Journal of Functional Analysis 287 (2024) 110558

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Weakly porous sets and Muckenhoupt A_p distance functions [☆]

Theresa C. Anderson [a], Juha Lehrbäck [b,*], Carlos Mudarra [b], Antti V. Vähäkangas [b]

[a] *Carnegie Mellon University, Wean Hall, Hammerschlag Dr., Pittsburgh, PA 15213, USA*
[b] *University of Jyvaskyla, Department of Mathematics and Statistics, P.O. Box 35, Jyvaskyla, FI-40014, Finland*

A R T I C L E   I N F O

Article history:
Received 8 January 2024
Accepted 8 June 2024
Available online 3 July 2024
Communicated by Camil Muscalu

MSC:
42B99
28A75
28A80
42B25

Keywords:
Muckenhoupt weight
Weak porosity
Distance function
Fractals

A B S T R A C T

We consider the class of weakly porous sets in Euclidean spaces. As our first main result we show that the distance weight $w(x) = \operatorname{dist}(x, E)^{-\alpha}$ belongs to the Muckenhoupt class A_1, for some $\alpha > 0$, if and only if $E \subset \mathbb{R}^n$ is weakly porous. We also give a precise quantitative version of this characterization in terms of the so-called Muckenhoupt exponent of E. When E is weakly porous, we obtain a similar quantitative characterization of $w \in A_p$, for $1 < p < \infty$, as well. At the end of the paper, we give an example of a set $E \subset \mathbb{R}$ which is not weakly porous but for which $w \in A_p \setminus A_1$ for every $0 < \alpha < 1$ and $1 < p < \infty$.

☆ T.C.A. was supported in part by an NSF graduate research fellowship, NSF DMS-2231990 and NSP DMS-1954407. She thanks Tuomas Hytönen for an invitation to visit the University of Helsinki where this project originated. C.M. was supported by the Academy of Finland via the projects *Geometric Aspects of Sobolev Space Theory* (grant No. 314789) and *Incidences on Fractals* (grant No. 321896).

* Corresponding author.
E-mail addresses: tanders2@andrew.cmu.edu (T.C. Anderson), juha.lehrback@jyu.fi (J. Lehrbäck), carlos.mudarra@ntnu.no (C. Mudarra), antti.vahakangas@iki.fi (A.V. Vähäkangas).

https://doi.org/10.1016/j.jfa.2024.110558
0022-1236/

1. Introduction

Let $E \subsetneq \mathbb{R}^n$, $n \in \mathbb{N}$, be a nonempty set. We are interested in the Muckenhoupt A_p properties of the weights

$$w(x) = w_{\alpha,E}(x) = \operatorname{dist}(x, E)^{-\alpha}, \qquad x \in \mathbb{R}^n,$$

where $\alpha \in \mathbb{R}$. Previously, these properties have been studied, for instance, in [1–4,8,12]. It is known, by [4, Corollary 3.8(b)], that if the set E is porous, then $w_{\alpha,E}$ belongs to the Muckenhoupt class A_1 if and only if $0 \le \alpha < n - \dim_{\mathrm{A}}(E)$; here $\dim_{\mathrm{A}}(E)$ is the Assouad dimension of E. Since $\dim_{\mathrm{A}}(E) < n$ if and only if $E \subset \mathbb{R}^n$ is porous (see e.g. [11, Section 5]), it follows in particular that for each porous set $E \subset \mathbb{R}^n$ there exists some $\alpha > 0$ such that $w_{\alpha,E}$ is an A_1 weight.

The results in [4] do not apply for nonporous sets, but the bound $0 \le \alpha < n - \dim_{\mathrm{A}}(E)$ for admissible α might suggest that $w_{\alpha,E}$ cannot be an A_1 weight for any $\alpha > 0$ if $E \subset \mathbb{R}^n$ is not porous, since then $\dim_{\mathrm{A}}(E) = n$. However, Vasin showed in [13] that if E is a subset of the unit circle $\mathbb{T} \subset \mathbb{R}^2$, then the weight $w_{\alpha,E}$ belongs to the class $A_1(\mathbb{T})$, for some $\alpha > 0$, if and only if E is *weakly porous*; see Section 3 for the definition and commentary concerning this condition.

The definition of weak porosity in [13] is rather specific to the one-dimensional case. Our first goal in this paper is to extend both this condition and the related characterization of the A_1 property of the weight $\operatorname{dist}(\cdot, E)^{-\alpha}$. The underlying ideas are in principle similar to those in Vasin [13], but the higher dimensional case requires several nontrivial modifications. In particular, we use dyadic definitions and tools, including a type of dyadic iteration, that lead to efficient and natural proofs.

Our first main result can be stated as follows.

Theorem 1.1. *Let $E \subsetneq \mathbb{R}^n$ be a nonempty set. Then $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_1$, for some $\alpha > 0$, if and only if E is weakly porous.*

One consequence of Theorem 1.1 is that if $E \subsetneq \mathbb{R}^n$ is weakly porous, then $\operatorname{dist}(\cdot, E)^{-\alpha}$ is locally integrable for some $\alpha > 0$. This implies that the upper Minkowski dimension of $E \cap B(x, r)$ is strictly less than n for every $x \in \mathbb{R}^n$ and $r > 0$; see Remark 6.8 for more details.

Theorem 1.1 is quantitative in the sense that α and the constants in the A_1 and weak porosity conditions only depend on each other and n. More precise dependencies are given in Lemma 4.1 and Lemma 5.3, which prove the necessity and sufficiency in Theorem 1.1, respectively.

A closely related question is to quantify the precise range of exponents $\alpha \in \mathbb{R}$ for which the weight $w_{\alpha,E}(x) = \operatorname{dist}(x, E)^{-\alpha}$ belongs to the Muckenhoupt class A_p for a given $1 \le p < \infty$. If $E \subset \mathbb{R}^n$ is porous, then it follows from [4, Corollary 3.8] that $w_{\alpha,E} \in A_1$ if and only if $0 \le \alpha < n - \dim_{\mathrm{A}}(E)$, and $w_{\alpha,E} \in A_p$, for $1 < p < \infty$, if and only if

$$(1-p)(n-\dim_{\mathrm{A}}(E)) < \alpha < n-\dim_{\mathrm{A}}(E).$$

In this paper we obtain the following extension of [4, Corollary 3.8] for weakly porous sets, given in terms of the *Muckenhoupt exponent* $\mathrm{Mu}(E)$ that we introduce in Definition 6.1. For a porous set $E \subset \mathbb{R}^n$ it holds that $\mathrm{Mu}(E) = n - \dim_{\mathrm{A}}(E)$, see Section 6 for details.

Theorem 1.2. *Assume that $E \subset \mathbb{R}^n$ is a weakly porous set. Let $\alpha \in \mathbb{R}$ and define $w(x) = \operatorname{dist}(x,E)^{-\alpha}$ for every $x \in \mathbb{R}^n$. Then*

(i) $w \in A_1$ if and only if $0 \le \alpha < \mathrm{Mu}(E)$.
(ii) $w \in A_p$, for $1 < p < \infty$, if and only if

$$(1-p)\,\mathrm{Mu}(E) < \alpha < \mathrm{Mu}(E). \tag{1}$$

If we omit the special case $\alpha = 0$, in which the connection to the geometry of E is lost, then in part (i) of Theorem 1.2 the assumption that E is weakly porous is actually superfluous, and we have the following full characterization.

Theorem 1.3. *Assume that $E \subset \mathbb{R}^n$ is a nonempty set. Let $\alpha \in \mathbb{R} \setminus \{0\}$ and define $w(x) = \operatorname{dist}(x,E)^{-\alpha}$ for every $x \in \mathbb{R}^n$. Then $w \in A_1$ if and only if $0 < \alpha < \mathrm{Mu}(E)$.*

By combining Theorems 1.1 and 1.3, we see that E is weakly porous if and only if $\mathrm{Mu}(E) > 0$; cf. Corollary 6.6 and Remark 6.7 for related comments.

Theorem 1.3 raises the question whether also (1) could provide a full characterization of $w_{\alpha,E} \in A_p$ when $\alpha \neq 0$ and $1 < p < \infty$. In Section 8 we show that this is *not* the case, by giving a nontrivial construction of a set $E \subset \mathbb{R}^n$ which is not weakly porous (whence $\mathrm{Mu}(E) = 0$) but still $w_{\alpha,E} \in A_p$ for all $0 < \alpha < 1$ and all $1 < p < \infty$. This set illustrates the delicate interplay between the Muckenhoupt conditions and the distance functions, and also gives a novel type of an example of weights which are in A_p for all $1 < p < \infty$ but not in A_1. Nevertheless, a full characterization of sets $E \subset \mathbb{R}^n$ for which $w_{\alpha,E} \in A_p$ for some (or all) $1 < p < \infty$ remains an open question.

Another interesting consequence of Theorem 1.2 is the following strong self-improvement property of A_p-distance weights for weakly porous sets: if $\alpha \ge 0$ and E is weakly porous, then $w_{\alpha,E} \in A_p$ for some $1 < p < \infty$ (i.e. $w_{\alpha,E} \in A_\infty$) if and only if $w_{\alpha,E} \in A_1$. The example in Section 8 shows that this is not true for general sets.

The outline for the rest of the paper is as follows. In Section 2 we introduce notation and recall some definitions and properties of dyadic decompositions and Muckenhoupt weights. Weakly porous sets are defined in Section 3, where we also examine some of their basic properties. Theorem 1.1 is proved in Sections 4 and 5. Section 6 contains the definition of the Muckenhoupt exponent and the proofs of Theorems 1.2 and 1.3, together with some related results. In Section 7, we give an example of a weakly porous set $E \subset \mathbb{R}^n$ which is not porous and compute explicitly the Muckenhoupt exponent of

E. Finally, in Section 8 we construct the set $E \subset \mathbb{R}$ which is not weakly porous, but still $w_{\alpha,E} \in A_p$ for all $0 < \alpha < 1$ and $1 < p < \infty$.

2. Preliminaries

Throughout this paper, we consider $\mathbb{R}^n$ equipped with the Euclidean distance and the n-dimensional Lebesgue (outer) measure. The diameter of a set $E \subset \mathbb{R}^n$ is denoted by $\operatorname{diam}(E)$ and $|E|$ is the Lebesgue (outer) measure of E. If $x \in \mathbb{R}^n$, then $d_E(x) = \operatorname{dist}(x, E)$ denotes the distance from x to the set E, and $\operatorname{dist}(E, F)$ is the distance between the sets E and F, that is,

$$\operatorname{dist}(E, F) = \inf\{|x - y| : x \in E,\, y \in F\}.$$

The open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is

$$B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}.$$

In this paper, we only consider cubes which are half-open and have sides parallel to the coordinate axes. That is, a cube in $\mathbb{R}^n$ is a set of the form

$$Q = [a_1, b_1) \times \cdots \times [a_n, b_n),$$

with side-length $\ell(Q) = b_1 - a_1 = \cdots = b_n - a_n$. For $x \in \mathbb{R}^n$ and $r > 0$, the cube with center x and side length $2r$ is

$$Q(x, r) = \big\{y \in \mathbb{R}^n : -r \le y_j - x_j < r \text{ for all } j = 1, \dots, n\big\}. \tag{2}$$

Clearly,

$$|Q(x, r)| = (2r)^n \quad \text{and} \quad \operatorname{diam}(Q(x, r)) = (2\sqrt{n})r.$$

The dyadic decomposition of a cube $Q_0 \subset \mathbb{R}^n$ is

$$\mathcal{D}(Q_0) = \bigcup_{j=0}^{\infty} \mathcal{D}_j(Q_0),$$

where each $\mathcal{D}_j(Q_0)$ consists of the 2^{jn} pairwise disjoint (half-open) cubes Q, with side length $\ell(Q) = 2^{-j}\ell(Q_0)$, such that

$$Q_0 = \bigcup_{Q \in \mathcal{D}_j(Q_0)} Q$$

for every $j = 0, 1, 2, \dots$. The cubes in $\mathcal{D}(Q_0)$ are called dyadic cubes (with respect to Q_0) and they satisfy following properties:

(D1) Let $j \geq 1$ and $Q \in \mathcal{D}_j(Q_0)$. Then there exists a unique dyadic cube $\pi Q \in \mathcal{D}_{j-1}(Q_0)$ satisfying $Q \subset \pi Q$. The cube πQ is called the dyadic parent of Q, and Q is called a dyadic child of πQ.
(D2) Every dyadic cube $Q \in \mathcal{D}(Q_0)$ has 2^n dyadic children.
(D3) Nestedness property: $P \cap Q \in \{P, Q, \emptyset\}$ for every $P, Q \in \mathcal{D}(Q_0)$.

A locally integrable function w in $\mathbb{R}^n$, with $w(x) > 0$ for almost every $x \in \mathbb{R}^n$, is called a weight in $\mathbb{R}^n$.

Definition 2.1. A weight w in $\mathbb{R}^n$ belongs to the Muckenhoupt class A_1 if there exists a constant C such that

$$\fint_Q w(x)\,dx \leq C \operatorname*{ess\,inf}_{x \in Q} w(x), \tag{3}$$

for every cube $Q \subset \mathbb{R}^n$. The smallest possible constant C in (3) is called the A_1 constant of w, and it is denoted by $[w]_{A_1}$.

Above, we have used the notation

$$\fint_A w(x)\,dx = \frac{1}{|A|} \int_A w(x)\,dx$$

for the mean value integral over a measurable set $A \subset \mathbb{R}^n$ with $0 < |A| < \infty$.

For $1 < p < \infty$, the class A_p is defined as follows.

Definition 2.2. A weight w in $\mathbb{R}^n$ belongs to the Muckenhoupt class A_p, for $1 < p < \infty$, if there exists a constant C such that

$$\fint_Q w(x)\,dx \left(\fint_Q w(x)^{\frac{1}{1-p}}\,dx \right)^{p-1} \leq C \tag{4}$$

for every cube $Q \subset \mathbb{R}^n$. The smallest possible constant C in (4) is called the A_p constant of w, and it is denoted by $[w]_{A_p}$.

We recall that the inclusions $A_1 \subset A_p \subset A_q$ hold for $1 \leq p \leq q$. Also, it is immediate that $w \in A_p$, for $1 < p < \infty$, if and only if $w^{1-p'} \in A_{p'}$, and then $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{1/(p-1)}$. Here $p' = \frac{p}{p-1}$ is the conjugate exponent of $1 < p < \infty$. See [6, Chapter IV] for an introduction to the theory of Muckenhoupt weights.

The following elementary property will be useful in Section 6.

