**Research Article**

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# Curvature exponent and geodesic dimension on Sard-regular Carnot groups

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**Abstract:** In this study, we characterize the geodesic dimension $N_{\mathrm{GEO}}$ and give a new lower bound to the curvature exponent $N_{\mathrm{CE}}$ on Sard-regular Carnot groups. As an application, we give an example of step-two Carnot group on which $N_{\mathrm{CE}} > N_{\mathrm{GEO}}$; this answers a question posed by Rizzi (*Measure contraction properties of Carnot groups*. Calc. Var. Partial Differential Equations 55 (2016), no. 3, Art. 60, 20).

**Keywords:** Carnot groups; curvature exponent; geodesic dimension; sub-Riemannian geometry

**MSC 2020:** 53C17; 53C23

## 1 Introduction

### 1.1 Geodesic dimension and curvature exponent

In a geodesic metric measure space $(G, d, \mathrm{vol})$, for a point $p \in G$, a set $E \subset G$, and $t \in [0, 1]$, define the *set of t-intermediate points*

$$Z_t(p, E) \coloneqq \{z \in G : \exists q \in E : d(p, z) = td(p, q), d(z, q) = (1 - t)d(p, q)\}. \tag{1.1}$$

Although our definition of set of intermediate points is not the same as in [1,21], we will clarify in Remark 3.5 that they are interchangeable in our study of geodesic dimension and curvature exponent.

We are interested in the behaviour of $t \mapsto \mathrm{vol}(Z_t(p, E))$ for $E$ measurable with $0 < \mathrm{vol}(E) < \infty$. In particular, we have two characteristic exponents. First, the *geodesic dimension at p* is

$$N_{\mathrm{GEO}}(p) \coloneqq \inf \left\{N > 0 : \sup_{E \in \mathscr{F}} \limsup_{t \to 0} \frac{\mathrm{vol}(Z_t(p, E))}{t^N \mathrm{vol}(E)} = \infty\right\},$$

where $\mathscr{F} \coloneqq \{E \subset G \text{ bounded, measurable with } 0 < \mathrm{vol}(E) < \infty\}$.

Second, the *curvature exponent at p* is

$$N_{\mathrm{CE}}(p) \coloneqq \inf\{N > 0 : \mathrm{vol}(Z_t(p, E)) \geq t^N \mathrm{vol}(E), \forall t \in [0, 1], \quad \forall E \in \mathscr{F}\}.$$

The geodesic dimension was originally introduced in [1]. The curvature exponent was originally introduced in [19]. See also [21]. Note that we have $N_{\mathrm{CE}}(p) \geq N_{\mathrm{GEO}}(p)$ by definition.

If $G$ is a Lie group, $d$ is left-invariant, and vol is a Haar measure, the choice of $p$ does not play any role and thus, we can focus on $p = e$, the identity element of $G$. Consequently, we have the *geodesic dimension* $N_{\mathrm{GEO}}$ of $G$ and the *curvature exponent* $N_{\mathrm{CE}}$ of $G$.

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## 1.2 Sard-regular Carnot group

We will give estimates for the geodesic dimension and the curvature exponent of Carnot groups. Let $G$ be a Carnot group with stratified Lie algebra $\mathfrak{g} = \oplus_{j=1}^{s} V_j$ and a fixed scalar product $\langle\cdot,\cdot\rangle$ on $V_1$. See Section 3 for details.

The sub-Riemannian exponential map (based at the identity element $e$) SExp is an analytic function from $T_e^*G = \mathfrak{g}^*$ to $G$, where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$. We denote by Jac(SExp) the Jacobian determinant of SExp. For more details, see Sections 2 and 3.

**Definition 1.1.** Define $\mathscr{D} \subset \mathfrak{g}^*$ as the open set of all $\xi \in \mathfrak{g}^*$ such that $\mathrm{Jac}(\mathrm{SExp})(t\xi) \neq 0$ for all $t \in (0,1]$ and $t \mapsto \mathrm{SExp}(t\xi)$ is the unique constant-speed length-minimizing curve $[0,1] \to G$ from $e$ to $\mathrm{SExp}(\xi)$.

Next define $\mathcal{S}_e \subset G$ as the image $\mathrm{SExp}(\mathscr{D})$. Note that SExp is a diffeomorphism from $\mathscr{D}$ to $\mathcal{S}_e$. We denote by $\rho : \mathcal{S}_e \to \mathscr{D}$, the inverse of SExp on $\mathcal{S}_e$.

It is well known that $\mathcal{S}_e$ is dense in $G$, see [3] and references therein. However, it is not known whether $\mathcal{S}_e$ has always full measure in $G$, see for instance [14]. In our study, we need $\mathcal{S}_e$ to have full measure.

**Definition 1.2.** (Sard-regular) We say that a Carnot group $G$ is *Sard-regular* if the set $\mathcal{S}_e$ has full measure in $G$.

Note that Carnot groups of step two are Sard-regular by [19, Proposition 15]. We stress that it is an open question whether all Carnot groups are Sard-regular, i.e., this hypothesis might be superfluous.

## 1.3 Sub-Riemannian exponential map

We will use the fact that the sub-Riemannian exponential map is analytic to estimate both $N_{\mathrm{GEO}}$ and $N_{\mathrm{CE}}$.

The stratification $\mathfrak{g} = \oplus_{j=1}^{s} V_j$ of the Lie algebra of $G$ induces a splitting $\mathfrak{g}^* = \oplus_{j=1}^{s} V_j^*$ of the dual space $\mathfrak{g}^*$, where $V_j^* = \{\alpha \in \mathfrak{g}^* : V_i \subset \ker\alpha, \quad \forall i \neq j\}$.

For $\lambda \in \mathbb{R}$, define $\zeta_\lambda : \mathfrak{g}^* \to \mathfrak{g}^*$ as

$$\zeta_\lambda\left(\sum_{j=1}^{s} \xi_j\right) = \sum_{j=1}^{s} \lambda^{j-1}\xi_j.$$

Note that, when $\lambda \to 0$, we have $\zeta_\lambda(\xi) \to \xi_1 \in V_1^*$. By insight to the Hamiltonian system, we obtain in Section 3.2 that the Jacobian determinant of the sub-Riemannian exponential map satisfies

$$\mathrm{Jac}(\mathrm{SExp})(\lambda\xi) = \lambda^{2Q-2n}\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi)). \tag{1.2}$$

Here $n = \dim(G)$ and $Q = \sum_{j=1}^{s} j \dim(V_j)$ are the topological and homogeneous dimensions of $G$, respectively. In Proposition 3.8, we prove that, if $G$ is Sard-regular, then, for every measurable $E \subset G$,

$$\mathrm{vol}(Z_\lambda(e,E)) = \lambda^{2Q-n} \int_{\rho(E\cap\mathcal{S}_e)} |\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi))| d\xi. \tag{1.3}$$

Formula (1.3) is crucial for our estimates of $N_{\mathrm{GEO}}$ and $N_{\mathrm{CE}}$.

By analyticity of the sub-Riemannian exponential map, there are analytic functions $P_k : \mathfrak{g}^* \to \mathbb{R}$ such that for every $\xi \in \mathfrak{g}^*$, there exists $\lambda_\xi > 0$ with

$$\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi)) = \sum_{k=0}^{\infty} P_k(\xi)\lambda^k, \tag{1.4}$$

for $|\lambda| < \lambda_\xi$. One can take $\xi \mapsto \lambda_\xi$ continuous. Define

$$\begin{aligned}\Gamma(\xi) &\coloneqq \min\{k : P_k(\xi) \neq 0\},\\ \Gamma(G) &\coloneqq \min\{\Gamma(\xi) : \xi \in \mathfrak{g}^*\}, \quad \text{and}\\ \hat{\Gamma}(G) &\coloneqq \sup\{\Gamma(\xi) : \xi \in \mathfrak{g}^* \quad \text{with } \Gamma(\xi) < \infty\}.\end{aligned} \tag{1.5}$$

## 1.4 Main results

Our main results are the following two Theorems A and B, which we then summarize in Theorem C.

**Theorem A.** *If $G$ is a Sard-regular Carnot group, then*

$$N_{\mathrm{GEO}} = 2Q - n + \Gamma(G),$$

*where $n = \dim(G)$ is the topological dimension, $Q = \sum_{j=1}^{s} j \dim(V_j)$ is the homogeneous dimension, and $\Gamma(G)$ is defined in* (1.5).

See Section 4 for the proof of Theorem A.

**Theorem B.** *In a Sard-regular Carnot group, we have*

$$2Q - n + \hat{\Gamma}(G) \leq N_{\mathrm{CE}},$$

*where $n = \dim(G)$ is the topological dimension, $Q = \sum_{j=1}^{s} j \dim(V_j)$ is the homogeneous dimension, while $\hat{\Gamma}(G)$ is defined in* (1.5).

See Section 5 for a proof of Theorem B.

We summarize the results of Theorems A and B in the following statement:

**Theorem C.** *In a Sard-regular Carnot group, we have*

$$n \leq Q \leq N_{\mathrm{GEO}} = 2Q - n + \Gamma(G) \leq 2Q - n + \hat{\Gamma}(G) \leq N_{\mathrm{CE}},$$

*where $n = \dim(G)$ is the topological dimension, $Q = \sum_{j=1}^{s} j \dim(V_j)$ is the homogeneous dimension, while $\Gamma(G)$ and $\hat{\Gamma}(G)$ are defined in* (1.5).

It is known that $N_{\mathrm{CE}}$ is finite on ideal Carnot groups by Rifford [19] and by Barilari and Rizzi [7]. Then, it was generalized to the class of so-called Lipschitz Carnot groups, which includes step-two Carnot groups [4]. The fact that $Q \leq N_{\mathrm{GEO}} \leq N_{\mathrm{CE}}$ was already known, see [21] or [1, Proposition 5.49]. To our best knowledge, all known examples of sub-Riemannian Carnot groups satisfy $N_{\mathrm{GEO}} = N_{\mathrm{CE}}$. In particular, Juillet [11] showed that $N_{\mathrm{CE}} = N_{\mathrm{GEO}} = 2Q - n$ on the Heisenberg group $\mathbb{H}^n$. Later on, the equality $N_{\mathrm{GEO}} = N_{\mathrm{CE}}$ has been proven for all corank 1 Carnot groups in [21] and for generalized H-type groups in [6].

In Carnot groups of step two, we will give a constructive method to compute both $\Gamma(G)$ and $\hat{\Gamma}(G)$. As a consequence, we will provide examples of Carnot groups of step two where $\Gamma(G) < \hat{\Gamma}(G)$. In such cases, we have $N_{\mathrm{GEO}} < N_{\mathrm{CE}}$, which answers a question posed by Rizzi [21].

**Corollary 1.3.** *There are sub-Riemannian Carnot groups of step two where $N_{\mathrm{GEO}} < N_{\mathrm{CE}}$.*

Borza and Tashiro have recently given examples of sub-Finsler Carnot groups with $N_{\mathrm{GEO}} < N_{\mathrm{CE}}$ [8]. Similarly with what we do, they study asymptotic behaviours of the Jacobian of the sub-Riemannian, or sub-Finsler, exponential map. In their case, since they use $\ell^p$-norms instead of the $\ell^2$-norm we use, the value of $\Gamma(\xi)$ may be fractional.

**Remark 1.4.** After this study was completed, Rizzi informed us of the following results in [1]. In our framework of Sard-regular Carnot groups, it follows from [1, Lemma 6.27] that our $2Q + n + \Gamma(\lambda)$ in this study coincides with $\mathcal{N}_\lambda$ there, whose value by [1, Definition 5.44] is given by geodesic growth vector $\mathcal{G}_\lambda$. We refer to Remark 6.9 for more details about the value of $\mathcal{N}_\lambda$ on step-two Carnot groups. Furthermore, in [1, Definition 5.47], the geodesic dimension is actually defined by the minimum of those $\mathcal{N}_\lambda$, which is exactly our Theorem A. However, our Theorem B and Corollary 1.3 remain new.

### 1.5 Summary

In Section 2, we give a brief description of the Hamiltonian formalism that defines the sub-Riemannian exponential map. We then introduce Carnot groups in Section 3. Section 4 contains the proof of Theorem A, while Section 5 contains the proof of Theorem B. In Section 6, we study more closely Carnot groups of step two, and the sub-Riemannian exponential map thereof. Finally, in Section 7 we compute several explicit examples.

## 2 Hamiltonian systems on Lie groups

In this section, $G$ denotes a Lie group with Lie algebra $\mathfrak{g}$. For the sake of completeness, we will describe the standard construction of Hamiltonian systems on $G$ given by left-invariant Hamiltonians $H : T^*G \to \mathbb{R}$. We will then apply this formalism to sub-Riemannian Carnot groups in Section 3.

We identify $\mathfrak{g}$ with the tangent space $T_eG$ of $G$ at the identity element $e \in G$. For any function $v : U \to \mathfrak{g}$ on an open subset $U \subset G$, we define the vector field $\tilde{v} \in \Gamma(TU)$ on $U$ as

$$\tilde{v}(p) \coloneqq DL_p|_e[v] \in T_pG.$$

Similarly, if $\alpha : U \to \mathfrak{g}^*$, we define $\tilde{\alpha} \in \Gamma(T^*U)$ as

$$\tilde{\alpha}(p) = DL_{p^{-1}}|_p^*[\alpha] \in T_p^*G.$$

Note that the vector field $\tilde{v}$ is left-invariant if and only if the function $v$ is constant, and similarly $\tilde{\alpha}$ is left-invariant if and only if $\alpha$ is constant.

We will denote by $\langle\cdot|\cdot\rangle$ the pairing of a vector space with its dual, or, more generally, the pairing between linear maps and their domain. Scalar products are usually denoted by $\langle\cdot,\cdot\rangle$.

### 2.1 Differential forms

For a vector space $V$ and an open set $U \subset G$, we define

$$\Omega_L^k(U;\ V) \coloneqq C^\infty(U;\ \mathrm{Alt}^k(\mathfrak{g};\ V)),$$

where $\mathrm{Alt}^k(\mathfrak{g};\ V)$ is the space of $k$-multilinear alternating maps from $\mathfrak{g}$ to $V$. Elements in $\Omega_L^k(U;\ V)$ are identified with differential forms on $U$ as follows. Define $\mathrm{MC} : \Omega^k(U;\ V) \to \Omega_L^k(U;\ V)$ by

$$\langle \mathrm{MC}(\tilde{\alpha})(p)|v_1 \wedge \cdots \wedge v_k\rangle = \langle\tilde{\alpha}(p)|\tilde{v}_1(p) \wedge \cdots \wedge \tilde{v}_k(p)\rangle,$$

for $p \in U$, $\tilde{\alpha} \in \Omega^k(U;\ V)$, and $v_j \in \mathfrak{g}$. *Vice versa*, if $\alpha \in \Omega_L^k(U;\ V)$, we denote by $\tilde{\alpha}$ the only element of $\Omega^k(U;\ V)$ such that $\mathrm{MC}(\tilde{\alpha}) = \alpha$.

We use the map MC to push the exterior derivative from $\Omega^k(U;\ V)$ to $\Omega_L^k(U;\ V)$. We define $d : \Omega_L^k(U;\ V) \to \Omega_L^{k+1}(U;\ V)$ as $\mathrm{d}\alpha \coloneqq \mathrm{MC}(d\tilde{\alpha})$. Using standard formulas for the exterior differential, we obtain for $\alpha \in \Omega_L^k(U;\ V)$ and $v_0, \dots, v_k \in \mathfrak{g}$,

$$\begin{aligned}\langle \mathrm{d}\alpha | v_0 \wedge \cdots \wedge v_k\rangle = &\sum_{j=0}^{k} (-1)^j \tilde{v}_j \langle \alpha(\cdot) | v_0 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k\rangle \\ &+ \sum_{i<j}^{k} (-1)^{i+j} \langle \alpha | [v_i, v_j] \wedge v_0 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k\rangle.\end{aligned} \tag{2.1}$$

## 2.2 Cotangent bundle and Hamiltonian mechanics

The cotangent bundle $T^*G$ of a Lie group $G$ with Lie algebra $\mathfrak{g}$ has a (left-)canonical group structure as direct product $G \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is seen as abelian Lie group. More precisely, we make $T^*G$ into a Lie group isomorphic to $G \times \mathfrak{g}^*$ via a map $\Phi_L : G \times \mathfrak{g}^* \to T^*G$ defined by

$$\Phi_L(g, \alpha) := \tilde{\alpha}(g) = DL_{g^{-1}}\,|_g^*[\alpha].$$

This group structure allows us to use the notation from Section 2.1 for differential forms on $T^*G$.

