JYU DISSERTATIONS 810

Tapio Kurkinen

Harnack's Inequalities and Boundary Regularity for a General Nonlinear Parabolic Equation in Non-Divergence Form

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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Foreword

I would like to mainly thank my supervisor Mikko Parviainen for countless interesting discussions and invaluable guidance. He has shown a lot of patience in dealing with my mistakes and made me a better student in many ways. This thesis would not be ready without our weekly meetings. I would also like to thank my co-author Jarkko Siltakoski for showing me the advantages and enjoyment of working together on a problem. I hope we get to work more in the future. Thank you as well to the Department of Mathematics and Statistics of the University of Jyväskylä for providing me with great colleagues and a relaxed work environment. I have gotten many opportunities to improve my teaching and outreach skills.

I want to thank Benny Avelin for agreeing to be my defense opponent and Ugo Gianazza and Vincenzo Vespri for their nice words and for taking the time to evaluate my thesis.

Final thanks go to my family and friends who helped me to have a life outside work. One goes through highs and lows while doing research work and I have had a great network of people to help me through the lows.

Jyväskylä, June 2024
Tapio Kurkinen

List of included articles

This dissertation consists of an introduction and the following three articles:

[A] Tapio Kurkinen and Jarkko Siltakoski. *Intrinsic Harnack's inequality for a general nonlinear parabolic equation in non-divergence form.* To appear in *Potential Analysis.*

[B] Tapio Kurkinen, Mikko Parviainen and Jarkko Siltakoski. *Elliptic Harnack's inequality for a singular nonlinear parabolic equation in non-divergence form. Bulletin of the London Mathematical Society Volume 55, Issue 1 (2023).*

[C] Tapio Kurkinen. *Boundary regularity for a general nonlinear parabolic equation in non-divergence form.* Preprint (April 2024), arXiv:2404.12848.

In the introduction these articles are referred to as [A], [B], and [C], whereas other references will be referred by [AS22], [BBG17] etc.

The author of this dissertation has actively taken part in the research of the joint articles [A] and [B].

Abstract

This thesis studies a nonlinear parabolic equation that generalizes both the usual p-parabolic equation and the normalized p-parabolic equation arising from stochastic game theory. Apart from special cases, the equation is in non-divergence form and we use the concept of viscosity solutions.

The articles [A] and [B] focus on Harnack's inequalities. We prove that all non-negative viscosity solutions satisfy a parabolic Harnack's inequality with intrinsic scaling. Intrinsic scaling here means that the needed waiting time between time slices depends on the value of the solution. We also show that for a singular range, this waiting time is not needed and a so-called elliptic Harnack's inequality, where we get the estimate on both sides without the waiting time, holds. Exponent ranges for both inequalities are optimal as shown by counterexamples. We also show that for very singular exponents, all solutions vanish in finite time.

The article [C] examines boundary regularity for this equation. We prove that there exists a barrier family at a boundary point if and only if that point is regular. We use this characterization to prove geometric conditions that also guarantee regularity. These include an exterior ball condition and a result that shows that all locally time-wise earliest points are regular.

Tiivistelmä

Tässä väitöskirjassa tutkitaan epälineaarista parabolista yhtälöä, jonka erikoistapauksina saadaan p-parabolinen yhtälö ja normalisoitu p-parabolinen yhtälö. Yhtälö poikkeustapauksia lukuunottamatta ei ole divergenssimuotoinen ja tämän takia sopiva ratkaisun käsite saadaan viskositeettiratkaisujen teoriasta.

Artikkeleissa [A] ja [B] tutkitaan Harnackin epäyhtälöitä. Artikkelissa [A] todistetaan, että kaikki positiiviset viskositeettiratkaisut toteuttavat parabolisen Harnackin epäyhtälön, jossa epäyhtälön odotusaika riippuu ratkaisun arvosta tarkastelupisteessä. Artikkelissa [B] todistetaan, että singulaarisille eksponenteille epäyhtälö pätee myös ilman odotusaikaa ja saadaan niin sanottu elliptinen Harnackin epäyhtälö. Näytämme vastaesimerkeillä, että epäyhtälöiden eksponettiehdot ovat optimaaliset.

Artikkelissa [C] tutkitaan yhtälön reuna-arvosäännöllisyyttä. Artikkelissa todistetaan että tietynlaisen funktioperheen olemassaolo reunapisteessä on karakterisaatio pisteen säännöllisyydelle. Käyttäen tätä karakterisaatiota artikkelissa todistetaan geometrisia ehtoja, joista jokaisesta seuraa säännöllisyys. Näitä ovat muun muassa ulkopalloehto ja tulos, jonka mukaan aikasuunnassa lokaalisti ensimmäiset pisteet ovat aina säännöllisiä.

Contents

1. Introduction

This thesis studies the viscosity solutions of the following nonlinear parabolic equation in non-divergence type

$$\partial_t u = |\nabla u|^{q-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = |\nabla u|^{q-2} \left(\Delta u + (p-2)\Delta_\infty^N u\right) \quad \text{in } \Omega_T, \tag{1.1}$$

where $q > 1$ and $p > 1$. When $q = p$, this reduces to the usual p-parabolic equation

$$\partial_t u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) \quad \text{in } \Omega_T, \tag{1.2}$$

and when $q = 2$, we get the normalized or game theoretic p-parabolic equation

$$\partial_t u = \Delta u + (p-2)\Delta_\infty^N u \quad \text{in } \Omega_T.$$

Here the normalized or game theoretic infinity Laplace operator is given by

$$\Delta_\infty^N u := \sum_{i,j=1}^{n} \frac{\partial_{x_i} u \, \partial_{x_j} u \, \partial_{x_i x_j} u}{|\nabla u|^2}.$$

This latter equation gets its name from the connection to stochastic tug-of-war games with noise as shown by Manfredi, Parviainen, and Rossi in [MPR10].

The p-parabolic equation has been the focus of many books and research papers in the past fifty years and the normalized equation has had recent interest for example in [BG14], [JS17],[BBP19], [HL19], [DPZZ20] and [AS22]. The generalized parabolic equation (1.1) apart from special cases is fully nonlinear, not uniformly elliptic, and not in divergence form. This means that we do not have access to energy estimates among many other common tools in our proofs. Most theory for a general equation of this type remain undeveloped. The general form of (1.1) has been examined for example by Imbert, Jin, and Silvestre [IJS19] and Parviainen and Vázquez [PV20] and falls into the more general form examined by Ohnuma and Sato [OS97]. Articles [A] and [B] examine Harnack's inequalities for viscosity solutions of the generalized parabolic equation. Article [A] proves the so-called intrinsic form of parabolic Harnack's inequality for viscosity solutions of the generalized parabolic equation. Article [B] focuses on the singular case $q < 2$ and shows that here we get the so-called elliptic Harnack's inequality. Article [C] examines boundary regularity for the generalized parabolic equation using a barrier family characterization.

2. Viscosity solutions

Given a partial differential equation in some set with a given boundary data, there does not necessarily exist a solution in a classical sense. For example, solving the one-dimensional eikonal equation

$$
\begin{cases} |u'| - 1 = 0 & \text{in } (-1,1) \\ u(\pm 1) = 0 \end{cases} \tag{2.1}
$$

we can quickly see that there is no hope for the existence of a smooth solution so we need a more general concept of solutions. When an equation is in divergence form, we can define weak solutions by partial integration against a smooth test function. Apart from the special case $q = p$, equation (1.1) is in non-divergence form and hence we cannot use the theory of weak solutions. Evans, Crandall, and Lions introduced the concept of viscosity solutions in their papers [Eva80] and [CL83]. Evans initially added a so-called "vanishing viscosity" term into the equation to have smooth solutions and then defined viscosity solution as a limit when the added term vanishes. For practical purposes, this definition is often not easy to work with and the definition given by Crandall and Lions that uses touching test functions has become standard. A function φ touches u from above at point x if $u(x) = \varphi(x)$ and $u(y) < \varphi(y)$ when $y \neq x$. Touching from below is defined analogously. We test solution u point-wise by touching it with smooth test functions from above and below and checking if these test functions satisfy the partial differential equation at the level of an inequality. For example one can verify that $u(x) = 1 - |x|$ is the unique viscosity solution to (2.1) by verifying that all φ touching from above satisfy

$$
\left|\varphi'\right| - 1 \leq 0
$$

and touching from below the opposite inequality. Notice that there are no smooth test functions that can touch u from below at $x = 0$, so the condition is automatically satisfied. One resource for the basics of this theory is [Cra97]. This definition coincides with the classical solution when one exists. Viscosity solutions are also exactly the same as the corresponding continuous weak solutions for many equations such as (1.2) for all $p \in (1, \infty)$, see [JLM01, PV20, Sil21].

Because we do not assume the order of the exponents p and q, equation (1.1) can be highly singular and thus the definition at singular points is not immediate. A suitable definition that takes into account these singularities was established first in [IS95] for a different class of equations and by Ohnuma and Sato [OS97] for our setting. Compared to the usual viscosity definition this is done by restricting the class of test functions to retain good priori control on the behavior near the singularities and to ensure the limits remain well defined when approaching critical points. To be more exact we require that for each test function φ and a critical point (x_0, t_0), there are $\delta > 0$, $f \in C^2([0, \infty))$ and $\sigma \in C^1(\mathbb{R})$ suitably well-behaving functions such that

$$
|\varphi(x,t) - \varphi(x_0,t_0) - \partial_t \varphi(x_0,t_0)(t - t_0)| \leq f(|x - x_0|) + \sigma(t - t_0),
$$

for all $(x,t) \in B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$. The key feature we need is that for $g(x) = f(|x|)$, it holds that

$$
\lim_{\substack{x \to 0 \\ x \neq 0}} |\nabla g|^{q-p} \operatorname{div}\left(|\nabla g|^{p-2} \nabla u\right) = 0.
$$

This definition originates from Ohnuma and Sato [OS97] but we use slightly different assumptions on the function σ used first by Juutinen, Lindqvist, and Manfredi [JLM01]. These two definitions are equivalent.

2.1. Radial Equivalence.

Radial Equivalence. One of the main tools used in our proofs is the radial equivalence proven by Parviainen and Vázquez [PV20]. Assume for a moment that u is smooth and radial classical solution to (1.1) and $d \in \mathbb{N}$ for

$$
d := \frac{(n-1)(q-1)}{p-1} + 1.
$$

Assuming that $\nabla u \neq 0$, a short calculation gives us

$$\begin{aligned}\partial_t u(r,t) &= \left|u'(r,t)\right|^{q-2}\left(u''(r,t) + \frac{n-1}{r}u'(r,t) + (p-2)u''(r,t)\right)\\ &= \frac{p-1}{q-1}\left|u'(r,t)\right|^{q-2}\left((q-1)u''(r,t) + \frac{d-1}{r}u'(r,t)\right)\\ &= \frac{p-1}{q-1}\Delta_q^d u(r,t),\end{aligned}$$

where $\Delta_q^d u$ is the usual q-Laplacian taken over d spacial dimensions. We now see that u is a classical solution to the scaled q-parabolic equation. So at least for smooth radial solutions with no critical points, we have a connection between our non-divergence form equation (1.1) and divergence form (1.2).

In more generality, Parviainen and Vázquez showed that radial viscosity solutions to (1.1) are equivalent to continuous weak solutions of

$$\partial_t u = \frac{p-1}{q-1}\left|u'\right|^{q-2}\left((q-1)u'' + \frac{d-1}{r}u'\right) \text{ in } (-R,R)\times(0,T) \tag{2.2}$$

with the time scaling $\frac{p-1}{q-1}$ for all d and if $d \in \mathbb{N}$, these are moreover equivalent to radial weak solutions of the time-scaled q-parabolic equation in $B_R \times (0,T) \subset \mathbb{R}^{d+1}$, see [PV20, Section 3].

Most comparison and barrier functions used in the proofs in this thesis are radial, and hence we could use this result to directly translate many known results proven for the q-parabolic equation to (1.1). One of the problems is that d is usually not an integer. The equivalence to the one-dimensional equation holds for all d and this equation is in divergence form which gives us tools from the weak theory. We also get good heuristics on what should be true for (1.1) by restricting to integer d and comparing to the q-parabolic case. In some cases, it is easier to prove results for the one-dimensional equation (2.2) and then use radial equivalence to transfer the estimates to the n-dimensional setting. We follow this plan when proving finite extinction of solutions.

3. Harnack's inequalities

3.1. **Background.** In his book [Har87], Carl Gustav Axel Harnack proved that any non-negative solution to

$$\Delta u = 0 \quad \text{in } \Omega$$

satisfies the following inequality

$$u(x_0)\frac{R-r}{R+r} \leq u(x) \leq u(x_0)\frac{R+r}{R-r}$$

for any $x \in B_R(x_0)$. This result was given the name Harnack's inequality and it implies many powerful results, Liouville theorem for example, even though the inequality follows almost directly from the Poisson formula. Harnack's results were expanded to other equations and generalized in the following years. For our approach the relevant formulation follows from [Kel29, Chapter X, Theorem VII]. This says that a non-negative harmonic function satisfies

$$\gamma^{-1}\sup_{B_r(x_0)} u(\cdot) \leq u(x_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot) \tag{3.1}$$

for a constant $\gamma := \gamma(n)$, as long as $B_{2r}(x_0) \subset \Omega$.

The first parabolic Harnack's results were proven 67 years after [Har87] for the heat equation. For non-negative solutions of the standard parabolic heat equation

$$\partial_t u = \Delta u \quad \text{in } \Omega_T,$$

Hadamard [Had54] and Pini [Pin54] individually proved that there exists a constant $\gamma := \gamma(n)$, such that

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0 - r^2) \leq u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, t_0 + r^2), \tag{3.2}$$

as long as $B_{2r}(x_0) \times (t_0 - (2r)^2, t_0 + (2r)^2) \subset \Omega_T$. The value of the function is bounded between the infimum taken in the future and the supremum taken in the past. The term r^2 in the time variable is often called a waiting time. Moser shows by counterexample [Mos64] that this waiting time is necessary as

$$u(x,t) = t^{-\frac{1}{2}} e^{-\frac{(x+\xi)^2}{4t}}$$

solves the heat equation for $n = 1$ but for any fixed positive x,

$$0 < u(0,1) = e^{\frac{x^2}{4}} e^{\frac{x\xi}{2}} e^{-\frac{(x+\xi)^2}{4}} = e^{\frac{x^2}{4}} e^{\frac{x\xi}{2}} u(x,1)$$

for any $\xi \in \mathbb{R}$ and thus

$$\lim_{\xi \to -\infty} \frac{u(0,1)}{u(x,1)} = \lim_{\xi \to -\infty} e^{\frac{x^2}{4}} e^{\frac{x\xi}{2}} = 0.$$

For more early history about Harnack's inequalities, we refer to [Kas07].

A nonlinear generalization of the Laplace equation that has been a focus of much research in the past fifty years is the p-Laplace equation

$$\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0 \quad \text{in } \Omega, \tag{3.3}$$

and its parabolic counterpart (1.2). If $p = 2$, these reduce back to the Laplace equation and the heat equation respectively. These are both nonlinear but it turns out that a non-negative weak solution $u \in W^{1,p}(\Omega)$ of (3.3) also satisfies Harnack's inequality (3.1) for all $p > 1$, see for example [Ser64].

The parabolic problem turns out to be more delicate. It might seem possible that weak solutions of (1.2) satisfy an inequality similar to (3.2) with a waiting time cr^p for some constant c depending on data but this turns out to be false. If we assume $p > 2$ and take the usual Barenblatt solution

$$\mathcal{B}(x,t) = t^{-\frac{n}{\lambda}} \left(1 - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{\frac{1}{\lambda}}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}} \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{3.4}$$

where $\lambda = n(p-2) + p > 2$ and $(\cdot)_+$ denotes the non-negative part. This is a non-negative solution to (1.2) in $\mathbb{R}^n \times (0,\infty)$ by a direct calculation.

Now we can take (x_0, t_0) to satisfy $t_0 = |x_0|^\lambda$ so that $\mathcal{B}(x_0, t_0) = 0$. But for any $r > 0$ and $c > 0$, we can increase t_0 to be large enough so that $B_r(x_0) \times \{t_0 - cr^p\}$ intersects with the support of $x \mapsto B(x, t_0 - cr^p)$ and thus Harnack's inequality in the form used above would imply

$$0 < \sup_{B_r(x_0)} \mathcal{B}(\cdot, t_0 - cr^p) \leq \mathcal{B}(x_0, t_0) = 0.$$

This is illustrated in Figure 1 below.

What solves this problem is a Harnack's inequality with so-called intrinsic scaling. Assuming that

$$p > \frac{2n}{n+1} \tag{3.5}$$

there exists $\gamma := \gamma(n,p)$ and $c := c(n,p)$ such that all non-negative weak solutions to (1.2) satisfy

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0 - \theta r^p) \leq u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^p),$$

as long as $B_{4r}(x_0) \times (t_0 - \theta(4r)^p, t_0 + \theta(4r)^p) \subset \Omega_T$ for $\theta = cu(x_0, t_0)^{2-p}$. Now the waiting time intrinsically depends on the value of the solution. This was proven by DiBenedetto in the degenerate case $p > 2$ in [DiB88] and in the singular case $p < 2$ by DiBenedetto and Kwong

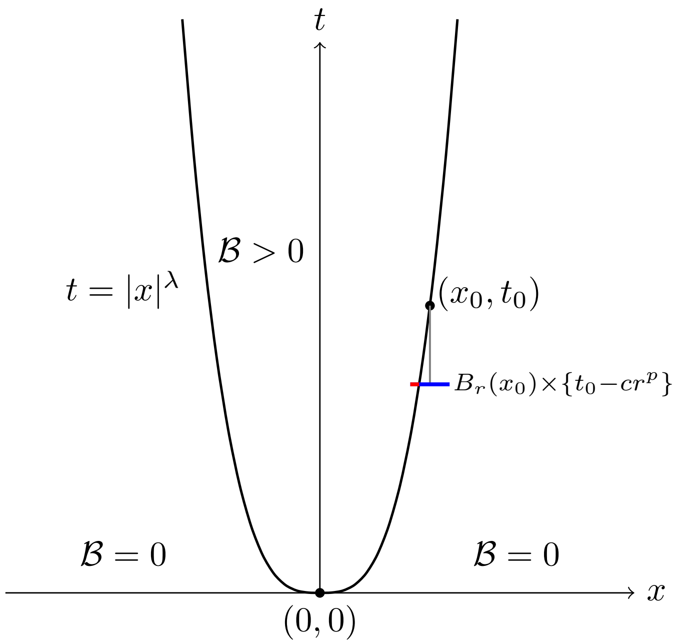

FIGURE 1. The support of the Barenblatt function for $n = 1$ and an illustration of the counterexample.

in [DK92]. These two use similar strategies establishing a small set of positivity and then constructing explicit comparison functions and using the comparison principle. The comparison functions blow up as $p \to 2$ and hence, to get stable constants c and γ for all p, DiBenedetto proves a third case of $p \approx 2$ separately. These results were generalized for equations with growth of order p by DiBenedetto, Gianazza, and Vespri [DGV08] and by Kuusi [Kuu08]. The range condition (3.5) is optimal because DiBenedetto proves in [DiB93] that all p-parabolic functions vanish in finite time for the so-called subcritical range $p \leq \frac{2n}{n+1}$ and hence there is no way for Harnack's inequality of this type to hold.

Unlike the heat equation case with the counterexample by Moser, it turns out that the waiting time is not necessary if

$$2 > p > \frac{2n}{n+1}.$$

In his book [DiB93], DiBenedetto proves that there exists $\gamma := \gamma(n, p)$ and $c := c(n, p)$ such that all non-negative weak solutions of (1.2), satisfy the inequality without waiting time

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0) \leq u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, t_0),$$

as long as $B_{4r}(x_0) \times (t_0 - \theta(4r)^p, t_0 + \theta(4r)^p) \subset \Omega_T$ for $\theta = cu(x_0, t_0)^{2-q}$. This is often called the elliptic Harnack's inequality. This is not true for $p > 2$ as the Barenblatt solution (3.4) has compact support and thus works as a counterexample as taking (x_0, t_0) close to the boundary of the support of $\mathcal{B}$ would imply

$$0 < u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, t_0) = 0.$$

We generalize these results for (1.1) in Articles [A] and [B]. Because equation (1.1) is in non-divergence form, unless $p = q$, we do not have the same tools for our proofs as DiBenedetto and others. The proof in [DK92] uses a weak Harnack's type estimate which is not accessible to us because it is proven by using a specific test function in the weak formulation. DiBenedetto also uses this weak formulation when proving finite extinctions of solutions in the subcritical range and here we use the same strategy using the radial equivalence result proven by Parviainen and Vázquez.

3.2. **Intrinsic Harnack's inequality for equation (1.1).** Article [A] deals with establishing all the remaining unproven cases for the intrinsic Harnack's inequality for equation (1.1). Using the radial equivalence, the range condition (3.5) becomes $q > \frac{2d}{d+1}$ which can be rewritten as

$$q > \begin{cases} 1 & \text{if } p \geq \frac{1+n}{2}, \\ \frac{2(n-p)}{n-1} & \text{if } 1 < p < \frac{1+n}{2}. \end{cases} \tag{3.6}$$

This turns out to be the optimal range where Harnack's inequality of this type can hold. The following is the main result of this article.

Theorem 3.1 (Theorem 1.1 in [A])**.** *Let $u \geq 0$ be a viscosity solution to* (1.1) *in Ω_T and let the range condition* (3.6) *hold. Fix $(x_0, t_0) \in \Omega_T$ such that $u(x_0, t_0) > 0$. Then there exist $\gamma = \gamma(n, p, q)$, $c = c(n, p, q)$ and $\sigma = \sigma(n, p, q) > 1$ such that*

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0 - \theta r^q) \leq u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q)$$

where

$$\theta = cu(x_0, t_0)^{2-q},$$

whenever $B_{\sigma r}(x_0) \times (t_0 - \theta(\sigma r)^q, t_0 + \theta(\sigma r)^q) \subset \Omega_T$.

These parabolic inequalities are usually established by separately proving the forward inequality

$$u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q),$$

and then suitably using this to prove the backward inequality. The standard idea of the proof is to first scale the equation and then establish a positive lower bound in some small ball $B_\rho(x_0) \times \{t_0\}$. This can be done for example using energy estimates, weak Harnack's inequalities, or oscillation estimates depending on the equation. After this, you construct a subsolution that is under this lower bound in the small ball and you use the comparison principle to expand the set of positivity over a larger ball in a future time slice.

Parviainen and Vázquez proved the forward inequality for $q > 2$ in their paper [PV20] using a Barenblatt-type solution for (1.1) as a comparison function after establishing the initial set of positivity using an oscillation estimate similar to DiBenedetto in [DiB88]. We prove the remaining two cases in Article [A]. Similarly to the p-parabolic case, one comparison function is not enough to get the proper estimate when $q < 2$ and we need two different viscosity subsolutions. The intuition behind these is to use the radial equivalence to transfer them from the q-parabolic case, but because we cannot assume that $d \in \mathbb{N}$, we needed to prove their validity by hand. We also use the Hölder continuity for viscosity solutions of (1.1) proven by [IJS19] to prove an oscillation estimate in the singular case $q < 2$ as well and use this instead of the integral Harnack's inequality we do not have access to used by DiBenedetto and Kwong [DK92] to prove the initial positivity. The backward inequality

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0 - \theta r^q) \leq u(x_0, t_0)$$

is also proven in three cases. We prove the cases $q > 2$ and $q = 2$ in Article [A] and $q < 2$ in Article [B]. These are all proven using the forward inequality in different space-time cylinders in the past which requires room to ensure that assumptions are satisfied for each of these and then we separately prove a covering argument to show that this extra room is not needed.

3.3. **Finite extinction.** To prove that the range condition (3.6) is optimal, we prove in Article [A] that all viscosity solutions to (1.1) eventually vanish after a finite time if the range condition does not hold.

Proposition 3.2 (Proposition 7.4. in [A]). *Assume q does not satisfy the range condition* (3.6). *Let u be a viscosity solution of*

$$\begin{cases} \partial_t u = |\nabla u|^{q-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) & \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ u(\cdot, 0) = u_0(\cdot) \geq 0 & \text{where radial } u_0 \in C_0(B_R) \text{ for some } R > 0. \end{cases} \tag{3.7}$$

There exists a finite time $T^* := T^*(n, p, q, u_0)$, such that

$$u(\cdot, t) \equiv 0 \quad \text{for all } t \geq T^*$$

and

$$0 < T^* \leq C \, ||u_0||^{2-q}_{L^s(r^{d-1},(0,R))}$$

where $C := C(n, p, q)$, $s = \frac{d(2-q)}{q}$ and

$$||v||_{L^q(r^{d-1},(0,R))} := \left(\int_0^R |v|^q \, r^{d-1} \, dr \right)^{\frac{1}{q}}.$$

DiBenedetto has a similar result for the p-parabolic equation and their proof uses the divergence structure. We use the radial equivalence result proven by Parviainen and Vázquez to transfer the problem to the one-dimensional equation (2.2) which is in divergence form and then use the weak formulation for that equation with suitable mollified test functions to get the estimates we need. Another difference between these two proofs is that because the standard formulation of the Gagliardo-Nirenberg inequality requires $q < n$ and we are working with one spacial dimension, we needed to prove a separate weighted radial Sobolev's inequality to replace this estimate.

This finite extinction phenomenon is enough to show that an intrinsic Harnack's inequality of this type cannot hold. Let u be a viscosity solution to (3.7) and T^* the finite extinction time given by Proposition 3.2. Now choose $(x_0, t_0) \in \mathbb{R}^n \times (0, T^*)$ close enough to satisfy

$$T^* - t_0 < \frac{t_0}{\sigma^q},$$

and choose $r > 0$ to satisfy

$$cu(x_0, t_0)^{2-q} r^q = T^* - t_0$$

where c and σ are the constants given by the intrinsic Harnack's inequality Theorem 3.1. By these choices

$$t_0 - cu(x_0, t_0)^{2-q} (\sigma r)^q = t_0 - \sigma^q \left(T^* - t_0\right) > 0$$

and therefore $B_{\sigma r}(x_0) \times (t_0 - \theta(\sigma r)^q, t_0 + \theta(\sigma r)^q) \subset \mathbb{R}^n \times \mathbb{R}^+$ and thus we can use Harnack's inequality Theorem 3.1 to obtain

$$0 < u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, T^*) = 0,$$

which is a contradiction. There is potential for different kinds of Harnack's type results in this range as well. There are some known results for the p-parabolic equation in the subcritical range, see for example [DGV09, Proposition 1.1] and [BIV10, Section 8].

3.4. **Elliptic Harnack's inequality for equation (1.1).** Our main result of Article [B] is that the waiting time is not necessary for $q < 2$ and the non-negative viscosity solutions to (1.1) satisfy the following *elliptic Harnack's inequality*. This is possible because equation (1.1) has infinite speed of propagation in this range caused by the singular diffusion term $|\nabla u|^{q-2}$. This means that every non-negative viscosity solution u in Ω_T is either identically zero or strictly positive on each time slice $\Omega \times \{t\}$ for $t \in [0, T]$.

Theorem 3.3 (Theorem 2.1 in [B]). *Let $u \geq 0$ be a viscosity solution to* (1.1) *in Ω_T and the range condition* (3.6) *holds and $q < 2$. Fix $(x_0, t_0) \in \Omega_T$. Then for any $\sigma > 1$ there exist $\gamma = \gamma(n, p, q, \sigma)$ and $c = c(n, p, q, \sigma)$ such that*

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0) \leq u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, t_0),$$

whenever $B_{\sigma r}(x_0) \times (t_0 - \theta(\sigma r)^q, t_0 + \theta(\sigma r)^q) \subset \Omega_T$ where

$$\theta = cu(x_0, t_0)^{2-q}.$$

Similarly to the p-parabolic case, such inequality cannot hold for $q > 2$ as the standard Barenblatt solution of (1.1) also has compact support. On the lower bound of the range condition (3.6), we can let $p < \frac{n+1}{2}$, $q = \frac{2(n-p)}{n-1}$, $\kappa = \frac{p-1}{q-1}$ and

$$u(x,t) = \left(|x|^{\frac{2d}{d-1}} + e^{\kappa bt}\right)^{-\frac{d-1}{2}}, \qquad \text{where } b = \frac{2d}{d-1}\frac{2d}{d+1}.$$

Short calculation shows that this is a weak solution to the one-dimensional equation (2.2) and thus is a radial viscosity solution to (1.1) by the radial equivalence. But this u does not satisfy the elliptic Harnack's inequality in a similar way as Moser's counterexample since

$$\lim_{t \to -\infty} \frac{u(1,t)}{u(0,t)} = \lim_{t \to -\infty} \left(\frac{\left(e^{\kappa bt}\right)}{\left(1 + e^{\kappa bt}\right)} \right)^{\frac{d-1}{2}} = 0.$$

DiBenedetto's proof for the elliptic Harnack's inequality in [DiB93] uses the divergence structure of (1.2) and thus we cannot use the same type of proof. In our proof we first use the intrinsic form of Harnack's inequality to get an estimate over a past time step. We construct an explicit viscosity supersolution under this known value inside the cylinder and with infinite boundary values which is only possible when $q < 2$. We can then use the comparison principle over a space-time cylinder to get the final estimate at our original time level. To prove the right inequality, we repeat the same steps but for a space-time cylinder centered at the point where $\inf_{B_r(x_0)} u(\cdot, t_0)$ is attained.

4. Boundary regularity

A point on a boundary of a set is called *regular* with respect to a partial differential equation if all solutions to the Dirichlet problem with continuous boundary values attain their boundary values continuously. To be more precise, let $\Theta \subset \mathbb{R}^{n+1}$ be open and bounded. We define Perron solutions in the usual way for bounded boundary data $f : C(\partial\Theta) \to \mathbb{R}$ by setting upper class $\mathcal{U}_f$ to be the class of all viscosity supersolutions u to equation (1.1) in Θ which are bounded from below and such that

$$\liminf_{\Theta \ni \eta \to \xi} u(\eta) \geq f(\xi) \quad \text{for all } \xi \in \partial\Theta.$$

We define the upper Perron solution of f to be

$$\overline{H} f(\xi) = \inf_{u \in \mathcal{U}_f} u(\xi), \quad \xi \in \Theta,$$

and lower class and lower Perron solution analogously except taking viscosity subsolutions and reversing the inequalities. In Article [C], we prove many basic results for Perron solutions and also prove an elliptic form of the comparison principle for equation (1.1). With bounded continuous boundary data, we show that both $\overline{H} f(\xi)$ and $\underline{H} f(\xi)$ are viscosity solutions to (1.1) which can be used to prove existence results. Our focus is on the boundary behavior.

We call $\xi_0 \in \partial\Theta$ *regular* to equation (1.1) if

$$\liminf_{\Theta \ni \xi \to \xi_0} \overline{H} f(\xi) = f(\xi_0)$$

for every $f : C(\partial\Theta) \to \mathbb{R}$. Characterizing boundary regularity for different equations has a long history. The approach through barrier functions seems to date back to Poincaré [Poi90] but were named by Lebesgue in [Leb24] where he characterizes regularity for the Laplace equation using barriers. For the elliptic p-Laplace equation, boundary regularity can be characterized by a barrier condition as proven by Granlund, Lindqvist and Martio [GLM86]. There also exists a Wiener criterion, which is sufficient by [Maz76] and necessary by [LM85] and [KM94].

The parabolic case is quite delicate compared to the elliptic case and not as well understood. Petrovskiĭ criterion for the one-dimensional heat equation, presented in [Pet34] and proven in [Pet35], shows that a boundary point that is regular for the equation

$$\partial_t u = \Delta u$$

turns out to be irregular for the multiplied equation

$$2\partial_t u = \Delta u.$$

However surprisingly boundary points remain regular for all multiplied p-parabolic equation when $p \neq 2$ as proven in [BBGP15]. We prove a similar result for equation (1.1) when $q \neq 2$.

Theorem 4.1 (Theorem 6.3 in [C]). *Let $\xi_0 \in \partial\Theta$ and $a > 0$. If $q \neq 2$, the ξ_0 is regular if and only if it is regular to the multiplied equation*

$$a\partial_t u = |\nabla u|^{q-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right).$$

The barrier approach in the parabolic setting was first used for the p-parabolic equation by Kilpeläinen and Lindqvist [KL96] where they established the suitable parabolic Perron method and suggested a barrier approach. Later Björn, Björn, Gianazza, and Parviainen proved a characterization using barrier families in [BBGP15] and showed that a single barrier is not enough for singular exponents $p < 2$ in [BBG17] unlike in the elliptic case. Characterization using a single barrier remains an open problem for $p > 2$. In Article [C] we establish these results for (1.1). We call a family of functions $w_j : \Theta \to (0, \infty]$, $j = 1, 2, \ldots$, a *barrier family* to (1.1) in Θ at point $\xi_0 \in \partial\Theta$ if for each j, we have

(a) w_j is positive viscosity supersolution to equation (1.1) in Θ,
(b) $\liminf_{\Theta \ni \zeta \to \xi_0} w_j(\zeta) = 0$,
(c) for each $k = 1, 2, \ldots$, there is a j such that

$$\liminf_{\Theta \ni \zeta \to \xi} w_j(\zeta) \geq k \quad \text{for all } \xi \in \partial\Theta \text{ with } |\xi - \xi_0| \geq \frac{1}{k}.$$

This is slightly different from the definition often seen in the literature. Often condition (a) requires w_j to be a positive lower semicontinuous function that satisfies the parabolic comparison principle in all space-time cylinders or boxes. This class of functions has different names depending on the equation and context. For our proofs, it turned out to be easier to work directly with viscosity supersolutions and prove the equivalence between these definitions separately. For p-parabolic and normalized p-parabolic equations such functions are also the same as the corresponding viscosity solutions as shown in [JLM01] and [BG14] respectively. Barriers used in most proofs are radial and thus we can use the radial equivalence connection between (1.1) and the d-dimensional q-Laplacian to gain heuristics and construct barriers.

The main theorem of Article [C] proves that the existence of a barrier family at a point is equivalent to that point being regular.

Theorem 4.2 (Theorem 5.5 in [C]). *Let $\xi_0 \in \partial\Theta$. The point ξ_0 is regular if and only if there exists a barrier family at ξ_0.*

This result is known to be true for the p-parabolic equation [BBGP15, Theorem 3.3]. Notice that this does not say whether the existence of a single barrier function is enough to characterize regularity which remains an interesting still open problem. For the normalized p-parabolic equation, the existence of a single barrier implies the existence of a barrier family, and thus a single barrier is enough to guarantee boundary regularity [BG14, Theorem 4.5]. When $p < 2$, there is a known counterexample to show that a single barrier is not enough for the p-parabolic equation [BBG17, Theorem 1.1 and Proposition 1.2]. We modify their counterexample to fit equation (1.1) and get the following result.

Theorem 4.3 (Theorem 6.2 in [C])**.** *Let $1 < q < 2$, $K > 0$ and $0 < s < \frac{1}{q}$. Then there exists a single barrier w at $(0,0)$ for the domain*

$$\Theta = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x| \leq K(-t)^s \text{ and } -1 < t < 0\}$$

despite $(0,0)$ being irregular.

There are no known counterexamples or proofs for either equation when $p > 2$ or $q > 2$ respectively. Theorem 4.3 is proven by first taking an explicit viscosity supersolution that is continuous on the boundary but jumps when we approach the origin along the axis $x = 0$. Setting this as the boundary data guarantees irregularity but we show by calculation that there still exists a barrier function at the origin.

By using Theorem 4.2 and constructing suitable barrier families, we also prove three geometric conditions that imply boundary regularity. One of these is the exterior ball condition. It turns out that if it is possible to touch the boundary point with an exterior ball where the touching point is neither the north nor the south pole of the ball, this implies regularity. As an example, the restriction in the theorem excludes the tops and bottoms of every usual space-time cylinder Ω_T. Excluding the top is as expected, since it is well known that the solution of the Dirichlet problem

$$\begin{cases} \partial_t u = |\nabla u|^{q-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) & \text{in } \Omega_T \\ u = g & \text{on } \partial_p \Omega_T. \end{cases}$$

will uniquely determine the values on the top of the cylinder $\Omega \times \{T\}$, so no point in this set can be regular.

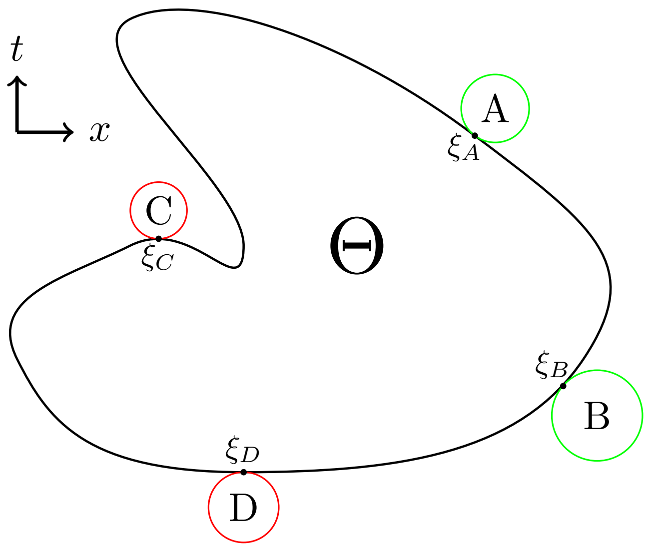

FIGURE 2. A set Θ with four exterior balls.

Theorem 4.4 (Lemma 7.1 in [C])**.** *Let $\xi_0 = (x_0, t_0) \in \partial\Theta$. Suppose that there exists a $\xi_1 = (x_1, t_1) \in \Theta^c$ and a radii $R_1 > 0$ such that $B_{R_1}(\xi_1) \cap \Theta = \emptyset$ and $\xi_0 \in \partial B_{R_1}(\xi_1) \cap \partial\Theta$. If $x_1 \neq x_0$, then ξ_0 is regular with respect to Θ.*

In Figure 2, we see an example set $\Theta \subset \mathbb{R}^{1+1}$ with four chosen boundary points ξ_A, ξ_B, ξ_C, and ξ_D and one corresponding exterior ball A, B, C and D for each point respectively.

Theorem 4.4 can verify that points ξ_A and ξ_B are regular with respect to (1.1) but it does not say anything about the regularity of ξ_C and ξ_D. Regularity of ξ_D follows from Lemma 4.5 below.

We prove Theorem 4.4 by constructing a suitable barrier family at ξ_0 with respect to the set $\Theta \cap B$, where B is a small ball containing ξ_0 and then use a result about the locality of boundary regularity to conclude that ξ_0 is also regular with respect to the set Θ.

Finally, it turns out that the earliest points time-wise are always regular. This automatically implies regularity for all points on the bottom of the set if that is flat like the cylinder Ω_T for example.

Lemma 4.5 (Lemma 7.3 in [C]). *Let $\xi_0 = (x_0, t_0) \in \partial\Theta$. If $\xi_0 \notin \partial\Theta_-$ for*

$$\Theta_- = \{(x,t) \in \Theta \mid t < t_0\},$$

then ξ_0 is regular with respect to Θ. In particular, this holds if $\Theta_- = \emptyset$.

To be more precise this gives regularity for ξ_0 as long as it is locally the time-wise earliest point of the boundary. We illustrate the case where $\Theta_- \neq \emptyset$ below in Figure 3. The proof follows by constructing a suitable barrier family.

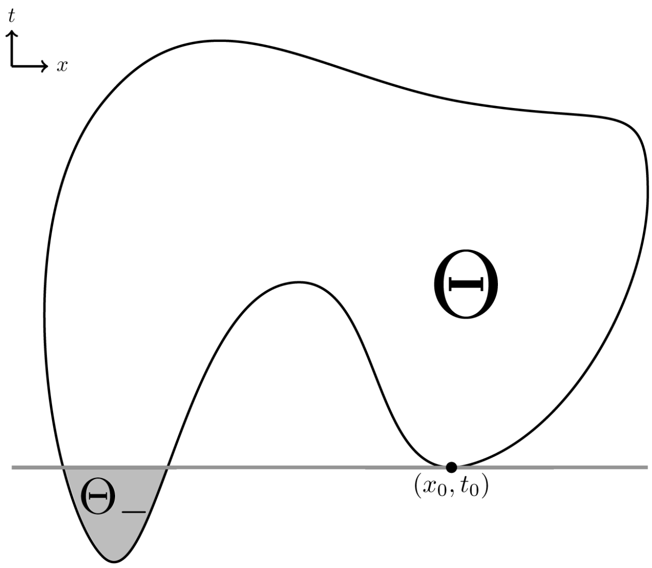

Figure 3. Illustration of Lemma 4.5 showing a case $\Theta_- \neq \emptyset$ and implying regularity of (x_0, t_0).

References

[AS22] P. D. S. Andrade and M. S. Santos. Improved regularity for the parabolic normalized p-Laplace equation. *Calculus of Variations and Partial Differential Equations*, 61(5):196, 2022.

[BBG17] A. Björn, J. Björn, and U. Gianazza. The Petrovskiĭ criterion and barriers for degenerate and singular p-parabolic equations. *Mathematische Annalen*, 368:885–904, 2017.

[BBGP15] A. Björn, J. Björn, U. Gianazza, and M. Parviainen. Boundary regularity for degenerate and singular parabolic equations. *Calculus of Variations and Partial Differential Equations*, 52(3):797–827, 2015.

[BBP19] A. Björn, J. Björn, and M. Parviainen. The tusk condition and Petrovskiĭ criterion for the normalized p-parabolic equation. *Journal of the London Mathematical Society*, 100(2):623–643, 2019.

[BG14] A. Banerjee and N. Garofalo. On the Dirichlet boundary value problem for the normalized p-laplacian evolution. *Communications on Pure and Applied Analysis*, 14:1–21, 2014.

[BIV10] M. Bonforte, R. G. Iagar, and J. L. Vázquez. Local smoothing effects, positivity, and Harnack inequalities for the fast p-Laplacian equation. *Adv. Math. (N. Y).*, 224(5):2151–2215, 2010.

[CL83] M. G. Crandall and P.-L. Lions. Viscosity Solutions of Hamilton-Jacobi Equations. *Transactions of the American Mathematical Society*, 277(1):1–42, 1983.

[Cra97] M. G. Crandall. Viscosity solutions: A Primer. *Lecture notes in Mathematics*, 1660:1–43, 1997.

[DGV08] E. DiBenedetto, U. Gianazza, and V. Vespri. Harnack estimates for quasi-linear degenerate parabolic differential equations. *Acta Math.*, 200(2):181–209, 2008.

[DGV09] E. DiBenedetto, U. Gianazza, and V. Vespri. Alternative Forms of the Harnack Inequality for Non-Negative Solutions to Certain Degenerate and Singular Parabolic Equations. *Rend. Lincei - Mat. E Appl.*, 20:369–377, 2009.

[DiB88] E. DiBenedetto. Intrinsic Harnack type inequalities for solutions of certain degenerate parabolic equations. *Arch. Ration. Mech. Anal.*, 100(2):129–147, 1988.

[DiB93] E. DiBenedetto. *Degenerate Parabolic Equations*. Universitext. Springer New York, 1993.

[DK92] E. DiBenedetto and Y. C. Kwong. Harnack Estimates and Extinction Profile for Weak Solutions of Certain Singular Parabolic Equations. *Trans. Am. Math. Soc.*, 330(2):783–811, 1992.

[DPZZ20] H. Dong, F. Peng, Y. R.-Y. Zhang, and Y. Zhou. Hessian estimates for equations involving p-Laplacian via a fundamental inequality. *Adv. Math. (N. Y).*, 370:40 pp, 2020.

[Eva80] L. C. Evans. On solving certain nonlinear partial differential equations by accretive operator methods. *Israel Journal of Mathematics*, 36(3):225–247, 1980.

[GLM86] S. Granlund, P. Lindqvist, and O. Martio. Note on the PWB-method in the nonlinear case. *Pacific Journal of Mathematics*, 125, 1986.

[Had54] J. Hadamard. Extension à l'équation de la chaleur d'un théorème de A. Harnack. *Rend. del Circ. Mat. di Palermo*, 3(3):337–346, 1954.

[Har87] C. G. A. Harnack. *Die Grundlagen der Theorie des logarithmischen Potentiales und der ein- deutigen Potentialfunktion in der Ebene*. Leibzig, Germany, 1887.

[HL19] F. A. Høeg and P. Lindqvist. Regularity of solutions of the parabolic normalized p-Laplace equation. *Adv. Nonlinear Anal.*, 9(1):7–15, 2019.

[IJS19] C. Imbert, T. Jin, and L. Silvestre. Hölder gradient estimates for a class of singular or degenerate parabolic equations. *Adv. Nonlinear Anal.*, 8(1):845–867, 2019.

[IS95] H. Ishii and P. Souganidis. Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor. *Tôhoku Math. J.*, 47:227–250, 1995.

[JLM01] P. Juutinen, P. Lindqvist, and J. J. Manfredi. On the Equivalence of Viscosity Solutions and Weak Solutions for a Quasi-Linear Equation. *SIAM J. Math. Anal.*, 33(3):699–717, 2001.