Lemma 2.3. *Let $w \in A_p$ for some $1 < p < \infty$. If $w^\beta \in A_1$ for some $\beta > 0$, then $w \in A_1$.*

Proof. Let $q \geq p$ be large enough so that $s = \frac{1}{q-1} \leq \beta$ and $w \in A_q$. Then we have $w^s \in A_1$ as well, thanks to Jensen's inequality. The A_q condition on a cube $Q \subset \mathbb{R}^n$ for w yields

$$\fint_Q w \leq [w]_{A_q} \left(\fint_Q w^{\frac{1}{1-q}} \right)^{1-q} = [w]_{A_q} \left(\fint_Q w^{-s} \right)^{-1/s} \leq [w]_{A_q} \left(\fint_Q w^{s} \right)^{1/s}$$
$$\leq [w]_{A_q} \left([w^s]_{A_1} \operatorname*{ess\,inf}_Q w^s \right)^{1/s} = [w]_{A_q} [w^s]_{A_1}^{1/s} \operatorname*{ess\,inf}_Q w,$$

and thus $w \in A_1$. $\square$

3. Weakly porous sets

Recall that a set $E \subset \mathbb{R}^n$ is *porous* if there exists a constant $c > 0$ such that for every $x \in \mathbb{R}^n$ and $r > 0$ there exists $y \in \mathbb{R}^n$ satisfying $B(y, cr) \subset B(x, r) \setminus E$. Equivalently, E is porous if and only if there is a constant $c > 0$ such that for all cubes $Q_0 \subset \mathbb{R}^n$ there is a dyadic subcube $Q \in \mathcal{D}(Q_0)$ such that $Q \cap E = \emptyset$ and $|Q| \geq c|Q_0|$.

In [13] Vasin defined *weak porosity* in the unit circle $\mathbb{T} \subset \mathbb{R}^2$ as follows: a set $E \subset \mathbb{T}$ is weakly porous, if there are constants $c, \delta > 0$ such that if $I \subset \mathbb{T}$ is an arbitrary arc, then

$$\sum |J_k| \geq c|I|,$$

where the sum is taken over all (pairwise disjoint) subarcs $J_k \subset I$ that contain no points of E and satisfy $|J_k| \geq \delta|J|$, where $J \subset I$ is a lengthwise largest subarc without points of E. The subarcs that do not intersect E are called *free arcs*.

We consider an extension of the above definition to $\mathbb{R}^n$.

Definition 3.1. Let $E \subset \mathbb{R}^n$ be a nonempty set.

(i) When $P \subset \mathbb{R}^n$ is a cube, a dyadic subcube $Q \subset \mathcal{D}(P)$ is called *E-free* if $E \cap Q = \emptyset$. We denote by $\mathcal{M}(P) \in \mathcal{D}(P)$ a largest E-free dyadic subcube of P, that is, $\ell(\mathcal{M}(P)) \geq \ell(R)$ if $R \in \mathcal{D}(P)$ is an E-free dyadic subcube of P. Such a cube need not be unique, but we fix one of them.

(ii) The set $E \subset \mathbb{R}^n$ is *weakly porous*, if there are constants $0 < c, \delta < 1$ such that for all cubes $P \subset \mathbb{R}^n$ there exist $N \in \mathbb{N}$ and pairwise disjoint E-free cubes $Q_k \in \mathcal{D}(P)$, $k = 1, \ldots, N$, such that $|Q_k| \geq \delta|\mathcal{M}(P)|$ for all $k = 1, \ldots, N$ and

$$\sum_{k=1}^{N} |Q_k| \geq c|P|. \tag{5}$$

Instead of dyadic cubes, also general subcubes of P could be used in the definition of weak porosity. However, the dyadic formulation is convenient from the point of view of our proofs. Notice also that inequality (5) can be written as

$$\left|\bigcup_{k=1}^{N} Q_k\right| \geq c|P|,$$

since the cubes $Q_1, \ldots, Q_N$ are pairwise disjoint. Hence, the weak porosity of a set E can roughly be described as follows: for every cube P, the union of those disjoint E-free subcubes that are not too small (compared to the largest E-free cube in P) has measure comparable to that of P.

The following properties are easy to verify using the definition of weak porosity:

- If $E \subset \mathbb{R}^n$ is porous, then E is weakly porous.
- $E \subset \mathbb{R}^n$ is weakly porous if and only if the closure $\overline{E}$ is weakly porous.
- If $E \subset \mathbb{R}^n$ is weakly porous, then $|E| = 0$. This is a consequence of the Lebesgue differentiation theorem.
- Weak porosity implicitly implies that for every cube $P \subset \mathbb{R}^n$ there exists an E-free dyadic subcube $Q \in \mathcal{D}(P)$.

Let $E \subset \mathbb{R}^n$ be a nonempty set. Given a cube $P \subset \mathbb{R}^n$ and $\delta > 0$, we write

$$\widehat{\mathcal{F}_\delta}(P) = \{Q \in \mathcal{D}(P) \,:\, |Q| \geq \delta|\mathcal{M}(P)| \text{ and } Q \cap E = \emptyset\}.$$

We denote by $\mathcal{F}_\delta(P)$ the maximal subfamily of the cubes in $\widehat{\mathcal{F}_\delta}(P)$. That is, each $R \in \widehat{\mathcal{F}_\delta}(P)$ is contained in some cube $Q \in \mathcal{F}_\delta(P)$ and if $Q \in \mathcal{F}_\delta(P)$, then Q is not strictly contained in another cube in $\widehat{\mathcal{F}_\delta}(P)$. Observe that the cubes in $\mathcal{F}_\delta(P)$ are pairwise disjoint, since two dyadic cubes are either disjoint, or one of them is strictly contained in the other one. The weak porosity of E can now be formulated in terms of the sets $\mathcal{F}_\delta$, since E is weakly porous if and only if there are constants $0 < c, \delta < 1$ such that

$$\sum_{Q \in \mathcal{F}_\delta(P)} |Q| \geq c|P| \qquad \text{for all cubes } P \subset \mathbb{R}^n. \tag{6}$$

Indeed, it is clear that (6) implies weak porosity of E. Conversely,

$$c|P| \leq \sum_{k=1}^{N} |Q_k| \leq \sum_{Q \in \mathcal{F}_\delta(P)} \sum_{k=1}^{N} \mathbf{1}_{Q_k \subset Q} |Q_k| \leq \sum_{Q \in \mathcal{F}_\delta(P)} |Q|,$$

whenever c, δ, P and Q_k, $k = 1, \ldots, N$, are as in Definition 3.1 (ii).

Part (ii) of the next lemma will be important when proving that weak porosity implies the A_1-property for $\operatorname{dist}(\cdot, E)^{-\alpha}$, for some $\alpha > 0$; see the proof of Lemma 5.2.

Lemma 3.2. *Assume that $E \subset \mathbb{R}^n$ is weakly porous set, with constants $0 < c, \delta < 1$. Then the following statements hold.*

(i) Assume that $Q \subset R$ are two cubes such that $E \cap Q \neq \emptyset$ and $|\mathcal{M}(Q)| < 4^{-n}\delta|\mathcal{M}(R)|$. Then

$$|Q| \leq (1 - 2^{-n}c)|R|.$$

(ii) Assume that $Q \subset R$ are two cubes such that $|R| = 2^n|Q|$. Then there exists a number $k = k(n, c) \in \mathbb{N}$ such that

$$|\mathcal{M}(R)| \leq 4^{nk}\delta^{-k}|\mathcal{M}(Q)|.$$

(iii) Assume that $Q \subset R$ are two cubes. Then there exist constants $C = C(n, c, \delta)$ and $\sigma = \sigma(n, c, \delta) > 0$ such that

$$|\mathcal{M}(R)| \leq C \left(\frac{\ell(R)}{\ell(Q)}\right)^{\sigma} |\mathcal{M}(Q)|.$$

Proof. We first remark that the dyadic grids $\mathcal{D}(Q)$ and $\mathcal{D}(R)$ need not be compatible, and this is taken into account in the arguments below.

First we show (i). Fix $S \in \mathcal{F}_\delta(R)$. We claim that the center $x_S \in R$ of S belongs to $R \setminus Q$. Assume the contrary, namely, that $x_S \in Q$. Since S is E-free and Q intersects E, there exists an E-free dyadic cube $T \in \mathcal{D}(Q)$ such that $\ell(T) \geq \ell(S)/4$. It follows that

$$|\mathcal{M}(Q)| \geq |T| \geq 4^{-n}|S| \geq 4^{-n}\delta|\mathcal{M}(R)|.$$

This is a contradiction, since $|\mathcal{M}(Q)| < 4^{-n}\delta|\mathcal{M}(R)|$ by assumption. We have shown that $x_S \in R \setminus Q$, and therefore there exists a cube $S' \subset S \setminus Q$ such that $|S'| = 2^{-n}|S|$. Since $\{S' : S \in \mathcal{F}_\delta(R)\}$ is a pairwise disjoint family of cubes contained in $R \setminus Q$, we obtain that

$$|R| - |Q| = |R \setminus Q| \geq \sum_{S \in \mathcal{F}_\delta(R)} |S'| = 2^{-n} \sum_{S \in \mathcal{F}_\delta(R)} |S|.$$

By weak porosity, the last term above is bounded below by $2^{-n}c|R|$, and reorganizing the terms gives $(1 - 2^{-n}c)|R| \geq |Q|$ as claimed in (i).

Next we show (ii). If $E \cap Q = \emptyset$, then

$$|\mathcal{M}(R)| \leq |R| = 2^n|Q| \leq 4^n\delta^{-1}|Q| = 4^n\delta^{-1}|\mathcal{M}(Q)|.$$

In this case, we may take $k = 1$. In the sequel we assume that $E \cap Q \neq \emptyset$. Choose $k = k(n, c)$ such that $2^{n/k} < \frac{1}{1-2^{-n}c}$. Then there exists a finite sequence

$$Q = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_k = R$$

of cubes such that $|R_i| \cdot |R_{i-1}|^{-1} = 2^{n/k}$. Observe that

$$2^n = (2^{n/k})^k = \prod_{i=1}^{k} \frac{|R_i|}{|R_{i-1}|} = \frac{|R_k|}{|R_0|} = \frac{|R|}{|Q|}.$$

Fix $1 \le i \le k$. We have $\emptyset \neq E \cap Q \subset E \cap R_{i-1}$ and $R_{i-1} \subset R_i$. Moreover,

$$(1 - 2^{-n}c)|R_i| = (1 - 2^{-n}c)2^{n/k}|R_{i-1}| < |R_{i-1}|$$

and therefore the contrapositive of part (i) implies that

$$|\mathcal{M}(R_{i-1})| \ge 4^{-n}\delta|\mathcal{M}(R_i)|$$

for all $i = 1, 2, \dots, k$. This allows us to conclude that

$$|\mathcal{M}(R_0)| \ge 4^{-n}\delta|\mathcal{M}(R_1)| \ge (4^{-n}\delta)^2|\mathcal{M}(R_2)| \ge \cdots \ge (4^{-n}\delta)^k|\mathcal{M}(R_k)|.$$

The desired conclusion follows, since $R_0 = Q$ and $R_k = R$.

Finally, we prove (iii). An easy computation shows that $R \subset \lambda Q$, for $\lambda = 3\ell(R)/\ell(Q)$. Here λQ denotes the cube with the same center as Q and side-length equal to $\lambda\ell(Q)$. Then, for

$$m = 1 + \left\lfloor \log_2 \left(\frac{3\ell(R)}{\ell(Q)} \right) \right\rfloor,$$

we have that $R \subset 2^m Q$. Hence $|\mathcal{M}(R)| \le C(n)|\mathcal{M}(2^m Q)|$. Denote by $C_1 = 4^{nk}\delta^{-k}$ the constant in (ii). Then, by iterating (ii) we obtain

$$\begin{aligned} |\mathcal{M}(2^m Q)| &\le C_1^m |\mathcal{M}(Q)| \le C_1^{1+\log_2\left(\frac{3\ell(R)}{\ell(Q)}\right)} |\mathcal{M}(Q)| \\ &= C(n,c,\delta) \left(\frac{\ell(R)}{\ell(Q)} \right)^{\sigma} |\mathcal{M}(Q)|, \end{aligned}$$

where $\sigma = \sigma(n,c,\delta)$. The claim (iii) follows by combining the above estimates. $\square$

Example 3.3. Unlike for porous sets, inclusions do not preserve weak porosity: there are sets $F \subset E$ such that E is weakly porous but F is not. For instance, $\mathbb{Z}$ is clearly a weakly porous subset of $\mathbb{R}$, but $\mathbb{N} \subset \mathbb{Z}$ is not a weakly porous subset of $\mathbb{R}$. Indeed, assume for the contrary that $\mathbb{N}$ is weakly porous in $\mathbb{R}$ with constants $0 < c, \delta < 1$. Consider cubes $Q_j = [0, 2^j)$, $j \in \mathbb{N}$. Observe that $Q_j \subset R_j = [-2^j, 2^j)$. Lemma 3.2 (ii) implies that there is a constant $C = C(c,\delta) > 0$ such that $2^j = |\mathcal{M}(R_j)| \le C|\mathcal{M}(Q_j)| = C$. By choosing j large enough, we get a contradiction.

4. A_1 implies weak porosity

This section and the following Section 5 contain the proof of Theorem 1.1. We begin by proving the necessity part of the equivalence in the theorem, that is, if $\operatorname{dist}(\cdot, E)^{-\alpha}$ is an A_1 weight, then E is a weakly porous set. The straight-forward proof illustrates in a nice way the connection between the A_1 condition and the definition of weak porosity.

Lemma 4.1. *Let $E \subset \mathbb{R}^n$ be a nonempty set, let $\alpha > 0$, and write $w(x) = \operatorname{dist}(x, E)^{-\alpha}$ for all $x \in \mathbb{R}^n$. If $w \in A_1$, then E is weakly porous with constants depending on n, α and $[w]_{A_1}$.*

Proof. Since $\operatorname{dist}(\cdot, E) = \operatorname{dist}(\cdot, \overline{E})$ and E is weakly porous if and only if $\overline{E}$ is weakly porous, quantitatively, we may assume that E is closed. Assume that $w \in A_1$ and fix $0 < \delta < 1$ to be chosen later. Let $P \subset \mathbb{R}^n$ be a cube and write $\ell = \ell(\mathcal{M}(P))$ for the sidelength of $\mathcal{M}(P)$.

