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# Shape optimization for the Stokes system with threshold leak boundary conditions

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ABSTRACT

This paper discusses the process of optimizing the shape of systems that are controlled by the Stokes flow with threshold leak boundary conditions. In the theoretical part it focuses on studying the stability of solutions to the state problem in relation to a specific set of domains. In order to facilitate computation, the slip term and impermeability condition are regulated. In the computational part, the optimized portion of the boundary is defined using Bézier polynomials, in order to create a finite dimensional optimization problem. The paper also includes numerical examples to demonstrate the computational efficiency of this approach.

## 1. Introduction

Control and optimization of fluid mechanics models including shape optimization is nowadays well established discipline with many practical applications, see [12,19] and references therein. Typically the behavior of the controlled system is governed by generally nonlinear partial differential equations comprising appropriate boundary conditions. Their solutions are usually smooth functions of control parameters. Some thirty years ago, mathematicians introduced into fluid models the so-called threshold boundary conditions which are well-known in contact mechanics of solids as unilateral and friction conditions. Fujita in his pioneering paper [8] studied two types of such conditions in the Stokes and Navier–Stokes model, namely slip and leak boundary conditions of Tresca type, when slip, leak on the boundary may occur only if the shear, and normal stress, respectively, attains a threshold bound given a-priori. A possible way how to express these conditions is to write them in the form of inclusions involving multivalued mappings which represent the subdifferential of appropriate nonsmooth convex functions. The whole mathematical model then leads to an inequality type problem whose complexity depends partly on the flow model and partly on the choice of the slip/leak law see [2,3,18], e.g. Optimization of systems governed by nonsmooth state relations gives rise to possible nonsmoothness of the whole optimization problem. This fact creates some difficulties from the computational point of view. If we use the original nonsmooth formulation then (to be correct) discretized models should be solved by methods which are tailored just for this type of problems [21]. But their successful application needs some elementary knowledge of tools of nonsmooth analysis. On the other hand, classical gradient type methods when used for solving nonsmooth problems usually fail or give unsatisfactory results. One of ways how to overcome these difficulties is to replace the original state problem by a sequence of smooth ones and to use them as the new state relation in optimization. The resulting problem becomes smooth (provided that the cost function is smooth, too) and so it can be solved by standard methods. Just this way is used in this paper.

The present paper deals with a class of 2D shape optimization problems governed by the Stokes equations with threshold leak boundary conditions of Tresca–Navier type which are prescribed on an optimized part of the boundary. It extends the previous

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papers [15,16] which are devoted to the Stokes system but with the threshold slip conditions and, in addition it improves several results obtained there. To simplify our presentation we shall consider a very simple geometry of admissible domains. Moreover, the optimized part of the boundary will be represented by the graph of $C^{1,1}$ functions which will play the role of the design variables. The velocity formulation of the state relation leads to a variational inequality of the 2nd kind using the terminology from [10] due to the presence of the nonsmooth leak term $j$. To regularize the problem, $j$ is replaced by an appropriate sequence of smooth functionals $j_\varepsilon$, $\varepsilon \to 0+$. There is yet another troublesome thing from the computational point of view: namely the zero tangential velocity condition $u_\tau = 0$ prescribed on the optimized part of the boundary. This condition is realized in computations by a smooth penalty technique. Thus we use simultaneously a penalty and regularization approach for solving the state problem.

The paper is organized as follows: in Section 2, the state and shape optimization problems in their original, i.e. nonsmooth form, are defined together with the assumptions guaranteeing the existence of a solution. The most important result needed in the existence analysis is the proof of a stability of solutions with respect to domains. i.e. to show that the solutions to the state problem considered as a function of domains depend continuously (in an appropriate sense) on domain variations. To this end one needs another very important property: to show that any test function used in the weak formulation to the Stokes equations on any admissible domain can be approximated by functions which can be used as test functions on close domains. In [16] this property has been proven for functions satisfying the impermeability condition $v_\nu = 0$ on the slip part of the boundary. In the present paper this result is extended to the more general boundary condition of the form $\boldsymbol{v} \cdot \boldsymbol{s} = 0$ prescribed on the optimized part of the boundary, where $\boldsymbol{s}$ is a sufficiently smooth vector field depending continuously on the boundary variations. Shape optimization problems with the penalized/regularized state equations are introduced in Section 3. Their solutions now depend on the regularization/penalization parameter $\varepsilon$. It is shown that if $\varepsilon \to 0+$, they tend on subsequences to a solution of the original nonsmooth optimization problem. Also this convergence result is stronger than these ones in [15,16]. Section 5 deals with computational aspects. Optimized part of the boundary with the leak conditions is parametrized by Bézier polynomials, while the regularized-penalized state problem is discretized by stable P1-bubble/P1 elements. The gradient of the cost function is evaluated using the algebraic adjoint state approach. Finally, Section 6 presents computational results for two model problems.

The paper uses the following notation. If $Q$ is a bounded domain in $\mathbb{R}^n$, $n = 1,2$ then $H^k(Q)$, $k \geq 0$ integer, denotes the standard Sobolev space of functions defined in $Q$ which are together with their derivatives up to order $k$ square integrable in $Q$. We set $H^0(Q) = L^2(Q)$. The norm in $H^k(Q)$ will be denoted by $\|\cdot\|_{k,Q}$ and the scalar product by $(\ ,\ )_{k,Q}$. If $X$ is an ordered vector space then $X_+$ stands for the cone of its non-negative elements. Algebraic vectors and vector functions will be denoted by bold characters. If $\mathbf{a}, \mathbf{b}$ are two vectors from $R^d$, $d = 1,2,\ldots$ their scalar product is denoted by $\mathbf{a} \cdot \mathbf{b}$. If $\mathbf{A}=(a_{ij})$, $\mathbf{B}=(b_{ij})$ are two $n \times n$ matrices then $\mathbf{A} : \mathbf{B} := a_{ij}b_{ij}$ (the summation convention is used). The symbol $c$ stands for a generic positive constant, which may take different values at different places of its occurrence.

## 2. State problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary $\partial\Omega = \overline{\Gamma} \cup \overline{\Gamma_{\mathrm{N}}} \cup \overline{S}$, where $\Gamma$, $\Gamma_{\mathrm{N}}$, and $S$ are non-empty, disjoint parts open in $\partial\Omega$. The classical formulation of the state problem reads as follows: find the velocity vector $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ and the pressure $p : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -2\mu \operatorname{div}(\mathbb{D}\boldsymbol{u}) + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \\ \boldsymbol{\sigma\nu} = \boldsymbol{\sigma}_N & \text{on } \Gamma_{\mathrm{N}}, \\ u_\tau = 0 & \text{on } S, \\ |\sigma_\nu + \kappa u_\nu| \leq g, \quad (\sigma_\nu + \kappa u_\nu)u_\nu + g|u_\nu| = 0 & \text{on } S. \end{cases} \tag{2.1}$$

Here $\mu > 0$ is the dynamic viscosity of the fluid, $\boldsymbol{f} \in (L^2(\Omega))^2$, $\boldsymbol{\sigma}_N \in (L^2(\Gamma_{\mathrm{N}}))^2$, $g, \kappa : S \to \mathbb{R}_+$ denote an external force, a given value of the stress vector, a non-negative leak threshold, and leak coefficient, respectively. Further $\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}})$ is the symmetric part of the gradient of $\boldsymbol{u}$, $\boldsymbol{\nu}$, $\boldsymbol{\tau}$ are the unit normal, and tangential vector, respectively, to $\partial\Omega$. Finally, $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $v_\tau = \boldsymbol{v} \cdot \boldsymbol{\tau}$ are the normal, and tangential components of a vector $\boldsymbol{v} \in \mathbb{R}^2$ on $\partial\Omega$, respectively, $\boldsymbol{\sigma} = 2\mu(\mathbb{D}\boldsymbol{u}) - p\boldsymbol{I}$ is the stress tensor, and $\sigma_\nu = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}$ is the normal component of the stress vector $\boldsymbol{\sigma\nu}$ on $\partial\Omega$.

From $(2.1)_6$ it follows:

$$\left.\begin{array}{l} \bullet \text{ if } u_\nu(x) = 0 \text{ then } |\sigma_\nu(x)| \leq g(x),\ x \in S, \\ \bullet \text{ if } u_\nu(x) \neq 0 \text{ then } \sigma_\nu(x) = -\kappa(x)u_\nu(x) - g(x)\operatorname{sign} u_\nu(x),\ x \in S. \end{array}\right\} \tag{2.2}$$

The relation between $\sigma_\nu$ and $-u_\nu$ is depicted in Fig. 1. Thus a leak at $x \in S$ occurs only if $|\sigma_\nu(x) + \kappa(x)u_\nu(x)| = g(x)$.

The weak formulation of (2.1) reads:

$$\begin{cases} \text{Find } \boldsymbol{u} \in \mathbb{V}(\Omega),\ p \in L^2(\Omega) \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + b(\boldsymbol{v} - \boldsymbol{u}, p) + j(v_\nu, u_\nu) - j(u_\nu, u_\nu) \geq L(\boldsymbol{v} - \boldsymbol{u}) \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega) \\ b(\boldsymbol{u}, q) = 0 \quad \forall q \in L^2(\Omega), \end{cases} \tag{$\mathcal{P}$}$$

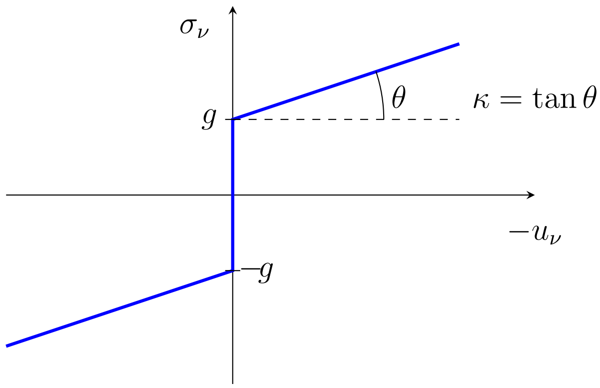


**Fig. 1.** Relation between normal velocity and normal stress on $S$.

where

$$\begin{aligned}
\mathbb{V}(\Omega) &= \{\boldsymbol{v} \in (H^1(\Omega))^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma,\ v_\tau = 0 \text{ on } S\},\\
a(\boldsymbol{u},\boldsymbol{v}) &= 2\mu \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v}\,dx, \quad \boldsymbol{u},\boldsymbol{v} \in (H^1(\Omega))^2,\\
b(\boldsymbol{v},q) &= -\int_\Omega q \operatorname{div} \boldsymbol{v}\,dx, \quad \boldsymbol{v} \in (H^1(\Omega))^2, q \in L^2(\Omega),\\
L(\boldsymbol{v}) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx + \int_{\Gamma_\mathrm{N}} \boldsymbol{\sigma}_N \cdot \boldsymbol{v}\,ds, \quad \boldsymbol{f} \in (L^2(\Omega))^2,\ \boldsymbol{\sigma}_N \in (L^2(\Gamma_\mathrm{N}))^2,\ \boldsymbol{v} \in (H^1(\Omega))^2,\\
j(v_\nu,u_\nu) &= \int_S (g|v_\nu| + \kappa\, v_\nu u_\nu)\,ds, \quad g,\kappa \in L^\infty_+(S),\ \boldsymbol{u},\boldsymbol{v} \in (H^1(\Omega))^2.
\end{aligned}$$

Problem $(\mathcal{P})$ has been studied in [8,9] provided that $\Gamma_\mathrm{N} = \emptyset$ and $\kappa \equiv 0$ on $S$. Fujita proved that $\boldsymbol{u}$ is unique, whereas $p$ is determined up to an additive constant which is subject to appropriate constraints arising from the leak conditions $(2.1)_6$. In our case the pressure is unique since the boundary condition of $\Gamma_\mathrm{N}$ fixes the value of $p$ on $S$.

In what follows we shall suppose that $\mu = \frac{1}{2}$. Concerning the existence and uniqueness of the solution to $(\mathcal{P})$ we have the following

**Theorem 2.1.** *Problem $(\mathcal{P})$ has a unique solution $(\boldsymbol{u},p)$ for any $\boldsymbol{f} \in (L^2(\Omega))^2$, $\boldsymbol{\sigma}_N \in (L^2(\Gamma_\mathrm{N}))^2$, and $g,\kappa \in L^\infty_+(S)$. In addition,*

$$\|\boldsymbol{u}\|_{1,\Omega} \le \frac{1}{c_K}\|L\|_* \le \frac{1}{c_K}\left(\|\boldsymbol{f}\|_{0,\Omega} + c_{tr}\|\boldsymbol{\sigma}_N\|_{0,\Gamma_\mathrm{N}}\right) \tag{2.3}$$

*and*

$$\beta\|p\|_{0,\Omega} \le \|a\|\|\boldsymbol{u}\|_{1,\Omega} + c_{tr}\|g\|_{\infty,S}|\text{length } S|^{1/2} + c_{tr}^2\|\kappa\|_{\infty,S}\|\boldsymbol{u}\|_{1,\Omega}, \tag{2.4}$$

*where $c_K > 0$ is the constant in Korn's inequality, $c_{tr} > 0$ is the norm of the trace mapping $tr : \mathbb{V}(\Omega) \to (L^2(\partial\Omega))^2$, $\|a\|$, $\|L\|_*$ is the norm of $a$, and $L$, respectively, and $\beta > 0$ is the constant in the inf-sup condition for $b$.*

**Proof.** The existence and uniqueness of the solution to $(\mathcal{P})$ follows from $\mathbb{V}(\Omega)$-ellipticity of the bilinear form $a$ which is a consequence of Korn's inequality

$$\exists c_K = \text{const.} > 0 : \quad \int_\Omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{v}\,dx \ge c_K\|\boldsymbol{v}\|^2_{1,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega) \tag{2.5}$$

and the inf-sup condition satisfied by the form $b$ on $\mathbb{V}(\Omega) \times L^2(\Omega)$ [17]:

$$\exists \beta = \text{const.} > 0 : \quad \sup_{\boldsymbol{v}\in\mathbb{V}(\Omega)\setminus\{\boldsymbol{0}\}} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{1,\Omega}} \ge \beta\|q\|_{0,\Omega} \quad \forall q \in L^2(\Omega). \tag{2.6}$$

Inserting $\boldsymbol{v} = \boldsymbol{0}, 2\boldsymbol{u}$ into $(\mathcal{P})_1$, we obtain:

$$a(\boldsymbol{u},\boldsymbol{u}) + j(u_\nu,u_\nu) = L(\boldsymbol{u}), \tag{2.7}$$

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) + j(u_\nu,v_\nu) \ge L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega). \tag{2.8}$$

From (2.7) and (2.5) we have:

$$c_K\|\boldsymbol{u}\|^2_{1,\Omega} \le a(\boldsymbol{u},\boldsymbol{u}) + j(u_\nu,u_\nu) \le \|L\|_*\|\boldsymbol{u}\|_{1,\Omega}. \tag{2.9}$$

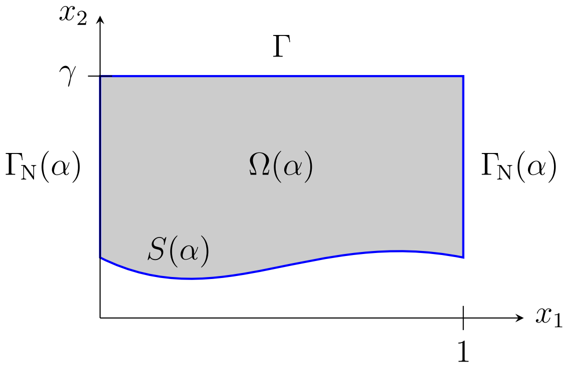


**Fig. 2.** Decomposition of the boundary of $\Omega(\alpha)$.

It is readily seen that

$$\|L\|_* \le \|\boldsymbol{f}\|_{0,\Omega} + c_{tr}\|\boldsymbol{\sigma}_N\|_{0,\Gamma_N}.$$

From this and (2.9) the estimate (2.3) follows.