On $T^*G$, we have the tautological 1-form $\tau \in \Omega^1(T^*G)$,

$$\tau(\xi)[w] = \langle \xi | D\pi_{T^*G} w\rangle, \quad \text{for } \xi \in T^*G \quad \text{and} \quad w \in T_\xi(T^*G),$$

where $\pi_{T^*G} : T^*G \to G$ is the bundle projection. We pull back $\tau$ to $G \times \mathfrak{g}^*$ via $\Phi_L$ and we take its left version $\tau_L$. In other words, we define $\tau_L \in \Omega_L^1(G \times \mathfrak{g}^*; \mathbb{R})$ as

$$\tau_L = \mathrm{MC}(\Phi_L^* \tau).$$

It might look abstract and complicated, but the point of this reasoning is to obtain the following formula right, i.e., we really want to be sure that we are dealing with the standard tautological form and later with the standard symplectic form. Indeed, the above formula and the definition of the exterior derivative on $\Omega_L^1(G \times \mathfrak{g}^*; \mathbb{R})$ imply

$$\omega_L := -\mathrm{d}\tau_L = -\mathrm{MC}(\Phi_L^* \mathrm{d}\tau).$$

A short computation gives us, for all $(g, \alpha, v, \mu) \in G \times \mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^*$,

$$\langle \tau_L(g, \alpha) | (v, \mu)\rangle = \langle \alpha | v\rangle. \tag{2.2}$$

Indeed,

$$\begin{aligned}\langle \tau_L(g, \alpha) | (v, \mu)\rangle &= \left\langle \Phi_L^* \tau(g, \alpha) \middle| \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} (g \exp(\varepsilon v), \alpha + \varepsilon\mu) \right\rangle \\ &= \left\langle \tau(\tilde{\alpha}(g)) \middle| \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \Phi_L(g \exp(\varepsilon v), \alpha + \varepsilon\mu) \right\rangle \\ &= \left\langle \tilde{\alpha}(g) \middle| D\pi_{T^*G} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \Phi_L(g \exp(\varepsilon v), \alpha + \varepsilon\mu) \right\rangle \\ &= \left\langle \tilde{\alpha}(g) \middle| \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \pi_{T^*G} \Phi_L(g \exp(\varepsilon v), \alpha + \varepsilon\mu) \right\rangle \\ &= \left\langle \tilde{\alpha}(g) \middle| \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} g \exp(\varepsilon v) \right\rangle \\ &= \langle \tilde{\alpha}(g) | \tilde{v}(g)\rangle = \langle \alpha | v\rangle.\end{aligned}$$

Now, we can compute the symplectic form $\omega_L = -d\tau_L$ as

$$\langle \omega_L(g, \alpha) | (v_0, \mu_0) \wedge (v_1, \mu_1)\rangle = \langle \mu_1 | v_0\rangle - \langle \mu_0 | v_1\rangle + \langle \alpha | [v_0, v_1]\rangle. \tag{2.3}$$

Indeed, using (2.1), we easily compute

$$\begin{aligned}\langle\omega_L(g,\alpha)|(v_0,\mu_0)\wedge(v_1,\mu_1)\rangle &= -(v_0,\mu_0)^{\sim}\langle\tau_L(\cdot)|(v_1,\mu_1)\rangle + (v_1,\mu_1)^{\sim}\langle\tau_L(\cdot)|(v_0,\mu_0)\rangle + \langle\tau_L(g,\alpha)|[(v_0,\mu_0),(v_1,\mu_1)]\rangle\\ &= -\langle\mu_0|v_1\rangle + \langle\mu_1|v_0\rangle + \langle\alpha|[v_0,v_1]\rangle.\end{aligned}$$

If $H : G\times\mathfrak{g}^*\to\mathbb{R}$ is a smooth function, we define $X_H : G\times\mathfrak{g}^*\to\mathfrak{g}\times\mathfrak{g}^*$ by the formula

$$\langle\omega_L(g,\alpha)|X_H(g,\alpha)\wedge(v_1,\mu_1)\rangle = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} H(g\exp(\varepsilon v_1),\alpha+\varepsilon\mu_1), \tag{2.4}$$

which is required to hold for all $(g,\alpha,v_1,\mu_1)\in G\times\mathfrak{g}^*\times\mathfrak{g}\times\mathfrak{g}^*$. To compute $X_H$, we write $X_H=(v_H,\mu_H)$ with $v_H : G\times\mathfrak{g}^*\to\mathfrak{g}$ and $\mu_H : G\times\mathfrak{g}^*\to\mathfrak{g}^*$. By linearity, we obtain that (2.4) is equivalent to

$$\begin{cases}\langle\mu_1|v_H\rangle = \dfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} H(g,\alpha+\varepsilon\mu_1),\\ \langle\mu_H|v_1\rangle - \langle\alpha|[v_H,v_1]\rangle = -\dfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} H(g\exp(\varepsilon v_1),\alpha),\end{cases} \tag{2.5}$$

for all $(g,\alpha,v_1,\mu_1)\in G\times\mathfrak{g}^*\times\mathfrak{g}\times\mathfrak{g}^*$.

A solution to the *Hamiltonian equations* is a curve $t\mapsto(g(t),\alpha(t))$ such that

$$\begin{cases}DL_{g(t)}^{-1}\dot g(t) = v_H(g(t),\alpha(t)),\\ \dot\alpha(t) = \mu_H(g(t),\alpha(t)).\end{cases} \tag{2.6}$$

## 2.3 Sub-Riemannian Hamiltonian system

Let $V_1\subset\mathfrak{g}$ be a bracket-generating linear subspace of $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle$ be a scalar product on $V_1$. The scalar product on $V_1$ induces a scalar product $\langle\cdot,\cdot\rangle^*$ on the dual space $V_1^*$. We will use the standard notation $\alpha\mapsto\alpha^\sharp$ to denote the canonical isomorphism $V_1^*\to V_1$ induced by the scalar product, and its inverse $V_1\to V_1^*, v\mapsto v^\flat$. For example, by definition for every $\alpha,\beta\in V_1^*$ we have $\alpha^\sharp,\beta^\sharp\in V_1$ with

$$\langle\alpha,\beta\rangle^* = \langle\alpha|\beta^\sharp\rangle = \langle\alpha^\sharp,\beta^\sharp\rangle.$$

The Hamiltonian we are interested in is

$$H : G\times\mathfrak{g}^*\to\mathbb{R},\quad H(g,\alpha) = \frac{1}{2}\langle\alpha\,|_{V_1},\alpha\,|_{V_1}\rangle^*. \tag{2.7}$$

We have, for all $(g,\alpha,v,\mu)\in G\times\mathfrak{g}^*\times\mathfrak{g}\times\mathfrak{g}^*$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} H(g\exp(\varepsilon v),\alpha) = 0,\quad\text{and}\\ &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} H(g,\alpha+\varepsilon\mu) = \langle\alpha|_{V_1},\mu|_{V_1}\rangle^*.\end{aligned}$$

Thus, equations (2.5) defining the Hamiltonian vector field $X_H$ become

$$\begin{cases}\langle\mu|v_H\rangle = \langle\alpha|_{V_1},\mu|_{V_1}\rangle^*,\\ \langle\mu_H|v\rangle - \langle\alpha|[v_H,v]\rangle = 0.\end{cases}$$

Solving these equations in $v_H$ and $\mu_H$, we obtain that

$$\begin{cases}v_H(g,\alpha) = (\alpha|_{V_1})^\sharp\in V_1,\\ \mu_H(g,\alpha) = \alpha\circ\mathrm{ad}_{(\alpha|_{V_1})^\sharp}.\end{cases} \tag{2.8}$$

Therefore, the Hamiltonian flow is given by curves $(g(t),\alpha(t))$ solving (2.6), that is,

$$\begin{cases}DL_{g(t)}^{-1}\dot g(t) = (\alpha(t)|_{V_1})^\sharp,\\ \dot\alpha(t) = \alpha(t)\circ\mathrm{ad}_{(\alpha(t)|_{V_1})^\sharp}.\end{cases} \tag{2.9}$$

**Proposition 2.1.** *Let* $(g, \alpha) : I \to G \times \mathfrak{g}^*$ *be a solution to* (2.9) *with* $g(0) = e$*, the identity element of* $G$*. Then,* $\alpha(t)$ *is the restriction to the curve* $g$ *of a right-invariant 1-form. In other words, for all* $t \in I$,

$$\alpha(0) = \alpha(t) \circ \mathrm{Ad}_{g(t)^{-1}}.$$

**Proof.** We show that the derivative in $t$ of $\alpha(t)\mathrm{Ad}_{g(t)^{-1}}$ is zero. So, we first see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \alpha(t+\varepsilon)\mathrm{Ad}_{g(t+\varepsilon)^{-1}} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \alpha(t+\varepsilon)\mathrm{Ad}_{g(t)^{-1}} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \alpha(t)\mathrm{Ad}_{g(t+\varepsilon)^{-1}} \\ &= \alpha(t) \circ \mathrm{ad}_{(\alpha(t)|_{V_1})^\sharp}\mathrm{Ad}_{g(t)^{-1}} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \alpha(t)\mathrm{Ad}_{g(t+\varepsilon)^{-1}g(t)}\mathrm{Ad}_{g(t)^{-1}}.\end{aligned}$$

Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} g(t+\varepsilon)^{-1}g(t) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} (g(t)^{-1}g(t+\varepsilon))^{-1} \\ &= -\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} g(t)^{-1}g(t+\varepsilon) = -DL_{g(t)^{-1}}\dot{g}(t) = -(\alpha(t)|_{V_1})^\sharp,\end{aligned}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \alpha(t)\mathrm{Ad}_{g(t+\varepsilon)^{-1}g(t)}\mathrm{Ad}_{g(t)^{-1}} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \alpha(t)\mathrm{Ad}_{\exp(-\varepsilon(\alpha(t)|_{V_1})^\sharp)}\mathrm{Ad}_{g(t)^{-1}} = -\alpha(t)\mathrm{ad}_{(\alpha(t)|_{V_1})^\sharp}\mathrm{Ad}_{g(t)^{-1}}.$$

We conclude that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \alpha(t+\varepsilon)\mathrm{Ad}_{g(t+\varepsilon)^{-1}} = 0$, and thus

$$\alpha(t)\mathrm{Ad}_{g(t)^{-1}} = \alpha(0)\mathrm{Ad}_{g(0)^{-1}} = \alpha(0).$$

□

Proposition 2.1 gives a reinterpretation of the ordinary differential equation (ODE) (2.9). Indeed, given $\alpha_0 \in \mathfrak{g}^*$, we first take the right-invariant 1-form $\alpha(g) \coloneqq \mathrm{Ad}_g^*\alpha_0$, then we define the horizontal vector field $v_H(g) = (\alpha(g)|_{V_1})_\flat$, and finally we integrate the vector field $\tilde{v}_H$ starting from $e$. Explicitly, $\tilde{v}_H$ is

$$\tilde{v}_H(g) = DL_g\,|_e((\mathrm{Ad}_g^*\alpha_0)|_{V_1})^\sharp.$$

The ODE (2.9) has a few useful symmetries that we want to highlight.

**Lemma 2.2.** (Symmetries of the sub-Riemannian Hamiltonian flow: change of speed) *If* $(g, \alpha) : I \to G \times \mathfrak{g}^*$ *is a solution to* (2.9), *then* $t \mapsto (g(\lambda t), \lambda\alpha(\lambda t))$ *is also a solution to* (2.9), *for every* $\lambda > 0$.

**Proof.** Define $h(t) = g(\lambda t)$ and $\beta(t) = \lambda\alpha(\lambda t)$. Then,

$$DL_{h(t)}^{-1}\dot{h}(t) = \lambda DL_{g(\lambda t)}^{-1}\dot{g}(\lambda t) = \lambda(\alpha(\lambda t)|_{V_1})^\sharp = (\beta(t)|_{V_1})^\sharp$$

and

$$\dot{\beta}(t) = \lambda^2\dot{\alpha}(\lambda t) = \lambda^2\alpha(\lambda t) \circ \mathrm{ad}_{(\alpha(\lambda t)|_{V_1})^\sharp} = \beta(t) \circ \mathrm{ad}_{(\beta(t)|_{V_1})^\sharp}.$$

Therefore, $(h, \beta)$ is a solution to (2.9). □

**Lemma 2.3.** (Symmetries of the sub-Riemannian Hamiltonian flow: left translations) *If* $p \in G$ *and if* $(g, \alpha) : I \to G \times \mathfrak{g}^*$ *is a solution to* (2.9), *then*

$$t \mapsto (L_p(g(t)), \alpha(t))$$

*is also a solution to* (2.9).

**Proof.** The second equation in (2.9) does not depend on $g(t)$. In the first equation, we have

$$DL_{pg(t)}^{-1}(DL_p\dot{g}(t)) = DL_{g(t)}^{-1}\dot{g}(t),$$

and thus, the curve $(L_p(g(t)), \alpha(t))$ is still a solution to (2.9). □

**Lemma 2.4.** (Symmetries of the sub-Riemannian Hamiltonian flow: homotheties) *Let $L : G \to G$ be a Lie group automorphism with Lie algebra automorphism $\ell : \mathfrak{g} \to \mathfrak{g}$. Assume that $\ell(V_1) = V_1$ and that there exists $\lambda \in \mathbb{R}$ with $\langle \ell v, \ell w\rangle = \lambda^2\langle v, w\rangle$ for all $v, w \in V_1$.*

*If $(g, \alpha) : I \to G \times \mathfrak{g}^*$ is a solution to* (2.9)*, then*

$$t \mapsto (L(g(t)), \lambda^2\alpha(t) \circ \ell^{-1}) \tag{2.10}$$

*is also a solution to* (2.9).

In fact, the existence of a homothety like in Lemma 2.4 implies that the group is a Carnot group [13].

**Proof.** First of all, note that, if $\alpha \in V_1^*$, then for all $w \in V_1$, we have

$$\langle \ell\alpha^\sharp, w\rangle = \lambda^2\langle \alpha^\sharp, \ell^{-1}w\rangle = \lambda^2\langle \alpha|\ell^{-1}w\rangle = \langle \lambda^2\alpha \circ \ell^{-1}|w\rangle.$$

Therefore,

$$\ell\alpha^\sharp = \lambda^2(\alpha \circ \ell^{-1})^\sharp. \tag{2.11}$$

Next define $h(t) = L(g(t))$ and $\beta(t) = \lambda^2\alpha(t) \circ \ell^{-1}$. Then, using both (2.11) and (2.9), we have

$$DL^{-1}_{h(t)}\dot h(t) = \ell DL^{-1}_{g(t)}\dot g(t) = \ell(\alpha(t)|_{V_1})^\sharp = (\lambda^2\alpha(t)|_{V_1} \circ \ell^{-1})^\sharp = (\beta(t)|_{V_1})^\sharp.$$

Similarly,

$$\begin{aligned}\dot\beta(t) &= \lambda^2\dot\alpha(t) \circ \ell^{-1} = \lambda^2\alpha(t) \circ \mathrm{ad}_{(\alpha(t)|_{V_1})^\sharp} \circ \ell^{-1}\\ &= \lambda^2\alpha(t) \circ \ell^{-1} \circ \mathrm{ad}_{\ell(\alpha(t)|_{V_1})^\sharp}\\ &= \lambda^2\alpha(t) \circ \ell^{-1} \circ \mathrm{ad}_{(\lambda^2\alpha(t) \circ \ell^{-1}|_{V_1})^\sharp}\\ &= \beta(t) \circ \mathrm{ad}_{(\beta(t)|_{V_1})^\sharp}.\end{aligned}$$

Therefore, $(h, \beta)$ is a solution to (2.9). □

## 2.4 Sub-Riemannian exponential map

Note that the Hamiltonian vector field $\mathcal{X}_H$ defined in (2.4) is complete. Indeed, by Proposition 2.1, we only need to show that if $(g, \alpha) : (a, b) \to G \times \mathfrak{g}^*$ is an integral curve of $\mathcal{X}_H$, then $g : (a, b) \to G$ can be continuously extended to the closed interval $[a, b]$. We know that the curve $g$ is a length-minimizing curve parametrized by constant speed with respect to a left-invariant sub-Riemannian distance on $G$, see for instance [2,17]. Since such a distance is complete, the curve $g$ has a continuous extension to the closed interval $[a, b]$. It follows that $\mathcal{X}_H$ is a complete vector field.