Observe that the set E is of measure zero, since w is locally integrable and $w(x) = \infty$ in E. Since E is closed, for every $x \in P \setminus E$ we have $\operatorname{dist}(x, E) > 0$ and therefore there exists an E-free dyadic cube $Q \in \mathcal{D}(P)$ such that $x \in Q$. As a consequence, we can write $P \setminus E$ as a disjoint union of maximal E-free dyadic cubes $Q \in \mathcal{D}(P)$. Let $x \in P \setminus E$ such that $x \notin \bigcup_{Q \in \mathcal{F}_\delta(P)} Q$. Then the maximal E-free dyadic cube $Q \in \mathcal{D}(P)$ containing x satisfies

$$|Q| < \delta |\mathcal{M}(P)| = \delta \ell^n.$$

Since $\pi Q \in \mathcal{D}(P)$ is not E-free, we have

$$\operatorname{dist}(x, E) \leq \operatorname{diam}(\pi Q) < \delta^{1/n} 2\sqrt{n}\ell.$$

It follows that

$$\ell^{-\alpha} < C(n, \alpha) \delta^{\alpha/n} \operatorname{dist}(x, E)^{-\alpha}$$

for every $x \in (P \setminus E) \setminus \bigcup_{Q \in \mathcal{F}_\delta(P)} Q$. By integrating, and using the fact that E is of measure zero, we obtain

$$\begin{aligned}
\ell^{-\alpha} \frac{|P \setminus \bigcup_{Q \in \mathcal{F}_\delta(P)} Q|}{|P|} &\leq C(n, \alpha) \delta^{\alpha/n} \frac{1}{|P|} \int\limits_{P \setminus \bigcup_{Q \in \mathcal{F}_\delta(P)} Q} \operatorname{dist}(x, E)^{-\alpha} \, dx \\
&\leq C(n, \alpha) \delta^{\alpha/n} \fint_P \operatorname{dist}(x, E)^{-\alpha} \, dx \\
&\leq C(n, \alpha) \delta^{\alpha/n} [w]_{A_1} \operatorname*{ess\,inf}_{x \in P} \operatorname{dist}(x, E)^{-\alpha}.
\end{aligned}$$

Denote by y the center of $\mathcal{M}(P) \subset P$. Then

$$\operatorname*{ess\,inf}_{x\in P} \operatorname{dist}(x, E)^{-\alpha} \le \operatorname{dist}(y, E)^{-\alpha} \le 2^{\alpha}\ell(\mathcal{M}(P))^{-\alpha} = 2^{\alpha}\ell^{-\alpha}.$$

Simplifying, we get

$$|P| - \sum_{Q\in\mathcal{F}_\delta(P)} |Q| = \left| P \setminus \bigcup_{Q\in\mathcal{F}_\delta(P)} Q \right| \le C(n, \alpha)\delta^{\alpha/n}[w]_{A_1}|P|.$$

It remains to choose $\delta = \delta(n, \alpha, [w]_{A_1}) > 0$ so small that $C(n, \alpha)\delta^{\alpha/n}[w]_{A_1} < 1$, and condition (6) follows. □

5. Weak porosity implies A_1

Next, we turn to the sufficiency part of the equivalence in Theorem 1.1, that is, the weak porosity of E implies that $\operatorname{dist}(\cdot, E)^{-\alpha}$ is an A_1 weight; see Lemma 5.3. The proof applies an iteration scheme, which is built on an efficient use of the dyadic definition of weak porosity; see the proof of Lemma 5.2. The following sets $\mathcal{F}_\delta^k$ and $\mathcal{G}_\delta^k$ will be important in the iteration.

Fix a weakly porous closed set $E \subset \mathbb{R}^n$ with constants $0 < c, \delta < 1$ and a cube $P_0 \subset \mathbb{R}^n$. Recall that $\mathcal{F}_\delta(P_0)$ is the maximal subfamily of the collection

$$\widehat{\mathcal{F}_\delta}(P_0) = \big\{Q \in \mathcal{D}(P_0) \,:\, |Q| \ge \delta|\mathcal{M}(P_0)| \text{ and } Q \cap E = \emptyset\big\}.$$

We will need also the complementary family $\mathcal{G}_\delta(P_0)$, which is defined to be the maximal subfamily of the collection

$$\widehat{\mathcal{G}_\delta}(P_0) = \left\{P \in \mathcal{D}(P_0) \,:\, P \subset P_0 \setminus \bigcup_{Q\in\mathcal{F}_\delta(P_0)} Q\right\}.$$

Due to the lattice properties of dyadic cubes, we have $|Q| \ge \delta|\mathcal{M}(P_0)|$ for all $Q \in \mathcal{G}_\delta(P_0)$. Indeed, such a cube $Q \in \mathcal{G}_\delta(P_0)$ cannot be contained in any cube belonging to $\mathcal{F}_\delta(P_0)$, but, on the other hand, the dyadic parent $\pi Q \in \mathcal{D}(P_0)$ of Q must intersect some $R \in \mathcal{F}_\delta(P_0)$. Consequently $R \subsetneq \pi Q$, and

$$|Q| = 2^{-n}|\pi Q| \ge |R| \ge \delta|\mathcal{M}(P_0)|.$$

We let $\mathcal{G}_\delta^0 = \{P_0\}$, $\mathcal{F}_\delta^1 = \mathcal{F}_\delta(P_0)$, $\mathcal{G}_\delta^1 = \mathcal{G}_\delta(P_0)$,

$$\mathcal{F}_\delta^2 = \bigcup_{R\in\mathcal{G}_\delta^1} \mathcal{F}_\delta(R), \qquad \mathcal{G}_\delta^2 = \bigcup_{R\in\mathcal{G}_\delta^1} \mathcal{G}_\delta(R),$$

and in general, for $k = 3, 4, \ldots$, we define

$$\mathcal{F}_\delta^k = \bigcup_{R\in\mathcal{G}_\delta^{k-1}} \mathcal{F}_\delta(R), \qquad \mathcal{G}_\delta^k = \bigcup_{R\in\mathcal{G}_\delta^{k-1}} \mathcal{G}_\delta(R).$$

Lemma 5.1. *Assume that $E \subset \mathbb{R}^n$ is a weakly porous closed set with constants $0 < c, \delta < 1$. Let $P_0 \subset \mathbb{R}^n$ be a cube, and let sets $\mathcal{F}_\delta^k$, for $k = 1, 2, \dots$, be as above. Then*

$$P_0 \setminus E = \bigcup_{k=1}^{\infty} \bigcup_{Q\in\mathcal{F}_\delta^k} Q.$$

Proof. Let $x \in P_0 \setminus E$. Because E is closed, there exists a dyadic cube $Q \in \mathcal{D}(P_0)$ such that $x \in Q$ and $Q \cap E = \emptyset$. We claim that $Q \subset \bigcup_{k=1}^\infty \bigcup \mathcal{F}_\delta^k$. Suppose, for the sake of contradiction, that Q is not a subset of this union. Because $Q \not\subset \bigcup \mathcal{F}_\delta^1$, there exists $R_1 \in \mathcal{G}_\delta^1$ containing Q. Now $Q \not\subset \bigcup \mathcal{F}_\delta(R_1)$, as $Q \not\subset \bigcup \mathcal{F}_\delta^2$. Thus there exists $R_2 \in \mathcal{G}_\delta(R_1)$ containing Q, and again, $Q \not\subset \bigcup \mathcal{F}_\delta(R_2)$. Repeating this argument, for every k we obtain cubes

$$R_1 \supset R_2 \supset \cdots \supset R_k \supset Q$$

with $R_j \in \mathcal{G}_\delta(R_{j-1})$ and such that $Q \not\subset \bigcup \mathcal{F}_\delta(R_k)$. Also, because each R_j is strictly contained in R_{j-1}, we must have $|R_j| \leq 2^{-n}|R_{j-1}|$. Then Q satisfies

$$|Q| < \delta|\mathcal{M}(R_k)| \leq \delta|R_k| \leq \frac{\delta}{2^{n(k-1)}}|R_1| \leq \frac{\delta}{2^{nk}}|P_0|.$$

Letting $k \to \infty$, we derive a contradiction. $\square$

Lemma 5.2. *Assume that $E \subset \mathbb{R}^n$ is a weakly porous closed set with constants $0 < c, \delta < 1$. Let $P_0 \subset \mathbb{R}^n$ be a cube and let sets $\mathcal{F}_\delta^k$, for $k = 1, 2, \dots$, be as above. Then there are constants $0 < \gamma = \gamma(c, \delta, n) < \frac{1}{n}$ and $C = C(c, \delta, n) > 0$ such that*

$$\sum_{k=1}^{\infty} \sum_{Q\in\mathcal{F}_\delta^k} |Q|^{1-\gamma} \leq C|P_0||\mathcal{M}(P_0)|^{-\gamma}.$$

Proof. Let $0 < \gamma < \frac{1}{n}$, whose exact value will be fixed later; we remark that both inequalities $\gamma > 0$ and $\gamma < \frac{1}{n}$ are needed in Lemma 5.3 below. By the definition of $\mathcal{F}_\delta^k$, we obtain

$$\begin{aligned}\sum_{Q\in\mathcal{F}_\delta^k} |Q|^{1-\gamma} &\leq \sum_{R\in\mathcal{G}_\delta^{k-1}} \sum_{Q\in\mathcal{F}_\delta(R)} \delta^{-\gamma}|\mathcal{M}(R)|^{-\gamma}|Q| \\ &\leq \delta^{-\gamma} \sum_{R\in\mathcal{G}_\delta^{k-1}} |\mathcal{M}(R)|^{-\gamma}|R|,\end{aligned} \tag{7}$$

for every $k = 1, 2, \dots$.

Next, we show by induction that

$$\sum_{R\in\mathcal{G}_\delta^{k-1}} |\mathcal{M}(R)|^{-\gamma}|R| \le ((1-c)(\sigma\delta)^{-\gamma})^{k-1}|\mathcal{M}(P_0)|^{-\gamma}|P_0| \tag{8}$$

for every $k \in \mathbb{N}$. If $k = 1$, this is immediate since $\mathcal{G}_\delta^{k-1} = \{P_0\}$.

Then we assume that (8) holds for some $k \in \mathbb{N}$. Fix $R \in \mathcal{G}_\delta^{k-1}$ and let $P \in \mathcal{G}_\delta(R)$. Since P is a maximal dyadic cube in $R \setminus \bigcup_{Q\in\mathcal{F}_\delta(R)} Q$ and $\mathcal{F}_\delta(R) \neq \emptyset$ by weak porosity, the dyadic parent πP intersects a cube Q in $\mathcal{F}_\delta(R)$.

Since $\pi P, Q \in \mathcal{D}(R)$, we have $\pi P \subset Q$ or $Q \subset \pi P$ by the nestedness property (D3) of dyadic cubes. Clearly $\pi P \subset Q$ is not possible, as this would lead to the contradiction $P \subset \pi P \subset Q \subset \bigcup_{Q'\in\mathcal{F}_\delta(R)} Q'$. Therefore $Q \subset \pi P$. By Lemma 3.2 (ii), there exists a constant $\sigma = \sigma(c, \delta, n) > 0$ such that

$$|\mathcal{M}(P)| \ge \sigma|\mathcal{M}(\pi P)|.$$

Using also the definition of $\mathcal{F}_\delta(R)$, we get

$$|\mathcal{M}(P)| \ge \sigma|\mathcal{M}(\pi P)| \ge \sigma|Q| \ge \sigma\delta|\mathcal{M}(R)|.$$

On the other hand, since E is weakly porous, we have by (6) that

$$\sum_{P\in\mathcal{G}_\delta(R)} |P| = \left(|R| - \sum_{Q\in\mathcal{F}_\delta(R)} |Q|\right) \le (1-c)|R|.$$

Applying the two estimates above and the induction hypothesis (8) for k, we obtain

$$\begin{aligned}
\sum_{P\in\mathcal{G}_\delta^{k}} |\mathcal{M}(P)|^{-\gamma}|P| &\le \sum_{R\in\mathcal{G}_\delta^{k-1}} \sum_{P\in\mathcal{G}_\delta(R)} (\sigma\delta)^{-\gamma}|\mathcal{M}(R)|^{-\gamma}|P| \\
&\le (\sigma\delta)^{-\gamma} \sum_{R\in\mathcal{G}_\delta^{k-1}} |\mathcal{M}(R)|^{-\gamma} \sum_{P\in\mathcal{G}_\delta(R)} |P| \\
&\le (\sigma\delta)^{-\gamma} \sum_{R\in\mathcal{G}_\delta^{k-1}} |\mathcal{M}(R)|^{-\gamma}(1-c)|R| \\
&\le (1-c)(\sigma\delta)^{-\gamma}((1-c)(\sigma\delta)^{-\gamma})^{k-1}|\mathcal{M}(P_0)|^{-\gamma}|P_0| \\
&\le ((1-c)(\sigma\delta)^{-\gamma})^{k}|\mathcal{M}(P_0)|^{-\gamma}|P_0|.
\end{aligned}$$

This proves (8) for $k+1$, and thus the claim holds for every $k \in \mathbb{N}$, by the principle of induction.

Now choose $\gamma = \gamma(c, \delta, n) \in (0, 1/n)$ to be such that $(1-c)(\sigma\delta)^{-\gamma} < 1$. Observe that

$$\sum_{k=1}^{\infty}((1-c)(\sigma\delta)^{-\gamma})^{k-1} = C(c, \sigma, \delta, \gamma) = C(c, \delta, n) < \infty.$$

Hence, by using also (7) and (8), we have

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{Q\in\mathcal{F}_\delta^k}|Q|^{1-\gamma} &\le \sum_{k=1}^{\infty}\delta^{-\gamma}\sum_{R\in\mathcal{G}_\delta^{k-1}}|\mathcal{M}(R)|^{-\gamma}|R|\\ &\le \sum_{k=1}^{\infty}\delta^{-\gamma}((1-c)(\sigma\delta)^{-\gamma})^{k-1}|\mathcal{M}(P_0)|^{-\gamma}|P_0|\\ &\le \delta^{-\gamma}|\mathcal{M}(P_0)|^{-\gamma}|P_0|\sum_{k=1}^{\infty}((1-c)(\sigma\delta)^{-\gamma})^{k-1}\\ &\le C(c,\delta,n)|P_0||\mathcal{M}(P_0)|^{-\gamma}. \quad \square\end{aligned}$$

Lemma 5.3. *Assume that $E\subset\mathbb{R}^n$ is a weakly porous set with constants $0<c,\delta<1$. Then there are constants $0<\alpha=\alpha(c,\delta,n)<1$ and $C=C(n,c,\delta)$ such that $\operatorname{dist}(\cdot,E)^{-\alpha}\in A_1(\mathbb{R}^n)$ and $[\operatorname{dist}(\cdot,E)^{-\alpha}]_{A_1}\le C$.*

Proof. Observe that the closure $\overline{E}$ is also weakly porous. Since $\operatorname{dist}(\cdot,E)=\operatorname{dist}(\cdot,\overline{E})$, we may assume in the sequel that E is a weakly porous closed set. Throughout this proof C denotes a constant that can depend on n, c and δ. Let $0<\gamma=\gamma(n,c,\delta)<\frac{1}{n}$ be as in Lemma 5.2. Fix a cube $P_0\subset\mathbb{R}^n$, and assume first that P_0 is not an E-free cube. Let sets $\mathcal{F}_\delta^k$, for P_0 and $k=1,2,\dots$, be defined as above.