To prove (2.4) we use the inf-sup condition (2.6) and (2.8). It is easy to see that

$$\frac{(\operatorname{div}\boldsymbol{v}, q)_{0,\Omega}}{\|\boldsymbol{v}\|_{1,\Omega}} \le \|a\|\|\boldsymbol{u}\|_{1,\Omega} + c_{tr}\|g\|_{\infty,S}|\text{length } S|^{1/2} + c_{tr}^2\|\kappa\|_{\infty,S}\|\boldsymbol{u}\|_{1,\Omega}$$

holds for any $\boldsymbol{v} \in \mathbb{V}(\Omega)$, $\boldsymbol{v} \ne \boldsymbol{0}$. From this and (2.6) the estimate (2.4) follows. □

## 3. Optimal shape design problem: definition and existence analysis

The aim of this section is to present and analyze a class of shape optimization problems with the state problem introduced in Section 2.

To this end we use the following system of admissible domains:

$$\mathcal{O} = \{\Omega(\alpha) \mid \alpha \in \mathcal{U}_{ad}\}$$

where

$$\Omega(\alpha) = \{(x_1, x_2) \mid x_2 \in (0,1),\ \alpha(x_1) < x_2 < \gamma\}$$

and

$$\mathcal{U}_{ad} = \{\alpha \in C^{1,1}([0,1]) \mid -\gamma + \Delta < \alpha_{\min} \le \alpha \le \alpha_{\max} < \gamma \text{ in } [0,1],$$
$$|\alpha^{(j)}| \le C_j \text{ a.e. in } [0,1],\ j = 1,2\}. \tag{3.1}$$

Here $\alpha_{\min}$ is a real constant and $\alpha_{\max}, \gamma, C_1, C_2, \Delta$ are positive constants such that $\mathcal{U}_{ad} \ne \emptyset$. By $\widehat{\Omega} = (0,1) \times (-\gamma, \gamma)$ we denote the hold-all domain, i.e. $\Omega(\alpha) \subseteq \widehat{\Omega}\ \forall \alpha \in \mathcal{U}_{ad}$. The boundary of any $\Omega(\alpha)$ will be decomposed as follows: $\partial\Omega(\alpha) = \overline{\Gamma} \cup \overline{\Gamma_N(\alpha)} \cup \overline{S(\alpha)}$, where

$$\Gamma = (0,1) \times \{1\},\ S(\alpha) = \text{graph of } \alpha,\ \Gamma_N(\alpha) = \partial\Omega(\alpha) \setminus (\overline{\Gamma} \cup \overline{S(\alpha)})$$

(see Fig. 2).

Since the state problem on $\Omega(\alpha)$ will be defined for variable $\alpha \in \mathcal{U}_{ad}$ we shall suppose that $\boldsymbol{f} \in (L^2(\widehat{\Omega}))^2$ and $\boldsymbol{\sigma}_N \in (L^2(\hat{\Gamma}_N))^2$, where $\hat{\Gamma}_N$ is the union of the vertical sides of $\widehat{\Omega}$. To define the functions $g, \kappa$ appearing in the leak term $j$ on any $S(\alpha), \alpha \in \mathcal{U}_{ad}$ we use functions $\tilde{g}, \tilde{\kappa} \in L^\infty_+((0,1))$ and set

$$g(x_1, x_2) = \tilde{g}(x_1), \quad \kappa(x_1, x_2) = \tilde{\kappa}(x_1) \quad \forall (x_1, x_2) \in \widehat{\Omega}.$$

Hence

$$\|g\|_{\infty,S(\alpha)} = \|\tilde{g}\|_{\infty,(0,1)}, \quad \|\kappa\|_{\infty,S(\alpha)} = \|\tilde{\kappa}\|_{\infty,(0,1)} \quad \forall \alpha \in \mathcal{U}_{ad}. \tag{3.2}$$

On any $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ we consider the following state problem:

$$\begin{cases} \text{Find } (\boldsymbol{u}(\alpha), p(\alpha)) \in \mathbb{V}(\Omega(\alpha)) \times L^2(\Omega(\alpha)) \text{ such that} \\ a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v}-\boldsymbol{u}(\alpha)) + b_\alpha(\boldsymbol{v}-\boldsymbol{u}(\alpha), p(\alpha)) + j_\alpha(\boldsymbol{v}\cdot\boldsymbol{v}^\alpha, \boldsymbol{u}(\alpha)\cdot\boldsymbol{v}^\alpha) \\ \qquad - j_\alpha(\boldsymbol{u}(\alpha)\cdot\boldsymbol{v}^\alpha, \boldsymbol{u}(\alpha)\cdot\boldsymbol{v}^\alpha) \ge L_\alpha(\boldsymbol{v}-\boldsymbol{u}(\alpha)) \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega(\alpha)) \\ b_\alpha(\boldsymbol{u}(\alpha), q) = 0 \quad \forall q \in L^2(\Omega(\alpha)), \end{cases} \tag{$\mathcal{P}(\alpha)$}$$

where $\mathbb{V}(\Omega(\alpha))$ is the space $\mathbb{V}(\Omega)$ defined in Section 2 with $\Omega := \Omega(\alpha)$, $S := S(\alpha)$, and $\Gamma_N := \Gamma_N(\alpha)$. To point out that the forms $a, b, L$ and the leak term $j$ depend on $\alpha \in \mathcal{U}_{ad}$, we use notation $a_\alpha, b_\alpha, L_\alpha$, and $j_\alpha$, respectively in what follows. The same convention holds for the vectors $\boldsymbol{v}^\alpha, \boldsymbol{\tau}^\alpha$.

Finally, let $J : \mathcal{U}_{ad} \times (H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega}) \to \mathbb{R}$ be a cost functional. The optimal shape design problem we shall study reads as follows:

$$\begin{cases} \text{Find } \alpha^* \in \mathcal{U}_{ad} \text{ such that} \\ J(\alpha^*, \boldsymbol{u}(\alpha^*), p(\alpha^*)) \le J(\alpha, \boldsymbol{u}(\alpha), p(\alpha)) \quad \forall \alpha \in \mathcal{U}_{ad}, \end{cases} \tag{$\mathbb{P}$}$$

where $(\boldsymbol{u}(\alpha), p(\alpha))$ is the solution to $(\mathcal{P}(\alpha))$.

Our aim is to show that under appropriate assumptions on $J$, problem $(\mathbb{P})$ has at least one solution. We start with

**Lemma 3.1.** *Solutions to $(\mathcal{P}(\alpha))$ are uniformly bounded with respect to $\alpha \in \mathcal{U}_{ad}$: there exists a positive constant $c$, which does not depend on $\alpha \in \mathcal{U}_{ad}$ such that*

$$\|\boldsymbol{u}(\alpha)\|_{1,\Omega(\alpha)} + \|p(\alpha)\|_{0,\Omega(\alpha)} \le c \quad \forall \alpha \in \mathcal{U}_{ad}. \tag{3.3}$$

**Proof.** From (2.3) and the assumptions on $\boldsymbol{f}$ and $\boldsymbol{\sigma}_N$ it follows:

$$\|\boldsymbol{u}(\alpha)\|_{1,\Omega(\alpha)} \le \frac{1}{c_K}\left(\|\boldsymbol{f}\|_{0,\hat{\Omega}} + c_{tr^\alpha}\|\boldsymbol{\sigma}_N\|_{0,\hat{\Gamma}_N}\right). \tag{3.4}$$

The constant $c_K$ of Korn's inequality can be chosen to be independent of $\alpha \in \mathcal{U}_{ad}$ (see [20]). Further $tr^\alpha$ stands for the norm of the trace mapping $tr^\alpha : \mathbb{V}(\Omega(\alpha)) \to (L^2(\partial\Omega(\alpha)))^2$. It is readily seen that $tr^\alpha$ can be chosen to be independent of $\alpha \in \mathcal{U}_{ad}$ [14]. From this and (3.4) uniform boundedness of $\|\boldsymbol{u}(\alpha)\|_{1,\Omega(\alpha)}$ follows. The prove the same for $\|p(\alpha)\|_{0,\Omega(\alpha)}$ we use (2.4):

$$\begin{aligned} \|p(\alpha)\|_{0,\Omega(\alpha)} \le \frac{1}{\beta}\big(&\|a_\alpha\|\|\boldsymbol{u}(\alpha)\|_{1,\Omega(\alpha)} + c_{tr^\alpha}\|g\|_{\infty,S(\alpha)}|\text{lenght } S(\alpha)|^{1/2} \\ &+ c_{tr^\alpha}^2\|\kappa\|_{\infty,S(\alpha)}\|\boldsymbol{u}(\alpha)\|_{1,\Omega(\alpha)}\big), \end{aligned} \tag{3.5}$$

The constant $\beta > 0$ of the inf-sup condition can be chosen again to be independent of $\alpha \in \mathcal{U}_{ad}$ [4] and the same holds for $\|a_\alpha\|$. Finally $|\text{length } S(\alpha)|^{1/2} \le \sqrt{1 + C_1^2}$ $\forall \alpha \in \mathcal{U}_{ad}$ as follows from the definition of $\mathcal{U}_{ad}$. Taking into account all these facts together with (3.2), (3.4), (3.5), we obtain uniform boundedness of $\|p(\alpha)\|_{0,\Omega(\alpha)}$ with respect to $\alpha \in \mathcal{U}_{ad}$. □

The solution $(\boldsymbol{u}(\alpha), p(\alpha))$ to $(\mathcal{P}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$ will be extended from $\Omega(\alpha)$ on the hold-all domain $\hat{\Omega}$ and denoted as $(\hat{\boldsymbol{u}}(\alpha), \hat{p}(\alpha)) \in (H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega})$ in what follows. We can use any extension mapping which preserves the uniform boundedness property of $(\hat{\boldsymbol{u}}(\alpha), \hat{p}(\alpha))$ with respect to $\alpha \in \mathcal{U}_{ad}$:

$$\exists c > 0 : \ \|\hat{\boldsymbol{u}}(\alpha)\|_{1,\hat{\Omega}} + \|\hat{p}(\alpha)\|_{0,\hat{\Omega}} \le c\left(\|\boldsymbol{u}(\alpha)\|_{1,\Omega(\alpha)} + \|p(\alpha)\|_{0,\Omega(\alpha)}\right) \overset{(3.3)}{\le} c \quad \forall \alpha \in \mathcal{U}_{ad}, \tag{3.6}$$

where $c$ is a positive constant which does not depend on $\alpha \in \mathcal{U}_{ad}$.

For the pressure $p(\alpha) \in L^2(\Omega(\alpha))$ we simply use the extension by zero on $\hat{\Omega} \setminus \overline{\Omega(\alpha)}$. The extension of $\boldsymbol{u}(\alpha) \in (H^1(\Omega(\alpha)))^2$ is more involved. One can use either a general result from [5] on the uniform extension property of domains satisfying the uniform cone property or to construct himself an extension mapping taking advantage of a simple shape of $\Omega(\alpha) \in \mathcal{O}$ and the condition $|\alpha'| \le C_1$ in $[0, 1]$.

The key role in the existence analysis plays

**Theorem 3.1.** *For any sequence $\{(\alpha_n, \boldsymbol{u}_n, p_n)\}$, where $\alpha_n \in \mathcal{U}_{ad}$ and $(\boldsymbol{u}_n, p_n) := (\boldsymbol{u}(\alpha_n), p(\alpha_n)) \in \mathbb{V}(\Omega(\alpha_n)) \times L^2(\Omega(\alpha_n))$ solves $(\mathcal{P}(\alpha_n))$, $n \to \infty$ there exist: its subsequence (denoted by the same symbol) and functions $\alpha \in \mathcal{U}_{ad}$, $(\overline{\boldsymbol{u}}, \overline{p}) \in (H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega})$ such that*

$$\begin{cases} \alpha_n \to \alpha & \text{in } C^1([0,1]), \\ \hat{\boldsymbol{u}}_n \rightharpoonup \overline{\boldsymbol{u}} & \text{(weakly) in } (H^1(\hat{\Omega}))^2, \\ \hat{p}_n \rightharpoonup \overline{p} & \text{in } L^2(\hat{\Omega}),\ n \to \infty. \end{cases} \tag{3.7}$$

*In addition, $(\overline{\boldsymbol{u}}, \overline{p})_{|\Omega(\alpha)} = (\boldsymbol{u}(\alpha), p(\alpha))$ solves $(\mathcal{P}(\alpha))$.*

**Proof.** The existence of a subsequence satisfying (3.7) results from compactness of $\mathcal{U}_{ad}$ in $C^1([0,1])$ and (3.6). To prove that $(\overline{\boldsymbol{u}}, \overline{p})_{|\Omega(\alpha)}$ solves $(\mathcal{P}(\alpha))$ we first verify that $\overline{\boldsymbol{u}}_{|\Omega(\alpha)} \in \mathbb{V}(\Omega(\alpha))$. To this end it is sufficient to show that

$$\overline{\boldsymbol{u}}_{|S(\alpha)} \cdot \boldsymbol{\tau}^\alpha = 0 \quad \text{on } S(\alpha). \tag{3.8}$$

From Lemma 2.21 in [14] we know that

$$\hat{\boldsymbol{u}}_n \circ \alpha_n := \hat{\boldsymbol{u}}_n(x_1, \alpha_n(x_1)) \to \overline{\boldsymbol{u}} \circ \alpha \quad \text{in } (L^2((0,1)))^2,\ n \to \infty$$

and also

$$0 = \hat{\boldsymbol{u}}_n \circ \alpha_n \cdot \boldsymbol{\tau}^{\alpha_n} \circ \alpha_n \to \overline{\boldsymbol{u}} \circ \alpha \cdot \boldsymbol{\tau}^\alpha \circ \alpha \quad \text{in } L^2((0,1))$$

making use of $(3.7)_1$. Thus (3.8) holds and so $\overline{\boldsymbol{u}}_{|\Omega(\alpha)} \in \mathbb{V}(\Omega(\alpha))$.