[JS17] T. Jin and L. Silvestre. Hölder gradient estimates for parabolic homogeneous p-Laplacian equations. *J. Math. Pures Appl.*, 108(1):63–87, 2017.

[Kas07] M. Kassmann. Harnack Inequalities: An Introduction. *Boundary Value Problems*, 2007(1):081415, 2007.

[Kel29] O. D. Kellogg. *Foundations of Potential Theory*. Springer Berlin, Heidelberg, 1929.

[KL96] T. Kilpeläinen and P. Lindqvist. On the Dirichlet Boundary Value Problem for a Degenerate Parabolic Equation. *SIAM Journal on Mathematical Analysis*, 27(3):661–683, 1996.

[KM94] T. Kilpeläinen and J. Malý. The Wiener test and potential estimates for quasilinear elliptic equations. *Acta Mathematica*, 172(1):137 – 161, 1994.

[Kuu08] T. Kuusi. Harnack estimates for weak supersolutions to nonlinear degenerate parabolic equations. *Ann. della Sc. Norm. Super. di Pisa - Cl. di Sci.*, 7(4):673–716, 2008.

[Leb24] H. Lebesgue. On the Dirichlet Boundary Value Problem for a Degenerate Parabolic Equation. *Comptes Rendus de l'Académie des Sciences*, 178:349–354, 1924.

[LM85] P. Lindqvist and O. Martio. Two theorems of N. Wiener for solutions of quasilinear elliptic equations. *Acta Mathematica*, 155(none):153 – 171, 1985.

[Maz76] V. G. Maz'ya. On the continuity at a boundary point of solutions of quasilinear elliptic equations (English translation). *Vestnik Leningrad University, Mathematics*, 3:225–242, 1976. Original in Vestnik Leningradskogo Universiteta 25 (1970), 42–55 (in Russian).

[Mos64] J. Moser. A harnack inequality for parabolic differential equations. *Commun. Pure Appl. Math.*, 17(1):101–134, 1964.

[MPR10] J. J. Manfredi, M. Parviainen, and J. D. Rossi. An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. *SIAM J. Math. Anal.*, 42(5):2058–2081, 2010.

[OS97] M. Ohnuma and K. Sato. Singular degenerate parabolic equations with applications. *Commun. Partial Differ. Equations*, 22(3-4):381–441, 1997.

[Pet34] I. Petrovskiĭ. Über die Lösungen der ersten Randwertaugabe der Wärmerleitungsgleichung. *Uch. Zap. Mosk. Gos. Univ.*, 2:55–59, 1934.

[Pet35] I. Petrovskiĭ. Zur ersten Randwertaufgabe der Wärmeleitungsgleichung. *Compositio Mathematica*, 1:383–419, 1935.

[Pin54] B. Pini. Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico. Technical report, 1954.

[Poi90] H. Poincaré. Sur les Equations aux Dérivées Partielles de la Physique Mathématique. *American Journal of Mathematics*, 12(3):211–294, 1890.

[PV20] M. Parviainen and J. L. Vázquez. Equivalence between radial solutions of different parabolic gradient-diffusion equations and applications. *Ann. della Sc. Norm. Super. Di Pisa - Cl. di Sci.*, 21(Special Issue):303–359, 2020.

[Ser64] J. Serrin. Local behavior of solutions of quasi-linear equations. *Acta Mathematica*, 111:247 – 302, 1964.

[Sil21] J. Siltakoski. Equivalence of viscosity and weak solutions for a p-parabolic equation. *J. Evol. Equations*, 21(2):2047–2080, 2021.

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Intrinsic Harnack's inequality for a general nonlinear parabolic equation in non-divergence form

Tapio Kurkinen and Jarkko Siltakoski

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RESEARCH

Intrinsic Harnack's Inequality for a General Nonlinear Parabolic Equation in Non-divergence Form

Tapio Kurkinen[1] **· Jarkko Siltakoski**[1]

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Abstract
We prove the intrinsic Harnack's inequality for a general form of a parabolic equation that generalizes both the standard parabolic p-Laplace equation and the normalized version arising from stochastic game theory. We prove each result for the optimal range of exponents and ensure that we get stable constants.

Keywords Intrinsic Harnack's inequality · Viscosity solutions · Nonlinear equation · p-parabolic equation

Mathematics Subject Classification (2010) 35K55 (primary) · 35K67, 35D40 (secondary)

1 Introduction

We prove the intrinsic Harnack's inequality for the following general non-divergence form version of the nonlinear parabolic equation

$$\partial_t u = |\nabla u|^{q-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = |\nabla u|^{q-2} (\Delta u + (p-2)\Delta_\infty^N u), \tag{1.1}$$

for the optimal range of exponents. The theorem states that a non-negative viscosity solution satisfies the following local a priori estimate

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0 - \theta r^q) \leq u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q) \tag{1.2}$$

for a scaling constant θ which depends on the value of u. This intrinsic waiting time is the origin of the name and is required apart from the singular range of exponents where the elliptic Harnack's inequality holds [19]. We also establish stable constants at the vicinity of $q = 2$.

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When $q = p$, the Eq. 1.1 is the standard p-parabolic equation for which the intrinsic Harnack's inequality was proven by DiBenedetto [7] and Kwong [9], see also [8]. These results were generalized for equations with growth of order p by DiBenedetto, Gianazza, and Vespri [4] and by Kuusi [21]. When $q \neq p$, the Eq. 1.1 is in non-divergence form. For non-divergence form equations parabolic Harnack's inequalities and related Hölder regularity results were first studied by Cordes [3] and Landis [22]. Parabolic Harnack's inequality for a non-divergence form equation with bounded and measurable coefficients was proven by Krylov and Safonov [20]. Further regularity results for general fully nonlinear equations were proven by Wang [28], see also [15]. To the best of our knowledge, our proof is partly new even in the special case of the p-parabolic equation since it does not rely on the divergence structure.

The idea of the proof of the right inequality in Eq. 1.2 is to first locate a local supremum and establish a positive lower bound in some small ball around this point. Then we use specific subsolutions as comparison functions to expand the set of positivity over the unit ball for a specific time slice using the comparison principle. Our proof uses the connection of Eq. 1.1 and the p-parabolic equation established by Parviainen and Vázquez in [26] to construct suitable comparison functions. Heuristically, radial solutions to the non-divergence form problem can be interpreted as solutions to divergence form p-parabolic equation in a fictitious dimension d, which does not need to be an integer. The proof of the left inequality is based on estimating the values of a function in the specific time slice by using the other inequality with suitable radii and scaling of constants.

Our proofs often are split into three different cases because the behavior of solutions to Eq. 1.1 depends on the value of q. For the degenerate case $q > 2$, the right-side inequality is proven in [26] and we prove the singular case $q < 2$ as well as the case of q near 2. This is done separately to obtain stable constants as $q \to 2$. For the left-side inequality, the singular case was proven in [19] and we prove the remaining cases.

DiBenedetto's proof uses the theory of weak solutions but since the Eq. 1.1 is in non-divergence form, unless $q = p$, we use the theory of viscosity solutions instead. Because of this, we cannot directly use energy estimates as in [4] or in [8]. Even defining solutions is non-trivial for this type of equations. A suitable definition taking singularities of the problem into account was established by Ohnuma and Sato [25]. When $q = 2$, we get the normalized p-parabolic equation arising from game theory which was first examined in the parabolic setting in [23]. This problem has had recent interest for example in [11, 12, 18] and [1]. We also point out that normalized equations have been studied in connection to image processing [10], economics [24] and machine learning [2]. The general form of Eq. 1.1 has been examined for example in [13] and [26] in addition to [19].

1.1 Results

We work with the exponent range

$$q > \begin{cases} 1 & \text{if } p \geq \frac{1+n}{2}, \\ \frac{2(n-p)}{n-1} & \text{if } 1 < p < \frac{1+n}{2}, \end{cases} \tag{1.3}$$

which is optimal for the intrinsic Harnack's inequality as we prove in Section 7. For the elliptic version of the inequality where we get both estimates without waiting time, the optimal range is to assume (1.3) and $q < 2$, as we proved in [19]. The notation used for space-time cylinders is defined in the next section.

Theorem 1.1 *Let $u \geq 0$ be a viscosity solution to Eq. 1.1 in $Q_1^-(1)$ and let the range condition Eq. 1.3 hold. Fix $(x_0, t_0) \in Q_1^-(1)$ such that $u(x_0, t_0) > 0$. Then there exist $\gamma = \gamma(n, p, q)$, $c = c(n, p, q)$ and $\sigma = \sigma(n, p, q) > 1$ such that*

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0 - \theta r^q) \leq u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q)$$

where

$$\theta = cu\,(x_0, t_0)^{2-q}\,,$$

whenever $(x_0, t_0) + Q_{\sigma r}(\theta) \subset Q_1^-(1)$.

We prove this theorem in Sections 4 and 5 after first introducing prerequisites and proving auxiliary results in Sections 2 and 3. The theorem is proven by first establishing the right inequality, from now on called the forward Harnack's inequality, and then using this result to prove the left inequality, henceforth called backward Harnack's inequality. These names are standard in the literature. We prove the forward inequality by first locating the local supremum of our function and establishing a positive lower bound in some small ball around the supremum point. This differs from the integral Harnack's inequality used by DiBenedetto for weak solutions at this step [8, Chapter VII]. The proof of this integral inequality uses the divergence form structure of the p-parabolic equation and thus is not available to us without a new proof. Next, we expand the positivity set around the obtained supremum point by using suitably constructed viscosity subsolutions and the comparison principle. In the singular case, we first expand the set in the time direction using one comparison function and then expand it sidewise for a specific time slice using another one. In the degenerate case, a single Barenblatt-type function is enough to get a similar result. Yet we need a different comparison function to handle exponents near $q = 2$ if we wish to have stable constants as $q \to 2$. We construct these viscosity subsolutions in Section 3.

For the backward Harnack's inequality, the singular case is proven as [19, Theorem 5.2], and we prove the remaining cases in Section 5. The case $q = 2$ is a direct consequence of the forward inequality as we do not have to deal with intrinsic scaling. The proof of the degenerate case follows the proof of the similar result for the p-parabolic equation [6, Section 5.2] and uses the forward inequality and proceeds by contradiction that the backward inequality has to hold. In Section 6 we prove covering arguments that take the intrinsic scaling into account. We do this by repeatedly iterating Harnack's inequality and choosing points and radii taking the intrinsic scaling into account. In the last Section 7, we prove that if q does not satisfy the range condition (1.3), it must vanish in finite time and thus cannot satisfy the intrinsic Harnack's inequality. Thus the range condition is optimal.

2 Prerequisites

When $\nabla u \neq 0$, we denote

$$\Delta_p^q u := |\nabla u|^{q-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = |\nabla u|^{q-2} \left(\Delta u + (p-2)\Delta_\infty^N u\right),$$

where $p > 1$ and $q > 1$ are real parameters and the normalized or game theoretic infinity Laplace operator is given by

$$\Delta_\infty^N u := \sum_{i,j=1}^{n} \frac{\partial_{x_i} u\, \partial_{x_j} u\, \partial_{x_i x_j} u}{|\nabla u|^2}.$$

Thus the Eq. 1.1 can be written as

$$\partial_t u = \Delta_p^q u.$$

Let $\Omega \subset \mathbb{R}^n$ be a domain and denote $\Omega_T = \Omega \times (0, T)$ the space-time cylinder and

$$\partial_p \Omega := (\Omega \times \{0\}) \cup (\partial\Omega \times [0, T])$$

its parabolic boundary. We will mainly work with the following type of cylinders

$$\begin{aligned} Q_r^-(\theta) &:= B_r(0) \times (-\theta r^q, 0], \\ Q_r^+(\theta) &:= B_r(0) \times (0, \theta r^q) \end{aligned}$$

where θ is a positive parameter that determines the time-wise length of the cylinder relative to r^q. We denote the union of these cylinders as

$$Q_r(\theta) := Q_r^+(\theta) \cup Q_r^-(\theta)$$

and when not located at the origin, we denote

$$\begin{aligned} (x_0, t_0) + Q_r^-(\theta) &:= B_r(x_0) \times (t_0 - \theta r^q, t_0], \\ (x_0, t_0) + Q_r^+(\theta) &:= B_r(x_0) \times (t_0, t_0 + \theta r^q), \\ (x_0, t_0) + Q_r(\theta) &:= B_r(x_0) \times (t_0 - \theta r^q, t_0 + \theta r^q). \end{aligned}$$

Apart from the case $p = q$, the Eq. 1.1 is in non-divergence form and thus the standard theory of weak solutions is not available, and we will use the concept of viscosity solutions instead. Moreover, the equation is singular for $2 > q > 1$, and thus we need to restrict the class of test function in the definition to retain good a priori control on the behavior of solutions near the singularities. We use the definition first introduced in [14] for a different class of equations and in [25] for our setting. This is the standard definition in this context and it naturally lines up with the p-parabolic equation ($p = q$), where notions of weak and viscosity solutions are equivalent for all $p \in (1, \infty)$[17, 26, 27]. See also [16].

Denote

$$F(\eta, X) = |\eta|^{q-2} \operatorname{Tr}\left(X + (p-2)\frac{\eta \otimes \eta}{|\eta|^2} X \right)$$

where $(a \otimes b)_{ij} = a_i b_j$, so that

$$F(\nabla u, D^2 u) = |\nabla u|^{q-2} (\Delta u + (p-2)\Delta_\infty^N u) = \Delta_p^q u$$

whenever $\nabla u \neq 0$. Let $\mathcal{F}(F)$ be the set of functions $f \in C^2([0, \infty))$ such that

$$f(0) = f'(0) = f''(0) = 0 \text{ and } f''(r) > 0 \text{ for all } r > 0,$$

and also require that for $g(x) := f(|x|)$, it holds that

$$\lim_{\substack{x \to 0 \\ x \neq 0}} F(\nabla g(x), D^2 g(x)) = 0.$$

This set $\mathcal{F}(F)$ is never empty because it is easy to see that $f(r) = r^\beta \in \mathcal{F}(F)$ for any $\beta > \max(q/(q-1), 2)$. Note also that if $f \in \mathcal{F}(F)$, then $\lambda f \in \mathcal{F}(F)$ for all $\lambda > 0$.

Define also the set

$$\Sigma = \{\sigma \in C^1(\mathbb{R}) \mid \sigma \text{ is even}, \sigma(0) = \sigma'(0) = 0, \text{ and } \sigma(r) > 0 \text{ for all } r > 0\}.$$

We use $\mathcal{F}(F)$ and Σ to define an admissible set of test functions for viscosity solutions.

Definition 2.1 A function $\varphi \in C^2(\Omega_T)$ is admissible at a point $(x_0, t_0) \in \Omega_T$ if either $\nabla\varphi(x_0, t_0) \neq 0$ or there are $\delta > 0$, $f \in \mathcal{F}(F)$ and $\sigma \in \Sigma$ such that

$$|\varphi(x, t) - \varphi(x_0, t_0) - \partial_t\varphi(x_0, t_0)(t - t_0)| \leq f(|x - x_0|) + \sigma(t - t_0),$$

for all $(x, t) \in B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$. A function is admissible in a set if it is admissible at every point of the set.

Note that by definition a function φ is automatically admissible in Ω_T if either $\nabla\varphi(x, t) \neq 0$ in Ω_T or the function $-\varphi$ is admissible in Ω_T.

Definition 2.2 A function $u : \Omega_T \to \mathbb{R} \cup \{\infty\}$ is a viscosity supersolution to

$$\partial_t u = \Delta_p^q u \quad \text{in } \Omega_T$$

if the following three conditions hold.

1. u is lower semicontinuous,
2. u is finite in a dense subset of Ω_T,
3. whenever an admissible $\varphi \in C^2(\Omega_T)$ touches u at $(x, t) \in \Omega_T$ from below, we have

$$\begin{cases} \partial_t\varphi(x, t) - \Delta_p^q\varphi(x, t) \geq 0 & \text{if } \nabla\varphi(x, t) \neq 0, \\ \partial_t\varphi(x, t) \geq 0 & \text{if } \nabla\varphi(x, t) = 0. \end{cases}$$

A function $u : \Omega_T \to \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution if $-u$ is a viscosity supersolution. A function $u : \Omega_T \to \mathbb{R}$ is a viscosity solution if it is a supersolution and a subsolution.

The existence and uniqueness for viscosity solutions of Eq. 1.1 is proven in [25, Theorem 4.8]. In our proof of the forward Harnack's inequality for the singular range we need a version of the corollary proven in [26, Corollary 7.2] for the case $q < 2$. The lemma remains largely the same except we change the signs of the exponents. We present the proof here for the convenience of the reader.

Lemma 2.3 *Let u be a viscosity solution to Eq. 1.1 in $Q_{4r}(1)$ and let the range condition (1.3) hold and assume $q < 2$. For any $\delta \in (0, 1)$, there exists $C := C(n, p, q, \delta) > 1$ such that the following holds. Suppose that $\omega_0 > 1$ is such that for $a_0 := \omega_0^{q-2} < 1$ we have*

$$\operatorname{osc}_{Q_r(a_0)} u \leq \omega_0,$$

and define the sequences

$$r_i := C^{-i} r, \quad \omega_i := \delta\omega_{i-1}, \quad a_i := \omega_i^{q-2}$$

where $i = 1, 2, \ldots$. Then it holds that

$$Q_{r_{i+1}}(a_{i+1}) \subset Q_{r_i}(a_i) \quad \text{and} \quad \operatorname{osc}_{Q_{r_i}(a_i)} u \leq \omega_i.$$

Proof Observe that $Q_{r_{i+1}}(a_{i+1}) \subset Q_{r_i}(a_i) \subset Q_r(1)$ holds as long as in the time-direction we have

$$a_{i+1}r_{i+1}^q = \omega_{i+1}^{q-2} C^{-(i+1)q} r^q = \delta^{q-2} C^{-q} \omega_i^{q-2} (C^{-i} r)^q \leq a_i r_i^q$$

which holds if we choose C to satisfy $C^q \delta^{2-q} \geq 1$. To prove the second claim we will use induction.

The case $i = 0$ holds by assumption. Suppose that the claim holds for some $i = k$ meaning

$$\operatorname{osc}_{Q_{r_k}(a_k)} u \leq \omega_k$$

and define

$$u_k(x,t) := \frac{u(r_k x, a_k r_k^q t) - \inf_{Q_{r_k}(a_k)} u}{\omega_k}.$$

By induction assumption $\sup_{Q_1(1)} u_k \leq 1$. By change of variables, we can rewrite

$$\begin{aligned} \operatorname{osc}_{Q_{r_{k+1}}(a_{k+1})} \frac{u}{\omega_k} &= \operatorname{osc}_{(x,t)\in Q_1(1)} \frac{u(r_{k+1}x, a_{k+1}r_{k+1}^q t)}{\omega_k} \\ &= \operatorname{osc}_{(x,t)\in Q_1(1)} \frac{u(C^{-1}r_k x, \delta^{q-2}C^{-q}a_k r_k^q t) - \inf_{Q_{r_k}(a_k)} u}{\omega_k} \\ &= \operatorname{osc}_{Q_{C^{-1}}(\delta^{q-2}C^{-q})} u_k \end{aligned} \tag{2.1}$$

Next, we will use the Hölder estimates proved in [13] to estimate the oscillation. By [13, Lemma 3.1], there exists a constant $C_1 := C_1(n, p, q, ||u_k||_{L^\infty(Q_{4r}(1))})$ such that

$$\sup_{\substack{t,s\in[-1,1]\\ t\neq s}} \frac{|u_k(x,t) - u_k(x,s)|}{|t-s|^{\frac{1}{2}}} \leq C_1 \tag{2.2}$$

and by using [13, Lemma 2.3] for $y = x_0$ and $t = t_0$, there exists a constant $C_2 := C_2(n, p, q, ||u_k||_{L^\infty(Q_{16r}(1))})$ such that

$$u_k(x,t) - u_k(y,t) \leq C_2\left(|x-y| + |x-y|^2\right). \tag{2.3}$$

By our induction assumption and the definition of u_k, $||u_k||_{L^\infty(Q_{4r})(1)} \leq 1$ and thus C_1 and C_2 can be chosen independent of the solution.

Now Eq. 2.1 can be estimated with Eqs. 2.2 and 2.3 in the following way: Denote $A := Q_{C^{-1}}(\delta^{q-2}C^{-q})$ and let $(\bar{x}, \bar{t}) \in A$ be the point where $\sup_A u_k$ is obtained and $(\bar{y}, \bar{s}) \in A$ be the point where $\inf_A u_k$ is obtained. Now for $C_3 = \max\{C_1, C_2\}$, we have

$$\begin{aligned} \operatorname{osc}_{Q_{C^{-1}}(\delta^{q-2}C^{-q})} u_k &\leq u_k(\hat{x}, \hat{t}) - u_k(\bar{y}, \bar{s}) + u_k(\bar{y}, \bar{t}) - u_k(\bar{y}, \bar{t}) \\ &\leq C_1 \left|\bar{t} - \bar{s}\right|^{\frac{1}{2}} + C_2\left(|\bar{x} - \bar{y}| + |\bar{x} - \bar{y}|^2\right) \\ &\leq C_3\left(\left[\delta^{q-2}C^{-q}\right]^{\frac{1}{2}} + C^{-1} + C^{-2}\right) \\ &\leq C_3\left(\frac{\delta}{3C_3} + \frac{\delta}{3C_3} + \frac{\delta}{3C_3}\right) = \delta \end{aligned} \tag{2.4}$$

where the last inequality holds if we choose

$$C = \max\left\{\frac{3C_3}{\delta}, \frac{(3C_3)^{\frac{2}{q}}}{\delta^{\frac{4-q}{q}}}, \delta^{-1}\right\}.$$

Thus by combining Eqs. 2.1 and 2.4, we get

$$\operatorname{osc}_{Q_{r_{k+1}}(a_{k+1})} u \leq \delta\omega_k = \omega_{k+1}$$

as desired. □

A standard argument (see [26, Corollary 7.2]) together with the assumption $C \geq \delta^{-1}$ now yields the following oscillation estimate.

Corollary 2.4 *Let u be a viscosity solution to Eq. 1.1 in $Q_{4r}(1)$ and let the range condition Eq. 1.3 hold. For any given $\omega_0 > 1$ such that $a_0 := \omega_0^{q-2}$ satisfies*

$$\operatorname{osc}_{Q_r(a_0)} u \leq \omega_0,$$

there exist constants $\hat{C} = \hat{C}(n, p, q) > 1$ and $\nu = \nu(n, p, q) \in (0, 1)$ such that for any $0 < \rho < r$ it hold

$$\operatorname{osc}_{Q_\rho(a_0)} u \leq \hat{C}\omega_0 \left(\frac{\rho}{r}\right)^\nu .$$

The proof of this well-known result is a direct calculation. This lemma directly generalizes for functions with time dependence and also for functions u where $u(x - \bar{x})$ is radial for some vector $\bar{x} \in \mathbb{R}^n$. When dealing with functions $v : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, we denote the spacial derivative by $v'(r, t)$ and the time derivative by $\partial_t v(r, t)$.

Our proofs use the following comparison principle, which is Theorem 3.1 in [25].

Theorem 2.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that u is a viscosity supersolution and v is a viscosity subsolution to Eq. 1.1 in Ω_T. If*

$$\infty \neq \limsup_{\Omega_T \ni (y,s) \to (x,t)} v(y, s) \leq \liminf_{\Omega_T \ni (y,s) \to (x,t)} u(y, s) \neq -\infty$$

for all $(x, t) \in \partial_p \Omega_T$, then $v \leq u$ in Ω_T.

3 Comparison Functions

Comparison functions are used in the standard proof for the intrinsic Harnack's inequality for the divergence form equation to expand the positivity set around the supremum point using the comparison principle. In the degenerate case, a single Barenblatt-type solution is enough to get the estimate but in the singular case, we need two separate subsolutions. The Barenblatt solutions do not have compact support in the singular range and thus we need to find another type of comparison function. Because of the connection of the Eq. 1.1 and the usual p-parabolic equation examined in [26], we can use similar comparison functions as DiBenedetto in his proof for the singular range. We will need three different subsolutions to handle the singular case and values of q near $q = 2$. We denote throughout this section

$$\eta := \frac{p-1}{q-1}$$

which is the time-scaling constant connecting Eq. 1.1 to the usual q-parabolic equation in the radial case.

Assume $q < 2$. We will use the following subsolution which is a time-rescaled version of the solution used in the p-parabolic case by DiBenedetto [8, VII.7]. Let

$$\Phi(x, t) := \frac{\kappa \rho^{q\xi}}{R(t)^\xi} \left(1 - \left(\frac{|x|^q}{R(t)}\right)^{\frac{1}{q-1}}\right)_+^2, \tag{3.1}$$

where

$$R(t) := \eta\kappa^{q-2}t + \rho^q$$

and κ and ρ are positive parameters and $\xi > 1$ is chosen independent of κ and ρ. By $(\cdot)_+$ we denote the positive part of the function inside the bracket.

By construction $\operatorname{supp}\Phi(\cdot,0)=B_\rho(0)$ and for $t\geq 0$, we get the expanding balls

$$\operatorname{supp}\Phi(\cdot,t)=B_{R(t)^{\frac{1}{q}}}(0)$$

and the estimate

$$\Phi(x,0)\leq\Phi(x,t)\leq\kappa\quad\text{for } t\geq 0. \tag{3.2}$$

We examine Φ in the domains

$$\mathcal{P}_{\kappa,\xi}:=B_{R(t)^{\frac{1}{q}}}(0)\times\left(0,\frac{\kappa^{2-q}\rho^q}{\eta\xi}\right).$$

We have $\Phi\in C^\infty(\mathcal{P}_{\kappa,\xi})\cap C(\overline{\mathcal{P}_{\kappa,\xi}})$ and as we see in the following lemma, we can choose the constant ξ to make Φ a viscosity subsolution to Eq. 1.1 in this set.

Lemma 3.1 *Let the range condition* (1.3) *hold and* $q<2$. *There exists a constant* $\xi:=\xi(n,p,q)$ *so that* Φ *is a viscosity subsolution to Eq. 1.1 in* $\mathbb{R}^n\times\left(0,\frac{\kappa^{2-q}\rho^q}{\eta\xi}\right)$.

Proof The function $\Phi\equiv 0$ outside $\mathcal{P}_{\kappa,\xi}$, so it is enough for us to check that Φ is a viscosity subsolution on the boundary and inside this set. Let us first look at the points where $\nabla\Phi\neq 0$, because here we can use the radiality of Φ in spacial coordinates and a simple calculation to simplify our statement to the form

$$\partial_t\phi-\left|\phi'\right|^{q-2}\left((p-1)\phi''+\phi'\frac{n-1}{r}\right)\leq 0\quad\text{in }\mathcal{P}'_{\kappa,\xi}:=\left(0,R(t)^{\frac{1}{q}}\right)\times\left(0,\frac{\kappa^{2-q}\rho^q}{\eta\xi}\right) \tag{3.3}$$

where

$$\phi(r,t):=\frac{\kappa\rho^{q\xi}}{R(t)^\xi}\left(1-\left(\frac{r^q}{R(t)}\right)^{\frac{1}{q-1}}\right)_+^2.$$

We use the following notation during the calculation

$$R(t):=\eta\kappa^{q-2}t+\rho^q,\quad \mathcal{F}:=1-z^{\frac{1}{q-1}},\quad z:=\frac{r^q}{R(t)},\quad A:=\frac{\kappa\rho^{q\xi}}{R(t)^\xi}.$$

By direct calculation inside $\mathcal{P}'_{\kappa,\xi}$, we have

$$\begin{aligned}
\mathcal{F}'&=-\frac{1}{q-1}z^{\frac{1}{q-1}-1}\frac{qr^{q-1}}{R(t)}=-\frac{q}{q-1}\frac{z^{\frac{1}{q-1}}}{r}\\
\mathcal{F}''&=-\frac{q}{q-1}\left(\frac{q}{q-1}\frac{z^{\frac{1}{q-1}}}{r^2}-\frac{z^{\frac{1}{q-1}}}{r^2}\right)=-\frac{q}{(q-1)^2}\frac{z^{\frac{1}{q-1}}}{r^2}\\
\phi'&=2A\mathcal{F}\mathcal{F}'=-2A\mathcal{F}\frac{q}{q-1}\frac{z^{\frac{1}{q-1}}}{r}\\
\phi''&=2A\left((\mathcal{F}')^2+\mathcal{F}\mathcal{F}''\right)=2A\left(\frac{q^2}{(q-1)^2}\frac{z^{\frac{2}{q-1}}}{r^2}-\mathcal{F}\frac{q}{(q-1)^2}\frac{z^{\frac{1}{q-1}}}{r^2}\right)\\
&=2A\frac{q}{(q-1)^2}\left(qz^{\frac{1}{q-1}}-\mathcal{F}\right)\frac{z^{\frac{1}{q-1}}}{r^2}
\end{aligned}$$

Moreover,

$$\begin{aligned}\partial_t\phi &= -\frac{\xi\kappa\rho^{q\xi}}{R(t)^{\xi+1}}\mathcal{F}^2\eta\kappa^{q-2} + 2A\mathcal{F}\frac{1}{q-1}z^{\frac{1}{q-1}-1}\frac{r^q}{R(t)^2}\eta\kappa^{q-2} \\ &= -\frac{\xi\eta\kappa^{q-1}\rho^{q\xi}}{R(t)^{\xi+1}}\mathcal{F}^2 + \frac{\eta\kappa^{q-1}\rho^{q\xi}}{R(t)^{\xi+1}}\mathcal{F}\frac{2}{q-1}z^{\frac{1}{q-1}}. \qquad (3.4)\end{aligned}$$

Define an operator $\mathcal{L} : C^2(\mathbb{R}) \to \mathbb{R}$ by

$$\mathcal{L}(\phi) := \frac{R(t)^{\xi+1}}{\eta\kappa^{q-1}\rho^{q\xi}\mathcal{F}}\left(\partial_t\phi - \left|\phi'\right|^{q-2}((p-1)\phi'' + \phi'\frac{n-1}{r})\right).$$

By the calculations above, we have

$$\begin{aligned}\mathcal{L}(\phi) &= -\xi\mathcal{F} + \frac{2}{q-1}z^{\frac{1}{q-1}} - \left|\phi'\right|^{q-2}2\frac{\kappa^{2-q}R(t)}{\eta\mathcal{F}}\frac{q}{q-1}\left(\eta\left(qz^{\frac{1}{q-1}} - \mathcal{F}\right)\frac{z^{\frac{1}{q-1}}}{r^2} - \mathcal{F}\frac{z^{\frac{1}{q-1}}}{r}\frac{n-1}{r}\right) \\ &= -\xi\mathcal{F} + \frac{2}{q-1}z^{\frac{1}{q-1}} + \left|\phi'\right|^{q-2}2\frac{\kappa^{2-q}R(t)}{\eta}\frac{q}{q-1}\frac{z^{\frac{1}{q-1}}}{r^2}\left(\eta\left(1 - \frac{qz^{\frac{1}{q-1}}}{\mathcal{F}}\right) + n - 1\right) \\ &= -\xi\mathcal{F} + \frac{2}{q-1}z^{\frac{1}{q-1}} + \left|2A\mathcal{F}\frac{q}{q-1}\frac{z^{\frac{1}{q-1}}}{r}\right|^{q-2}2\kappa^{2-q}R(t)\frac{q}{q-1}\frac{z^{\frac{1}{q-1}}}{r^2}\left(\frac{n-1}{\eta} + 1 - \frac{qz^{\frac{1}{q-1}}}{\mathcal{F}}\right) \\ &= -\xi\mathcal{F} + \frac{2}{q-1}z^{\frac{1}{q-1}} + (2A\mathcal{F})^{q-2}\left(\frac{q}{q-1}\right)^{q-1}2\kappa^{2-q}R(t)\frac{z^{\frac{q-2}{q-1}}}{r^{q-2}}\frac{z^{\frac{1}{q-1}}}{r^2}\left(C_1 - \frac{qz^{\frac{1}{q-1}}}{\mathcal{F}}\right) \\ &= -\xi\mathcal{F} + \frac{2}{q-1}z^{\frac{1}{q-1}} + \left(\frac{2q}{q-1}\right)^{q-1}\left(\frac{\rho^{q\xi}}{R(t)^\xi}\mathcal{F}\right)^{q-2}\left(\frac{n-1}{\eta} + 1 - \frac{qz^{\frac{1}{q-1}}}{\mathcal{F}}\right).\end{aligned}$$

Introduce the two sets

$$\mathcal{E}_1 := \left\{(r,t) \in \mathcal{P}'_{k,\xi} \mid \mathcal{F} < \delta\right\}, \qquad \mathcal{E}_2 := \left\{(r,t) \in \mathcal{P}'_{k,\xi} \mid \mathcal{F} \geq \delta\right\}$$

where $\delta > 0$ is a constant to be chosen. Now inside $\mathcal{E}_1$ we can estimate what is inside the last brackets from above

$$\left(\frac{n-1}{\eta} + 1 - \frac{qz^{\frac{1}{q-1}}}{\mathcal{F}}\right) \leq \frac{n-1}{\eta} + 1 - \frac{q}{\mathcal{F}} \leq \frac{n-1}{\eta} + 1 - \frac{q}{\delta} < 0$$

if we choose δ small enough that the last inequality holds. Notice also that both $\mathcal{F} \in [0, 1]$ and $\frac{\rho^{q\xi}}{R(t)^\xi} \in [0, 1]$ and thus

$$\left(\frac{\rho^{q\xi}}{R(t)^\xi}\mathcal{F}\right)^{q-2} \geq 1$$

by our assumption $q < 2$. Thus inside $\mathcal{E}_1$

$$\mathcal{L}(\phi) \leq \frac{2}{q-1} + \left(\frac{2q}{q-1}\right)^{q-1}\left(\frac{n-1}{\eta} + 1 - \frac{q}{\delta}\right) < 0. \qquad (3.5)$$

Here we can choose δ to be small enough to guarantee that the right side of the equation is negative and this can be done without dependence on ξ.

Let us next focus on $\mathcal{E}_2$. By the range of t, we have

$$\left(\frac{R(t)^\xi}{\rho^{q\xi}}\frac{1}{\mathcal{F}}\right)^{2-q} \le \left(\frac{\left(\eta\kappa^{q-2}\frac{\kappa^{2-q}\rho^q}{\eta\xi}+\rho^q\right)^\xi}{\rho^{q\xi}}\frac{1}{\mathcal{F}}\right)^{2-q} \le \left(\frac{\xi+1}{\xi}\right)^{\xi(2-q)}\delta^{q-2} \le \left(\frac{e}{\delta}\right)^{q-2}$$

and thus for δ we chose above, we have

$$\begin{aligned}\mathcal{L}(\phi) &\le -\xi\mathcal{F}+\frac{2}{q-1}\left(\frac{R(t)^\xi}{\rho^{q\xi}}\frac{1}{\mathcal{F}}\right)^{2-q}\left(\frac{n-1}{\eta}+1\right)\\ &\le -\xi\mathcal{F}+\frac{2}{q-1}\left(\frac{e}{\delta}\right)^{q-2}\left(\frac{n-1}{\eta}+1\right)\end{aligned} \tag{3.6}$$

in $\mathcal{E}_2$. We can now choose ξ to be large enough to guarantee that the right side is negative. Thus combining the estimates Eqs. 3.5 and 3.6, we have that $\mathcal{L}(\phi) \le 0$ in the entire $\mathcal{P}_{k,\xi}$ and thus by just multiplying the positive scaling factor in the definition of $\mathcal{L}$ away, we have that ϕ is a classical subsolution.

We still need to check the points where $\nabla\Phi = 0$, because there the simplification we did earlier in Eq. 3.3 does not hold. Also because of the singular nature of the Eq. 1.1, the concept of a classical solution does not really make sense at these points and we need to use the definition of viscosity solutions. By similar calculation to the radial case using the same notation, we have

$$|\nabla\Phi(x,t)| = \left|-2A\mathcal{F}\left(\frac{q}{q-1}\right)\frac{|x|^{\frac{q}{q-1}-2}}{R(t)^{\frac{1}{q-1}}}x\right| = \frac{2A}{R(t)^{\frac{1}{q-1}}}\left(\frac{q}{q-1}\right)\mathcal{F}\,|x|^{\frac{1}{q-1}}$$

and thus the gradient vanishes at the origin as $\frac{1}{q-1} > 0$ and in the set $\partial B_{R(t)^{\frac{1}{q}}}(0)\times\left(0,\frac{\kappa^{2-q}\rho^q}{\xi}\right)$ as there $\mathcal{F} = 0$. This latter set happens to be the lateral boundary of the support of Φ. By our previous calculation (3.4), the time derivative of Φ is

$$\partial_t\Phi = -\frac{\xi\kappa^{q-1}\rho^{q\xi}}{R(t)^{\xi+1}}\mathcal{F}^2 + \frac{\kappa^{q-1}\rho^{q\xi}}{R(t)^{\xi+1}}\mathcal{F}\frac{2}{q-1}z^{\frac{1}{q-1}},$$

which clearly satisfies $\partial_t\Phi \le 0$ at the critical points as the first term is negative and the second is zero if either $z = 0$ or $\mathcal{F} = 0$. Let $\varphi \in C^2$ be an admissible test function touching Φ at a critical point (x,t) from above. For any such function $\partial_t\varphi(x,t) = \partial_t\Phi(x,t) \le 0$ and thus Φ is a viscosity subsolution in $\mathcal{P}_{\kappa,\xi}$. The zero function is also a viscosity subsolution so Φ is a viscosity subsolution in the entire $\mathbb{R}^n \times \left(0,\frac{\kappa^{2-q}\rho^q}{\eta\xi}\right)$ as we already verified the boundary. □

The comparison function Φ defined in Eq. 3.1 does not give us stable constants as $q \to 2$ because the radius we use it for blows up. We can extend the proof of the degenerate case slightly below $q = 2$ with a different comparison function and use this to get stable constants in our inequality for the whole range (1.3). Let ρ and κ be positive parameters and define the function

$$\mathcal{G}(x,t) := \frac{\kappa\rho^{\frac{\nu}{\lambda(\nu)}}}{\Sigma(t)^\nu}\left(1-\left(\frac{|x|}{\Sigma(t)^{\lambda(\nu)}}\right)^{\frac{q}{q-1}}\right)_+^{\frac{q}{q-1}}, \tag{3.7}$$

where

$$\Sigma(t) := \eta\kappa^{q-2}\rho^{(q-2)\frac{\nu}{\lambda(\nu)}}t+\rho^{\frac{1}{\lambda(\nu)}}, \quad t \ge 0.$$

Here $\nu > 1$ is a constant and

$$\lambda(\nu) := \frac{1-\nu(q-2)}{q}. \tag{3.8}$$

The function (3.7) is a time-rescaled version of the comparison function introduced by DiBenedetto in [8, VII 3(i)]. We also introduce a number

$$q(\nu) := \frac{4(1+2\nu)}{1+4\nu}.$$

This number $q(\nu)$ will define the size of the interval around $q = 2$, where $\mathcal{G}$ is a viscosity subsolution.

Lemma 3.2 *Let $q \in (4-q(\nu), 7/3)$. There exists a $\nu := \nu(n, p) > 1$ independent of q such that $\mathcal{G}$ is a viscosity subsolution to Eq. 1.1 in $\mathbb{R}^n \times \mathbb{R}^+$.*

Proof We prove this statement by first showing that $\mathcal{G}$ is a classical subsolution in the support of this function

$$\mathcal{S} := \operatorname{supp} \mathcal{G} = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \middle| \, |x| < \Sigma(t)^{\lambda(\nu)}, t > 0 \right\}$$

apart from the points where $\nabla \mathcal{G} = 0$ and dealing with the boundary and rest of the space afterward. The function $\mathcal{G}$ is radial with respect to space and thus we can perform our calculations in radial coordinates. Define

$$g(r,t) := \frac{\kappa \rho^{\frac{\nu}{\lambda(\nu)}}}{\Sigma(t)^{\nu}} \left(1 - \left(\frac{r}{\Sigma(t)^{\lambda(\nu)}} \right)^{\frac{q}{q-1}} \right)_+^{\frac{q}{q-1}},$$

and

$$z := \frac{r}{\Sigma(t)^{\lambda(\nu)}}, \quad \mathcal{F} := (1 - z^{\frac{q}{q-1}})_+, \quad a := \left(\frac{q}{q-1} \right)^2, \quad A := \frac{\kappa \rho^{\frac{\nu}{\lambda(\nu)}}}{\Sigma(t)^{\nu}}, \quad \mathcal{S}' := \operatorname{supp} g.$$

Again whenever $g' \neq 0$, we can use the radiality and a quick calculation to simplify our statement to the form

$$\partial_t g - \left| g' \right|^{q-2} \left((p-1) g'' + g' \frac{n-1}{r} \right) \leq 0 \quad \text{in } \mathbb{R} \times (0, \infty).$$

Inside $\mathcal{S}'$, we have

$$\partial_t A = -\nu \frac{\kappa \rho^{\frac{\nu}{\lambda(\nu)}}}{\Sigma(t)^{\nu+1}} \eta \kappa^{q-2} \rho^{(q-2)\frac{\nu}{\lambda(\nu)}} = -\nu \frac{\rho^{(q-1)\frac{\nu}{\lambda(\nu)}}}{\Sigma(t)^{\nu+1}} \eta \kappa^{q-1}$$

$$\partial_t \mathcal{F} = -\frac{q}{q-1} z^{\frac{1}{q-1}} r \frac{-\lambda(\nu)}{\Sigma(t)^{\lambda(\nu)+1}} \eta \kappa^{q-2} \rho^{(q-2)\frac{\nu}{\lambda(\nu)}}$$

and thus

$$\begin{aligned}
\partial_t g &= -\nu \frac{\rho^{(q-1)\frac{\nu}{\lambda(\nu)}}}{\Sigma(t)^{\nu+1}} \kappa^{q-1} \eta \mathcal{F}^{\frac{q}{q-1}} + A \frac{q}{q-1} \mathcal{F}^{\frac{1}{q-1}} \left(-\frac{q}{q-1} z^{\frac{1}{q-1}} r \frac{-\lambda(\nu)}{\Sigma(t)^{\lambda(\nu)+1}} \eta \kappa^{q-2} \rho^{(q-2)\frac{\nu}{\lambda(\nu)}} \right) \\
&= -\nu \frac{\left(\kappa \rho^{\frac{\nu}{\lambda(\nu)}} \right)^{q-1}}{\Sigma(t)^{\nu+1}} \eta \mathcal{F}^{\frac{q}{q-1}} + A a \mathcal{F}^{\frac{1}{q-1}} z^{\frac{q}{q-1}} \frac{\lambda(\nu)}{\Sigma(t)} \eta \left(\kappa \rho^{\frac{\nu}{\lambda(\nu)}} \right)^{q-2} \\
&= \frac{\left(\kappa \rho^{\frac{\nu}{\lambda(\nu)}} \right)^{q-1}}{\Sigma(t)^{\nu+1}} \left(-\nu \eta \mathcal{F} + a \lambda(\nu) \eta \mathcal{F}^{\frac{1}{q-1}} z^{\frac{q}{q-1}} \right).
\end{aligned} \tag{3.9}$$

For the spatial derivatives, we have

$$
\begin{aligned}
g' &= A\frac{q}{q-1}\mathcal{F}^{\frac{1}{q-1}}\mathcal{F}' = -Aa\mathcal{F}^{\frac{1}{q-1}}\frac{z^{\frac{1}{q-1}}}{\Sigma(t)^{\lambda(\nu)}}\\
g'' &= A\frac{q}{q-1}\left(\frac{1}{q-1}\mathcal{F}^{\frac{1}{q-1}-1}(\mathcal{F}')^2 + \mathcal{F}^{\frac{1}{q-1}}\mathcal{F}''\right)\\
&= A\frac{q}{q-1}\left(\frac{q^2}{(q-1)^3}\mathcal{F}^{\frac{1}{q-1}-1}\frac{z^{\frac{2}{q-1}}}{\Sigma(t)^{2\lambda(\nu)}} - \mathcal{F}^{\frac{1}{q-1}}\frac{q}{(q-1)^2}\frac{z^{\frac{1}{q-1}-1}}{\Sigma(t)^{2\lambda(\nu)}}\right)\\
&= \frac{Aa}{q-1}\left(\frac{q}{q-1}z^{\frac{q}{q-1}} - \mathcal{F}\right)z^{\frac{1}{q-1}-1}\frac{\mathcal{F}^{\frac{1}{q-1}-1}}{\Sigma(t)^{2\lambda(\nu)}}
\end{aligned}
$$

so finally

$$
\begin{aligned}
&|g'|^{q-2}\left((p-1)g'' + g'\frac{n-1}{r}\right)\\
&= \left(\frac{Aa}{\Sigma(t)^{\lambda(\nu)}}\right)^{q-2}(\mathcal{F}z)^{\frac{q-2}{q-1}}\left(Aa\frac{p-1}{q-1}\left(\frac{q}{q-1}z^{\frac{q}{q-1}} - \mathcal{F}\right)z^{\frac{1}{q-1}-1}\frac{\mathcal{F}^{\frac{1}{q-1}-1}}{\Sigma(t)^{2\lambda(\nu)}} - Aa\mathcal{F}^{\frac{1}{q-1}}\frac{z^{\frac{1}{q-1}}}{\Sigma(t)^{\lambda(\nu)}}\frac{n-1}{r}\right)\\
&= \frac{(Aa)^{q-1}}{\Sigma(t)^{(q-2)\lambda(\nu)}}\left(\frac{p-1}{q-1}\left(\frac{q}{q-1}z^{\frac{q}{q-1}} - \mathcal{F}\right)\frac{1}{\Sigma(t)^{2\lambda(\nu)}} - \mathcal{F}\frac{r}{\Sigma(t)^{2\lambda(\nu)}}\frac{n-1}{r}\right)\\
&= \frac{(Aa)^{q-1}}{\Sigma(t)^{q\lambda(\nu)}}\left(C_2 z^{\frac{q}{q-1}} - C_1\mathcal{F}\right)
\end{aligned}
$$

for constants $C_1 = \frac{(n-1)(q-1)+p-1}{q-1}$ and $C_2 = \frac{q(p-1)}{(q-1)^2}$. We define an operator $\mathcal{L} : C^2(\mathbb{R}) \to \mathbb{R}$ by

$$
\mathcal{L}(g) := \frac{\Sigma(t)^{\nu+1}}{\left(\kappa\rho^{\frac{\nu}{\lambda(\nu)}}\right)^{q-1}}\left(\partial_t g - |g'|^{q-2}\left((p-1)g'' + g'\frac{n-1}{r}\right)\right).
$$

Therefore by the calculation above, we have

$$
\begin{aligned}
\mathcal{L}(g) &= -\nu\eta\mathcal{F}^{\frac{q}{q-1}} + a\lambda(\nu)\eta\mathcal{F}^{\frac{1}{q-1}}z^{\frac{q}{q-1}} - \frac{\Sigma(t)^{\nu+1}}{\left(\kappa\rho^{\frac{\nu}{\lambda(\nu)}}\right)^{q-1}}\frac{a^{q-1}}{\Sigma(t)^{q\lambda(\nu)}}\left(\frac{\kappa\rho^{\frac{\nu}{\lambda(\nu)}}}{\Sigma(t)^{\nu}}\right)^{q-1}\left(C_2 z^{\frac{q}{q-1}} - C_1\mathcal{F}\right)\\
&= -\nu\eta\mathcal{F}^{\frac{q}{q-1}} + a\lambda(\nu)\eta\mathcal{F}^{\frac{1}{q-1}}z^{\frac{q}{q-1}} + \Sigma(t)^{(2-q)\nu+1-q\lambda(\nu)}a^{q-1}\left(C_1\mathcal{F} - C_2 z^{\frac{q}{q-1}}\right)\\
&= -\nu\eta\mathcal{F}^{\frac{q}{q-1}} + a\lambda(\nu)b\mathcal{F}^{\frac{1}{q-1}}z^{\frac{q}{q-1}} + a^{q-1}\left(C_1\mathcal{F} - C_2 z^{\frac{q}{q-1}}\right) \qquad (3.10)
\end{aligned}
$$

where the exponent of $\Sigma(t)$ is zero because of Eq. 3.8. We introduce two sets

$$
\begin{aligned}
\mathcal{E}_1 &:= \left\{(x,t) \in \mathbb{R} \times (0,\infty) \mid z^{\frac{q}{q-1}} \geq \frac{1}{2}\left(1 + \frac{C_1}{C_1 + C_2}\right)\right\}\\
\mathcal{E}_2 &:= \left\{(x,t) \in \mathbb{R} \times (0,\infty) \mid z^{\frac{q}{q-1}} < \frac{1}{2}\left(1 + \frac{C_1}{C_1 + C_2}\right)\right\}
\end{aligned}
$$

and note that as λ is decreasing with respect to q, we have

$$
\frac{1}{4} = \lambda(q(\nu)) \leq \lambda(\nu) \leq \lambda(4 - q(\nu)) = \frac{6\nu+1}{8\nu} \leq \frac{7}{8} \quad \text{for} \quad q \in [4 - q(\nu), q(\nu)]. \qquad (3.11)
$$

Inside $\mathcal{E}_1$, the first term can be small depending on the data but the lower bound we chose for $z^{\frac{q}{q-1}}$ ensures that the rest of the terms are negative on their own without dependence on

ν. Using Eq. 3.10 and estimate Eq. 3.11, it follows that in $\mathcal{E}_1$ it holds

$$
\begin{aligned}
\mathcal{L}(g) &\leq -\nu\eta\mathcal{F}^{\frac{q}{q-1}} + a\lambda(\nu)\eta\mathcal{F}^{\frac{1}{q-1}}z^{\frac{q}{q-1}} + a^{q-1}\left(C_1(1-z^{\frac{q}{q-1}})_+ - C_2 z^{\frac{q}{q-1}}\right) \\
&\leq a\lambda(\nu)\eta\mathcal{F}^{\frac{1}{q-1}} + a^{q-1}\left(C_1 - (C_1+C_2)z^{\frac{q}{q-1}}\right) \\
&\leq \hat{a}\left(\lambda(\nu)\eta + C_1 - \frac{1}{2}(C_1+C_2)\left(1+\frac{C_1}{C_1+C_2}\right)\right) \\
&= \hat{a}\left(\lambda(\nu)\eta - \frac{q(p-1)}{2(q-1)^2}\right) \\
&\leq \hat{a}\eta\left(\frac{7}{8} - \frac{q}{2(q-1)}\right) \leq 0
\end{aligned} \tag{3.12}
$$

where $\hat{a} = \max\{a, a^{q-1}\}$. The last inequality holds because we assumed that $q < \frac{7}{3}$. Notice that this estimate holds for all ν but only for $q \in (4 - q(\nu), \frac{7}{3})$ depending on the ν we pick.