**Definition 2.5.** Given a Lie group $G$ with Lie algebra $\mathfrak{g}$ and a bracket generating subspace $V_1 \subset \mathfrak{g}$ endowed with a scalar product $\langle\cdot,\cdot\rangle$. We consider the left-invariant Hamiltonian function $H : G \times \mathfrak{g}^* \to \mathbb{R}$ defined as in (2.7). The *sub-Riemannian exponential map* is the function

$$\mathrm{SExp} : \mathfrak{g}^* \to G$$

that maps every $\xi \in \mathfrak{g}^* = T_e^*G$ to the end point $\mathrm{SExp}(\xi) = g(1)$ of the solution $(g, \alpha) : [0, 1] \to T^*G$ of the Hamiltonian system (2.6), which becomes in this case (2.9), with $g(0) = e$ and $\alpha(0) = \xi$.

Note that SExp is an analytic function defined on the whole space $\mathfrak{g}^*$. Indeed, since the Hamiltonian $H$ is analytic, the Hamiltonian vector field $\mathcal{X}_H$ defined in (2.4) is also analytic. By the Cauchy-Kovalevskaya theorem, the flow of $\mathcal{X}_H$ on $T^*G$ is analytic. Since SExp is the restriction of the flow of $\mathcal{X}_H$ to $T_e^*G$ composed with the bundle projection $T^*G \to G$, we conclude that SExp is an analytic function.

# 3 Preliminaries on Carnot groups

## 3.1 Carnot groups

For a more detailed introduction to Carnot groups, we suggest to consult [12]. A *Carnot group* is a connected and simply connected Lie group $G$ whose Lie algebra $\mathfrak{g}$ has a fixed stratification $\mathfrak{g} = \oplus_{j=1}^{s} V_j$ and a fixed scalar product $\langle\cdot,\cdot\rangle$ on $V_1$. A *stratification* is a linear splitting $\mathfrak{g} = \oplus_{j=1}^{s} V_j$ where $[V_1, V_j] = V_{j+1}$ for $j \in \{1, \dots, s-1\}$ and $[V_1, V_s] = \{0\}$.

Since $V_1$ Lie generates $\mathfrak{g}$, the left-invariant horizontal vector bundle $\tilde{V}_1 \subset TG$ is bracket generating and together with the scalar product $\langle\cdot,\cdot\rangle$ on $V_1$, a sub-Riemannian distance is determined on $G$ [12]. Although we will not directly deal with distances in this article, when we will speak of length-minimizing curves we mean with respect to the sub-Riemannian distance induced by the choice of $V_1$ and of $\langle\cdot,\cdot\rangle$ on $V_1$.

Since $G$ turns out to be a nilpotent group, the group exponential map $\exp: \mathfrak{g} \to G$ is a global diffeomorphism. Furthermore, the Haar measure vol on $G$ is just the pushforward measure of the Lebesgue measure on $\mathfrak{g}$ by exp.

We assume $V_s \neq \{0\}$, and so $G$ has *step* $s$. The *(topological) dimension* of $G$ is $n = \sum_{j=1}^{s} \dim(V_j)$; the *homogeneous dimension* of $G$ is $Q = \sum_{j=1}^{s} j \dim(V_j)$.

The dual space $\mathfrak{g}^*$ inherits a splitting $\mathfrak{g}^* = \oplus_{j=1}^{s} V_j^*$, where $V_j^* = \{\alpha \in \mathfrak{g}^* : V_i \subset \ker \alpha, \quad \forall i \neq j\}$.

For $\lambda \in \mathbb{R}$, *dilation of factor* $\lambda$ on $G$ is the group automorphism $\delta_\lambda : G \to G$ whose induced Lie algebra automorphism $(\delta_\lambda)_* : \mathfrak{g} \to \mathfrak{g}$ is the linear map $(\delta_\lambda)_* v = \lambda^j v$ for $v \in V_j$. We usually denote $(\delta_\lambda)_*$ again by $\delta_\lambda$.

## 3.2 Symmetries of the sub-Riemannian exponential map on Carnot groups

Lemma 2.4 translates to symmetries of the sub-Riemannian exponential map on Carnot groups. For $\lambda \in \mathbb{R}\setminus\{0\}$, define $\eta_\lambda : \mathfrak{g}^* \to \mathfrak{g}^*$ as

$$\eta_\lambda\left(\sum_{j=1}^{s} \xi_j\right) := \sum_{j=1}^{s} \lambda^{2-j} \xi_j.$$

From Lemma 2.4, we obtain, for all $\xi \in \mathfrak{g}^*$ and $\lambda \in \mathbb{R}\setminus\{0\}$,

$$\mathrm{SExp}(\eta_\lambda \xi) = \delta_\lambda \mathrm{SExp}(\xi). \tag{3.1}$$

Taking the Jacobian determinant in (3.1), we also obtain for $\lambda \in \mathbb{R}\setminus\{0\}$

$$\lambda^{2n-Q} \mathrm{Jac}(\mathrm{SExp})(\eta_\lambda(\xi)) = \lambda^Q \mathrm{Jac}(\mathrm{SExp})(\xi), \tag{3.2}$$

where we used the fact that $\det(\eta_\lambda) = \lambda^{2n} \det(\delta_{1/\lambda}) = \lambda^{2n-Q}$.

**Remark 3.1.** For the precise meaning of the Jacobian determinant here, we first fix coordinates on $\mathfrak{g}^*$ and $\mathfrak{g}$ (thus on $G$ by the group exponential map exp) which preserve the Carnot group stratification, respectively. Then the Jacobian determinant can be calculated in the usual Euclidean sense. Although different choices of the coordinates will change the value of the Jacobian determinant by multiplying a nonzero constant, it turns out that our definitions of $\Gamma(\xi)$, $\Gamma(G)$, and $\hat{\Gamma}(G)$ in (1.5) are independent of the choice of the coordinates and our proofs below remain the same. Furthermore, noting by definition the geodesic dimension $N_{\mathrm{GEO}}$ and the curvature exponent $N_{\mathrm{CE}}$ are independent of the choice of the Haar measure, in the following we can assume the Haar measure vol on $G$ is exactly the pushforward measure of the Lebesgue measure induced by the fixed coordinates on $\mathfrak{g}$ without loss of generality.

For $\lambda \in \mathbb{R}$, define $\zeta_\lambda : \mathfrak{g}^* \to \mathfrak{g}^*$ as

$$\zeta_\lambda\left(\sum_{j=1}^{s}\xi_j\right) \coloneqq \sum_{j=1}^{s}\lambda^{j-1}\xi_j.$$

Then, $\lambda\xi = \eta_\lambda\zeta_\lambda(\xi)$ for all $\lambda \in \mathbb{R}\setminus\{0\}$. If we substitute $\xi$ with $\zeta_\lambda(\xi)$ in (3.2), we obtain

$$\mathrm{Jac}(\mathrm{SExp})(\lambda\xi) = \lambda^{2Q-2n}\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi)), \tag{3.3}$$

for all $\lambda \in \mathbb{R}$. Note that, when $\lambda \to 0$, we have $\zeta_\lambda(\xi) \to \xi_1 \in V_1^*$.

Recall Definition 1.1: $\mathscr{D} \subset \mathfrak{g}^*$ is the open set of all $\xi \in \mathfrak{g}^*$ such that $\mathrm{Jac}(\mathrm{SExp})(t\xi) \neq 0$ for all $t \in (0,1]$ and $t \mapsto \mathrm{SExp}(t\xi)$ is the unique constant-speed length-minimizing curve $[0,1] \to G$ from $e$ to $\mathrm{SExp}(\xi)$. From this definition, it follows that if $\xi \in \mathscr{D}$, then $t\xi \in \mathscr{D}$ for all $t \in (0,1]$.

We have defined $\mathcal{S}_e \subset G$ as the image $\mathrm{SExp}(\mathscr{D})$, so that SExp is a diffeomorphism from $\mathscr{D}$ to $\mathcal{S}_e$. We denote by $\rho : \mathcal{S}_e \to \mathscr{D}$ the inverse of SExp on $\mathcal{S}_e$. It is well known that $\mathcal{S}_e$ is dense in $G$, although it is not known whether it has full measure.

**Lemma 3.2.** *If $\xi \in \mathscr{D}$, then $\zeta_\lambda(\xi) \in \mathscr{D}$ for all $\lambda \in (0,1]$.*

**Proof.** We will apply Definition 1.1 itself. Fix $\xi \in \mathscr{D}$ and $\lambda \in (0,1]$.

First, from (3.3), we have for all $t \in (0,1]$,

$$\mathrm{Jac}(\mathrm{SExp})(t\zeta_\lambda(\xi)) = \mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(t\xi)) = \frac{\mathrm{Jac}(\mathrm{SExp})(\lambda t\xi)}{\lambda^{2Q-2n}} \neq 0$$

because $0 < \lambda t \leq 1$.

Second, suppose that $\eta : [0,1] \to G$ is a constant-speed length-minimizing curve from $e$ to $\mathrm{SExp}(\zeta_\lambda(\xi))$. Then, $\gamma(t) \coloneqq \delta_\lambda(\eta(t))$ is also a constant-speed length-minimizing curve from $e$ to

$$\delta_\lambda(\mathrm{SExp}(\zeta_\lambda(\xi))) = \mathrm{SExp}(\eta_\lambda\zeta_\lambda(\xi)) = \mathrm{SExp}(\lambda\xi),$$

where we have used (3.1). Since $\lambda\xi \in \mathscr{D}$, then $\gamma(t) = \mathrm{SExp}(t\lambda\xi)$. Therefore,

$$\eta(t) = \delta_{1/\lambda}\gamma(t) = \delta_{1/\lambda}\mathrm{SExp}(t\lambda\xi) = \mathrm{SExp}(\eta_{1/\lambda}\eta_\lambda\zeta_\lambda(t\xi)) = \mathrm{SExp}(t\zeta_\lambda(\xi)).$$

We have thus shown that $\zeta_\lambda(\xi) \in \mathscr{D}$. □

Since SExp is an analytic map, the function $\mathfrak{g}^* \times \mathbb{R} \to \mathbb{R}$, $(\xi,\lambda) \mapsto \mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi))$, is also analytic. In particular, there are analytic functions $P_k : \mathfrak{g}^* \to \mathbb{R}$ such that for every $U \subset \mathfrak{g}^*$ bounded there is $\lambda_0 > 0$ such that for all $\xi \in U$ and $\lambda \in (-\lambda_0, \lambda_0)$,

$$\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi)) = \sum_{k=0}^{\infty} P_k(\xi)\lambda^k, \tag{3.4}$$

where the series converges absolutely and uniformly in $\xi \in U$.

**Lemma 3.3.** *Each $P_k$ is a homogeneous polynomial of degree $2Q - 2n + k$, i.e., for every $\xi \in \mathfrak{g}^*$ and $\mu \in \mathbb{R}$,*

$$P_k(\mu\xi) = \mu^{2Q-2n+k}P_k(\xi). \tag{3.5}$$

*In particular, we have*

$$\Gamma(\mu\xi) = \Gamma(\xi), \quad \forall\mu \neq 0, \xi \in \mathfrak{g}^*. \tag{3.6}$$

**Proof.** Using the same notation we had for (3.4), together with the linearity of the maps $\zeta_\lambda$ and the group property $\zeta_\mu\zeta_\lambda = \zeta_{\mu\lambda}$, we obtain for $\lambda$ and $\mu$ small enough,

$$\sum_{k=0}^{\infty} P_k(\mu\xi)\lambda^k = \mathrm{Jac}(\mathrm{SExp})(\mu\zeta_\lambda(\xi)) = \mu^{2Q-2n}\mathrm{Jac}(\mathrm{SExp})(\zeta_\mu\zeta_\lambda(\xi))$$
$$= \mu^{2Q-2n}\mathrm{Jac}(\mathrm{SExp})(\zeta_{\mu\lambda}(\xi)) = \sum_{k=0}^{\infty} \mu^{2Q-2n+k} P_k(\xi)\lambda^k.$$

Analyticity implies that (3.5) holds for all $\xi \in \mathfrak{g}^*$, all $\mu \in \mathbb{R}$, and all $k \in \mathbb{N}$. □

## 3.3 Intermediate points of a negligible set are negligible

The goal of this section is to show the following Proposition 3.4 after two auxiliary lemmas. We will then have two consequences. First, the equivalence of $\mathrm{vol}(Z_\lambda(e, E))$ with standard definitions in the literature, see Remark 3.5. Next, formula (3.8) in Proposition 3.8 for the volume of $Z_\lambda(e, E)$.

**Proposition 3.4.** *Let $G$ be a Sard-regular Carnot group. If $E \subset G$ has measure zero, then $Z_t(e, E)$ has also measure zero, for all $t \in [0, 1]$.*

**Remark 3.5.** We obtain from Proposition 3.4 that, in Sard-regular Carnot groups, our definition of intermediate points (1.1) is “almost equivalent” to the definition given in [1,21].

Indeed, the set $E_{e,\lambda}$ defined in [1, Definition 5.43] or [21, Eq.(1)] corresponds, in our notation, to $Z_\lambda(e, E \cap \mathcal{S}_e)$. However, Proposition 3.4 implies that $\mathrm{vol}(Z_\lambda(e, E)) = \mathrm{vol}(Z_\lambda(e, E \cap \mathcal{S}_e)) = \mathrm{vol}(E_{e,\lambda})$.

The following lemma is well known to experts. For the notions of *End point map*, *regular*, and *strictly normal* curves, see [2,20]

**Lemma 3.6.** *Let $\gamma : [0, 1] \to G$ be a length-minimizing curve parametrized with constant speed. Suppose that there exist $\xi \in \mathscr{D}$ and $t$ such that $\gamma(ut) = \mathrm{SExp}(u\xi)$ for all $u \in [0, 1]$. Then, there exists $\eta \in \mathfrak{g}^*$ such that $\gamma(u) = \mathrm{SExp}(u\eta)$ for all $u \in [0, 1]$.*

**Proof.** Since $\xi \in \mathscr{D}$, then the restriction $\gamma|_{[0,t]}$ is *regular* for the End point map, because the image of the differential of the End point map contains the image of the differential of the sub-Riemannian exponential map (see the proof of Lemma 2.31 in [14]). It follows that $\gamma$ is also regular for the End point map, and thus *strictly normal*. In particular, there exists $\eta \in \mathfrak{g}^*$ such that $\gamma(u) = \mathrm{SExp}(u\eta)$ for all $u \in [0, 1]$. □

**Lemma 3.7.** *If $E \subset G$ has zero measure, then $\mathrm{SExp}^{-1}(E) \subset \mathfrak{g}^*$ has also zero measure.*

**Proof.** Suppose that $\mathrm{SExp}^{-1}(E) \subset \mathfrak{g}^*$ has positive measure. Since $\mathrm{Jac}(\mathrm{SExp})$ is an analytic function, there is a Lebesgue's density point $\xi$ of $\mathrm{SExp}^{-1}(E)$ ([10, Theorem 1.8]) such that $\mathrm{Jac}(\mathrm{SExp})(\xi) \neq 0$. Hence, SExp is a diffeomorphism on a neighbourhood of $\xi$ and thus $\mathrm{SExp}(\mathrm{SExp}^{-1}(E))$ has positive measure. Since $\mathrm{SExp}(\mathrm{SExp}^{-1}(E)) \subset E$, $E$ has positive measure too. □

**Proof of Proposition 3.4.** Set $\mathrm{Sing} = G \backslash \mathcal{S}_e$. We decompose $Z_t(e, E)$ into the following three sets:

$$Z_t(e, E) = Z_t(e, E \cap \mathcal{S}_e) \cup (Z_t(e, E \cap \mathrm{Sing}) \cap \mathrm{Sing}) \cup (Z_t(e, E \cap \mathrm{Sing}) \cap \mathcal{S}_e).$$

Case 1: volume of $Z_t(e, E \cap \mathcal{S}_e)$. In this case, we have

$$Z_t(e, E \cap \mathcal{S}_e) = \mathrm{SExp}(t\rho(E \cap \mathcal{S}_e)),$$

where $\mathrm{vol}(E \cap \mathcal{S}_e) = 0$ and where $p \mapsto \mathrm{SExp}(t\rho(p))$ is a diffeomorphism in a neighbourhood of $E \cap \mathcal{S}_e$. Since $\mathrm{vol}(E) = 0$, $\mathrm{vol}(Z_t(e, E \cap \mathcal{S}_e)) = 0$.