Let $\chi_n, \chi_\alpha$ be the characteristic functions of $\Omega(\alpha_n)$, and $\Omega(\alpha)$, respectively and $\hat{\chi}_n, \hat{\chi}_\alpha \in L^2(\hat{\Omega})$ their extensions by zero on $\hat{\Omega}$. From $(3.7)_1$ it easily follows that

$$\hat{\chi}_n \to \hat{\chi}_\alpha \quad \text{in } L^2(\hat{\Omega}),\ n \to \infty. \tag{3.9}$$

The definition of $(\mathcal{P}(\alpha_n))$, $(3.7)_2$ and (3.9) yield:

$$0 = b_n(\boldsymbol{u}_n, q) = b_{\hat{\Omega}}(\hat{\boldsymbol{u}}_n, \hat{\chi}_n q) \to b_{\hat{\Omega}}(\overline{\boldsymbol{u}}, \hat{\chi}_\alpha q) = b_\alpha(\overline{\boldsymbol{u}}, q) \quad \forall q \in L^2(\hat{\Omega}), \tag{3.10}$$

where for brevity of notation $b_n := b_{\alpha_n}$ and similarly for other forms in the sequel. Hence $\operatorname{div} \overline{\boldsymbol{u}}_{|\Omega(\alpha)} = 0$.

To accomplish the proof it remains to show that the couple $(\overline{\boldsymbol{u}}, \overline{p})_{|\Omega(\alpha)}$ satisfies the inequality in $(\mathcal{P}(\alpha))$.

Let $\boldsymbol{v} \in \mathbb{V}(\Omega(\alpha))$ be given. Then accordingly to Theorem A.1 and Remark A.2 from Appendix there exist: a sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k \in (H^1(\hat{\Omega}))^2$ and a function $\overline{\boldsymbol{v}} \in (H^1(\hat{\Omega}))^2$ such that $\overline{\boldsymbol{v}}_{|\Omega(\alpha)} = \boldsymbol{v}$ and

$$\boldsymbol{v}_k \to \overline{\boldsymbol{v}} \quad \text{in } (H^1(\hat{\Omega}))^2,\ k \to \infty. \tag{3.11}$$

Moreover, for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$\boldsymbol{v}_{k\,|\Omega(\alpha_{n_k})} \in \mathbb{V}(\Omega(\alpha_{n_k})) \tag{3.12}$$

and consequently $\boldsymbol{v}_{k\,|\Omega(\alpha_{n_k})}$ can be used as a test function in $(\mathcal{P}(\alpha_{n_k}))$:

$$\begin{aligned} &a_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) + b_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}, p_{n_k}) + \\ &j_{n_k}(\boldsymbol{v}_k \cdot \boldsymbol{v}^{n_k}, \boldsymbol{u}_{n_k} \cdot \boldsymbol{v}^{n_k}) - j_{n_k}(\boldsymbol{u}_{n_k} \cdot \boldsymbol{v}^{n_k}, \boldsymbol{u}_{n_k} \cdot \boldsymbol{v}^{n_k}) \ge L_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}) \end{aligned} \tag{3.13}$$

holds for any $k \in \mathbb{N}$, where $a_{n_k} := a_{\alpha_{n_k}}$, $\boldsymbol{v}^{n_k} := \boldsymbol{v}^{\alpha_{n_k}}$, etc.

Next, we pass to the limit with $k \to \infty$ in (3.13). From $(3.7)_2$, (3.9) and (3.11) we obtain:

$$\begin{aligned} &\limsup_{k\to\infty} a_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) = \limsup_{k\to\infty} \int_{\hat{\Omega}} \hat{\chi}_{n_k} \mathbb{D}\hat{\boldsymbol{u}}_{n_k} : \mathbb{D}(\boldsymbol{v}_k - \hat{\boldsymbol{u}}_{n_k})\, dx \\ &\le \int_{\hat{\Omega}} \hat{\chi}_\alpha \mathbb{D}\overline{\boldsymbol{u}} : \mathbb{D}(\overline{\boldsymbol{v}} - \overline{\boldsymbol{u}})\, dx = a_\alpha(\overline{\boldsymbol{u}}, \boldsymbol{v} - \overline{\boldsymbol{u}}) \end{aligned} \tag{3.14}$$

using weak lower semicontinuity of $a_{\hat{\Omega}}$ and the fact that $\overline{\boldsymbol{v}}_{|\Omega(\alpha)} = \boldsymbol{v}$.

Similarly

$$\lim_{k\to\infty} b_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}, p_{n_k}) = \lim_{k\to\infty} b_{n_k}(\boldsymbol{v}_k, p_{n_k}) = b_\alpha(\boldsymbol{v} - \overline{\boldsymbol{u}}, \overline{p}) \tag{3.15}$$

as follows from (3.10) and

$$\lim_{k\to\infty} L_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}) = L_\alpha(\boldsymbol{v} - \overline{\boldsymbol{u}}). \tag{3.16}$$

Finally, the leak term:

$$\begin{aligned} &j_{n_k}(\boldsymbol{v}_k \cdot \boldsymbol{v}^{n_k}, \boldsymbol{u}_{n_k} \cdot \boldsymbol{v}^{n_k}) = \int_{S(\alpha_{n_k})} \kappa(\boldsymbol{v}_k \cdot \boldsymbol{v}^{n_k})(\boldsymbol{u}_{n_k} \cdot \boldsymbol{v}^{n_k})\, ds + \int_{S(\alpha_{n_k})} g|\boldsymbol{v}_k \cdot \boldsymbol{v}^{n_k}|\, ds \\ &= \int_0^1 \tilde{\kappa}(\boldsymbol{v}_k \cdot \boldsymbol{v}^{n_k}) \circ \alpha_{n_k} (\boldsymbol{u}_{n_k} \cdot \boldsymbol{v}^{n_k}) \circ \alpha_{n_k} \sqrt{1 + (\alpha'_{n_k})^2}\, dx_1 \\ &+ \int_0^1 \tilde{g}|\boldsymbol{v}_k \cdot \boldsymbol{v}^{n_k}| \circ \alpha_{n_k} \sqrt{1 + (\alpha'_{n_k})^2}\, dx_1 \to j_\alpha(\boldsymbol{v} \cdot \boldsymbol{v}^\alpha, \overline{\boldsymbol{u}} \cdot \boldsymbol{v}^\alpha) \end{aligned} \tag{3.17}$$

using $(3.7)_{1,2}$, (3.11) and convergence of $\{\boldsymbol{v}^{n_k} \circ \alpha_{n_k}\}$ to $\boldsymbol{v}^\alpha \circ \alpha$ in $(L^2([0,1]))^2$. Similarly for the second leak term.

From (3.13)–(3.17) we arrive at the assertion of the theorem. □

**Remark 3.1.** Besides $(3.7)_2$ one can prove strong convergence of $\boldsymbol{u}_n$ to $\boldsymbol{u}(\alpha)$ in the $H^1_{loc}(\Omega(\alpha))$-norm. Indeed, from (2.7) it follows that

$$\|\hat{\chi}_n \mathbb{D}\hat{\boldsymbol{u}}_n : \mathbb{D}\hat{\boldsymbol{u}}_n\|_{0,\hat{\Omega}} \to \|\hat{\chi}_\alpha \mathbb{D}\overline{\boldsymbol{u}} : \mathbb{D}\overline{\boldsymbol{u}}\|_{0,\hat{\Omega}}.$$

From this, $(3.7)_2$, and the fact that we already know that $\overline{\boldsymbol{u}}_{|\Omega(\alpha)} = \boldsymbol{u}(\alpha)$, we have

$$\|\boldsymbol{u}(\alpha) - \boldsymbol{u}_n\|_{1,D} \to 0,\ n \to \infty, \tag{3.18}$$

that holds for any subdomain $D \subset \Omega(\alpha)$ such that $\operatorname{dist}(\overline{D}, S(\alpha)) > 0$. The same result has been proven in [16, Remark 3] for the Stokes system with the threshold slip boundary condition of Tresca type.

Next we show that the sequence $\{p_n\}$ tends strongly to $p(\alpha)$ in the $L^2_{loc}(\Omega(\alpha))$-norm. To this end we introduce the set $G_\delta(\alpha) = \Omega(\alpha) \setminus \overline{\mathrm{BL}_\delta(\alpha)}$, where

$$\mathrm{BL}_\delta(\alpha) = \{(x_1, x_2) \in \Omega(\alpha) \mid x_1 \in (0,1),\ \alpha(x_1) < x_2 < \alpha(x_2) + \delta\}$$

is the boundary layer along $S(\alpha)$ and $\delta > 0$ is an arbitrary but sufficiently small. Let such $\delta > 0$ be fixed. On any $G_\delta(\alpha)$ we consider the spaces

$$\mathbb{V}_0(G_\delta(\alpha)) = \{\boldsymbol{v} \in (H^1(G_\delta(\alpha)))^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma \cup S_\delta(\alpha),\ S_\delta(\alpha) = S(\alpha) + \delta\}$$

and

$$\widehat{\mathbb{V}}_0(G_\delta(\alpha)) = \{\hat{\boldsymbol{v}} \in (H^1(\widehat{\Omega}))^2 \mid \hat{\boldsymbol{v}}_{|G_\delta(\alpha)} = \boldsymbol{v} \in \mathbb{V}_0(G_\delta(\alpha)),\ \hat{\boldsymbol{v}} = \boldsymbol{0} \text{ in } \widehat{\Omega} \setminus \overline{G_\delta(\alpha)}\}.$$

Owing to $(3.7)_1$ there exists $n_1 := n_1(\delta)$ such that $\|\alpha_n - \alpha\|_{C([0,1])} < \delta/2\ \forall n \geq n_1$. Hence any function from $\widehat{\mathbb{V}}_0(G_\delta(\alpha))$ can be used as a test function in $(\mathcal{P}(\alpha))$, and $(\mathcal{P}(\alpha_n))$, $n \geq n_1$. The inequality (2.8) corresponding to $(\mathcal{P}(\alpha_n))$ with test functions $\hat{\boldsymbol{v}} \in \widehat{\mathbb{V}}_0(G_\delta(\alpha))$ changes into the equation

$$a_{\widehat{\Omega}}(\hat{\boldsymbol{u}}_n, \hat{\boldsymbol{v}}) + b_{\widehat{\Omega}}(\hat{\boldsymbol{v}}, \hat{p}_n) = L_{\widehat{\Omega}}(\hat{\boldsymbol{v}}) \quad \hat{\boldsymbol{v}} \in \widehat{\mathbb{V}}_0(G_\delta(\alpha)). \tag{3.19}$$

Since $\boldsymbol{v} = \boldsymbol{0}$ on $S(\alpha_n)$, $n \geq n_1$, the leak term $j_{\alpha_n}$ disappears. From the definition of $\widehat{\mathbb{V}}_0(G_\delta(\alpha))$ we see that (3.19) is equivalent to

$$a_{G_\delta(\alpha)}(\boldsymbol{u}_n, \boldsymbol{v}) + b_{G_\delta(\alpha)}(\boldsymbol{v}, p_n) = L_{G_\delta(\alpha)}(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbb{V}_0(G_\delta(\alpha)),\ n \geq n_1.$$

The same holds for the solution $(\boldsymbol{u}(\alpha), p(\alpha))$ to $(\mathcal{P}(\alpha))$:

$$a_{G_\delta(\alpha)}(\boldsymbol{u}(\alpha), \boldsymbol{v}) + b_{G_\delta(\alpha)}(\boldsymbol{v}, p(\alpha)) = L_{G_\delta(\alpha)}(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbb{V}_0(G_\delta(\alpha)).$$

Subtracting the second equation from the first one we obtain

$$\int_{G_\delta(\alpha)} \operatorname{div} \boldsymbol{v}\,(p_n - p(\alpha))\,dx = a_{G_\delta(\alpha)}(\boldsymbol{u}_n - \boldsymbol{u}(\alpha), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbb{V}_0(G_\delta(\alpha)).$$

Finally from this and the inf-sup condition we obtain

$$\beta\|p_n - p(\alpha)\|_{0,G_\delta(\alpha)} \leq c\|\boldsymbol{u}_n - \boldsymbol{u}(\alpha)\|_{1,G_\delta(\alpha)} \overset{(3.18)}{\longrightarrow} 0,$$

where $c = \text{const.} > 0$ which does not depend on $n$. □

To guarantee the existence of a minimizer of $J$ in the optimal shape design problem $(\mathbb{P})$, we shall suppose that $J$ is *lower semicontinuous* in the following sense: for any sequence $\{(\alpha_n, \boldsymbol{y}_n, z_n)\}$, $\alpha_n \in \mathcal{U}_{ad}$, $\boldsymbol{y}_n \in (H^1(\widehat{\Omega}))^2$ and $z_n \in L^2(\widehat{\Omega})$ such that

$$\begin{aligned} &\alpha_n \to \alpha && \text{in } C^1([0,1]),\\ &\boldsymbol{y}_n \rightharpoonup \boldsymbol{y} && \text{in } (H^1(\widehat{\Omega}))^2,\\ &z_n \rightharpoonup z && \text{in } L^2(\widehat{\Omega}), \quad n \to \infty, \end{aligned}$$

it holds that

$$\liminf_{n\to\infty} J(\alpha_n, \boldsymbol{y}_{n|\Omega(\alpha_n)}, z_{n|\Omega(\alpha_n)}) \geq J(\alpha, \boldsymbol{y}_{|\Omega(\alpha)}, z_{|\Omega(\alpha)}). \tag{3.20}$$

**Theorem 3.2.** *Problem* $(\mathbb{P})$ *has a solution.*

**Proof.** The result follows from (3.20) using compactness arguments stated in Theorem 3.1. □

**Remark 3.2.** In the next computational section we shall use two cost functionals:

$$J_1(\alpha) = \frac{1}{2}\int_0^1 (u_\nu(\alpha)\circ\alpha - \tilde{u})^2\,dx_1, \quad \tilde{u} \in L^\infty((0,1)) \text{ given},$$

and

$$J_2(\alpha) = \frac{1}{2}a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{u}(\alpha)),$$

where $\boldsymbol{u}(\alpha)$ is the velocity component of the solution to $(\mathcal{P}(\alpha))$. It is easy to see that on the basis of Theorem 3.1 both cost functionals satisfy (3.20) ($J_1$ is in fact even continuous).

## 4. Shape optimization with penalized/regularized state problem

Problem $(\mathbb{P})$ studied in the previous section posseses two inconveniences from the computational point of view. First of all, the problem is nonsmooth since the state relation is represented by the variational inequality $(\mathcal{P}(\alpha))$. This fact restricts the use of numerical minimization methods. Secondly, the tangential no-slip condition $u_\tau = 0$ is prescribed on the designed part $S(\alpha)$. To avoid these drawbacks we use a penalization to release this condition on $S(\alpha)$ and a regularization of the nonsmooth leak term.