In $\mathcal{E}_2$, we have

$$
\mathcal{F} \geq \frac{C_2}{2(C_1+C_2)}
$$

and we can ensure that $\mathcal{L}(g)$ is negative by choosing a suitably large ν. We again estimate using Eqs. 3.10 and 3.11 that inside $\mathcal{E}_2$, it holds

$$
\begin{aligned}
\mathcal{L}(g) &\leq -\nu\eta\mathcal{F}^{\frac{q}{q-1}} + a\lambda(\nu)\eta\mathcal{F}^{\frac{1}{q-1}}z^{\frac{q}{q-1}} + a^{q-1}\left(C_1\mathcal{F} - C_2 z^{\frac{q}{q-1}}\right) \\
&\leq -\nu\eta\left(\frac{C_2}{2(C_1+C_2)}\right)^{\frac{q}{q-1}} + \hat{a}\left(\frac{7}{8}b + C_1\right).
\end{aligned} \tag{3.13}
$$

Choose

$$
\nu := \max_{q\in[8/5,7/3]} \hat{a}\left(\frac{7}{8} + \frac{C_1}{\eta}\right)\left(\frac{C_2}{2(C_1+C_2)}\right)^{-\frac{q}{q-1}}
$$

so that for this ν, we have $\mathcal{L}(g) \leq 0$ in $\mathcal{E}_2$ by Eq. 3.13. Notice that this choice of ν depends on n and p but not q and that $4 - q(\nu) > \frac{8}{5}$ for all $\nu \geq 1$. Thus for this choise of ν, we get that by Eqs. 3.12 and 3.13, we have $\mathcal{L}(g) \leq 0$ in the classical sense in $\mathcal{S}'$ for all $q \in (4 - q(\nu), \frac{7}{3})$.

We still have to check the points where $\nabla\mathcal{G} = 0$. The gradient for the original function (3.7) is

$$
\nabla\mathcal{G}(x,t) = A\frac{q}{q-1}\mathcal{F}^{\frac{1}{q-1}}\mathcal{F}' = -Aa\mathcal{F}^{\frac{1}{q-1}}z^{\frac{1}{q-1}}\frac{x}{|x|\,\Sigma(t)^{\lambda(\nu)}} = -Aa\mathcal{F}^{\frac{1}{q-1}}|x|^{\frac{1}{q-1}-1}\frac{x}{\Sigma(t)^{q\lambda(\nu)}}
$$

which exists and vanishes at the origin as $\frac{1}{q-1} > 0$ and also vanishes when $\mathcal{F} = 0$, that is when $x \in \partial\mathcal{R}$. Using the time derivative we calculated in Eq. 3.9, we have that for $x \in \partial\mathcal{R}$ it holds $\partial_t\mathcal{G} = 0$ and for $x = 0$, we have

$$
\partial_t\mathcal{G} = -\nu\frac{\left(\kappa\rho^{\frac{\nu}{\lambda(\nu)}}\right)^{q-1}}{\Sigma(t)^{\nu+1}} \leq 0.
$$

Let $\varphi \in C^2$ be an admissible test function touching Φ at a critical point (x,t) from above. For any such function $\partial_t\varphi(x,t) = \partial_t\mathcal{G}(x,t) \leq 0$ and thus Φ is a viscosity subsolution in $\mathcal{S}$. In $\left(\mathbb{R}^n \times \mathbb{R}^+\right) \setminus \mathcal{S}$, any admissible test function touching $\mathcal{G}$ from above must have zero time-derivative and thus $\mathcal{G}$ is a viscosity subsolution in the entire $\mathbb{R}^n \times \mathbb{R}^+$. □

We need one more comparison function to handle expanding the sidewise positivity set in our proof of the singular forward Harnack's inequality in Theorem 4.1. This differs from the degenerate case where only one Barenblatt type comparison function is used [26, Theorem 7.3].

Let k and ν be positive parameters and consider cylindrical domains with annular cross-section

$$\mathcal{C}(\theta) := \{\nu < |x| < 1\} \times (0, \theta). \tag{3.14}$$

For these parameters and a constant ζ, we define

$$\Psi(x,t) := k\left(1-|x|^2\right)_+^{\frac{q}{q-1}} \left(1 + k^{\frac{2-q}{q-1}} \zeta \left(\frac{|x|^q}{\eta t}\right)^{\frac{1}{q-1}}\right)^{-\frac{q-1}{2-q}}. \tag{3.15}$$

This is a rescaled version of the comparison function introduced by DiBenedetto in [8, VII 6]. Our set (3.14) has different scaling compared to DiBenedetto's as we feel this slightly simplifies the roles of parameters. After finding a suitable ζ to ensure that Ψ is a subsolution, we can pick k to set what value Ψ attains on the inner lateral boundary and finally pick ν to set the size of the hole in the annular cross-section of our cylinder to be of suitable radius. In our proof of the forward inequality these are picked in Eqs. 4.5 and 4.6. We present the proof in detail for the ease of the reader and to fix some typos in the literature.

Lemma 3.3 *Let the range condition (1.3) hold and $q < 2$. There exist constants $\zeta := \zeta(n,p,q)$ and $\Theta := \Theta(n,p,q)$ such that for every $0 < \nu < 1$ and $k > 0$, Eq. 3.15 is a viscosity subsolution to the equation 1.1 in $C(\theta)$ for*

$$\theta = \nu^q k^{2-q} \Theta. \tag{3.16}$$

Proof The function Ψ is radial and thus we will again do our calculations in radial coordinates. Define

$$\psi(r,t) := k\left(1-r^2\right)_+^{\frac{q}{q-1}} \left(1 + k^{\frac{2-q}{q-1}} \zeta \left(\frac{r^q}{\eta t}\right)^{\frac{1}{q-1}}\right)^{-\frac{q-1}{2-q}}$$

and denote

$$z := k^{\frac{2-q}{q-1}} \zeta \left(\frac{r^q}{\eta t}\right)^{\frac{1}{q-1}}, \quad \mathcal{F} := 1 + z, \quad w := \frac{k}{\mathcal{F}^{\frac{q-1}{2-q}}}, \quad v := (1-r^2)^{\frac{q}{q-1}},$$

so that $\psi = vw$. Whenever $\psi' \neq 0$, we can simplify our statement to the form

$$\partial_t \psi - \left|\psi'\right|^{q-2} \left((p-1)\psi'' + \psi' \frac{n-1}{r}\right) \le 0 \quad \text{in } \mathcal{C}'(\theta) := \{\nu < r < 1\} \times (0,\theta).$$

We have

$$\begin{aligned}
v' &= -\frac{2rq}{q-1}\left(1-r^2\right)^{\frac{1}{q-1}} = -\frac{2rq}{q-1} v^{\frac{1}{q}} \\
v'' &= \frac{4r^2 q}{(q-1)^2}\left(1-r^2\right)^{\frac{2-q}{q-1}} - \frac{2q}{q-1}\left(1-r^2\right)^{\frac{1}{q-1}} = \frac{4r^2 q}{(q-1)^2} v^{\frac{2-q}{q}} - \frac{2q}{q-1} v^{\frac{1}{q}} \\
w' &= k\frac{1-q}{2-q} \mathcal{F}^{-\frac{1}{2-q}} \frac{q}{q-1} k^{\frac{2-q}{q-1}} \zeta \left(\frac{r}{\eta t}\right)^{\frac{1}{q-1}} = -\frac{q}{2-q} \frac{w}{\mathcal{F}} \frac{z}{r}
\end{aligned}$$

$$w'' = -\left(\frac{q}{2-q}\right)\left[-\left(\frac{q}{2-q}\right)\frac{w}{\mathcal{F}^2}\frac{z^2}{r^2} + \left(\frac{q}{q-1}\right)\frac{w}{\mathcal{F}}\frac{z}{r^2} - \left(\frac{q}{q-1}\right)\frac{w}{\mathcal{F}}\frac{z^2}{r^2} - \frac{w}{\mathcal{F}}\frac{z}{r^2}\right]$$
$$= \left(\frac{q}{2-q}\right)\left[\left(\frac{q}{(2-q)(q-1)}\right)\frac{w}{\mathcal{F}^2}\frac{z^2}{r^2} - \frac{1}{q-1}\frac{w}{\mathcal{F}}\frac{z}{r^2}\right]$$
$$= \frac{q^2}{(2-q)^2(q-1)}\frac{w}{\mathcal{F}^2}\frac{z^2}{r^2} - \frac{q}{(2-q)(q-1)}\frac{w}{\mathcal{F}}\frac{z}{r^2}.$$

Define operators $\mathcal{Q} : C^2(\mathbb{R}) \to \mathbb{R}$ and $\mathcal{R} : C^2(\mathbb{R}) \to \mathbb{R}$ by

$$\mathcal{Q}(\psi) := \partial_t \psi - \left|\psi'\right|^{q-2}\left((p-1)\psi'' + \psi'\frac{n-1}{r}\right)$$

and

$$\mathcal{R}(\psi) := -(p-1)\psi'' - \frac{n-1}{r}\psi'$$

so that $\mathcal{Q}(\psi) = \partial_t \psi + \left|\psi'\right|^{q-2}\mathcal{R}(\psi)$. Using $\psi' = w'v + wv'$ and $\psi'' = w''v + 2w'v' + wv''$, we can estimate $\mathcal{R}(\psi)$ to obtain

$$\begin{aligned}
\mathcal{R}(\psi) =& -(p-1)(w''v + 2w'v' + wv'') - \frac{n-1}{r}(w'v + wv') \\
=& -(p-1)\Bigg[\left(\frac{q^2}{(2-q)^2(q-1)}\frac{w}{\mathcal{F}^2}\frac{z^2}{r^2} - \frac{q}{(2-q)(q-1)}\frac{w}{\mathcal{F}}\frac{z}{r^2}\right)v + \frac{4q^2}{(2-q)(q-1)}\frac{w}{\mathcal{F}}\frac{z}{r}v^{\frac{1}{q}} \\
& + w\left(\frac{4r^2 q}{(q-1)^2}v^{\frac{2-q}{q}} - \frac{2q}{q-1}v^{\frac{1}{q}}\right)\Bigg] - \frac{n-1}{r}\left(-\frac{q}{2-q}\frac{w}{\mathcal{F}}\frac{z}{r}v - \frac{2rq}{q-1}v^{\frac{1}{q}}w\right) \\
=& -\left[\frac{(p-1)q^2}{(2-q)^2(q-1)}\frac{z}{\mathcal{F}} - \frac{(p-1)q}{(2-q)(q-1)} - \frac{(n-1)q}{2-q}\right]\frac{w}{\mathcal{F}}\frac{z}{r^2}v \\
& - \frac{4q^2(p-1)}{(2-q)(q-1)}\frac{z}{\mathcal{F}r}wv^{\frac{1}{q}} - \frac{4r^2 q(p-1)}{(q-1)^2}wv^{\frac{2-q}{q}} + \frac{2q(p-1)}{q-1}wv^{\frac{1}{q}} + \frac{2(n-1)q}{q-1}wv^{\frac{1}{q}} \\
=:& \frac{q}{2-q}\eta\left[d - \frac{q}{(2-q)}\frac{z}{\mathcal{F}}\right]\frac{wz}{\mathcal{F}r^2}v + A.
\end{aligned} \tag{3.17}$$

Here $d = \frac{(n-1)(q-1)}{(p-1)} + 1$ and A consists of four latter terms. Next, we will prove that A is negative for suitably large z and we prove the technical part of this as a separate lemma after finishing this proof. Let $Z(p,q,N)$ be the positive constant given by the Lemma 3.4 proven below and note that to use this lemma, we will need to restrict θ to make sure that

$$z \geq Z \text{ for all } (r,t) \in C(\theta).$$

The correct choice turns out to be

$$\theta \leq \frac{\zeta^{q-1}\nu^q k^{2-q}}{\eta}\frac{1}{Z^{q-1}} \tag{3.18}$$

as plugging this into the definition of z, we get

$$z = k^{\frac{2-q}{q-1}}\zeta\left(\frac{r^q}{\eta t}\right)^{\frac{1}{q-1}} \geq k^{\frac{2-q}{q-1}}\zeta\left(\frac{\nu^q}{\eta t}\right)^{\frac{1}{q-1}} \geq k^{\frac{2-q}{q-1}}\zeta\left(\frac{\nu^q}{\zeta^{q-1}\nu^q k^{2-q}\frac{1}{Z^{q-1}}}\right)^{\frac{1}{q-1}} = Z.$$

Thus by Lemma 3.4

$$A = -wv^{\frac{1}{q}}\frac{4q^2(p-1)}{(2-q)(q-1)}\frac{z}{\mathcal{F}r} - wv^{\frac{2-q}{q}}\frac{4r^2 q(p-1)}{(q-1)^2} + wv^{\frac{1}{q}}\frac{2q(p-1)}{q-1} + wv^{\frac{1}{q}}\frac{2(n-1)q}{q-1}$$

$$
\begin{aligned}
&= \frac{2q}{q-1} w v^{\frac{1}{q}} \left(-2 \frac{q(p-1)}{2-q} \frac{z}{\mathcal{F} r} - 2 \frac{p-1}{q-1} v^{\frac{1-q}{q}} r^2 + p + n - 2 \right) \\
&= \frac{2q}{q-1} w v^{\frac{1}{q}} \left(-2 \frac{q(p-1)}{2-q} \frac{z}{(1+z)} \frac{1}{r} - 2 \frac{p-1}{q-1} \frac{r^2}{1-r^2} + p + n - 2 \right) \\
&\leq 0
\end{aligned}
$$

and thus combining this with Eq. 3.17, we get

$$
\mathcal{R}(\psi) \leq \frac{q}{2-q} \eta v \frac{w}{\mathcal{F}} \frac{z}{r^2} \left[d - \frac{q}{2-q} \frac{z}{\mathcal{F}} \right]. \tag{3.19}
$$

Next, we estimate

$$
\begin{aligned}
\left|\psi'\right| = -\psi' = -w'v - wv' &= \frac{q}{2-q} \frac{w}{\mathcal{F}} \frac{z}{r} v + w \frac{2rq}{q-1} v^{\frac{1}{q}} \\
&= \frac{w}{r} \left(\frac{q}{2-q} \frac{z}{\mathcal{F}} (1-r^2)^{\frac{q}{q-1}} + \frac{2q}{q-1} r^2 (1-r^2)^{\frac{1}{q-1}} \right) \\
&\leq \frac{w}{r} \left(\frac{q}{2-q} + \frac{2q}{q-1} \right) =: C_1 \frac{w}{r}
\end{aligned}
$$

and by direct calculation

$$
\partial_t \psi = v \left(\frac{1-q}{2-q} \frac{w}{\mathcal{F}} \left(-\frac{1}{q-1} \frac{z}{t} \right) \right) = \frac{1}{2-q} v \frac{w}{\mathcal{F}} \frac{z}{t}
$$

and thus we get

$$
\left|\psi'\right|^{2-q} \partial_t \psi \leq \frac{C_1^{2-q}}{2-q} \left(\frac{w}{r} \right)^{2-q} v \frac{w}{\mathcal{F}} \frac{z}{t}. \tag{3.20}
$$

Set

$$
\mathcal{L}(\psi) = \frac{(2-q)\mathcal{F} r^2}{vwz} \left|\psi'\right|^{2-q} \mathcal{Q}(\psi) = \frac{(2-q)\mathcal{F} r^2}{vwz} \left(\left|\psi'\right|^{2-q} \partial_t \psi + \mathcal{R}(\psi) \right)
$$

and plug in our estimates Eqs. 3.19 and 3.20 to get

$$
\begin{aligned}
\mathcal{L}(\psi) &\leq \frac{(2-q)\mathcal{F} r^2}{vwz} \left(\frac{C_1^{2-q}}{2-q} \left(\frac{w}{r} \right)^{2-q} v \frac{w}{\mathcal{F}} \frac{z}{t} + \frac{q}{2-q} \eta v \frac{w}{\mathcal{F}} \frac{z}{r^2} \left[d - \frac{q}{2-q} \frac{z}{\mathcal{F}} \right] \right) \\
&= C_1^{2-q} w^{2-q} \frac{r^q}{t} + \eta q \left[d - \frac{q}{2-q} \frac{z}{\mathcal{F}} \right].
\end{aligned} \tag{3.21}
$$

By our definition of w and z

$$
w^{2-q} \frac{r^q}{t} = \left(\frac{z}{1+z} \right)^{q-1} \frac{k^{2-q}}{\left(k^{\frac{2-q}{q-1}} \zeta \left(\frac{r^q}{\eta t} \right)^{\frac{1}{q-1}} \right)^{q-1}} \frac{r^q}{t} \leq \frac{\eta}{\zeta^{q-1}}
$$

and

$$
\begin{aligned}
\eta q \left[d - \frac{q}{2-q} \frac{z}{\mathcal{F}} \right] &= \eta \frac{q}{2-q} \left[d(2-q) - \frac{qz}{1+z} \right] = \eta \frac{q}{2-q} \left[d(2-q) - q + \frac{q}{1+z} \right] \\
&=: \eta \frac{q}{2-q} \left[-\lambda + \frac{q}{\mathcal{F}} \right].
\end{aligned}
$$

Using these we can further estimate Eq. 3.21 to get

$$\mathcal{L}(\psi) \leq \frac{C_1^{2-q}\eta}{\zeta^{q-1}} + \eta\frac{q}{2-q}\left[-\lambda + \frac{q}{\mathcal{F}}\right]. \tag{3.22}$$

Now finally if we assume

$$z > \frac{2q}{\lambda} \tag{3.23}$$

we have $\frac{q}{\mathcal{F}} = \frac{q}{1+z} < \frac{\lambda}{2}$ and can choose the ζ that satisfies

$$\frac{C_1^{2-q}\eta}{\zeta^{q-1}} - \eta\frac{q}{2-q}\frac{\lambda}{2} \leq 0.$$

For this ζ, the estimate Eq. 3.22 becomes $\mathcal{L}(\psi) \leq 0$ and we have that ψ is a classical subsolution. To ensure that only z satisfying both $z \geq Z$ and Eq. 3.23 are in our annulus $\mathcal{C}'(\theta)$, we need to further restrict θ we picked in Eq. 3.18 to make sure that

$$t < k^{2-q}\zeta^{q-1}r^q\left(\frac{\lambda}{2q}\right)^{q-1}\eta^{-1}$$

in the set. By the definition of $\mathcal{C}'(\theta)$, we have $r \geq \nu$ so it suffices to choose

$$\theta \leq \left(\frac{\lambda}{2q}\right)^{q-1}\frac{\zeta^{q-1}\nu^q k^{2-q}}{\eta}$$

so picking

$$\theta := \frac{\zeta^{q-1}}{\eta}\min\left\{\left(\frac{\lambda}{2q}\right)^{q-1}, \frac{1}{Z^{q-1}}\right\}\nu^q k^{2-q} =: \Theta(n,p,q)\nu^q k^{2-q}$$

all estimates hold and ψ is a classical subsolution in $\mathcal{C}'(\theta)$. We still have to check the points where $\nabla\Psi = 0$. By direct calculation, denoting

$$v = (1-|x|^2)^{\frac{q}{q-1}} \quad \text{and} \quad w = k\left(1 + k^{\frac{2-q}{q-1}}\zeta\left(\frac{|x|^q}{\eta t}\right)^{\frac{1}{q-1}}\right)^{-\frac{q-1}{2-q}},$$

we have

$$\nabla\Psi(x) = \frac{2q}{q-1}v^{\frac{1}{q}}xw + v\left(-\frac{q}{2-q}\right)k^{\frac{1}{q-1}}\left(1 + k^{\frac{2-q}{q-1}}\zeta\left(\frac{|x|^{\frac{q}{q-1}}}{(\eta t)^{\frac{1}{q-1}}}\right)\right)^{-\frac{1}{2-q}}\zeta\left(\frac{|x|^{\frac{2-q}{q-1}}}{(\eta t)^{\frac{1}{q-1}}}\right)x$$

and it is easy to see that $\nabla\Psi(x) = 0$ if and only $|x| = 1$ or $x = 0$. The origin is outside our domain so let (y, s) be an arbitrary point such that $|y| = 1$ and $s \in (0, \theta)$ and let $\varphi \in C^2$ be an admissible test function touching Ψ from above at (y, s). At such point

$$\partial_t\varphi(y,s) = \partial_t\Psi(y,s) = \frac{1}{2-q}(1-|y|^2)^{\frac{q}{q-1}}\frac{w}{\mathcal{F}}\frac{z}{t} = 0$$

and same trivially holds when touching a point in $(\mathbb{R}^n \setminus B_1(0)) \times (0, \theta)$ and thus Ψ is a viscosity subsolution in $(\mathbb{R}^n \setminus B_\nu(0)) \times (0, \theta)$. This finishes the proof of Lemma 3.3. □

Next, we will prove Lemma 3.4 that we used in the above proof to show that A was negative.

Lemma 3.4 *There exists a constant $Z = Z(p, q, n)$ such that for all $z \geq Z$ and all $r \in (0, 1)$ we have*

$$E(r) := 2\frac{q(p-1)}{2-q}\frac{z}{(1+z)}\frac{1}{r} + 2\frac{p-1}{q-1}\frac{r^2}{1-r^2} - p - n + 2 \geq 0.$$

Proof Let $K := \max\{K_1, K_2\}$ where

$$K_1 := \frac{n-1}{p-1}\frac{2-q}{2q}, \quad K_2 := \left(1 - \frac{n-1}{p-1}\right)\frac{2-q}{3q}.$$

We begin by showing that $K < 1$ using the range condition (1.3).
We first consider the case where $p < \frac{n+1}{2}$ and $q > \frac{2(n-p)}{n-1}$. Since the latter inequality implies

$$\frac{q}{2-q} > \frac{\frac{2(n-p)}{n-1}}{2 - 2\frac{(n-p)}{n-1}} = \frac{n-p}{p-1},$$

we obtain

$$K_1 = \frac{n-1}{p-1}\frac{2-q}{2q} < \frac{1}{2}\frac{n-1}{n-p} < \frac{1}{2}\frac{n-1}{n-\frac{n+1}{2}} = \frac{n-1}{2n-n-1} = 1$$

using the upper bound on p. Similarly, we estimate

$$K_2 \leq \frac{1}{3}\frac{p-1}{n-p}\left(1 + \frac{n-1}{p-1}\right) = \frac{1}{3}\left(\frac{p+n-2}{n-p}\right) < \frac{1}{3}\frac{\frac{n+1}{2}+n-2}{n-\frac{n+1}{2}} = 1.$$

In the case $p > \frac{n+1}{2}$, we have directly

$$K_1 = \frac{n-1}{p-1}\frac{2-q}{2q} \leq \frac{n-1}{\frac{n+1}{2}-1}\frac{2-q}{2q} = 2\frac{n-1}{n-1}\frac{2-q}{2q} = \frac{2-q}{q} < 1$$

and

$$K_2 = \frac{1}{3}\frac{2-q}{q}\left(1 + \frac{n-1}{p-1}\right) \leq \frac{1}{3}\frac{2-q}{q}\left(1 + \frac{n-1}{\frac{n+1}{2}-1}\right) = \frac{2-q}{q} < 1$$

so hence we have $K < 1$ for all exponents satisfying (1.3). Now observe that this implies

$$\frac{z}{z+1} \geq K$$

if and only if

$$z \geq \frac{K}{1-K}.$$

Denote $Z = \frac{K}{1-K}$ so that by above we have

$$\frac{z}{z+1} \geq K \quad \text{for all } z \geq Z. \tag{3.24}$$

Now, we estimate $E(r)$ separately in the cases $r \geq \frac{2}{3}$ and $r < \frac{2}{3}$. If we first assume $r \geq \frac{2}{3}$, this implies $\frac{r^2}{1-r^2} \geq \frac{4}{5} =: a$ so using Eq. 3.24, we can estimate

$$\begin{aligned}
E(r) &\geq 2\frac{q(p-1)}{2-q}K_1\frac{1}{r} + 2\frac{p-1}{q-1}a - p - n + 2 \\
&= (p-1)\left(\frac{2q}{2-q}K_1 + \frac{2a}{q-1} - 1 - \frac{n-1}{p-1}\right) \\
&= (p-1)\left(\frac{2q}{2-q}K_1 + \frac{2a+1-q}{q-1} - \frac{n-1}{p-1}\right) \\
&\geq (p-1)\left(\frac{2q}{2-q}K_1 - \frac{n-1}{p-1}\right) \\
&= 0,
\end{aligned}$$

where the last identity follows from the definition of K_1. If $r \leq \frac{2}{3}$, we discard the second term with r and estimate again using Eq. 3.24 to get

$$\begin{aligned}
E(r) &\leq 2\frac{q(p-1)}{2-q}\frac{3}{2}K_2 - p - n + 2 \\
&= (p-1)\left(\frac{1}{3}\frac{q}{2-q}K_2 - 1 - \frac{n-1}{p-1}\right) \\
&= 0,
\end{aligned}$$

where we used the definition of K_2. □

4 Forward Intrinsic Harnack's Inequality

In their paper [26], Parviainen and Vázquez prove the forward Harnack's inequality for viscosity solutions of Eq. 1.1 in the degenerate case $q > 2$. In this section, we prove the remaining singular case $q < 2$ and the case of q near 2. For the proof of the same results for the standard singular p-parabolic equation see [8, VII.9]. In the proof we first rescale the equation into a simpler form, locate the local supremum of the function in some specific cylinder and, use oscillation estimates to show that there exists some small ball on a time slice where the function is strictly larger than the value depending on the singularity of the Eq. 1.1. Barenblatt-type solutions have an infinite speed of propagation for $q < 2$ and hence do not work as comparison functions similarly to the degenerate case. In the strictly singular case, we next use a comparison function constructed in Lemma 3.1 to expand the set of positivity in the time direction to get a similar lower bound extended from one time slice to a space-time cylinder. Finally, we use a second comparison function constructed in Lemma 3.3 to widen the set of positivity in the spacial direction to fill the entire ball we are interested in and get the final estimate. At the end of this section, we prove the inequality for values of q near 2. This case is similar to the degenerate case and only requires one comparison function but here we use one constructed in Lemma 3.2 instead of the Barenblatt solution used in the degenerate case. This method gives us stable constants as $q \to 2$ from either side.

Theorem 4.1 *Let $u \geq 0$ be a viscosity solution to Eq. 1.1 in $Q_1^-(1)$ and let the range condition (1.3) hold. Fix $(x_0, t_0) \in Q_1^-(1)$ such that $u(x_0, t_0) > 0$. Then there exist $\mu = \mu(n, p, q)$*

and $c = c(n, p, q)$ *such that*

$$u(x_0, t_0) \le \mu \inf_{B_r(x_0)} u\left(\cdot, t_0 + \theta r^q\right)$$

where

$$\theta = cu(x_0, t_0)^{2-q},$$

whenever $(x_0, t_0) + Q_{4r}(\theta) \subset Q_1^-(1)$.

Remark 4.2 The constants μ and c can be picked to be stable as $q \to 2$ from either side as we show in the proof. As q approaches the lower bound in Eq. 1.3, μ tends to infinity, and c tends to zero. As $q \to \infty$, both μ and c tend to infinity.

Proof of Theorem 4.1 The proof for the degenerate case $q > 2$ is given as [26, Theorem 7.3] and thus we can focus on the singular case $q < 2$. Consider the rescaled equation

$$v(x, t) = \frac{1}{u(x_0, t_0)} u(x_0 + rx, t_0 + u(x_0, t_0)^{2-q} t r^q)$$

which solves

$$\begin{cases} \partial_t v = |\nabla v|^{q-p} \operatorname{div}\left(|\nabla v|^{p-2} \nabla v\right) \text{ in } Q \\ v(0, 0) = 1, \end{cases}$$

where $Q = B_4(0) \times (-4^q, 4^q)$. Now it is enough to show that there exists positive constants c_0 and μ_0 such that

$$\inf_{B_1(0)} v(\cdot, c_0) \ge \mu_0 \tag{4.1}$$

because then by the definition of v, we have

$$\begin{aligned} \mu_0 u(x_0, t_0) &\le u(x_0, t_0) \inf_{x \in B_1(0)} \frac{1}{u(x_0, t_0)} u(x_0 + rx, t_0 + c_0 u(x_0, t_0)^{2-q} r^q) \\ &= \inf_{B_r(x_0)} u(\cdot, t_0 + c_0 u(x_0, t_0)^{2-q} r^q). \end{aligned}$$

For the first part of the proof, we make the extra assumption $q < 2$ and deal with values near $q = 2$ afterward. Proof for $q = 2$ is easy but we need to deal with values near it separately to ensure that we get stable constants as $q \to 2$ from either side. We will prove Eq. 4.1 in the following steps.

Step 1: Locating the supremum. First, we will need to locate the supremum of v in Q and establish a positive lower bound for v in some small ball around the supremum point. We do this by using Hölder continuity results. For all $\tau \in [0, 1)$ and $\sigma \in (0, 1)$ to be chosen later define nested expanding cylinders

$$Q_\tau := \{(x, t) \in Q \mid |x| < \tau, t \in (-\sigma\tau, 0)\}$$

and the numbers

$$M_\tau := \sup_{Q_\tau} v, \qquad N_\tau := (1 - \tau)^{-\frac{q}{2-q}}.$$

Notice that $M_0 = 1 = N_0$ and

$$\lim_{\tau \nearrow 1} N_\tau = \infty \quad \text{and} \quad \lim_{\tau \nearrow 1} M_\tau < \infty$$

as v is bounded. Therefore by continuity, the equation $M_\tau = N_\tau$ must have a largest root $\tau_0 \in [0, 1)$, which satisfies

$$M_{\tau_0} = (1-\tau_0)^{-\frac{q}{2-q}} \quad \text{and} \quad \sup_{Q_\tau} v = M_\tau \le N_\tau \text{ for all } 1 > \tau > \tau_0.$$

Especially for $\hat\tau := \frac{1+\tau_0}{2}$ we have

$$M_{\hat\tau} \le N_{\hat\tau} = 2^{\frac{q}{2-q}}(1-\tau_0)^{-\frac{q}{2-q}}.$$

By continuity of v, it achieves the value M_{τ_0} at some point $(\hat x, \hat t) \in \overline{Q}_{\tau_0}(1)$ and for the radius $R = \frac{1-\tau_0}{2}$ we have $(\hat x, \hat t) + Q_R(1) \subset Q_{\hat\tau}(1)$ as $R + \left|(\hat x, \hat t)\right| \le \frac{1-\tau_0}{2} + \tau_0 = \hat\tau$.

Thus the supremum can be estimated

$$\sup_{(\hat x,\hat t)+Q_R(1)} v \le \sup_{Q_{\hat\tau}(1)} v = M_{\hat\tau} \le 2^{\frac{q}{2-q}}(1-\tau_0)^{-\frac{q}{2-q}} =: \omega_0 > 1. \tag{4.2}$$

Let $a_0 = \omega_0^{q-2}$ and note that $a_0 R^q = \omega_0^{q-2} R^q \le R^q$ and thus by Eq. 4.2, we have

$$\operatorname{osc}_{(\hat x,\hat t)+Q_R(a_0)} v \le \operatorname{osc}_{(\hat x,\hat t)+Q_R(1)} v \le \omega_0.$$

Thus we can use Corollary 2.4 to find $\hat C := \hat C(n, p, q) > 1$ and $\nu := \nu(n, p, q) \in (0, 1)$ such that

$$\operatorname{osc}_{B_\rho(x_0)} v(\cdot, \hat t) \le \hat C \omega_0 \left(\frac{\rho}{R}\right)^\nu$$

for any $0 < \rho < R$. Pick $\rho = \delta R$ for some $\delta := \delta(n, p, q)$ small enough to satisfy $1 - \hat C \delta^\nu 2^{\frac{q}{2-q}} \ge \frac{1}{2}$ so that

$$\begin{aligned} v(x, \hat t) &\ge \inf_{B_{\delta R}(\hat x)} v(\cdot, \hat t) = \sup_{B_{\delta R}(\hat x)} v(\cdot, \hat t) - \operatorname{osc}_{B_{\delta R}(\hat x)} v(\cdot, \hat t) \\ &\ge v(\hat x, \hat t) - \hat C \left(\frac{\delta R}{R}\right)^\nu 2^{\frac{q}{2-q}}(1-\tau_0)^{-\frac{q}{2-q}} \\ &= \left(1 - \hat C \delta^\nu 2^{\frac{q}{2-q}}\right)(1-\tau_0)^{-\frac{q}{2-q}} \ge \frac{1}{2}(1-\tau_0)^{-\frac{q}{2-q}} =: \kappa \end{aligned} \tag{4.3}$$

for all $x \in B_\rho(\hat x)$.

Step 2: Time expansion of positivity. We have managed to prove the positivity of v in a small ball for time $\hat t$ and now we intend to improve this estimate to get positivity in a time cylinder. Consider the translated comparison function $\Phi(x - \hat x, t - \hat t)$ introduced in Eq. 3.1 for choices κ and ρ introduced above. By Lemma 3.1, we have thus that Φ is a viscosity subsolution to Eq. 1.1 in

$$(\hat x, \hat t) + \mathcal{P}_{\kappa,\xi} := B_{(\hat R(t-\hat t))^{\frac{1}{q}}}(\hat x) \times \left(\hat t, \hat t + \frac{\kappa^{2-q}\rho^q}{\eta\xi}\right)$$

for time dependent radius $\hat R(t) := \eta\kappa^{q-2} t + \rho^q$. We choose

$$3\sigma := \frac{\kappa^{2-q}\rho^q}{\eta\xi} = \frac{(1-\tau_0)^{-q}\rho^q}{2^{2-q}\eta\xi}$$

where σ is the constant we did not yet choose in the definition of our cylinders Q_ρ. Now by Eqs. 4.3 and 3.2, it holds

$$v(x, \hat t) \ge \kappa \ge \Phi(x - \hat x, \hat t - \hat t)$$

and by positivity $v \geq \Phi$ on the spatial boundary. Thus by the comparison principle Theorem 2.5 (See Fig. 1 below)

$$v \geq \Phi \text{ in } \left\{ \left| x - \hat{x} \right|^q < \hat{R}(3\sigma - \hat{t}) \right\} \times \left\{ 0 < t - \hat{t} < 3\sigma \right\}$$

so in particular as $\rho \leq (\hat{R}(3\sigma - \hat{t}))^{\frac{1}{q}}$, we get for $t - \hat{t} \in (\sigma, 3\sigma)$ and $|x| \leq \rho$

$$\begin{aligned} v(x,t) &\geq \frac{\kappa \rho^{q\xi}}{\left(\eta \kappa^{q-2}(3\sigma) + \rho^q\right)^{\xi}} \left(1 - \left(\frac{\rho^q}{\eta \kappa^{q-2}\sigma + \rho^q}\right)^{\frac{1}{q-1}}\right)_+^2 \\ &= \frac{\frac{1}{2}(1-\tau_0)^{-\frac{q}{2-q}}}{\left(\frac{1}{\xi}+1\right)^{\xi}} \left(1 - \left(\frac{3\xi}{3\xi+1}\right)^{\frac{1}{q-1}}\right)_+^2 \\ &=: \hat{c}(n,p,q)(1-\tau_0)^{-\frac{q}{2-q}}. \end{aligned} \tag{4.4}$$

We do not have a way to know the exact location of $\hat{t}$ inside Q_{τ_0} but we know that $\hat{t} \in (-1, 0)$ and $\sigma \in (0, 1)$. Thus as

$$(\sigma, 2\sigma) \subset \bigcap_{\hat{t} \in (-1,0)} (\hat{t} + \sigma, \hat{t} + 3\sigma),$$

we have estimate Eq. 4.4 for all $(x, t) \in B_\rho(\hat{x}) \times (\sigma, 2\sigma)$. As $q \nearrow 2$, we have $\sigma \searrow 0$ and hence the set converges towards an empty set. To get the estimate for values of q near 2, we repeat a similar argument but with a different comparison function.

Step 3: Sidewise expansion of positivity. We will next expand the positivity set of v over $B_1(\hat{x})$ for a specific time slice using yet another comparison function to finally get the estimate Eq. 4.1. Choose

$$k = \hat{c}(1-\tau_0)^{-\frac{q}{2-q}}, \tag{4.5}$$

we got from Eq. 4.4,

$$\nu := \frac{\rho}{3} \tag{4.6}$$

and let θ be given by Eq. 3.16 for this k and ν. We have Eq. 4.4 for all $(x, t) \in B_\rho(\hat{x}) \times (\sigma, 2\sigma)$ so we have the same estimate with a smaller constant $\hat{\sigma} = \min\{\theta, \sigma\}$. We want to use a translated and scaled version of the comparison function

$$\Psi\left(\frac{x - \hat{x}}{3}, \frac{t - \hat{\sigma}}{3^q}\right)$$

in the annular cylindrical domain

$$\hat{\mathcal{C}} := \left\{ \rho < \left| x - \hat{x} \right| < 3 \right\} \times (\hat{\sigma}, 2\hat{\sigma})$$

where we introduced Ψ in Eq. 3.15. This rescaled Ψ is a viscosity subsolution to Eq. 1.1 in $\hat{\mathcal{C}}$ by Lemma 3.3. Notice that this Ψ vanishes for $x \in \partial B_3(\hat{x})$ or $t = \hat{\sigma}$ and that

$$\Psi\left(\frac{x - \hat{x}}{3}, \frac{t - \hat{\sigma}}{3^q}\right) \leq k = \hat{c}(1-\tau_0)^{-\frac{q}{2-q}}$$

everywhere in $\hat{\mathcal{C}}$. Combining this estimate with Eq. 4.4 we have

$$\Psi\left(\frac{x - \hat{x}}{3}, \frac{t - \hat{\sigma}}{3^q}\right) \leq v(x,t)$$

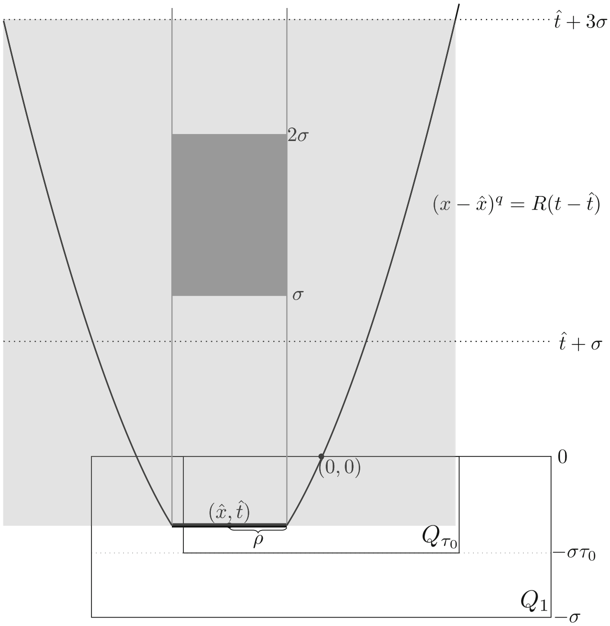

Fig. 1 Illustration of the time expansion. We use the comparison principle over the light gray cylinder and get the final estimate over the gray cylinder

for all $(x,t) \in \{\left|x-\hat{x}\right| = \rho\} \times (\hat{\sigma}, 2\hat{\sigma})$ by continuity of v. Thus we can again use the comparison principle to get $\Psi \leq v$ in the entire set $\hat{\mathcal{C}}$.
In $B_2(\hat{x}) \times \{t = 2\hat{\sigma}\}$, for our chosen $k = \hat{c}(1-\tau_0)^{-\frac{q}{2-q}}$, we have

$$\begin{aligned} v(x,2\hat{\sigma}) &\geq \hat{c}(1-\tau_0)^{-\frac{q}{2-q}} \left(1-\left(\frac{2}{3}\right)^2\right)_+^{\frac{q}{q-1}} \left(1+\left[\hat{c}(1-\tau_0)^{-\frac{q}{2-q}}\right]^{\frac{2-q}{q-1}} \zeta \left(\frac{\left(\frac{2}{3}\right)^q}{2\eta\hat{\sigma}}\right)^{\frac{1}{q-1}}\right)^{-\frac{q-1}{2-q}} \\ &\geq \inf_{0\leq\tau\leq 1} \hat{c}(1-\tau)^{-\frac{q}{2-q}} \left(1-\left(\frac{2}{3}\right)^2\right)_+^{\frac{q}{q-1}} \left(1+\left[\hat{c}(1-\tau)^{-\frac{q}{2-q}}\right]^{\frac{2-q}{q-1}} \zeta \left(\frac{\left(\frac{2}{3}\right)^q}{2\eta\hat{\sigma}}\right)^{\frac{1}{q-1}}\right)^{-\frac{q-1}{2-q}} \\ &=: \mu_0(n,p,q). \end{aligned}$$

Thus by taking infimum over $B_1(0) \subset B_2(\hat{x})$, we get

$$\inf_{B_1(0)} v(\cdot, 2\hat{\sigma}) \geq \mu_0$$

so we have proved estimate Eq. 4.1 for $c_0 := 2\hat{\sigma}$.