Case 2: volume of $Z_t(e, E \cap \mathrm{Sing}) \cap \mathrm{Sing}$. Since we assume $G$ to be Sard-regular, $\mathrm{vol}(\mathrm{Sing}) = 0$ and thus $\mathrm{vol}(Z_t(e, E \cap \mathrm{Sing}) \cap \mathrm{Sing}) = 0$.

Case 3: volume of $Z_t(e, E \cap \mathrm{Sing}) \cap \mathcal{S}_e$. We claim that

$$Z_t(e, E \cap \mathrm{Sing}) \cap \mathcal{S}_e \subset \mathrm{SExp}(t\,\mathrm{SExp}^{-1}(E)). \tag{3.7}$$

Indeed, let $z \in Z_t(e, E \cap \mathrm{Sing}) \cap \mathcal{S}_e$. Then, there are a point $q \in E \cap \mathrm{Sing}$ and a length-minimizing geodesic $\gamma : [0, 1] \to G$ parametrized with constant speed with $\gamma(0) = e$, $\gamma(1) = q$, and $\gamma(t) = z$. Since $z \in \mathcal{S}_e$, there is $\xi \in \mathscr{D}$ such that $\mathrm{SExp}(\xi) = z$. By the definition of $\mathscr{D}$, we have $\mathrm{SExp}(u\xi) = \gamma(ut)$ for all $u \in [0, 1]$. From Lemma 3.6, it follows that there is $\eta \in \mathfrak{g}^*$ such that $\mathrm{SExp}(u\eta) = \gamma(u)$. Thus, the claim (3.7) is proven.

By Lemma 3.7, we have $\mathrm{vol}(\mathrm{SExp}(t\,\mathrm{SExp}^{-1}(E))) = 0$ and thus, (3.7) implies that $\mathrm{vol}(Z_t(e, E \cap \mathrm{Sing}) \cap \mathcal{S}_e) = 0$. □

**Proposition 3.8.** *If $G$ is a Sard-regular Carnot group, then for every measurable $E \subset G$,*

$$\mathrm{vol}(Z_\lambda(e, E)) = \lambda^{2Q-n} \int_{\rho(E\cap\mathcal{S}_e)} |\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi))|\mathrm{d}\xi. \tag{3.8}$$

**Proof.** From Definition 1.1 and the definition of intermediate points, we obtain that

$$\begin{aligned} Z_\lambda(e, E) &= Z_\lambda(e, E \cap \mathcal{S}_e) \cup Z_\lambda(e, E\setminus\mathcal{S}_e) \\ &= \mathrm{SExp}(\lambda\rho(E \cap \mathcal{S}_e)) \cup Z_\lambda(e, E\setminus\mathcal{S}_e). \end{aligned}$$

Proposition 3.4 says that $\mathrm{vol}(Z_\lambda(e, E\setminus\mathcal{S}_e)) = 0$.

Thus, by Remark 3.1, the area formula, and the identity (3.3), we conclude

$$\begin{aligned} \mathrm{vol}(Z_\lambda(e, E)) &= \mathrm{vol}(\mathrm{SExp}(\lambda\rho(E \cap \mathcal{S}_e))) \\ &= \int_{\lambda\rho(E\cap\mathcal{S}_e)} |\mathrm{Jac}(\mathrm{SExp})(\xi)|\mathrm{d}\xi \\ &= \lambda^n \int_{\rho(E\cap\mathcal{S}_e)} |\mathrm{Jac}(\mathrm{SExp})(\lambda\xi)|\mathrm{d}\xi \\ &= \lambda^{2Q-n} \int_{\rho(E\cap\mathcal{S}_e)} |\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi))|\mathrm{d}\xi. \end{aligned}$$

□

# 4 Proof of Theorem A

Thanks to Remark 3.5, we can apply the following result by Agrachev-Barilari-Rizzi.

**Proposition 4.1.** [1, Theorem D, page 58] *For any bounded, measurable set $E \subset \mathcal{S}_e$ with $0 < \mathrm{vol}(E) < +\infty$ we have $\mathrm{vol}(Z_\varepsilon(e, E)) \sim \varepsilon^{N_{\mathrm{GEO}}}$ for $\varepsilon \to 0$.*

**Proof of Theorem A.** We will give a measurable set $E \subset G$ with $0 < \mathrm{vol}(E) < \infty$ such that

$$\lim_{\varepsilon\to 0} \frac{\mathrm{vol}(Z_\varepsilon(e, E))}{\varepsilon^{2Q-n+\Gamma(G)}} \in (0, +\infty), \tag{4.1}$$

and then we will apply Proposition 4.1 (i.e., [1, Theorem D, page 58]) to conclude that $N_{\mathrm{GEO}} = 2Q - n + \Gamma(G)$.

Since $\mathscr{E} := \{\xi : P_{\Gamma(G)}(\xi) \neq 0\}$ is open and dense in $\mathfrak{g}^*$, the set $\mathscr{D} \cap \mathscr{E}$ is open and non-empty. As a result, we can choose $\xi_0 \in \mathscr{D} \cap \mathscr{E}$ and a compact connected neighbourhood $U \subset \mathscr{D} \cap \mathscr{E}$ of $\xi_0$ and an open interval $I \subset \mathbb{R}$ with $0 \in I$ such that for $\varepsilon \in I$ and $\xi \in U$, we have

$$\mathrm{Jac}(\mathrm{SExp})(\zeta_\varepsilon(\xi)) = \sum_{k=\Gamma(G)}^{\infty} P_k(\xi)\varepsilon^k,$$

where $P_k : U \to \mathbb{R}$ are analytic functions, the series is absolutely convergent and $P_{\Gamma(G)}(\xi) \neq 0$ for all $\xi \in U$. In particular, if $P_{\Gamma(G)}(\xi_0) > 0$ ($P_{\Gamma(G)}(\xi_0) < 0$, resp.), then there is $\eta > 0$ such that $P_{\Gamma(G)}(\xi) > \eta$ ($P_{\Gamma(G)}(\xi) < -\eta$, resp.) for all $\xi \in U$.

Let us assume $P_{\Gamma(G)}(\xi_0) > 0$, as the other case is similar. Since $U$ is compact and since $(\varepsilon, \xi) \mapsto \frac{\mathrm{Jac}(\mathrm{SExp})(\zeta_\varepsilon(\xi))}{\varepsilon^{\Gamma(G)}}$ is analytic in a neighbourhood of $\{0\} \times U$, the limit

$$\lim_{\varepsilon \to 0} \frac{\mathrm{Jac}(\mathrm{SExp})(\zeta_\varepsilon(\xi))}{\varepsilon^{\Gamma(G)}} = P_{\Gamma(G)}(\xi)$$

is uniform for $\xi \in U$. Therefore, there exists $\varepsilon_0 > 0$ such that

$$\mathrm{Jac}(\mathrm{SExp})(\zeta_\varepsilon(\xi)) > \frac{\eta}{2} \varepsilon^{\Gamma(G)} > 0$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Set $E \coloneqq \mathrm{SExp}(U)$. For $\varepsilon \in (0, \varepsilon_0)$, we obtain from Proposition 3.8

$$\begin{aligned} \mathrm{vol}(Z_\varepsilon(e, E)) &= \varepsilon^{2Q-n} \int_U |\mathrm{Jac}(\mathrm{SExp})(\zeta_\varepsilon(\xi))| \mathrm{d}\xi \\ &= \varepsilon^{2Q-n} \int_U \mathrm{Jac}(\mathrm{SExp})(\zeta_\varepsilon(\xi)) \mathrm{d}\xi \\ &= \varepsilon^{2Q-n+\Gamma(G)} \sum_{k=\Gamma(G)}^{\infty} \varepsilon^{k-\Gamma(G)} \int_U P_k(\xi) \mathrm{d}\xi. \end{aligned}$$

Since $\int_U P_{\Gamma(G)}(\xi) \mathrm{d}\xi > 0$ from our choice of $U$, we have

$$\lim_{\varepsilon \to 0} \frac{\mathrm{vol}(Z_\varepsilon(e, E))}{\varepsilon^{2Q-n+\Gamma(G)}} = \int_U P_{\Gamma(G)}(\xi) \mathrm{d}\xi \in (0, \infty),$$

i.e., (4.1). □

# 5 Proof of Theorem B

**Proposition 5.1.** *On a Sard-regular Carnot group, the following statements are equivalent for every* $N > 0$:

(*i*) $N \geq N_{\mathrm{CE}}$;

(*ii*) $\lambda^{2Q-n} |\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi))| \geq \lambda^N |\mathrm{Jac}(\mathrm{SExp})(\xi)|$ *for all* $\xi \in \mathscr{D}$ *and* $\lambda \in [0, 1]$;

(*iii*) $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=1} \left[\frac{\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi))}{\lambda^{N-2Q+n}}\right]^2 \leq 0$ *for all* $\xi \in \mathscr{D}$;

(*iv*) $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} \left[\frac{\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi))}{\lambda^{N-2Q+n}}\right]^2 \leq 0$ *for all* $\xi \in \mathscr{D}$ *and* $\lambda_0 \in (0, 1]$.

**Proof.** (i) ⇒ (ii) If $N \geq N_{\mathrm{CE}}$, then

$$\mathrm{vol}(Z_\lambda(p, E)) \geq \lambda^N \mathrm{vol}(E) \quad \forall \lambda \in [0, 1],$$

for all measurable $E \subset G$ with $0 < \mathrm{vol}(E) < \infty$. By Proposition 3.8, we then have, for every $U \subset \mathscr{D}$ open and bounded,

$$\lambda^{2Q-n} \int_U |\mathrm{Jac}(\mathrm{SExp})(\zeta_\lambda(\xi))| \mathrm{d}\xi \geq \lambda^N \int_U |\mathrm{Jac}(\mathrm{SExp})(\xi)| \mathrm{d}\xi, \quad \forall \lambda \in [0, 1].$$

By the Lebesgue differentiation theorem, and by the continuity in $\xi$ of both left- and right-hand integrand functions, (ii) follows.

(ii) ⇒ (i) By a direct application of Proposition 3.8, the pointwise estimate for Jac(SExp) implies the volume estimate of the definition of $N_{\mathrm{CE}}$.

(ii) ⇒ (iii) Inequality (ii) implies that the function

$$f_\xi(\lambda) \coloneqq \frac{\mathrm{Jac(SExp)}(\zeta_\lambda(\xi))}{\lambda^{N-2Q+n}}$$

satisfies $f_\xi^2(\lambda) \geq f_\xi^2(1)$ for $\lambda \in [0,1]$. Since $f_\xi$ is smooth, we conclude $\frac{\mathrm{d}}{\mathrm{d}\lambda}\big|_{\lambda=1} f_\xi^2(\lambda) \leq 0$.

(iii) ⇒ (iv) Fix $\xi \in \mathscr{D}$ and $\lambda_0 \in (0,1]$. Note that, since $\zeta_{\lambda_0\lambda} = \zeta_{\lambda_0} \circ \zeta_\lambda$, the function $f_\xi(\lambda)$ defined above satisfies

$$f_{\zeta_{\lambda_0}\xi}(\lambda) = \lambda_0^{N-2Q+n} f_\xi(\lambda_0\lambda),$$

for all $\lambda \in (0,1]$. Moreover, by Lemma 3.2, we have $\zeta_{\lambda_0}\xi \in \mathscr{D}$. Therefore, from (iii) we obtain

$$0 \geq \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=1} f^2_{\zeta_{\lambda_0}\xi}(\lambda) = \lambda_0^{2N-4Q+2n}\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=1} f_\xi^2(\lambda_0\lambda) = \lambda_0^{2N-4Q+2n+1}\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=\lambda_0} f_\xi^2(\lambda).$$

We conclude that (iv) holds.

(iv) ⇒ (ii) Hypothesis (iv) implies that $f_\xi^2$ is non increasing on $(0,1]$, whenever $\xi \in \mathscr{D}$. It follows that $f_\xi^2(\lambda) \geq f_\xi^2(1)$ for all $\xi \in \mathscr{D}$ and $\lambda \in (0,1]$, which is equivalent to (ii). □

**Lemma 5.2.** *On Sard-regular Carnot group G, we have*

$$\hat\Gamma(G) = \sup\{\Gamma(\xi) : \xi \in \mathscr{D}\}.$$

**Proof.** Let us denote the number on the right-hand side (RHS) by $K$. It follows from the definition of $\mathscr{D}$ that if $\xi \in \mathscr{D}$, then $\Gamma(\xi) < \infty$. Then, it follows from the original definition of $\hat\Gamma(G)$ (cf. (1.5)) that $K \leq \hat\Gamma(G)$.

To prove the converse inequality, by (3.6), it suffices to prove that if $\xi \in \mathfrak{g}^*$ with $\Gamma(\xi) < \infty$, then for $\mu > 0$ small enough, we have $\mu\xi \in \mathscr{D}$. We check Definition 1.1 for $\mu\xi$ with $\mu$ small enough. Since $\Gamma(\xi) < \infty$, for $\mu$ small and $t \in (0,1]$, we have

$$\mathrm{Jac(SExp)}(\zeta_{t\mu}(\xi)) = \sum_{k=\Gamma(\xi)}^{\infty} P_k(\xi)(t\mu)^k \neq 0.$$

It follows from (3.3) that

$$\mathrm{Jac(SExp)}(t\mu\xi) = (t\mu)^{2Q-2n}\mathrm{Jac(SExp)}(\zeta_{t\mu}(\xi)) \neq 0, \quad \forall t \in (0,1].$$

By the local minimality (cf. [2, Theorem 4.65]), for $\mu$ small enough, $t \mapsto \mathrm{SExp}(t\mu\xi)$ is the unique constant-speed length-minimizing curve $[0,1] \to G$ from $e$ to $\mathrm{SExp}(\mu\xi)$. Thus, we have proven that $\mu\xi \in \mathscr{D}$ and this ends the proof of the lemma. □

**Proof of Theorem B.** Fix $\xi \in \mathscr{D}$ and assume

$$N < 2Q - n + \Gamma(\xi). \tag{5.1}$$

For $\lambda \in (0,1]$, define the analytic function

$$f_\xi(\lambda) \coloneqq \frac{\mathrm{Jac(SExp)}(\zeta_\lambda(\xi))}{\lambda^{N-2Q+n}}.$$

By (1.4), there is $\lambda_\xi > 0$ such that, for $\lambda \in [0,\lambda_\xi)$,

$$f_\xi(\lambda) = \lambda^{\Gamma(\xi)-N+2Q-n}\sum_{k\geq 0} P_{\Gamma(\xi)+k}(\xi)\lambda^k.$$

By standard rules of calculus, we have, for $\lambda \in [0,\lambda_\xi)$,

$$f'_\xi(\lambda) = \lambda^{\Gamma(\xi)-N+2Q-n-1} \sum_{k\geq 0} (\Gamma(\xi) + k - N + 2Q - n) P_{\Gamma(\xi)+k}(\xi)\lambda^k.$$

Since $\Gamma(\xi) + k - N + 2Q - n > 0$ by (5.1), as $\lambda \to 0^+$, $f_\xi(\lambda)$ and $f'_\xi(\lambda)$ have the same sign as $P_{\Gamma(\xi)}(\xi)$.

We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f^2_\xi(\lambda) = 2f_\xi(\lambda) f'_\xi(\lambda) > 0$$

for $\lambda$ positive and small enough. Proposition 5.1 implies that $N < N_{\mathrm{CE}}$.

Since $N$ is arbitrary in $(0, 2Q - n + \Gamma(\xi))$ and $\xi$ is arbitrary in $\mathscr{D}$, we complete the proof of Theorem B by Lemma 5.2. □

**Remark 5.3.** From the definition in (1.5) and Lemma 3.3, $\Gamma(\xi)$ attains its minimum value $\Gamma(G)$ on an open, nonempty Zariski subset of $\mathfrak{g}^*$ (see also Remark 1.4 and [1, Proposition 5.46]), which implies it is constant almost everywhere. It seems that the curvature exponent $N_{\mathrm{CE}}$ should not detect higher values of $\Gamma(\xi)$ by definition. However, (ii) of Proposition 5.1 shows that the curvature exponent $N_{\mathrm{CE}}$ provides a uniform bound for the Jacobian determinant of the sub-Riemannian exponential map while $\Gamma(\xi)$ is defined in a pointwise way. Thus, when $\xi$ moves in the set $\{\xi : P_{\Gamma(G)}(\xi) \neq 0\}$, the coefficient of the first nonzero term in (1.4), or equivalently $P_{\Gamma(G)}(\xi)$, may become small and create an obstacle for the uniformity. This helps to explain why the curvature exponent $N_{\mathrm{CE}}$ could be strictly larger than the geodesic dimension $N_{\mathrm{GEO}}$.