To simplify the presentation, the smooth part of the leak functional $j : (H^1(\Omega(\alpha)))^2 \times (H^1(\Omega(\alpha)))^2 \to \mathbb{R}_+$ defined in Section 2 will be added to the bilinear form $a_\alpha, \alpha \in \mathcal{U}_{ad}$ and the new form will be denoted as

$$a_{\kappa,\alpha}(\boldsymbol{u}, \boldsymbol{v}) := a_\alpha(\boldsymbol{u}, \boldsymbol{v}) + (\kappa\boldsymbol{u}\cdot\boldsymbol{\nu}^\alpha, \boldsymbol{v}\cdot\boldsymbol{\nu}^\alpha)_{0,S(\alpha)}, \quad \boldsymbol{u}, \boldsymbol{v} \in (H^1(\Omega(\alpha)))^2 \tag{4.1}$$

and set

$$j_\alpha(\boldsymbol{v}\cdot\boldsymbol{v}^\alpha) := \int_{S(\alpha)} g|\boldsymbol{v}\cdot\boldsymbol{v}^\alpha|\,ds, \quad \boldsymbol{v}\in (H^1(\Omega(\alpha)))^2. \tag{4.2}$$

We use the simplest penalty functional

$$t_\varepsilon^\alpha(\boldsymbol{v}\cdot\boldsymbol{\tau}^\alpha) = \frac{1}{2\varepsilon}\|\boldsymbol{v}\cdot\boldsymbol{\tau}^\alpha\|^2_{0,S(\alpha)}, \quad \boldsymbol{v}\in (H^1(\Omega(\alpha)))^2. \tag{4.3}$$

Regularization of $j_\alpha$ defined by (4.2) consists in its approximation by an appropriate sequence $\{j_\alpha^\varepsilon\}, \varepsilon\to 0+$ of smooth functionals $j_\alpha^\varepsilon$. We do not specify their particular choice at the moment, only summarize their properties which will be needed in what follows:

- $j_\alpha^\varepsilon : L^2(S(\alpha)) \to \mathbb{R}_+$ are convex, $C^2$-functionals $\forall \varepsilon > 0, \alpha\in\mathcal{U}_{ad}$, (4.4)
- $\left.\begin{array}{l}\alpha_n \to \alpha \text{ in } C^1([0,1]), \alpha_n, \alpha \in \mathcal{U}_{ad} \\ \boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \text{ in } (H^1(\widehat{\Omega}))^2,\ n\to\infty\end{array}\right\} \Longrightarrow j_{\alpha_n}^{\varepsilon_n}(\boldsymbol{v}_n\cdot\boldsymbol{v}^{\alpha_n}) \underset{\varepsilon_n\to 0+}{\longrightarrow} j_\alpha(\boldsymbol{v}\cdot\boldsymbol{v}^\alpha)$, (4.5)
- $\exists c_0 > 0\ \exists \varepsilon_0 > 0 :\ j_\alpha^\varepsilon(0) \le c_0 \quad \forall \varepsilon\in[0,\varepsilon_0], \alpha\in\mathcal{U}_{ad}$. (4.6)

On any $\Omega(\alpha), \alpha\in\mathcal{U}_{ad}$ and $\varepsilon > 0$ we define the following penalized/regularized state problem:

$$\begin{cases}\text{Find } (\boldsymbol{u}^\varepsilon(\alpha), p^\varepsilon(\alpha)) \in \widetilde{\mathbb{V}}(\Omega(\alpha)) \times L^2(\Omega(\alpha)) \text{ such that}\\ a_{\kappa,\alpha}(\boldsymbol{u}^\varepsilon(\alpha),\boldsymbol{v}) + b_\alpha(\boldsymbol{v},p^\varepsilon(\alpha)) + (\nabla j_\alpha^\varepsilon(\boldsymbol{u}^\varepsilon(\alpha)\cdot\boldsymbol{v}^\alpha), \boldsymbol{v}\cdot\boldsymbol{v}^\alpha)_{0,S(\alpha)}\\ \qquad\qquad + \frac{1}{\varepsilon}(\boldsymbol{u}^\varepsilon(\alpha)\cdot\boldsymbol{\tau}^\alpha, \boldsymbol{v}\cdot\boldsymbol{\tau}^\alpha)_{0,S(\alpha)} = L_\alpha(\boldsymbol{v}) \quad \forall \boldsymbol{v}\in\widetilde{\mathbb{V}}(\Omega(\alpha))\\ b(\boldsymbol{u}^\varepsilon(\alpha),q) = 0 \quad \forall q\in L^2(\Omega(\alpha)),\end{cases} \tag{$\mathcal{P}_\varepsilon(\alpha)$}$$

where

$$\widetilde{\mathbb{V}}(\Omega(\alpha)) = \{\boldsymbol{v}\in (H^1(\Omega(\alpha)))^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma\}.$$

From (4.4) and (4.5) it follows that $\mathcal{P}_\varepsilon(\alpha)$ has a unique solution for any $\varepsilon > 0$ and $\alpha\in\mathcal{U}_{ad}$.[1]

We define the new shape optimization problem in which the state equation $(\mathcal{P}_\varepsilon(\alpha))$ instead of $(\mathcal{P}(\alpha))$ is used: given $\varepsilon > 0$,

$$\begin{cases}\text{Find } \alpha_\varepsilon^* \in \mathcal{U}_{ad} \text{ such that}\\ J(\alpha_\varepsilon^*, \boldsymbol{u}^\varepsilon(\alpha_\varepsilon^*), p^\varepsilon(\alpha_\varepsilon^*)) \le J(\alpha, \boldsymbol{u}^\varepsilon(\alpha), p^\varepsilon(\alpha)) \quad \forall \alpha\in\mathcal{U}_{ad},\end{cases} \tag{$\mathbb{P}_\varepsilon$}$$

where $J$ is the same cost functional as in $(\mathbb{P})$ and $(\boldsymbol{u}^\varepsilon(\alpha), p^\varepsilon(\alpha))$ solves $(\mathcal{P}_\varepsilon(\alpha))$.

In what follows we shall study if there is a relation between $(\mathbb{P})$ and $(\mathbb{P}_\varepsilon)$ as $\varepsilon\to 0+$. We start with

**Lemma 4.1.** *Solutions to $(\mathcal{P}_\varepsilon(\alpha))$ are uniformly bounded with respect to $\varepsilon > 0$ and $\alpha\in\mathcal{U}_{ad}$:*

$$\exists c = \text{const.} > 0 : \quad \|\boldsymbol{u}^\varepsilon(\alpha)\|_{1,\Omega(\alpha)} + \|p^\varepsilon(\alpha)\|_{0,\Omega(\alpha)} + \frac{1}{2\varepsilon}\|\boldsymbol{u}^\varepsilon(\alpha)\cdot\boldsymbol{\tau}^\alpha\|^2_{0,S(\alpha)} \le c, \tag{4.7}$$

*where $c$ does not depend on $\varepsilon > 0$ and $\alpha\in\mathcal{U}_{ad}$.*

**Proof.** It is well-known that the velocity component $\boldsymbol{u}^\varepsilon(\alpha)$ solves the following minimization problem:

$$\begin{cases}\text{Find } \boldsymbol{u}^\varepsilon(\alpha) \in \widetilde{\mathbb{V}}_{\text{div}}(\Omega(\alpha)) \text{ such that}\\ \mathcal{J}_\alpha^\varepsilon(\boldsymbol{u}^\varepsilon(\alpha)) = \min\limits_{\boldsymbol{v}\in\widetilde{\mathbb{V}}_{\text{div}}(\Omega(\alpha))} \mathcal{J}_\alpha^\varepsilon(\boldsymbol{v})\end{cases} \tag{4.8}$$

where

$$\mathcal{J}_\alpha^\varepsilon(\boldsymbol{v}) = \frac{1}{2}a_{\kappa,\alpha}(\boldsymbol{v},\boldsymbol{v}) + j_\alpha^\varepsilon(\boldsymbol{v}\cdot\boldsymbol{v}^\alpha) + \frac{1}{2\varepsilon}\|\boldsymbol{v}\cdot\boldsymbol{\tau}^\alpha\|^2_{0,S(\alpha)} - L_\alpha(\boldsymbol{v}) \tag{4.9}$$

and

$$\widetilde{\mathbb{V}}_{\text{div}}(\Omega(\alpha)) = \{\boldsymbol{v}\in\widetilde{\mathbb{V}}(\Omega(\alpha)) \mid \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega(\alpha)\}. \tag{4.10}$$

From (4.8), (4.9), nonnegativeness of $j_\alpha^\varepsilon$, $t_\alpha$, Korn's inequality and (4.6) we have:

$$\begin{aligned}&\frac{1}{2}a_{\kappa,\alpha}(\boldsymbol{u}^\varepsilon(\alpha),\boldsymbol{u}^\varepsilon(\alpha)) + \frac{1}{2\varepsilon}\|\boldsymbol{u}^\varepsilon(\alpha)\cdot\boldsymbol{\tau}^\alpha\|^2_{0,S(\alpha)} \le \mathcal{J}_\alpha^\varepsilon(\boldsymbol{u}^\varepsilon(\alpha)) + L_\alpha(\boldsymbol{u}^\varepsilon(\alpha))\\ &\le \mathcal{J}_\alpha^\varepsilon(\boldsymbol{0}) + \left(\|\boldsymbol{f}\|_{0,\widehat{\Omega}} + c_{tr^\alpha}\|\boldsymbol{\sigma}_N\|_{0,\widehat{\Gamma}_{\text{N}}}\right)\|\boldsymbol{u}^\varepsilon(\alpha)\|_{1,\Omega(\alpha)} \le c,\end{aligned} \tag{4.11}$$

where the meaning of $\widehat{\Omega}$, $c_{tr^\alpha}$, and $\widehat{\Gamma}_{\text{N}}$ is the same as in Section 3.

[1] In fact, the existence and uniqueness of the solution can be established under weaker assumptions (see [10]). The stronger assumptions (4.4)–(4.6) are needed because $j_\alpha^\varepsilon$ depends also on $\alpha\in\mathcal{U}_{ad}$.

To prove boundedness of $\|p^\varepsilon(\alpha)\|_{0,\Omega(\alpha)}$ we proceed as follows. Let

$$\widetilde{\mathbb{V}}_0(\Omega(\alpha)) = \{\boldsymbol{v} \in \widetilde{\mathbb{V}}(\Omega(\alpha)) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } S(\alpha)\}.$$

Then the definition of $(\mathcal{P}_\varepsilon(\alpha))$ with test functions $\boldsymbol{v} \in \widetilde{\mathbb{V}}_0(\Omega(\alpha))$ yields:

$$a_{\kappa,\alpha}(\boldsymbol{u}^\varepsilon(\alpha), \boldsymbol{v}) + b_\alpha(\boldsymbol{v}, p^\varepsilon(\alpha)) = L_\alpha(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \widetilde{\mathbb{V}}_0(\Omega(\alpha)).$$

From this, the inf-sup condition for $b_\alpha$ on $\widetilde{\mathbb{V}}_0(\Omega(\alpha)) \times L^2(\Omega(\alpha))$, and (4.11) uniform boundedness of $\|p^\varepsilon(\alpha)\|_{0,\Omega(\alpha)}$ and hence (4.7) follows. $\square$

As before the solutions $(\boldsymbol{u}^\varepsilon(\alpha), p^\varepsilon(\alpha)) \in \widetilde{\mathbb{V}}(\Omega(\alpha)) \times L^2(\Omega(\alpha))$ to $(\mathcal{P}_\varepsilon(\alpha))$ will be extented from $\Omega(\alpha)$ to $\widehat{\Omega}$ and then denoted by $(\hat{\boldsymbol{u}}^\varepsilon(\alpha), \hat{p}^\varepsilon(\alpha)) \in (H^1(\widehat{\Omega}))^2 \times L^2(\widehat{\Omega})$. We use again the extension mappings which preserve uniform boundedness with respect to $\alpha \in \mathcal{U}_{ad}$ and $\varepsilon > 0$:

$$\exists c = \text{const.} > 0 : \quad \|\hat{\boldsymbol{u}}^\varepsilon(\alpha)\|_{1,\widehat{\Omega}} + \|\hat{p}^\varepsilon(\alpha)\|_{0,\widehat{\Omega}} \le c \quad \forall \alpha \in \mathcal{U}_{ad}, \varepsilon > 0. \tag{4.12}$$

Next we prove a stability type result which is parallel to Theorem 3.1.

**Theorem 4.1.** *For any sequence $\{(\alpha_k, \boldsymbol{u}_k^{\varepsilon_k}, p_k^{\varepsilon_k})\}$, where $\alpha_k \in \mathcal{U}_{ad}$ and $(\boldsymbol{u}_k^{\varepsilon_k}, p_k^{\varepsilon_k}) := (\boldsymbol{u}^{\varepsilon_k}(\alpha_k), p^{\varepsilon_k}(\alpha_k))$ is a solution to $(\mathcal{P}_{\varepsilon_k}(\alpha_k))$, $\varepsilon_k \to 0+$ as $k \to \infty$, there exist its subsequence (denoted by the same symbol) and functions $\alpha \in \mathcal{U}_{ad}$, $(\overline{\boldsymbol{u}}, \overline{p}) \in (H^1(\widehat{\Omega}))^2 \times L^2(\widehat{\Omega})$ such that*

$$\begin{cases} \alpha_k \to \alpha & \text{in } C^1([0,1]), \\ \hat{\boldsymbol{u}}_k^{\varepsilon_k} \rightharpoonup \overline{\boldsymbol{u}} & \text{in } (H^1(\widehat{\Omega}))^2, \\ \hat{p}_k^{\varepsilon_k} \rightharpoonup \overline{p} & \text{in } L^2(\widehat{\Omega}), \quad \text{as } k \to \infty. \end{cases} \tag{4.13}$$

*In addition, $(\overline{\boldsymbol{u}}, \overline{p})_{|\Omega(\alpha)} = (\boldsymbol{u}(\alpha), p(\alpha))$ solves $(\mathcal{P}(\alpha))$.*

**Proof.** The existence of a subsequence and a couple $(\overline{\boldsymbol{u}}, \overline{p})$ satisfying (4.13) is obvious. It is readily seen that $\overline{\boldsymbol{u}}_{|\Omega(\alpha)}$ is divergence free in $\Omega(\alpha)$ and $\overline{\boldsymbol{u}} \cdot \boldsymbol{\tau}^\alpha = 0$ on $S(\alpha)$. Indeed,

$$\|\hat{\boldsymbol{u}}_k^{\varepsilon_k} \cdot \boldsymbol{\tau}^{\alpha_k}\|_{0,S(\alpha_k)} = \|\boldsymbol{u}_k^{\varepsilon_k} \cdot \boldsymbol{\tau}^{\alpha_k}\|_{0,S(\alpha_k)} \to \|\overline{\boldsymbol{u}} \cdot \boldsymbol{\tau}^\alpha\|_{0,S(\alpha)} = 0$$

taking into account $(4.13)_{1,2}$ and (4.7). Hence

$$\overline{\boldsymbol{u}}_{|\Omega(\alpha)} \in \mathbb{V}(\Omega(\alpha)) \quad \text{and} \quad b_\alpha(\overline{\boldsymbol{u}}, q) = 0 \quad \forall q \in L^2(\Omega(\alpha)). \tag{4.14}$$

It remains to verify that $(\overline{\boldsymbol{u}}, \overline{p})_{|\Omega(\alpha)}$ satisfies the first inequality in $(\mathcal{P}(\alpha))$.