Since $\gamma n<1$, we have for every E-free cube Q the estimate

$$\int_Q \operatorname{dist}(x,E)^{-\gamma n}\,dx \le \int_Q \operatorname{dist}(x,\partial Q)^{-\gamma n}\,dx = C(\gamma,n)\ell(Q)^{n-\gamma n} = C|Q|^{1-\gamma}. \tag{9}$$

In particular, the upper bound $\gamma n<1$ implies that the second integral in (9) is finite. Bearing in mind that $|E|=0$, using Lemma 5.1 and combining (9) with Lemma 5.2, we obtain

$$\begin{aligned}\fint_{P_0}\operatorname{dist}(x,E)^{-\gamma n}\,dx &= \frac{1}{|P_0|}\int_{P_0\setminus E}\operatorname{dist}(x,E)^{-\gamma n}\,dx = \frac{1}{|P_0|}\sum_{k=1}^{\infty}\sum_{Q\in\mathcal{F}_\delta^k}\int_Q \operatorname{dist}(x,E)^{-\gamma n}\,dx\\ &\le \frac{C}{|P_0|}\sum_{k=1}^{\infty}\sum_{Q\in\mathcal{F}_\delta^k}|Q|^{1-\gamma}\le C|\mathcal{M}(P_0)|^{-\gamma}.\end{aligned}$$

Let $x\in P_0\setminus E$. Since E is closed, the point x is contained in a maximal E-free dyadic cube $Q\in\mathcal{D}(P_0)$. Recall that P_0 is not E-free, and so Q is a strict subcube of P_0. Furthermore πQ is not E-free due to maximality of Q. This implies that

$$\operatorname{dist}(x,E)\le \operatorname{diam}(\pi Q) = 2\operatorname{diam}(Q) = 2\sqrt{n}\,\ell(Q)\le 2\sqrt{n}\,\ell(\mathcal{M}(P_0)).$$

Hence,

$$\operatorname*{ess\,inf}_{x\in P_0} \operatorname{dist}(x,E)^{-\gamma n} \geq (2\sqrt{n})^{-\gamma n}\ell(\mathcal{M}(P_0))^{-\gamma n} = C(n,c,\delta)|\mathcal{M}(P_0)|^{-\gamma},$$

and we conclude that

$$\fint_{P_0} \operatorname{dist}(x,E)^{-\gamma n}\,dx \leq C \operatorname*{ess\,inf}_{x\in P_0} \operatorname{dist}(x,E)^{-\gamma n}. \tag{10}$$

It remains to consider the case where P_0 is an E-free cube. We study two situations separately. If $\operatorname{dist}(P_0,E) < 2\operatorname{diam}(P_0)$, then we have $\operatorname{dist}(x,E) \leq 3\operatorname{diam}(P_0)$ for every $x \in P_0$, and so

$$\operatorname*{ess\,inf}_{x\in P_0} \operatorname{dist}(x,E)^{-\gamma n} \geq \big(3\operatorname{diam}(P_0)\big)^{-\gamma n} \geq C|P_0|^{-\gamma}.$$

Using (9), together with this observation, we obtain

$$\fint_{P_0} \operatorname{dist}(x,E)^{-\gamma n}\,dx \leq C|P_0|^{-\gamma} \leq C \operatorname*{ess\,inf}_{x\in P_0} \operatorname{dist}(x,E)^{-\gamma n}. \tag{11}$$

Finally, we consider the case $\operatorname{dist}(P_0,E) \geq 2\operatorname{diam}(P_0)$. If $x,y \in P_0$, then

$$\begin{aligned}\operatorname{dist}(x,E) &\geq \operatorname{dist}(y,E) - |x-y| \geq \operatorname{dist}(y,E) - \operatorname{diam}(P_0)\\ &\geq \operatorname{dist}(y,E) - \tfrac{1}{2}\operatorname{dist}(P_0,E) \geq \tfrac{1}{2}\operatorname{dist}(y,E).\end{aligned}$$

Hence,

$$\operatorname{dist}(x,E)^{-\gamma n} \leq C \operatorname*{ess\,inf}_{y\in P_0} \operatorname{dist}(y,E)^{-\gamma n}$$

for all $x \in P_0$, and so

$$\fint_{P_0} \operatorname{dist}(x,E)^{-\gamma n}\,dx \leq C \operatorname*{ess\,inf}_{y\in P_0} \operatorname{dist}(y,E)^{-\gamma n}. \tag{12}$$

By combining estimates (10), (11), and (12), we see that $\operatorname{dist}(\cdot,E)^{-\gamma n} \in A_1(\mathbb{R}^n)$, and this proves the theorem with $\alpha = \gamma n$. $\square$

6. Muckenhoupt exponent

In this section, we introduce the concept of Muckenhoupt exponent and explore its connections to weak porosity and the A_p properties of distance weights, for $1 \leq p < \infty$. In particular, we prove Theorems 1.2 and 1.3 at the end of this section.

For a bounded set $A \subset \mathbb{R}^n$ and $r > 0$, we let $N(A, r)$ denote the minimal number of open balls of radius r that are needed to cover the set A. Recall that the *Assouad dimension* $\dim_{\mathrm{A}}(E)$ of $E \subset \mathbb{R}^n$ is then the infimum of $\lambda \geq 0$ such that

$$N\big(E \cap B(x, R), r\big) \leq C\left(\frac{R}{r}\right)^{\lambda}$$

for every $x \in E$ and $0 < r < R$. Equivalently, $\dim_{\mathrm{A}}(E) = n - \operatorname{codim}_{\mathrm{A}}(E)$, where the *Assouad codimension* $\operatorname{codim}_{\mathrm{A}}(E)$ is the supremum of $\alpha \geq 0$ such that

$$\frac{|E_r \cap B(x, R)|}{|B(x, R)|} \leq C\left(\frac{R}{r}\right)^{-\alpha} \tag{13}$$

for every $x \in E$ and $0 < r < R$. Here

$$E_r = \{y \in \mathbb{R}^n : \operatorname{dist}(y, E) < r\}$$

is the *open r-neighborhood of E*. See e.g. [9, (3.11)] for more details concerning this equivalence, which also follows from Lemma 6.2.

It is well-known that a set $E \subset \mathbb{R}^n$ is porous if and only if $\dim_{\mathrm{A}}(E) < n$, or equivalently $\operatorname{codim}_{\mathrm{A}}(E) > 0$, as was already pointed out in the introduction. See e.g. [11, Section 5] or [10, Theorem 10.25] for details. The following Muckenhoupt exponent can be seen as a refinement of the Assouad codimension: for porous sets these two agree but the Muckenhoupt exponent can be nonzero also for nonporous sets; see the comment after Definition 6.1.

Definition 6.1. Let $E \subset \mathbb{R}^n$.

(i) If $B(x, r)$ is a ball in $\mathbb{R}^n$, we denote by $h_E(B(x, r))$ the supremum of all $t > 0$ such that $B(y, t) \subset B(x, r) \setminus E$ for some $y \in B(x, r)$. If there is no such number $t > 0$, then we set $h_E(B(x, r)) = 0$.

(ii) If $h_E(B(x, R)) > 0$ for every $x \in E$ and $R > 0$, then the *Muckenhoupt exponent* $\mathrm{Mu}(E)$ is the supremum of the numbers $\alpha \in \mathbb{R}$ for which there exists a constant C such that

$$\frac{|E_r \cap B(x, R)|}{|B(x, R)|} \leq C\left(\frac{h_E(B(x, R))}{r}\right)^{-\alpha} \tag{14}$$

for every $x \in E$ and $0 < r < h_E(B(x, R)) \leq R$. If $h_E(B(x, R)) = 0$ for some $x \in E$ and $R > 0$, then we set $\mathrm{Mu}(E) = 0$.

Observe that $h_E(B(x, R)) \leq R/2$ if $x \in E$. It is clear from the definition that $\mathrm{Mu}(E) \geq 0$ for all sets $E \subset \mathbb{R}^n$, since (14) always holds with $\alpha = 0$ if $h_E(B(x, R)) > 0$. If $E \subset \mathbb{R}^n$ is porous, then $cR \le h_E(B(x,R)) \le R/2$ for all $x \in E$ and $R > 0$, showing that $\mathrm{Mu}(E) = \operatorname{codim}_{\mathrm{A}}(E)$. On the other hand, if $E \subset \mathbb{R}^n$ is not porous, then $\operatorname{codim}_{\mathrm{A}}(E) = 0 \le \mathrm{Mu}(E)$, and thus always $\operatorname{codim}_{\mathrm{A}}(E) \le \mathrm{Mu}(E)$. This inequality is strict if and only if E is weakly porous but not porous since the weak porosity of E is characterized by $\mathrm{Mu}(E) > 0$, see Corollary 6.6. As an example, it is straightforward to see that $\operatorname{codim}_{\mathrm{A}}(\mathbb{Z}) = 0$ and $\mathrm{Mu}(\mathbb{Z}) = 1$. See also Section 7 for other examples of such sets.

In Lemma 6.3 below we give for the Muckenhoupt exponent an alternative characterization, which resembles the definition of the Assouad dimension. The following estimate will be applied in the proof of Lemma 6.3.

Lemma 6.2. *Let $E \subset \mathbb{R}^n$, $x \in E$ and $0 < r < R$. Then*

$$C_1(n)N\big(E \cap B(x,R/2),r\big) \le \frac{|E_r \cap B(x,R)|}{r^n} \le C_2(n)N\big(E \cap B(x,2R),r\big).$$

Proof. Let $\{B(x_i,r)\}_{i=1}^N$ be a cover of $E \cap B(x,2R)$, with $N = N\big(E \cap B(x,2R),r\big)$. Then

$$E_r \cap B(x,R) \subset \bigcup_{i=1}^N B(x_i,2r),$$

and thus

$$|E_r \cap B(x,R)| \le C(n)N(2r)^n = C_2(n)r^n N\big(E \cap B(x,2R),r\big).$$

This proves the second inequality in the claim.

Conversely, let $\{B(x_i,r)\}_{i=1}^N$ be a cover of $E \cap B(x,R/2)$ such that $x_i \in E \cap B(x,R/2)$ for all $i = 1,\dots,N$ and the balls $B(x_i,r/2)$ are pairwise disjoint (such a cover can be found by choosing $\{x_i\}_{i=1}^N$ to be a maximal r-net in $E \cap B(x,R/2)$, see [7, p. 101]). Then

$$E_r \cap B(x,R) \supset \bigcup_{i=1}^N B(x_i,r/2),$$

and thus

$$|E_r \cap B(x,R)| \ge C(n)N(r/2)^n \ge C_1(n)r^n N\big(E \cap B(x,R/2),r\big).$$

This proves the first inequality in the claim. $\square$

Lemma 6.3. *Let $E \subset \mathbb{R}^n$ be such that $h_E(B(x,R)) > 0$ for every $x \in E$ and $R > 0$. Then $\mathrm{Mu}(E)$ is the supremum of the numbers $\alpha \ge 0$ for which there exists a constant C such that*

$$N\big(E \cap B(x,R),r\big) \le C\left(\frac{R}{r}\right)^n \left(\frac{h_E(B(x,R))}{r}\right)^{-\alpha} \tag{15}$$

for every $x \in E$ *and* $0 < r < h_E(B(x,R)) \le R$.

Proof. Assume first that $\alpha \ge 0$ is such that (15) holds for every $x \in E$ and $0 < r < h_E(B(x,R)) \le R$ with a constant C_1. Let $x \in E$ and $0 < r < h_E(B(x,R)) \le R$. Then $0 < r < h_E(B(x,R)) \le h_E(B(x,2R)) \le R < 2R$, and by Lemma 6.2 and (15) we have

$$\begin{aligned}\frac{|E_r \cap B(x,R)|}{|B(x,R)|} &\le C(n)\left(\frac{r}{R}\right)^n N\big(E \cap B(x,2R), r\big)\\ &\le C_1 C(n)\left(\frac{r}{R}\right)^n \left(\frac{2R}{r}\right)^n \left(\frac{h_E(B(x,2R))}{r}\right)^{-\alpha}\\ &\le C(n,C_1)\left(\frac{h_E(B(x,R))}{r}\right)^{-\alpha}.\end{aligned}$$

Thus $\alpha \le \mathrm{Mu}(E)$.

By the definition of Muckenhoupt exponent, we always have $\mathrm{Mu}(E) \ge 0$. If $\mathrm{Mu}(E) = 0$ and (15) holds for $\alpha \ge 0$, the preceding computation shows that $\alpha = 0$ as well, and the result follows. Then assume that $0 \le \alpha < \mathrm{Mu}(E)$ and let $x \in E$ and $0 < r < h_E(B(x,R)) \le R$. By Lemma 6.2 and (14), for α and a constant C_α, we have

$$\begin{aligned}N\big(E \cap B(x,R), r\big) &\le C(n)\frac{|E_r \cap B(x,2R)|}{r^n}\\ &\le C(n)C_\alpha\left(\frac{2R}{r}\right)^n \left(\frac{h_E(B(x,2R))}{r}\right)^{-\alpha}\\ &\le C(n,C_\alpha)\left(\frac{R}{r}\right)^n \left(\frac{h_E(B(x,R))}{r}\right)^{-\alpha}.\end{aligned}$$

Since this holds for every $0 \le \alpha < \mathrm{Mu}(E)$, we conclude that $\mathrm{Mu}(E)$ is indeed the supremum of α for which (15) holds for all $x \in E$ and $0 < r < h_E(B(x,R)) \le R$. $\square$

Next, we turn to the relations between the Muckenhoupt exponent and A_1 weights. Lemma 6.4 and Theorem 6.5 together characterize the property $\mathrm{dist}(\cdot,E)^{-\alpha} \in A_1$, for $\alpha \ne 0$, in terms of the Muckenhoupt exponent of E; see the proof of Theorem 1.3 after the proof of Theorem 6.5.