# 6 Carnot groups of step two

The following construction of the sub-Riemannian exponential map in Carnot groups of step two, and in particular the splitting given in Definition 6.6, is linked to the techniques used in [15,16] for the study of phase function of Fourier integral operators on Carnot groups. In fact, it has been explained to the first-named author by Alessio Martini.

## 6.1 Preliminary observations on Carnot groups of step two

Let $\mathfrak{g} = V_1 \oplus V_2$ be a stratified Lie algebra of step two, with a scalar product $\langle\cdot,\cdot\rangle$ on $V_1$.

If $\mu \in V_2^*$, let $J_\mu : V_1 \to V_1$ be the linear map defined by

$$\langle J_\mu v, w\rangle = \langle \mu | [v, w]\rangle,$$

for all $v, w \in V_1$. Note that $J_\mu$ is skew-symmetric, i.e.,

$$J_\mu \in \mathfrak{so}(V_1, \langle\cdot,\cdot\rangle) = \{J : V_1 \to V_1 \text{ linear}, \langle Jx, y\rangle = -\langle x, Jy\rangle, \quad \forall x, y \in V_1\}.$$

We will fix a scalar product $\langle\cdot,\cdot\rangle$ on the whole $\mathfrak{g}$ that extends the given one on $V_1$ and such that $V_1$ and $V_2$ are orthogonal. We will use this scalar product to identify $\mathfrak{g}$ with $\mathfrak{g}^*$, via $v \mapsto v^\flat$. For instance, if $u \in V_2$, we have $J_u \coloneqq J_{u^\flat}$. It follows that, for all $v, w \in V_1$ and $u \in V_2$,

$$\langle u, [v, w]\rangle = \langle u^\flat | [v, w]\rangle = \langle J_{u^\flat} v, w\rangle = \langle J_u v, w\rangle. \tag{6.1}$$

There is a special choice of scalar product on $V_2$: we keep this result for completeness.

**Lemma 6.1.** *Denote by $\langle\cdot,\cdot\rangle_{HS}$ the Hilbert–Schmidt scalar product in $\mathfrak{so}(V_1, \langle\cdot,\cdot\rangle)$, i.e.,*

$$\langle A, B\rangle_{HS} = \mathrm{trace}(AB^*),$$

*where $B^*$ is the conjugate of $B$ with respect to the scalar product $\langle\cdot,\cdot\rangle$ on $V_1$.*

*There exists a unique extension of $\langle\cdot,\cdot\rangle$ from $V_1$ to a scalar product on $\mathfrak{g}$ such that $V_1$ and $V_2$ are orthogonal and, for every $v, w \in V_2$, we have*

$$\langle v, w\rangle = \langle J_{v^\flat}, J_{w^\flat}\rangle_{HS}, \tag{6.2}$$

*where* $V_2 \ni x \mapsto x^\flat \in V_2^*$ *is the linear isomorphism induced by the scalar product.*

**Proof.** By definition and the fact $V_2 = [V_1, V_1]$, it is direct to check that the map $V_2^* \to \mathfrak{so}(V_1, \langle\cdot,\cdot\rangle)$ defined by $\mu \mapsto J_\mu$ is linear and injective. Then, there is a unique scalar product $\langle\cdot,\cdot\rangle^*$ on $V_2^*$ such that for every $\alpha, \beta \in V_2^*$,

$$\langle \alpha, \beta\rangle^* = \langle J_\alpha, J_\beta\rangle_{HS}. \tag{6.3}$$

The scalar product $\langle\cdot,\cdot\rangle^*$ on $V_2^*$ induces a scalar product $\langle\cdot,\cdot\rangle$ on $V_2$: combining this with the original $\langle\cdot,\cdot\rangle$ on $V_1$, we obtain a scalar product on the whole $\mathfrak{g}$ such that $V_1$ and $V_2$ are orthogonal and (6.2) holds by definition.

Uniqueness is ensured because (6.2) is equivalent to (6.3), which in turn uniquely determines the dual scalar product $\langle\cdot,\cdot\rangle^*$ on $V_2^*$, and thus on $V_2$. □

## 6.2 Sub-Riemannian exponential map

To integrate the first of the two equations in (2.9), we will give the following construction of $G$: First, we denote by $V = V_1 \oplus V_2$ the vector space underlying the Lie algebra $\mathfrak{g}$; second, we define the Lie group $G$ as the smooth manifold $V$ endowed with the group operation

$$a * b = a + b + \frac{1}{2}[a, b], \quad a, b \in \mathfrak{g}.$$

It follows that 0 is the identity element $e$, and $g^{-1} = -g$. The advantage of this construction is that we will take derivatives as we do in the vector space $V$. For instance, the differential at $0 = e$ of the left translation $L_g : V \to V$ is the linear map $DL_g|_0 : V \to V$,

$$DL_g|_0[x] = x + \frac{1}{2}[g, x].$$

**Lemma 6.2.** *The ODE* (2.9) *is equivalent to*

$$\begin{cases} \dot{x} = \xi, \\ \dot{u} = \frac{1}{2}[x, \xi], \\ \dot{\xi} = J_\mu \xi, \\ \dot{\mu} = 0, \end{cases} \tag{6.4}$$

*for curves* $((x, u), (\xi, \mu)) : I \to (V_1 \oplus V_2) \oplus (V_1 \oplus V_2)$. *In other words,* $((x, u), (\xi, \mu))$ *is a solution to* (6.4) *if and only if* $((x, u), (\xi + \mu)^\flat)$ *is solution to* (2.9).

**Proof.** If $g = x + u \in G$ with $x \in V_1$ and $u \in V_2$, and if $\alpha = (\xi + \mu)^\flat = \xi^\flat + \mu^\flat \in \mathfrak{g}^*$ with $\xi \in V_1$ and $\mu \in V_2$, then $(\alpha|_{V_1})^\sharp = \xi$ and

$$DL_g|_e[(\alpha|_{V_1})^\sharp] = \xi + \frac{1}{2}[x, \xi].$$

Thus, we have that the first two equations in (6.4) are equivalent to the first equation in (2.9).

Next, if $v = v_1 + v_2$ with $v_j \in V_j$, then

$$\langle \alpha \circ \mathrm{ad}_{(\alpha|_{V_1})^\sharp} | v\rangle = \langle \alpha \circ \mathrm{ad}_\xi | v\rangle = \langle \alpha | [\xi, v]\rangle = \langle \mu, [\xi, v_1]\rangle = \langle J_\mu \xi, v_1\rangle,$$

i.e., $\alpha \circ \mathrm{ad}_{(\alpha|_{V_1})^\sharp} = (J_\mu \xi)^\flat$. It follows that the second two equations in (6.4) are equivalent to the second equation in (2.9). □

**Proposition 6.3.** *Given $\xi_0 \in V_1$ and $\mu_0 \in V_2$, the analytic solution $((x,u),(\xi,\mu)) : \mathbb{R} \to (V_1 \oplus V_2) \oplus (V_1 \oplus V_2)$ to* (6.4) *with $x(0) = 0$, $u(0) = 0$, $\xi(0) = \xi_0$, and $\mu(0) = \mu_0$ is*

$$\begin{aligned}
x(t) &= \frac{e^{tJ_{\mu_0}} - \mathrm{Id}}{J_{\mu_0}}\xi_0,\\
u(t) &= t^3\sum_{k=1}^{\infty} B_k(\xi_0,\mu_0)t^{k-1},\\
\xi(t) &= e^{tJ_{\mu_0}}\xi_0,\\
\mu(t) &= \mu_0,
\end{aligned}$$

*where*

$$B_k(\xi_0,\mu_0) \coloneqq \sum_{m=0}^{k} \frac{[J_{\mu_0}^m\xi_0, J_{\mu_0}^{k-m}\xi_0]}{2(m+1)!(k-m)!(k+2)}. \tag{6.5}$$

**Proof.** Since $\dot\mu = 0$, then $\mu(t) = \mu_0$. Since $\dot\xi = J_{\mu_0}\xi$, then $\xi(t) = e^{tJ_{\mu_0}}\xi_0$. Since $\dot x = \xi$ and $x(0) = 0$, then

$$x(t) = \frac{e^{tJ_{\mu_0}} - \mathrm{Id}}{J_{\mu_0}}\xi_0 = \sum_{k=0}^{\infty}\frac{t^{k+1}}{(k+1)!}J_{\mu_0}^k\xi_0.$$

Finally, since $\dot u = \frac{1}{2}[x,\xi]$ and $u(0) = 0$, then

$$\begin{aligned}
u(t) &= \int_0^t \dot u(s)\mathrm{d}s = \int_0^t \frac{[x(s),\xi(s)]}{2}\mathrm{d}s\\
&= \int_0^t \frac{1}{2}\left[\sum_{a=0}^{\infty}\frac{s^{a+1}J_{\mu_0}^a}{(a+1)!}\xi_0, \sum_{b=0}^{\infty}\frac{s^bJ_{\mu_0}^b}{b!}\xi_0\right]\mathrm{d}s\\
&= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{1}{2}\left[\frac{J_{\mu_0}^a}{(a+1)!}\xi_0, \frac{J_{\mu_0}^b}{b!}\xi_0\right]\int_0^t s^{a+b+1}\mathrm{d}s\\
&= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{[J_{\mu_0}^a\xi_0, J_{\mu_0}^b\xi_0]}{2(a+1)!b!(a+b+2)}t^{a+b+2}\\
&= t^2\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\frac{[J_{\mu_0}^m\xi_0, J_{\mu_0}^{k-m}\xi_0]}{2(m+1)!(k-m)!(k+2)}\right)t^k.
\end{aligned}$$

Using $B_k$ as defined in (6.5), note that $B_0 = 0$, thus $u(t) = t^3\sum_{k=1}^{\infty}B_kt^{k-1}$.

We claim that the series defining $u$ is absolutely convergent for all $t \in \mathbb{R}$. Indeed, if $\|\cdot\|$ is any norm on $V$ and $C > 0$ is such that $\|[x,y]\| \le C\|x\|\cdot\|y\|$ for all $x, y \in V_1$, then

$$\begin{aligned}
\|B_k\| &\le \frac{C\|J_{\mu_0}\|^k\|\xi_0\|^2}{2(k+2)}\sum_{m=0}^{k}\frac{1}{(m+1)!(k-m)!}\\
&= \frac{C\|J_{\mu_0}\|^k\|\xi_0\|^2}{2(k+2)(k+1)!}\sum_{m=0}^{k}\binom{k+1}{m+1}\\
&= \frac{C\|J_{\mu_0}\|^k\|\xi_0\|^2}{2(k+2)!}\left(\sum_{m=0}^{k+1}\binom{k+1}{m} - 1\right)\\
&= \frac{C\|J_{\mu_0}\|^k\|\xi_0\|^2}{2(k+2)!}((1+1)^{k+1} - 1)\\
&= \frac{C\|J_{\mu_0}\|^k\|\xi_0\|^2(2^{k+1}-1)}{2(k+2)!}.
\end{aligned}$$

Therefore, the series defining $u$ is absolutely convergent for all $t \in \mathbb{R}$. □

**Theorem 6.4.** *The analytic function* SExp : $\mathfrak{g} \to \mathfrak{g}$ *defined by*

$$\mathrm{SExp}(\xi, \mu) = \left( \sum_{k=0}^{\infty} \frac{J_\mu^k}{(k+1)!} \xi, \sum_{k=1}^{\infty} B_k(\xi, \mu) \right)$$

*with*

$$B_k(\xi, \mu) := \frac{1}{2(k+2)} \sum_{m=0}^{k} \frac{[J_\mu^m \xi, J_\mu^{k-m} \xi]}{(m+1)!(k-m)!}$$

*is the sub-Riemannian exponential map, after the identification* $\mathfrak{g} \simeq \mathfrak{g}^* = T_e^*G$ *via the scalar product that extends the given one on* $V_1$ *and such that* $V_1$ *and* $V_2$ *are orthogonal, and after the identification* $\mathfrak{g} \simeq G$ *via the group exponential map.*

**Proof.** This is a direct consequence of Proposition 6.3. □

## 6.3 Differential of the exponential map

**Theorem 6.5.** *The differential of the function* SExp *from Theorem* 6.4 *at a point* $(\xi, \mu) \in V_1 \oplus V_2$ *in the direction* $(w, v) \in V_1 \oplus V_2$, *that is* $D\mathrm{SExp}(\xi, \mu)[w, v]$, *is*

$$\begin{pmatrix} \displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left( J_\mu^k w + \sum_{m=1}^{k} J_\mu^{m-1} J_\nu J_\mu^{k-m} \xi \right) \\ \displaystyle\sum_{k=1}^{\infty} \frac{1}{2(k+2)} \sum_{m=0}^{k} \frac{k-2m}{(m+1)!(k-m+1)!} \left[ J_\mu^m w + \sum_{j=1}^{m} J_\mu^{j-1} J_\nu J_\mu^{m-j} \xi, J_\mu^{k-m} \xi \right] \end{pmatrix},$$

*where we use the conventions* $\sum_{j=1}^{0} = 0$ *and* $J_0^0 = \mathrm{Id}$.

**Proof.** Using the notation of Proposition 6.3, we write $\mathrm{SExp}(\xi, \mu) = (x(\xi, \mu), u(\xi, \mu))$. The derivative $\frac{\partial x}{\partial \xi}(\xi, \mu) : V_1 \to V_1$ is the linear map

$$\frac{\partial x}{\partial \xi}(\xi, \mu) = \sum_{k=0}^{\infty} \frac{J_\mu^k}{(k+1)!}.$$

An elementary computation shows that the derivative $\frac{\partial x}{\partial \mu}$ at $(\xi, \mu)$ is the linear map $V_2 \to V_1$

$$\frac{\partial x}{\partial \mu}(\xi, \mu) : v \mapsto \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \left( \sum_{m=1}^{k} J_\mu^{m-1} J_\nu J_\mu^{k-m} \right) \xi.$$

The derivatives of the second component of SExp are

$$\frac{\partial u}{\partial \xi}(\xi, \mu) = \sum_{k=1}^{\infty} \frac{\partial}{\partial \xi} B_k(\xi, \mu), \quad \frac{\partial u}{\partial \mu}(\xi, \mu) = \sum_{k=1}^{\infty} \frac{\partial}{\partial \mu} B_k(\xi, \mu).$$

Since $B_k(\xi, \mu)$ is bilinear in $\xi$, the derivative $\frac{\partial}{\partial \xi} B_k(\xi, \mu)$ is the linear map $V_1 \to V_2$ that maps $w \in V_1$ to

$$\begin{aligned} \frac{\partial}{\partial \xi} B_k(\xi, \mu)[w] &= \sum_{m=0}^{k} \frac{[J_\mu^m w, J_\mu^{k-m} \xi] + [J_\mu^m \xi, J_\mu^{k-m} w]}{2(m+1)!(k-m)!(k+2)} \\ &= \frac{1}{2(k+2)} \sum_{m=0}^{k} \frac{k-2m}{(m+1)!(k-m+1)!} [J_\mu^m w, J_\mu^{k-m} \xi]. \end{aligned}$$

We can compute the derivative $\frac{\partial}{\partial\mu}B_k(\xi,\mu)$ as a linear map $V_2 \to V_2$ that maps $v \in V_2$ to

$$\begin{aligned}\frac{\partial}{\partial\mu}B_k(\xi,\mu)[v] &= \sum_{m=0}^{k}\frac{1}{2(m+1)!(k-m)!(k+2)}\\ &\quad\times\left(\left[\left(\sum_{j=1}^{m}J_\mu^{j-1}J_vJ_\mu^{m-j}\right)\xi, J_\mu^{k-m}\xi\right]+\left[J_\mu^m\xi,\left(\sum_{j=1}^{k-m}J_\mu^{j-1}J_vJ_\mu^{k-m-j}\right)\xi\right]\right)\\ &= \frac{1}{2(k+2)}\sum_{m=0}^{k}\frac{k-2m}{(m+1)!(k-m+1)!}\left[\sum_{j=1}^{m}J_\mu^{j-1}J_vJ_\mu^{m-j}\xi, J_\mu^{k-m}\xi\right].\end{aligned}$$

□

## 6.4 Jacobian of the sub-Riemannian exponential map

**Definition 6.6.** For every fixed pair $(\xi,\mu)\in V_1\oplus V_2$, we define the following vector spaces. First, we define the following increasing sequence $U^\ell$ of subspaces of $V_1$: for $\ell=0$, we set $U^0 := \{0\}\subset V_1$, for $\ell=1$, we set $U^1 := \mathbb{R}\xi$, and for $\ell>1$,

$$U^\ell := \operatorname{span}\{\xi, J_\mu\xi, \ldots, J_\mu^{\ell-1}\xi\}\subset V_1.$$

Second, we take the dual decreasing sequence $U_\ell$ of subspaces of $V_2$: for all $\ell\geq 0$, set

$$U_\ell := \{v\in V_2 : J_v(U^\ell)=\{0\}\}.$$

Third, we define the orthogonal splitting $V_2=\oplus_{j=0}^{\infty}W_j\oplus W_\infty$ by setting

$$W_\infty := \{v\in V_2 : J_v(J_\mu^\ell\xi)=0, \forall \ell\geq 0\},$$

and by requiring

$$U_\ell = U_{\ell+1}\oplus W_\ell.$$

For example, $W_0$ is the orthogonal complement of $U_1=\{v\in V_2 : J_v\xi=0\}$, while $W_1$ is the orthogonal complement of $U_2=\{v\in V_2 : J_v\xi=J_vJ_\mu\xi=0\}$ in $U_1$. See Section 7.1 for an explicit example in the Heisenberg group.