Let $\boldsymbol{v} \in \mathbb{V}(\Omega(\alpha))$ be arbitrary but fixed and $\overline{\boldsymbol{v}}, \boldsymbol{v}_k, k \to \infty$ be functions from $(H^1(\widehat{\Omega}))^2$ satisfying (3.11) and (3.12). Since[2] $\boldsymbol{v}_{k|\Omega_{n_k}} \in \mathbb{V}(\Omega_{n_k}) \subseteq \widetilde{\mathbb{V}}(\Omega_{n_k})$ it can be used as a test function in $(\mathcal{P}_{\varepsilon_k}(\alpha_{n_k}))$ which can be equivalently written as follows [10]:

$$\begin{aligned} &a_{\kappa,n_k}(\boldsymbol{u}_{n_k}^{\varepsilon_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}^{\varepsilon_k}) + b_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}^{\varepsilon_k}, p_{n_k}^{\varepsilon_k}) \\ &+ j_{n_k}^{\varepsilon_k}(\boldsymbol{v}_k \cdot \boldsymbol{\nu}^{n_k}) - j_{n_k}^{\varepsilon_k}(\boldsymbol{u}_{n_k}^{\varepsilon_k} \cdot \boldsymbol{\nu}^{n_k}) \ge L_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}^{\varepsilon_k}) \quad \forall k, \end{aligned} \tag{4.15}$$

where $\{\varepsilon_k\}, \varepsilon_k > 0$ is an arbitrary sequence tending to zero. Here we used the fact that the penalty term

$$\tfrac{1}{2\varepsilon_k}\|\boldsymbol{v}_k \cdot \boldsymbol{\tau}^{n_k}\|^2_{0,S_{n_k}} - \tfrac{1}{2\varepsilon_k}\|\boldsymbol{u}_{n_k}^{\varepsilon_k} \cdot \boldsymbol{\tau}^{n_k}\|^2_{0,S_{n_k}} = -\tfrac{1}{2\varepsilon_k}\|\boldsymbol{u}_{n_k}^{\varepsilon_k} \cdot \boldsymbol{\tau}^{n_k}\|^2_{0,S_{n_k}} \le 0$$

since $\boldsymbol{v}_{k|\Omega_{n_k}} \in \mathbb{V}(\Omega(\alpha_{n_k}))$. Passing to the limit with $k \to \infty$ and using (3.14)–(3.16) and (4.5), we arrive at

$$a_{\kappa,\alpha}(\overline{\boldsymbol{u}}, \boldsymbol{v} - \overline{\boldsymbol{u}}) + b_\alpha(\boldsymbol{v} - \overline{\boldsymbol{u}}, \overline{p}) + j_\alpha(\boldsymbol{v} \cdot \boldsymbol{\nu}^\alpha) - j_\alpha(\overline{\boldsymbol{u}} \cdot \boldsymbol{\nu}^\alpha) \ge L_\alpha(\boldsymbol{v} - \overline{\boldsymbol{u}})$$

holds for any $\boldsymbol{v} \in \mathbb{V}(\Omega(\alpha))$ using that $\overline{\boldsymbol{v}}_{|\Omega(\alpha)} = \boldsymbol{v}$. From this and (4.14) we may conclude that $(\overline{\boldsymbol{u}}, \overline{p})_{|\Omega(\alpha)} = (\boldsymbol{u}(\alpha), p(\alpha))$ solves $(\mathcal{P}(\alpha))$. $\square$

**Remark 4.1.** Similarly to Section 3 one can show that there exists a subsequence of $\{(\boldsymbol{u}_k^{\varepsilon_k}, p_k^{\varepsilon_k})\}$ (denoted by the same symbol) such that

$$(\boldsymbol{u}_k^{\varepsilon_k}, p_k^{\varepsilon_k}) \to (\boldsymbol{u}(\alpha), p(\alpha)) \quad \text{in } (H^1_{loc}(\Omega(\alpha)))^2 \times L^2_{loc}(\Omega(\alpha)), \quad \text{as } k \to \infty. \tag{4.16}$$

Indeed, from $(4.13)_2$ it follows:

$$\liminf_{k\to\infty} \int_{\widehat{\Omega}} \hat{\chi}_k \mathbb{D}\hat{\boldsymbol{u}}_k^{\varepsilon_k} : \mathbb{D}\hat{\boldsymbol{u}}_k^{\varepsilon_k}\, dx \ge \int_{\Omega} \hat{\chi}_\alpha \mathbb{D}\hat{\boldsymbol{u}}(\alpha) : \mathbb{D}\hat{\boldsymbol{u}}(\alpha)\, dx. \tag{4.17}$$

[2] To simplify notation we shall write $\Omega_{n_k} := \Omega(\alpha_{n_k})$, $\boldsymbol{\tau}^{n_k} = \boldsymbol{\tau}^{\alpha_{n_k}}$, $a_{\kappa,n_k} := a_{\kappa,\alpha_{n_k}}$, etc.

From Theorem A.1 in Appendix we know that there exists a function $\boldsymbol{z} \in (H^1(\widehat{\Omega}))^2$ and a sequence $\{\boldsymbol{z}_k\}$, $\boldsymbol{z}_k \in (H^1(\widehat{\Omega}))^2$ such that $\boldsymbol{z}_{|\Omega(\alpha)} = \boldsymbol{u}(\alpha)$ and

$$\begin{cases} \boldsymbol{z}_k \to \boldsymbol{z} & \text{in } (H^1(\widehat{\Omega}))^2 \\ \boldsymbol{z}_{k|\Omega_{n_k}} \in \mathbb{V}(\Omega_{n_k}) \end{cases} \tag{4.18}$$

for an appropriate $n_k \in \mathbb{N}$. Using $\boldsymbol{z}_k$ instead of $\boldsymbol{v}_k$ in (4.15) we get:

$$\begin{aligned} &a_{\kappa,n_k}(\boldsymbol{u}_{n_k}^{\varepsilon_k}, \boldsymbol{z}_k - \boldsymbol{u}_{n_k}^{\varepsilon_k}) \geq L_{n_k}(\boldsymbol{z}_k - \boldsymbol{u}_{n_k}^{\varepsilon_k}) - b_{n_k}(\boldsymbol{z}_k - \boldsymbol{u}_{n_k}^{\varepsilon_k}, p_{n_k}^{\varepsilon_k}) \\ &-j_{n_k}^{\varepsilon_k}(\boldsymbol{z}_k \cdot \boldsymbol{v}^{n_k}) + j_{n_k}^{\varepsilon_k}(\boldsymbol{u}_{n_k}^{\varepsilon_k} \cdot \boldsymbol{v}^{n_k}) \to 0 \text{ as } k \to \infty \end{aligned}$$

making use of the properties of $\boldsymbol{z}$, $(4.18)_1$, and (4.5). Therefore

$$\limsup_{k\to\infty} a_{\kappa,n_k}(\boldsymbol{u}_{n_k}^{\varepsilon_k}, \boldsymbol{u}_{n_k}^{\varepsilon_k}) \leq \lim_{k\to\infty} a_{\kappa,n_k}(\boldsymbol{u}_{n_k}^{\varepsilon_k}, \boldsymbol{z}_k) = a_{\kappa,\alpha}(\boldsymbol{u}(\alpha), \boldsymbol{u}(\alpha)).$$

From this and the definition of $a_{\kappa,n_k}$ it easily follows that

$$\limsup_{k\to\infty} \int_{\widehat{\Omega}} \hat{\chi}_{n_k} \mathbb{D}\hat{\boldsymbol{u}}_{n_k}^{\varepsilon_k} : \mathbb{D}\hat{\boldsymbol{u}}_{n_k}^{\varepsilon_k}\, dx \leq \int_{\widehat{\Omega}} \hat{\chi}_\alpha \mathbb{D}\hat{\boldsymbol{u}}(\alpha) : \mathbb{D}\hat{\boldsymbol{u}}(\alpha)\, dx$$

which together with (4.17) gives

$$\lim_{k\to\infty} \int_{\widehat{\Omega}} \hat{\chi}_{n_k} \mathbb{D}\hat{\boldsymbol{u}}_{n_k}^{\varepsilon_k} : \mathbb{D}\hat{\boldsymbol{u}}_{n_k}^{\varepsilon_k}\, dx = \int_{\widehat{\Omega}} \hat{\chi}_\alpha \mathbb{D}\hat{\boldsymbol{u}}(\alpha) : \mathbb{D}\hat{\boldsymbol{u}}(\alpha)\, dx.$$

From this, local convergence of $\{\boldsymbol{u}_k^{\varepsilon_k}\}$ to $\boldsymbol{u}(\alpha)$ in $(H^1_{loc}(\Omega(\alpha)))^2$ follows. To prove that $p_k^{\varepsilon_k}$ tends to $p(\alpha)$ in $L^2_{loc}(\Omega(\alpha))$ we proceed exactly as in Remark 3.1. □

Due to the choice of the penalty term $\iota^\alpha$ and the regularization functional $j_\alpha^\varepsilon$ which satisfies (4.4), state problem $(\mathcal{P}_\varepsilon(\alpha))$ is smooth, i.e. the control-to-state mapping $\alpha \mapsto (\boldsymbol{u}^\varepsilon(\alpha), p^\varepsilon(\alpha))$ is differentiable with respect to $\alpha \in \mathcal{U}_{ad}$. Hence, problem $(\mathbb{P}_\varepsilon)$ is smooth, provided that $J$ is smooth, too. This makes it possible to use classical, gradient-type methods for numerical minimization of $J$.

It remains to establish a relation between $(\mathbb{P}_\varepsilon)$ and $(\mathbb{P})$ for $\varepsilon \to 0+$. We start with

**Theorem 4.2.** *Let (3.20) be satisfied. Then $(\mathbb{P}_\varepsilon)$ has a solution for any $\varepsilon > 0$.*

**Proof.** By reason of its simplicity we only sketch it. The existence of a solution follows from Lemma 4.1 and the stability type result for the solution to $(\mathcal{P}_\varepsilon(\alpha))$ with respect to $\alpha \in \mathcal{U}_{ad}$ keeping $\varepsilon > 0$ fixed. In addition, since functions from $\widetilde{\mathbb{V}}(\Omega(\alpha))$ are not subject to any kinematic constraint on $S(\alpha)$, there is no need to use Theorem A.1 from Appendix. □

Before we pass to the final result we shall need an additional continuity assumption on the cost functional $J : \mathcal{U}_{ad} \times (H^1(\widehat{\Omega}))^2 \times L^2(\widehat{\Omega}) \to \mathbb{R}$. We shall suppose that for any $\alpha \in \mathcal{U}_{ad}$ fixed, $J$ is continuous function of the remaining two variables. More precisely, if $(\boldsymbol{y}_n, z_n)$, $n = 1, 2, \ldots$ and $(\boldsymbol{y}, z)$ are elements of $(H^1(\widehat{\Omega}))^2 \times L^2(\widehat{\Omega})$ such that

$$\begin{cases} (\boldsymbol{y}_n, z_n)_{|\Omega(\alpha)} \to (\boldsymbol{y}, z)_{|\Omega(\alpha)} & \text{in } (H^1(\Omega(\alpha)))^2 \times L^2(\Omega(\alpha)) \quad \text{then} \\ \lim_{n\to\infty} J(\alpha, \boldsymbol{y}_{n|\Omega(\alpha)}, z_{n|\Omega(\alpha)}) = J(\alpha, \boldsymbol{y}_{|\Omega(\alpha)}, z_{|\Omega(\alpha)}) \end{cases} \tag{4.19}$$

holds for any $\alpha \in \mathcal{U}_{ad}$.

**Theorem 4.3.** *Let (3.20) and (4.19) be satisfied. Then for any sequence $\{(\alpha_\varepsilon^*, \boldsymbol{u}^\varepsilon(\alpha_\varepsilon^*), p^\varepsilon(\alpha_\varepsilon^*))\}$, $\varepsilon \to 0+$, where $\alpha_\varepsilon^*$ solves $(\mathbb{P}_\varepsilon)$ and $(\boldsymbol{u}^\varepsilon(\alpha_\varepsilon^*), p^\varepsilon(\alpha_\varepsilon^*))$ is the solution to $(\mathcal{P}(\alpha_\varepsilon^*))$, one can find a subsequence (denoted by the same symbol) and a triplet $(\alpha^*, \boldsymbol{u}^*, p^*) \in \mathcal{U}_{ad} \times (H^1(\widehat{\Omega}))^2 \times L^2(\widehat{\Omega})$ such that*

$$\begin{cases} \alpha_\varepsilon^* \to \alpha^* & \text{in } C^1([0,1]), \\ \hat{\boldsymbol{u}}^\varepsilon(\alpha_\varepsilon^*) \rightharpoonup \boldsymbol{u}^* & \text{in } (H^1(\widehat{\Omega}))^2, \\ \boldsymbol{u}^\varepsilon(\alpha_\varepsilon^*) \to \boldsymbol{u}^* & \text{in } (H^1_{loc}(\Omega(\alpha^*)))^2, \\ \hat{p}^\varepsilon(\alpha_\varepsilon^*) \rightharpoonup p^* & \text{in } L^2(\widehat{\Omega}), \\ p^\varepsilon(\alpha_\varepsilon^*) \to p^* & \text{in } L^2_{loc}(\Omega(\alpha^*)), \quad \text{as } \varepsilon \to 0+. \end{cases} \tag{4.20}$$

*In addition, $\alpha^*$ is a solution to $(\mathbb{P})$ and $(\boldsymbol{u}^*, p^*)_{|\Omega(\alpha^*)} = (\boldsymbol{u}(\alpha^*), p(\alpha^*))$ solves $(\mathcal{P}(\alpha^*))$. Any accumulation point of $\{(\alpha_\varepsilon^*, \boldsymbol{u}^\varepsilon(\alpha_\varepsilon^*), p^\varepsilon(\alpha_\varepsilon^*))\}$ in the sense of (4.20) has this property.*

**Proof.** The existence of a subsequence and a triplet $(\alpha^*, \boldsymbol{u}^*, p^*)$ such that $(u^*, p^*)_{|\Omega(\alpha^*)} = (\boldsymbol{u}(\alpha^*), p(\alpha^*))$ solves $(\mathcal{P}(\alpha^*))$ and (4.20) holds follows from Theorem 4.1 and Remark 4.1. Only what we need to show is that $\alpha^*$ solves $(\mathbb{P})$.