Lemma 6.4. *Let* $E \subset \mathbb{R}^n$ *be a nonempty set and let* $\alpha \in \mathbb{R}$ *be such that* $\mathrm{dist}(\cdot,E)^{-\alpha} \in A_1$. *Then* $0 \le \alpha \le \mathrm{Mu}(E)$.

Proof. Assume first that $\alpha < 0$. Let $x \in E$ and $r > 0$. Then

$$\fint_{Q(x,r)} \mathrm{dist}(y,E)^{-\alpha}\,dy \le C \operatorname*{ess\,inf}_{y \in Q(x,r)} \mathrm{dist}(y,E)^{-\alpha} = 0;$$

here the cube $Q(x,r)$ is as in (2). Thus $\operatorname{dist}(y,E)^{-\alpha}=0$ for almost every $y\in Q(x,r)$, which is a contradiction since $\operatorname{dist}(\cdot,E)^{-\alpha}$ is a weight. Hence $\alpha\geq 0$.

The claim holds if $\alpha=0$, and so we may assume that $\alpha>0$. Then $h_E(B(x,R))>0$ for every $x\in E$ and $R>0$. Indeed, otherwise there exists a ball $B(x,R)$ such that $\operatorname{dist}(y,E)=0$ for every $y\in B(x,R)$, and therefore $\operatorname{dist}(\cdot,E)^{-\alpha}$ is not locally integrable. This is again a contradiction since $\operatorname{dist}(\cdot,E)^{-\alpha}$ is a weight.

Let $x\in E$ and $0<r<h_E(B(x,R))\leq R$, and write $F=E_r\cap B(x,R)$. Let C_1 be the constant in the A_1 condition (3) for $\operatorname{dist}(\cdot,E)^{-\alpha}$. Observe from $B(x,R)\subset Q(x,R)$ that

$$h_E(B(x,R))\leq \operatorname*{ess\,sup}_{y\in Q(x,R)}\operatorname{dist}(y,E),$$

and hence

$$\operatorname*{ess\,inf}_{y\in Q(x,R)}\operatorname{dist}(y,E)^{-\alpha}\leq h_E(B(x,R))^{-\alpha}.$$

Since $\operatorname{dist}(y,E)<r$ for every $y\in F$ and $F\subset B(x,R)\subset Q(x,R)$, using the A_1 condition (3) we obtain

$$\begin{aligned}|F|&\leq r^{\alpha}\int_F \operatorname{dist}(y,E)^{-\alpha}\,dy\leq r^{\alpha}\int_{Q(x,R)}\operatorname{dist}(y,E)^{-\alpha}\,dy\\&\leq C_1 r^{\alpha}|Q(x,R)|h_E(B(x,R))^{-\alpha}=C(n,C_1)R^n\left(\frac{h_E(B(x,R))}{r}\right)^{-\alpha}.\end{aligned}$$

Thus

$$\frac{|E_r\cap B(x,R)|}{|B(x,R)|}=\frac{|F|}{|B(x,R)|}\leq C(n,C_1)\left(\frac{h_E(B(x,R))}{r}\right)^{-\alpha},$$

and the claim $\operatorname{Mu}(E)\geq\alpha$ follows. □

Theorem 6.5. *Let $E\subset\mathbb{R}^n$ be a nonempty set and assume that $0\leq\alpha<\operatorname{Mu}(E)$. Then $\operatorname{dist}(\cdot,E)^{-\alpha}\in A_1$.*

Proof. It suffices to show that there exists a constant $C>0$ such that

$$\fint_{B(x,r)}\operatorname{dist}(y,E)^{-\alpha}\,dy\leq C\operatorname*{ess\,inf}_{y\in B(x,r)}\operatorname{dist}(y,E)^{-\alpha}\tag{16}$$

for all $x\in E$ and $r>0$. Indeed, if $\operatorname{dist}(Q,E)<2\operatorname{diam}(Q)$ for a cube $Q\subset\mathbb{R}^n$, then the desired A_1 property (3) for $w=\operatorname{dist}(\cdot,E)^{-\alpha}$ follows easily from (16) by considering a ball $B=B(x,r)$ such that $x\in E$, $Q\subset B$ and $|B|\leq C(n)|Q|$. On the other hand, if $\operatorname{dist}(Q, E) \geq 2\operatorname{diam}(Q)$, then an argument similar to the one leading to (12) shows that (3) holds, and thus $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_1$.

Let $\lambda > 0$ with $\operatorname{Mu}(E) > \lambda > \alpha$, and let $x \in E$ and $r > 0$. Observe from inequality $\operatorname{Mu}(E) > 0$ that $0 < h_E(B(x, 2r)) \leq r$. Hence, there is $j_0 \in \mathbb{N}$ such that

$$2^{-j_0} r < h_E(B(x, 2r)) \leq 2^{1-j_0} r.$$

Define

$$F_j = \{y \in B(x, r) : \operatorname{dist}(y, E) \leq 2^{1-j} r\} \quad \text{and} \quad A_j = F_j \setminus F_{j+1},$$

for $j \geq j_0$. Since $\lambda < \operatorname{Mu}(E)$, there is a constant $C_1 = C_1(E, \lambda, n)$ such that

$$\begin{aligned} \frac{|F_j|}{|B(x,r)|} &\leq \frac{2^n |E_{2^{2-j}r} \cap B(x, 2r)|}{|B(x, 2r)|} \\ &\leq C_1 \left(\frac{h_E(B(x, 2r))}{2^{-j} r} \right)^{-\lambda} = C_1 2^{-j\lambda} \left(\frac{h_E(B(x, 2r))}{r} \right)^{-\lambda}. \end{aligned} \tag{17}$$

Since $\lambda > 0$ and $\overline{E} \cap B(x, r) \subset F_j$ for every $j \geq j_0$, by letting $j \to \infty$ we see in particular that $|\overline{E} \cap B(x, r)| = 0$. Here $r > 0$ is arbitrary, and thus $|\overline{E}| = 0$.

If $y \in B(x, r) \setminus \overline{E}$, then $\operatorname{dist}(y, E) \leq |y - x| < r$. Hence,

$$B(y, \operatorname{dist}(y, E)) \subset B(x, 2r) \setminus E,$$

and therefore $0 < \operatorname{dist}(y, E) \leq h_E(B(x, 2r)) \leq 2^{1-j_0} r$. It follows that the union of sets A_j with $j \geq j_0$ covers $B(x, r)$ up to the set $\overline{E} \cap B(x, r)$, which has measure zero. If $y \in A_j$, then $2^{-j} r < \operatorname{dist}(y, E) \leq 2^{1-j} r$. In addition, $A_j \subset F_j$ for every $j \geq j_0$. By combining the above observations and using (17) we obtain

$$\begin{aligned} \fint_{B(x,r)} \operatorname{dist}(y, E)^{-\alpha}\, dy &\leq \frac{1}{|B(x,r)|} \sum_{j=j_0}^{\infty} \int_{A_j} \operatorname{dist}(y, E)^{-\alpha}\, dy \leq \sum_{j=j_0}^{\infty} \frac{|F_j|}{|B(x,r)|} (2^{-j} r)^{-\alpha} \\ &\leq C_1 \sum_{j=j_0}^{\infty} (2^{-j} r)^{-\alpha} 2^{-j\lambda} \left(\frac{h_E(B(x, 2r))}{r} \right)^{-\lambda} \\ &\leq C_1 r^{-\alpha} \left(\frac{h_E(B(x, 2r))}{r} \right)^{-\lambda} \sum_{j=j_0}^{\infty} (2^{-j})^{\lambda - \alpha} \\ &\leq C(C_1, \lambda, \alpha) r^{-\alpha} \left(\frac{h_E(B(x, 2r))}{r} \right)^{-\lambda} \left(\frac{h_E(B(x, 2r))}{r} \right)^{\lambda - \alpha} \\ &\leq C(C_1, \lambda, \alpha) h_E(B(x, 2r))^{-\alpha} \\ &\leq C(C_1, \lambda, \alpha) \operatorname*{ess\,inf}_{y \in B(x,r)} \operatorname{dist}(y, E)^{-\alpha}. \end{aligned}$$

This shows that (16) holds, and the claim follows. $\square$

Recall that Theorem 1.3 states, for a nonempty set $E \subset \mathbb{R}^n$ and $\alpha \neq 0$, that $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_1$ if and only if $0 < \alpha < \operatorname{Mu}(E)$. We are now ready to prove this.

Proof of Theorem 1.3. If $0 < \alpha < \operatorname{Mu}(E)$, then $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_1$ by Theorem 6.5. Conversely, assume that $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_1$. Since $\alpha \neq 0$ by assumption, Lemma 6.4 implies that $\alpha > 0$. By the self-improvement of A_1 weights (see [6, pp. 399–400]), there exists $s > 1$ such that $\operatorname{dist}(\cdot, E)^{-s\alpha} \in A_1$. Thus we obtain from Lemma 6.4 that $0 < \alpha < s\alpha \leq \operatorname{Mu}(E)$. $\square$

Since $\operatorname{dist}(\cdot, E)^0 \in A_1$ holds for all (nonempty) sets $E \subset \mathbb{R}^n$ (under the interpretation that $0^0 = 1$), Theorem 1.3 implies that

$$\operatorname{Mu}(E) = \sup\{\alpha \geq 0 : \operatorname{dist}(\cdot, E)^{-\alpha} \in A_1\}$$

for all nonempty sets $E \subset \mathbb{R}^n$. On the other hand, by Theorem 1.1 we have $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_1$, for some $\alpha > 0$, if and only if E is weakly porous. This, together with Theorem 1.3, gives the following corollary.

Corollary 6.6. *A nonempty set $E \subset \mathbb{R}^n$ is weakly porous if and only if* $\operatorname{Mu}(E) > 0$.

Using Theorem 1.3 and Corollary 6.6, we can prove Theorem 1.2, as follows.

Proof of Theorem 1.2. Since E is weakly porous, we have $\operatorname{Mu}(E) > 0$ by Corollary 6.6. Therefore, the equivalences in both (i) and (ii) hold if $\alpha = 0$, and so we may assume from now on that $\alpha \neq 0$. In this case the claim in (i) follows directly from Theorem 1.3.

In part (ii), let $1 < p < \infty$ and assume first that $w \in A_p$. Because E is weakly porous, Lemma 5.3 provides us with some $\sigma > 0$ for which $\operatorname{dist}(\cdot, E)^{-\sigma} \in A_1(\mathbb{R}^n)$. If $\alpha > 0$, we can use Lemma 2.3 with $\beta = \sigma/\alpha$ to deduce that $w = \operatorname{dist}(\cdot, E)^{-\alpha} \in A_1$. Then Theorem 1.3 implies $\operatorname{Mu}(E) > \alpha$, and so (1) holds. On the other hand, if $\alpha < 0$, then we have

$$\operatorname{dist}(\cdot, E)^{-\left(\frac{-\alpha}{p-1}\right)} = w^{1-p'} \in A_{p'},$$

where $\frac{-\alpha}{p-1} > 0$. Hence the previous case, for a positive power and the class $A_{p'}$, shows that

$$(1 - p') \operatorname{Mu}(E) < 0 < \frac{-\alpha}{p-1} < \operatorname{Mu}(E), \tag{18}$$

which is equivalent to (1).

Conversely, assume that (1) holds for some $\alpha \neq 0$. If $\alpha > 0$, then $w = \operatorname{dist}(\cdot, E)^{-\alpha} \in A_1 \subset A_p$ by Theorem 1.3. Finally, if $\alpha < 0$, we observe that (1) is equivalent to (18), where $\frac{-\alpha}{p-1} > 0$. Thus we may apply the preceding case for the exponent $\frac{-\alpha}{p-1} > 0$ and the class $A_{p'}$ to conclude that $\operatorname{dist}(\cdot, E)^{\alpha/(p-1)} \in A_{p'}$. Hence $w = \operatorname{dist}(\cdot, E)^{-\alpha} \in A_p$, proving part (ii). $\square$

Remark 6.7. Note that in part (i) of Theorem 1.2 the explicit assumption that E is weakly porous is needed in the necessity part, since for $\alpha = 0$ the claim $w \in A_1$ holds for all (nonempty) sets $E \subset \mathbb{R}^n$. However, if $\alpha > 0$, then we know by Theorem 1.1 that $w \in A_1$ can only hold if E is weakly porous, which in turn is equivalent to $\operatorname{Mu}(E) > 0$.

In part (ii) the case $\alpha = 0$ again shows that (1) is not necessary for $w \in A_p$, for general sets $E \subset \mathbb{R}^n$. Moreover, if we do not assume weak porosity of E, then even in the case $\alpha \neq 0$ the requirement (1) is not necessary for $w \in A_p$. This follows from Theorem 8.1, which gives a set $E \subset \mathbb{R}$ with $\operatorname{Mu}(E) = 0$, i.e. E is not weakly porous, such that $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_p$ for all $0 < \alpha < 1$ and all $1 < p < \infty$.

Remark 6.8. When $E \subset \mathbb{R}^n$ is a bounded set, the *upper Minkowski* (or *box*) *dimension* $\overline{\dim}_{\mathrm{M}}(E)$ is the infimum of all $\lambda \geq 0$ for which there is a constant C such that

$$N(E, r) \leq C r^{-\lambda} \tag{19}$$

for every $0 < r < \operatorname{diam}(E)$. Note that (19) is equivalent to the condition that there is a constant C such that $|E_r| \leq C r^{n-\lambda}$ for every $0 < r < \operatorname{diam}(E)$; this follows from Lemma 6.2.

If a set $E \subset \mathbb{R}^n$ is weakly porous and $0 < \alpha < \operatorname{Mu}(E)$, then $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_1$ by Theorem 1.3, and so $\int_{B(x,R)} \operatorname{dist}(y, E)^{-\alpha}\, dy < \infty$ for every $x \in E$ and $R > 0$. Hence, if $x \in E$ and $R > 0$, then it holds for all $0 < r < \operatorname{diam}(E \cap B(x, R)) \leq 2R$ that

$$|(E \cap B(x, R))_r| \leq r^{\alpha} \int\limits_{B(x,3R)} \operatorname{dist}(y, E)^{-\alpha}\, dy \leq C(x, R, E) r^{n-(n-\alpha)}.$$

Thus

$$\overline{\dim}_{\mathrm{M}}(E \cap B(x, R)) \leq n - \alpha < n.$$

Since this holds for all $0 < \alpha < \operatorname{Mu}(E)$, we obtain $\overline{\dim}_{\mathrm{M}}(E \cap B(x, R)) \leq n - \operatorname{Mu}(E)$. In particular, if $E \subset \mathbb{R}^n$ is bounded, then $0 \leq \operatorname{Mu}(E) \leq n - \overline{\dim}_{\mathrm{M}}(E)$.