Note that

$$\text{if } v\in W_\ell\backslash\{0\}, \quad \text{then } J_v(J_\mu^\ell\xi)\neq 0. \tag{6.6}$$

Since $V_2$ is finite dimensional, only finitely many $W_j$'s are non-trivial. For $(\xi,\mu)$ fixed, define

$$d := \max\{\ell : W_\ell\neq\{0\}\}\in\mathbb{N}\cup\{\infty\}$$

and

$$N_{\mathrm{SExp}}(\xi,\mu) := 2\sum_{j=0}^{d} j\dim(W_j). \tag{6.7}$$

If $W_\infty\neq\{0\}$, then $d=\infty$ and $N_{\mathrm{SExp}}(\xi,\mu)=\infty$. Note also that $N_{\mathrm{SExp}}(\xi,\mu)=N_{\mathrm{SExp}}(\xi,s\mu)$ for all $s\neq 0$.

**Theorem 6.7.** *Let $G$ be a step-two Carnot group as above, and fix $(\xi,\mu)\in V$.*
(*a*) *If $v\in W_\infty$, then* $D\mathrm{SExp}(\xi,\mu)[0,v]=0$. *In particular, if $W_\infty\neq\{0\}$, then* $\mathrm{Jac}(\mathrm{SExp})(\xi,\varepsilon\mu)=0$ *for all $\varepsilon\in\mathbb{R}$.*
(*b*) *If $W_\infty=\{0\}$, then, for all $\varepsilon\in\mathbb{R}$,*

$$\mathrm{Jac}(\mathrm{SExp})(\xi,\varepsilon\mu)=\varepsilon^{N_{\mathrm{SExp}}(\xi,\mu)}\det(a(\varepsilon)), \tag{6.8}$$

*where $a(\varepsilon)$ is a $n\times n$ matrix, depending on $(\xi,\mu)$, analytic in $\varepsilon$ and with* $\det(a(0))>0$, *and where $N_{\mathrm{SExp}}(\xi,\mu)$ is as in* (6.7).

*In conclusion,*

$$\Gamma(\xi,\mu) = N_{\mathrm{SExp}}(\xi,\mu), \tag{6.9}$$

*where* $\Gamma$ *was defined in* (1.5).

A direct consequence of Theorem 6.7 is the following corollary.

**Corollary 6.8.** *In the setting of step-two Carnot groups, we have*

$$\mathrm{Jac}(\mathrm{SExp})(\xi) > 0, \quad \forall \xi \in \mathscr{D}.$$

**Remark 6.9.** Fix a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ as before and identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Considering Remark 1.4, the value of $\Gamma(\xi,\mu)$ (or equivalently $\mathcal{N}_\lambda$) can also be computed out by geodesic growth vector $\mathcal{G}_\lambda$ with $\lambda = (\xi,\mu)$. See [1, Definition 5.44], where "ample" is equivalent to strictly normal in our setting. In fact, it can be shown that the geodesic growth vector is given by $\mathcal{G}_\lambda = (k_1, \dots, k_{d+2})$ with $k_{\ell+1} = n - \dim(U_\ell)$ for $0 \le \ell \le d+1$ and this approach gives the same number.

**Proof of Theorem 6.7.** By inspection of the formula in Theorem 6.5, one can easily check the statement (a).

Suppose now that $W_\infty = \{0\}$. Define $A(\varepsilon) \coloneqq D\mathrm{SExp}(\xi, \varepsilon\mu)$, which is a linear map from $V$ to $V$. Using orthogonal projections in $V$, we decompose these linear maps into $A_{V_1}^{V_1}(\varepsilon) : V_1 \to V_1$, $A_{W_\ell}^{V_1}(\varepsilon) : V_1 \to W_\ell$, $A_{V_1}^{W_\ell}(\varepsilon) : W_\ell \to V_1$, and $A_{W_s}^{W_r}(\varepsilon) : W_r \to W_s$, for $0 \le \ell, r, s \le d$. For example, $A_{W_s}^{W_r}(\varepsilon) = \pi_s \circ A(\varepsilon)|_{W_r}$. Here $\pi_s$ denotes the orthogonal projection onto $W_s$.

Note that if $w \in V_1$, $v' \in W_\ell$, $k \ge 1$, $0 \le m \le k$ are such that

$$0 \ne \langle [J_\mu^m(w), J_\mu^{k-m}\xi], v' \rangle = -\langle J_\mu^m(w), J_{v'}J_\mu^{k-m}\xi \rangle,$$

then $k - m \ge \ell$, and thus $k \ge \ell$. Therefore, if $\ell \ge 1$, we have

$$\begin{aligned} A_{W_\ell}^{V_1}(\varepsilon)w &= \pi_\ell\left(\sum_{k=1}^{\infty} \frac{\varepsilon^k}{2(k+2)} \sum_{m=0}^{k} \frac{k-2m}{(m+1)!(k-m+1)!}[J_\mu^m(w), J_\mu^{k-m}\xi]\right) \\ &= \varepsilon^\ell \pi_\ell\left(\sum_{k=\ell}^{\infty} \frac{\varepsilon^{k-\ell}}{2(k+2)} \sum_{m=0}^{k} \frac{k-2m}{(m+1)!(k-m+1)!}[J_\mu^m(w), J_\mu^{k-m}\xi]\right); \end{aligned}$$

while if $\ell = 0$, then

$$\begin{aligned} A_{W_0}^{V_1}(\varepsilon)w &= \pi_0\left(\sum_{k=1}^{\infty} \frac{\varepsilon^k}{2(k+2)} \sum_{m=0}^{k} \frac{k-2m}{(m+1)!(k-m+1)!}[J_\mu^m(w), J_\mu^{k-m}\xi]\right) \\ &= \varepsilon \pi_0\left(\sum_{k=1}^{\infty} \frac{\varepsilon^{k-1}}{2(k+2)} \sum_{m=0}^{k} \frac{k-2m}{(m+1)!(k-m+1)!}[J_\mu^m(w), J_\mu^{k-m}\xi]\right). \end{aligned}$$

Note that if $v \in W_\ell$, $k \ge 1$, and $1 \le m \le k$ are such that $J_v J_\mu^{k-m}\xi \ne 0$, and $k - m \ge \ell$, then $k \ge \ell + 1$. Therefore,

$$A_{V_1}^{W_\ell}(\varepsilon)v = \sum_{k=1}^{\infty} \frac{\varepsilon^{k-1}}{(k+1)!}\left(\sum_{m=1}^{k} J_\mu^{m-1} J_{(v)} J_\mu^{k-m}\right)\xi = \varepsilon^\ell \sum_{k=\ell+1}^{\infty} \frac{\varepsilon^{k-\ell-1}}{(k+1)!}\left(\sum_{m=1}^{k} J_\mu^{m-1} J_{(v)} J_\mu^{k-m}\right)\xi.$$

Note that if $v \in W_r$, $v' \in W_s$, $k \ge 1$, $1 \le m \le k$, and $1 \le j \le m$ are such that

$$0 \ne \langle [J_\mu^{j-1} J_{(v)} J_\mu^{m-j}\xi, J_\mu^{k-m}\xi], v' \rangle = -\langle J_{v'} J_\mu^{k-m}\xi, J_\mu^{j-1} J_{(v)} J_\mu^{m-j}\xi \rangle,$$

then $J_v J_\mu^{m-j}\xi \ne 0$ and $J_{v'} J_\mu^{k-m}\xi \ne 0$, which implies $k - m \ge s$ then $m - j \ge r$, and $m \ge r + 1$ and $k \ge r + s + 1$. Therefore,

$$\begin{aligned} A_{W_s}^{W_r}(\varepsilon)v &= \pi_s\left(\sum_{k=1}^{\infty}\frac{\varepsilon^{k-1}}{2(k+2)}\sum_{m=1}^{k}\frac{k-2m}{(m+1)!(k-m+1)!}\left[\left[\left(\sum_{j=1}^{m}J_\mu^{j-1}J_vJ_\mu^{m-j}\right)\xi, J_\mu^{k-m}\xi\right]\right]\right) \\ &= \varepsilon^{r+s}\pi_s\left(\sum_{k=r+s+1}^{\infty}\frac{\varepsilon^{k-r-s-1}}{2(k+2)}\sum_{m=1}^{k}\frac{k-2m}{(m+1)!(k-m+1)!}\left[\left[\left(\sum_{j=1}^{m}J_\mu^{j-1}J_vJ_\mu^{m-j}\right)\xi, J_\mu^{k-m}\xi\right]\right]\right). \end{aligned}$$

Define the matrix $a(\varepsilon)$ as

$$\begin{aligned} a_{V_1}^{V_1}(\varepsilon) &= A_{V_1}^{V_1}(\varepsilon), &\quad a_{V_1}^{W_\ell}(\varepsilon) &= A_{V_1}^{W_\ell}(\varepsilon)/\varepsilon^\ell, \\ a_{W_\ell}^{V_1}(\varepsilon) &= A_{W_\ell}^{V_1}(\varepsilon)/\varepsilon^\ell, &\quad a_{W_s}^{W_r}(\varepsilon) &= A_{W_s}^{W_r}(\varepsilon)/\varepsilon^{r+s}, \end{aligned}$$

where $0 \le \ell, r, s \le d$. Clearly, $\varepsilon \mapsto a(\varepsilon)$ is an analytic map with $a(0)$ given by

$$\begin{aligned} a_{V_1}^{V_1}(0) &= \mathrm{Id}_{V_1} \\ a_{W_0}^{V_1}(0) &= 0 \\ a_{W_\ell}^{V_1}(0) &= \pi_\ell\left(\frac{1}{2(\ell+2)}\frac{\ell}{(\ell+1)!}[(\cdot), J_\mu^\ell\xi]\right) = \frac{\ell}{2(\ell+2)!}\pi_\ell[(\cdot), J_\mu^\ell\xi] \\ a_{V_1}^{W_\ell}(0) &= \frac{1}{(\ell+2)!}J_{(\cdot)}J_\mu^\ell\xi \\ a_{W_s}^{W_r}(0) &= \frac{1}{2(r+s+3)}\frac{s-r-1}{(r+2)!(s+1)!}\pi_s[J_{(\cdot)}J_\mu^r\xi, J_\mu^s\xi]. \end{aligned} \tag{6.10}$$

The proof that $\det(a(0)) > 0$ is long, we will show it in Section 6.5: see Lemma 6.13. Finally, statement (b) follows from the relation $\det(A(\varepsilon)) = \varepsilon^{N_{\mathrm{SExp}}(\xi,\mu)}\det(a(\varepsilon))$. □

## 6.5 Determinant of $a(0)$

**Lemma 6.10.** *For $0 \le \ell, r, s \le d$, define the following maps*:

$$\begin{aligned} M_\ell &: V_1 \to W_\ell, &\quad M_\ell(w) &\coloneqq \pi_\ell[w, J_\mu^\ell\xi], \\ M^\ell &: W_\ell \to V_1, &\quad M^\ell(v) &\coloneqq -J_vJ_\mu^\ell\xi, \\ M_s^r &: W_r \to W_s, &\quad M_s^r(v) &\coloneqq -\pi_s[J_vJ_\mu^r\xi, J_\mu^s\xi]. \end{aligned}$$

*The following identities hold*:

$$\begin{aligned} (M_\ell)^* &= M^\ell, \\ (M_s^r)^* &= M_r^s, \\ M_s^r &= M_s \circ M_r^* = M_s \circ M^r, \end{aligned}$$

*where $\cdot^*$ denotes the conjugate with respect to the scalar product $\langle\cdot,\cdot\rangle$.*

**Proof.** Let $w \in V_1$ and $v \in W_\ell$. Then,

$$\langle M_\ell w, v\rangle = \langle [w, J_\mu^\ell\xi], v\rangle = -\langle w, J_vJ_\mu^\ell\xi\rangle = \langle w, M^\ell v\rangle.$$

Therefore, $(M_\ell)^* = M^\ell$.

Let $v_r \in W_r$ and $v_s \in W_s$. Then,

$$\langle M_s^r v_r, v_s\rangle = -\langle [J_{v_r}J_\mu^r\xi, J_\mu^s\xi], v_s\rangle = \langle J_{v_r}J_\mu^r\xi, J_{v_s}J_\mu^s\xi\rangle = -\langle v_r, [J_{v_s}J_\mu^s\xi, J_\mu^r\xi]\rangle = \langle v_r, M_r^s v_s\rangle.$$

Therefore, $(M_s^r)^* = M_r^s$.

Let $v_r \in W_r$. Then,

$$M_s \circ M^r v_r = M_s(-J_{v_r} J_\mu^r \xi) = -\pi_s[J_{v_r} J_\mu^r \xi, J_\mu^s \xi] = M_s^r v_r.$$

So, the last equality is also proved. □

**Lemma 6.11.** *Define*

$$\mathscr{M} := \left( \frac{M_s^r}{r+s+3} \right)_{r,s=0}^{d} : V_2 \to V_2.$$

*The matrix $\mathscr{M}$ is symmetric, positive definite, and non-singular. In particular,* $\det(\mathscr{M}) > 0$.

**Proof.** Note that for all $v, v' \in V_2$,

$$\langle \mathscr{M} v, v' \rangle = \sum_{r,s=0}^{d} \frac{\langle M_s^r(\pi_r v), \pi_s v' \rangle}{r+s+3}.$$

We apply Lemma 6.14 to the following spaces: $V_1$ with the scalar product $g = \langle \cdot, \cdot \rangle$ and $\mathbb{R}^{d+1}$ with the scalar product $h$ whose matrix with respect to the standard basis $(e_0, \dots, e_d)$ of $\mathbb{R}^{d+1}$ is

$$h_{ij} = h(e_i, e_j) = \frac{1}{i+j+3} = \frac{1}{(i+2)+(j+2)-1},$$

which is a minor of a Hilbert matrix $\left( \frac{1}{i+j-1} \right)_{i,j \geq 1}$. By Lemma 6.14, the bilinear form $b = g \otimes h$ is a scalar product on $V_1 \otimes \mathbb{R}^{d+1}$.