The definition of $(\mathbb{P}_\varepsilon)$ yields:

$$J(\alpha_\varepsilon^*, \boldsymbol{u}^\varepsilon(\alpha_\varepsilon^*), p^\varepsilon(\alpha_\varepsilon^*)) \leq J(\alpha, \boldsymbol{u}^\varepsilon(\alpha), p^\varepsilon(\alpha)) \quad \forall \alpha \in \mathcal{U}_{ad}. \tag{4.21}$$

Let $\bar{\alpha} \in \mathcal{U}_{ad}$ be fixed. Then it is well-known that the sequence $\{(\boldsymbol{u}^\varepsilon(\bar{\alpha}), p^\varepsilon(\bar{\alpha}))\}$, $\varepsilon \to 0+$ of solutions to $(\mathcal{P}_\varepsilon(\bar{\alpha}))$ tends to the solution $(\boldsymbol{u}(\bar{\alpha}), p(\bar{\alpha}))$ of $(\mathcal{P}(\bar{\alpha}))$:

$$\begin{cases} \boldsymbol{u}^\varepsilon(\bar{\alpha}) \to \boldsymbol{u}(\bar{\alpha}) & \text{in } (H^1(\Omega(\bar{\alpha})))^2, \\ p^\varepsilon(\bar{\alpha}) \to p(\bar{\alpha}) & \text{in } L^2(\Omega(\bar{\alpha})). \end{cases} \tag{4.22}$$

Letting $\varepsilon \to 0+$ in (4.21) we obtain:

$$J(\alpha^*, \boldsymbol{u}(\alpha^*), p(\alpha^*)) \le \liminf_{\varepsilon \to 0+} J(\alpha^*_\varepsilon, \boldsymbol{u}^\varepsilon(\alpha^*_\varepsilon), p^\varepsilon(\alpha^*_\varepsilon))$$
$$\le \lim_{\varepsilon \to 0+} J(\bar{\alpha}, u^\varepsilon(\bar{\alpha}), p^\varepsilon(\bar{\alpha})) = J(\bar{\alpha}, \boldsymbol{u}(\bar{\alpha}), p(\bar{\alpha}))$$

making use of (3.20), (4.19) and (4.22). □

## 5. Approximation and numerical realization of $(\mathbb{P}_\varepsilon)$

In this section we describe how to discretize and realize shape optimization problems governed by the regularized and penalized Stokes system. The admissible domains $\Omega$ are determined by functions $\alpha \in \mathcal{U}_{ad}$. The control variable $\alpha \in \mathcal{U}_{ad}$ will be discretized by Bézier functions, while a stable mixed finite element method will be used to discretize the state equation $(\mathcal{P}_\varepsilon(\alpha))$.

### 5.1. Discrete design parametrization and a finite element approximation of the state problem

We define the following finite dimensional parametrization of the leak boundary

$$S(\alpha_m) = \{(x_1, x_2) \mid x_1 \in [0,1],\ x_2 = \alpha_m(x_1)\}, \quad \alpha_m \in \mathcal{U}_{ad}$$

using the $m$th degree Bézier functions:

$$\alpha_m(x_1) = \sum_{i=0}^{m} a_i B_i^{(m)}(x_1), \tag{5.1}$$

where $B_i^{(m)}(t) = \binom{m}{i} t^i (1-t)^{m-i}$, $i=0,\dots,m$ are the Bernstein polynomials on $[0,1]$. Then the discrete design variable vector $\mathbf{a} = (a_0, a_1, \dots, a_m)$ consists of the coefficients in the linear combination (5.1).

Next we discretize the state problem $(\mathcal{P}_\varepsilon(\alpha_m))$ using the P1-bubble/P1 elements satisfying the LBB condition [1]. Let $\overline{\Omega_h(\alpha_m)}$ be a polygonal approximation of $\Omega(\alpha_m)$ and let $\mathcal{T}_h$ be its triangulation. We define

$$\widetilde{\mathcal{V}}_h(\alpha_m) = \{v_h \in C(\overline{\Omega}_h(\alpha_m)) \mid v_{h|T} \in P_1(T)\ \forall T \in \mathcal{T}_h,\ v_h = 0 \text{ on } \Gamma\},$$
$$\mathcal{B}_h(\alpha_m) = \left\{ v_h \in C(\overline{\Omega}_h(\alpha_m)) \mid v_{h|T} \in \operatorname{span}(b_T)\ \forall T \in \mathcal{T}_h \right\},$$

where $b_T \in P_3(T)$ are the "bubble" functions satisfying $b_T = 0$ on $\partial T$. The construction of the reference mesh $\widehat{\mathcal{T}}_h$ and its deformation in $x_2$-direction $\widehat{\mathcal{T}}_h \to \mathcal{T}_h := \mathcal{T}_h(\alpha_m)$ are constructed in the same way as in [15]. Then we introduce the following finite element spaces:

$$\widetilde{\mathbb{V}}_h(\alpha_m) = [\widetilde{\mathcal{V}}_h(\alpha_m) + \mathcal{B}_h(\alpha_m)]^2,$$
$$Q_h(\alpha_m) = \left\{ q_h \in C(\overline{\Omega}_h(\alpha_m)) \mid q_{h|T} \in P_1(T)\ \forall T \in \mathcal{T}_h \right\},$$

which are the discretizations of the spaces $\mathbb{V}(\Omega(\alpha_m))$ and $L^2(\Omega(\alpha_m))$, respectively.

The finite element approximation of the regularized/penalized state problem in the parametrized domain $\Omega(\alpha_m)$ then reads

$$\begin{cases} \text{Find } (\boldsymbol{u}_h^\varepsilon(\alpha_m), p_h^\varepsilon(\alpha_m)) \in \widetilde{\mathbb{V}}_h(\alpha_m) \times Q_h(\alpha_m) \text{ such that} \\ a_{\kappa,\alpha_m}(\boldsymbol{u}_h^\varepsilon, \boldsymbol{v}_h) + b_{\alpha_m}(\boldsymbol{v}_h, p_h^\varepsilon) + (\nabla j^\varepsilon_{\alpha_m}(\boldsymbol{u}_h^\varepsilon \cdot \boldsymbol{v}^{\alpha_m}), \boldsymbol{v}_h \cdot \boldsymbol{v}^{\alpha_m})_{0,S(\alpha_m)} \\ \qquad + \frac{1}{\varepsilon}(\boldsymbol{u}_h^\varepsilon \cdot \boldsymbol{\tau}^{\alpha_m}, \boldsymbol{v}_h \cdot \boldsymbol{\tau}^{\alpha_m})_{0,S(\alpha_m)} = L_{\alpha_m}(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \widetilde{\mathbb{V}}_h(\alpha_m) \\ b_{\alpha_m}(\boldsymbol{u}_h^\varepsilon, q_h) = 0 \quad \forall q_h \in Q_h(\alpha_m). \end{cases} \tag{$\mathcal{P}_\varepsilon^h(\alpha_m)$}$$

### 5.2. Finite dimensional optimization problem and its sensitivity analysis

After performing the finite element discretization of $(\mathcal{P}_\varepsilon^h(\alpha_m))$, its algebraic form is given by the following system of nonlinear algebraic equations:

$$\boldsymbol{r}([\mathbf{u}, \mathbf{p}]^{\mathrm{T}}) := \begin{bmatrix} \mathbf{A} + \mathbf{L}_\varepsilon(\mathbf{u}) + \frac{1}{\varepsilon}\mathbf{T} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} - \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \tag{5.2}$$

where $\mathbf{u} \in \mathbb{R}^{n_u}, \mathbf{p} \in \mathbb{R}^{n_p}$ is the vector of the nodal values of the velocity $\boldsymbol{u}$ and the pressure $p$, respectively, $\mathbf{A} \in \mathbb{R}^{n_u \times n_u}$ is a symmetric and positive definite matrix, $\mathbf{B} \in \mathbb{R}^{n_p \times n_u}$ is the velocity–pressure coupling matrix, $\frac{1}{\varepsilon}\mathbf{T} \in \mathbb{R}^{n_c \times n_u}$ is a matrix representation of the penalized no-slip condition $u_\tau = 0$, $\mathbf{L}_\varepsilon(\mathbf{u}) \in \mathbb{R}^{n_c \times n_u}$ is a matrix function representation of the smoothed leak term, and $\mathbf{f}$ is the discretization of the forcing term $L(\cdot)$. Further $n_p$ is the total number of the nodes in $\mathcal{T}_h$, $n_c$ is the number of the nodes lying on the leak boundary $\overline{S(\alpha_m)}$, and $n_u$ is the dimension of the solution component representing the velocity. The system (5.2) can be solved iteratively by using Newton's method with line search, e.g. Let

$$\mathcal{U} = \Big\{ \mathbf{a}\in\mathbb{R}^{m+1} \mid \alpha_{\min}\le a_i \le \alpha_{\max},\ i=0,\dots,m;\ |a_{i+1}-a_i| \le \tfrac{C_1}{m},\ i=0,\dots,m-1,$$
$$|a_{i+2}-2a_{i+1}+a_i| \le \tfrac{C_2}{m^2},\ i=0,\dots,m-2 \Big\},$$

where $C_1$ and $C_2$ are the same as in (3.1), be the set of admissible discrete design variables. From the properties of the Bernstein polynomials [6] it easily follows that if $\mathbf{a}\in\mathcal{U}$ then $\alpha_m \in \mathcal{U}_{ad}$, where $\alpha_m$ is defined by (5.1).

As the residual vector $\boldsymbol{r}$ in (5.2) depends also on the design variable $\mathbf{a}$, we write the algebraic state problem (5.2) in the form

$$\mathbf{r}(\mathbf{a},\mathbf{u}(\mathbf{a})) = \mathbf{0}, \quad \mathbf{q}(\mathbf{a}) = [\mathbf{u}(\mathbf{a}),\mathbf{p}(\mathbf{a})]^{\mathrm{T}}.$$

Denote $\mathfrak{J}_i : \mathcal{U}\to\mathbb{R}$, $\mathfrak{J}_i(\mathbf{a}) := \mathcal{I}_i(\mathbf{a},\mathbf{q}(\mathbf{a}))$, where $\mathcal{I}_i$ is a discretization of the cost functional $J_i$, $i=1,2$, mentioned in Remark 3.2. Then the discrete optimization problem to be realized reads as follows:

$$\mathbf{a}^* \in \operatorname*{argmin}_{\mathbf{a}\in\mathcal{U}} \left\{ \mathfrak{J}_i(\mathbf{a}) \mid \mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a})) = \mathbf{0} \right\}. \tag{5.3}$$

In order to be able to use gradient-based nonlinear programming algorithms for solving (5.3) we need to evaluate the gradient of $\mathfrak{J}_i$ with respect to the design variable vector $\mathbf{a}$. The cost function $\mathfrak{J}_i$ is continuously differentiable provided that $\mathcal{I}_i$ is so owing to the fact that $\mathcal{T}_h$ is a smooth topologically equivalent deformation of $\widehat{\mathcal{T}}_h$ (see [14]). Then, it is well-known that the partial derivatives of $\mathfrak{J}_i$ with respect to the design variables are given by

$$\frac{\mathrm{d}\mathfrak{J}_i(\mathbf{a})}{\mathrm{d}a_k} = \frac{\partial \mathcal{I}_i(\mathbf{a},\mathbf{q}(\mathbf{a}))}{\partial a_k} + \boldsymbol{\eta}^{\mathrm{T}}\left[\frac{\partial \mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a}))}{\partial a_k}\right], \quad k=0,\dots,m, \tag{5.4}$$

where $\boldsymbol{\eta}$ is the solution to the adjoint equation

$$\left[\frac{\partial \mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a}))}{\partial \mathbf{q}}\right]^{\mathrm{T}} \boldsymbol{\eta} = \nabla_{\mathbf{q}} \mathcal{I}_i(\mathbf{a},\mathbf{q}(\mathbf{a})). \tag{5.5}$$

The computation of partial derivatives in (5.4) and (5.5) can be done by hand or by using automatic differentiation of computer programs. For further details, see, e.g., [11,14].

## 6. Numerical examples

The MATLAB programming language was employed to implement both the state solver and the cost function evaluation [22]. The partial derivatives required in Eqs. (5.4) and (5.5) were straightforward enough to be manually computed and implemented. The minimization process was executed through the utilization of the sequential quadratic programming algorithm (SQP). Specifically, we employed the SQP implementation provided by the MATLAB Optimization Toolbox, utilizing the wrapper function 'fmincon' with the 'sqp' option. The parameters governing the stopping criterion were selected as TolX=$10^{-4}$, TolFun=$10^{-5}$, and TolCon=$10^{-5}$.

*Example 1*

Let $\mu=\frac{1}{2}$, $g=15$, and $\kappa=30$. Functions $\boldsymbol{f}$ and $\boldsymbol{\sigma}_N$ appearing on the right hand side of (2.1) are given by $\boldsymbol{f} = -2\mu\,\mathrm{div}\,\mathbb{D}(\boldsymbol{u}_{\exp}) + \nabla p_{\exp}$ and $\boldsymbol{\sigma}_N = 2\mu\mathbb{D}(\boldsymbol{u}_{\exp})\boldsymbol{\nu} - p_{\exp}\boldsymbol{\nu}$, where

$$\boldsymbol{u}_{\exp}(x) = \big[(1-\cos(2\pi x_1))\sin(2\pi x_2),\ \sin(2\pi x_1)(\cos(2\pi x_2)-1)\big],$$
$$p_{\exp}(x) = 2\pi(1-\cos(2\pi x_1) + 2\cos(2\pi x_2)).$$

With $\alpha\equiv 0$, the state problem is then the one used as a test problem in paper [13].

The parameters defining $\mathcal{U}$ are $m=20$, $\alpha_{\min}=-0.1$, $\alpha_{\max}=0.2$, $C_1=5$, and $C_2=10$. As the objective function we use $J_1(\alpha)$ with the (fixed) target profile $\tilde{u}$ defined by the monotonic $C^1$ cubic spline [7] interpolating the following set of datapoints $(X_i,Y_i)$, $i=1,\dots,10$, where

$$X_i \in \{0,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1\},$$
$$Y_i \in \{0,\ 0,\ 0.06,\ 0.16,\ 0.24,\ 0.26,\ 0.22,\ 0.12,\ 0,\ 0\}$$

(see Fig. 3).

We solved the discretized shape optimization problem using a reference mesh $\widehat{\mathcal{T}}_h$ consisting of 19604 elements. For regularization/penalization parameters we used three different values $\varepsilon = 10^{-3}, 10^{-4}, 10^{-5}$. In all cases $\alpha_m \equiv 0$ was used as the initial guess. These regularization/penalization parameter values represent a compromise. Larger parameter values result in poorly enforced constraints, while smaller values lead to ill-conditioned state and optimization problems. The selected parameter values strike a good compromise between accuracy and ill-conditioning.

The optimized $\alpha_m^*$ and convergence histories of the objective function values are shown in Fig. 4. The behavior with respect to $\varepsilon$ is reasonably stable justifying the regularization/penalization approach used. The velocity and pressure as well as the normal velocity component $u_\nu$ and the normal stress $\sigma_\nu$ on $S(\alpha_m^*)$ for $\varepsilon = 10^{-5}$ are shown in Figs. 5 and 6. It is well-known that the used descent type optimization method is only guaranteed to find an approximate local minimum. However, by examining Fig. 6, it can be concluded that the computed solution is close to the global optimum of $\mathfrak{J}_1$ in $\mathcal{U}$.