Case of q near 2. The case of q near 2 is quite similar to the proof we presented above. Let $\varepsilon > 0$ be a small number to be fixed later and assume that $q \in (2-\varepsilon, 2+\varepsilon)$. This time we define a family of nested expanding cylinders by

$$Q_\tau := \left\{(x,t) \in Q \mid |x| < \tau, t \in (-\tau^q, 0)\right\}$$

so they no longer depend on the constant σ. We define

$$M_\tau := \sup_{Q_\tau} v, \qquad N_\tau := (1-\tau)^{-\beta}$$

for $\beta > 0$ to be chosen later similar to the proof of the degenerate case. Again let $\tau_0 \in [0, 1)$ be the largest root of equation $M_\tau = N_\tau$ to ensure

$$M_{\tau_0} = (1-\tau_0)^{-\beta} \quad \text{and} \quad M_{\hat{\tau}} \le 2^\beta (1-\tau_0)^\beta$$

for $\hat{\tau} = \frac{1+\tau_0}{2}$. By continuity of v, it achieves the value M_{τ_0} at some point $(\hat{x}, \hat{t}) \in \overline{Q}_{\tau_0}(1)$ and for radii $R = \frac{1-\tau_0}{2}$, we have

$$\sup_{Q_R(1)} v \le \sup_{Q_{\hat{\tau}}(1)} v = M_{\hat{\tau}} \le 2^\beta (1-\tau_0)^{-\beta} =: \omega_0 > 1.$$

For $q < 2$, we can repeat the same steps as we used to obtain Eq. 4.3 to find $\rho := \delta R$ for some $\delta := \delta(n, p, q)$ small enough so that

$$\begin{aligned} v(x, \hat{t}) &\ge v(\hat{x}, \hat{t}) - \hat{C} \left(\frac{\delta R}{R}\right)^\nu 2^\beta (1-\tau_0)^{-\beta} \\ &= \left(1 - \hat{C}\delta^\nu 2^\beta\right)(1-\tau_0)^{-\beta} \ge \frac{1}{2}(1-\tau_0)^{-\beta} =: \kappa \end{aligned}$$

for all $x \in B_\rho(\hat{x})$. For $q \ge 2$, we repeat the same steps but use [26, Corollary 7.2] instead of Corollary 2.4 and we get the same estimate.

This is where we need the special subsolution constructed in Lemma 3.2. Let κ and ρ be the constants we set above and

$$\mathcal{G}(x, t) := \frac{\kappa \rho^{\frac{\nu}{\lambda(\nu)}}}{\Sigma(t)^\nu} \left(1 - \left(\frac{|x|}{\Sigma(t)^{\lambda(\nu)}}\right)^{\frac{q}{q-1}}\right)_+^{\frac{q}{q-1}}.$$

Consider the translated version $\mathcal{G}(x - \hat{x}, t - \hat{t})$ which is a viscosity subsolution to Eq. 1.1 in $\mathbb{R}^n \times \mathbb{R}^+$ as long as our exponent q is close enough to 2. Let ν be the constant given by Lemma 3.2 and pick $\varepsilon = \min\left\{\frac{4(1+2\nu)}{1+4\nu} - 2, \frac{1}{3}, \frac{1}{\nu}\right\}$. The first two numbers ensure that $\mathcal{G}$ is a viscosity solution by Lemma 3.2 and the restriction $\varepsilon \le \frac{1}{\nu}$ is here to ensure that $\lambda(\nu) \ge 0$ for all q in our range.

At time level $t = c_0$, the support of $\mathcal{G}(x - \hat{x}, c_0 - \hat{t})$ is the set

$$\operatorname{supp} \mathcal{G}(x - \hat{x}, c_0 - \hat{t}) = \left\{x \in \mathbb{R}^n \,\middle|\, |x - \hat{x}| < \Sigma(c_0 - \hat{t})^{\lambda(\nu)}\right\}$$

where

$$\lambda(\nu) = \frac{1 - \nu(q-2)}{q}$$

and

$$\begin{aligned} \Sigma(c_0 - \hat{t}) &= \left(\frac{p-1}{q-1}\right) \kappa^{q-2} \rho^{(q-2)\frac{\nu}{\lambda(\nu)}} \left(c_0 - \hat{t}\right) + \rho^{\frac{1}{\lambda(\nu)}}. \\ &= \left(\frac{p-1}{q-1}\right) \left(\frac{1}{2}(1-\tau_0)^{-\beta}\right)^{q-2} (\delta R)^{(q-2)\frac{\nu}{\lambda(\nu)}} \left(c_0 - \hat{t}\right) + \rho^{\frac{1}{\lambda(\nu)}} \\ &= A(1-\tau_0)^{(q-2)\left(\frac{\nu}{\lambda(\nu)} - \beta\right)} \left(c_0 - \hat{t}\right) + \rho^{\frac{1}{\lambda(\nu)}}. \end{aligned}$$

Here we used $R = \frac{1-\tau_0}{2}$ and defined

$$A := \left(\frac{p-1}{q-1}\right)\left(\frac{1}{2}\left(\frac{\delta}{2}\right)^{\frac{\nu}{\lambda(\nu)}}\right)^{q-2}.$$

We choose

$$\beta = \frac{\nu}{\lambda(\nu)} \quad \text{and} \quad c_0 = \frac{3^{\frac{1}{\lambda(\nu)}}}{A} + \hat{t}$$

and since $|\hat{x}| < 1$ and $\hat{t} \in (-1, 0]$, these choices ensure

$$\operatorname{supp} \mathcal{G}(x-\hat{x}, c_0-\hat{t}) = \left\{ |x-\hat{x}| < \left(A\left(\frac{3^{\frac{1}{\lambda(\nu)}}}{A} + \hat{t} - \hat{t}\right) + \rho^{\frac{1}{\lambda(\nu)}}\right)^{\lambda(\nu)} \right\} \supset B_3(\hat{x})$$

and thus $B_2(0) \subset \operatorname{supp} \mathcal{G}(x-\hat{x}, c_0-\hat{t})$. In the set $\operatorname{supp} \mathcal{G}(x-\hat{x}, \hat{t}-\hat{t}) = B_\rho(\hat{x})$, we have

$$\mathcal{G}(x-\hat{x}, \hat{t}-\hat{t}) = \kappa\left(1-\left(\frac{|x-\hat{x}|}{\rho}\right)^{\frac{q}{q-1}}\right)_+^{\frac{q}{q-1}} \leq \kappa \leq v(x,t)$$

and similarly $\mathcal{G} \leq v$ on the rest of $\partial_p\left(B_2(\hat{x}) \times [\hat{t}, c_0]\right)$ because we assumed v to be positive. Hence by the comparison principle Theorem 2.5

$$\begin{aligned}
\inf_{B_1(0)} v(\cdot, c_0) &\geq \inf_{B_1(0)} \mathcal{G}(\cdot, c_0) \\
&= \frac{\kappa\rho^{\frac{\nu}{\lambda(\nu)}}}{\left(3^{\frac{1}{\lambda(\nu)}} + \rho^{\frac{1}{\lambda(\nu)}}\right)^{\nu}}\left(1-\left(\frac{1}{\left(3^{\frac{1}{\lambda(\nu)}} + \rho^{\frac{1}{\lambda(\nu)}}\right)^{\lambda(\nu)}}\right)^{\frac{q}{q-1}}\right)_+^{\frac{q}{q-1}} \\
&\geq \frac{1}{2}\left(\frac{\delta}{2}\right)^{\frac{\nu}{\lambda(\nu)}} \frac{1}{\left(3^{\frac{1}{\lambda(\nu)}} + \rho^{\frac{1}{\lambda(\nu)}}\right)^{\nu}}\left(1-\left(\frac{1}{3}\right)^{\frac{q}{q-1}}\right)_+^{\frac{q}{q-1}} \\
&\geq 3^{-\frac{\nu}{\lambda(\nu)}} 2^{-(1+2\nu)}\left(\frac{\delta}{2}\right)^{\frac{\nu}{\lambda(\nu)}}\left(1-\left(\frac{1}{3}\right)^{\frac{q}{q-1}}\right)_+^{\frac{q}{q-1}} \\
&=: \mu_0(n, p, q),
\end{aligned}$$

so we have proven Eq. 4.1. Notice that all constants used here are stable as $q \to 2$ from either side. ◻

5 Backward Intrinsic Harnack's Inequality

In this section, we will prove the backward intrinsic Harnack's inequality for the optimal range of exponents (1.3). We proved the singular case as Theorem 5.2 in [19] but the degenerate case has not been proven before to the best of our knowledge for Eq. 1.1. The degenerate case is proven for the standard p-parabolic equation [6, Section 5.3]. All proofs are based on using the forward inequality in a specific way taking into account the intrinsic scaling. In the degenerate case, we move backward in time centered at x_0 seeking for a time where the function obtains a value larger than $\mu u(x_0, t_0)$. We handle the case of such time existing and not existing separately and show that in both cases we get the backward inequality using the forward inequality. The main difference to the singular case is that when $q \geq 2$, we have to assume that $u(x_0, t_0) > 0$ or the inequality will not hold. The case $q = 2$ follows directly from forward Harnack's inequality as we do not have to worry about the intrinsic scaling. In the singular case, the amount of space needed around our space-time cylinder depends on n, p, and q but we improve this result using covering arguments in the next section.

Theorem 1.1 *Let $u \geq 0$ be a viscosity solution to Eq. 1.1 in $Q_1^-(1)$ and let the range condition (1.3) hold. Fix $(x_0, t_0) \in Q_1^-(1)$ such that $u(x_0, t_0) > 0$. Then there exist $\gamma = \gamma(n, p, q)$, $c = c(n, p, q)$ and $\sigma = \sigma(n, p, q) > 1$ such that*

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0 - \theta r^q) \leq u(x_0, t_0) \leq \gamma \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q)$$

where

$$\theta = cu\,(x_0, t_0)^{2-q}\,,$$

whenever $(x_0, t_0) + Q_{\sigma r}(\theta) \subset Q_1^-(1)$.

Proof Let c and μ be the constants we get from Theorem 4.1 and let $\theta = cu\,(x_0, t_0)^{2-q}$.
Case 1 (q<2)**:** The case $q < 2$ is Theorem 5.2. in [19] where we get the theorem for constant $\sigma = \frac{6}{\alpha}$ where $\alpha = (2\mu)^{\frac{q-2}{q}} < 1$.

Apart from the non-emptyness of $\mathcal{U}_\alpha$ this proof extends directly to the case $q = 2$ but this can be done easier as we do not have intrinsic time scaling in this case.

Case 2 (q=2)**:** Let $\bar{t} = t_0 - cr^2$ and $y \in B_r(x_0)$. Now by Theorem 4.1 at the point $(y, \bar{t})$, we get an estimate

$$u(y, \bar{t}) \leq \mu \inf_{B_r(y)} u(\cdot, \bar{t} + cr^2) = \mu \inf_{B_r(y)} u(\cdot, t_0) \leq \mu u(x_0, t_0). \tag{5.1}$$

This holds for any $y \in B_r(x_0)$ and thus by taking supremum over all of them we get

$$\sup_{B_r(x_0)} u(\cdot, t - cr^2) \leq \mu u(x_0, t_0), \tag{5.2}$$

as desired. The use of Harnack's inequality in Eq. 5.1 is justified because in the space direction $B_{4r}(y) \subset B_{5r}(x_0) \subset B_{\frac{6}{\alpha}r}(x_0)$ and in the time direction we have

$$\bar{t} - c(4r)^2 \geq t_0 - cr^2 - c(4r)^2 = t_0 - c(5r^2) \geq t_0 - c(\sigma r^2),$$

for any $\sigma > 5$.

Inequality (5.2) combined with Theorem 4.1 proves the inequality in the case $q = 2$.

Case 3 (q>2)**:** Finally, let $q > 2$ where we again have to deal with the time-scaling. Let ρ be a radius such that $(x_0, t_0) + Q_{6\rho}(\theta) \subset Q_1^-(1)$ for $\theta = cu(x_0, t_0)^{2-q}$ and define the set

$$\mathcal{T} = \{t \in (t_0 - \theta(4\rho)^q, t_0) \mid u(x_0, t) = 2\mu u(x_0, t_0)\}.$$

Now $\mathcal{T}$ is either empty or non-empty. If it happens that $\mathcal{T} \neq \emptyset$, there exists a largest $\tau \in \mathcal{T}$ by continuity of u. For a time like this, it must hold that

$$t_0 - \tau > cu(x_0, \tau)^{q-2} \rho^q = c\,(2\mu u(x_0, t_0))^{2-q}\, \rho^q, \tag{5.3}$$

because otherwise we can choose $\hat{\beta} \in (0, 1)$ such that

$$\tau + cu(x_0, \tau)^{q-2} (\hat{\beta}\rho)^q = t_0$$

and use Theorem 4.1 on the point (x_0, τ) for radius $\hat{\beta}\rho$ to get

$$2\mu u(x_0, t_0) = u(x_0, \tau) \le \mu \inf_{B_{\hat{\beta}\rho}(x_0)} u(\cdot, t_0) \le u(x_0, t_0).$$

This is a contradiction assuming that we have suitable space to use the forward Harnack's inequality. This is automatically satisfied in space as we are centered at x_0 and in time we have

$$\begin{aligned}\tau - cu(x_0, \tau)^{2-q}(4\hat{\beta}\rho)^q &> \tau - cu(x_0, \tau)^{2-q}(4\rho)^q = \tau - c(2\mu)^{2-q}u(x_0, t_0)^{2-q}(4\rho)^q \\ &> t_0 - \theta(4\rho)^q - (2\mu)^{2-q}\theta(4\rho)^q = t_0 - \left(1 + (2\mu)^{2-q}\right)\theta(4\rho)^q \\ &> t_0 - \theta(6\rho)^q,\end{aligned}$$

where the last inequality holds for all $\mu > 1$ as $1 + (2\mu)^{2-q}4^q < 2 \cdot 4^q < 6^q$ for $q \ge 2$. Here we used the fact that $\tau > t_0 - \theta(4\rho)^q$ and $u(x_0, \tau) = 2\mu u(x_0, t_0)$ by the definition of $\mathcal{T}$. Set

$$s = t_0 - c(2\mu u(x_0, t_0))^{2-q}\rho^q \tag{5.4}$$

and notice that by Eq. 5.3, it holds $s \in (\tau, t_0)$ and

$$u(x_0, s) \le 2\mu u(x_0, t_0).$$

Assume thriving for a contradiction that there exists $y \in B_\rho(x_0)$ such that

$$u(y, s) = 2\mu u(x_0, t_0), \tag{5.5}$$

and note that

$$s + cu(y, s)^{2-q}\rho^q = t_0.$$

Therefore assuming there is enough room to use Theorem 4.1, we get

$$2\mu u(x_0, t_0) = u(y, s) \le \mu \inf_{B_\rho(y)} u(\cdot, s + cu(x_0, t_0)^{2-q}\rho^q) = \mu \inf_{B_\rho(y)} u(\cdot, t_0) \le \mu u(x_0, t_0).$$

We have enough room in space as $B_{4\rho}(y) \subset B_{5\rho}(x_0)$ and by Eqs. 5.4 and 5.5 in time, it holds

$$\begin{aligned}s - cu(y, s)^{2-q}(4\rho)^q &= t_0 - c(2\mu u(x_0, t_0))^{2-q}\rho^q - c(2\mu u(x_0, t_0))^{2-q}(4\rho)^q \\ &= t_0 - \theta\left[\left(2\mu + 4^{\frac{q}{2-q}}2\mu\right)^{\frac{2-q}{q}}\rho\right]^q > t_0 - \theta(6\rho)^q,\end{aligned}$$

where the last inequality holds for all $\mu > 1$ because for $q > 2$ we have

$$\left(2\mu + 4^{\frac{q}{2-q}}2\mu\right)^{\frac{2-q}{q}} < (2\mu)^{\frac{2-q}{q}} < 1.$$

Therefore such $y \in B_\rho(x_0)$ cannot exist and we have

$$u(y, s) < 2\mu u(x_0, t_0) \quad \text{for all } y \in B_\rho(x_0)$$

and thus by definition of s

$$\sup_{B_\rho(x_0)} u(\cdot, t_0 - c(2\mu u(x_0, t_0))^{2-q}\rho^q) \le 2\mu u(x_0, t_0) \tag{5.6}$$

Let $r = (2\mu)^{\frac{2-q}{q}} \rho \leq \rho$ and rewrite Eq. 5.6 as

$$u(x_0, t_0) \geq (2\mu)^{-1} \sup_{B_\rho(x_0)} u\left(\cdot, t_0 - \theta\left((2\mu)^{\frac{2-q}{q}} \rho\right)^q\right) \geq (2\mu)^{-1} \sup_{B_r(x_0)} u\left(\cdot, t_0 - \theta r^q\right).$$

This combined with Theorem 4.1 for radius r and taking 2μ gives

$$(2\mu)^{-1} \sup_{B_r(x_0)} u\left(\cdot, t_0 - \theta r^q\right) \leq u(x_0, t_0) \leq \mu \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q) \leq 2\mu \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q) \tag{5.7}$$

which is what we wanted. If it happens that $\mathcal{T} = \emptyset$, we have

$$u(x_0, t) < 2\mu u(x_0, t_0) \text{ for all } t \in (t_0 - \theta(4\rho)^q, t_0) \tag{5.8}$$

by continuity of u. Assume thriving for a contradiction that

$$\sup_{B_r(x_0)} u(\cdot, t_0 - \theta r^q) > 2\mu^2 u(x_0, t_0) \tag{5.9}$$

which implies by continuity that there exists a point $x_* \in B_r(x_0)$ such that

$$u(x_*, t_0 - \theta r^q) = 2\mu^2 u(x_0, t_0). \tag{5.10}$$

Assuming we have enough room to use Theorem 4.1 around the point $(x_*, t_0 - \theta r^q)$, we have

$$u(x_*, t_0 - \theta r^q) \leq \mu \inf_{B_r(x_*)} u(\cdot, t_0 - \theta r^q + cu(x_*, t_0 - \theta r^q)^{2-q} r^q). \tag{5.11}$$

The required space here is $(x_0, t_0) + Q_{5r} \subset Q_1^-$ because we need to make sure that $B_{4r}(x_*) \subset B_1$. In time we do not need more room because

$$\begin{aligned} t_0 - \theta r^q - cu(x_*, t_0 - \theta r^q)^{2-q}(4r)^q &= t_0 - c\left[u(x_0, t_0)^{2-q} + 4^q\left(2\mu^2 u(x_0, t_0)\right)^{2-q}\right] r^q \\ &= t_0 - \left[1 + 4^q(2\mu^2)^{2-q}\right] cu(x_0, t_0)^{2-q} r^q \\ &= t_0 - \theta\left(\left[1 + 4^q(2\mu^2)^{2-q}\right]^{\frac{1}{q}} r\right)^q \\ &\geq t_0 - \theta(6r)^q \end{aligned}$$

where the last inequality holds assuming $q \geq 2$ and

$$1 + 4^q(2\mu^2)^{2-q} < 6^q$$

which is true for any $\mu \geq 1$ as $(2\mu^2)^{2-q} \leq 1$. We can estimate the time level by using Eq. 5.10 to get

$$\begin{aligned} t_0 - \theta r^q - cu(x_*, t_0 - \theta r^q)^{2-q} r^q &= t_0 - c\left(u(x_0, t_0)^{2-q} - u(x_*, t_0 - \theta r^q)^{2-q}\right) r^q \\ &= t_0 - c\left(u(x_0, t_0)^{2-q} - \left(2\mu^2 u(x_0, t_0)\right)^{2-q}\right) r^q \\ &= t_0 - \left(1 - (2\mu)^{2-q}\right)\theta r^q < t_0, \end{aligned}$$

where the last inequality follows from $q > 2$ and taking $\mu > 1$. Therefore because $x_0 \in B_r(x_*)$, combining Eqs. 5.11 and 5.8 we get a contradiction

$$2\mu^2 u(x_0, t_0) = u(x_*, t_0 - \theta r^q) \leq \mu u(x_0, \cdot, t_0 - \theta r^q + cu(x_*, t_0 - \theta r^q)^{2-q} r^q) < 2\mu^2 u(x_0, t_0).$$

Thus inequality Eq. 5.9 cannot hold and we have

$$\sup_{B_r(x_0)} u(\cdot, t_0 - \theta r^q) \le 2\mu^2 u(x_0, t_0).$$

Dividing both sides by $2\mu^2$ and combining this with Theorem 4.1 gives us

$$(2\mu^2)^{-1} \sup_{B_r(x_0)} u(\cdot, t_0 - \theta r^q) \le u(x_0, t_0) \le \mu \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q) \le 2\mu^2 \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q). \tag{5.12}$$

as desired. Because $\mathcal{T}$ has to be either empty or non-empty, combining Eqs. 5.7 and 5.12 gives us Harnack's inequality for constant $\gamma = 2\mu^2$.

Our space requirement $(x_0, t_0) + Q_{6\rho}(\theta) \subset Q_1^-(1)$ becomes $(x_0, t_0) + Q_{\frac{6}{\alpha}r}(\theta) \subset Q_1^-(1)$ for $\alpha = (2\mu)^{\frac{2-q}{q}} < 1$ so we have the result for $\sigma = \frac{6}{\alpha}$.

□

6 The Covering Argument

The intrinsic Harnack requires a lot of room around the target cylinder if μ from Theorem 4.1 happens to be large. The amount of needed room can be reduced by using a covering argument but details about this are hard to find in the literature. In the time-independent case, this can be done easily by covering by small balls but in our case, the intrinsic scaling in the time direction can cause problems with the sets. We apply the covering argument in two steps: We first prove, in Lemma 6.1 below, that we can reduce the needed room as much as we want in the time variable by relaxing our constant in the space variable. Then by a second covering argument, we prove that we can gain back what we lost in the space variable without relaxing the time direction.

We only consider the forward Harnack's inequality as the presented proof can be directly modified for the backward version. We point out, however, that since the argument iteratively applies Harnack's inequality, it yields different constants c and μ for the backward and forward versions.

We use right-angled paths connecting two points to deal with space and time variables separately. Given $(x, t), (y, s) \in \mathbb{R}^{n+1}$, we denote by $\gamma_{(x,t)}^{(y,s)} : [0, 1) \to \mathbb{R}^{n+1}$ a path from (x, t) to (y, s) such that

$$\gamma([0, 1)) = [(x, t), (y, t)] \cup [(y, t), (y, s)).$$

That is, $\gamma_{(x,t)}^{(y,s)}$ first moves from (x, t) to (y, t) in space, and then from (y, t) to (y, s) in time. For $a \in \mathbb{R}$, we denote by $\lceil a \rceil \in \mathbb{Z}$ the number a rounded up to the nearest integer.

Lemma 6.1 *Let $u \ge 0$ be a viscosity solution to Eq. 1.1 in $Q_1^-(1)$ and let the range condition Eq. 1.3 hold. Fix $(x_0, t_0) \in Q_1^-(1)$ such that $u(x_0, t_0) > 0$. Then for any $\sigma_t > 1$ there exist $\mu = \mu(n, p, q, \sigma_t)$, $\alpha = \alpha(n, p, q, \sigma_t)$, $c = c(n, p, q, \sigma_t)$ and $\sigma_x = \sigma_x(n, p, q, \sigma_t)$ such that*

$$u(x_0, t_0) \le \mu \inf_{B_{\alpha r}(x_0)} u(\cdot, t_0 + \theta r^q)$$

whenever

$$(x_0, t_0) + B_{\sigma_x r}(x_0) \times (t_0 - \theta(\sigma_t r)^q, t_0 + \theta(\sigma_t r)^q) \subset Q_1^-(1),$$

where $\theta := cu(x_0, t_0)^{2-q}$.

Proof Let $\tilde{c}$, $\tilde{\mu}$ and $\tilde{\sigma}$ be the constants given by Theorem 1.1. That is, we have

$$u(x,t) \le \tilde{\mu} \inf_{B_l(x)} u(\cdot, t + \tilde{c}u(x,t)^{2-q} l^q), \tag{6.1}$$

whenever

$$|x| + \tilde{\sigma} l < 1,$$
$$t \pm \tilde{c}u(x,t)^{2-q}(\tilde{\sigma} l)^q \subset [0,1).$$

We may assume that $\sigma_t < \tilde{\sigma}$ as otherwise the claim holds by Eq. 6.1. We denote

$$\kappa := \frac{\tilde{\sigma}^q - 1}{\sigma_t^q - 1}$$

and set

$$\alpha := \kappa^{-\frac{1}{q}},$$
$$\sigma_x := \alpha(\tilde{\sigma} \max(1, \mu^{\frac{2-q}{q}\lceil \kappa \rceil}) + 1),$$
$$c := \tilde{c}(\lceil \kappa \rceil + 1)\kappa^{-1},$$
$$\rho_i := \left(\mu^{(2-q)i}\kappa^{-1}\right)^{\frac{1}{q}} r = \mu^{\frac{(2-q)i}{q}} \alpha r.$$

Let $(\hat{x}, \hat{t})$ be the target point, that is, $\hat{x} \in B_{\alpha r}(x_0)$ and

$$\hat{t} := t_0 + cu(x_0, t_0)^{2-q} r^q.$$

It now suffices to prove that $u(x_0, t_0) \le \mu u(\hat{x}, \hat{t})$. We proceed by iteration.

Initial step: Set

$$t_1^* := t_0 + \tilde{c}u(x_0, t_0)^{2-q} \rho_0^q.$$

Then, since $|x - x_0| \le \alpha r = \rho_0$, we have by Harnack's inequality

$$u(x_0, t_0) \le \tilde{\mu} u(\hat{x}, t_1^*).$$

Now, let $(\hat{x}, t_1)$ be the first point along the path $\gamma_{(\hat{x}, t_1^*)}^{(\hat{x}, \hat{t})}$ such that

$$u(x_0, t_0) = \tilde{\mu} u(\hat{x}, t_1).$$

If no such point exists, then by continuity we must have

$$u(x_0, t_0) \le \tilde{\mu} u(\hat{x}, \hat{t})$$

and the claim already holds.

Iteration step: Let $i \in \{2, \ldots\}$ and suppose that we have already chosen $(\hat{x}, t_{i-1})$ such that

$$u(x_0, t_0) = \tilde{\mu}^{i-1} u(\hat{x}, t_{i-1}).$$

Set

$$t_i^* := t_{i-1} + \tilde{c}u(\hat{x}, t_{i-1})^{2-q} \rho_i^q.$$

If $u(\hat{x}, t_{i-1}) < \tilde{\mu} u(\hat{x}, t_i^*)$, then we move along the path $\gamma_{(\hat{x}, t_i^*)}^{(\hat{x}, \hat{t})}$ until we find a point $(\hat{x}, t_i)$ such that $u(\hat{x}, t_{i-1}) = \tilde{\mu} u(\hat{x}, t_i)$ so that

$$u(x_0, t_0) = \tilde{\mu}^{i-1} u(\hat{x}, t_{i-1}) = \tilde{\mu}^i u(\hat{x}, t_i).$$

If no such point exists, then by continuity we must have

$$u(x_0, t_0) = \tilde{\mu}^{i-1} u(\hat{x}, t_{i-1}) \le \tilde{\mu}^{i} u(\hat{x}, \hat{t}),$$

and the claim of the lemma follows.

If the iteration does not end prematurely, we continue until $t_i^* \ge \hat{t}$. When that happens, we apply Harnack's inequality one more time with a radius $\rho \le \rho_{i-1}$ so that we obtain an estimate at the exact time level $\hat{t}$. We define $i_{\hat{t}}$ as the smallest natural number such that $t^*_{i_{\hat{t}}+1} \ge \hat{t}$. We have

$$i_{\hat{t}} \le \lceil \kappa \rceil$$

as otherwise

$$\begin{aligned} t^*_{i_{\hat{t}}} \ge t^*_{\lceil \kappa \rceil + 1} &= t_{\lceil \kappa \rceil} + \tilde{c} u(\hat{x}, t_{\lceil \kappa \rceil})^{2-q} \rho^q_{\lceil \kappa \rceil} \\ &\ge t^*_{\lceil \kappa \rceil} + \tilde{c} \tilde{\mu}^{(q-2)\lceil \kappa \rceil} u(x_0, t_0)^{2-q} \rho^q_{\lceil \kappa \rceil} \\ &\ge t_0 + \sum_{i=0}^{\lceil \kappa \rceil} \tilde{c} \tilde{\mu}^{(q-2)i} u(x_0, t_0)^{2-q} \rho_i^q \\ &= t_0 + \tilde{c} u(x_0, t_0)^{2-q} \sum_{i=0}^{\lceil \kappa \rceil} \kappa^{-1} r^q \\ &= t_0 + \tilde{c} u(x_0, t_0)^{2-q} (\lceil \kappa \rceil + 1) \kappa^{-1} \\ &= \hat{t}, \end{aligned}$$

which would be against the definition of $i_{\hat{t}}$. Consequently, the procedure yields the estimate

$$u(x_0, t_0) \le \tilde{\mu}^{\lceil \kappa \rceil + 1} u(\hat{x}, \hat{t}).$$

We still need to verify that there is enough room to apply Harnack's inequality throughout the iteration. To this end, notice that the biggest jump in the time direction that we can do is

$$J = \tilde{c} u(x_0, t_0)^{2-q} \kappa^{-1} r^q.$$

Therefore we always have room in time direction, since in the worst case the jump starts from $\hat{t} - J$, and then we have (using that $\tilde{c} \le c$)

$$\begin{aligned} \hat{t} - J + J\tilde{\sigma}^q &= t_0 + c u(x_0, t_0)^{2-q} r^q + (\tilde{\sigma}^q - 1) \tilde{c} u(x_0, t_0)^{2-q} \kappa^{-1} r^q \\ &\le t_0 + c u(x_0, t_0)^{2-q} r^q + \tilde{c} (\sigma_t^q - 1) u(x_0, t_0)^{2-q} r^q \\ &\le t_0 + c u(x_0, t_0)^{2-q} (\sigma_t r)^q < 1. \end{aligned}$$

We also have enough room in space direction since

$$\begin{aligned} \rho_i \tilde{\sigma} + |\hat{x} - x_0| &\le \tilde{\sigma} \max(\rho_0, \rho_{\lceil \kappa \rceil}) + \alpha r \\ &= \alpha (\tilde{\sigma} \max(1, \mu^{\frac{2-q}{q} \lceil \kappa \rceil}) + 1) r \\ &= \sigma_x r < 1. \end{aligned}$$

□

We are now ready to prove the general form of Harnack's inequality. We remark that as the space required around the intrinsic cylinder $(x_0, t_0) + Q_{\sigma r}(\theta) \subset Q_1$ tends to zero (i.e. when $\sigma \to 1$), the waiting time coefficient c blows up if $q > 2$, and tends to zero if $q < 2$.

Theorem 6.2 *Let $u \geq 0$ be a viscosity solution to Eq. 1.1 in $Q_1^-(1)$ and let the range condition (1.3) hold. Fix $(x_0, t_0) \in Q_1^-(1)$ such that $u(x_0, t_0) > 0$. Then for any $\sigma > 1$ there exist $\mu = \mu(n, p, q, \sigma)$ and $c = c(n, p, q, \sigma)$ such that*

$$u(x_0, t_0) \leq \mu \inf_{B_r(x_0)} u(\cdot, t_0 + cu(x_0, t_0)^{2-q} r^q),$$

whenever

$$(x_0, t_0) + B_{\sigma r}(x_0) \times (t_0 - \theta(\sigma r)^q, t_0 + \theta(\sigma r)^q) \subset Q_1^-(1),$$

where $\theta := cu(x_0, t_0)^{2-q}$.

Proof Let $\tilde{c}$, σ_x, α and $\tilde{\mu}$ be the constants that we get from Lemma 6.1 for $\sigma_t := \sigma$. Then we have

$$u(x, t) \leq \tilde{\mu} \inf_{B_{\alpha l}(x)} u(\cdot, t + \tilde{c}u(x, t)^{2-q} l^q) \tag{6.2}$$

whenever

$$(x, t) + B_{\sigma_x l}(x) \times (t - \tilde{c}u(x, t)^{2-q}(\sigma l)^q, t + \tilde{c}u(x, t)^{2-q}(\sigma l)^q) \subset Q_1^-(1),$$

i.e.

$$\begin{aligned} |x| + \sigma_x l &< 1, \\ t \pm \tilde{c}u(x, t)^{2-q}(\sigma l)^q &\subset [0, 1). \end{aligned}$$

We denote

$$\varrho := \frac{\sigma - 1}{\sigma_x}$$

and

$$c := \tilde{c}\varrho^q \sum_{k=1}^{\lceil (\alpha\varrho)^{-1} \rceil + 1} \tilde{\mu}^{(q-2)(k-1)}.$$

Let $(\hat{x}, \hat{t})$ be the target point, that is, $\hat{x} \in B_r(x_0)$ and

$$\hat{t} := t_0 + cu(x_0, t_0)^{2-q} r^q.$$

We now proceed by iteration (see Fig. 2 below).

Initial step: Let $\rho := \varrho r$ and set

$$t_1^* := t_0 + \tilde{c}u(x_0, t_0)^{2-q} \rho^q.$$

By the Harnack's inequality in Eq. 6.2 we have

$$u(x_0, t_0) \leq \tilde{\mu} u(x_1^*, t_1^*),$$

where x_1^* is the point in $\overline{B}_{\alpha\rho}(x_0)$ that is closest to $\hat{x}$. Now, if $u(x_0, t_0) < \tilde{\mu} u(x_1^*, t_1^*)$, then we move along the path $\gamma_{(x_1^*, t_1^*)}^{(\hat{x}, \hat{t})}$ until we find a point (x_1, t_1) such that

$$u(x_0, t_0) = \tilde{\mu} u(x_1, t_1).$$

If no such point exists, then by continuity we must have

$$u(x_0, t_0) \leq \tilde{\mu} u(\hat{x}, \hat{t}),$$

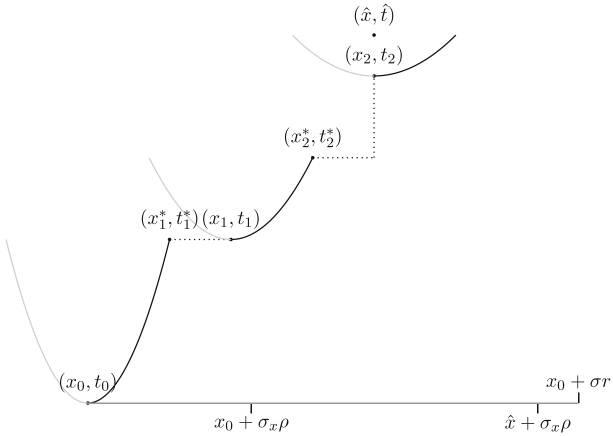

Fig. 2 Illustration of the Harnack chain in the proof of Theorem 6.2 when $q < 2$. If $q > 2$, the paraboloids get steeper instead

and we end the iteration.

Iteration step: Let $i \in \{2, \ldots\}$ and suppose that we have already chosen (x_{i-1}, t_{i-1}) such that

$$u(x_0, t_0) = \tilde{\mu}^{i-1} u(x_{i-1}, t_{i-1}).$$

Set

$$t_i^* := t_{i-1} + \tilde{c}u(x_{i-1}, t_{i-1})^{2-q} \rho^q.$$

By Harnack's inequality in Eq. 6.2 we have

$$u(x_{i-1}, t_{i-1}) \leq \tilde{\mu} u(x_i^*, t_i^*),$$

where x_i^* is the point in $\overline{B}_\rho(x_{i-1})$ that is closest to $\hat{x}$. Now, if $u(x_{i-1}, t_{i-1}) < \tilde{\mu} u(x_i^*, t_i^*)$, then we move along the path $\gamma_{(x_i^*, t_i^*)}^{(\hat{x}, \hat{t})}$ until we find a point (x_i, t_i) such that $u(x_{i-1}, t_{i-1}) = \tilde{\mu} u(x_i, t_i)$ so that

$$u(x_0, t_0) = \tilde{\mu}^{i-1} u(x_{i-1}, t_{i-1}) = \tilde{\mu}^i u(x_i, t_i).$$

If no such point exists, then by continuity we must have

$$u(x_0, t_0) = \tilde{\mu}^{i-1} u(x_{i-1}, t_{i-1}) \leq \tilde{\mu}^i u(\hat{x}, \hat{t}),$$

and we end the iteration.

If the iteration does not end prematurely, we continue until $t_i^* \geq \hat{t}$. When that happens, we apply the Harnack's inequality (6.2) one more time with a radius smaller or equal to ρ so that we hit $\hat{t}$. We define $i_{\hat{t}}$ as the smallest natural number such that $t_{i_{\hat{t}}+1}^* \geq \hat{t}$. That is, $t_{i_{\hat{t}}}$ is the time from which it remains to apply the Harnack's inequality (6.2) one more time to reach the target time $\hat{t}$. Next, we show that our selections of the constants ensure the finiteness of $i_{\hat{t}}$ and that $x_{i_{\hat{t}}} = \hat{x}$. For finiteness, we observe that

$$i_{\hat{t}} \leq \lceil (\alpha \varrho)^{-1} \rceil.$$

Indeed, otherwise

$$
\begin{aligned}
t^*_{i_{\hat{t}}} \geq t^*_{\lceil(\varrho\alpha)^{-1}\rceil+1} &\geq t_0 + \sum_{k=1}^{\lceil(\varrho\alpha)^{-1}\rceil+1} \tilde{c}u(x_{k-1}, t_{k-1})^{2-q}\rho^q \\
&= t_0 + \sum_{k=1}^{\lceil(\varrho\alpha)^{-1}\rceil+1} \tilde{c}\frac{1}{\tilde{\mu}^{(k-1)(2-q)}}u(x_0, t_0)^{2-q}\rho^q \\
&= t_0 + \tilde{c}u(x_0, t_0)^{2-q}\rho^q \sum_{k=1}^{\lceil(\varrho\alpha)^{-1}\rceil+1} \tilde{\mu}^{(q-2)(k-1)} \\
&= t_0 + cu(x_0, t_0)^{2-q}r^q \\
&= \hat{t},
\end{aligned}
$$

which would be against the definition of $i_{\hat{t}}$. Next we estimate the smallest $i_{\hat{x}} \in \{1, \ldots\}$ such that $x_{i_{\hat{x}}} = \hat{x}$ (observe that the construction ensures that $x_i = \hat{x}$ for all $i \geq i_{\hat{x}}$). At each iteration step, unless we have already reached $\hat{x}$, we move at least $\alpha\rho$ closer towards $\hat{x}$. Since $\left|x_0 - \hat{x}\right| \leq r$, we thus have

$$
i_{\hat{x}} \leq \left\lceil \frac{r}{\alpha\rho} \right\rceil = \left\lceil \frac{r}{\alpha\varrho r} \right\rceil = \lceil(\alpha\varrho)^{-1}\rceil .
$$

We want to show that $i_{\hat{x}} \leq i_{\hat{t}}$, as this implies that $x_{i_{\hat{t}}} = \hat{x}$. For this end, we may assume that the Harnack chain does not skip in time direction using the paths $\gamma^{(\hat{x},\hat{t})}_{(x_i,t_i)}$, as otherwise the chain automatically reaches $\hat{x}$. Using this we conclude that $i_{\hat{x}} \leq i_{\hat{t}}$ must hold since otherwise

$$
\begin{aligned}
t^*_{i_{\hat{t}}+1} \leq t^*_{i_{\hat{x}}} &\leq t^*_{\lceil(\alpha\varrho)^{-1}\rceil} \\
&= t_0 + \tilde{c}u(x_0, t_0)^{2-q}\rho^q \sum_{k=1}^{\lceil(\rho\alpha)^{-1}\rceil} \tilde{\mu}^{(q-2)(k-1)} \\
&< t_0 + \tilde{c}u(x_0, t_0)^{2-q}\rho^q \sum_{k=1}^{\lceil(\rho\alpha)^{-1}\rceil+1} \tilde{\mu}^{(q-2)(k-1)} \\
&= \hat{t},
\end{aligned}
$$

which is against the definition of $i_{\hat{t}}$. Thus the procedure reaches $\hat{x}$ before we apply Harnack's inequality one last time. This yields the estimate

$$
u(x_0, t_0) \leq \tilde{\mu}^{\lceil(\alpha\varrho)^{-1}\rceil+1}u(\hat{x}, \hat{t}).
$$

We still need to check that we have room to use Harnack's inequality. The room in space is clear from the definition of ρ since

$$
\left|\hat{x} - x_0\right| + \sigma_x\rho = r + \sigma_x\frac{\sigma - 1}{\sigma_x}r \leq r + (\sigma - 1)r = \sigma r < 1.
$$

For the room in time, observe that we always end the Harnack at most the time level $\hat{t}$. Therefore, the worst-case scenario would be if our biggest possible jump in Harnack's inequality ended up at $\hat{t}$. Since the sequence $u(x_i, t_i)$ is decreasing, the biggest possible jump is

$$J := \begin{cases} \tilde{c}u(x_0, t_0)^{2-q}\rho^q, & \text{if } q < 2, \\ \tilde{c}u(x_0, t_{\lceil(\alpha\varrho)^{-1}\rceil})^{2-q}\rho^q, & \text{if } q > 2. \end{cases}$$

To land on $\hat{t}$, the jump would have to start from $\hat{t} - J$. Thus it suffices to ensure that

$$(\hat{t} - J) + J\sigma^q = \hat{t} + (\sigma^q - 1)J < 1.$$

This holds, since if $q < 2$, we have

$$\begin{aligned} \hat{t} + (\sigma^q - 1)J &= \hat{t} + (\sigma^q - 1)\tilde{c}u(x_0, t_0)^{2-q}\rho^q \\ &= t_0 + cu(x_0, t_0)^{2-q}r^q + (\sigma^q - 1)cu(x_0, t_0)^{2-q}(\frac{\tilde{c}\varrho^q}{c}) \\ &\leq t_0 + cu(x_0, t_0)^{2-q}(\sigma\rho)^q < 1, \end{aligned}$$

and if $q > 2$, we have

$$\begin{aligned} \hat{t} + (\sigma^q - 1)J &= \hat{t} + \tilde{c}u(x_0, t_k)^{2-q}\rho^q(\sigma^q - 1) \\ &= t_0 + cu(x_0, t_0)^{2-q}r^q + cu(x_0, t_0)^{2-q}r^q(\sigma^q - 1)\left(\frac{\tilde{c}\varrho^q\mu^{(q-2)(\lceil(\alpha\varrho)^{-1}\rceil)}}{c}\right) \\ &\leq t_0 + cu(x_0, t_0)^{2-q}(\sigma r)^q < 1. \end{aligned}$$

□

7 Optimality of the Range of Exponents

Intrinsic Harnack's inequality may fail outside of the range condition (1.3) as for such exponents, viscosity solutions of Eq. 1.1 vanish in finite time as we will prove in this section. The solutions of the standard p-parabolic equation in the corresponding subcritical exponent range behave in a similar way. Idea, behind the proof is to use the equivalence result proven by Parviainen and Vázquez [26] to transfer the problem onto a one-dimensional divergence form equation and then to prove that a solution to this equation vanishes. We use the weak formulation for a time-mollified solution with a suitable test function after first proving that this formulation holds for all weak solutions as the separate lemma. Next, we simplify both sides of the formulation, estimate using Sobolev's inequality and ultimately get a vanishing upper bound for the norm of the solution. We do this first in bounded domains and then prove the global result using convergence and stability results. This global result Proposition 7.5 gives us a counterexample to the intrinsic Harnack's inequality 1.1 and thus proves that range (1.3) is optimal.

