n>0\}}| \tag{3-4}$$

in the realization Z. This immediately implies that

$$\frac{\mathcal{H}^{n-1}\llcorner\mathcal{F}E}{\mathcal{H}^{n-1}\llcorner\mathcal{F}E(B_{r_i}(x))}\rightharpoonup\mathcal{H}^{n-1}\llcorner\{x_n=0\}$$

because $\mathcal{H}^{n-1}\llcorner\{x_n=0\}$ is the surface measure on $\{x_n=0\}$ that gives measure 1 to the unit ball.

Hence we have shown that, for every $x\in\mathcal{D}$ and every sequence $r_i\to 0$, there is a realization Z in which one has the convergence

$$\left(\mathsf{X},\frac{\mathsf{d}}{r_i},\frac{\mathcal{H}^{n-1}\llcorner\mathcal{F}E}{\mathcal{H}^{n-1}\llcorner\mathcal{F}E(B_{r_i}(x))},x\right)\to(\mathbb{R}^{n-1},\mathsf{d}_e,\mathcal{H}^{n-1},0),$$

which is exactly what one needed to show in order to verify (A-1) (recall [Bate 2022, Proposition 2.13]). Hence the application of Theorem A.1 gives the $(n-1)$-rectifiability of $\mathcal{F}E$.

[2] Notice that in the following equation we are considering $\mathcal{H}^{n-1}\llcorner\mathcal{F}E$ in the original space X and not in the rescaled space

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