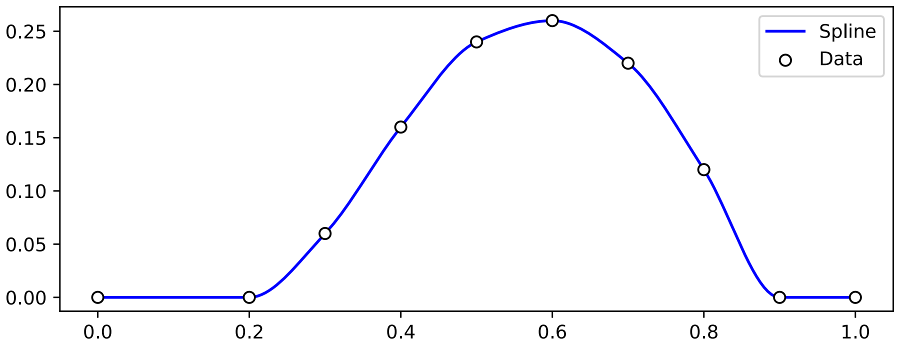


**Fig. 3.** Target profile $\tilde{u}$ defined by the monotonic cubic spline interpolating given data.

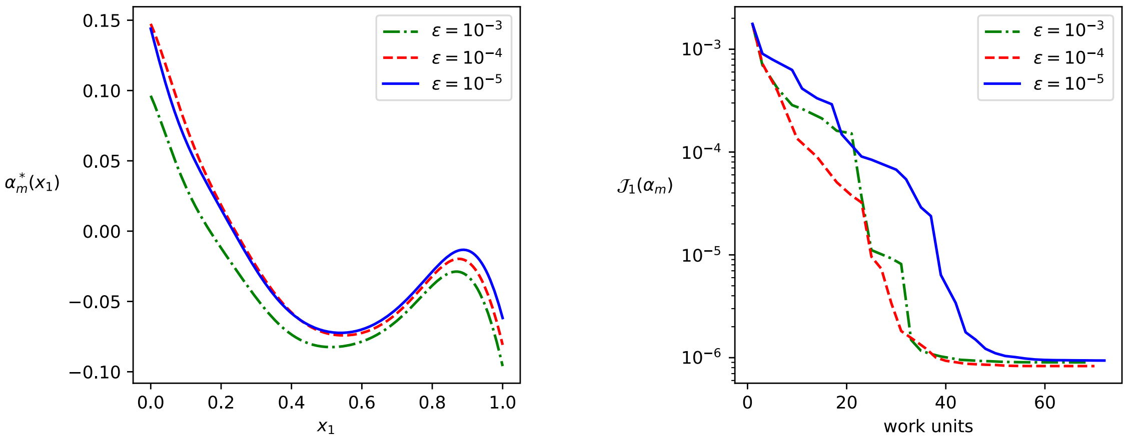


**Fig. 4.** Optimized shapes (left) and convergence histories (right) for different values of parameter $\varepsilon$.

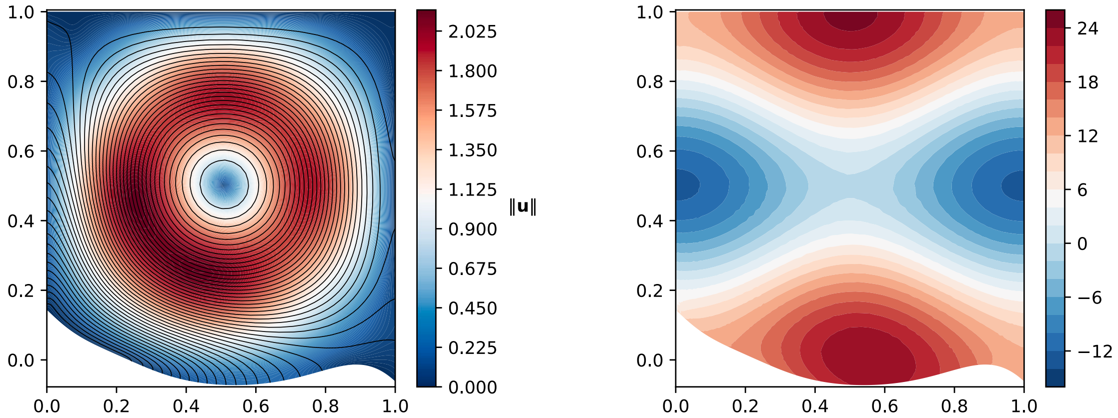


**Fig. 5.** Streamlines/velocity (left) and pressure (right) in optimized domain ($\varepsilon = 10^{-5}$).

*Example 2*

In this example we minimize the objective function $J_2(\alpha) = \frac{1}{2}a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{u}(\alpha))$. Let the state problem be defined by the data $\mu = \frac{1}{2}$, $g = 15, \kappa = 0$, $\boldsymbol{\sigma}_N = \mathbf{0}$, and

$$\boldsymbol{f}(x) = \begin{cases} (0,\ [(x_2 - 0.2)(1 - x_2)]^2) & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \times [\frac{2}{10}, 1] \\ (0,\ 0) & \text{otherwise.} \end{cases}$$

The parameters defining $\mathcal{U}$ are $m = 40$, $\alpha_{\min} = -0.1$, $\alpha_{\max} = 0.1$, $C_1 = 5$, and $C_2 = 10$. As the objective function contains a domain integral (that can be reduced simply by shrinking the area) we add an additional area constraint meas $\Omega(\alpha) = 1$ to the problem, i.e.

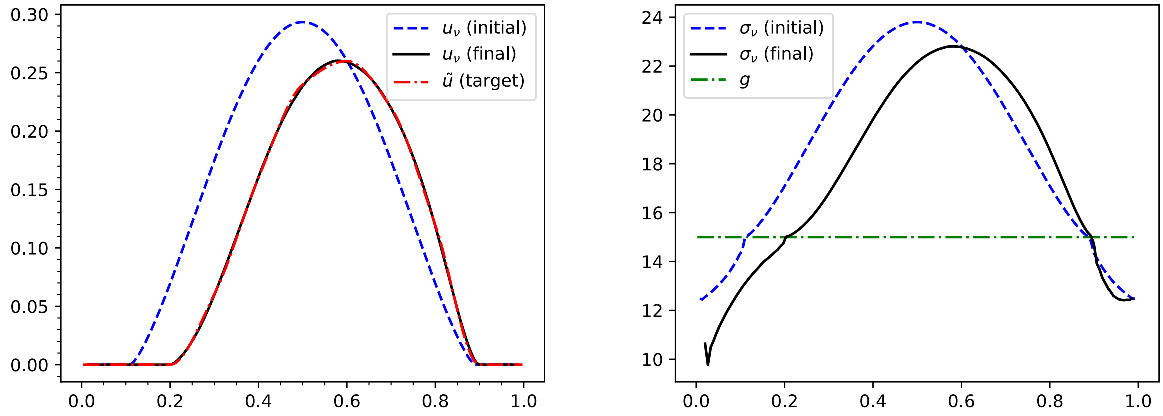


**Fig. 6.** Normal velocity $u_\nu$ and normal stress $\sigma_\nu$ ($\varepsilon = 10^{-5}$).

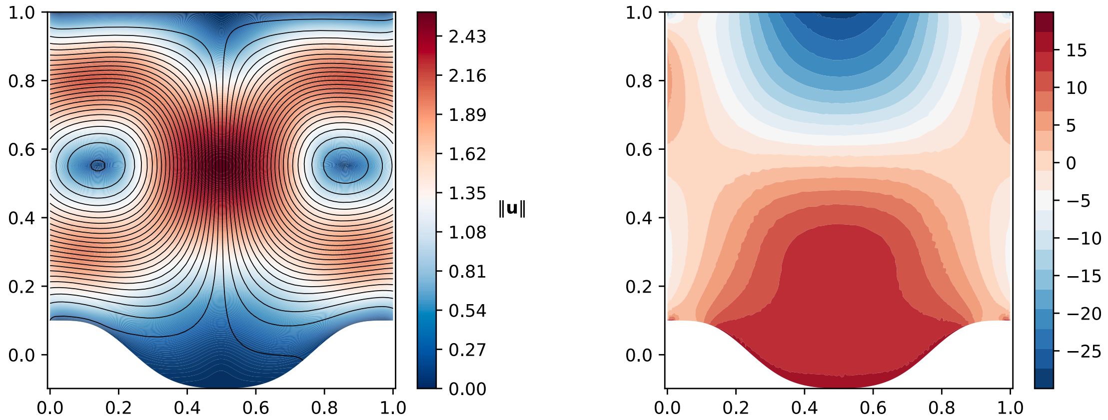


**Fig. 7.** Streamlines/velocity (left) and pressure (right) in optimized domain.

we minimize over the set $\mathcal{U}_{ad}^{\dagger} := \mathcal{U}_{ad} \cap \{\alpha \mid \int_0^1 \alpha(x_1)\,dx_1 = 0\}$. This added constraint does not cause any significant problems in the theoretical or numerical analysis.

We solved the discretized shape optimization problem using the same reference mesh as in the previous example. The value of the regularization parameter was $\varepsilon = 10^{-4}$ and $\alpha \equiv 0$ was again used as the initial guess.

The objective function value corresponding to the initial guess was 26.14. After two iterations (and five function evaluations) the objective value was reduced to 23.16. The low iteration count might be due to the bang–bang like nature of the optimized shape, i.e. many constraints become immediately active. The velocity and pressure contours in the optimized domain are depicted in Fig. 7. The normal velocity component $u_\nu$ and the normal stress $\sigma_\nu$ on $S(\alpha_m^*)$ corresponding the initial and optimized domain are shown in Fig. 8. Examining the initial and final normal stress distributions, it can be observed that the minimization of $J_2$ tends to produce a more uniform stress distribution. This kind of behavior appears also in shape optimization of an elastic body governed by the Signorini state problem (see [14, Section 3.3]) and is a consequence of the constant volume constraint in the definition of $\mathcal{U}_{ad}^{\dagger}$.

## 7. Conclusions

In the present paper we have considered shape optimization with the state constraint given by the Stokes system with the threshold leak boundary conditions on a part of the computational domain. In numerical realization, the part of boundary to be optimized is parametrized using a Bézier function. The state problem is discretized by stable finite elements of the lowest order. The leak boundary condition is realized approximately using a combination of the penalty method and smoothing of the nondifferentiable leak term. The numerical examples demonstrate the computational feasibility of our approach.

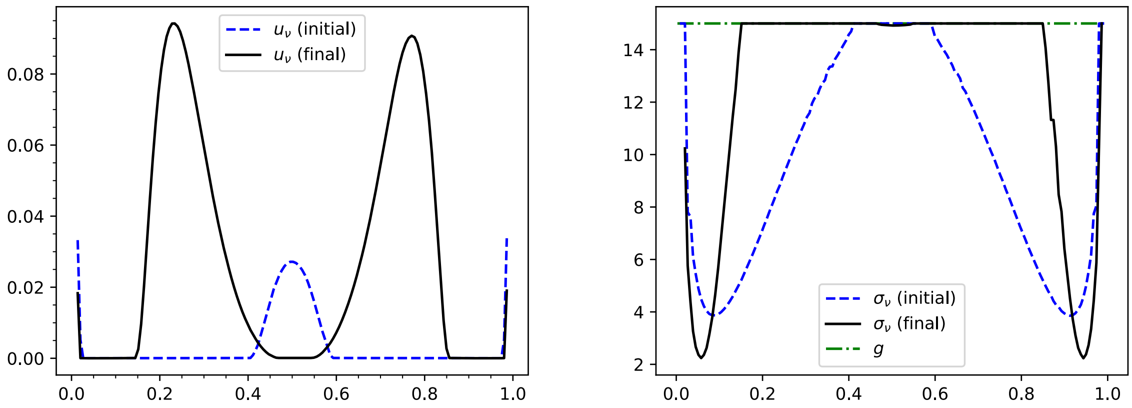


**Fig. 8.** Normal velocity $u_\nu$ and normal stress $\sigma_\nu$.

## Appendix

The aim of this part is to justify (3.11) and (3.12) in Theorem 3.1 which plays the key role in the existence analysis. Roughly speaking, we want to prove that any function from the function space on the limit domain which is used in the weak formulation can be approximated by functions from the same type of spaces on close domains.

Before we start, let us recall some notation which will be used in the sequel. If $\boldsymbol{\varphi}, \boldsymbol{\xi} : Q \mapsto \mathbb{R}^2$, $Q \subseteq \mathbb{R}^d$, $d = 1, 2$, are two vector functions, then

$$\|\boldsymbol{\varphi}\| := \|\boldsymbol{\varphi}\|_{\infty,Q} = \operatorname*{ess\,sup}_{x\in Q} \|\boldsymbol{\varphi}(x)\|,$$

where $\|\boldsymbol{\varphi}(x)\|$ denotes the Euclidian norm of $\boldsymbol{\varphi}(x) \in \mathbb{R}^2$ and

$$\boldsymbol{\varphi} \cdot \boldsymbol{\xi} : Q \to \mathbb{R}^1, \quad (\boldsymbol{\varphi} \cdot \boldsymbol{\xi})(x) = \boldsymbol{\varphi}(x) \cdot \boldsymbol{\xi}(x) \quad \forall x \in Q.$$

The system of admissible domains is exactly the same as in Section 3. Unlike domains $\Omega(\alpha) \in \mathcal{O}$ considered in Section 3 the boundaries of which are decomposed into $\Gamma, \Gamma_N(\alpha), S(\alpha)$, boundaries of domains considered in this appendix are split into two parts: $\partial\Omega(\alpha) = \overline{\Gamma(\alpha)} \cup \overline{S(\alpha)}$, $S(\alpha) =$ graph of $\alpha$.