As proven by Parviainen and Vázquez, radial viscosity solutions to Eq. 1.1 are equivalent to weak solutions of the one-dimensional equation

$$\partial_t u - \frac{p-1}{q-1}\Delta_{q,d}u = 0 \quad \text{in } (-R, R) \times (0, T). \tag{7.1}$$

Here, denoting by u' the radial derivative of u,

$$\Delta_{q,d}u := |u'|^{q-2}\left((q-1)u'' + \frac{d-1}{r}u'\right)$$

is heuristically the usual radial q-Laplacian in a fictitious dimension

$$d := \frac{(n-1)(q-1)}{p-1} + 1.$$

If d happens to be an integer, then solutions to Eq. 7.1 are equivalent to radial weak solutions of the q-parabolic equation in $B_R \times (0, T) \subset \mathbb{R}^{d+1}$ by [26, Section 3]. If $d \notin \mathbb{N}$, we still have an equivalence between radial viscosity solutions of Eq. 1.1 and continuous weak solutions of Eq. 7.1 as proven in [26, Theorem 4.2].

A weak solution of Eq. 7.1 is in a weighted Sobolev space but we are only interested in continuous solutions and thus will assume this in the following definition. The description of the exact definition in the elliptic case is in [27, Definition 2.2]. The following definition is written in a slightly different form but is equivalent to the definition given by Parviainen and Vazquez [26, Definition 4.1]. We use the notation $dz := r^{d-1}\, dr\, dt$ for the natural parabolic measure for this problem and denote the distributional derivative of v by v' and define it by

$$\int_0^R v'\varphi\, dr = -\int_0^R v\varphi'\, dr$$

for all $\varphi \in C_0^\infty((0, R))$ so it coincides with standard derivative for differentiable functions.

Definition 7.1 Let $0 < T \le \infty$ and $0 < R \le \infty$. A function $u \in C((-R, R) \times (0, T))$ such that $u' \in C((-R, R) \times (0, T))$ and $u'(0, t) = 0$ is a continuous weak solution to Eq. 7.1 if we have

$$\int_{t_1}^{t_2}\int_0^R u\partial_t\phi - \frac{p-1}{q-1}\left(\left|u'\right|^{q-2} u'\right)\phi'\, dz = 0$$

for all $0 < t_1 < t_2 < T$ and $\phi \in C_0^\infty((-R, R) \times (0, T))$.

We define time-mollification and prove a basic result for it in Lemma 7.2 below for the convenience of the reader. Let $\varepsilon > 0$ and $\eta_\varepsilon : \mathbb{R} \to [0, \infty)$ be the standard mollifier such that $\operatorname{supp} \eta_\varepsilon \subset (-\varepsilon, \varepsilon)$. The time-mollification of $u \in L^1((0, R) \times (0, T))$ is defined by

$$u_\varepsilon(r, t) := \eta_\varepsilon * u(r, t) = \int_0^T \eta_\varepsilon(t-s)u(r, s)\, ds. \tag{7.2}$$

Lemma 7.2 *Let u be a continuous weak solution to Eq. 7.1 and let u_ε denote the mollification (7.2). Then for all $0 < t_1 < t_2 < T$ and $\phi \in C^\infty((0, R) \times (0, T))$ such that* $\operatorname{supp}\phi(\cdot, t) \Subset (-R, R)$ *for any $t \in (t_1, t_2)$, we have*

$$\int_{t_1}^{t_2}\int_0^R u_\varepsilon\partial_t\phi - \frac{p-1}{q-1}\left(\left|u'\right|^{q-2} u'\right)_\varepsilon \phi'\, dz = \int_0^R \left(u_\varepsilon(r, t_2)\phi(r, t_2) - u_\varepsilon(r, t_1)\phi(r, t_1)\right) r^{d-1}\, dr$$

Proof Let first $\varepsilon > 0$ and $\varphi \in C_0^\infty((-R, R) \times (0, T))$ be such that $\operatorname{supp}\varphi \Subset (-R, R) \times (\varepsilon, T-\varepsilon)$. Because η_ε is even, we have by partial integration (the boundary terms vanish since $\varphi(0, \cdot) \equiv \varphi(T, \cdot) \equiv 0$)

$$\begin{aligned}\partial_t\varphi_\varepsilon(t, r) = \partial_t\int_0^T \eta_\varepsilon(t-s)\varphi(r, s)\, ds &= \int_0^T -\partial_t\eta_\varepsilon(t-s)\varphi(r, s)\, ds \\ &= \int_0^T \eta_\varepsilon(t-s)\partial_s\varphi(r, s)\, ds.\end{aligned}$$

Thus by Fubini's theorem

$$\int_0^T\int_0^R u(r,t)\partial_t\varphi_\varepsilon(r,t)r^{d-1}\,dr\,dt = \int_0^T\int_0^R\int_0^T u(r,t)\eta_\varepsilon(t-s)\partial_s\varphi(r,s)r^{d-1}\,ds\,dr\,dt$$
$$= \int_0^T\int_0^R\int_0^T u(r,t)\eta_\varepsilon(t-s)\,dt\,\partial_s\varphi(r,s)r^{d-1}\,dr\,ds$$
$$= \int_0^T\int_0^R u_\varepsilon(r,s)\partial_s\varphi(r,s)r^{d-1}\,dr\,ds.$$

Similarly, for the space derivative we have

$$\int_0^T\int_0^R \left|u'(r,t)\right|^{q-2}u'(r,t)\varphi_\varepsilon'(r,t)r^{d-1}\,dr\,dt$$
$$= \int_0^T\int_0^R\int_0^T \eta_\varepsilon(t-s)\partial_r\varphi(r,s)\left|u'(r,t)\right|^{q-2}u'(r,t)\,ds r^{d-1}\,dr\,dt$$
$$= \int_0^T\int_0^R\int_0^T \eta_\varepsilon(t-s)\left|u'(r,t)\right|^{q-2}u'(r,t)\,dt\partial_r\varphi(r,s)r^{d-1}\,dr\,ds$$
$$= \int_0^T\int_0^R (\left|u'\right|^{q-2}u')_\varepsilon(r,s)\varphi'(r,s)r^{d-1}\,dr\,ds.$$

By the last two displays, we obtain

$$\int_0^T\int_0^R u_\varepsilon\partial_t\varphi - \frac{p-1}{q-1}(\left|u'\right|^{q-2}u')_\varepsilon\varphi'\,dz = 0. \tag{7.3}$$

Let now $0 < t_1 < t_2 < T$ and $\phi \in C^\infty((-R,R)\times(0,T))$ be such that $\phi(\cdot,t) \Subset (-R,R)$ for any $t \in (t_1,t_2)$. Define the cut-off function

$$\xi_h(t) := \begin{cases} 0, & t \in (0, t_1-h) \\ \frac{1}{h}(t-t_1), & t \in [t_1-h, t_1), \\ 1, & t \in [t_1,t_2), \\ 1-\frac{1}{h}(t-t_2), & t \in [t_2, t_2+h), \\ 0, & t \in [t_2+h, T). \end{cases}$$

Since $\varphi_h := \xi_h\phi$ is Lipschitz, it satisfies (7.3) by the first part of the proof and a simple approximation argument. Since by continuity all $t_1, t_2 \in (0,T)$ satisfy

$$\int_0^T\int_0^R u_\varepsilon\partial_t\varphi_h\,dz \to \int_{t_1}^{t_2}\int_0^R u_\varepsilon\partial_t\phi\,dz + \int_0^R (u_\varepsilon(r,t_2)\phi(r,t_2) - u_\varepsilon(r,t_1)\phi(r,t_1))r^{d-1}\,dr$$

as $h \to 0$, the claim of the lemma follows. □

Our proof of finite extinction uses the following Sobolev's inequality, which is heuristically speaking the Gagliardo-Nirenberg inequality for radial functions in the fictitious dimension d. The standard formulation of the Gagliardo-Nirenberg inequality requires $q < n$ and hence does not work for our one-dimensional case.

Theorem 7.3 (Radial Sobolev's inequality) *Suppose that* $1 \le q < d$. *Let* $v \in C^\infty(0,\infty) \cap C[0,\infty)$ *be such that* $v(r) \equiv 0$ *for all large* $r > 0$. *Then there exists* $C = C(d,q)$ *such that*

$$\left(\int_0^\infty |v(r)|^{\frac{dq}{d-q}} r^{d-1}\,dr\right)^{\frac{d-q}{dq}} \le C\left(\int_0^\infty \left|v'(r)\right|^q r^{d-1}\,dr\right)^{\frac{1}{q}}.$$

Proof Suppose first that $q = 1$. We denote

$$g(r) := |v(r)|^{\frac{d}{d-1}} = \left| \int_r^\infty v'(s)\, ds \right|^{\frac{d}{d-1}}.$$

Since $d/(d-1) > 1$, we have $g \in C^1(\mathbb{R})$ and

$$\begin{aligned} g'(r) =& \frac{d}{d-1} |v(r)|^{\frac{1}{d-1}} v'(r) \operatorname{sgn}(v(r)) \\ =& \frac{d}{d-1} \left| \int_r^\infty v'(s)\, ds \right|^{\frac{1}{d-1}} v'(r) \operatorname{sgn}(v(r)). \end{aligned}$$

Integrating by parts and using that $g(r) = 0$ for large r, we obtain

$$\begin{aligned} \int_0^\infty |v(r)|^{\frac{d}{d-1}} r^{d-1}\, dr &= \lim_{k\to\infty} \int_0^k g(r) r^{d-1}\, dr \\ &= \lim_{k\to\infty} \left(g(r) \frac{r^d}{d} \Big|_{r=0}^{r=k} - \int_0^k g'(r) \frac{r^d}{d} \right) \\ &= \frac{1}{d} \int_0^\infty g'(r) r^d\, dr \\ &= \frac{1}{d-1} \int_0^\infty \left| \int_r^\infty v'(s)\, ds \right|^{\frac{1}{d-1}} v'(r) \operatorname{sgn}(v(r)) r^d\, dr \\ &\le \frac{1}{d-1} \int_0^\infty \left(\int_r^\infty \left| v'(s) \right|\, ds \right)^{\frac{1}{d-1}} r \cdot \left| v'(r) \right| r^{d-1}\, dr. \end{aligned}$$

This we can further estimate as

$$\begin{aligned} &\frac{1}{d-1} \int_0^\infty \left(\int_r^\infty \left| v'(s) \right| s^{d-1} \underbrace{\frac{r^{d-1}}{s^{d-1}}}_{\le 1} ds \right)^{\frac{1}{d-1}} \left| v'(r) \right| r^{d-1}\, dr \\ &\le \frac{1}{d-1} \int_0^\infty \left(\int_0^\infty \left| v'(s) \right| s^{d-1}\, ds \right)^{\frac{1}{d-1}} \left| v'(r) \right| r^{d-1}\, dr \\ &= \frac{1}{d-1} \left(\int_0^\infty \left| v'(s) \right| s^{d-1}\, ds \right)^{\frac{1}{d-1}} \int_0^\infty \left| v'(r) \right| r^{d-1}\, dr \\ &= \frac{1}{d-1} \left(\int_0^\infty \left| v'(r) \right| r^{d-1}\, dr \right)^{\frac{d}{d-1}} \end{aligned}$$

so that

$$\left(\int_0^\infty |v(r)|^{\frac{d}{d-1}} r^{d-1}\, dr \right)^{\frac{d-1}{d}} \le C \int_0^\infty \left| v'(r) \right| r^{d-1}\, dr. \tag{7.4}$$

Suppose then that $1 < q < d$. Using Eq. 7.4 with $v := u^{\frac{dq-q}{d-q}}$, we obtain

$$\left(\int_0^\infty |u(r)|^{\frac{dq}{d-q}} r^{d-1}\, dr \right)^{\frac{d-1}{d}} \le C \int_0^\infty \left| u'(r) \right| |u(r)|^{\frac{d(q-1)}{d-q}} r^{d-1}\, dr$$

$$\leq C\left(\int_0^\infty \left|u'(r)\right|^q r^{d-1}\,dr\right)^{\frac{1}{q}}\left(\int_0^\infty |u(r)|^{\frac{dq}{d-q}}\,r^{d-1}\,dr\right)^{\frac{q-1}{q}},$$

which implies the desired inequality. □

Now we have the needed tools to state and prove the finite extinction of solutions. We do this by first proving the result for solutions of a Dirichlet problem in simple cylinders and then expanding this result to the entire space by convergence results. The existence of global solutions with extinction in finite time is a counterexample for the intrinsic Harnack's inequality as we show at the end of this section. The proof uses the following notation for the weighted Lebesgue norm

$$||v||_{L^q(r^{d-1},(0,R))} := \left(\int_0^R |v|^q\,r^{d-1}\,dr\right)^{\frac{1}{q}}.$$

We only consider radially symmetric initial data in what follows. The finite extinction holds in the general situation by comparison principle.

Proposition 7.4 *Assume q does not satisfy the range condition (1.3) and let $R > 0$. Let u be a viscosity solution of*

$$\begin{cases} \partial_t u = |\nabla u|^{q-p}\,\mathrm{div}\left(|\nabla u|^{p-2}\,\nabla u\right) & \text{in } B_R \times (0,T),\\ u(\cdot,0) = u_0(\cdot) \geq 0 & \text{where } u_0 \in L^\infty(B_R)\cap C(B_R) \text{ is radial},\\ u(\cdot,t) = 0 & \text{on } \partial B_R \text{ for any } t \in (0,T). \end{cases} \tag{7.5}$$

There exists a finite time $T^* := T^*(n,p,q,u_0)$, such that

$$u(\cdot,0) \equiv 0 \quad \text{for all } t \geq T^*$$

and

$$0 < T^* \leq C\,||u_0||^{2-q}_{L^s(r^{d-1},(0,R))}$$

where $C := C(n,p,q)$ and $s = \frac{d(2-q)}{q}$.

Proof The existence of a solution $u \in C(\overline{B}_R \times [0,T])$ to the Cauchy-Dirichlet problem (7.5) can be proven for example by modified Perron's method (see [26, Theorem 2.6]) and the comparison principle ensures that it is radial. Therefore, by the equivalence result [26, Theorem 4.2], u is a continuous weak solution to

$$\begin{cases} \partial_t u - \frac{p-1}{q-1}\left|u'\right|^{q-2}\left((q-1)u'' + \frac{d-1}{r}u'\right) = 0 & \text{in } (-R,R)\times(0,T),\\ u(\cdot,0) = u_0(\cdot) \geq 0 & \text{where } u_0 \in L^\infty((-R,R)),\\ u(-R,t) = u(R,t) = 0 & \text{for any } t \in (0,T). \end{cases}$$

Let

$$s = \frac{d(2-q)}{q}$$

and notice that $s > 1$ because we assumed $q < \frac{2d}{d+1}$. We define the test function $\varphi := u_{\varepsilon,h}^{s-1} - h^{s-1}$, where $u_{\varepsilon,h} := u_\varepsilon + h$ for $\varepsilon, h > 0$ and u_ε denotes the time-mollification. We add this h to ensure that our function remains strictly positive as we have negative exponents during the calculation. Then φ is an admissible test function and by Lemma 7.2 we have

$$\int_{t_1}^{t_2}\int_0^R u_\varepsilon \partial_t \varphi \, dz - \int_0^R (u_\varepsilon \varphi(r,t_2) - u_\varepsilon \varphi(r,t_1)) r^{d-1}\, dr = \frac{p-1}{q-1}\int_{t_1}^{t_2}\int_0^R (|u'|^{q-2}u')_\varepsilon \partial_r (u_{\varepsilon,h}^{s-1})\, dz$$
$$=: A_{\varepsilon,h} \tag{7.6}$$

for all $0 < t_1 < t_2 < T$. We rewrite the first term on the left-hand side using integration by parts and Fubini's theorem

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^R u_\varepsilon \partial_t \varphi \, dz &= \int_{t_1}^{t_2}\int_0^R u_\varepsilon \partial_t u_{\varepsilon,h}^{s-1}\, dz\\ &= -\int_{t_1}^{t_2}\int_0^R u_{\varepsilon,h}^{s-1}\partial_t u_\varepsilon \, dz + \int_0^R (u_\varepsilon u_{\varepsilon,h}^{s-1}(r,t_2) - u_\varepsilon u_{\varepsilon,h}^{s-1}(r,t_1)) r^{d-1}\, dr\\ &= -\int_{t_1}^{t_2}\int_0^R \frac{1}{s}\partial_t u_{\varepsilon,h}^{s}\, dz + \int_0^R (u_\varepsilon u_{\varepsilon,h}^{s-1}(r,t_2) - u_\varepsilon u_{\varepsilon,h}^{s-1}(r,t_1)) r^{d-1}\, dr\\ &= -\frac{1}{s}\int_{t_1}^{t_2}\partial_t \left\| u_{\varepsilon,h}\right\|_{L^s(r^{d-1},(0,R))}^s dt + \int_0^R (u_\varepsilon u_{\varepsilon,h}^{s-1}(r,t_2) - u_\varepsilon u_{\varepsilon,h}^{s-1}(r,t_1)) r^{d-1}\, dr.\end{aligned}$$

Hence, since $u_{\varepsilon,h}^{s-1} - \varphi = h^{s-1}$, the Eq. 7.6 becomes

$$\frac{1}{s}\int_0^R (u_{\varepsilon,h}^s(r,t_1) - u_{\varepsilon,h}^s(r,t_2)) r^{d-1}\, dr - h^{s-1}\int_0^R (u_\varepsilon(r,t_2) - u_\varepsilon(r,t_1)) r^{d-1}\, dr = A_{\varepsilon,h}.$$

Since we eliminated the time derivative, we may let $\varepsilon \to 0$ to obtain

$$\frac{1}{s}\int_0 \left(u_h^s(r,t_1) - u_h^s(r,t_2)\right) r^{d-1}\, dr - h^{s-1}\int_0^R (u(r,t_2) - u(r,t_1)) r^{d-1}\, dr = A_h. \tag{7.7}$$

Next, we rewrite A_h as follows

$$\begin{aligned}A_h = \frac{p-1}{q-1}\int_{t_1}^{t_2}\int_0^R |u'|^{q-2}u'\partial_r(u_h^{s-1})\, dz &= \frac{p-1}{q-1}\int_{t_1}^{t_2}\int_0^R |u_h'|^{q-2}u_h'\partial_r(u_h^{s-1})\, dz\\ &= \frac{(p-1)(s-1)}{q-1}\int_{t_1}^{t_2}\int_0^R |u_h'|^q u_h^{s-2}\, dz,\end{aligned}$$

where by Sobolev's inequality in Theorem 7.3 forcing vanishing boundary values

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^R |u_h'|^q u_h^{s-2}\, dz &= \int_{t_1}^{t_2}\int_0^R \left| u' u_h^{\frac{s-2}{q}}\right|^q dz\\ &= \left(\frac{q}{s+q-2}\right)^q \int_{t_1}^{t_2}\int_0^R \left|\partial_r(u_h^{\frac{s+q-2}{q}}(r,t) - u_h^{\frac{s+q-2}{q}}(R,t))\right|^q dz\\ &\geq C_1\left(\frac{q}{s+q-2}\right)^q \int_{t_1}^{t_2}\left(\int_0^R \left| u_h^{\frac{s+q-2}{q}}(r,t) - u_h^{\frac{s+q-2}{q}}(R,t)\right|^{\frac{dq}{d-q}} dz\right)^{\frac{d-q}{d}}.\end{aligned}$$

Here C_1 is the constant in Sobolev's inequality. Since $s = \frac{d(2-q)}{q}$, we have by the last two displays

$$\liminf_{h\to 0} A_h \geq C_1 \frac{(p-1)(s-1)}{q-1} \left(\frac{q}{s+q-2}\right)^q \int_{t_1}^{t_2} \left(\int_0^R \left|u^{\frac{s+q-2}{q}}\right|^{\frac{dq}{d-q}} dz\right)^{\frac{d-q}{d}}$$

$$=: C_2 \int_{t_1}^{t_2} \left(\int_0^R |u|^s \, dz\right)^{\frac{d-q}{d}}.$$

Consequently, letting $h \to 0$ in Eq. 7.7, we obtain

$$\frac{1}{s}\int_0^R (u^s(r,t_1) - u^s(r,t_2)) r^{d-1}\, dr \geq C_2 \int_{t_1}^{t_2} \left(\int_0^R |u|^s \, dz\right)^{\frac{d-q}{d}}.$$

Denoting $v(t) := \|u(\cdot,t)\|_{L^s(r^{d-1},(0,R))}$ and multiplying the inequality by $-s$, we have

$$v^s(t_2) - v^s(t_1) \leq -C_2 s \int_{t_1}^{t_2} v(t)^{s\frac{d-q}{d}}\, dt. \tag{7.8}$$

Observe that this implies in particular that v is decreasing. Next, we derive a distributional inequality which implies that v must in fact vanish for large times. For this end, let $\kappa := q-2+s$ and observe that for any $0 < a < b$ we have

$$\begin{aligned} a^{2-q} - b^{2-q} = -\frac{1}{b^\kappa}\int_a^b (2-q) b^\kappa t^{1-q}\, dt &\leq -\frac{1}{b^\kappa}\int_a^b (2-q) t^\kappa t^{1-q}\, dt \\ &= -\frac{1}{b^\kappa}\int_a^b (2-q) t^{s-1}\, dt \\ &= \frac{2-q}{s}\frac{1}{b^\kappa}(a^s - b^s). \end{aligned}$$

Let then $\varphi \in C_0^\infty(0,T)$ be non-negative. Next, we apply the integration by parts formula for difference quotients and the fact that v is decreasing together with the above elementary inequality. This way, we obtain by dominated convergence theorem

$$\begin{aligned} -\int_0^T v^{2-q}(t)\varphi'(t)\, dt &= -\lim_{\delta\to 0}\int_0^T v^{2-q}(t)\frac{\varphi(t-\delta)-\varphi(t)}{-\delta}\, dt \\ &= \lim_{\delta\to 0}\int_0^T \varphi(t)\frac{v^{2-q}(t+\delta)-v^{2-q}(t)}{\delta}\, dt \\ &\leq \lim_{\delta\to 0}\int_0^T \varphi(t)\frac{1}{v^\kappa(t+\delta)}\frac{2-q}{s}\frac{v^s(t+\delta)-v^s(t)}{\delta}\, dt \end{aligned}$$

Here we can use the estimate (7.8)

$$\begin{aligned} &(2-q)\lim_{\delta\to 0}\int_0^T \varphi(t)\frac{1}{v^\kappa(t+\delta)}\frac{1}{s}\frac{v^s(t+\delta)-v^s(t)}{\delta}\, dt \\ &\leq -(2-q)C_2\lim_{\delta\to 0}\int_0^T \varphi(t)\frac{1}{v^\kappa(t+\delta)}\frac{1}{\delta}\int_t^{t+\delta} v(l)^{s\frac{d-q}{d}}\, dl \\ &= -(2-q)C_2\int_0^T \varphi(t)\frac{v(t)^{s\frac{d-q}{d}}}{v^\kappa(t)}\, dt \end{aligned}$$

$$= -(2-q)C_2 \int_0^T \varphi(t)\,dt,$$

where the last two identities follow from continuity and the computation

$$s\frac{d-q}{d} - \kappa = s\frac{d-q}{d} + 2 - q - s = \frac{(2-q)(d(d-q)+dq-d^2)}{dq} = 0.$$

Hence we have established the distributional inequality

$$\int_0^T -v^{2-q}(t)\varphi'(t) + (2-q)C_2\varphi(t)\,dt \le 0 \quad \text{for all non-negative } \varphi \in C_0^\infty(0,T).$$

Since v is continuous up to the boundary, this yields

$$v^{2-q}(t) - v^{2-q}(0) + (2-q)C_2 t \le 0 \quad \text{for all } t \in [0,T],$$

which is, recalling $v(t) = \|u(\cdot,t)\|_{L^s(r^{d-1},(0,R))}$, equivalent with

$$||u(\cdot,t)||_{L^s(r^{d-1},(0,R))} \le ||u_0(\cdot)||_{L^s(r^{d-1},(0,R))}\left(1-(2-q)C_2\,||u_0(\cdot)||^{q-2}_{L^s(r^{d-1},(0,R))}\,t\right)^{\frac{1}{2-q}}.$$

Thus as long as the original $T > 0$ is large enough, u vanishes for time T^* satisfying

$$0 < T^* \le C\,||u_0(\cdot)||^{2-q}_{L^s(r^{d-1},(0,R))} \quad \text{for } C = ((2-q)C_2)^{-1}.$$

□

Next, we expand this local result to a global result.

Proposition 7.5 *Assume q does not satisfy the range condition (1.3). Let u be a viscosity solution of*

$$\begin{cases} \partial_t u = |\nabla u|^{q-p}\,\mathrm{div}\left(|\nabla u|^{p-2}\,\nabla u\right) & \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ u(\cdot,0) = u_0(\cdot) \ge 0 & \text{where radial } u_0 \in C_0(B_R) \text{ for some } R > 0. \end{cases} \tag{7.9}$$

There exists a finite time $T^* := T^*(n,p,q,u_0)$, such that

$$u(\cdot,0) \equiv 0 \quad \text{for all } t \ge T^*$$

and

$$0 < T^* \le C\,||u_0||^{2-q}_{L^s(r^{d-1},(0,R))} \tag{7.10}$$

where $C := C(n,p,q)$ and $s = \frac{d(2-q)}{q}$.

Proof Let u_i be the radial viscosity solution to the bounded problem (7.5) for $R = i \in \mathbb{N}$. Now by Proposition 7.4 there exists a finite time T_i^* satisfying

$$0 < T_i^* \le C\,||u_0(\cdot)||^{2-q}_{L^s(r^{d-1},(0,i))},$$

such that $u_i(\cdot,t) \equiv 0$ for $t \ge T_i^*$. By the comparison principle 2.5 we have $u_{i+1} \ge u_i$ in $B_i \times (0,i)$ which implies that $T_{i+1}^* \ge T_i^*$ and because we assumed that u_0 has compact support this sequence of extinction times has a limit $T^*_{\lceil R\rceil}$.

Using the Hölder estimates proven in [13], we have that each u_i is Hölder continuous in both variables and the Hölder constant only depends on n, p, q and $||u_i||_{L^\infty(B_i\times(0,i))}$. By the comparison principle these L^∞-norms are bounded from above by $||u_0||_{L^\infty(\mathbb{R}^n\times\mathbb{R}^+)}$ and thus the sequence $(u_i)_{i=1}^\infty$ is uniformly equicontinuous. By construction, $u_i \to u$ converges pointwise as $i \to \infty$ passing to a subsequence if necessary and because of the equicontinuity, the Arzelà-Ascoli theorem ensures that the convergence is uniform.

For any compact subset $A \subset \mathbb{R}^n \times \mathbb{R}^+$, u_i is a viscosity solution to Eq. 1.1 in A for i large enough and thus u is also a viscosity solution in this set by stability result proven by Ohnuma and Sato [25, Theorem 6.1, Proposition 6.2]. Because A is arbitrary, u is a viscosity solution in the entire space and by construction, it has the correct initial value. This solution is unique as proven by [25, Corollary 4.10] and this proves that u vanishes after finite time $T^*_{\lceil R \rceil}$ satisfying (7.10). □

Now we have the tools needed to show that intrinsic Harnack's inequality does not hold for q not satisfying the range condition (1.3). Let u be a viscosity solution to Eq. 7.9 and T^* the finite extinction time given by Proposition 7.5. Choose $(x_0, t_0) \in \mathbb{R}^n \times (0, T^*)$ close enough to satisfy

$$T^* - t_0 < \frac{t_0}{\sigma^q},$$

and choose $r > 0$ to satisfy

$$cu(x_0, t_0)^{2-q} r^q = T^* - t_0$$

where c and σ are the constants given by Harnack's inequality. By these choices

$$t_0 - cu(x_0, t_0)^{2-q} (\sigma r)^q = t_0 - \sigma^q \left(T^* - t_0\right) > 0$$

and therefore $(x_0, t_0) + Q_{\sigma r}(\theta) \subset \mathbb{R}^n \times \mathbb{R}^+$ and thus we can use the Harnack's inequality to obtain

$$0 < u(x_0, t_0) \le \gamma \inf_{B_r(x_0)} u(\cdot, T^*) = 0,$$

which is a contradiction.

There are some known Harnack-type results with additional assumptions for the p-parabolic equation in the subcritical range, see for example [5, Proposition 1.1].

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References

1. Andrade, P.D.S., Santos, M.S.: Improved regularity for the parabolic normalized p-Laplace equation. Calc. Var. Partial. Differ. Equ. **61**(5), 196 (2022)
2. Calder, J.: The game theoretic p-Laplacian and semi-supervised learning with few labels. Nonlinearity **32**(1), 301 (2019)
3. Cordes, H.O.: Über die erste Randwertaufgabe bei quasilinearen Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen. Math. Ann. **131**(3), 278–312 (1956)
4. DiBenedetto, E., Gianazza, U., Vespri, V.: Harnack estimates for quasi-linear degenerate parabolic differential equations. Acta Math. **200**(2), 181–209 (2008)
5. DiBenedetto, E., Gianazza, U., Vespri, V.: Alternative forms of the Harnack inequality for non-negative solutions to certain degenerate and singular parabolic equations. Rend. Lincei - Mat. E Appl. **20**, 369–377 (2009)
6. DiBenedetto, E., Gianazza, U., Vespri, V.: Harnack's Inequality for Degenerate and Singular Parabolic Equations, vol. 165. Springer (2012)
7. DiBenedetto, E.: Intrinsic Harnack type inequalities for solutions of certain degenerate parabolic equations. Arch. Ration. Mech. Anal. **100**(2), 129–147 (1988)
8. DiBenedetto, E.: Degenerate Parabolic Equations. Universitext. Springer, New York (1993)
9. DiBenedetto, E., Kwong, Y.C.: Harnack Estimates and Extinction Profile for Weak Solutions of Certain Singular Parabolic Equations. Trans. Am. Math. Soc. **330**(2), 783–811 (1992)
10. Does, K.: An evolution equation involving the normalized p-Laplacian. Commun. Pure Appl. Anal. **10**(1), 361–396 (2011)
11. Dong, H., Peng, F., Zhang, Y.R.-Y., Zhou, Y.: Hessian estimates for equations involving p-Laplacian via a fundamental inequality. Adv. Math. (N. Y). **370**, 40 (2020)
12. Høeg, F.A., Lindqvist, P.: Regularity of solutions of the parabolic normalized p-Laplace equation. Adv. Nonlinear Anal. **9**(1), 7–15 (2019)
13. Imbert, C., Jin, T., Silvestre, L.: Hölder gradient estimates for a class of singular or degenerate parabolic equations. Adv. Nonlinear Anal. **8**(1), 845–867 (2019)
14. Ishii, H., Souganidis, P.: Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor. Tôhoku Math. J. **47**, 227–250 (1995)
15. Imbert, C., Silvestre, L.: Introduction to fully nonlinear parabolic equations To cite this version : HAL Id : hal-00798300 Introduction to fully nonlinear parabolic equations (2013)
16. Julin, V., Juutinen, P.: A new proof for the equivalence of weak and viscosity solutions for the p-Laplace equation. Comm. Partial Differential Equations **37**, 934–946 (2011)
17. Juutinen, P., Lindqvist, P., Manfredi, J.J.: On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. SIAM J. Math. Anal. **33**(3), 699–717 (2001)
18. Jin, T., Silvestre, L.: Hölder gradient estimates for parabolic homogeneous p-Laplacian equations. J. Math. Pures Appl. **108**(1), 63–87 (2017)
19. Kurkinen, T., Parviainen, M., Siltakoski, J.: Elliptic Harnack's inequality for a singular nonlinear parabolic equation in non-divergence form. Bull. Lond. Math. Soc. **55**(1), 470–489 (2023)
20. Krylov, N.V., Safonov, M.V.: A certain property of solutions of parabolic equations with measurable coefficients. Math. USSR Izv. **16**(1), 151–164 (1981)
21. Kuusi, T.: Harnack estimates for weak supersolutions to nonlinear degenerate parabolic equations. Ann. della Sc. Norm. Super. di Pisa - Cl. di Sci. **7**(4), 673–716 (2008)
22. Landis, E.M.: Second Order Equations of Elliptic and Parabolic Type. American Mathematical Society (1971)
23. Manfredi, J.J., Parviainen, M., Rossi, J.D.: An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. SIAM J. Math. Anal. **42**(5), 2058–2081 (2010)
24. Nyström, K., Parviainen, M.: Tug-of-war, market manipulation, and option pricing. Math. Finance **27**(2), 279–312 (2014)
25. Ohnuma, M., Sato, K.: Singular degenerate parabolic equations with applications. Commun. Partial Differ. Equations **22**(3–4), 381–441 (1997)
26. Parviainen, M., Vázquez, J.L.: Equivalence between radial solutions of different parabolic gradient-diffusion equations and applications. Ann. della Sc. Norm. Super. Di Pisa - Cl. di Sci. **21**(Special Issue), 303–359 (2020)
27. Siltakoski, J.: Equivalence of viscosity and weak solutions for a p-parabolic equation. J. Evol. Equations **21**(2), 2047–2080 (2021)

28. Wang, L.: On the regularity theory of fully nonlinear parabolic equations. Bull. Am. Math. Soc. **22**(1), 107–114 (1990)

[B]

Elliptic Harnack's inequality for a singular nonlinear parabolic equation in non-divergence form

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RESEARCH ARTICLE

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Elliptic Harnack's inequality for a singular nonlinear parabolic equation in non-divergence form

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Abstract
We prove an elliptic Harnack's inequality for a general form of a parabolic equation that generalizes both the standard parabolic p-Laplace equation and the normalized version that has been proposed in stochastic game theory. This version of the inequality does not require the intrinsic waiting time and we get the estimate with the same time level on both sides of the inequality.

MSC 2020
35K55 (primary), 35K67, 35D40 (secondary)

1 | INTRODUCTION

In his monograph, DiBenedetto [5, Theorem VII.1.2] proved elliptic Harnack's inequality for the divergence form p-parabolic equation in the supercritical case. In this case, the intrinsic waiting time required for degenerate parabolic equations is no longer needed. Instead he established Harnack's inequality with the same time level on both sides of the estimate akin to the elliptic case.

In this paper, we prove elliptic Harnack's inequality for the following general non-divergence form version of the non-linear parabolic equation:

$$\partial_t u = |\nabla u|^{q-p} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |\nabla u|^{q-2}(\Delta u + (p-2)\Delta_\infty^N u), \tag{1.1}$$

for a natural range of exponents. When $q = 2$, we get the normalized p-parabolic equation arising from the game theory, and when $q = p$, it is the standard p-parabolic equation.

Elliptic Harnack's inequality, Theorem 2.1, states that a non-negative solution satisfies the following local a priori estimate:

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0) \leqslant u(x_0, t_0) \leqslant \gamma \inf_{B_r(x_0)} u(\cdot, t_0).$$

DiBenedetto's proof uses the theory of weak solutions. Since the equation is in a non-divergence form, unless $q = p$, the usual weak theory based on integration by parts is not available in our case. Our proof uses the parabolic (forward) Harnack's inequality proven by Parviainen and Vázquez [26] to estimate the solution in the past, constructing an explicit supersolution with infinite boundary values and using the comparison principle to get an estimate at our original time level. The idea both in the proof of the forward Harnack as well as in the derivation of the explicit supersolution is based on an equivalence result. Heuristically speaking, radial solutions to the original non-divergence form problem can be interpreted as solutions to the divergence form p-parabolic equation, but in a fictitious space dimension d.

Nash discussed the possibility of elliptic Harnack's inequality for a parabolic equation in [24]. Later Moser [22] pointed out that such an estimate does not hold for the heat equation. For the p-parabolic equation, elliptic Harnack's inequality is obviously false if $p > 2$, and holds for $\frac{2n}{n+1} < p < 2$. In addition to [5], Harnack's inequalities in the singular range have been studied, for example, by Dibenedetto, Gianazza and Vespri in [7, 8] and [9]. The intrinsic forward Harnack's inequality for weak solutions of the p-parabolic equation was proven by Dibenedetto in [4] and [10], see also [5], and later for equations with growth of order p by Dibenedetto, Gianazza and Vespri in [6] and by Kuusi in [20]. For non-divergence form equations, parabolic Harnack's inequalities and related Hölder regularity under additional restrictions were studied by Cordes [3] and Landis [21]. With bounded and measurable coefficients parabolic Harnack's inequality was established by Krylov and Safonov [19].

Since the Equation (1.1) is in non-divergence form except in a special case, the solutions in this paper are understood in the viscosity sense. The suitable concept of viscosity solutions to the general equations (1.1) was established by Ohnuma and Sato [25]. In the special case $q = 2$, we get the normalized p-parabolic equation that arises from the stochastic game theory [23]. This non-divergence form special case as well as the general equation (1.1) have recently received attention in the works of Jin-Silvestre [17], Imbert-Jin-Silvestre [15], Høeg-Lindqvist [14], and Dong-Fa-Zhang-Zhou [12] in addition to [26].

2 | MAIN RESULTS

Denote

$$\Delta_p^q u := |\nabla u|^{q-p} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |\nabla u|^{q-2}(\Delta u + (p-2)\Delta_\infty^N u), \tag{2.1}$$

where $p > 1$ and $q > 1$ are real parameters and

$$\Delta_\infty^N u = \sum_{i,j=1}^{n} \frac{\partial_{x_i} u\, \partial_{x_j} u\, \partial_{x_i x_j} u}{|\nabla u|^2}$$

so the Equation (1.1) gets the form $\partial_t u = \Delta_p^q u$. Because the dimensions of the sets play part in some of the estimates, we shall denote

$$Q_r^-(\theta) = B_r(0) \times (-\theta r^q, 0],$$

$$Q_r^+(\theta) = B_r(0) \times (0, \theta r^q),$$

where θ is a positive parameter that determines the time-wise length of the cylinder relative to r^q. We denote the union of these sets as

$$Q_r(\theta) = Q_r^+(\theta) \cup Q_r^-(\theta)$$

and when not located at the origin, we denote

$$\begin{aligned}(x_0, t_0) + Q_r^-(\theta) &= B_r(x_0) \times (t_0 - \theta r^q, t_0],\\ (x_0, t_0) + Q_r^+(\theta) &= B_r(x_0) \times (t_0, t_0 + \theta r^q),\\ (x_0, t_0) + Q_r(\theta) &= B_r(x_0) \times (t_0 - \theta r^q, t_0 + \theta r^q).\end{aligned}$$

Our main result is that non-negative viscosity solutions to (1.1) satisfy the following *elliptic Harnack's inequality* if the following range condition holds:

$$2 > q > \begin{cases} 1 & \text{if } p \geqslant \frac{1+n}{2}, \\ \frac{2(n-p)}{n-1} & \text{if } 1 < p < \frac{1+n}{2}. \end{cases} \tag{2.2}$$

We inspect the optimality of this range after the formulation of the theorem.

Theorem 2.1 (Elliptic Harnack's inequality). *Let $u \geqslant 0$ be a viscosity solution to (1.1) in $Q_1^-(1)$ and the range condition (2.2) holds. Fix $(x_0, t_0) \in Q_1^-(1)$. Then, for any $\sigma > 1$, there exist $\gamma = \gamma(n, p, q, \sigma)$ and $c = c(n, p, q, \sigma)$ such that*

$$\gamma^{-1} \sup_{B_r(x_0)} u(\cdot, t_0) \leqslant u(x_0, t_0) \leqslant \gamma \inf_{B_r(x_0)} u(\cdot, t_0),$$

whenever $(x_0, t_0) + Q_{\sigma r}(\theta) \subset Q_1^-(1)$ where

$$\theta = cu(x_0, t_0)^{2-q}.$$

Our proof relies on comparison principle and parabolic Harnack's inequality proven for viscosity solutions of (1.1) in [26] and construction of an explicit viscosity supersolution with infinite boundary values. Existence of such solutions relies on the singularity of the equation and was proven for the p-parabolic case in [2, Theorem 4.1]. Here, we constructed a concrete solution in order to obtain an explicit proof at each step. If q approaches either end point of range (2.2), the constant γ tends to infinity and c approaches zero.

Elliptic Harnack's inequality may fail outside of the range condition (2.2). To illustrate this, recall a result by Parviainen and Vázquez [26] according to which radial viscosity solutions to (1.1) are equivalent to weak solutions of the one-dimensional equation

$$\partial_t u - \varkappa \Delta_{q,d} u = 0 \quad \text{in } (-R, R) \times (0, T). \tag{2.3}$$

Here, $\varkappa := (p-1)/(q-1)$ and (denoting by u_r the radial derivative of u)

$$\Delta_{q,d} u := |u_r|^{q-2} \left((q-1) u_{rr} + \frac{d-1}{r} u_r \right)$$

is heuristically the usual radial q-Laplacian in a fictitious dimension

$$d := \frac{(n-1)(q-1)}{p-1} + 1.$$

If d happens to be an integer, then solutions to (2.3) are equivalent to radial weak solutions of the q-parabolic equation in $B_R \times (0,T) \subset \mathbb{R}^{d+1}$. On the other hand, the counterexamples in [9, p. 140] show that elliptic Harnack's inequality for the q-parabolic equation in $\mathbb{R}^{d+1}$ holds only in the range $2d/(d+1) < q < 2$, from which one can derive the range condition (2.2) by recalling the definition of d. In fact, the counterexample in [9] directly translates into our context even when d is not an integer. To see this, suppose that $1 < p < (1+n)/2$ and set $q = 2(n-p)/(n-1)$. Then, in particular $q > 1$. We define in radial coordinates

$$u(r,t) := (|r|^{\frac{2d}{d-1}} + e^{\varkappa bt})^{-(d-1)/2} \quad \text{for all } r \in \mathbb{R}.$$

By a direct computation, u satisfies (2.3) classically in $(-R,-\delta) \cup (\delta,R)$ for any $R > 0$ and small $\delta > 0$. Letting $\delta \to 0$ then shows that u is a weak solution in the sense of [26] and therefore a viscosity solution to (1.1) in $\mathbb{R}^{n+1}$. However, u fails to satisfy elliptic Harnack's inequality since $u(0,t)/u(1,t) \to 0$ as $t \to -\infty$.

Finally, we point out that in the case $q = p$, the range condition becomes

$$2 > p > \frac{2n}{n+1} =: p_*$$

the so-called supercritical p-parabolic equation for which we have both intrinsic [6, 20] and elliptic Harnack's inequality [5]. As mentioned, in the subcritical case $p \leqslant p_*$ both of the inequalities fail [9] but there are some known Harnack-type results, see, for example, [7, Proposition 1.1].

3 | PRELIMINARIES

Apart from the case $q = p$, Equation (1.1) is in non-divergence form and we cannot use integration by parts to define standard weak solutions and will thus use the concept of viscosity solutions. Moreover the equation is singular when $2 > q > 1$, and thus we need to restrict the class of test functions to retain good priori control on the behavior near the singularities and make sure the limits remain well defined. We use the definition with admissible test functions introduced in [16] for a different class of equations and in [25] for our setting. This is the standard definition in this context. In the case of the p-parabolic equation, that is, $q = p$, the notions of weak and viscosity solution are equivalent for all $p \in (1,\infty)$ [18, 26, 27].

Let $\Omega \subset \mathbb{R}^n$ be a domain and denote $\Omega_T = \Omega \times (0,T)$ the space-time cylinder and

$$\partial_p\Omega = (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T])$$

its parabolic boundary. Denote

$$F(\eta, X) = |\eta|^{q-2}\,\mathrm{Tr}\left(X - (p-2)\frac{\eta \otimes \eta}{|\eta|^2}X\right) \tag{3.1}$$

so that

$$F(\nabla u, D^2 u) = |\nabla u|^{q-2}(\Delta u + (p-2)\Delta_\infty^N u) = \Delta_p^q u$$

whenever $\nabla u \neq 0$. Let $\mathcal{F}(F)$ be the set of functions $f \in C^2([0,\infty))$ such that

$$f(0) = f'(0) = f''(0) = 0 \text{ and } f''(r) > 0 \text{ for all } r > 0,$$

and also require that for $g(x) := f(|x|)$, it holds that

$$\lim_{\substack{x\to 0\\ x\neq 0}} F(\nabla g, D^2 g) = 0.$$

This set $\mathcal{F}(F)$ is never empty because it is easy to see that $f(r) = r^\beta \in \mathcal{F}(F)$ for any $\beta > \max(q/(q-1), 2)$. Note also that if $f \in \mathcal{F}(F)$, then $\lambda f \in \mathcal{F}(F)$ for all $\lambda > 0$.

Define also the set

$$\Sigma = \{\sigma \in C^1(\mathbb{R}) \mid \sigma \text{ is even}, \sigma(0) = \sigma'(0) = 0, \text{ and } \sigma(r) > 0 \text{ for all } r > 0\}.$$

We use these $\mathcal{F}(F)$ and Σ to define admissible set of test functions for viscosity solutions.

Definition 3.1. A function $\varphi \in C^2(\Omega_T)$ is admissible if for any $(x_0, t_0) \in \Omega_T$ with $\nabla\varphi(x_0, t_0) = 0$, there are $\delta > 0$, $f \in \mathcal{F}(F)$ and $\sigma \in \Sigma$ such that

$$\left|\varphi(x,t) - \varphi(x_0,t_0) - \partial_t\varphi(x_0,t_0)(t-t_0)\right| \leqslant f(\left|x - x_0\right|) + \sigma(t - t_0),$$

for all $(x,t) \in B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$.