Let $\mathbb{M} : V_2 \to V_1 \otimes \mathbb{R}^{d+1}$ be the map

$$\mathbb{M}(v) = \sum_{\ell=0}^{d} M^\ell(\pi_\ell v) \otimes e_\ell.$$

We claim that, for $v, v' \in V_2$,

$$\mathbb{M}^*(b)(v, v') = \langle \mathscr{M} v, v' \rangle. \tag{6.11}$$

Indeed,

$$\begin{aligned}
\mathbb{M}^*(b)(v, v') &= b(\mathbb{M} v, \mathbb{M} v') \\
&= \sum_{r,s=0}^{d} b(M^r(\pi_r v) \otimes e_r, M^s(\pi_s v') \otimes e_s) \\
&= \sum_{r,s=0}^{d} \frac{\langle M^r(\pi_r v), M^s(\pi_s v') \rangle}{r+s+3} \\
&= \sum_{r,s=0}^{d} \frac{\langle M_s M^r(\pi_r v), \pi_s v' \rangle}{r+s+3} \\
&= \langle \mathscr{M} v, v' \rangle.
\end{aligned}$$

Next, we claim that the map $\mathbb{M}$ is injective. Indeed, if $v \in V_2$ is such that $\mathbb{M} v = 0$, then $0 = M^\ell(\pi_\ell v) = -J_{\pi_\ell v} J_\mu^\ell \xi$ for all $\ell$. From (6.6), we obtain that $\pi_\ell v = 0$ for all $\ell$, and thus $v = 0$.

Since $b$ is positive definite and $\mathbb{M}$ is injective, we obtain from (6.11) that $\mathscr{M}$ is symmetric, non-singular, and positive definite. In particular, $\det(\mathscr{M}) > 0$. □

The next lemma is a standard result from Linear Algebra.

**Lemma 6.12.** *Let $A, B, C, D$ be matrices of suitable dimensions, or linear maps between suitable spaces. Suppose $A$ is invertible. Then,*

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\cdot \det(D - CA^{-1}B).$$

**Lemma 6.13.** *The determinant of* $a(0)$ *is*

$$\det(a(0)) = \left(\prod_{\ell=0}^{d}\left(\frac{\ell+1}{(\ell+2)!}\right)^{\dim W_\ell}\right)^2 \cdot \det(\mathscr{M}) > 0.$$

**Proof.** We can rewrite the map $a(0)$ from (6.10) as follows:

$$\begin{aligned}
a_{V_1}^{V_1}(0) &= \mathrm{Id}_{V_1} \\
a_{W_\ell}^{V_1}(0) &= \frac{\ell}{2(\ell+2)!}M_\ell \\
a_{V_1}^{W_\ell}(0) &= -\frac{1}{(\ell+2)!}M^\ell \\
a_{W_s}^{W_r}(0) &= \frac{1+r-s}{2(r+s+3)(r+2)!(s+1)!}M_s^r.
\end{aligned}$$

Since

$$a_{W_s}^{V_1}(0)a_{V_1}^{W_r}(0) = -\frac{s}{2(s+2)!}\frac{1}{(r+2)!}M_s \circ M^r = -\frac{s}{2(s+2)!(r+2)!}M_s^r,$$

we obtain from Lemma 6.12

$$\begin{aligned}
\det(a(0)) &= \det((a_{W_s}^{W_r}(0))_{r,s=0}^d - (a_{W_\ell}^{V_1}(0))_{\ell=0}^d(a_{V_1}^{W_\ell}(0))_{\ell=0}^d) \\
&= \det\left(\left(\frac{1+r-s}{2(r+s+3)(r+2)!(s+1)!}M_s^r\right)_{r,s=0}^d + \left(\frac{s}{2(s+2)!(r+2)!}M_s^r\right)_{r,s=0}^d\right) \\
&= \det\left(\left(\frac{(r+1)(s+1)}{(r+s+3)(r+2)!(s+2)!}M_s^r\right)_{r,s=0}^d\right) \\
&= \left(\prod_{\ell=0}^{d}\left(\frac{\ell+1}{(\ell+2)!}\right)^{\dim W_\ell}\right)^2 \cdot \det\left(\left(\frac{M_s^r}{r+s+3}\right)_{r,s=0}^d\right).
\end{aligned}$$

□

**Lemma 6.14.** (A side-note on tensor product of matrices) *Let $V$ and $W$ be two vector spaces and let $g$ and $h$ be two bilinear maps on $V$ and $W$, respectively. Let $b = g \otimes h$ be the bilinear map on $V \otimes W$ defined by $b(v_1 \otimes w_1, v_2 \otimes w_2) = g(v_1, v_2)h(w_1, w_2)$ for all $v_i \in V$ and $w_i \in W$.*

*If $g$ and $h$ are symmetric and positive definite (i.e., scalar products), then $b$ is also symmetric and positive definite (i.e., a scalar product).*

**Proof.** Clearly $b$ is symmetric.

Let $(e_1, \dots, e_d)$ be an orthonormal basis for $(W, h)$. Any element of $V \otimes W$ can be written as $\sum_{i=1}^d v_i \otimes e_i$ for some $v_i \in V$. Indeed $\sum_k u_k \otimes w_k = \sum_k u_k \otimes (\sum_{i=1}^d w_k^i e_i) = \sum_{i=1}^d (\sum_k w_k^i u_k) \otimes e_i$. So if $x = \sum_{i=1}^d v_i \otimes e_i \in V \otimes W$, then

$$b(x,x) = b\left(\sum_{i=1}^d v_i \otimes e_i, \sum_{j=1}^d v_j \otimes e_j\right) = \sum_{i,j=1}^d g(v_i, v_j)h(e_i, e_j) = \sum_{i=1}^d g(v_i, v_i) \geq 0,$$

and clearly $b(x,x) = 0$ if and only if $x = 0$. Therefore, $b$ is positive definite. □

# 7 Examples of Carnot groups of step two

In this section, we collect several examples of step-two Carnot groups. In Section 7.1, we recall the classical example of Heisenberg group. Then, we give several generalizations of the Heisenberg group in Sections 7.2–7.4. In particular, in Section 7.2, we give the examples of free step-two groups on which $N_{\mathrm{GEO}} > 2Q - n$. Then, in Section 7.3, we give the main examples of the work: groups on which $N_{\mathrm{CE}} > N_{\mathrm{GEO}}$. Finally, in Section 7.4, we provide more examples of step-two groups where the $N_{\mathrm{CE}}$ can be computed.

## 7.1 The Heisenberg group $\mathbb{H}$

Recall that the simplest non-abelian Carnot group is the Heisenberg group $\mathbb{H}$ whose Lie algebra is given by $\mathfrak{g} = V_1 \oplus V_2$ with

$$V_1 \coloneqq \operatorname{span}\{X_1, X_2\}, \quad V_2 \coloneqq \operatorname{span}\{Y\}.$$

Here $\{X_1, X_2\}$ is an orthonormal basis of $V_1$ and the only nontrivial bracket relation of $\mathfrak{g}$ is $[X_1, X_2] = Y$. The topological dimension of $\mathbb{H}$ is $n = 3$ and the homogeneous dimension is $Q = 4$.

By formula (6.1), we obtain that, for $a, b, c, d \in \mathbb{R}$ and $\mu \in V_2$,

$$(ad - bc)\langle \mu, Y\rangle = \langle \mu, [aX_1 + bX_2, cX_1 + dX_2]\rangle = \langle J_\mu(aX_1 + bX_2), cX_1 + dX_2\rangle.$$

Since $a, b, c, d \in \mathbb{R}$ are arbitrary, the matrix representation of $J_\mu$ with respect to the orthonormal basis $\{X_1, X_2\}$ is

$$J_\mu = \langle \mu, Y\rangle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.1}$$

As a result (Table 1), it is easy to check with Theorem 6.7 that

$$\Gamma(\xi, \mu) = \begin{cases} 0 & \text{if } \xi \neq 0, \\ \infty & \text{if } \xi = 0. \end{cases}$$

By Theorem A, the geodesic dimension of $\mathbb{H}$ is 5, in accordance with the literature. Furthermore, it follows from the result of [11] that $N_{\mathrm{CE}} = 5$ on $\mathbb{H}$. See also Proposition 7.7 in Section 7.4.

**Table 1:** Objects from Definition 6.6 for the first Heisenberg group $\mathbb{H}$

| | $\xi = 0$ | $\xi \neq 0, \quad \mu = 0$ | $\xi \neq 0, \quad \mu \neq 0$ |
|---|---|---|---|
| | $U^0 = \{0\}$ | $U^0 = \{0\}$ | $U^0 = \{0\}$ |
| $U^\ell$ | $U^1 = \{0\}$ | $U^1 = \mathbb{R}\xi$ | $U^1 = \mathbb{R}\xi$ |
| | $U^\ell = \{0\}, \quad \ell > 1$ | $U^\ell = \mathbb{R}\xi, \quad \ell > 1$ | $U^\ell = V_1, \quad \ell > 1$ |
| | $U_0 = V_2$ | $U_0 = V_2$ | $U_0 = V_2$ |
| $U_\ell$ | $U_1 = V_2$ | $U_1 = \{0\}$ | $U_1 = \{0\}$ |
| | $U_\ell = V_2, \quad \ell > 1$ | $U_\ell = \{0\}, \quad \ell > 1$ | $U_\ell = \{0\}, \quad \ell > 1$ |
| $W_\infty$ | $W_\infty = V_2$ | $W_\infty = \{0\}$ | $W_\infty = \{0\}$ |
| $W_j, \quad j < \infty$ | $W_0 = \{0\}$ | $W_0 = V_2$ | $W_0 = V_2$ |
| | $W_j = \{0\}, \quad 1 \leq j < \infty$ | $W_j = \{0\}, \quad 1 \leq j < \infty$ | $W_j = \{0\}, \quad 1 \leq j < \infty$ |
| $\Gamma(\xi, \mu) = N_{\mathrm{SExp}}$ | $\infty$ | 0 | 0 |

## 7.2 Free step-two Carnot group with $k$ generators $N_{k,2}$

One possible generalization of the Heisenberg group $\mathbb{H}$ is the free step-two group $N_{k,2}$. For every $k \geq 2$, the Lie algebra of $N_{k,2}$ is $\mathfrak{g} = V_1 \oplus V_2$ with

$$V_1 := \operatorname{span}\{X_1, \dots, X_k\}, \quad V_2 := \operatorname{span}\{Y_{1,2}, Y_{1,3}, \dots, Y_{k-1,k}\}.$$

Here $\{X_1, \dots, X_k\}$ is an orthonormal basis of $V_1$ with the property $[X_i, X_j] = Y_{i,j}$, $\forall 1 \leq i < j \leq k$. For $k = 2$, $N_{2,2}$ is exactly the Heisenberg group. As before, formula (6.1) yields for every $\mu \in V_2$:

$$\sum_{i<j}(a_ib_j - a_jb_i)\langle \mu, Y_{i,j}\rangle = \left\langle \mu, \left[\sum_{i=1}^{k} a_iX_i, \sum_{j=1}^{k} b_jX_j\right]\right\rangle = \sum_{1\leq i,j\leq k} a_ib_j\langle J_\mu X_i, X_j\rangle$$

and thus, under the orthonormal basis $\{X_1, \dots, X_k\}$

$$J_\mu = \begin{pmatrix} 0 & -\langle \mu, Y_{1,2}\rangle & -\langle \mu, Y_{1,3}\rangle & \cdots & -\langle \mu, Y_{1,k}\rangle \\ \langle \mu, Y_{1,2}\rangle & 0 & -\langle \mu, Y_{2,3}\rangle & \cdots & -\langle \mu, Y_{2,k}\rangle \\ \langle \mu, Y_{1,3}\rangle & \langle \mu, Y_{2,3}\rangle & 0 & \cdots & -\langle \mu, Y_{3,k}\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle \mu, Y_{1,k}\rangle & \langle \mu, Y_{2,k}\rangle & \langle \mu, Y_{3,k}\rangle & \cdots & 0 \end{pmatrix}.$$

Since $\{Y_{1,2}, Y_{1,3}, \dots, Y_{k-1,k}\}$ is a basis, the map $\mu \mapsto J_\mu$ gives a linear isomorphism between $V_2$ and $\mathfrak{so}(V_1, \langle\cdot,\cdot\rangle)$. Fix a pair $(\xi, \mu) \in V_1 \oplus V_2$. Recall that the spaces $U^\ell$, $U_\ell$, and $W_\ell$ are defined in Definition 6.6.

**Lemma 7.1.** *For $(\xi, \mu) \in V_1 \oplus V_2$ fixed, we have*

$$U_\ell \simeq \mathfrak{so}((U^\ell)^\perp, \langle\cdot,\cdot\rangle), \quad W_\ell \simeq \mathfrak{so}((U^\ell)^\perp, \langle\cdot,\cdot\rangle)/\mathfrak{so}((U^{\ell+1})^\perp, \langle\cdot,\cdot\rangle), \quad \forall \ell \geq 0.$$

*On the RHS, we regard an element of $\mathfrak{so}((U^{\ell+1})^\perp, \langle\cdot,\cdot\rangle)$ as an element of $\mathfrak{so}((U^\ell)^\perp, \langle\cdot,\cdot\rangle)$ by zero extension.*

**Proof.** As in Definition 6.6, $U_\ell := \{v \in V_2 : J_v(U^\ell) = \{0\}\}$, $\forall \ell \geq 0$. It is easy to see that if $v \in U_\ell$, then $J_v$ maps $(U^\ell)^\perp$ to itself, or equivalently $J_v|_{(U^\ell)^\perp} \in \mathfrak{so}((U^\ell)^\perp, \langle\cdot,\cdot\rangle)$. Conversely, if we start from an element in $\mathfrak{so}((U^\ell)^\perp, \langle\cdot,\cdot\rangle)$, by zero extension we obtain a unique element $v \in U_\ell$. This gives us the isomorphism between $U_\ell$ and $\mathfrak{so}((U^\ell)^\perp, \langle\cdot,\cdot\rangle)$. Then, the rest of the lemma follows from the following commutative diagram:

$$\begin{array}{ccc} U_{\ell+1} & \xrightarrow{\simeq} & \mathfrak{so}((U^{\ell+1})^\perp, \langle\cdot,\cdot\rangle) \\ \downarrow & & \downarrow \\ U_\ell & \xrightarrow{\simeq} & \mathfrak{so}((U^\ell)^\perp, \langle\cdot,\cdot\rangle) \end{array}$$

where the map from $\mathfrak{so}((U^{\ell+1})^\perp, \langle\cdot,\cdot\rangle)$ to $\mathfrak{so}((U^\ell)^\perp, \langle\cdot,\cdot\rangle)$ is given by zero extension. □

**Lemma 7.2.** *Fix a pair $(\xi, \mu) \in V_1 \oplus V_2$. If $U^\ell = U^{\ell+1}$, then $U^j = U^\ell$ for every $j \geq \ell$.*

**Proof.** *If $U^\ell = U^{\ell+1}$, then $J_\mu^\ell \xi = \sum_{i=0}^{\ell-1} a_iJ_\mu^i\xi$ for some $a_i \in \mathbb{R}$. Therefore, $J_\mu^{\ell+1}\xi = \sum_{i=0}^{\ell-1} a_iJ_\mu^{i+1}\xi \in U^{\ell+1} = U^\ell$, i.e., $U^{\ell+2} = U^\ell$. By induction, we obtain the lemma.* □

**Proposition 7.3.** *On free step-two group with k generators $N_{k,2}$ with $k \geq 3$, we have*

$$2Q - n = \frac{3k^2 - k}{2} < \frac{3k^2 - k}{2} + \frac{k(k-1)(k-2)}{3} = N_{\mathrm{GEO}} \leq N_{\mathrm{CE}}.$$

**Proof.** For every $\ell \geq 0$, if $U^\ell = U^{\ell+1}$, then $W_\infty = U_\ell \simeq \mathfrak{so}((U^\ell)^\perp, \langle\cdot,\cdot\rangle)$, by Lemmas 7.1 and 7.2. Since $\dim(U^\ell) \leq \ell$, if there is $\ell < k - 1$ with $U^\ell = U^{\ell+1}$, then $W_\infty \neq \{0\}$.