On any $\Omega(\alpha), \alpha \in \mathcal{U}_{ad}$ we define the space

$$\mathbb{V}(\alpha) = \{\boldsymbol{v} \in (H^1(\Omega(\alpha)))^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma(\alpha),\ \boldsymbol{v} \cdot \boldsymbol{s}^\alpha = 0 \text{ on } S(\alpha)\},$$

where $\boldsymbol{s}^\alpha : [0,1] \to \mathbb{R}^2$ is a given unit vector field defined on $S(\alpha)$:

$$\left.\begin{aligned} &\boldsymbol{s}^\alpha(x_1) := \boldsymbol{s}^\alpha(x_1, \alpha(x_1)) \\ &\|\boldsymbol{s}^\alpha(x_1)\| = 1 \end{aligned}\right\} \quad \forall x_1 \in (0,1)\ \forall \alpha \in \mathcal{U}_{ad}.$$

Let $\boldsymbol{r}^\alpha : [0,1] \to \mathbb{R}^2$ be another unit vector field on $S(\alpha)$ which is perpendicular to $\boldsymbol{s}^\alpha$ at any point of $S(\alpha)$:

$$\left.\begin{aligned} &\boldsymbol{s}^\alpha \cdot \boldsymbol{r}^\alpha = 0 \\ &\|\boldsymbol{r}^\alpha(x_1)\| = 1 \end{aligned}\right\} \quad \forall x_1 \in (0,1)\ \forall \alpha \in \mathcal{U}_{ad}. \tag{A.1}$$

Both these vector fields will be extended from $S(\alpha)$ on the hold-all domain $\hat{\Omega} = (0,1) \times (-\gamma, \gamma)$ as follows:

$$\boldsymbol{s}^\alpha(x_1, x_2) = \boldsymbol{s}^\alpha(x_1),\ \boldsymbol{r}^\alpha(x_1, x_2) = \boldsymbol{r}^\alpha(x_1) \quad \forall x = (x_1, x_2) \in \hat{\Omega}. \tag{A.2}$$

Convention: *from now on the symbols $\boldsymbol{s}^\alpha$ and $\boldsymbol{r}^\alpha$ will denote the vector fields defined by (A.2) in the whole $\hat{\Omega}$.*

Since the pair $(\boldsymbol{s}^\alpha, \boldsymbol{r}^\alpha)$ is the orthonormal basis at any $x \in \hat{\Omega}$ as follows from (A.1), any function $\boldsymbol{v} : \hat{\Omega} \to \mathbb{R}^2$ can be written in the form

$$\boldsymbol{v} = \boldsymbol{v}{\cdot}\boldsymbol{s}^\alpha \boldsymbol{s}^\alpha + \boldsymbol{v}_{\boldsymbol{r}^\alpha} = \boldsymbol{v}{\cdot}\boldsymbol{s}^\alpha \boldsymbol{s}^\alpha + \boldsymbol{v}{\cdot}\boldsymbol{r}^\alpha \boldsymbol{r}^\alpha \quad \text{in } \hat{\Omega}. \tag{A.3}$$

In the next theorem we shall need the following assumptions imposed on $\boldsymbol{s}^\alpha$ and $\boldsymbol{r}^\alpha$:

- $\boldsymbol{s}^\alpha, \boldsymbol{r}^\alpha \in (C^{0,1}(\overline{\hat{\Omega}}))^2 \quad \forall \alpha \in \mathcal{U}_{ad}$, (A.4)
- $\exists C_3 = \text{const.} > 0 : \ \|\nabla \boldsymbol{s}^\alpha\|_{\infty,\hat{\Omega}} \le C_3 \quad \forall \alpha \in \mathcal{U}_{ad}$, (A.5)
- $\alpha_n \to \alpha$ in $C^1([0,1])$, $\alpha_n, \alpha \in \mathcal{U}_{ad} \implies \boldsymbol{s}^{\alpha_n} \to \boldsymbol{s}^\alpha$ in $(C(\overline{\hat{\Omega}}))^2$. (A.6)

**Theorem A.1.** *Let (A.4)–(A.6) be satisfied and $\alpha_n, \alpha \in \mathcal{U}_{ad}$ be such that $\alpha_n \to \alpha$ in $C^1([0,1])$. Then for any $\boldsymbol{v} \in \mathbb{V}(\alpha)$ there exists a sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$ and a function $\overline{\boldsymbol{v}} \in (H^1(\widehat{\Omega}))^2$ such that $\overline{\boldsymbol{v}}_{|\Omega(\alpha)} = \boldsymbol{v}$ and*

$$\boldsymbol{v}_k \to \overline{\boldsymbol{v}} \quad \text{in } (H^1(\widehat{\Omega}))^2,\ k \to \infty. \tag{A.7}$$

*In addition, for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that*

$$\boldsymbol{v}_k|_{\Omega(\alpha_{n_k})} \in \mathbb{V}(\alpha_{n_k}). \tag{A.8}$$

**Proof.** Let $\boldsymbol{v} \in \mathbb{V}(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ be fixed and $\alpha_n \to \alpha$ in $C^1([0,1])$, as $n \to \infty$. We denote

$$\varphi := \boldsymbol{v} \cdot \boldsymbol{s}^\alpha, \quad \boldsymbol{\psi} := \boldsymbol{v}_{\boldsymbol{r}^\alpha} = \boldsymbol{v} \cdot \boldsymbol{r}^\alpha \boldsymbol{r}^\alpha \quad \text{in } \Omega(\alpha). \tag{A.9}$$

From the definition of $\mathbb{V}(\alpha)$ it follows that $\varphi \in H_0^1(\Omega(\alpha))$ and $\boldsymbol{\psi} \in (H^1(\Omega(\alpha)))^2$, $\boldsymbol{\psi} = \boldsymbol{0}$ on $\Gamma(\alpha)$. Using the density arguments we know that there exist sequences $\{\varphi_k\}$, $\varphi_k \in C_0^\infty(\Omega(\alpha))$ and $\{\boldsymbol{\psi}_k\}$, $\boldsymbol{\psi}_k \in (C^\infty(\overline{\Omega(\alpha)}))^2$ such that $\mathrm{dist}(\mathrm{supp}\,\boldsymbol{\psi}_k, \Gamma(\alpha)) > 0$ for all $k \in \mathbb{N}$ and

$$\begin{cases} \varphi_k \to \varphi & \text{in } H_0^1(\Omega(\alpha)), \\ \boldsymbol{\psi}_k \to \boldsymbol{\psi} & \text{in } (H^1(\Omega(\alpha)))^2. \end{cases} \tag{A.10}$$

Therefore[3]

$$\begin{cases} \hat{\varphi}_k \to \hat{\varphi} & \text{in } H_0^1(\widehat{\Omega}), \\ \hat{\boldsymbol{\psi}}_k \to \hat{\boldsymbol{\psi}} & \text{in } (H^1(\widehat{\Omega}))^2. \end{cases} \tag{A.11}$$

Moreover we may suppose that $\mathrm{dist}(\mathrm{supp}\,\hat{\boldsymbol{\psi}}_k, \hat{\Gamma}) > 0\ \forall k \in \mathbb{N}$, where

$$\hat{\Gamma} = \{0\} \times (-\gamma, \gamma) \cup \{1\} \times (-\gamma, \gamma).$$

To construct the sequence $\{\boldsymbol{v}_k\}$ which satisfies (A.7) and (A.8), let us suppose for the moment that for any $k \in \mathbb{N}$ there exist: $n_k \in \mathbb{N}$ such that

- $$S(\alpha_n) \cap \mathrm{supp}\hat{\varphi}_k = \emptyset \quad \forall n \ge n_k \tag{A.12}$$

and a function $\mathbf{N}_{n_k} \in (C^{0,1}(\overline{\widehat{\Omega}}))^2$ satisfying:

- $$\mathbf{N}_{n_k}|_{S(\alpha_{n_k})} = \boldsymbol{s}^{\alpha_{n_k}}|_{S(\alpha_{n_k})}, \tag{A.13}$$
- $$\mathbf{N}_{n_k} \to \boldsymbol{s}^\alpha \quad \text{in } (H^1(\widehat{\Omega}))^2 \text{ as } k \to \infty, \tag{A.14}$$
- $$\exists C_4 = \text{const.} > 0 : \ \|\mathbf{N}_{n_k}\|_{\infty,\widehat{\Omega}} + \|\nabla \mathbf{N}_{n_k}\|_{\infty,\widehat{\Omega}} \le C_4 \quad \forall k \in \mathbb{N}. \tag{A.15}$$

Then the sequence $\{\boldsymbol{v}_k\}$ is defined as follows:

$$\boldsymbol{v}_k = \hat{\varphi}_k \mathbf{N}_{n_k} + \hat{\boldsymbol{\psi}}_k - \hat{\boldsymbol{\psi}}_k \cdot \mathbf{N}_{n_k} \mathbf{N}_{n_k}. \tag{A.16}$$

Clearly $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$, $\boldsymbol{v}_k = \boldsymbol{0}$ on $\Gamma(\alpha_{n_k})$ and

$$\begin{aligned} &(\boldsymbol{v}_k \cdot \boldsymbol{s}^{\alpha_{n_k}})|_{S(\alpha_{n_k})} = (\hat{\varphi}_k \mathbf{N}_{n_k} \cdot \boldsymbol{s}^{\alpha_{n_k}})|_{S(\alpha_{n_k})} + (\hat{\boldsymbol{\psi}}_k \cdot \boldsymbol{s}^{\alpha_{n_k}})|_{S(\alpha_{n_k})} \\ &-(\hat{\boldsymbol{\psi}}_k \cdot \mathbf{N}_{n_k})|_{S(\alpha_{n_k})} (\mathbf{N}_{n_k} \cdot \boldsymbol{s}^{\alpha_{n_k}})|_{S(\alpha_{n_k})} \\ &= (\hat{\boldsymbol{\psi}}_k \cdot \boldsymbol{s}^{\alpha_{n_k}})|_{S(\alpha_{n_k})} - (\hat{\boldsymbol{\psi}}_k \cdot \mathbf{N}_{n_k})|_{S(\alpha_{n_k})} = 0 \end{aligned}$$

making use of (A.12) and (A.13). From (A.16), (A.11), (A.14), and (A.15) we see that

$$\boldsymbol{v}_k \underset{k\to\infty}{\longrightarrow} \hat{\varphi} \boldsymbol{s}^\alpha + \hat{\boldsymbol{\psi}} - \hat{\boldsymbol{\psi}} \cdot \boldsymbol{s}^\alpha \boldsymbol{s}^\alpha =: \overline{\boldsymbol{v}} \quad \text{in } (H^1(\widehat{\Omega}))^2.$$

Finally from (A.3) and (A.9) it follows that $\overline{\boldsymbol{v}}_{|\Omega(\alpha)} = \boldsymbol{v}$.

It remains to construct the functions $\mathbf{N}_{n_k}$ and the sequence $\{n_k\}$, $k \to \infty$ satisfying (A.13)–(A.15). Let $\xi_k \in C^\infty([0,\infty))$, $k \to \infty$ be functions such that $0 \le \xi_k \le 1$ in $[0,\infty)$, $\xi_k|_{[0,1/(2k)]} = 1$, $\xi_k|_{[1/k,\infty)} = 0\ \forall k \in \mathbb{N}$. For any $k, n \in \mathbb{N}$ we define

$$\mathbf{N}_{n,k}(x) = \xi_k(|x_2 - \alpha(x_1)|)(\boldsymbol{s}^{\alpha_n} - \boldsymbol{s}^\alpha) + \boldsymbol{s}^\alpha \quad \text{in } \widehat{\Omega}. \tag{A.17}$$

It is readily seen that $\mathbf{N}_{n_k} \in (C^{0,1}(\overline{\widehat{\Omega}}))^2$. Further

$$\|\mathbf{N}_{n,k}\|_{\infty,\widehat{\Omega}} \le 3 \quad \forall k, n \in \mathbb{N} \tag{A.18}$$

and from (A.17) and (A.6)

$$\|\mathbf{N}_{n,k} - \boldsymbol{s}^\alpha\|_{0,\widehat{\Omega}} \le \|\boldsymbol{s}^{\alpha_n} - \boldsymbol{s}^\alpha\|_{0,\widehat{\Omega}} \to 0, \quad n \to \infty \tag{A.19}$$

[3] Let us observe that for functions from $H_0^1(\Omega(\alpha))$ the symbol " ^ " above the function means its extension by zero from $\Omega(\alpha)$ on $\widehat{\Omega}$.

uniformly with respect to $k$.

Let $k \in \mathbb{N}$ be fixed. Since $\alpha_n \to \alpha$ in $C^1([0,1])$, there exists an index $n_0 := n_0(k)$ such that (A.12) holds for any $n \geq n_0$ and so $\mathbf{N}_{n,k}|_{S(\alpha_n)} = \boldsymbol{s}^{\alpha_n}|_{S(\alpha_n)}$ $\forall n \geq n_0$, i.e. (A.13) is satisfied. To estimate $\|\nabla \mathbf{N}_{n,k}\|_{\infty,\hat{\Omega}}$ we only need to estimate the term

$$\max_{x\in\hat{\Omega}}\left(\left\|\nabla\xi_k(|x_2-\alpha(x_1)|)\right\|\right)\|\boldsymbol{s}^{\alpha_n}-\boldsymbol{s}^{\alpha}\|_{\infty,\hat{\Omega}}\|\boldsymbol{s}^{\alpha}\|_{\infty,\hat{\Omega}}. \tag{A.20}$$

Since $\xi_k'$ is unbounded on $[1/(2k), 1/k]$ as $k \to \infty$, one has to compensate this fact by (A.6). Thus for $k$ fixed, there exists $n_1 := n_1(k)$ such that the expression (A.20) is bounded by (say) 2 for any $n \geq n_1$. The remaining terms appearing in $\nabla \mathbf{N}_{n,k}$ are uniformly bounded due to (A.5). This, together with (A.18) proves (A.15).

To verify (A.14) it remains to estimate $\|\nabla(\mathbf{N}_{n,k} - \boldsymbol{s}^{\alpha})\|_{0,\hat{\Omega}}$. From (A.17) and the definition of $\xi_k$ is follows:

$$\begin{aligned}
&\|\nabla(\mathbf{N}_{n,k}-\boldsymbol{s}^{\alpha})\|_{0,\hat{\Omega}} \leq \\
&\max_{x\in\hat{\Omega}}\|\nabla\xi_k(|x_2-\alpha(x_1)|)\|\|\boldsymbol{s}^{\alpha_n}-\boldsymbol{s}^{\alpha}\|_{0,\hat{\Omega}}+\|\nabla(\boldsymbol{s}^{\alpha_n}-\boldsymbol{s}^{\alpha})\|_{0,\{|x_2-\alpha(x_1)|<1/k\}}\\
&\leq \sqrt{1+C_1^2}\,\|\xi_k'\|_{\infty,[0,\infty)}\|\boldsymbol{s}^{\alpha_n}-\boldsymbol{s}^{\alpha}\|_{0,\hat{\Omega}}+\mathcal{O}(1/k),
\end{aligned}$$

where $C_1, C_3$ are the constants from (3.1) and (A.5). Then we proceed in the same way as in the estimation of (A.20). One can find $n_2 := n_2(k) \in \mathbb{N}$ such that $\|\nabla(\mathbf{N}_{n,k} - \boldsymbol{s}^{\alpha})\|_{0,\hat{\Omega}} = \mathcal{O}(1/k)$ for any $n \geq n_2$. This, together with (A.19) proves (A.14). The function $\mathbf{N}_{n_k} := \mathbf{N}_{n_k,k}$ having the required properties is defined by (A.17) with $n_k = \max\{n_0, n_1, n_2\}$. □

**Remark A.1.** It is easy to show that the assertion of Theorem A.1 remains valid also for the space $\mathbb{V}(\Omega(\alpha))$, $\Omega(\alpha) \in \mathcal{O}$ introduced in Section 3 when $\Gamma_N \neq \emptyset$.

**Remark A.2.** In the previous part of the paper we use Theorem A.1 with $\boldsymbol{s}^{\alpha} := \boldsymbol{\tau}^{\alpha}$, and $\boldsymbol{r}^{\alpha} := \boldsymbol{\nu}^{\alpha}$, where $\boldsymbol{\tau}^{\alpha}$, $\boldsymbol{\nu}^{\alpha}$ are the unit tangential, and outward normal vectors at points of $S(\alpha)$, $\alpha \in \mathcal{U}_{ad}$, respectively. Since

$$\boldsymbol{\tau}^{\alpha} = \left(\frac{1}{\sqrt{1+(\alpha')^2}}, \frac{\alpha'}{\sqrt{1+(\alpha')^2}}\right), \quad \boldsymbol{\nu}^{\alpha} = \left(\frac{\alpha'}{\sqrt{1+(\alpha')^2}}, \frac{-1}{\sqrt{1+(\alpha')^2}}\right),$$

it is effortless to show that (A.4)–(A.6) are satisfied. This justifies the use of Theorem A.1 to this particular choice of $\boldsymbol{s}^{\alpha}$, $\boldsymbol{r}^{\alpha}$.

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