On the other hand, the condition that $\overline{\dim}_{\mathrm{M}}(E \cap B(x, R)) \leq c < n$ for every $x \in E$ and $R > 0$ is not sufficient for the weak porosity of E. For instance, if $E \subset \mathbb{Z} \subset \mathbb{R}$ is not weakly porous (e.g. $E = \mathbb{N}$), then we have $\overline{\dim}_{\mathrm{M}}(E \cap B(x, R)) = 0 < 1 = n$ for every $x \in E$ and $R > 0$ since $E \cap B(x, R)$ is a finite set.

See also [14] and the references therein for much more elaborate connections between Minkowski dimensions and the integrability of distance functions.

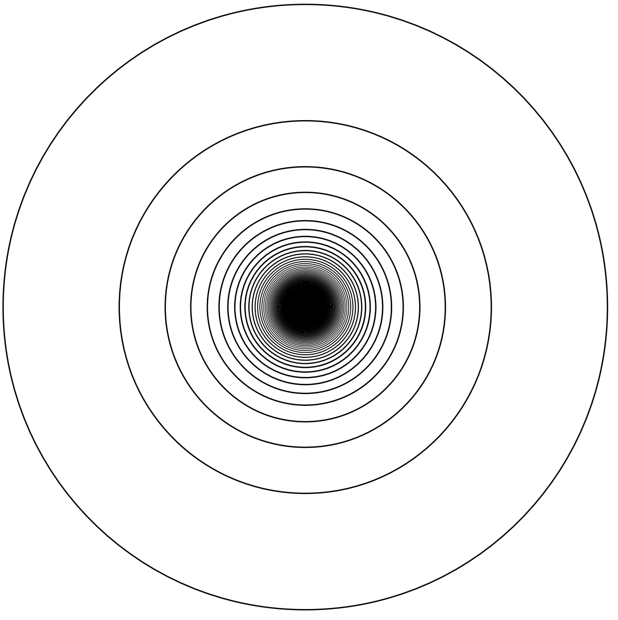

Fig. 1. The set E, with $n = 2$ and $\gamma = 0.7$.

7. Example of a weakly porous set

The notions of weak porosity and Muckenhoupt exponent are interesting only if there are (plenty of) weakly porous sets which are not porous. Below we construct a family of such sets in $\mathbb{R}^n$ and determine the Muckenhoupt exponents for different values of the parameter $\gamma > 0$. These sets are inspired by the often used one-dimensional example $\{j^{-\gamma} : j \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$. For instance, in [5, Section 6] such sets were applied to illustrate the so-called Assouad spectrum.

Theorem 7.1. *Let $n \in \mathbb{N}$ and $\gamma > 0$. Then the set*

$$E = \bigcup_{j=1}^{\infty} \partial B\big(0, j^{-\gamma}\big) \cup \{0\} \subset \mathbb{R}^n$$

is weakly porous with $\mathrm{Mu}(E) = \min\{1, \frac{n\gamma}{1+\gamma}\}$.

The origin is included in E in order to have a compact set, but for our purposes this does not make any essential difference. See Fig. 1 for an illustration of the set E.

By considering the balls $B(0, j^{-\gamma})$ as $j \to \infty$, it is straightforward to verify that E is not porous, and hence $\dim_{\mathrm{A}}(E) = n$. Moreover, special cases of the computations in the proof of Theorem 7.1 below can be used to show that $\overline{\dim}_{\mathrm{M}}(E) = \max\{n-1, \frac{n}{1+\gamma}\}$, and so in combination with Theorem 7.1 we obtain for the set E the identity $\mathrm{Mu}(E) = n - \overline{\dim}_{\mathrm{M}}(E)$; compare to Remark 6.8.

For the proof of Theorem 7.1, we define $S^t = \partial B(0,t)$ and $A_s^t = \overline{B}(0,t) \setminus B(0,s)$ for every $0 \le s \le t$, where we use the notation $B(0,0) = \emptyset$. We begin with the following lemma.

Lemma 7.2. *Let $B = B(x,R) \subset \mathbb{R}^n$ be a ball such that $x \in S^t$, with $t = j^{-\gamma}$ for some $j \in \mathbb{N}$, and $B \cap E = B \cap S^t$. Then (14) holds for B if and only if $\alpha \le 1$. Moreover, if $\alpha \le 1$, then the constant in (14) for B can be chosen to depend on n, γ and α only.*

Proof. We have $h_E(B) = R/2$, and given $0 < r < h_E(B)$, the set $A_{j^{-\gamma}-r}^{j^{-\gamma}+r} \cap B$ satisfies

$$(2r) \inf_{b\in[t-r,t+r]} \mathcal{H}^{n-1}\left(S^b \cap B\right) \le |A_{j^{-\gamma}-r}^{j^{-\gamma}+r} \cap B| \le (2r) \sup_{b\in[t-r,t+r]} \mathcal{H}^{n-1}\left(S^b \cap B\right), \quad (20)$$

where $\mathcal{H}^{n-1}$ is the normalized Hausdorff measure in $\mathbb{R}^n$. For each $b \in [t-r, t+r]$, the set $S^b \cap B$ is a hyperspherical cap within the sphere S^b, whose angle α_b satisfies, by virtue of the law of cosines, that $\cos(\alpha_b) = \frac{b^2+t^2-R^2}{2bt}$. Therefore

$$\sin\left(\frac{\alpha_b}{2}\right) = \left(\frac{R^2 - (b-t)^2}{4bt}\right)^{1/2}.$$

For a sufficiently small constant $c(\gamma)$, we have that $r \le c(\gamma)h_E(B)$ implies $\alpha_b \simeq C(\gamma)\left(\frac{R}{2t}\right)$ for every $b \in [t-r, t+r]$; here and below $a \simeq C(*)b$ means that $C(*)^{-1}b \le a \le C(*)b$. This leads us to

$$\mathcal{H}^{n-1}\big(S^b \cap B\big) \simeq C(n,\gamma)b^{n-1}(\alpha_b)^{n-1} \simeq C(n,\gamma)b^{n-1}\left(\frac{R}{2t}\right)^{n-1} \simeq C(n,\gamma)R^{n-1}, \quad (21)$$

for every $b \in [t-r, t+r]$. The sets $A_{(j-1)^{-\gamma}-r}^{(j-1)^{-\gamma}} \cap B$ and $A_{(j+1)^{-\gamma}}^{(j+1)^{-\gamma}+r} \cap B$ (meaning $A_{(j-1)^{-\gamma}-r}^{(j-1)^{-\gamma}} = \emptyset$ in the case $j = 1$) are also contained in $E_r \cap B$, but their measures are controlled by $C(n,\gamma)|A_{j^{-\gamma}-r}^{j^{-\gamma}+r} \cap B|$. Bearing in mind this observation and (20) and (21), we obtain

$$\left(\frac{h_E(B)}{r}\right)^{\alpha} \frac{|E_r \cap B|}{|B|} \le C(n,\gamma,\alpha)R^{\alpha-n}r^{-\alpha}|A_{j^{-\gamma}-r}^{j^{-\gamma}+r} \cap B| \le C(n,\gamma,\alpha)\left(\frac{r}{R}\right)^{1-\alpha}.$$

If $\alpha \le 1$, the last term is bounded by $C(n,\gamma,\alpha)$. On the other hand, if $\alpha > 1$, then (20) and (21) yield

$$\left(\frac{h_E(B)}{r}\right)^{\alpha} \frac{|E_r \cap B|}{|B|} \ge c(n,\gamma,\alpha)R^{\alpha-n}r^{1-\alpha}R^{n-1} \ge c(n,\gamma,\alpha)\left(\frac{r}{R}\right)^{1-\alpha},$$

and the last term tends to infinity as $r \to 0$. $\square$

Proof of Theorem 7.1. First we show that (14) holds for every α with $0 < \alpha < \min\{1, \frac{n\gamma}{1+\gamma}\}$. This implies that $\mathrm{Mu}(E) \geq \min\{1, \frac{n\gamma}{1+\gamma}\} > 0$, and thus E is weakly porous, by Corollary 6.6.

Fix $0 < \alpha < \min\{1, \frac{n\gamma}{1+\gamma}\}$ and let $B = B(x,R) \subset \mathbb{R}^n$ be a ball with $x \in E$, and let $0 < r < h_E(B)$. We suppose first that B is contained in $\overline{B}(0,1)$. Let k be the largest number in $\mathbb{N}$ and N be the smallest number in $\mathbb{N} \cup \{\infty\}$ such that $B \subset \overline{B}\,(0, k^{-\gamma}) \setminus B\,(0, N^{-\gamma})$. We interpret $N^{-\gamma} = 0$ and $B\,(0, N^{-\gamma}) = \emptyset$ when $0 \in \overline{B}$. It is clear that $N \geq k+2$, since the center x of B belongs to E. In the case $N = k+2$ we have $x \in S^{(k+1)^{-\gamma}}$, and (14) follows immediately from Lemma 7.2. Hence we may assume that $N \geq k+3$. Also, observe that

$$h_E(B) \leq \tfrac{1}{2}\left(k^{-\gamma} - (k+1)^{-\gamma}\right) \leq \tfrac{\gamma}{2} k^{-\gamma-1} \tag{22}$$

and

$$R \geq \tfrac{1}{2}\left((k+1)^{-\gamma} - (N-1)^{-\gamma}\right) \geq \tfrac{\gamma}{2}(N-k-2)(N-1)^{-\gamma-1}. \tag{23}$$

Now we study two cases.

(i) Suppose $\mathrm{dist}(\{0\}, B) > \mathrm{diam}(B)$. We have the estimates

$$(k+1)^{-\gamma} \leq \sup_{x\in B}|x| \leq \mathrm{dist}(\{0\}, B) + \mathrm{diam}(B) \leq 2\,\mathrm{dist}(\{0\}, B) \leq 2(N-1)^{-\gamma},$$

and so $N - 1 \leq C(\gamma)(k+1)$. Then we have

$$|E_r \cap B| \leq \sum_{j=k}^{N} |A_{j^{-\gamma}-r}^{j^{-\gamma}+r} \cap B| \leq C(n) \sum_{j=k}^{N} rR^{n-1} \leq C(n)(N-k+1)rR^{n-1}.$$

The previous observation, together with (22) and (23), leads us to

$$\begin{aligned}\left(\frac{h_E(B)}{r}\right)^{\alpha} \frac{|E_r \cap B|}{|B|} &\leq C(n,\gamma)k^{-(1+\gamma)\alpha}\, r^{1-\alpha} R^{-1}(N-k+1) \\ &\leq C(n,\gamma)k^{-(1+\gamma)\alpha}\, r^{1-\alpha}(N-1)^{1+\gamma} \\ &\leq C(n,\gamma)k^{-(1+\gamma)\alpha}\, r^{1-\alpha}(k+1)^{1+\gamma} \\ &\leq C(n,\gamma)\left(rk^{1+\gamma}\right)^{1-\alpha}.\end{aligned}$$

The last term is bounded by a constant $C(n,\gamma,\alpha)$ because $\alpha \leq 1$ and $r \leq C(\gamma)k^{-1-\gamma}$.

(ii) Now suppose $\mathrm{dist}(\{0\}, B) \leq \mathrm{diam}(B)$. Then we have

$$(2k)^{-\gamma} \leq (k+1)^{-\gamma} \leq \mathrm{dist}(\{0\}, B) + \mathrm{diam}(B) \leq 2\,\mathrm{diam}(B),$$

and hence $k^{-\gamma} \leq 2^{1+\gamma}\operatorname{diam}(B)$. Given $0 < r < h_E(B)$, denote by $j_0 \in \mathbb{N}$ the smallest number for which

$$2r \geq j_0^{-\gamma} - (j_0+1)^{-\gamma} \geq C(\gamma)(j_0+1)^{-\gamma-1}.$$

Notice that $k < j_0$ and, by the definition of j_0, we also have

$$r \leq (j_0-1)^{-\gamma} - j_0^{-\gamma} \leq C(\gamma)(j_0-1)^{-\gamma-1} \leq C(\gamma) j_0^{-\gamma-1}.$$

This observation permits us to write

$$\begin{aligned}|E_r \cap B| &\leq |B \cap A^{k^{-\gamma}}_{k^{-\gamma}-r}| + |B(0, j_0^{-\gamma}+r) \cap B| + \sum_{j=k+1}^{j_0-1} |A^{j^{-\gamma}+r}_{j^{-\gamma}-r} \cap B| \\ &\leq C(n)\left(\left(j_0^{-\gamma}+r\right)^n + \sum_{j=k}^{j_0-1} r\left(j^{-\gamma}+r\right)^{n-1}\right).\end{aligned}$$

Using the inequalities $0 < \alpha < \min\{1, \frac{n\gamma}{1+\gamma}\}$, $k^{-\gamma} \leq C(\gamma)R$, $c(\gamma) j_0^{-1-\gamma} \leq r \leq C(\gamma) j_0^{-1-\gamma}$, and $h_E(B) \leq C(\gamma) k^{-1-\gamma}$, we obtain

$$\begin{aligned}\left(\frac{h_E(B)}{r}\right)^{\alpha} \frac{|E_r \cap B|}{|B|} &\leq C(n,\gamma) k^{n\gamma-(1+\gamma)\alpha} r^{-\alpha}\left(\left(j_0^{-\gamma}+r\right)^n + \sum_{j=k}^{j_0-1} r\left(j^{-\gamma}+r\right)^{n-1}\right) \\ &\leq C(n,\gamma) k^{n\gamma-(1+\gamma)\alpha} r^{-\alpha}\left(j_0^{-n\gamma} + \sum_{j=k}^{j_0-1} r\left(j^{-\gamma}+r\right)^{n-1}\right) \\ &\leq C(n,\gamma)\left(\left(k j_0^{-1}\right)^{n\gamma-(1+\gamma)\alpha} + k^{n\gamma-(1+\gamma)\alpha} \sum_{j=k}^{j_0-1} r^{1-\alpha}\left(j^{-\gamma}+r\right)^{n-1}\right) \\ &\leq C(n,\gamma) + C(n,\gamma) k^{n\gamma-(1+\gamma)\alpha} \sum_{j=k}^{j_0-1} j^{-(1-\alpha)(1+\gamma)}\left(j^{-\gamma}+j^{-1-\gamma}\right)^{n-1} \\ &\leq C(n,\gamma) + C(n,\gamma) k^{n\gamma-(1+\gamma)\alpha} \sum_{j=k}^{\infty} j^{-1-n\gamma+(1+\gamma)\alpha} \leq C(n,\gamma,\alpha),\end{aligned}$$

where the last inequality follows by comparing the series to $\int_k^\infty t^{-1-n\gamma+(1+\gamma)\alpha}\,dt$, bearing in mind that $\alpha < \frac{n\gamma}{1+\gamma}$. The cases **(i)** and **(ii)** together show that (14) holds when $B \subset \overline{B}(0,1)$.