Note that by definition a function φ is automatically admissible in Ω_T if either $\nabla\varphi(x,t) \neq 0$ in Ω_T or the function $-\varphi$ is admissible in Ω_T.

Definition 3.2. A function $u : \Omega_T \to \mathbb{R} \cup \{\infty\}$ is a viscosity supersolution to

$$\partial_t u = \Delta_p^q u \quad \text{in } \Omega_T$$

if the following three conditions hold.

(1) u is lower semicontinuous,
(2) u is finite in a dense subset of Ω_T,
(3) whenever an admissible $\varphi \in C^2(\Omega_T)$ touches u at $(x,t) \in \Omega_T$ from below, we have

$$\begin{cases} \partial_t\varphi(x,t) - \Delta_p^q\varphi(x,t) \geqslant 0 & \text{if } \nabla\varphi(x,t) \neq 0, \\ \partial_t\varphi(x,t) \geqslant 0 & \text{if } \nabla\varphi(x,t) = 0. \end{cases}$$

A function $u : \Omega_T \to \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution if $-u$ is a viscosity supersolution. A function $u : \Omega_T \to \mathbb{R}$ is a viscosity solution if it is a supersolution and a subsolution.

Our proof uses the following comparison principle, which is Theorem 3.1 in [25].

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that u is viscosity supersolution and v is a viscosity subsolution to (1.1) in Ω_T. If*

$$\infty \neq \limsup_{\Omega_T \ni (y,s) \to (x,t)} v(y,s) \leqslant \liminf_{\Omega_T \ni (y,s) \to (x,t)} u(y,s) \neq -\infty$$

for all $(x,t) \in \partial_p \Omega_T$, then $v \leqslant u$ in Ω_T.

We also use the following forward Harnack's inequality, which is Theorem 7.3 in [26].

Theorem 3.4. *Let $u \geqslant 0$ be a viscosity solution to (1.1) in $Q_1^-(1)$ and the range condition (2.2) holds or $q \geqslant 2$. Fix $(x_0, t_0) \in Q_1^-(1)$ such that $u(x_0, t_0) > 0$. Then, there exist $\mu = \mu(n, p, q)$ and $c = c(n, p, q)$ such that*

$$u(x_0, t_0) \leqslant \mu \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q),$$

where

$$\theta = c u(x_0, t_0)^{2-q},$$

whenever $(x_0, t_0) + Q_{4r}(\theta) \subset Q_1^-(1)$.

Remark 3.5. Note that the assumption $u(x_0, t_0) > 0$ is needed only in the case $q \geqslant 2$. Assuming q satisfies the range condition (2.2), we can define $v(x,t) = u(x,t) + \varepsilon > 0$ for some small constant $\varepsilon > 0$. Using Theorem 3.4 for this v, we get

$$u(x_0, t_0) + \varepsilon \leqslant \mu \inf_{B_r(x_0)} u\left(\cdot, t_0 + c(u(x_0, t_0) + \varepsilon)^{2-q} r^q\right) + \varepsilon$$

and letting $\varepsilon \to 0$ gives us the intrinsic Harnack's inequality for u by continuity.

4 | A VISCOSITY SUPERSOLUTION WITH INFINITE BOUNDARY VALUES

In this section, we construct an explicit viscosity supersolution v to (1.1) in $B_R(0) \times (0, \infty)$ that takes infinite lateral boundary values and vanishes at the bottom of the cylinder. Recently infinite point source solutions have been constructed for the supercritical p-parabolic equation in [13]. While it is straightforward to check that our function is a supersolution, it may not be immediately clear how one obtains its expression and therefore we present the derivation. The construction is based on the equivalence result between radial viscosity solutions of (1.1) and weak solutions of (2.3), see [26, Theorem 4.2]. Solutions to the one-dimensional equation (2.3) can be at least formally obtained via the stationary equation

$$-\kappa \Delta_{q,d} v + \frac{v}{2-q} = 0. \tag{4.1}$$

Indeed, if v solves (4.1) and we set $u(r,t) = t^{\frac{1}{2-q}} v(r)$, then we have formally

$$\kappa \Delta_{q,d} u = \kappa |u_r|^{q-2} \left((q-1) u_{rr} + \frac{d-1}{r} u_r \right) = \kappa t^{\frac{1}{2-q} - 1} \Delta_{q,d} v = \frac{1}{2-q} t^{\frac{1}{2-q} - 1} v = \partial_t u,$$

so u solves (2.3). Now, Equation (4.1) can be seen as a radial version of the equation

$$-\kappa\Delta_q v + \frac{v}{2-q} = 0 \tag{4.2}$$

in a fictitious dimension d. Here, Δ_q denotes the usual q-Laplacian. Equations such as (4.2) have been widely studied in the literature when d is an integer. In particular, Díaz and Letelier [11] obtained the existence of local solutions with infinite boundary values to a large class of equations that includes (4.2). In their proof, they make use of an explicit radial supersolution with infinite boundary values (see [11, Theorem 5.1]). Our idea is to take this supersolution and use the above transformations to obtain a supersolution to (1.1). This way one arrives to the expression (4.3) below.

Lemma 4.1. *Suppose that $1 < q < 2$, $p > 1$ and let $R > 0$. Then, there exists a positive constant $\lambda = \lambda(n, p, q)$ such that the function*

$$v(x,t) := \lambda t^{\frac{1}{2-q}} \left(\frac{1}{R^{\frac{1}{1-q}} \left(R^{\frac{q}{q-1}} - |x|^{\frac{q}{q-1}}\right)} \right)^{\frac{q}{2-q}} \tag{4.3}$$

is a viscosity supersolution to (1.1) in $B_R(0) \times (0, \infty)$.

Proof. Let us first consider the case $R = 1$.

(Step 1) For $(r, t) \in [0, 1) \times (0, \infty)$, we set

$$w(r,t) := \lambda t^{\frac{1}{2-q}} (1 - r^{\frac{q}{q-1}})^{\frac{q}{q-2}},$$

where $\lambda = \lambda(n, p, q)$ is a large constant to be chosen later. We show that w satisfies

$$\partial_t w - |w'|^{q-2}\left((p-1)w'' + \frac{n-1}{r} w'\right) \geqslant 0 \quad \text{in } (0,1) \times (0, \infty). \tag{4.4}$$

We have

$$\begin{aligned} \partial_t w(r,t) &= \lambda \frac{1}{2-q} t^{\frac{1}{2-q}-1} (1 - r^{\frac{q}{q-1}})^{\frac{q}{q-2}}, \\ w'(r,t) &= -\lambda \frac{q^2}{(q-1)(q-2)} \cdot t^{\frac{1}{2-q}} r^{\frac{q}{q-1}-1} (1 - r^{\frac{q}{q-1}})^{\frac{q}{q-2}-1} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} w''(r,t) = &-\lambda \frac{q^2}{(q-1)(q-2)} \left(\frac{q}{q-1} - 1\right) \cdot t^{\frac{1}{2-q}} r^{\frac{q}{q-1}-2} (1 - r^{\frac{q}{q-1}})^{\frac{q}{q-2}-1} \\ &+ \lambda \frac{q^3}{(q-1)^2(q-2)} \left(\frac{q}{q-2} - 1\right) \cdot t^{\frac{1}{2-q}} r^{2(\frac{q}{q-1}-1)} (1 - r^{\frac{q}{q-1}})^{\frac{q}{q-2}-2}. \end{aligned} \tag{4.6}$$

Thus, by combining (4.5) and (4.6), we get

$$
\begin{aligned}
&(p-1)w''(r,t)+\frac{n-1}{r}w'(r,t)\\
&=-\lambda\frac{q^2(p-1)}{(q-1)^2(q-2)}t^{\frac{1}{2-q}}r^{\frac{q}{q-1}-2}(1-r^{\frac{q}{q-1}})^{\frac{q}{q-2}-1}\\
&\quad+\lambda\frac{2q^3(p-1)}{(q-1)^2(q-2)^2}t^{\frac{1}{2-q}}r^{2(\frac{q}{q-1}-1)}(1-r^{\frac{q}{q-1}})^{\frac{q}{q-2}-2}\\
&\quad-\lambda\frac{q^2(n-1)}{(q-1)(q-2)}t^{\frac{1}{2-q}}r^{\frac{q}{q-1}-2}(1-r^{\frac{q}{q-1}})^{\frac{q}{q-2}-1}\\
&\leqslant C(n,p,q)\lambda t^{\frac{1}{2-q}}r^{\frac{q}{q-1}-2}(1-r^{\frac{q}{q-1}})^{\frac{q}{q-2}-2}((1-r^{\frac{q}{q-1}})+r^{\frac{q}{q-1}}).\\
&\leqslant C(n,p,q)\lambda t^{\frac{1}{2-q}}r^{\frac{q}{q-1}-2}(1-r^{\frac{q}{q-1}})^{\frac{q}{q-2}-2}.
\end{aligned}
$$

Combining this with the formula (4.5), we obtain

$$
\begin{aligned}
&|w'|^{q-2}\left((p-1)w''+\frac{n-1}{r}w'\right)\\
&\leqslant C(n,p,q)\lambda^{q-2}t^{\frac{q-2}{2-q}}r^{(q-2)(\frac{q}{q-1}-1)}(1-r^{\frac{q}{q-1}})^{(q-2)(\frac{q}{q-2}-1)}\\
&\cdot\lambda t^{\frac{1}{2-q}}r^{\frac{q}{q-1}-2}(1-r^{\frac{q}{q-1}})^{\frac{q}{q-2}-2}\\
&=C(n,p,q)\lambda^{q-1}t^{\frac{1}{2-q}-1}(1-r^{\frac{q}{q-1}})^{\frac{q}{q-2}},
\end{aligned}
$$

where we used that $(q-2)(\frac{q}{q-1}-1)+(\frac{q}{q-1}-2)=\frac{q-2}{q-1}+\frac{2-q}{q-1}=0$ and $(q-2)(\frac{q}{q-2}-1)=2$. Hence,

$$
\begin{aligned}
&\partial_t w-|w'|^{q-2}\left((p-1)w''+\frac{n-1}{r}w'\right)\\
&\geqslant\lambda\frac{1}{2-q}t^{\frac{1}{2-q}-1}(1-r^{\frac{q}{q-1}})^{\frac{q}{q-2}}-C(n,p,q)\lambda^{q-1}t^{\frac{1}{2-q}-1}(1-r^{\frac{q}{q-1}})^{\frac{q}{q-2}}\\
&=\lambda^{q-1}t^{\frac{1}{2-q}-1}(1-r^{\frac{q}{q-1}})^{\frac{q}{q-2}}\left(\frac{\lambda^{2-q}}{2-q}-C(n,p,q)\right).
\end{aligned}
$$

By taking $\lambda=\lambda(n,p,q)$ large enough, the right-hand side of the above display can be made non-negative. This way we see that w satisfies (4.4).

(Step 2) We set

$$
v(x,t):=w(|x|,t)\quad\text{for all }(x,t)\in B_1\times(0,\infty).
$$

Suppose first that $(x,t)\in(B_1\setminus\{0\})\times(0,\infty)$ and denote $r=|x|$. Then, we have

$$
\begin{aligned}
\nabla v(x,t)&=\frac{x}{r}w'(r,t),\\
D^2v(x,t)&=\frac{x}{r}\otimes\frac{x}{r}w''(r,t)+\frac{1}{r}(I-\frac{x}{r}\otimes\frac{x}{r})w'(r,t).
\end{aligned}
$$

Therefore, since w satisfies (4.4), we have

$$\partial_t v - \Delta_p^q v = \partial_t v - |\nabla v|^{q-2}\operatorname{Tr}\left(D^2 v + (p-2)\frac{\nabla v \otimes \nabla v}{|\nabla v|^2} D^2 v\right)$$
$$= \partial_t w - \left|w'\right|^{q-2}((p-1)w'' + \frac{(n-1)}{r}w') \geqslant 0.$$

This means that v is a classical supersolution in $(B_1 \setminus \{0\}) \times (0,\infty)$. We still need to consider the set $\{0\} \times (0,\infty)$. Since $1 < q < 2$, it follows from the formulas (4.5) of w' and (4.6) of w'' that $v \in C^2(B_1 \times (0,\infty))$ with $\nabla v(0,t) = 0$ and $\partial_t v(0,t) \geqslant 0$ for all $t > 0$. Therefore, if $\varphi \in C^2$ touches v from below at $(0,t)$, we have $\nabla\varphi(0,t) = \nabla v(0,t) = 0$ and $\partial_t \varphi(0,t) = \partial_t v(0,t) \geqslant 0$, as required. Consequently v is a viscosity supersolution in $B_1 \times (0,\infty)$.

(Step 3) It remains to consider $R > 0$. Let v be the viscosity supersolution to

$$\partial_t v = \Delta_p^q v \quad \text{in } B_1 \times (0,\infty)$$

which we constructed in the previous steps. Set $\tilde{v}(x,t) := v(R^{-1}x, R^{-q}t)$. Then, for all $(x,t) \in B_R(0) \setminus \{0\} \times (0,\infty)$, we have

$$\partial_t \tilde{v}(x,t) - \Delta_p^q \tilde{v}(x,t) = R^{-q} v(R^{-1}x, R^{-q}t) - R^{-q}\Delta_p^q v(R^{-1}x, R^{-q}t) \geqslant 0,$$

so $\tilde{v}$ is a viscosity supersolution in $B_R(0) \times (0,\infty)$. Moreover,

$$\tilde{v}(x,t) = \lambda (R^{-q}t)^{\frac{1}{2-q}}\left(1 - \left|R^{-1}x\right|^{\frac{q}{q-1}}\right)^{\frac{q}{q-2}} = \lambda t^{\frac{1}{2-q}}(RR^{\frac{q}{1-q}}(R^{\frac{q}{q-1}} - |x|^{\frac{q}{q-1}}))^{\frac{q}{q-2}}$$
$$= \lambda t^{\frac{1}{2-q}}(R^{\frac{1}{1-q}}(R^{\frac{q}{q-1}} - |x|^{\frac{q}{q-1}}))^{\frac{q}{q-2}}$$

as desired. □

5 | A PARABOLIC HARNACK'S INEQUALITY

In this section, we prove a both-sided version of parabolic Harnack's inequality for Equation (1.1) which is of independent interest and needed for our proof of Theorem 2.1. The proof of the backwards estimate is an adaptation of section 6.9 in [9] apart from the non-emptyness of the set $\mathcal{U}_\alpha$ below, which we prove using the comparison principle and the explicit supersolution we constructed in Lemma 4.1. To this end, we need to reduce the waiting time in the forward Harnack inequality. This kind of reduction can be achieved by increasing the multiplier μ, as made precise in the following proposition.

Proposition 5.1. *Let $u \geqslant 0$ be a viscosity solution to (1.1) in $Q_1^-(1)$ and the range condition (2.2) holds. Fix $(x_0,t_0) \in Q_1^-(1)$ such that $u(x_0,t_0) > 0$. Let c be as in Theorem 3.4. Then, for any $\hat{c} \in (0,c)$, there exists $\hat{\mu} = \hat{\mu}(n,p,q,\hat{c})$ such that*

$$u(x_0,t_0) \leqslant \hat{\mu} \inf_{B_r(x_0)} u(\cdot, t_0 + \hat{\theta} r^q), \tag{5.1}$$

whenever $(x_0,t_0) + Q_{5r}(\hat{\theta}) \subset Q_1^-(1)$, where $\hat{\theta} = \hat{c}u(x_0,t_0)^{2-q}$.

We postpone the proof of Proposition 5.1 to the end of this section and consider the both-sided Harnack inequality next.

Theorem 5.2. *Let $u \geqslant 0$ be a viscosity solution to (1.1) in $Q_1^-(1)$ and the range condition (2.2) holds. Fix $(x_0, t_0) \in Q_1^-(1)$. Then, there exist $\mu = \mu(n, p, q)$, $c = c(n, p, q)$ and $\alpha = \alpha(n, p, q) \in (0, 1)$ such that*

$$\mu^{-1} \sup_{B_r(x_0)} u(\cdot, t_0 - \theta r^q) \leqslant u(x_0, t_0) \leqslant \mu \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^q),$$

where

$$\theta = cu(x_0, t_0)^{2-q},$$

whenever $(x_0, t_0) + Q_{\frac{6}{\alpha}r}(\theta) \subset Q_1^-(1)$.

Proof. Without loss of generality, we may assume $u(x_0, t_0) > 0$ as stated in Remark 3.5. Let $\hat{c}$ be a small positive constant to be chosen later. For this $\hat{c}$, let $\hat{\mu} > 2$ be given by Proposition 5.1. Let ρ be a radius such that $(x_0, t_0) + Q_{6\rho}(\hat{\theta}) \subset Q_1^-(1)$, $\hat{\theta} = \hat{c}u(x_0, t_0)^{2-q}$, and let

$$\bar{t} = t_0 - \hat{c}u(x_0, t_0)^{2-q}\rho^q. \tag{5.2}$$

Let $\alpha \in (0, 1)$ be a constant to be chosen later and define the sets

$$\mathcal{U}_\alpha = B_{\alpha\rho}(x_0) \cap \{x \in \overline{B}_{\alpha\rho}(x_0) \mid u(x, \bar{t}) \leqslant \hat{\mu}u(x_0, t_0)\} =: B_{\alpha\rho}(x_0) \cap D. \tag{5.3}$$

We will first show that α can be chosen to make $\mathcal{U}_\alpha$ open. Assume that $\mathcal{U}_\alpha$ is not empty and fix $z \in \mathcal{U}_\alpha$. Since u is continuous, we can choose a radius ε such that $B_\varepsilon(z) \subset B_{\alpha\rho}(x_0)$ and

$$u(y, \bar{t}) \leqslant 2\hat{\mu}u(x_0, t_0) \quad \text{for all } y \in B_\varepsilon(z). \tag{5.4}$$

For each $y \in B_\varepsilon(z)$, construct the intrinsic q-paraboloid

$$\mathcal{P}(y, \bar{t}) = \{(x, t) \in Q_1^-(1) \mid t - \bar{t} \geqslant \hat{c}u(y, \bar{t})^{2-q}|x - y|^q\}.$$

Selecting

$$\alpha := (2\hat{\mu})^{\frac{q-2}{q}}, \tag{5.5}$$

we have $(x_0, t_0) \in \mathcal{P}(y, \bar{t})$ whenever $y \in B_\varepsilon(z)$, since using (5.4) we can estimate

$$\begin{aligned}\hat{c}u(y, \bar{t})^{2-q}|y - x_0|^q \leqslant \hat{c}(2\hat{\mu})^{2-q}u(x_0, t_0)^{2-q}|y - x_0|^q \leqslant \hat{c}(2\hat{\mu})^{2-q}u(x_0, t_0)^{2-q}(\alpha\rho)^q \\ \leqslant \hat{c}u(x_0, t_0)^{2-q}\rho^q = t_0 - \bar{t}.\end{aligned}$$

Assume for a moment that $u(y,\bar{t}) \geqslant 2u(x_0,t_0)$ and pick a radius

$$\hat{\rho} = \frac{u(x_0,t_0)^{\frac{2-q}{q}}}{u(y,\bar{t})^{\frac{2-q}{q}}}\rho$$

so that

$$\bar{t} + \hat{c}u\big(y,\bar{t}\big)^{2-q}\hat{\rho}^q = \bar{t} + \hat{c}u(x_0,t_0)^{2-q}\rho^q = t_0.$$

Thus, by Proposition 5.1 we have

$$u\big(y,\bar{t}\big) \leqslant \hat{\mu} \inf_{B_{\hat{\rho}}(y)} u\Big(\cdot, \bar{t} + \hat{c}u\big(y,\bar{t}\big)^{2-q}\hat{\rho}^q\Big) = \hat{\mu} \inf_{B_{\hat{\rho}}(y)} u(\cdot,t_0) \leqslant \hat{\mu}u(x_0,t_0), \tag{5.6}$$

where the last inequality holds because from $(x_0,t_0) \in \mathcal{P}(y,\bar{t})$, it follows

$$|x_0 - y|^q \leqslant \frac{t_0 - \bar{t}}{\hat{c}u(y,\bar{t})^{2-q}} = \frac{\hat{c}u(x_0,t_0)^{2-q}\rho^q}{\hat{c}u(y,\bar{t})^{2-q}} = \hat{\rho}^q.$$

The use of Proposition 5.1 here is justified since $B_{5\hat{\rho}}(y) \subset B_{6\rho}(x_0)$ because

$$5\hat{\rho} + \rho = 5\frac{u(x_0,t_0)^{\frac{2-q}{q}}}{u(y,\bar{t})^{\frac{2-q}{q}}}\rho + \rho \leqslant 5\left(\frac{u(x_0,t_0)}{2u(x_0,t_0)}\right)^{\frac{2-q}{q}}\rho + \rho \leqslant 6\rho,$$

where we use our assumption $u(y,\bar{t}) \geqslant 2u(x_0,t_0)$ and $q < 2$. In the time direction it holds

$$\begin{aligned}
\bar{t} - \hat{c}u(y,\bar{t})^{2-q}(5\hat{\rho})^q &= \bar{t} - \hat{c}u(y,\bar{t})^{2-q}\left(\frac{u(x_0,t_0)}{u(y,\bar{t})}\right)^{2-q}(5\rho)^q \\
&= t_0 - \hat{c}u(x_0,t_0)^{2-q}\rho^q - \hat{c}u(x_0,t_0)^{2-q}(5\rho)^q \\
&= t_0 - (1+5^q)\hat{\theta}\rho^q > t_0 - \hat{\theta}(6\rho)^q
\end{aligned}$$

and thus there is enough room to use the proposition. The last inequality holds because $q > 1$.

If $u(y,\bar{t}) < 2u(x_0,t_0)$, then (5.6) holds automatically since $\hat{\mu} \geqslant 2$. We can get inequality (5.6) for any $y \in B_\varepsilon(z)$ and thus $B_\varepsilon(z) \subset \mathcal{U}_\alpha$ for a radius ε only depending on z. This can be repeated for any $z \in \mathcal{U}_\alpha$ and thus the set $\mathcal{U}_\alpha$ has to be open.

We still need to show that $\mathcal{U}_\alpha \neq \emptyset$. If we assume thriving for a contradiction that $\mathcal{U}_\alpha = \emptyset$, then

$$m := \inf_{B_{\alpha\rho}(x_0)} u(\cdot,\bar{t}) \geqslant \hat{\mu}u(x_0,t_0). \tag{5.7}$$

Consider the function

$$w(x,t) := -\lambda(t-\bar{t})^{\frac{1}{2-q}}\left(\frac{1}{(\alpha\rho)^{\frac{1}{1-q}}\left((\alpha\rho)^{\frac{q}{q-1}} - |x-x_0|^{\frac{q}{q-1}}\right)}\right)^{\frac{q}{2-q}} + m.$$

By Lemma 4.1, w is a viscosity subsolution to (1.1) in $B_{\alpha\rho}(x_0) \times (\bar{t}, \infty)$ and satisfies

$$\begin{cases} w(x,\bar{t}) \equiv m \leqslant u(x,\bar{t}) & \text{for all } x \in B_{\alpha\rho}(x_0), \\ \lim\limits_{\Omega_T \ni (x,t) \to (y,s)} w(x,t) = -\infty & \text{for all } (y,s) \in \partial B_{\alpha\rho}(x_0) \times (\bar{t}, \infty). \end{cases}$$

Thus, by comparison principle Theorem 3.3, we have $u \geqslant w$ in $B_{\alpha\rho}(x_0) \times [\bar{t}, \infty)$, so in particular we have

$$\begin{aligned} u(x_0,t_0) &\geqslant w(x_0,t_0) \\ &= -\lambda\left(t_0 - t_0 + \hat{c}u(x_0,t_0)^{2-q}\rho^q\right)^{\frac{1}{2-q}} \left(\frac{1}{(\alpha\rho)^{\frac{1}{1-q}}\left((\alpha\rho)^{\frac{q}{q-1}} - 0\right)} \right)^{\frac{q}{2-q}} + m \\ &= -\lambda \hat{c}^{\frac{1}{2-q}} \rho^{\frac{q}{2-q}} (\alpha\rho)^{-\frac{q}{2-q}} u(x_0,t_0) + m \\ &\geqslant -\lambda \hat{c}^{\frac{1}{2-q}} \alpha^{-\frac{q}{2-q}} u(x_0,t_0) + \hat{\mu} u(x_0,t_0) \\ &= \left(-2\lambda \hat{c}^{\frac{1}{2-q}} + 1\right) \hat{\mu} u(x_0,t_0) \\ &> 2\left(-2\lambda \hat{c}^{\frac{1}{2-q}} + 1\right) u(x_0,t_0), \end{aligned}$$

where the last two inequalities follow from our assumption (5.7) and that $\hat{\mu} > 2$. By taking $\hat{c}$ to be a small enough constant depending only on p, q and n, we can ensure that the coefficient of $u(x_0,t_0)$ on the right-hand side is larger than 1, yielding a contradiction. Thus, the set $\mathcal{U}_\alpha$ cannot be empty.

We have shown that the set $\mathcal{U}_\alpha = B_{\alpha\rho}(x_0) \cap D$ is open and non-empty. Because u is continuous, the set D is closed and thus for our α, we must have $B_{\alpha\rho}(x_0) \subset D$ and thus by definition of the set

$$\sup_{B_{\alpha\rho}(x_0)} u(\cdot,\bar{t}) \leqslant \hat{\mu} u(x_0,t_0).$$

Combining this with the right-hand side of the Harnack's inequality Proposition 5.1, we obtain

$$\hat{\mu}^{-1} \sup_{B_{\alpha\rho}(x_0)} u(\cdot, t_0 - \hat{c}u(x_0,t_0)^{2-q}\rho^q) \leqslant u(x_0,t_0) \leqslant \hat{\mu} \inf_{B_\rho(x_0)} u(\cdot, t_0 + \hat{c}u(x_0,t_0)^{2-q}\rho^q)$$

for the specific α chosen in (5.5). If we let $\tilde{c} = \alpha^{-q}\hat{c}$ and $r = \alpha\rho$, we have

$$\begin{aligned} \hat{\mu}^{-1} \sup_{B_r(x_0)} u(\cdot, t_0 - \tilde{c}u(x_0,t_0)^{2-q}r^q) \leqslant u(x_0,t_0) &\leqslant \hat{\mu} \inf_{B_\rho(x_0)} u(\cdot, t_0 + \tilde{c}u(x_0,t_0)^{2-q}r^q) \\ &\leqslant \hat{\mu} \inf_{B_r(x_0)} u(\cdot, t_0 + \tilde{c}u(x_0,t_0)^{2-q}r^q), \end{aligned}$$

which is what we wanted. The condition $(x_0, t_0) + Q_{6\rho}(\hat{\theta}) \subset Q_1^-(1)$ becomes the stated $(x_0, t_0) + Q_{\frac{6}{\alpha}r}(\theta) \subset Q_1^-(1)$. □

We conclude this section with the proof of Proposition 5.1.

Proof of Proposition 5.1. As discussed in Remark 3.5, we may assume that $u(x_0, t_0) > 0$. Let $\mu > 1$ and c be the constants in Theorem 3.4 and let $\hat{c} < c$. We prove (5.1) for $\hat{\mu} := \mu\tilde{\mu}$, where $\tilde{\mu} := (c/\hat{c})^{\frac{1}{2-q}}$. Denote $\hat{t} := t_0 + \hat{c}u(x_0, t_0)^{2-q} r^q$ and let $\hat{x} \in B_r(x_0)$ be an arbitrary point. It now suffices to prove that

$$u(x_0, t_0) \leqslant \tilde{\mu}\mu u(\hat{x}, \hat{t}). \tag{5.8}$$

To this end, we may suppose that $u(x_0, t_0) > \tilde{\mu}u(\hat{x}, \hat{t})$ because otherwise

$$u(\hat{x}, \hat{t}) \geqslant \frac{1}{\tilde{\mu}} u(x_0, t_0) > \frac{1}{\tilde{\mu}\mu} u(x_0, t_0),$$

which would already imply (5.8). Let $[(x_0, t_0), (\hat{x}, \hat{t})]$ be a segment from (x_0, t_0) to $(\hat{x}, \hat{t})$, that is,

$$[(x_0, t_0), (\hat{x}, \hat{t})] := \left\{ \left(x_0 + l\frac{\hat{x} - x_0}{|\hat{x} - x_0|}, t_0 + l\kappa \right) \mid l \in [0, |\hat{x} - x_0|] \right\}, \quad \kappa := \frac{\hat{t} - t_0}{|\hat{x} - x_0|}.$$

We have

$$u(\hat{x}, \hat{t}) < \frac{1}{\tilde{\mu}} u(x_0, t_0) < u(x_0, t_0).$$

Thus, by continuity there exists $(x_1, t_1) \in [(x_0, t_0), (\hat{x}, \hat{t})] \setminus \{(x_0, t_0), (\hat{x}, \hat{t})\}$ such that

$$u(x_1, t_1) = \frac{1}{\tilde{\mu}} u(x_0, t_0). \tag{5.9}$$

Moreover, since (x_1, t_1) lies on the segment, there is $l_1 \in (0, |\hat{x} - x_0|)$ such that

$$(x_1, t_1) = \left(x_0 + l_1 \frac{\hat{x} - x_0}{|\hat{x} - x_0|}, t_0 + l_1\kappa \right).$$

We now have

$$\begin{aligned} |x_1 - \hat{x}| &= \left| x_0 + l_1 \frac{\hat{x} - x_0}{|\hat{x} - x_0|} - x_0 - |\hat{x} - x_0| \frac{\hat{x} - x_0}{|\hat{x} - x_0|} \right| = (|\hat{x} - x_0| - l_1) \\ &= \left(\frac{\hat{t} - t_0}{\kappa} - \frac{t_1 - t_0}{\kappa} \right) = \frac{\hat{t} - t_1}{\kappa}. \end{aligned} \tag{5.10}$$

We set

$$\rho := \left(\frac{\hat{t} - t_1}{cu(x_1, t_1)^{2-q}} \right)^{\frac{1}{q}}$$

because then, since $\varkappa = (\hat{t} - t_0)/|\hat{x} - x_0|$, we obtain using (5.9)

$$\begin{aligned}\frac{\hat{t} - t_1}{\varkappa} &= \rho\frac{(\hat{t} - t_1)^{1-\frac{1}{q}}(cu(x_1, t_1)^{2-q})^{\frac{1}{q}}}{\varkappa}\\ &= \rho|\hat{x} - x_0|\frac{(\hat{t} - t_1)^{1-\frac{1}{q}}(cu(x_1, t_1)^{2-q})^{\frac{1}{q}}}{(\hat{t} - t_0)}\\ &< \rho r\left(\frac{cu(x_1, t_1)^{2-q}}{\hat{t} - t_0}\right)^{\frac{1}{q}}\\ &= \rho r\left(\frac{cu(x_1, t_1)^{2-q}}{\hat{c}u(x_0, t_0)^{2-q}r^q}\right)^{\frac{1}{q}}\\ &= \rho\left(\frac{c}{\hat{c}\tilde{\mu}^{2-q}}\right)^{\frac{1}{q}} = \rho. \end{aligned}\tag{5.11}$$

Combining (5.10) and (5.11) we see that $\hat{x} \in B_\rho(x_1)$. Moreover, by definition of ρ, we have $t_1 + cu(x_1, t_1)^{2-q}\rho^q = \hat{t}$. Consequently, assuming for the moment that we have enough space to apply Theorem 3.4 at (x_1, t_1) for radius ρ, we obtain

$$u(x_1, t_1) \leqslant \mu \inf_{B_\rho(x_1)} u(\cdot, t_1 + cu(x_1, t_1)^{2-q}\rho^q) \leqslant \mu u(\hat{x}, \hat{t}).$$

Hence by (5.9)

$$u(\hat{x}, \hat{t}) \geqslant \frac{1}{\mu}u(x_1, t_1) = \frac{1}{\mu\tilde{\mu}}u(x_0, t_0),$$

as desired.

Since we use Theorem 3.4 at (x_1, t_1), $t_1 > t_0$, we only need to check that the upper boundary of the cylinder $(x_1, t_1) + Q_{4\rho}(\theta)$ is within the domain of the solution. First, by (5.9), we have

$$\begin{aligned}|x_0 - x_1| + 4\rho &\leqslant r + 4\left(\frac{\hat{t} - t_1}{cu(x_1, t_1)^{2-q}}\right)^{\frac{1}{q}} \leqslant r + 4\left(\frac{\hat{t} - t_0}{cu(x_1, t_1)^{2-q}}\right)^{\frac{1}{q}}\\ &= r + 4\left(\frac{\hat{c}u(x_0, t_0)^{2-q}r^q}{cu(x_1, t_1)^{2-q}}\right)^{\frac{1}{q}} = r + 4\left(\frac{\hat{c}}{c}\tilde{\mu}^{2-q}\right)^{\frac{1}{q}}r = 5r.\end{aligned}$$

Further,

$$\begin{aligned}t_1 + cu(x_1, t_1)^{2-q}(4\rho)^q &= t_1 + cu(x_1, t_1)^{2-q}4^q\left(\frac{\hat{t} - t_1}{cu(x_1, t_1)^{2-q}}\right)\\ &= t_1 + 4^q(\hat{t} - t_1)\\ &= (4^q - 1)(t_0 - t_1) + t_0 + 4^q\hat{c}u(x_0, t_0)^{2-q}r^q\\ &\leqslant t_0 + \hat{c}u(x_0, t_0)^{2-q}(5r)^q.\end{aligned}$$

Thus, the upper boundary of $(x_1, t_1) + Q_{4\rho}(\theta)$ is contained in $(x_0, t_0) + Q_{5r}(\hat{\theta}) \subset Q_1^-(1)$. □

6 | PROOF OF THE ELLIPTIC HARNACK'S INEQUALITY

To prove Theorem 2.1, we first establish the following version where more space is required around the point (x_0, t_0). To prove this proposition, we first use the parabolic Harnack Theorem 5.2 to get an estimate at an earlier time level, use Lemma 4.1 to construct a super solution with infinite boundary values at this level and finally use the comparison principle Theorem 3.3 to get an estimate at our original time level. We repeat this process again around a local minimum of u to get the other side of the inequality.

Proposition 6.1. *Let $u \geqslant 0$ be a viscosity solution to (1.1) in $Q_1^-(1)$ and the range condition (2.2) holds. Fix $(x_0, t_0) \in Q_1^-(1)$. Then, there exist $\bar{\gamma} = \bar{\gamma}(n, p, q)$, $c = c(n, p, q)$ and $\alpha = \alpha(n, p, q) \in (0, 1)$ such that*

$$\bar{\gamma}^{-1} \sup_{B_r(x_0)} u(\cdot, t_0) \leqslant u(x_0, t_0) \leqslant \bar{\gamma} \inf_{B_r(x_0)} u(\cdot, t_0), \tag{6.1}$$

whenever $(x_0, t_0) + Q_{\frac{13}{\alpha}r}(\theta) \subset Q_1^-(1)$ where

$$\theta = cu(x_0, t_0)^{2-q}.$$

Proof. We can use parabolic Harnack (Theorem 5.2) for radius $2r$ to obtain constants $\mu = \mu(n, p, q)$ and $c = c(n, p, q)$ such that

$$u(x, t_0 - \theta(2r)^q) \leqslant \sup_{B_{2r}(x_0)} u(\cdot, t_0 - \theta(2r)^q) \leqslant \mu u(x_0, t_0) \tag{6.2}$$

for all $x \in B_{2r}(x_0)$, where $\theta = cu(x_0, t_0)^{2-q}$. This is justified because $\frac{6}{\alpha}(2r) < \frac{13}{\alpha}r$. Let

$$v(x, t) := \lambda(t - t_0 + \theta(2r)^q)^{\frac{1}{2-q}} \left(\frac{1}{(2r)^{\frac{1}{1-q}}((2r)^{\frac{q}{q-1}} - |x - x_0|^{\frac{q}{q-1}})} \right)^{\frac{q}{2-q}} + \mu u(x_0, t_0).$$

Then by Lemma 4.1, v is a viscosity supersolution in $B_{2r}(x_0) \times (t_0 - \theta(2r)^q, \infty)$ that satisfies

$$\begin{cases} v \geqslant \mu u(x_0, t_0) & \text{on } B_{2r}(x_0) \times \{t_0 - \theta(2r)^q\}, \\ \lim\limits_{\Omega_T \ni (x,t) \to (y,s)} v(x, t) = \infty & \text{for all } (y, s) \in \partial B_{2r}(x_0) \times (t_0 - \theta(2r)^q, \infty) \end{cases}$$

and we can use comparison principle Theorem 3.3 to get

$$u \leqslant v \text{ in } (x_0, t_0) + Q_{2r}(\theta) \tag{6.3}$$

because u is bounded in $(x_0, t_0) + Q_{2r}(\theta)$ and on the bottom of the cylinder we have by (6.2)

$$u(x, t_0 - \theta(2r)^q) \leqslant \mu u(x_0, t_0) \leqslant v(x, t_0 - \theta(2r)^q).$$

The estimate (6.3) and the definition of θ imply in particular that

$$
\begin{aligned}
\sup_{B_r(x_0)} u(\cdot,t_0) \leqslant \sup_{B_r(x_0)} v(\cdot,t_0) &= \lambda(\theta(2r)^q)^{\frac{1}{2-q}}\left((2r)^{\frac{1}{1-q}}((2r)^{\frac{q}{q-1}}-r^{\frac{q}{q-1}})\right)^{\frac{q}{q-2}}+\mu u(x_0,t_0)\\
&= \lambda(cu(x_0,t_0)^{2-q}2^q r^q)^{\frac{1}{2-q}}\left(r2^{\frac{1}{1-q}}(2^{\frac{q}{q-1}}-1)\right)^{\frac{q}{q-2}}+\mu u(x_0,t_0)\\
&= \left(\lambda c^{\frac{1}{2-q}}2^{\frac{q}{2-q}}\left(2^{\frac{1}{1-q}}(2^{\frac{q}{q-1}}-1)\right)^{\frac{q}{q-2}}+\mu\right)u(x_0,t_0)\\
&=: \bar{\gamma}(n,p,q)u(x_0,t_0). \qquad (6.4)
\end{aligned}
$$

Dividing by $\bar{\gamma}$ gives us the left-hand side of (6.1). The constant $\bar{\gamma}$ blows up in the limit cases because λ blows up when $q\to 2$ for all c, and μ does the same when q approaches the lower bound of (2.2).

Let $\hat{x}$ be a minimum point of $u(\cdot,t_0)$ in $\overline{B}_r(x_0)$. We will again use Theorem 5.2 to obtain

$$\sup_{B_{2r}(\hat{x})} u(\cdot,t_0-\hat{\theta}(2r)^q)\leqslant \mu u(\hat{x},t_0),$$

where $\hat{\theta}=c(u(\hat{x},t_0))^{2-q}$. The use of Harnack is justified because $\frac{6}{\alpha}(2r)+r<\frac{13}{\alpha}r$ because $\alpha\in(0,1)$. Let

$$\hat{v}(x,t)=\lambda(t-t_0+\hat{\theta}(2r)^q)^{\frac{1}{2-q}}\left(\frac{1}{(2r)^{\frac{1}{1-q}}((2r)^{\frac{q}{q-1}}-|x-\hat{x}|^{\frac{q}{q-1}})}\right)^{\frac{q}{2-q}}+\mu u(\hat{x},t_0).$$

Then again by Lemma 4.1, $\hat{v}$ is a viscosity supersolution in $B_{2r}(\hat{x})\times(t_0-\hat{\theta}(2r)^q,\infty)$ that satisfies

$$
\begin{cases}
\hat{v}\geqslant \mu u(\hat{x},t_0) & \text{on } B_{2r}(\hat{x})\times\{t_0-\hat{\theta}(2r)^q\},\\
\lim\limits_{\Omega_T\ni(x,t)\to(y,s)}\hat{v}(x,t)=\infty & \text{for all } (y,s)\in\partial B_{2r}(\hat{x})\times(t_0-\hat{\theta}(2r)^q,\infty)
\end{cases}
$$

and we can use comparison principle Theorem 3.3 to get

$$u\leqslant \hat{v} \text{ in } (\hat{x},t_0)+Q_{2r}(\hat{\theta})$$

and thus

$$
\begin{aligned}
u(x_0,t_0)\leqslant \sup_{B_r(\hat{x})}u(\cdot,t_0)\leqslant \sup_{B_r(\hat{x})}\hat{v}(\cdot,t_0) &= \left(\lambda c^{\frac{1}{2-q}}2^{\frac{q}{2-q}}\left(2^{\frac{1}{1-q}}(2^{\frac{q}{q-1}}-1)\right)^{\frac{q}{q-2}}+\mu\right)u(\hat{x},t_0)\\
&= \bar{\gamma}(n,p,q)\inf_{B_r(x)}u(\cdot,t_0), \qquad (6.5)
\end{aligned}
$$

which is the right-hand side of (6.1). Combining (6.4) and (6.5) proves the theorem. □

Next we combine Proposition 6.1 with a covering argument to prove Theorem 2.1. We first construct a suitable sequence of small balls along an arbitrary radial segment of our set. Then, we show by induction that there is enough room around cylinders defined on these balls to use Proposition 6.1 to get an Harnack-type estimate over any of these radial segments up arbitralily close to the boundary. Parabolic intrinsic Harnack chains for the p-parabolic equation have recently been examined in [1] in the degenerate case $p > 2$.

Proof of Theorem 2.1. By Proposition 6.1, there exist constants $\bar{\gamma}(n,p,q)$, $c'(n,p,q)$ and $\alpha(n,p,q) \in (0,1)$ such that the elliptic Harnack's inequality

$$\bar{\gamma}^{-1} \sup_{B_\tau(z)} u(\cdot, t_0) \leqslant u(z, t_0) \leqslant \bar{\gamma} \inf_{B_\tau(z)} u(\cdot, t_0) \tag{6.6}$$

holds whenever $B_{\frac{13}{\alpha}\tau}(z) \subset B_1$ and

$$t_0 \pm \left(\frac{13}{\alpha}\tau\right)^q c' u(z, t_0)^{2-q} \in (-1, 0]. \tag{6.7}$$

Fix an arbitrary $\hat{y} \in \partial B_r(x_0)$. Let $\rho := r\alpha(\sigma - 1)/13$. We define the points

$$y_k := x_0 + k\rho \frac{\hat{y} - x_0}{|\hat{y} - x_0|} \in B_r(x_0),$$

where $k = 0, \ldots, K$ and $K \geqslant 0$ is the smallest natural number such that $\hat{y} \in B_\rho(y_K)$. Since $\hat{y}$ is on the boundary of $B_r(x_0)$ and ρ is a scaling of r, the number K depends only on σ, n, p and q. We will apply the elliptic Harnack's inequality in the balls $B_\rho(y_k)$. Therefore, we need the corresponding intrinsic cylinders to be contained within $Q_1^-(1)$. Since the choice of ρ ensures that $B_{\frac{13}{\alpha}\rho}(y) \subset B_{\sigma r}(x_0) \subset B_1$ whenever $y \in B_r(x_0)$, it remains to show that (6.7) holds for $\tau = \rho$ and $z = y_k$, $k = 0, \ldots, K$. We choose

$$c := c'\left(\frac{\sigma - 1}{\sigma}\right)^q \bar{\gamma}^{K(2-q)}$$

and proceed by induction to check that we have enough space in the time direction to use Proposition 6.1 for each of the cylinders $(y_k, t_0) + Q_\rho(c'u(y_k, t_0)^{2-q})$. Note that the assumption $(x_0, t_0) + Q_{\sigma r}(\theta) \subset Q_1^-(1)$ implies

$$t_0 \pm (\sigma r)^q c u(x_0, t_0)^{2-q} \in (-1, 0]. \tag{6.8}$$

(Initial step) Since $\bar{\gamma} \geqslant 1$, we have

$$\begin{aligned} \left(\frac{13}{\alpha}\rho\right)^q c' u(y_0, t_0)^{2-q} = (r(\sigma - 1))^q c' u(x_0, t_0)^{2-q} &= (\sigma r)^q c u(x_0, t_0)^{2-q} \frac{c'(\sigma - 1)^q}{c\sigma^q} \\ &\leqslant (\sigma r)^q c u(x_0, t_0)^{2-q}. \end{aligned} \tag{6.9}$$

It follows from (6.9) and (6.8) that (6.7) holds with $z = y_0$ and $\tau = \rho$. Thus, the elliptic Harnack inequality (6.6) gives

$$\bar{\gamma}^{-1} \sup_{B_\rho(y_0)} u(\cdot, t_0) \leqslant u(x_0, t_0) \leqslant \bar{\gamma} \inf_{B_\rho(y_0)} u(\cdot, t_0).$$

(Induction step) Suppose that $1 \leqslant k \leqslant K$ and that we have

$$\bar{\gamma}^{-k} \sup_{B_\rho(y_{k-1})} u(\cdot, t_0) \leqslant u(x_0, t_0) \leqslant \bar{\gamma}^k \inf_{B_\rho(y_{k-1})} u(\cdot, t_0). \tag{6.10}$$

Since $y_k \in \overline{B}_\rho(y_{k-1})$, this implies in particular

$$u(y_k, t_0) \leqslant \bar{\gamma}^k u(x_0, t_0).$$

Therefore, by definition of ρ and c we have

$$\begin{aligned} \left(\frac{13}{\alpha}\rho\right)^q c' u(y_k, t_0)^{2-q} &\leqslant (r(\sigma-1))^q c' \bar{\gamma}^{k(2-q)} u(x_0, t_0)^{2-q} \\ &= (\sigma r)^q c u(x_0, t_0)^{2-q} \frac{c'(\sigma-1)^q \bar{\gamma}^{k(2-q)}}{c\sigma^q} \\ &\leqslant (\sigma r)^q c u(x_0, t_0)^{2-q}. \end{aligned} \tag{6.11}$$

It follows from (6.11) and (6.8) that (6.7) holds for $z = y_k$ and $\tau = \rho$. Consequently by the elliptic Harnack's inequality (6.6), we have

$$\bar{\gamma}^{-1} \sup_{B_\rho(y_k)} u(\cdot, t_0) \leqslant u(y_k, t_0) \leqslant \bar{\gamma} \inf_{B_\rho(y_k)} u(\cdot, t_0).$$

Since $y_k \in \overline{B}_\rho(y_{k-1})$, combining the above display with (6.10) yields

$$u(x_0, t_0) \geqslant \bar{\gamma}^{-k} \sup_{B_\rho(y_{k-1})} u(\cdot, t_0) \geqslant \bar{\gamma}^{-k} u(y_k, t_0) \geqslant \bar{\gamma}^{-(k+1)} \sup_{B_\rho(y_k)} u(\cdot, t_0)$$

and similarly

$$u(x_0, t_0) \leqslant \bar{\gamma}^k \inf_{B_\rho(y_{k-1})} u(\cdot, t_0) \leqslant \bar{\gamma}^k u(y_k, t_0) \leqslant \bar{\gamma}^{k+1} \inf_{B_\rho(y_k)} u(\cdot, t_0).$$

Thus,

$$\bar{\gamma}^{-(k+1)} \sup_{B_\rho(y_k)} u(\cdot, t_0) \leqslant u(x_0, t_0) \leqslant \bar{\gamma}^{k+1} \inf_{B_\rho(y_k)} u(\cdot, t_0) \tag{6.12}$$

and the induction step is complete.