Hence, if $W_\infty = \{0\}$, then $\dim(U^\ell) = \ell$ for all $\ell < k$, and $\dim(U^\ell) \in \{k, k-1\}$ for all $\ell \geq k$. It follows that $\dim(W_\ell) = 0$ for $\ell \geq k-1$ and, for $0 \leq \ell < k-1$,

$$\begin{aligned}\dim(W_\ell) &= \dim(\mathfrak{so}((U^\ell)^\perp, \langle\cdot,\cdot\rangle)) - \dim(\mathfrak{so}((U^{\ell+1})^\perp, \langle\cdot,\cdot\rangle))\\ &= \frac{(k-\ell)(k-\ell-1)}{2} - \frac{(k-\ell-1)(k-\ell-2)}{2} = k-\ell-1,\end{aligned}$$

and thus $d = k-2$. As a result, by (6.9) and Theorem A,

$$N_{\mathrm{GEO}} = \frac{3k^2-k}{2} + \sum_{\ell=1}^{k-2} 2\ell(k-\ell-1) = \frac{3k^2-k}{2} + \frac{k(k-1)(k-2)}{3}. \qquad \square$$

## 7.3 Step-two groups induced by star graphs $K_{1,k}$

In [9], the following step-two stratified Lie algebras are associated with star-shaped graphs. For every $k \geq 1$, the Lie algebra is given by $\mathfrak{g} = V_1 \oplus V_2$ with

$$V_1 \coloneqq \operatorname{span}\{X_0, X_1 \dots, X_k\}, \quad V_2 \coloneqq \operatorname{span}\{Y_1, \dots, Y_k\},$$

where the nontrivial bracket relations are $[X_0, X_j] = Y_j$, for all $1 \leq j \leq k$. We fix a scalar product on $\mathfrak{g}$ such that $\{X_0, X_1, \dots, X_k, Y_1, \dots, Y_k\}$ is an orthonormal basis. We remark that $k = 1$ case corresponds to the Heisenberg group $\mathbb{H}$.

By formula (6.1), if $\sum_{i=0}^k a_i X_i, \sum_{j=0}^k b_j X_j \in V_1$ and $\mu \in V_2$, then

$$\sum_{j=1}^k (a_0 b_j - a_j b_0)\langle \mu, Y_j\rangle = \left\langle \mu, \left[\sum_{i=0}^k a_i X_i, \sum_{j=0}^k b_j X_j\right]\right\rangle = \sum_{0\leq i,j\leq k} a_i b_j \langle J_\mu X_i, X_j\rangle.$$

Therefore, for every $\mu \in V_2$ we have

$$J_\mu = \begin{pmatrix} 0 & -\langle\mu, Y_1\rangle & \cdots & -\langle \mu, Y_k\rangle \\ \langle \mu, Y_1\rangle & & & \\ \vdots & & 0 & \\ \langle \mu, Y_k\rangle & & & \end{pmatrix}$$

with respect to the orthonormal basis $\{X_0, X_1, \dots, X_k\}$. The following proposition answers a question posed by Rizzi in [21], i.e., it shows that there are sub-Riemannian Carnot groups such that $N_{\mathrm{GEO}} \neq N_{\mathrm{CE}}$.

**Proposition 7.4.** *In the framework of step-two groups induced by star graphs $K_{1,k}$ with $k \geq 2$, the following is true*:

$$\Gamma(K_{1,k}) = 0 < 2k - 2 = \hat{\Gamma}(K_{1,k}). \tag{7.2}$$

*In particular, we have $N_{\mathrm{GEO}} < N_{\mathrm{CE}}$ in $K_{1,k}$ with $k \geq 2$*

**Proof.** Using the above bases for $V_1$ and $V_2$, we write $\xi = (\xi_0, \hat{\xi})$ with $\hat{\xi} \in \mathbb{R}^k$, and $\mu \in \mathbb{R}^k$. Then,

$$J_\mu \xi = (-\mu \cdot \hat{\xi}, \xi_0 \mu) \in V_1,$$

where $\cdot$ denotes the standard scalar product on $\mathbb{R}^k$. To compute $\Gamma(\xi, \mu)$, we consider three cases. First, if $\xi_0 \neq 0$, then $U_1 = \{v \in V_2 : J_v \xi = 0\} = \{0\}$. This implies $W_\infty = \{0\}$, $W_0 = V_2$, and $d = 0$.

Second, consider the case $\xi_0 = 0$ and $\mu \cdot \hat{\xi} \neq 0$. For a similar reason we have that $U_1$ has codimension 1 in $V_2$ and $U_2 = \{v \in V_2 : J_v J_\mu \xi = J_v \xi = 0\} = \{0\}$. This implies $W_\infty = \{0\}$ as well, but in this case, $d = 1$, $\dim(W_0) = 1$, and $\dim(W_1) = k - 1$.

In the remaining third case, when $\xi_0 = 0$ and $\mu \cdot \hat{\xi} = 0$, we have $J_\mu \xi = 0$ and thus $W_\infty \neq \{0\}$.

In conclusion, from Theorem 6.7, we obtain

$$\Gamma(\xi,\mu)=\begin{cases}0 & \text{if } \xi_0\neq 0,\\ 2k-2 & \text{if } \xi_0=0 \text{ and } \mu\cdot\hat{\xi}\neq 0,\\ \infty & \text{if } \xi_0=0 \text{ and } \mu\cdot\hat{\xi}=0.\end{cases}$$

This implies (7.2) by (1.5), and the we conclude by Theorems A and B.

## 7.4 Step-two groups $G_A$

In this section, we introduce a subclass of step-two groups which are again generalizations of Heisenberg group $\mathbb{H}$ but not ideal Carnot groups except for very special cases. Given a matrix of full-rank $A=A_{m\times k}=(A_{ij})_{1\le i\le m,1\le j\le k}$ with $m\le k$, the Lie algebra of $G_A$ is given by $\mathfrak{g}=V_1\oplus V_2$ with

$$V_1\coloneqq \operatorname{span}\{X_1,X_2,\ldots,X_{2k-1},X_{2k}\},\qquad V_2\coloneqq\operatorname{span}\{Y_1,\ldots,Y_m\},$$

where the nontrivial relations of the Lie algebra $\mathfrak{g}$ are $[X_{2j-1},X_{2j}]=\sum_{i=1}^m A_{ij}Y_i$, $\forall 1\le j\le k$. The topological dimension of $G_A$ is $n=2k+m$ and the homogeneous dimension is $Q=2k+2m$. This subclass of step-two groups is associated with CR manifolds [18]. We remark that for the case $m=k=1$ and $A=1$, it is nothing but the Heisenberg group $\mathbb{H}$.

We fix a scalar product on $\mathfrak{g}$ such that $\{X_1,X_2,\ldots,X_{2k-1},X_{2k},Y_1,\ldots,Y_m\}$ is an orthonormal basis. In the following, using the above basis, we write $\xi=(\xi_1,\ldots,\xi_{2k})\in\mathbb{R}^{2k}$ and $\mu=(\mu_1,\ldots,\mu_m)\in\mathbb{R}^m$. Moreover, we use $\cdot$ to denote the standard scalar product on $\mathbb{R}^m$ and $A_j\coloneqq(A_{1j},\ldots,A_{mj})$ the $j$th column of the matrix $A$. Using these notations, and by formula (6.1) again, we have

$$J_\mu=\operatorname{diag}\left\{\begin{pmatrix}0 & -\mu\cdot A_1\\ \mu\cdot A_1 & 0\end{pmatrix},\ldots,\begin{pmatrix}0 & -\mu\cdot A_k\\ \mu\cdot A_k & 0\end{pmatrix}\right\}.$$

Observe that from the above matrix, we have $J_\nu J_\mu=J_\mu J_\nu$ and as a consequence $W_\infty=\{\nu\in V_2: J_\nu\xi=0\}$. Thus, we obtain

$$\Gamma(\xi,\mu)=\begin{cases}0 & \text{if } \{\nu\in V_2: J_\nu\xi=0\}=\{0\},\\ \infty & \text{if } \{\nu\in V_2: J_\nu\xi=0\}\neq\{0\},\end{cases}$$

which implies $N_{\mathrm{GEO}}=2Q-n=2k+3m$ by Theorem A.

Now, we compute the explicit formula of the Jacobian of the sub-Riemannian exponential map with respect to the basis $\{X_1,\ldots,X_{2k},Y_1,\ldots,Y_m\}$, and we use this formula to compute the curvature exponent. For the computation on Heisenberg groups, we refer to [5,11]. In fact, using the matrix of $J_\mu$ above, and deducing from Lemma 6.2 again, the sub-Riemannian exponential map is represented by

$$\mathrm{SExp}(\xi,\mu)=\begin{pmatrix}\mathrm{E}_1(\xi_1,\xi_2,\mu\cdot A_1)\\ \vdots\\ \mathrm{E}_1(\xi_{2k-1},\xi_{2k},\mu\cdot A_k)\\ A\begin{pmatrix}\mathrm{E}_2(\xi_1,\xi_2,\mu\cdot A_1)\\ \vdots\\ \mathrm{E}_2(\xi_{2k-1},\xi_{2k},\mu\cdot A_k)\end{pmatrix}\end{pmatrix},$$

where

$$\mathrm{E}_1(w_1,w_2,\nu)\coloneqq\begin{pmatrix}\dfrac{\sin\nu}{\nu}w_1-\dfrac{1-\cos\nu}{\nu}w_2\\ \dfrac{1-\cos\nu}{\nu}w_1+\dfrac{\sin\nu}{\nu}w_2\end{pmatrix}\quad\text{and}\quad \mathrm{E}_2(w_1,w_2,\nu)\coloneqq\frac{1}{2}\frac{\nu-\sin\nu}{\nu^2}(w_1^2+w_2^2).$$

Define

$$\tilde{\mathrm{J}}_{1,1}(\mu) \coloneqq \operatorname{diag}\{\mathrm{J}_{1,1}(\mu\cdot A_1),\ldots,\mathrm{J}_{1,1}(\mu\cdot A_k)\},$$
$$\tilde{\mathrm{J}}_{i,j}(\xi,\mu) \coloneqq \operatorname{diag}\{\mathrm{J}_{i,j}(\xi_1,\xi_2,\mu\cdot A_1),\ldots,\mathrm{J}_{i,j}(\xi_{2k-1},\xi_{2k},\mu\cdot A_k)\},\quad \forall (i,j)\neq(1,1)$$

with

$$\mathrm{J}_{1,1}(v) \coloneqq \begin{pmatrix} \frac{\sin v}{v} & -\frac{1-\cos v}{v} \\ \frac{1-\cos v}{v} & \frac{\sin v}{v} \end{pmatrix},$$
$$\mathrm{J}_{1,2}(w_1,w_2,v) \coloneqq \begin{pmatrix} \frac{v\cos v-\sin v}{v^2}w_1 - \frac{v\sin v - 1+\cos v}{v^2}w_2 \\ \frac{v\sin v-1+\cos v}{v^2}w_1 + \frac{v\cos v-\sin v}{v^2}w_2 \end{pmatrix},$$
$$\mathrm{J}_{2,1}(w_1,w_2,v) \coloneqq \begin{pmatrix} \frac{v-\sin v}{v^2}w_1 & \frac{v-\sin v}{v^2}w_2 \end{pmatrix},$$
$$\mathrm{J}_{2,2}(w_1,w_2,v) \coloneqq \frac{1}{2}\frac{2\sin v - v - v\cos v}{v^3}(w_1^2+w_2^2).$$

Then, the differential is presented by

$$D\mathrm{SExp}(\xi,\mu) = \begin{pmatrix} \tilde{\mathrm{J}}_{1,1}(\mu) & \tilde{\mathrm{J}}_{1,2}(\xi,\mu)A^T \\ A\tilde{\mathrm{J}}_{2,1}(\xi,\mu) & A\tilde{\mathrm{J}}_{2,2}(\xi,\mu)A^T \end{pmatrix},$$

where $T$ denotes the transpose of the matrix. In the computation below, we assume $\tilde{\mathrm{J}}_{1,1}(\mu)$ is invertible and the final formula (7.4) holds for all $(\xi,\mu)$ by continuity. Then, Lemma 6.12 gives

$$\mathrm{Jac}(\mathrm{SExp})(\xi,\mu) = \det(A\tilde{\mathrm{J}}(\xi,\mu)A^T)\prod_{j=1}^{k}\det(\mathrm{J}_{1,1}(\mu\cdot A_j)),$$

where

$$\tilde{\mathrm{J}}(\xi,\mu) \coloneqq \operatorname{diag}\{\mathrm{J}(\xi_1,\xi_2,\mu\cdot A_1),\ldots,\mathrm{J}(\xi_{2k-1},\xi_{2k},\mu\cdot A_k)\},$$

with

$$\mathrm{J}(w_1,w_2,v) \coloneqq \mathrm{J}_{2,2}(w_1,w_2,v) - \mathrm{J}_{2,1}(w_1,w_2,v)\mathrm{J}_{1,1}(v)^{-1}\mathrm{J}_{1,2}(w_1,w_2,v).$$

In fact, we can write down the explicit formulas for $\mathrm{J}(\cdot)$ and $\det(\mathrm{J}_{1,1}(\cdot))$:

$$\mathrm{J}(w_1,w_2,v) = \frac{f_1(\frac{v}{2})}{4f_2(\frac{v}{2})}(w_1^2+w_2^2),\quad \text{and}\quad \det(\mathrm{J}_{1,1}(v)) = f_2\left(\frac{v}{2}\right)^2,$$

with two auxiliary functions $f_1$ and $f_2$ defined by

$$f_1(s) \coloneqq \frac{\sin s - s\cos s}{s^3},\quad f_2(s) \coloneqq \frac{\sin s}{s}. \tag{7.3}$$

Now, we need the following lemma from Linear Algebra.

**Lemma 7.5.** (Cauchy-Binet formula) *Let* $A = (A_1,\ldots,A_k)$ *be an* $m\times k$ *matrix and* $B = \begin{pmatrix} B_1 \\ \vdots \\ B_k \end{pmatrix}$, a $k\times m$ *matrix with* $m\le k$. *Then,*

$$\det(AB) = \sum_{1\le i_1<\cdots<i_m\le k}\det(A_{i_1},\ldots,A_{i_m})\det(B_{i_1}^T,\ldots,B_{i_m}^T).$$

Applying Lemma 7.5 to $B = \tilde{\mathrm{J}}(\xi,\mu)A^T$, we obtain

$$\mathrm{Jac}(\mathrm{SExp})(\xi,\mu) = 4^{-m} \sum_{1 \le i_1 < \cdots < i_m \le k} \det(A_{i_1}, \ldots, A_{i_m})^2 \mathrm{J}^{i_1,\ldots,i_m}(\xi,\mu), \tag{7.4}$$

where $\mathrm{J}^{i_1,\ldots,i_m}(\xi,\mu)$ is defined by

$$\prod_{j \notin \{i_1,\ldots,i_m\}} f_2\left(\frac{\mu \cdot A_j}{2}\right)^2 \prod_{j \in \{i_1,\ldots,i_m\}} f_1\left(\frac{\mu \cdot A_j}{2}\right) f_2\left(\frac{\mu \cdot A_j}{2}\right)(\xi_{2j-1}^2 + \xi_{2j}^2).$$

**Lemma 7.6.** *On step-two groups $G_A$, under the basis $\{X_1, \ldots, X_{2k}, Y_1, \ldots, Y_m\}$, the set $\mathscr{D}$ in* Definition 1.1 *satisfies*

$$\mathscr{D} \subset \mathbb{R}^{2k} \times \{\mu : |\mu \cdot A_j| < 2\pi, \quad \forall 1 \le j \le k\}.$$

**Proof.** From definition it suffices to prove that if $|\mu \cdot A_j| = 2\pi$ for some $j \in \{1, \ldots, k\}$, then $\mathrm{Jac}(\mathrm{SExp})(\xi,\mu) = 0$. Without loss of generality, we assume that $|\mu \cdot A_1| = 2\pi$. In fact, it follows from (7.3) that $f_2(\pm\pi) = 0$, which implies $\mathrm{Jac}(\mathrm{SExp})(\xi,\mu) = 0$ by (7.4). □

**Proposition 7.7.** *On step-two groups $G_A$, $N_{\mathrm{CE}} = 2Q - n = 2k + 3m$.*

**Proof.** By (ii) of Proposition 5.1 as well as Corollary 6.8, we only need to prove

$$\mathrm{Jac}(\mathrm{SExp})(\xi,\lambda\mu) \ge \mathrm{Jac}(\mathrm{SExp})(\xi,\mu), \quad \forall (\xi,\mu) \in \mathscr{D}, \lambda \in [0,1].$$

In fact, noting that the even function $f_2$ is decreasing on $[0,\pi]$, we have $f_2(\lambda s) \ge f_2(s)$ for $s \in [-\pi,\pi]$ and $\lambda \in [0,1]$. For the even function $f_1$, [6, Lemma 25] implies

$$f_1(\lambda s) \ge f_1(s), \quad \forall s \in [-\pi,\pi], \lambda \in [0,1].$$

Then, our proposition follows from the inequalities for $f_1, f_2$ above, Lemma 7.6, and (7.4). □

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