Now suppose that $B = B(x,R)$ is not contained in $\overline{B}(0,1)$. In the case $r \geq \frac{1-2^{-\gamma}}{2}$ we use the fact that $n - \alpha > 0$ to estimate

$$\left(\frac{h_E(B)}{r}\right)^{\alpha} \frac{|E_r \cap B|}{|B|} \leq C(n)|E_r| r^{-\alpha} R^{\alpha-n} \leq C(n)|\overline{B}(0,r+1)| r^{-\alpha} R^{\alpha-n}$$

$$\leq C(n,\gamma)\left(\frac{r}{R}\right)^{n-\alpha} \leq C(n,\gamma,\alpha).$$

In the sequel, we will assume that $r < \frac{1-2^{-\gamma}}{2}$.

If $x \in E \setminus S^1$, then $R \geq h_E(B) \geq c(\gamma)R \geq c(\gamma)$ and

$$|E_r \cap B| \leq |E_r \cap B(0,1)| + |E_r \setminus B(0,1)| \leq |E_r \cap B(0,1)| + C(n)r.$$

Therefore

$$\left(\frac{h_E(B)}{r}\right)^{\alpha} \frac{|E_r \cap B|}{|B|} \leq C(n)(r + |E_r \cap B(0,1)|)r^{-\alpha}R^{\alpha-n}$$
$$\leq C(n,\gamma,\alpha)R^{\alpha-n} \leq C(n,\gamma,\alpha),$$

where the second inequality follows by using the above case **(ii)** with $B = B(0,1)$. If $x \in S^1$ and $R \geq \frac{1-2^{-\gamma}}{2}$, then we can repeat the preceding argument to show that (14) holds, and finally, if $x \in S^1$ and $R < \frac{1-2^{-\gamma}}{2}$, then (14) holds by Lemma 7.2.

Next we show that $\mathrm{Mu}(E) \leq \min\{1, \frac{n\gamma}{1+\gamma}\}$. The bound $\mathrm{Mu}(E) \leq 1$ follows from Lemma 7.2. Let $\alpha > \frac{n\gamma}{1+\gamma}$ and consider the ball $B = B(0,1)$. Then $h_E(B) = \frac{1-2^{-\gamma}}{2} = C(\gamma)$. Given $0 < r < \frac{1}{100}$, let $j_0 \in \mathbb{N}$ be the smallest number for which $2r \geq j_0^{-\gamma} - (j_0+1)^{-\gamma}$. Then r is comparable to $c(\gamma)j_0^{-1-\gamma}$ and the annuli $\{A_{j^{-\gamma}-r}^{j^{-\gamma}+r}\}_{j=1}^{j_0}$ are pairwise disjoint. For sufficiently small r, we thus have

$$\left(\frac{h_E(B)}{r}\right)^{\alpha} \frac{|E_r \cap B|}{|B|} \geq c(n,\gamma,\alpha)r^{-\alpha}\sum_{j=2}^{j_0-1}\left(\left(j^{-\gamma}+r\right)^n - \left(j^{-\gamma}-r\right)^n\right)$$
$$\geq c(n,\gamma,\alpha)r^{1-\alpha}\sum_{j=2}^{j_0-1}\left(j^{-\gamma}-r\right)^{n-1}$$
$$\geq c(n,\gamma,\alpha)r^{1-\alpha}j_0\left((j_0-1)^{-\gamma}-r\right)^{n-1}$$
$$\geq c(n,\gamma,\alpha)r^{1-\alpha}j_0\left(j_0^{-\gamma}-j_0^{-\gamma-1}\right)^{n-1}$$
$$\geq c(n,\gamma,\alpha)r^{1-\alpha}j_0^{1-\gamma(n-1)}$$
$$\geq c(n,\gamma,\alpha)j_0^{(1-\alpha)(-1-\gamma)}j_0^{1-\gamma(n-1)}$$
$$= c(n,\gamma,\alpha)j_0^{(1+\gamma)\alpha-n\gamma}.$$

The last term goes to infinity as $r \to 0$, since $\alpha > \frac{n\gamma}{1+\gamma}$. Hence (14) does not hold if $\alpha > \frac{n\gamma}{1+\gamma}$, showing that $\mathrm{Mu}(E) \leq \frac{n\gamma}{1+\gamma}$. $\square$

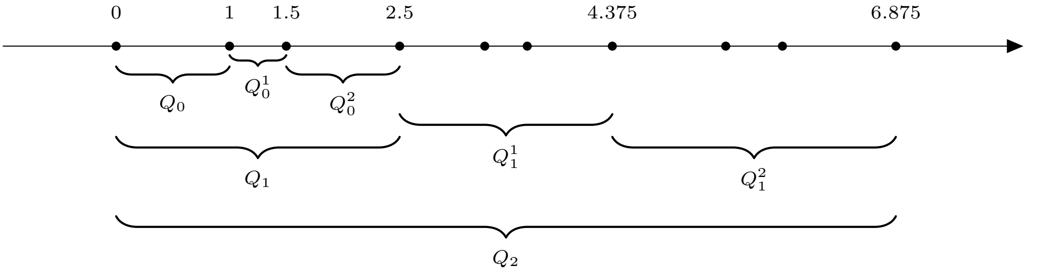

Fig. 2. First steps of the construction of the set E.

8. A_p-distance set that is not weakly porous

In this section we construct a set $E \subset \mathbb{R}$ such that $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_p \setminus A_1$ for all $0 < \alpha < 1$ and all $1 < p < \infty$; see Theorem 8.1. Recall that we abbreviate $d_E = \operatorname{dist}(\cdot, E)$.

Let $E_0 = \{0, 1\}$ and write $t_n = 1 - \frac{1}{2n}$ for every $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$, the set E_n is defined as $E_n = E_{n-1} \cup E_{n-1}^1 \cup E_{n-1}^2$, where:

- E_{n-1}^1 is a translation of E_{n-1} dilated by the factor t_n and whose first point is the last point of E_{n-1},
- E_{n-1}^2 is a translation of E_{n-1} whose first point is the last point of E_{n-1}^1.

Finally, we define $E^+ = \bigcup_{n=0}^{\infty} E_n$ and $E = E^+ \cup (-E^+)$. Here $-E^+$ is the reflection of E^+ with respect to the origin. We let Q_n, Q_n^1, and Q_n^2 denote the smallest intervals containing E_n, E_n^1, and E_n^2 respectively, for every $n \in \mathbb{N} \cup \{0\}$. See Fig. 2 for an illustration of the first steps of the construction.

During the rest of this section, we prove the following theorem for the set E.

Theorem 8.1. *Let $E \subset \mathbb{R}$ be as constructed above. Then it holds for all $0 < \alpha < 1$ and all $1 < p < \infty$ that $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_p \setminus A_1$. In particular, the set E is not weakly porous and $\operatorname{Mu}(E) = 0$.*

Proof. Let $0 < \alpha < 1$ and $1 < p < \infty$. We show in Lemma 8.4 that $\operatorname{dist}(\cdot, E)^{-\alpha} \notin A_1$, and the claim $\operatorname{dist}(\cdot, E)^{-\alpha} \in A_p$ follows from Lemma 8.7. Since $\operatorname{dist}(\cdot, E)^{-\alpha} \notin A_1$ for every $\alpha > 0$, the set E is not weakly porous by Theorem 1.1, and thus Corollary 6.6 implies that $\operatorname{Mu}(E) = 0$. $\square$

We say that a closed interval I is an *edge* of E if the endpoints of I are two consecutive points of E. For every $n \in \mathbb{N} \cup \{0\}$, the following properties hold:

- Each of the intervals Q_n, Q_n^1, and Q_n^2 has 3^n edges of E, of which the middle ones for $n \geq 1$ have lengths equal to $t_1 t_2 \cdots t_n$, $t_1 t_2 \cdots t_n t_{n+1}$, and $t_1 t_2 \cdots t_n$, respectively.

- Each of the intervals Q_n and Q_n^2 contains translated copies of the intervals $Q_0, \dots, Q_n$ distributed in a *palindromic* manner: both Q_n and Q_n^2 contain from left to right as well as from right to left intervals $Q_0^* \subset Q_1^* \subset \cdots \subset Q_n^*$ that are translated copies of $Q_0 \subset Q_1 \subset \cdots \subset Q_n$, respectively.
- Each interval Q_n^1 contains from left to right as well as from right to left intervals $t_{n+1}Q_0^* \subset t_{n+1}Q_1^* \subset \cdots \subset t_{n+1}Q_n^*$ that are translated copies of $Q_0 \subset Q_1 \subset \cdots \subset Q_n$ dilated by t_{n+1}.
- $d_E = d_{E_n}$ on Q_n.
- $|Q_n| = (2 + t_n)|Q_{n-1}|$ for every $n \in \mathbb{N}$.

Lemma 8.2. *For every $n \in \mathbb{N}$ and every $\beta > -1$, we have*

$$\fint_{Q_n} d_E(x)^\beta \, dx = \frac{2 + t_n^{1+\beta}}{2 + t_n} \fint_{Q_{n-1}} d_E(x)^\beta \, dx.$$

Proof. Let $n \in \mathbb{N}$ and $\beta > -1$. By the construction of E and the definition of Q_n, we obtain

$$\begin{aligned} \int_{Q_n} d_E^\beta = \int_{Q_n} d_{E_n}^\beta &= \int_{Q_{n-1}} d_{E_{n-1}}^\beta + \int_{Q_{n-1}^1} d_{E_{n-1}^1}^\beta + \int_{Q_{n-1}^2} d_{E_{n-1}^2}^\beta \\ &= \left(2 + t_n^{1+\beta}\right) \int_{Q_{n-1}} d_{E_{n-1}}^\beta = \left(2 + t_n^{1+\beta}\right) \int_{Q_{n-1}} d_E^\beta. \end{aligned}$$

The claim follows from the above identity and the equation $|Q_n| = (2 + t_n)|Q_{n-1}|$. □

Lemma 8.3. *For every $0 < \alpha < 1$ and $1 < p < \infty$, there exists $N_0 \in \mathbb{N}$, only depending on α and p, for which*

$$\log\left(\frac{2 + t_n^{1-\alpha}}{2 + t_n}\right) \geq \frac{\alpha}{12n} \quad \text{and} \quad \log\left[\left(\frac{2 + t_n^{1-\alpha}}{2 + t_n}\right)\left(\frac{2 + t_n^{1+\frac{\alpha}{p-1}}}{2 + t_n}\right)^{p-1}\right] \leq \frac{\alpha^2 p}{18(p-1)n^2}$$

for every $n \geq N_0$.

Proof. Consider the functions

$$f(t) = \log\left(\frac{2 + t^{1-\alpha}}{2 + t}\right), \qquad g(t) = \log\left[\left(\frac{2 + t^{1-\alpha}}{2 + t}\right)\left(\frac{2 + t^{1+\frac{\alpha}{p-1}}}{2 + t}\right)^{p-1}\right]$$

for $t > 0$. These functions satisfy $f(1) = 0$, $f'(1) = -\frac{\alpha}{3}$, $g(1) = g'(1) = 0$ and $g''(1) = \frac{2\alpha^2 p}{9(p-1)}$. Let $\varepsilon \in (0, 1/2)$ be small enough so that $|t - 1| \leq \varepsilon$ implies

$$|f(t) - f(1) - f'(1)(t-1)| \le \frac{\alpha}{6}|t-1|$$

and

$$|g(t) - g(1) - g'(1)(t-1) - \tfrac{1}{2}g''(1)(t-1)^2| \le \frac{\alpha^2 p}{9(p-1)}|t-1|^2.$$

Taking $N_0 \in \mathbb{N}$ large enough so that $N_0 \ge 1/(2\varepsilon)$ it follows that $|1 - t_n| \le \varepsilon$ for every $n \ge N_0$, and so the above estimates yield

$$f(t_n) \ge \frac{\alpha}{12n} \qquad \text{and} \qquad g(t_n) \le \frac{\alpha^2 p}{18(p-1)n^2}. \quad \square$$

Lemma 8.4. *For every $0 < \alpha < 1$, the weight $d_E^{-\alpha}$ does not belong to A_1.*

Proof. Let N_0 be the constant in Lemma 8.3 with, say, $p = 2$; the value of p is irrelevant here. Applying repeatedly Lemma 8.2, we obtain, for every $n \in \mathbb{N}$,

$$\fint_{Q_n} d_E^{-\alpha} = \left(\prod_{k=1}^{n} \frac{2 + t_k^{1-\alpha}}{2 + t_k}\right) \fint_{Q_0} d_E^{-\alpha} \ge \left(\prod_{k=N_0}^{n} \frac{2 + t_k^{1-\alpha}}{2 + t_k}\right) \fint_{Q_0} d_E^{-\alpha}.$$

By the first inequality of Lemma 8.3, we have

$$\log\left(\prod_{k=N_0}^{n} \frac{2 + t_k^{1-\alpha}}{2 + t_k}\right) = \sum_{k=N_0}^{n} \log\left(\frac{2 + t_k^{1-\alpha}}{2 + t_k}\right) \ge \sum_{k=N_0}^{n} \frac{\alpha}{12k},$$

and it follows that

$$\fint_{Q_n} d_E^{-\alpha} \ge \exp\left(\sum_{k=N_0}^{n} \frac{\alpha}{12k}\right) \fint_{Q_0} d_E^{-\alpha}.$$

Since the harmonic series diverges, we see that $\lim_{n\to\infty} \fint_{Q_n} d_E^{-\alpha} = \infty$. On the other hand, each Q_n contains edges of E of length equal to 1, and thus $\operatorname{ess\,inf}_{Q_n} d_E^{-\alpha} = 2^\alpha$. We conclude that $d_E^{-\alpha} \notin A_1$. $\square$

Lemma 8.5. *For every $0 < \alpha < 1$ and $1 < p < \infty$, there exists a constant $\widehat{C} = \widehat{C}(\alpha, p) > 0$ such that*

$$\fint_{Q_N} d_E(x)^{-\alpha}\, dx \left(\fint_{Q_N} d_E(x)^{\frac{\alpha}{p-1}}\, dx\right)^{p-1} \le \widehat{C}$$

for every $N \in \mathbb{N} \cup \{0\}$.