By the induction principle, the estimate (6.12) holds for all $k = 0, \dots, K$. Since $\hat{y} \in B_\rho(y_K)$, we have in particular

$$\bar{\gamma}^{-(K+1)} \sup_{[x,\hat{y}]} u(\cdot, t_0) \leqslant u(x_0, t_0) \leqslant \bar{\gamma}^{K+1} \inf_{[x,\hat{y}]} u(\cdot, t_0),$$

where $[x, \hat{y}]$ denotes the segment from x to $\hat{y}$. Since $\hat{y} \in \partial B_r(x_0)$ was arbitrary, the estimate of the theorem follows for $\gamma := \bar{\gamma}^{K+1}$. □

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REFERENCES

1. B. Avelin, T. Kuusi, and K. Nyströ, *Boundary behavior of solutions to the parabolic p-Laplace equation*, Anal. PDE **12** (2019), no. 1, 1–42.
2. M. Bonforte, R. G. Iagar, and J. L. Vázquez, *Local smoothing effects, positivity, and Harnack inequalities for the fast p-Laplacian equation*, Adv. Math. (N. Y). **224** (2010), no. 5, 2151–2215.
3. H. O. Cordes, *Über die erste Randwertaufgabe bei quasilinearen Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen*, Math. Ann. **131** (1956), no. 3, 278–312.
4. E. DiBenedetto, *Intrinsic Harnack type inequalities for solutions of certain degenerate parabolic equations*, Arch. Ration. Mech. Anal. **100** (1988), no. 2, 129–147.
5. E. DiBenedetto, *Degenerate parabolic equations*, Universitext. Springer, New York, 1993.
6. E. DiBenedetto, U. Gianazza, and V. Vespri, *Harnack estimates for quasi-linear degenerate parabolic differential equations*, Acta Math. **200** (2008), no. 2, 181–209.
7. E. DiBenedetto, U. Gianazza, and V. Vespri, *Alternative forms of the Harnack inequality for non-negative solutions to certain degenerate and singular parabolic equations*, Rend. Lincei - Mat. E Appl. **20** (2009), 369–377.
8. E. DiBenedetto, U. Gianazza, and V. Vespri, *Forward, backward and elliptic Harnack inequalities for non-negative solutions to certain singular parabolic partial differential equations*, Ann. della Sc. Norm. - Cl. di Sci. **9** (2010), no. 2, 385–422.
9. E. DiBenedetto, U. Gianazza, and V. Vespri, *Harnack's inequality for degenerate and singular parabolic equations*, vol. 165, Springer, New York, 2012.
10. E. DiBenedetto and Y. C. Kwong, *Harnack estimates and extinction profile for weak solutions of certain singular parabolic equations*, Trans. Am. Math. Soc. **330** (1992), no. 2, 783–811.
11. G. Díaz and R. Letelier, *Explosive solutions of quasilinear elliptic equations: existence and uniqueness*, Nonlinear Anal. Theory, Methods Appl. **20** (1993), no. 2, 97–125.
12. H. Dong, F. Peng, Y. R.-Y. Zhang, and Y. Zhou, *Hessian estimates for equations involving p-Laplacian via a fundamental inequality*, Adv. Math. (N. Y). **370** (2020), 40.
13. R. K. Giri, J. Kinnunen, and K. Moring, *Supercaloric functions for the parabolic p-Laplace equation in the fast diffusion case*, NoDEA Nonlinear Differential Equations Appl. **28** (2021), no. 3.
14. F. A. Høeg and P. Lindqvist, *Regularity of solutions of the parabolic normalized p-Laplace equation*, Adv. Nonlinear Anal. **9** (2019), no. 1, 7–15.
15. C. Imbert, T. Jin, and L. Silvestre, *Hölder gradient estimates for a class of singular or degenerate parabolic equations*, Adv. Nonlinear Anal. **8** (2019), no. 1, 845–867.
16. H. Ishii and P. Souganidis, *Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor*, Tôhoku Math. J. **47** (1995), 227–250.
17. T. Jin and L. Silvestre, *Hölder gradient estimates for parabolic homogeneous p-Laplacian equations*, J. Math. Pures Appl. **108** (2017), no. 1, 63–87.
18. P. Juutinen, P. Lindqvist, and J. J. Manfredi, *On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation*, SIAM J. Math. Anal. **33** (2001), no. 3, 699–717.
19. N. V. Krylov and M. V. Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, Math. USSR Izv. **16** (1981), no. 1, 151–164.
20. T. Kuusi, *Harnack estimates for weak supersolutions to nonlinear degenerate parabolic equations*, Ann. della Sc. Norm. Super. di Pisa - Cl. di Sci. **7** (2008), no. 4, 673–716.
21. E. M. Landis, Second order equations of elliptic and parabolic type, *Translations of Mathematical Monographs*, vol. 171, American Mathematical Society, Providence, RI, 1998. Translated from the 1971 Russian original by Tamara Rozhkovkaya, With a preface from Nina Uraltseva.

22. J. Moser, *A Harnack inequality for parabolic differential equations*, Commun. Pure Appl. Math. **17** (1964), no. 1, 101–134.
23. J. J. Manfredi, M. Parviainen, and J. D. Rossi, *An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games*, SIAM J. Math. Anal. **42** (2010), no. 5, 2058–2081.
24. J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Am. J. Math. **80** (1958), no. 4, 931.
25. M. Ohnuma and K. Sato, *Singular degenerate parabolic equations with applications*, Commun. Partial Differ. Equations **22** (1997), no. 3-4, 381–441.
26. M. Parviainen and J. L. Vázquez, *Equivalence between radial solutions of different parabolic gradient-diffusion equations and applications*, Ann. della Sc. Norm. Super. Di Pisa - Cl. di Sci. **21** (2020), no. Special Issue, 303–359.
27. J. Siltakoski, *Equivalence of viscosity and weak solutions for a p-parabolic equation*, J. Evol. Equations **21** (2021), no. 2, 2047–2080.

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Boundary regularity for a general nonlinear parabolic equation in non-divergence form

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BOUNDARY REGULARITY FOR A GENERAL NONLINEAR PARABOLIC EQUATION IN NON-DIVERGENCE FORM

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ABSTRACT. We characterize regular boundary points in terms of a barrier family for a general form of a parabolic equation that generalizes both the standard parabolic p-Laplace equation and the normalized version arising from stochastic game theory. Using this result we prove geometric conditions that ensure regularity by constructing suitable barrier families. We also prove that when $q < 2$, a single barrier does not suffice to guarantee regularity.

1. INTRODUCTION

We examine the boundary regularity for the following general non-divergence form version of the nonlinear parabolic equation

$$\partial_t u = |\nabla u|^{q-p} \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = |\nabla u|^{q-2}(\Delta u + (p-2)\Delta_\infty^N u), \tag{1.1}$$

where $q > 1$ and $p > 1$. When $q = p$, this reduces to the usual p-parabolic equation and when $q = 2$, we get the normalized p-parabolic equation arising from stochastic tug-of-war games.

A boundary point is called *regular* with respect to a partial differential equation if all solutions to the Dirichlet problem with continuous boundary values attain their boundary values continuously. Thus a given Dirichlet problem in a set is solvable in the classical sense if and only if all boundary points of the set are regular. Our main result is that the existence of a family of barrier functions at a point is equivalent to that point being regular. There are geometric conditions that imply the existence of barrier families and thus also imply boundary regularity by this characterization. We also show that the existence of a single barrier is not enough when $q < 2$. This problem remains open when $q > 2$. A key idea we use in the proofs is a radial connection to the p-parabolic case. The radial solutions of (1.1) solve in a suitable sense a weighted one dimensional q-parabolic equation as proven by Parviainen and Vázquez [PV20]. Thus the radial barrier functions and sets used in the proofs are similar between these two equations. Using barrier constructions, we prove that the exterior ball condition and two other geometric conditions guarantee boundary regularity.

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Key words and phrases. regular boundary point, barrier, Perron's method, viscosity solutions, nonlinear equation, p-parabolic equation, exterior ball condition.

Parabolic boundary regularity is delicate. Petrovskiĭ criterion for the one-dimensional heat equation, presented in [Pet34] and proven in [Pet35], shows that a boundary point that is regular for the equation

$$\partial_t u = \Delta u$$

turns out to be irregular for the multiplied equation

$$2\partial_t u = \Delta u.$$

However surprisingly boundary points remain regular for all multiplied p-parabolic equation when $p \neq 2$ as proven in [BBGP15]. We prove that similar phenomenon happens for equation (1.1) when $q \neq 2$. When $q = 2$, any multiple of a solution is also a solution and thus existence of a barrier implies the existence of a barrier family. It would seem that this might suggest that one barrier is not enough when $q > 2$, but we have not yet found a counterexample or a proof for the contrary.

Characterizing regular boundary points for different equations has a long history. One of the most celebrated of these is the original Wiener criterion proven by Norbert Wiener in 1924 [Wie24] for the Laplace equation. Wiener type criterion for the heat equation was proven by Evans and Gariepy [EG82] but remains open for the usual p-parabolic equation. For equations of p-parabolic type, the approach using barrier functions has proven fruitful. These seem to date back to Poincaré [Poi90] but were named by Lebesgue in [Leb24] where he characterizes regularity for the Laplace equation using barriers. Granlund, Lindqvist, and Martio extended the Perron method and established the barrier approach to the elliptic p-Laplacian in their paper [GLM86] and developed it in various papers. An overview of the elliptic results is given in the book by Heinonen, Kilpeläinen, and Martio [HKM06]. The theory for the p-parabolic case was initiated by Kilpeläinen and Lindqvist [KL96] where they established the parabolic Perron method and suggested a barrier approach. Björn, Björn, Gianazza, and Parviainen characterized boundary regularity using a family of barriers in [BBGP15] and proved that a single barrier does not suffice for singular exponents in [BBG17]. A single barrier turns out to be enough for the normalized p-parabolic equation as shown by Bannerjee and Garofalo [BG14]. Björn, Björn, and Parviainen proved a tusk condition and a Petrovskiĭ criterion for this equation in [BBP19]. Boundary regularity for the porous medium equation was examined in [BBGS18].

Since equation (1.1) is in non-divergence form except in the special case, we will use the concept of viscosity solutions. A suitable definition taking account potential singularities was established by Ohnuma and Sato in [OS97]. The normalized p-parabolic equation arises from game theory which was first examined in the parabolic setting in [MPR10]. This problem has attained recent interest for example in [JS17], [HL19], [DPZZ20] and [AS22] in addition to already mentioned [BBP19]. The general form of (1.1) has been examined for example in [IJS19],[PV20],[KPS23] and [KS23].

The structure of the paper is as follows. In Section 2 we present a suitable definition of viscosity solutions to equation (1.1) that takes into account the potential singularity of the equation and state some known results we need later. In Section 3 we present and prove an elliptic-type comparison principle for equation (1.1). In Section 4 we define Perron solutions and prove some basic properties. Sections 5 and 6 consist of defining boundary regularity and barriers and proving our main result. We also prove that regularity is a local property and show by a counterexample that a single barrier is not enough to prove regularity. In Section 7, we establish the exterior ball condition and a few other geometric conditions by constructing suitable barrier families. In Section 8, we analyze the connection of our definition for a barrier and the ones appearing in the literature for other equations.

2. Prerequisites

In this paper, we denote the dimension by n and let $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^{n+1}$ be open and bounded sets. Denote $\Omega_T = \Omega \times (0,T)$ and $\Omega_{t_1,t_2} = \Omega \times (t_1,t_2)$ the spacetime cylinders and a parabolic boundary by

$$\partial_p \Omega_{t_1,t_2} = (\Omega \times \{t_1\}) \cup (\partial\Omega \times [t_1,t_2]).$$

We denote the Euclidean ball of radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$ by $B_r(x_0)$ and Q_r denotes the scaled cylinder

$$Q_r = B_r(0) \times (-r^2, 0].$$

For $\xi \in \mathbb{R}^{n+1}$ and $A \subset \mathbb{R}^{n+1}$, we denote

$$\xi + A = \{\xi + a \mid a \in A\}.$$

When $\nabla u \neq 0$, we denote

$$\Delta_p^q u = |\nabla u|^{q-p} \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = |\nabla u|^{q-2}(\Delta u + (p-2)\Delta_\infty^N u), \tag{2.1}$$

where $p > 1$ and $q > 1$ are real parameters and the normalized or game theoretic infinity Laplace operator is given by

$$\Delta_\infty^N u = \sum_{i,j=1}^n \frac{\partial_{x_i} u\, \partial_{x_j} u\, \partial_{x_i x_j} u}{|\nabla u|^2}.$$

Thus equation (1.1) can be written as

$$\partial_t u = \Delta_p^q u.$$

Denote

$$F(\eta, X) = |\eta|^{q-2} \operatorname{Tr}\left(X + (p-2)\frac{\eta \otimes \eta}{|\eta|^2} X\right) \tag{2.2}$$

where $(a \otimes b)_{ij} = a_i b_j$, so that

$$F(\nabla u, D^2 u) = |\nabla u|^{q-2}(\Delta u + (p-2)\Delta_\infty^N u) = \Delta_p^q u$$

whenever $\nabla u \neq 0$. This F is *degenerate elliptic*, meaning that

$$F(\eta, X) \leq F(\eta, Y)$$

for all $\eta \in \mathbb{R}^n \setminus \{0\}$ and $X \leq Y$.

We will need to restrict the class of test functions in the definition of a viscosity solution to deal with the singularity of the equation near critical points. Let $\mathcal{F}(F)$ be the set of functions $f \in C^2([0,\infty))$ such that

$$f(0) = f'(0) = f''(0) = 0 \text{ and } f''(r) > 0 \text{ for all } r > 0,$$

and also require that for $g(x) = f(|x|)$, it holds that

$$\lim_{\substack{x\to 0\\ x\neq 0}} F(\nabla g(x), D^2 g(x)) = 0.$$

This set $\mathcal{F}(F)$ is never empty because it is easy to see that $f(r) = r^\beta \in \mathcal{F}(F)$ for any $\beta > \max\left(\frac{q}{q-1}, 2\right)$. Note also that if $f \in \mathcal{F}(F)$, then $\lambda f \in \mathcal{F}(F)$ for all $\lambda > 0$.

Additionally define the set

$$\Sigma = \{\sigma \in C^1(\mathbb{R}) \mid \sigma \text{ is even}, \sigma(0) = \sigma'(0) = 0, \text{ and } \sigma(r) > 0 \text{ for all } r > 0\}.$$

We use $\mathcal{F}(F)$ and Σ to define an admissible set of test functions for viscosity solutions.

Definition 2.1. *A function $\varphi \in C^2(\Theta)$ is admissible at a point $(x_0, t_0) \in \Theta$ if either $\nabla\varphi(x_0, t_0) \neq 0$ or there are $\delta > 0$, $f \in \mathcal{F}(F)$ and $\sigma \in \Sigma$ such that*

$$|\varphi(x,t) - \varphi(x_0,t_0) - \partial_t\varphi(x_0,t_0)(t-t_0)| \leq f(|x-x_0|) + \sigma(t-t_0),$$

for all $(x,t) \in B_\delta(x_0) \times (t_0-\delta, t_0+\delta)$.

Note that by definition a function φ is automatically admissible at a point (x_0, t_0) if either $\nabla\varphi(x_0, t_0) \neq 0$ or the function $-\varphi$ is admissible at a point (x_0, t_0).

Definition 2.2. *A function $u : \Theta \to \mathbb{R} \cup \{\infty\}$ is a viscosity supersolution to*

$$\partial_t u = \Delta_p^q u \quad \textit{in } \Theta$$

if the following three conditions hold.

(1) u is lower semicontinuous,
(2) u is finite in a dense subset of Θ,
(3) whenever an admissible $\varphi \in C^2(\Theta)$ touches u at $\xi \in \Theta$ from below, we have

$$\begin{cases} \partial_t\varphi(\xi) - \Delta_p^q\varphi(\xi) \geq 0 & \textit{if } \nabla\varphi(\xi) \neq 0,\\ \partial_t\varphi(\xi) \geq 0 & \textit{if } \nabla\varphi(\xi) = 0.\end{cases}$$

A function $u : \Theta \to \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution if $-u$ is a viscosity supersolution. A function $u : \Theta \to \mathbb{R}$ is a viscosity solution if it is a supersolution and a subsolution.

Note that if no admissible test function exists at a point ξ, the last condition is automatically satisfied. If $q \geq 2$, then viscosity solutions can be defined in a standard way by using semicontinuous extensions, see Proposition 2.2.8 in [Gig06].

Our proofs use two different comparison principles. The first is the standard parabolic comparison principle, which is proven as Theorem 3.1 in [OS97]. Here we assume that the solutions are ordered on the parabolic boundary of the set.

Theorem 2.3. *Suppose that u is a viscosity supersolution and v is a viscosity subsolution to (1.1) in Ω_T. If*

$$\infty \neq \limsup_{\Omega_T \ni (y,s) \to (x,t)} v(y,s) \leq \liminf_{\Omega_T \ni (y,s) \to (x,t)} u(y,s) \neq -\infty$$

for all $(x,t) \in \partial_p \Omega_T$, then $v \leq u$ in Ω_T.

The second is the elliptic-type comparison principle which holds for arbitrary Θ as long as we compare over the entire Euclidean boundary. We state and prove this in the next section.

Consider the Dirichlet problem

$$\begin{cases} \partial_t u = |\nabla u|^{q-p} \operatorname{div}\Big(|\nabla u|^{p-2} \nabla u\Big) & \text{in } \Omega_T \\ u = g & \text{on } \partial_p \Omega_T. \end{cases} \tag{2.3}$$

We have the following existence and uniqueness results for simple space-time cylinders.

Theorem 2.4. *Let $g \in C(\partial_p B_T)$. Then there exists a unique viscosity solution $u \in C(\overline{B}_T)$ to (2.3) with $\Omega_T = B_T$.*

This theorem follows from [Gig06, Theorem 2.4.9] and falls into the general framework studied by Ohnuma and Sato in [OS97, Section 4], where they prove the existence for the Cauchy problem in Corollary 4.10.

Imbert, Jin, and Silvestre proved the $C^{1,\alpha}$-regularity for solutions to (1.1). The time estimate we need follows from [IJS19, Lemma 3.1] and the space estimate follows from [IJS19, Corollary 2.4]. Combining these we get the following corollary.

Corollary 2.5. *Let u be a viscosity solution to (1.1) in Q_4 and $\alpha \in (0,1)$. Then there exists a two positive constants C_1 and C_2 depending only on n, p, q and $||u||_{L^\infty(Q_4)}$ such that*

$$|u(x,t) - u(y,s)| \leq C_1 |x-y|^\alpha + C_2 |t-s|^{\frac{1}{2}}$$

for all $(x,t), (y,s) \in Q_1$.

Various regularity results for the non-homogeneous version of (1.1) were proven by Attouchi in [Att20] and Attouchi and Ruosteenoja in [AR20].

In our proofs, we need the following stability result which is a special case of Theorem 5.2. in [IJS19] which follows from Theorem 6.1 in [OS97]. We provide a proof by modifying the proof used for the p-parabolic equation, see [KL96, Lemma 3.4].

Lemma 2.6. *Suppose that $(u_i)_{i=1}^\infty$ is a locally uniformly bounded sequence of viscosity solutions to (1.1) in Θ. Then there exists a subsequence that converges locally uniformly in Θ to a viscosity solution u.*

Proof. The proof is based on a diagonalization argument. Let $(K_i)_{i=1}^{\infty}$ be a sequence of compact sets in Θ such that $K_i \subset K_{i+1}$ for all i and

$$\bigcup_{i=1}^{\infty} K_i = \Theta.$$

Let $\Xi_i = \{\xi_1, \xi_2, \dots\}$ be the set of points with rational coordinates in K_i and define

$$d_i = \frac{d(K_i, \partial\Theta)}{5}$$

for every i. For each i, define the family of sets

$$U_i = \{\xi_j + Q_{d_i} \mid \xi_j \in \Xi_i\} = \{B_{d_i}(x_j) \times (t_j - d_i^2, 0] \mid (x_j, t_j) \in \Xi_i\}.$$

The family U_i is a countable cover of K_i and by compactness and construction has a finite subcover V_i formed over some finite index set $Z_i \subset \Xi_i$.

Because each u_k is a viscosity solution in each $\xi_j + Q_{d_i}$ and we chose d_i to have enough space around the set, we can use Hölder continuity result Corollary 2.5 to get the estimate

$$|u_k(x,t) - u_k(y,t)| \leq C|x-y|^{\alpha} \tag{2.4}$$

for any $(x,t), (y,t) \in \xi_j + Q_{d_i}$, $\alpha_{\xi_j} \in (0,1)$ and $k \in \mathbb{N}$, where $C = C(n, p, q, ||u_k||_{L^\infty(\xi_j + Q_{4d_i})})$. Because we assume that the sequence $(u_i)_{i=1}^{\infty}$ is locally uniformly bounded, we can pick a constants C independent of k and by taking maximum over all such C and α, we get that

$$|u_k(x,t) - u_k(y,t)| \leq \hat{C}|x-y|^{\hat{\alpha}} \tag{2.5}$$

holds for every k and some $\hat{C}$ and $\hat{\alpha}$. This estimate now holds for every $(x,t), (y,t) \in \bigcup_{\xi_j \in Z_i}(\xi_j + Q_{d_i})$ so especially in $\overline{K}_i$. Similarly by using Corollary 2.5 on each set and picking constants gets us the estimate

$$|u_k(x,t) - u_k(x,s)| \leq \hat{C}|t-s|^{\frac{1}{2}} \tag{2.6}$$

for every $(x,t), (x,s) \in \bigcup_{\xi_j \in Z_i}(\xi_j + Q_{d_i})$ so especially again in $\overline{K}_i$.

Estimates (2.5) and (2.6) give us that the sequence $(u_k)_{k=1}^{\infty}$ is equicontinuous with respect to both space and time in $\overline{K}_i$. Let $(u_k^i)_{k=1}^{\infty}$ be the subsequence given by Arzelà-Ascoli theorem that converges into a continuous function u^i in K_i. Define a new sequence $(v_k)_{k=1}^{\infty}$ such that $v_k = u_k^k$ for all k. Now v_k has a subsequence that converges locally uniformly in Θ to some continuous function u.

Let us show that u is a viscosity solution. Let $B_{t_1,t_2} \Subset \Theta$ for a ball B. Let v be a viscosity solution in B_{t_1,t_2}, continuous on $\overline{B}_{t_1,t_2}$ and taking the boundary values $v = u$ on $\partial_p B_{t_1,t_2}$. Such v exists by Theorem 2.4.

By convergence for any $\varepsilon > 0$

$$v - \varepsilon = u - \varepsilon < u_k < u + \varepsilon = v + \varepsilon$$

on $\partial_p B_{t_1,t_2}$ for large enough k. By comparison principle Theorem 2.3, we get

$$v - \varepsilon \leq u_k \leq v + \varepsilon$$

in B_{t_1,t_2} for each large k and thus taking the limit as $k \to \infty$ gives us

$$v - \varepsilon \leq u \leq v + \varepsilon$$

in B_{t_1,t_2}. Letting $\varepsilon \to 0$ gives us that u is a viscosity solution in B_{t_1,t_2} because the solution v given by Theorem 2.4 is unique. □

Later we prove that Perron solutions are actually viscosity solutions and for this proof, we need the concept of parabolic modification.

Definition 2.7. *Let $B_{t_1,t_2} \Subset \Theta$ and u be a viscosity supersolution to (1.1) in Θ and bounded in Ω_T. We define the parabolic modification of u in B_{t_1,t_2} as*

$$U = \begin{cases} v & \text{in } B_{t_1,t_2}, \\ u & \text{in } \Theta \setminus B_{t_1,t_2}, \end{cases}$$

where

$$v(\xi) = \sup\{h(\xi) \mid h \in C(\overline{B}_{t_1,t_2}) \text{ is a viscosity solution to (1.1) and } h \leq u \text{ on } \partial_p B_{t_1,t_2}\}.$$

Clearly $U \leq u$ in Θ because by comparison principle Theorem 2.3, each $h \leq u$ in B_{t_1,t_2} and thus also $v \leq u$ in B_{t_1,t_2}.

Lemma 2.8. *Let $B_{t_1,t_2} \Subset \Theta$ and u be a viscosity supersolution to (1.1) in Θ and bounded in B_{t_1,t_2}. Then the parabolic modification U is a viscosity supersolution in Θ and a viscosity solution in B_{t_1,t_2}.*

Proof. Let $(\theta_i)_{i=1}^{\infty}$ be an increasing sequence of continuous functions on $\partial_p B_{t_1,t_2}$ such that

$$u = \lim_{i \to \infty} \theta_i$$

on $\partial_p B_{t_1,t_2}$. By Theorem 2.4, there exists a sequence $(h_i)_{i=1}^{\infty}$ of viscosity solutions on B_{t_1,t_2} such that h_i coincides with θ_i on $\partial_p B_{t_1,t_2}$. Using the comparison principle Theorem 2.3 pairwise for each h_i, we get that the sequence $(h_i)_{i=1}^{\infty}$ is increasing on $\overline{B}_{t_1,t_2}$ and that the limit function is v in the definition of the parabolic modification. Moreover, since the sequence is bounded, the limit function v is also a viscosity solution to (1.1) in B_{t_1,t_2} by Lemma 2.6. The function U is a viscosity supersolution in $\Theta \setminus \overline{B}_{t_1,t_2}$ by definition, so only the boundary of two sets is left.

Let φ be an admissible test function touching U from below at $\xi \in \partial B_{t_1,t_2}$. We have $\varphi(\xi) = U(\xi)$ and $\varphi < U$ in some neighborhood V of ξ. This φ is also an admissible test function touching u from below at ξ because $\varphi(\xi) = U(\xi) = u(\xi)$ and $\varphi(\zeta) < U(\zeta) \leq u(\zeta)$ for all $\zeta \in V$. Because u is a viscosity supersolution in the entire Θ, we necessarily have

$$\begin{cases} \partial_t \varphi(\xi) - \Delta_p^q \varphi(\xi) \geq 0 & \text{if } \nabla \varphi(\xi) \neq 0, \\ \partial_t \varphi(\xi) \geq 0 & \text{if } \nabla \varphi(\xi) = 0. \end{cases}$$

which now implies that condition (3) of Definition 2.2 holds for U at ξ. Thus U is a viscosity solution in B_{t_1,t_2} and a viscosity supersolution in the entire Θ. □

3. Elliptic-type comparison principle

The comparison principle for general bounded open sets Θ has not been proven for equation (1.1) before and we need it for a few proofs in this paper. Hence we will provide a proof using the doubling of variables method and the Theorem on sums which is the standard strategy often used to prove comparison principle for viscosity solutions of equations of this type.

Theorem 3.1. *Suppose that u is a viscosity supersolution and v is a viscosity subsolution to (1.1) in Θ. If*

$$\infty \neq \limsup_{\Theta \ni (y,s) \to (x,t)} v(y,s) \leq \liminf_{\Theta \ni (y,s) \to (x,t)} u(y,s) \neq -\infty$$

for all $(x,t) \in \partial\Theta$, then $v \leq u$ in Θ.

Before the proof, we will define notation and prove some lemmas we need.

Lemma 3.2. *Assume $\varphi \in C^2(\Theta)$ is an admissible test function at $(x_0,t_0) \in \Theta$ and let $T = \sup\{t \in \mathbb{R} \mid (x,t) \in \Theta\}$. If $\nabla\varphi(x_0,t_0) = 0$, then*

$$\psi(x,t) = \varphi(x,t) + \frac{\gamma}{T-t}$$

is also admissible at (x_0,t_0) for all $\gamma > 0$.

Proof. We have $\partial_t\psi(x,t) = \partial_t\varphi(x,t) + \frac{\gamma}{(T-t)^2}$. Because we assumed that φ is admissible at (x_0,t_0), there exists a $\delta > 0$, $f \in \mathcal{F}(F)$ and $\sigma_1 \in \Sigma$ such that

$$\begin{aligned}
&|\psi(x,t) - \psi(x_0,t_0) - \partial_t\psi(x_0,t_0)(t-t_0)| \\
&= \left|\varphi(x,t) + \frac{\gamma}{T-t} - \varphi(x_0,t_0) - \frac{\gamma}{T-t_0} - \left(\partial_t\varphi(x_0,t_0) + \frac{\gamma}{(T-t_0)^2}\right)(t-t_0)\right| \\
&\leq f(|x-x_0|) + \sigma_1(t-t_0) + \left|\frac{\gamma}{T-t} - \frac{\gamma}{T-t_0} - \frac{\gamma(t-t_0)}{(T-t_0)^2}\right| \\
&\leq f(|x-x_0|) + \sigma_1(t-t_0) + \big|h(t) - h(t_0) - h'(t_0)(t-t_0)\big|,
\end{aligned} \tag{3.1}$$

for all $(x,t) \in B_\delta(x_0) \times (t_0-\delta, t_0+\delta) \subset \Theta$. Here

$$h(t) = \frac{\gamma}{T-t},$$

which is smooth for $t \in (t_0-\delta, t_0+\delta)$ as this interval does not contain T. By Taylor's theorem using the Lagrange form for the remainder, there exists $c \in (t,t_0)$ such that

$$h(t) = h(t_0) + h'(t_0)(t-t_0) + \frac{h''(c)}{2}(t-t_0)^2.$$

Because $h''(x)$ is bounded in $(t_0-\delta, t_0+\delta)$, we can estimate the last term of (3.1) by

$$\big|h(t) - h(t_0) - h'(t_0)(t-t_0)\big| = \left|\frac{h''(c)}{2}(t-t_0)^2\right| \leq \sup_{c \in (t_0-\delta, t_0+\delta)} h''(c)(t-t_0)^2 = \sigma_2(t-t_0).$$

This σ_2 is even, satisfies $\sigma_2(0) = \sigma_2'(0) = 0$ and $\sigma_2(r) > 0$ for all $r > 0$ and thus $\sigma_2 \in \Sigma$. Combining this with estimate (3.1), we have

$$|\psi(x,t) - \psi(x_0,t_0) - \partial_t\psi(x_0,t_0)(t-t_0)| \leq f(|x-x_0|) + \sigma(t-t_0), \tag{3.2}$$

for all $(x,t) \in B_\delta(x_0) \times (t_0-\delta, t_0+\delta)$. Here $f \in \mathcal{F}(F)$ and $\sigma = \sigma_1 + \sigma_2 \in \Sigma$ and thus ψ is admissible at point (x_0,t_0). □

Next we will define some notation used in the proof of the comparison principle. Let $T = \sup\{t \in \mathbb{R} \mid (x,t) \in \Theta\}$, $\varepsilon > 0$, $\gamma > 0$ and $f \in \mathcal{F}(F)$ and define

$$\Phi(x,t,y,s) = v(x,t) - u(y,s) - \Psi(x,t,y,s) \tag{3.3}$$

where

$$\Psi(x,t,y,s) = \frac{1}{\varepsilon}f(|x-y|) + \frac{1}{\varepsilon}(t-s)^2 + \frac{\gamma}{T-s} + \frac{\gamma}{T-t}. \tag{3.4}$$

By our assumptions for u and v, the function $v(x,t) - u(y,s)$ is upper semicontinuous and bounded from above by some constant M in $\overline{\Theta} \times \overline{\Theta}$. Thus it attains its maximum in this set and by continuity of Ψ, so does the function Φ.

Lemma 3.3. *Let $\xi_\varepsilon = (x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$ be the point where Φ attains its maximum in $\overline{\Theta} \times \overline{\Theta}$ and assume $v(x_\varepsilon,t_\varepsilon) - u(x_\varepsilon,t_\varepsilon) = \theta > 0$. Then there exists a constant $\gamma_0 > 0$, such that*

$$\lim_{\varepsilon\to 0}|x_\varepsilon - y_\varepsilon| = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}|t_\varepsilon - s_\varepsilon| = 0 \tag{3.5}$$

for all $\gamma < \gamma_0$. There also exists a constant $\varepsilon_0 = \varepsilon_0(\gamma_0) > 0$, such that $\xi_\varepsilon \in \Theta \times \Theta$ for all $\varepsilon < \varepsilon_0$.

Proof. We will first show a lower bound for $\Phi(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$. If we choose γ_0 to satisfy

$$\frac{2\gamma_0}{T-t_0} \leq \frac{\theta}{2},$$

we have that

$$\Phi(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon) \geq \Phi(x_\varepsilon, t_\varepsilon, x_\varepsilon, t_\varepsilon) = \theta - \frac{2\gamma}{T-t_0} \geq \frac{\theta}{2}, \tag{3.6}$$

for all $\gamma < \gamma_0$. By equation (3.6), we also have

$$\Psi(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon) < v(x_\varepsilon, t_\varepsilon) - u(y_\varepsilon, s_\varepsilon). \tag{3.7}$$

Because we took $f \in \mathcal{F}(F)$, we know it is necessarily monotone increasing in $\mathbb{R}^+$ by definition. Thus there exists a monotone increasing inverse function $f^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$. Using this and the fact that

$$f(|x-y|) \leq \varepsilon\Psi(x,t,y,s),$$

we can conclude by inequality (3.7)

$$|x_\varepsilon - y_\varepsilon| = f^{-1}(f(|x_\varepsilon - y_\varepsilon|)) \leq f^{-1}(\varepsilon\Psi(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)) \leq f^{-1}(\varepsilon M)$$

and

$$|t_\varepsilon - s_\varepsilon| \leq (\varepsilon\Psi(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon))^{\frac{1}{2}} \leq (\varepsilon M)^{\frac{1}{2}}.$$

Taking limits as $\varepsilon \to 0$ on both sides, these together imply

$$\lim_{\varepsilon\to 0}|x_\varepsilon - y_\varepsilon| = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}|t_\varepsilon - s_\varepsilon| = 0 \tag{3.8}$$

for all $\gamma < \gamma_0$.

We have all the tools we need to show that $\xi_0 \in \Theta \times \Theta$. Thriving for a contradiction, assume that ε_0 stated in the theorem does not exist. Then necessarily there exists sequences $(\varepsilon_i)_{i=1}^\infty$ and $(\gamma_i)_{i=1}^\infty \subset (0, \gamma_0)$, such that $\lim_{i\to\infty} \varepsilon_i = 0$ and Φ defined with $\varepsilon = \varepsilon_i$ and $\gamma = \gamma_i$ attains its maximum at a point $(x_i, t_i, y_i, s_i) \in \partial(\Theta \times \Theta)$. Because $\partial(\Theta \times \Theta)$ is compact and (3.8) holds, there exists a point $(\hat{x}, \hat{t}) \in \partial\Theta$, such that

$$\lim_{i\to\infty} x_i = \lim_{i\to\infty} y_i = \hat{x} \quad \text{and} \quad \lim_{i\to\infty} t_i = \lim_{i\to\infty} s_i = \hat{t}$$

passing to a subsequence if necessary. But now estimate (3.7) implies

$$\begin{aligned}
0 &< \limsup_{i\to\infty} \Psi(x_i, t_i, y_i, s_i) \\
&\le \limsup_{i\to\infty} (v(x_i, t_i) - u(y_i, s_i)) \\
&\le \limsup_{i\to\infty} v(x_i, t_i) - \liminf_{i\to\infty} u(y_i, s_i) \\
&\le \limsup_{\Theta \ni (y,s) \to (\hat{x}, \hat{t})} v(y, s) - \liminf_{\Theta \ni (y,s) \to (\hat{x}, \hat{t})} u(y, s) \le 0,
\end{aligned}$$

where we used our assumption about the functions on the boundary. This contradiction proves that there exists a $\varepsilon_0 > 0$, such that $\xi_0 \in \Theta \times \Theta$ for all $\varepsilon < \varepsilon_0$. □

Now we are ready to prove the elliptic-type comparison principle.

Proof of Theorem 3.1. Assume thriving for a contradiction that there exists a $(x_\varepsilon, t_\varepsilon) \in \Theta$, such that

$$v(x_\varepsilon, t_\varepsilon) - u(x_\varepsilon, t_\varepsilon) = \theta > 0.$$

Let Φ and Ψ be defined as before in (3.3) and (3.4) and let $\xi_\varepsilon = (x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$ be the point where Φ attains its maximum in $\overline{\Theta} \times \overline{\Theta}$. Note that this point depends on ε and δ. By Lemma 3.3, there exists constants γ_0 and ε_0 such that $\xi_\varepsilon \in \Theta \times \Theta$ for all $\varepsilon < \varepsilon_0$ and

$$\lim_{\varepsilon\to 0}|x_\varepsilon - y_\varepsilon| = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}|t_\varepsilon - s_\varepsilon| = 0 \tag{3.9}$$

for all $\gamma < \gamma_0$. Let

$$\varphi^+(x, t) = \frac{1}{\varepsilon} f(|x - y_\varepsilon|) + \frac{1}{\varepsilon}(t - s_\varepsilon)^2 + \frac{\gamma}{T - t},$$

and

$$\varphi^-(y, s) = -\frac{1}{\varepsilon} f(|x_\varepsilon - y|) - \frac{1}{\varepsilon}(t_\varepsilon - s)^2 - \frac{\gamma}{T - s}.$$

For every $\varepsilon < \varepsilon_0$ and $\gamma < \gamma_0$, there are two possible cases. First if $x_\varepsilon = y_\varepsilon$, we have $\nabla\varphi^+(x_\varepsilon, t_\varepsilon) = \nabla\varphi^-(y_\varepsilon, s_\varepsilon) = 0$. These are admissible test functions at points $(x_\varepsilon, t_\varepsilon)$ and

$(y_\varepsilon, s_\varepsilon)$ respectively by Lemma 3.2. The function φ^+, adding a constant if necessary, touches v from above at $(x_\varepsilon, t_\varepsilon)$, and hence by the definition of a viscosity subsolution, we have

$$\partial_t \varphi^+(x_\varepsilon, t_\varepsilon) = \frac{2}{\varepsilon}(t_\varepsilon - s_\varepsilon) + \frac{\gamma}{(T - t_\varepsilon)^2} \le 0.$$

Similarly, φ^- with a possible added constant touches u from below at $(y_\varepsilon, s_\varepsilon)$, and hence by the definition of a viscosity supersolution, we have

$$\partial_t \varphi^-(y_\varepsilon, s_\varepsilon) = \frac{2}{\varepsilon}(t_\varepsilon - s_\varepsilon) - \frac{\gamma}{(T - s_\varepsilon)^2} \ge 0.$$

Hence

$$0 \le \partial_t \varphi^-(y_\varepsilon, s_\varepsilon) - \partial_t \varphi^+(x_\varepsilon, t_\varepsilon) = -\frac{\gamma}{(T - s_\varepsilon)^2} - \frac{\gamma}{(T - t_\varepsilon)^2} < 0,$$

which is a contradiction.

In the second case, we have $x_\varepsilon \ne y_\varepsilon$. For such ε and γ, we have $\nabla \varphi^+(x_\varepsilon, t_\varepsilon) \ne 0$ and $\nabla \varphi^-(y_\varepsilon, s_\varepsilon) \ne 0$. We denote parabolic superjet by $\mathcal{P}^{2,+}$ and subjet by $\mathcal{P}^{2,-}$ and their closures by $\overline{\mathcal{P}}^{2,-}$ and $\overline{\mathcal{P}}^{2,-}$ respectively. For definitions of these, we direct the reader to see [CIL92]. We can use elliptic Theorem on sums in dimension $n+1$, see for example [Koi04, Lemma 3.6], with [OS97, Lemma 3.5]. By this, we conclude that there exist matrixes $X, Y \in S^n$, such that

$$(\partial_t \Psi(\xi_\varepsilon), \nabla_x \Psi(\xi_\varepsilon), X) \in \overline{\mathcal{P}}^{2,+} v(x_\varepsilon, t_\varepsilon) \text{ and } (-\partial_s \Psi(\xi_\varepsilon), -\nabla_y \Psi(\xi_\varepsilon), Y) \in \overline{\mathcal{P}}^{2,-} u(y_\varepsilon, s_\varepsilon),$$

and $X \le Y$. Notice that because we assumed $x_\varepsilon \ne y_\varepsilon$, we have

$$\nabla_x \Psi(\xi_0) = -\nabla_y \Psi(\xi_0) \ne 0 \tag{3.10}$$

and $F(\eta, X)$ is continuous in some neighborhood and we do not have to worry about the admissibility of test functions. Thus the viscosity solutions can be equivalently defined using semijets in this neighborhood, see [CIL92]. Since u is a viscosity supersolution, this definition with (3.10) implies

$$\begin{aligned} 0 &\le -\partial_s \Psi(\xi_\varepsilon) - F(-\nabla_y \Psi(\xi_\varepsilon), Y) = -\left(-\frac{2}{\varepsilon}(t_\varepsilon - s_\varepsilon) + \frac{\gamma}{(T - s_\varepsilon)^2}\right) - F(\nabla_x \Psi(\xi_\varepsilon), Y) \\ &= \frac{2}{\varepsilon}(t_\varepsilon - s_\varepsilon) - \frac{\gamma}{(T - s_\varepsilon)^2} - F(\nabla_x \Psi(\xi_\varepsilon), Y) \end{aligned} \tag{3.11}$$

and because v is a subsolution

$$0 \ge \partial_t \Psi(\xi_\varepsilon) - F(\nabla_x \Psi(\xi_\varepsilon), X) = \frac{2}{\varepsilon}(t_\varepsilon - s_\varepsilon) + \frac{\gamma}{(T - t_\varepsilon)^2} - F(\nabla_x \Psi(\xi_\varepsilon), X). \tag{3.12}$$

Subtracting (3.11) from (3.12), we get using degenerate ellipticity of F

$$\begin{aligned} 0 &\ge \frac{\gamma}{(T - t_\varepsilon)^2} + \frac{\gamma}{(T - s_\varepsilon)^2} + F(\nabla_x \Psi(\xi_\varepsilon), Y) - F(\nabla_x \Psi(\xi_\varepsilon), X) \\ &\ge \frac{\gamma}{(T - t_\varepsilon)^2} + \frac{\gamma}{(T - s_\varepsilon)^2} \\ &> 0. \end{aligned}$$

Both cases lead to a contradiction and this concludes the proof. ☐

4. Perron solutions

One common way of solving the Dirichlet problem in arbitrary domains is the Perron method. For our uses, it is enough to consider bounded domains with bounded boundary data. The idea is to construct an upper solution to the Dirichlet problem as a point-wise infimum over a suitable class of supersolutions. We prove that for bounded boundary data, this construction gives us a viscosity solution inside the set. We use the notation from [BBGP15] for Perron solutions.