Proof. For $N = 0$ the claim is clear. Assume that $N \geq 1$. By Lemma 8.2,

$$\fint_{Q_N} d_E^{-\alpha} \left(\fint_{Q_N} d_E^{\frac{\alpha}{p-1}} \right)^{p-1} = \left(\prod_{n=1}^{N} \frac{2+t_n^{1-\alpha}}{2+t_n} \right) \left(\prod_{n=1}^{N} \frac{2+t_n^{1+\frac{\alpha}{p-1}}}{2+t_n} \right)^{p-1} \fint_{Q_0} d_E^{-\alpha} \left(\fint_{Q_0} d_E^{\frac{\alpha}{p-1}} \right)^{p-1}$$

$$= \prod_{n=1}^{N} \left(\frac{2+t_n^{1-\alpha}}{2+t_n} \right) \left(\frac{2+t_n^{1+\frac{\alpha}{p-1}}}{2+t_n} \right)^{p-1} \fint_{Q_0} d_E^{-\alpha} \left(\fint_{Q_0} d_E^{\frac{\alpha}{p-1}} \right)^{p-1}.$$

Let $N_0 = N_0(\alpha, p) \in \mathbb{N}$ be as in Lemma 8.3. Then

$$\log \prod_{n=1}^{N} \left(\frac{2+t_n^{1-\alpha}}{2+t_n} \right) \left(\frac{2+t_n^{1+\frac{\alpha}{p-1}}}{2+t_n} \right)^{p-1}$$
$$\leq \sum_{n=1}^{N_0-1} \log \left[\left(\frac{2+t_n^{1-\alpha}}{2+t_n} \right) \left(\frac{2+t_n^{1+\frac{\alpha}{p-1}}}{2+t_n} \right)^{p-1} \right] + \sum_{n=N_0}^{N} \frac{\alpha^2 p}{18(p-1)n^2},$$

where the right-hand side is bounded from above by a constant $C_1 = C_1(\alpha, p)$ independent of N. Hence,

$$\fint_{Q_N} d_E^{-\alpha} \left(\fint_{Q_N} d_E^{\frac{\alpha}{p-1}} \right)^{p-1} \leq e^{C_1} \fint_{Q_0} d_E^{-\alpha} \left(\fint_{Q_0} d_E^{\frac{\alpha}{p-1}} \right)^{p-1},$$

and the claim follows. $\square$

Lemma 8.6. *For every $0 < \alpha < 1$ and $1 < p < \infty$, there exists a constant $C = C(\alpha, p) > 0$ such that*

$$\fint_{Q} d_E(x)^{-\alpha}\, dx \left(\fint_{Q} d_E(x)^{\frac{\alpha}{p-1}}\, dx \right)^{p-1} \leq C \tag{24}$$

for every interval $Q \subset [0, +\infty)$.

Proof. Observe that $Q \subset Q_N$ for some $N \in \mathbb{N}$. When Q contains at most 4 points of E, it is straightforward to see that the distance d_E satisfies (24) for Q and with some constant C_1 only depending on α and p. This includes the case where Q is contained in Q_1.

We prove by induction on N that d_E satisfies (24) for every interval $Q \subset Q_N$ with the constant $C = \max\{12^p \widehat{C}, C_1\}$, where $\widehat{C}$ is the constant in Lemma 8.5. The case $N = 1$ has already been proved since $C \geq C_1$. Hence, we assume that the claim holds for all $n = 1, \ldots, N-1$, and we need to verify the claim for all intervals Q contained in Q_N.

The case where $Q \subset Q_{N-1}$ follows from the induction hypothesis. Thus we may and do assume that Q is not contained in Q_{N-1}. We do a case study.

(i): Q is contained in one of the intervals Q^1_{N-1}, Q^2_{N-1}. In the first case, the interval $Q \subset Q^1_{N-1}$ can be written as $Q = t_N Q^*$, where Q^* is a translation of an interval $\widehat{Q}$ contained in Q_{N-1}. Then $|Q| = t_N|\widehat{Q}|$ and $\int_Q d_E^\beta = t_N^{1+\beta} \int_{\widehat{Q}} d_E^\beta$ for every $\beta > -1$. This gives

$$\fint_Q d_E^{-\alpha} \left(\fint_Q d_E^{\frac{\alpha}{p-1}} \right)^{p-1} = \left(t_N^{-\alpha} \fint_{\widehat{Q}} d_E^{-\alpha} \right) \left(t_N^{\frac{\alpha}{p-1}} \fint_{\widehat{Q}} d_E^{\frac{\alpha}{p-1}} \right)^{p-1}$$

$$= \fint_{\widehat{Q}} d_E^{-\alpha} \left(\fint_{\widehat{Q}} d_E^{\frac{\alpha}{p-1}} \right)^{p-1} \leq C,$$

where the last inequality holds by the induction hypothesis. In the second case we have $Q \subset Q^2_{N-1}$, and inequality (24) follows from the induction hypothesis since Q is now translation of an interval $\widehat{Q}$ contained in Q_{N-1}.

(ii): Q intersects both Q_{N-1} and Q^2_{N-1}. This implies that Q contains Q^1_{N-1}, and so

$$|Q| \geq |Q^1_{N-1}| = t_N|Q_{N-1}| = \frac{t_N}{2+t_N}|Q_N| \geq \frac{1}{6}|Q_N|.$$

Using this estimate together with Lemma 8.5, we obtain

$$\fint_Q d_E^{-\alpha} \left(\fint_Q d_E^{\frac{\alpha}{p-1}} \right)^{p-1} \leq \left(\frac{6}{|Q_N|} \int_Q d_E^{-\alpha} \right) \left(\frac{6}{|Q_N|} \int_Q d_E^{\frac{\alpha}{p-1}} \right)^{p-1}$$

$$\leq 6^p \fint_{Q_N} d_E^{-\alpha} \left(\fint_{Q_N} d_E^{\frac{\alpha}{p-1}} \right)^{p-1} \leq 6^p \widehat{C}.$$

(iii): Q contains one of the intervals Q_{N-1}, Q^1_{N-1}, Q^2_{N-1}. In this case $|Q| \geq t_N|Q_{N-1}| \geq \frac{1}{6}|Q_N|$. Using that $Q \subset Q_N$, the desired estimate follows as in the case **(ii)**.

(iv): Assume that $Q \cap Q_{N-1} \neq \emptyset \neq Q \cap Q^1_{N-1}$ but $Q \cap Q^2_{N-1} = \emptyset$. By the construction of Q_{N-1}, we can find $m \in \{-1, 0, \dots, N-2\}$ so that $Q^*_m \subset Q \cap Q_{N-1} \subset Q^*_{m+1}$, where Q^*_m and Q^*_{m+1} are translations of Q_m and Q_{m+1} respectively, and we use the notation $Q^*_{-1} = \emptyset$. This implies $|Q \cap Q_{N-1}| \geq |Q_m|$. Similarly, by the construction of Q^1_{N-1}, there exists $n \in \{-1, 0, \dots, N-2\}$ so that $t_N Q^*_n \subset Q \cap Q^1_{N-1} \subset t_N Q^*_{n+1}$, where Q^*_n and Q^*_{n+1} are translations of Q_n and Q_{n+1}, respectively, and so $|Q \cap Q^1_{N-1}| \geq t_N|Q_n|$. Now define $M = \max\{m, n\}$. If $M = -1$, then Q intersects at most 2 edges of E, and the desired estimate follows with the constant C_1 from the beginning of the proof. If $M \geq 0$, then we have $Q \cap Q_{N-1} \subset Q^*_{M+1}$ and $Q \cap Q^1_{N-1} \subset t_N Q^*_{M+1}$, and so

$$\int_Q d_E^\beta \leq \int_{Q_{M+1}} d_E^\beta + t_N^{1+\beta} \int_{Q_{M+1}} d_E^\beta = \left(1 + t_N^{1+\beta}\right) \int_{Q_{M+1}} d_E^\beta \leq 2 \int_{Q_{M+1}} d_E^\beta,$$

for every $\beta > -1$. On the other hand,

$$\begin{aligned} |Q| &= |Q \cap Q_{N-1}| + |Q \cap Q^1_{N-1}| \geq |Q_m| + t_N |Q_n| \\ &\geq t_N |Q_M| = \frac{t_N}{2 + t_{M+1}} |Q_{M+1}| \geq \frac{|Q_{M+1}|}{6}. \end{aligned}$$

This leads us to

$$\fint_Q d_E^{-\alpha} \left(\fint_Q d_E^{\frac{\alpha}{p-1}} \right)^{p-1} \leq 12^p \fint_{Q_{M+1}} d_E^{-\alpha} \left(\fint_{Q_{M+1}} d_E^{\frac{\alpha}{p-1}} \right)^{p-1} \leq 12^p \widehat{C},$$

where the last inequality follows from Lemma 8.5.

(v): Assume that $Q \cap Q^1_{N-1} \neq \emptyset \neq Q \cap Q^2_{N-1}$ but $Q \cap Q_{N-1} = \emptyset$. Recall that Q^2_{N-1} is a translation of Q_{N-1} that contains, from left to right, translated copies $Q^*_0 \subset Q^*_1 \subset \cdots \subset Q^*_{N-2} \subset Q^*_{N-1}$ of $Q_0 \subset Q_1 \subset \cdots \subset Q_{N-2} \subset Q_{N-1}$, respectively. In addition, Q^1_{N-1} contains, from right to left, translated copies $t_N Q^*_{N-1} \supset t_N Q^*_{N-2} \supset \cdots \supset t_N Q^*_1 \supset t_N Q^*_0$ of $Q_{N-1} \supset Q_{N-2} \supset \cdots \supset Q_1 \supset Q_0$ dilated by t_N. Now, the argument is identical to the case **(iv)**. □

Lemma 8.7. *Let $0 < \alpha < 1$ and $1 < p < \infty$, and let $C = C(\alpha, p)$ be the constant in Lemma 8.6. Then*

$$\fint_Q d_E(x)^{-\alpha}\, dx \left(\fint_Q d_E(x)^{\frac{\alpha}{p-1}}\, dx \right)^{p-1} \leq 2^p C \tag{25}$$

for every interval $Q \subset \mathbb{R}$, and so $d_E^{-\alpha} \in A_p$.

Proof. Given an interval $Q \subset \mathbb{R}$, we write $Q^+ = Q \cap [0, +\infty)$ and $Q^- = Q \cap (-\infty, 0]$. Let Q^* be the largest of the intervals Q^+ and $-Q^-$, that is, $Q^* \in \{Q^+, -Q^-\}$ and $Q^+ \cup -Q^- \subset Q^*$. Here $-Q^-$ denotes the reflection of Q^- with respect to the origin. Because E is symmetric with respect to the origin, we can write

$$\int_Q d_E^{-\alpha} = \int_{Q^+} d_E^{-\alpha} + \int_{Q^-} d_E^{-\alpha} = \int_{Q^+} d_E^{-\alpha} + \int_{-Q^-} d_E^{-\alpha} \leq 2 \int_{Q^*} d_E^{-\alpha}.$$

The same argument shows that $\fint_Q d_E^{\frac{\alpha}{p-1}} \leq 2 \fint_{Q^*} d_E^{\frac{\alpha}{p-1}}$. Because $|Q| \geq |Q^*|$ and Q^* is contained in $[0,\infty)$, we can use Lemma 8.6 to conclude that

$$\fint_Q d_E^{-\alpha} \left(\fint_Q d_E^{\frac{\alpha}{p-1}} \right)^{p-1} \leq 2^p \fint_{Q^*} d_E^{-\alpha} \left(\fint_{Q^*} d_E^{\frac{\alpha}{p-1}} \right)^{p-1} \leq 2^p C. \quad \square$$

Data availability

No data was used for the research described in the article.

References

[1] H. Aikawa, Quasiadditivity of Riesz capacity, Math. Scand. 69 (1) (1991) 15–30.
[2] H. Aimar, M. Carena, R. Durán, M. Toschi, Powers of distances to lower dimensional sets as Muckenhoupt weights, Acta Math. Hung. 143 (1) (2014) 119–137.
[3] R.G. Durán, F. López García, Solutions of the divergence and analysis of the Stokes equations in planar Hölder-α domains, Math. Models Methods Appl. Sci. 20 (1) (2010) 95–120.
[4] B. Dyda, L. Ihnatsyeva, J. Lehrbäck, H. Tuominen, A.V. Vähäkangas, Muckenhoupt A_p-properties of distance functions and applications to Hardy-Sobolev–type inequalities, Potential Anal. 50 (1) (2019) 83–105.
[5] J.M. Fraser, H. Yu, New dimension spectra: finer information on scaling and homogeneity, Adv. Math. 329 (2018) 273–328.
[6] J. García-Cuerva, J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland Mathematics Studies., vol. 116, North-Holland Publishing Co., Amsterdam, 1985, Notas de Matemática [Mathematical Notes], vol. 104.
[7] J. Heinonen, Lectures on Analysis on Metric Spaces, Universitext., Springer-Verlag, New York, 2001.
[8] T. Horiuchi, The imbedding theorems for weighted Sobolev spaces. II, Bull. Fac. Sci. Ibaraki Univ., Ser. A. 23 (1991) 11–37.
[9] A. Käenmäki, J. Lehrbäck, M. Vuorinen, Dimensions, Whitney covers, and tubular neighborhoods, Indiana Univ. Math. J. 62 (6) (2013) 1861–1889.
[10] J. Kinnunen, J. Lehrbäck, A. Vähäkangas, Maximal Function Methods for Sobolev Spaces, Mathematical Surveys and Monographs, vol. 257, American Mathematical Society, Providence, RI, 2021.
[11] J. Luukkainen, Assouad dimension: antifractal metrization, porous sets, and homogeneous measures, J. Korean Math. Soc. 35 (1) (1998) 23–76.
[12] S. Semmes, On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞-weights, Rev. Mat. Iberoam. 12 (2) (1996) 337–410.
[13] A.V. Vasin, The limit set of a Fuchsian group and the Dyn′kin lemma, Zap. Nauč. Semin. POMI 303 (Issled. po Lineĭn. Oper. i Teor. Funkts. 31) (2003) 89–101, 322 (in Russian); English translation in J. Math. Sci. (N.Y.) 129 (4) (2005) 3977–3984.
[14] D. Žubrinić, Analysis of Minkowski contents of fractal sets and applications, Real Anal. Exch. 31 (2) (2005/2006) 315–354.