Definition 4.1. *Let $f : \partial\Theta :\to \mathbb{R}$ be a bounded function. The upper class $\mathcal{U}_f$ is defined to be the class of all viscosity supersolutions u to equation (1.1) in Θ which are bounded from below and such that*

$$\liminf_{\Theta \ni \eta \to \xi} u(\eta) \geq f(\xi) \quad \text{for all } \xi \in \partial\Theta. \tag{4.1}$$

The upper Perron solution of f is defined as

$$\overline{H} f(\xi) = \inf_{u \in \mathcal{U}_f} u(\xi), \quad \xi \in \Theta.$$

Similarly, the lower class $\mathcal{L}_f$ is defined to be the class of all viscosity subsolutions u to equation (1.1) in Θ which are bounded from above and such that

$$\limsup_{\Theta \ni \eta \to \xi} u(\eta) \leq f(\xi) \quad \text{for all } \xi \in \partial\Theta, \tag{4.2}$$

and define the lower Perron solution of f by

$$\underline{H} f(\xi) = \sup_{u \in \mathcal{L}_f} u(\xi), \quad \xi \in \Theta.$$

Note that $\mathcal{L}_f$ and $\mathcal{U}_f$ are always non-empty for bounded f. We in fact have $u \in \mathcal{U}_f$ for any constant function larger than $\sup_{\xi \in \partial\Theta} f(\xi)$ and similar result for the lower class. It is also clear that for bounded f, the definition of Perron solutions does not change if we restrict $\mathcal{L}_f$ and $\mathcal{U}_f$ only to bounded functions.

In the next theorem, we prove that Perron solutions are in fact viscosity solutions. This result is quite classical and a similar proof works for different equations as it only uses the stability result, parabolic modification, and basic properties of Perron solutions. We provide a proof for the convenience of the reader.

Theorem 4.2. *If the boundary function $f : \partial\Theta \to \mathbb{R}$ is bounded, then Perron solutions $\overline{H} f$ and $\underline{H} f$ are viscosity solutions to (1.1) in Θ.*

Proof. We mainly follow the argument in Kilpeläinen and Lindqvist in [KL96, Theorem 5.1]. Fix a space-time cylinder $B_{t_1,t_2} \Subset \Theta$. Choose a countable, dense subset

$$\Xi = \{\xi_1, \xi_2, \dots\}$$

of B_{t_1,t_2}. For each $j = 1, 2, \dots$, we choose a sequence of functions $u_{i,j} \in \mathcal{U}_f$ with $i = 1, 2, \dots$, such that

$$\lim_{i\to\infty} u_{i,j}(\xi_j) = \overline{H} f(\xi_j).$$

We may assume that each $u_{i,j}$ is bounded. Now define

$$v_{i,j}(\xi) = \min_{1\leq m\leq j}\{u_{i,m}(\xi)\}$$

for each j and i. The minimum of two viscosity supersolutions is also a viscosity supersolution by standard arguments and by iterating this, we get that each $v_{i,j}$ is a viscosity supersolution to (1.1) in Θ and $v_{i,j} \in \mathcal{U}_f$. By definition $v_{i,j}(\xi) \geq \overline{H}f(\xi)$ for all i and j and now by construction $v_{i,k}(\xi) \geq v_{i,j}(\xi)$ for each $k = 1, 2, \ldots, j$. Thus for these indexes, we have

$$\overline{H}f(\xi_k) \leq v_{i,j}(\xi_k) \leq v_{i,k}(\xi_k)$$

and taking limits as $i \to \infty$, we get that for any j, the sequence we now have satisfies

$$\lim_{i\to\infty} v_{i,j}(\xi_k) = \overline{H}f(\xi_k) \tag{4.3}$$

for each $k = 1, 2, \ldots, j$.

Let $V_{i,j}$ be the parabolic modification of $v_{i,j}$ in B_{t_1,t_2} according to Definition 2.7. Now

$$\overline{H}f \leq V_{i,j} \leq v_{i,j}$$

by definition and $V_{i,j}$ is a viscosity solution in B_{t_1,t_2}.

By passing to a subsequence if necessary, we get from Lemma 2.6 that for any j, the sequence $(V_{i,j})_{i=1}^{\infty}$ converges locally uniformly to a viscosity solution v_j in B_{t_1,t_2}. Again by Lemma 2.6, the sequence $(v_j)_{j=1}^{\infty}$ has a subsequence that converges locally uniformly to a viscosity solution h in B_{t_1,t_2}. By the construction, it holds

$$h \geq \overline{H}f$$

in B_{t_1,t_2} and by equation (4.3), the equality $h = \overline{H}f$ holds in the dense subset $\Xi \subset B_{t_1,t_2}$. Take $v \in \mathcal{U}_f$ and let V be its parabolic modification in B_{t_1,t_2}. By definition $v \geq V$ and $V \geq \overline{H}f$ in B_{t_1,t_2}. Also because $h = \overline{H}f$ in a dense subset, we have by continuity of V and h $v \geq V \geq h$ in B_{t_1,t_2} and thus taking infimum over all $v \in \mathcal{U}_f$, we get

$$h \leq \overline{H}f$$

in B_{t_1,t_2}. It follows that $\overline{H}f = h$, so it is a viscosity solution for any cylinder B_{t_1,t_2} and thus in Θ. The lower Perron solution $\underline{H}f$ is treated analogously. □

To finish this section, we will prove the so-called pasting lemma that plays a key role in our proofs. It is kind of similar to the parabolic modification we used before but defined for arbitrary open sets. This is a useful tool when constructing suitable new viscosity supersolutions to be used as barriers.

Lemma 4.3. *(Pasting lemma) Let $G \subset \Theta$ be open. Also let u and v be viscosity supersolutions to* (1.1) *in Θ and G respectively, and let*

$$w = \begin{cases} \min\{u, v\} & \text{in } G, \\ u & \text{in } \Theta \setminus G. \end{cases}$$

If w is lower semicontinuous, then w is a viscosity supersolution to (1.1) *in Θ.*

Proof. By assumption, w is lower semicontinuous and by construction w is finite in a dense subset of Θ. Because the minimum of two viscosity supersolutions is a viscosity supersolution by standard arguments, we only need to verify the condition (3) of the Definition 2.2 for $\xi \in \partial G$.

Let φ be an admissible test function touching w from below at $\xi \in \partial G$. We have $\varphi(\xi) = w(\xi)$ and $\varphi < w$ in some neighborhood V of ξ. This φ is also an admissible test function touching u from below at ξ because $\varphi(\xi) = w(\xi) = u(\xi)$ and $\varphi(\zeta) < w(\zeta) \leq u(\zeta)$ for all $\zeta \in V$. Because u is a viscosity supersolution, we necessarily have

$$\begin{cases} \partial_t \varphi(\xi) - \Delta_p^q \varphi(\xi) \geq 0 & \text{if } \nabla \varphi(\xi) \neq 0, \\ \partial_t \varphi(\xi) \geq 0 & \text{if } \nabla \varphi(\xi) = 0. \end{cases}$$

which now implies that condition (3) holds for w at ξ. Thus w is a viscosity supersolution in the entire Θ. □

5. Barriers and boundary regularity

In this section, we define regular boundary points and barrier functions and prove how barriers can be used to characterize boundary regularity. It turns out that a boundary point is regular if and only if there exists a family of barrier functions. We show by a counterexample that a single barrier does not suffice when $q < 2$ and this remains an open problem for $q > 2$. This problem is open even for the usual p-parabolic equation where the barrier approach has been examined in [BBGP15] and [KL96]. We start with definitions.

Definition 5.1. *A boundary point $\xi_0 \in \partial\Theta$ is regular to equation (1.1) if*

$$\liminf_{\Theta \ni \xi \to \xi_0} \overline{H} f(\xi) = f(\xi_0)$$

for every $f : C(\partial\Theta) \to \mathbb{R}$. If the set is ambiguous from the context, we will specify that a point is regular with respect to the set Θ.

Since $\underline{H} f = -\overline{H}(-f)$, regularity can be equivalently defined using lower Perron solutions.

Next, we will define barriers and barrier families.

Definition 5.2. *Let $\xi_0 \in \partial\Theta$. A function $w : \Theta \to (0, \infty]$ is a barrier to (1.1) in Θ at point ξ_0 if*

(a) *w is a positive viscosity supersolution to equation (1.1) in Θ,*
(b) $\liminf_{\Theta \ni \zeta \to \xi_0} w(\zeta) = 0$*,*
(c) $\liminf_{\Theta \ni \zeta \to \xi} w(\zeta) > 0$ *for $\xi \in \partial\Theta \setminus \{\xi_0\}$.*

We define barriers to be viscosity supersolutions which is not the standard definition in the recent literature, where barriers are often defined through the comparison principle. These two definitions are equivalent as we will prove in Lemma 8.2.

Definition 5.3. *Let $\xi_0 \in \partial\Theta$. A family of functions $w_j : \Theta \to (0, \infty]$, $j = 1, 2, \dots,$ is a barrier family to (1.1) in Θ at point ξ_0 if for each j,*

(a) *w_j is positive viscosity supersolution to equation (1.1) in Θ,*
(b) $\liminf_{\Theta \ni \zeta \to \xi_0} w_j(\zeta) = 0$,
(c) *for each $k = 1, 2, \dots,$ there is a j such that*

$$\liminf_{\Theta \ni \zeta \to \xi} w_j(\zeta) \geq k \quad \text{for all } \xi \in \partial\Theta \text{ with } |\xi - \xi_0| \geq \frac{1}{k}.$$

We say that the family w_j is a strong barrier family in Θ at the point ξ_0 if, in addition, the following conditions hold:

(d) *w_j is continuous in Θ,*
(e) *there is a non-negative function $d \in C(\overline{\Theta})$, with $d(z) = 0$ if and only if $z = \xi_0$ such that for each $k = 1, 2, \dots,$ there is a $j = j(k)$ such that $w_j \geq kd$ in Θ.*

In the following lemma, we will evaluate the operators for a prototype barrier function. Most barriers we use consist of sums of these functions and thus these formulas make our calculations in the proofs easier.

Lemma 5.4. *Let $C > 0$, $\beta \in \mathbb{R}$ and*

$$v(x,t) = C|x|^{\frac{q}{q-1}} t^{\beta}.$$

Then for all $t > 0$, we have

$$\Delta_p^q v(x,t) = \Big(C\frac{q}{q-1}\Big)^{q-1}\Big(n + \frac{p-q}{q-1}\Big)t^{\beta(q-1)}.$$

and

$$\partial_t v(x,t) = C\beta|x|^{\frac{q}{q-1}} t^{\beta-1}.$$

Proof. The proof is a direct calculation but is included for the convenience of the reader. Denote $\alpha = \frac{q}{q-1}$. We have

$$\nabla v(x,t) = Ct^{\beta}\alpha|x|^{\alpha-2}x$$

and thus

$$\begin{aligned}
\Delta_p v(x,t) &= \operatorname{div}\Big(|\nabla v(x,t)|^{p-2}\nabla v(x,t)\Big) \\
&= \operatorname{div}\Big(\Big|Ct^{\beta}\alpha|x|^{\alpha-2}x\Big|^{p-2} Ct^{\beta}\alpha|x|^{\alpha-2}x\Big) \\
&= \Big(Ct^{\beta}\alpha\Big)^{p-1}\operatorname{div}\Big(|x|^{(\alpha-1)(p-2)+\alpha-2}x\Big) \\
&= \Big(Ct^{\beta}\alpha\Big)^{p-1}\sum_{i=1}^{n}\Big([(\alpha-1)(p-2)+\alpha-2]|x|^{(\alpha-1)(p-2)+\alpha-4}x_i^2 + |x|^{(\alpha-1)(p-2)+\alpha-2}\Big) \\
&= \Big(Ct^{\beta}\alpha\Big)^{p-1}(n + [(\alpha-1)(p-1)-1])|x|^{(\alpha-1)(p-1)-1}.
\end{aligned}$$

It also follows that

$$\begin{aligned}
\Delta_p^q v(x,t) &= |\nabla v(x,t)|^{q-p} \operatorname{div}\Big(|\nabla v(x,t)|^{p-2}\nabla v(x,t)\Big) \\
&= \Big|Ct^\beta \alpha |x|^{\alpha-2}x\Big|^{q-p}\Big(Ct^\beta\alpha\Big)^{p-1}(n+[(\alpha-1)(p-2)-\alpha-2])|x|^{(\alpha-1)(p-2)+\alpha-2} \\
&= \Big(Ct^\beta\alpha\Big)^{q-1}(n+[(\alpha-1)(p-2)+\alpha-2])|x|^{(\alpha-1)(p-2)+\alpha-2+(\alpha-1)(q-p)} \\
&= \Big(Ct^\beta\alpha\Big)^{q-1}(n+[(\alpha-1)(p-1)-1])|x|^{(\alpha-1)(q-1)-1}. \qquad (5.1)
\end{aligned}$$

Finally

$$(\alpha-1)(q-1)-1 = \Big(\frac{q}{q-1}-1\Big)(q-1)-1 = 0$$

and

$$(\alpha-1)(p-1)-1 = \Big(\frac{q}{q-1}-1\Big)(p-1)-1 = \frac{p-q}{q-1}.$$

Substituting these and α into (5.1), we get exactly what was stated. The time derivative is clear. □

In the next theorem, we prove that regularity of a boundary point is characterized by a barrier family existing at that point. This is our main tool when considering geometric approaches to characterizing regularity.

Theorem 5.5. *Let $\xi_0 \in \partial\Theta$. The point ξ_0 is regular if and only if there exists a barrier family at ξ_0.*

Proof. First, assume that there exists a barrier family at $\xi_0 \in \partial\Theta$. Take continuous function $f \in C(\partial\Theta)$. By continuity, for each $\varepsilon > 0$ there exists a constant $\delta > 0$ such that $|f(\xi) - f(\xi_0)| < \varepsilon$ whenever $|\xi - \xi_0| < \delta$, $\xi \in \partial\Theta$. Thus if $|\xi - \xi_0| < \delta$, we get that

$$f(\xi) - f(\xi_0) - \varepsilon < 0 \le \liminf_{\Theta\ni\zeta\to\xi} w_j(\zeta) \qquad (5.2)$$

because w_j are assumed to be positive. If $|\xi - \xi_0| \ge \delta$, we pick $k \in \mathbb{N}$ such that

$$k > f(\xi) - f(\xi_0) - \varepsilon \quad \text{ and } \quad \delta \ge \frac{1}{k}.$$

Now by Definition 5.3 condition (c) we know that there exists a j such that

$$\liminf_{\Theta\ni\zeta\to\xi} w_j(\zeta) \ge k > f(\xi) - f(\xi_0) - \varepsilon. \qquad (5.3)$$

Combining estimates (5.2) and (5.3), we get that for some $j \ge 1$

$$\liminf_{\Theta\ni\zeta\to\xi} w_j(\zeta) + f(\xi_0) + \varepsilon > f(\xi) \quad \text{ for all } \xi \in \partial\Theta.$$

Thus because this is also a supersolution, we have $w_j + f(\xi_0) + \varepsilon \in \mathcal{U}_f$, and hence

$$\limsup_{\Theta\ni\zeta\to\xi} \overline{H} f(\zeta) \le \liminf_{\Theta\ni\zeta\to\xi} w_j(\zeta) + f(\xi_0) + \varepsilon = f(\xi_0) + \varepsilon. \qquad (5.4)$$

By similar calculation as above, $-w_j - \varepsilon + f(\xi_0) \in \mathcal{L}_f$ and we obtain that for some j

$$\liminf_{\Theta \ni \zeta \to \xi} \overline{H} f(\zeta) \geq \liminf_{\Theta \ni \zeta \to \xi} \underline{H} f(\zeta) \geq -\varepsilon + f(\varepsilon_0), \tag{5.5}$$

where the first inequality follows from the fact that $\overline{H} f \geq \underline{H} f$ in Θ. This is because we can use the elliptic-type comparison principle Theorem 3.1 for any pair of $u \in \mathcal{U}_f$ and $v \in \mathcal{L}_f$ in the definitions of Perron solutions. Letting $\varepsilon \to 0$, combining (5.4) and (5.5) gives us that ξ_0 is regular.

For the other direction, let us assume that $\xi_0 \in \partial\Theta$ is regular. Without loss of generality, we may assume that ξ_0 is the origin. For all $(x, t) \in \mathbb{R}^{n+1}$ we define

$$\psi_j(x, t) = j \frac{q-1}{q} |x|^{\frac{q}{q-1}} + j^{q-1} \frac{n + \frac{p-q}{q-1}}{2 \operatorname{diam} \Theta} t^2$$

By Lemma 5.4, we have

$$\Delta_p^q \psi_j(x, t) = \left(j \frac{q-1}{q} \frac{q}{q-1} \right)^{q-1} \left[n + \frac{p-q}{q-1} \right]$$

and

$$\partial_t \psi_j(x, t) = j^{q-1} \frac{n + \frac{p-q}{q-1}}{\operatorname{diam} \Theta} t.$$

Thus

$$\partial_t \psi_j(x, t) - \Delta_p^q \psi_j(x, t) = j^{q-1} \left[n + \frac{p-q}{q-1} \right] \left(\frac{t}{\operatorname{diam} \Theta} - 1 \right) \leq 0$$

for all $(x, t) \in \Theta$ making ψ_j a subsolution. We will verify that $w_j = \underline{H} \psi_j$ gives us a barrier family at ξ_0 by checking the conditions from Definition 5.3. We have

(a): From $\psi_j \geq 0$, it follows also that $w_j \geq 0$. Because the set Θ is bounded, we get

$$\psi_j(x, t) \leq j \frac{q-1}{q} \operatorname{diam} \Theta^{\frac{q}{q-1}} + j^{q-1} \frac{cn}{2} \operatorname{diam} \Theta < \infty$$

for every j. Thus w_j is a viscosity supersolution by Theorem 4.2.

(b): Follows directly from the regularity of ξ_0 because $\psi_j(\xi_0) = 0$ for all j.

(c): Because ψ_j is a viscosity subsolution bounded above by itself on the boundary, we have $\psi_j \in \mathcal{L}_{\psi_j}$ and thus by definition $\underline{H} \psi_j \geq \psi_j$. Using this, let $k = 1, 2, \ldots$, and pick $r = \frac{1}{k}$. For any $\xi = (x, t) \in \Theta \setminus B_r(\xi_0)$, we have by continuity of ψ_j,

$$\liminf_{\Theta \ni \zeta \to \xi} w_j(\zeta) \geq \liminf_{\Theta \ni \zeta \to \xi} \psi_j(\zeta) \geq \psi_j(\xi) \geq j \frac{q-1}{q} r^{\frac{q}{q-1}} + j^{q-1} \frac{n + \frac{p-q}{q-1}}{2 \operatorname{diam} \Theta} r^2 \geq k,$$

where the last inequality holds for large enough j. This implies condition (c) of Definition 5.3. □

In some cases, the additional conditions satisfied by strong barrier families prove useful in practice. Note that condition (e) gives information over the entire $\overline{\Theta}$ and implies condition (c), which gives information only over $\partial\Theta$. It turns out that the existence of one type of barrier family implies the other.

Proposition 5.6. *Let $\xi_0 \in \partial\Theta$. There exists a barrier family at ξ_0 if and only if there exists a strong barrier family at ξ_0.*

Proof. A strong barrier family satisfies conditions $(a) - (c)$ by definition and thus is also a barrier family. We will prove the other direction.

Assume that there exists a barrier family at ξ_0. By Theorem 5.5, the point ξ_0 is regular. Just as in the proof of Theorem 5.5, we define

$$\psi_j(x,t) = j\frac{q-1}{q}|x|^{\frac{q}{q-1}} + j^{q-1}\frac{n+\frac{p-q}{q-1}}{2\operatorname{diam}\Theta}t^2$$

for all $(x,t) \in \mathbb{R}^{n+1}$ and let $w_j = \underline{H}\psi_j$. We will prove that w_j in fact forms a strong barrier family. In the proof of Theorem 5.5, we already proved conditions $(a) - (c)$.

- (d): Because w_j is a viscosity solution by Theorem 4.2, it is continuous for every j.
- (e): Let

  $$d(x,t) = \frac{q-1}{q}|x|^{\frac{q}{q-1}} + \frac{n+\frac{p-q}{q-1}}{2\operatorname{diam}\Theta}t^2. \tag{5.6}$$

  Notice that $\psi_j(x,t) \geq \min(j, j^{q-1})d(x,t)$. This is continuous and non-negative and $d(x,t) = 0$ if and only if $(x,t) = (0,0)$. Pick any $k \in \mathbb{N}$. Now by picking $j > \max\{k, k^{\frac{1}{q-1}}\}$, we get

  $$w_j = \underline{H}\psi_j \geq \psi_j \geq \min\{j, j^{q-1}\}d(x,t) \geq kd(x,t)$$

  as desired. □

We get the following restriction result that turns out to be useful in later proofs as a direct corollary of Theorem 5.5.

Corollary 5.7. *Let $\xi_0 \in \partial\Theta$ and let $G \subset \Theta$ be open and such that $\xi_0 \in \partial G$. If ξ_0 is regular with respect to Θ, then ξ_0 is regular with respect to G.*

Proof. Because ξ_0 is regular with respect to Θ, we have that by Theorem 5.5 and Proposition 5.6, there exists a strong barrier family $\{w_j\}_{j=1}^{\infty}$ in Θ at point ξ_0. Condition (e) from Definition 5.3 gives us a non-negative function d. Define $\tilde{w}_j = w_j\,|_G$ and $\tilde{d} = d\,|_G$.

Now $\{\tilde{w}_j\}_{j=1}^{\infty}$ is a barrier family in G at ξ_0 because it clearly satisfies conditions (a), (b) and (e) now with respect to the smaller set G and condition (e) implies (c). Thus by using Theorem 5.5 for this barrier family, we have that ξ_0 is regular with respect to G. □

We will also prove the following proposition to show that regularity is a local property. This is needed later in the proof of the exterior ball condition.

Proposition 5.8. *Let $\xi_0 \in \partial\Theta$ and $B \subset \mathbb{R}^{n+1}$ be any ball containing ξ_0. Then ξ_0 is regular with respect to Θ if and only if ξ_0 is regular with respect to $\Theta \cap B$.*

Proof. Using Corollary 5.7, we know that regularity of ξ_0 with respect to Θ implies regularity with respect to $\Theta \cap B$.

Assume ξ_0 is regular with respect to $\Theta \cap B$. By Theorem 5.5 and Proposition 5.6, there exists a strong barrier family w_j in $\Theta \cap B$. By condition (e) of Definition 5.3, there now exists a non-negative function $d \in C(\overline{\Theta \cap B})$ such that for each $k \in \mathbb{N}$, there exists a $j = j(k)$ such that $w_j \geq kd$ in $\Theta \cap B$. Now if we denote $m = \inf_{\overline{\Theta} \cap \partial B} d > 0$ and define

$$w'_k(\xi) = \begin{cases} \min\{w_{j(k)}(\xi), km\} & \text{in } \Theta \cap B \\ km & \text{in } \Theta \setminus B \end{cases}$$

and

$$d'(\xi) = \begin{cases} \min\{d(\xi), m\} & \text{in } \Theta \cap B \\ m & \text{in } \mathbb{R}^{n+1} \setminus B. \end{cases}$$

Now w'_k is lower semicontinuous, so it is a viscosity supersolution by Lemma 4.3. It also satisfies $w'_k \geq kd'$ in Θ which can be used to prove remaining condition (c).

Let $l = 1, 2, \dots$, and take $\xi \in \partial\Theta$ such that $|\xi - \xi_0| \geq \frac{1}{l}$. Now by picking k large enough to satisfy $kd'(\xi) > l$, we have

$$\liminf_{\Theta \ni \zeta \to \xi} w'_j(\zeta) \geq \liminf_{\Theta \ni \zeta \to \xi} kd'(\zeta) \geq l.$$

Thus w'_k forms a barrier family in Θ. This implies regularity with respect to Θ by Theorem 5.5. □

6. A Counterexample and the multiplied equation

In this section, we will prove that a single barrier is not enough to guarantee the regularity of a boundary point when $q < 2$. We construct a set where the origin is irregular but we can still find a barrier function on that point. We also prove that multiplying one side of (1.1) by a constant does not affect boundary regularity. We need the following scaling lemma for our proof.

Lemma 6.1. *(Scaling lemma) Let $q \neq 2$, $a > 0$ and $\Theta \subset \mathbb{R}^{n+1}$ be a domain such that $(0,0) \in \partial\Theta$. Set*

$$\tilde{\Theta} = \Big\{(ax, t) \in \mathbb{R}^{n+1} : (x, t) \in \Theta\Big\}.$$

Then $(0,0)$ is regular with respect to Θ if and only if it is regular to (1.1) with respect to $\tilde{\Theta}$.

Proof. The proof is almost identical to the p-parabolic case in [BBG17, Proposition 4.1]. Take $\tilde{u} : \tilde{\Theta} \to \mathbb{R}$ and define $u : \Theta \to \mathbb{R}$ by

$$u(x, t) = K\tilde{u}(ax, t),$$

where $K = a^{-\frac{q}{q-2}}$. By direct calculation $\partial_t u(x,t) = K\partial_t \tilde{u}(ax,t)$ and

$$\begin{aligned}\Delta_p^q u(x,t) &= |\nabla u(x,t)|^{q-p}\Delta_p u\\ &= |Ka\nabla \tilde{u}(ax,t)|^{q-p}K^{p-1}a^p\Delta_p\tilde{u}(ax,t)\\ &= K^{q-1}a^q|\nabla\tilde{u}(ax,t)|^{q-p}\Delta_p\tilde{u}(ax,t)\\ &= K\Delta_p^q\tilde{u}(ax,t).\end{aligned}$$

Thus u is a viscosity supersolution to (1.1) in Θ if and only if $\tilde{u}$ is a viscosity supersolution to (1.1) in $\tilde{\Theta}$. Take arbitrary $\tilde{f} \in C(\partial\tilde{\Theta})$ and define $f : \partial\Theta \to \mathbb{R}$ by

$$f(x,t) = K\tilde{f}(ax,t).$$

Denote by $\overline{H}_A f(x,t)$ the Perron solution defined over set A for bounded $f : \partial A \to \mathbb{R}$. By the calculation above we have

$$\overline{H}_\Theta f(x,t) = \overline{H}_{\tilde{\Theta}}(K\tilde{f})(ax,t)$$

for all $(x,t) \in \Theta$ and thus regularity of the origin with respect to Θ implies the same with respect to $\tilde{\Theta}$. Converse is proven by swapping the roles of Θ and $\tilde{\Theta}$ and replacing a with $\frac{1}{a}$. □

In Theorem 5.5, we proved that the existence of a barrier family is a sufficient condition for regularity and next, we will prove that the existence of a single barrier is not enough in the singular case. This corresponds to the same result for the p-parabolic equation and similarly, the existence of a such counterexample remains open for the degenerate case. The proof is based on constructing suitable boundary values to prove irregularity of the origin and then constructing a barrier at that point.

Theorem 6.2. *Let $1 < q < 2$, $K > 0$ and $0 < s < \frac{1}{q}$. Then there exists a single barrier w at $(0,0)$ for the domain*

$$\Theta = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x| \le K(-t)^s \text{ and } -1 < t < 0\}$$

despite $(0,0)$ being irregular.

Proof. We prove irregularity of $(0,0)$ first. We do this by constructing an explicit viscosity supersolution that is continuous on the boundary but jumps when we approach the origin along the axis $x = 0$. This is called an irregularity barrier by [BBG17], [KL96], and [Pet35]. Existence of such function ensures that the boundary point cannot be regular directly from the definition. Let

$$u(x,t) = \begin{cases} \frac{|x|^{\frac{q}{q-1}}}{(-t)^{\frac{qs}{q-1}}} - \frac{n+\frac{p-q}{q-1}}{1-qs}\left(\frac{q}{q-1}\right)^{q-1}(-t)^{1-qs}, & \text{for } (x,t) \in \overline{\Theta} \setminus \{(0,0)\},\\ 1, & \text{for } (x,t) = (0,0).\end{cases}$$

Using Lemma 5.4, we have

$$\Delta_p^q u(x,t) = \left(\frac{q}{q-1}(-t)^{-\frac{qs}{q-1}}\right)^{q-1}\left(n + \frac{p-q}{q-1}\right) = \left(\frac{q}{q-1}\right)^{q-1}\frac{n+\frac{p-q}{q-1}}{(-t)^{qs}}.$$

and by direct calculation

$$\partial_t u(x,t) = \frac{qs}{q-1}\frac{|x|^{\frac{q}{q-1}}}{(-t)^{\frac{qs}{q-1}+1}} + \left(\frac{q}{q-1}\right)^{q-1}\frac{n+\frac{p-q}{q-1}}{(-t)^{qs}}.$$

Thus $\partial_t u \geq \Delta_p^q u$ and hence u is viscosity supersolution to (1.1) in Θ.

Let $f = u\mid_{\partial\Theta} \in C(\partial\Theta)$ and let $v \in \mathcal{L}_f(\Theta)$. By definition of the lower class, we can use the elliptic-type comparison principle Theorem 3.1 to ensure $v \leq u$ in Θ, and thus also $\underline{H}f \leq u$. But now

$$\liminf_{\Theta\ni(x,t)\to(0,0)} \underline{H}f(x,t) \leq \liminf_{\Theta\ni(x,t)\to(0,0)} u(x,t) \leq \liminf_{t\to 0-} u(0,t) = 0 < 1 = f(0,0).$$

Hence $(0,0)$ is irregular to equation (1.1) with respect to the set Θ.

Next, we will show that there still exists a barrier at $(0,0)$. Assume first that $K=1$ and let

$$v(x,t) = (-t)^{\frac{1}{2-q}}\left(B - |x|^{\frac{q}{q-1}}\right)$$

where $B = \min\left\{\left(n+\frac{p-q}{q-1}\right)\left(\frac{q}{q-1}\right)^{q-1}(2-q), 1\right\}$. Again by Lemma 5.4, we have

$$\begin{aligned}\Delta_p^q v(x,t) &= -\left((-t)^{\frac{1}{2-q}}\frac{q}{q-1}\right)^{q-1}\left(n+\frac{p-q}{q-1}\right)\\ &= -(-t)^{\frac{q-1}{2-q}}\left(\frac{q}{q-1}\right)^{q-1}\left(n+\frac{p-q}{q-1}\right).\end{aligned}$$

For the time derivative, we have

$$\partial_t v(x,t) = -\frac{1}{2-q}(-t)^{\frac{q-1}{2-q}}\left(B-|x|^{\frac{q}{q-1}}\right) \geq -\frac{B}{2-q}(-t)^{\frac{q-1}{2-q}}$$

and thus by our choice of B, we have

$$\partial_t v(x,t) - \Delta_p^q v(x,t) \geq \left(\left(\frac{q}{q-1}\right)^{q-1}\left(n+\frac{p-q}{q-1}\right) - \frac{B}{2-q}\right)(-t)^{\frac{q-1}{2-q}} \geq 0$$

and thus v is a viscosity supersolution to (1.1) in Θ. Next, we define

$$\tilde{\Theta} = \left\{(x,t)\in\Theta : |x|^{\frac{q}{q-1}} < \frac{B}{2}\right\}$$

and

$$M = \inf_{(x,t)\in\partial\tilde{\Theta}} v(x,t) = \left(\frac{B}{2}\right)^{1+\frac{q-1}{sq(2-q)}} > 0.$$

By the pasting lemma (Lemma 4.3), we know that

$$w(x,t) = \begin{cases}\min\{v(x,t), M\} & \text{if } (x,t)\in\tilde{\Theta},\\ M & \text{if } (x,t)\in\Theta\setminus\tilde{\Theta},\end{cases}$$

is a viscosity supersolution to (1.1) in Θ because it is lower semicontinuous. It also clearly satisfies the other two conditions of being a barrier and thus we have found a barrier at $(0,0)$ despite this point being irregular. The result for general $K>0$ follows from Lemma 6.1. $\square$

If we take a viscosity solution u to equation (1.1) and a constant $c>0$, a simple calculation shows that the function cu is not a viscosity solution unless $q=2$. This also happens for the usual p-parabolic equation with $p\neq 2$. We get similar phenomena where cu now solves the multiplied equation

$$a\partial_t u = \Delta_p^q u. \tag{6.1}$$

for $a=c^{q-2}$. It quite surprisingly turns out that regular boundary points are the same for all multiplied equations of this type as long as $q\neq 2$ which we will prove next. This is known to be false for the heat equation by the Petrovskiĭ condition, see [Pet35].

Theorem 6.3. *Let $\xi_0\in\partial\Theta$ and $a>0$. If $q\neq 2$, the ξ_0 is regular if and only if it is regular to the multiplied equation (6.1).*

Proof. Let w be a viscosity supersolution to (1.1) and let $\tilde{w}=a^{\frac{1}{q-2}}w$. Then

$$a\partial_t\tilde{w}-\Delta_p^q\tilde{w}=a^{1+\frac{1}{q-2}}\partial_t w-a^{1+\frac{1}{q-2}}\Delta_p^q w\geq 0$$

and thus $\tilde{w}$ is a viscosity supersolution to the multiplied equation (6.1). We get equivalence by replacing a by a^{-1}.

From this it follows that if $u\in\mathcal{U}_f$ if and only if $a^{\frac{1}{q-2}}u\in\mathcal{U}_f^a$, where $\mathcal{U}_f^a$ is the upper class with respect to equation (6.1). The equivalence of regularity of ξ_0 with respect to equation (1.1) and with respect to equation (6.1) follow directly from the definition. $\square$

When $q=2$, our equation becomes the normalized p-parabolic equation, and we know that a single barrier is enough to characterize the regularity of a boundary point as proven by [BG14, Theorem 4.2]. This seems to be the case because invariance with regard to multiplication means that the existence of a single barrier implies the existence of a barrier family. We will prove this result to end this section. Based on this it would seem that a single barrier is not enough when $q\neq 2$. We know this to be the case for $q<2$ by Theorem 6.2 but the degenerate case $q>2$ remains an open problem.

Proposition 6.4. *Let $\xi_0\in\partial\Theta$ and $q=2$. There exists a barrier at ξ_0 if and only if there exists a barrier family at ξ_0.*

Proof. The existence of a barrier family clearly implies the existence of a single barrier. For the other direction assume that w is a barrier at ξ_0 and define

$$w_j = jw$$

for $j\in\mathbb{N}$. Now w_j are all positive viscosity supersolutions to (1.1) by simple calculation because $q=2$ and still clearly satisfy condition (b) for all j. Take $k\in\mathbb{N}$ and $\xi\in\partial\Theta$ such that $|\xi - \xi_0| \geq \frac{1}{k}$. Because w is a barrier, we know that

$$\liminf_{\Theta \ni \zeta \to \xi} w(\zeta) = a > 0$$

and thus by choosing $j > \frac{k}{a}$, we have

$$\liminf_{\Theta \ni \zeta \to \xi} w_j(\zeta) = k.$$

This proves condition (c) and thus we have a barrier family. □

7. Exterior ball condition

In this section, we will state and prove an exterior ball condition, which gives a simple geometric criterion for regularity. It turns out that the existence of an exterior ball touching the domain at a boundary point implies the existence of a suitable barrier family apart from a few exceptions. Consider the Dirichlet problem (2.3) set in the usual time cylinder Ω_T. It is well known that in this case, the solution will determine values on the top of the cylinder $\Omega \times \{T\}$, so none of these points can be regular. It turns out that if the tangent point of the exterior ball is its north pole or south pole the argument does not work. After this, we prove a different geometric condition that works for the north pole case and end the section by showing that any point that is time-wise earliest in the set, is always regular.

Lemma 7.1 (Exterior ball condition)**.** *Let $\xi_0 = (x_0, t_0) \in \partial\Theta$. Suppose that there exists a $\xi_1 = (x_1, t_1) \in \Theta^c$ and a radii $R_1 > 0$ such that $B_{R_1}(\xi_1) \cap \Theta = \emptyset$ and $\xi_0 \in \partial B_{R_1}(\xi_1) \cap \partial\Theta$. If $x_1 \neq x_0$ then ξ_0 is regular with respect to Θ.*

***Proof*.** The case $q = 2$ is proven in [BBP19, Lemma 4.2], so we may assume $q \neq 2$. Without loss of generality, assume $\xi_0 = (0,0)$ and $\partial B_{R_1}(\xi_1) \cap \partial\Theta = \{\xi_0\}$. Let $\xi_2 = (x_2, t_2) = \frac{1}{2}\xi_1$ and $R_2 = \frac{1}{2}R_1$. Also pick $\delta = \frac{1}{2}|x_2| > 0$ and let $\Theta_0 = \Theta \cap B_\delta(\xi_0)$. Positivity of δ follows from our assumption $x_1 \neq x_0$. For $\xi = (x,t) \in \overline{\Theta}_0$ and $R = |\xi - \xi_2| \leq 2R_2$, define

$$w_j(\xi) = \gamma\Big(e^{-jR_2^2} - e^{-jR^2}\Big)$$

where we will choose j and $\gamma = \gamma(j) > 0$ later. We will show that for suitable constants, w_j are a barrier family at ξ_0. We have

$$\begin{aligned}
\partial_t w_j(\xi) &= 2j\gamma e^{-jR^2}(t - t_2) \geq -4j\gamma R_2 e^{-jR^2}, \\
\nabla w_j(\xi) &= 2j\gamma e^{-jR^2}(x - x_2),
\end{aligned}$$

which means that, ξ^i denoting the i:th coordinate of vector ξ,

$$
\begin{aligned}
\Delta_p w_j(\xi) &= \operatorname{div}\Big(\Big|2j\gamma e^{-jR^2}(x-x_2)\Big|^{p-2} 2j\gamma e^{-jR^2}(x-x_2)\Big) \\
&= (2j\gamma)^{p-1}\sum_{i=1}^{n}\Big[\Big(-2j(p-1)e^{-j(p-1)R^2}|x-x_2|^{p-2}(x^i-x_2^i)^2\Big) \\
&\quad +\Big((p-2)e^{-j(p-1)R^2}|x-x_2|^{p-4}(x^i-x_2^i)^2\Big)+\Big(e^{-j(p-1)R^2}|x-x_2|^{p-2}\Big)\Big] \\
&= (2j\gamma)^{p-1}e^{-j(p-1)R^2}|x-x_2|^{p-2}\Big[-2j(p-1)|x-x_2|^2+p-2+n\Big]
\end{aligned}
$$

and further

$$
\begin{aligned}
\Delta_p^q w_j(\xi) &= \Big|2j\gamma e^{-jR^2}(x-x_2)\Big|^{q-p}\Delta_p w_j(\xi) \\
&= (2j\gamma)^{q-1}e^{-j(q-1)R^2}|x-x_2|^{q-2}\Big[-2j(p-1)|x-x_2|^2+p-2+n\Big] \\
&\le (2j\gamma)^{q-1}e^{-j(q-1)R^2}|x-x_2|^{q-2}\Big[-2j(p-1)\delta^2+p-2+n\Big],
\end{aligned}
$$

because $|x-x_2|\ge\delta$. Now choose $j_0>\frac{p-2+n}{(p-1)\delta^2}$ to be an integer, so that for any $j\ge j_0$, we have

$$
-2j(p-1)\delta^2+p-2+n\le -j(p-1)\delta^2,
$$

which implies

$$
\begin{aligned}
\Delta_p^q w_j(\xi) &\le -(2j\gamma)^{q-1}j(p-1)\delta^2 e^{-j(q-1)R^2}|x-x_2|^{q-2} \\
&\le -C_0(2j\gamma)^{q-1}j(p-1)e^{-j(q-1)R^2},
\end{aligned}
$$

where

$$
C_0=\begin{cases}(2R_2)^{q-2}\delta^2, & 1<q<2,\\ \delta^q, & q>2.\end{cases}
$$

Based on these calculations we have that w_j is a viscosity supersolution to (1.1) for all $j\ge j_0$ if

$$
4j\gamma R_2 e^{-jR^2}\le C_0(2j\gamma)^{q-1}j(p-1)e^{-j(q-1)R^2},
$$

which is equivalent to

$$
\gamma^{q-2}\ge\frac{j^{1-q}R_2e^{j(q-2)R^2}}{2^{q-3}(q-1)C_0}=C_1j^{1-q}e^{j(q-2)R^2} \tag{7.1}
$$

for $C_1=\frac{R_2}{2^{q-3}(q-1)C_0}$. Now we choose

$$
\gamma=\gamma(j)=\begin{cases}\big(C_1j^{1-q}\big)^{\frac{1}{q-2}}e^{jR_2^2}, & 1<q<2,\\ \big(C_1j^{1-q}\big)^{\frac{1}{q-2}}e^{4jR_2^2}, & q>2.\end{cases} \tag{7.2}
$$

Because $B_{R_2}(\xi_2)\cap\Theta_0$ is empty, we necessarily have $R\in(R_2,2R_2)$ and thus for this γ, the estimate (7.1) holds and thus w_j is a positive viscosity supersolution to (1.1) for all $j\ge j_0$.

We still need to check the rest of the conditions of Definition 5.3 to ensure that w_j forms a barrier family. Condition (b) clearly holds by continuity of w_j at ξ_0. For condition (c), let β

be the angle between vectors $-\xi_1$ and $\xi - \xi_1$ and denote $r_0 = |\xi|$ and $r_1 = |\xi - \xi_1|$. Using the cosine theorem, we get the equalities

$$R^2 = r_1^2 + \left(\frac{R_1}{2}\right)^2 - r_1 R_1 \cos\beta \quad \text{and} \quad r_0^2 = r_1^2 + R_1^2 - 2r_1 R_1 \cos\beta.$$

Using these one after another and lastly the inequality $r_1 \geq R_1$, we get

$$\begin{aligned} R^2 - R_2^2 &= \left(\frac{R_1}{2}\right)^2 + r_1^2 - r_1 R_1 \cos\beta - \left(\frac{R_1}{2}\right)^2 \\ &= r_1^2 - \frac{1}{2}(r_1^2 + R_1^2 - r_0^2) = \frac{1}{2}r_1^2 - \frac{R_1}{2}^2 + \frac{1}{2}r_0^2 \\ &\geq \frac{1}{2}r_0^2. \end{aligned}$$

Using this we can estimate the value of the barrier function. For any $r > 0$ and $\xi \in \overline{\Theta}_0 \setminus B_r(x_0)$,

$$w_j(\xi) = \gamma e^{-jR_2^2}(1 - e^{j(R_2^2 - R^2)}) \geq \gamma e^{-jR_2^2}(1 - e^{-\frac{j}{2}r^2}).$$

Inserting our choices of γ from equation (7.2), we have an estimate

$$w_j(\xi) \geq \begin{cases} (C_1 j^{1-q})^{\frac{1}{q-2}}(1 - e^{-\frac{j}{2}r^2}), & 1 < q < 2, \\ (C_1 j^{1-q})^{\frac{1}{q-2}} e^{3jR_2^2}(1 - e^{-\frac{j}{2}r^2}), & q > 2. \end{cases} \tag{7.3}$$

In either case, the right-hand side tends to ∞ as $j \to \infty$ for any fixed r. Thus for any $k \in \mathbb{N}$ we can pick $r = \frac{1}{k}$ and equation (7.3) implies

$$\liminf_{\Theta \ni \zeta \to \xi} w_j(\zeta) \geq k$$

for some large j and any $\xi \in \overline{\Theta}_0 \setminus B_r(x_0)$. Thus condition (c) holds and w_j forms a barrier family. Thus by Theorem 5.5 the point ξ_0 is regular to equation (1.1) with respect to the set Θ_0. The regularity with respect to the set Θ follows from Proposition 5.8. □

Similarly to the p-parabolic case, this proof does not work when ξ_0 is the north pole or the south pole of the ball. In the north pole case, we get a result matching to [BBGP15, Proposition 4.2].

Proposition 7.2. *Let $\Theta \in \mathbb{R}^{n+1}$ be open and $(x_0, t_0) \in \partial\Theta$. Assume that for some $\theta > 0$, we have*

$$\Theta \subset \{(x,t) \mid t - t_0 > -\theta|x - x_0|^l\},$$

where $l \geq \frac{q}{q-1}$ if $1 < q < 2$, and $l > q$ if $q > 2$. Then (x_0, t_0) is regular with respect to Θ.

Proof. Without loss of generality, we may assume $(x_0, t_0) = (0, 0)$. Let

$$\Theta_0 = \Big\{(x,t) \mid t > -\theta|x|^l \text{ and } -1 < t < 0\Big\}$$

and

$$G^j = \left\{(x,t) \in \Theta_0 \,\middle|\, |x| < \frac{1}{j^{\frac{1}{s}}} \text{ and } -\frac{\theta}{j^{\frac{l}{s}}} < t < 0\right\},$$

where $s > 0$ will be fixed later. Note that $G_{j+1} \subset G_j$ for all j. Now let

$$f_j(x,t) = j\frac{q-1}{q}|x|^{\frac{q}{q-1}} + \Big(n + \frac{p-q}{q-1}\Big)j^{q-1}t$$

and use Lemma 5.4 to conclude that

$$\Delta_p^q f_j(x,t) = \Big(j\frac{q-1}{q}\frac{q}{q-1}\Big)^{q-1}\Big(n + \frac{p-q}{q-1}\Big) = j^{q-1}\Big(n + \frac{p-q}{q-1}\Big)$$

and

$$\partial_t f_j(x,t) = \Big(n + \frac{p-q}{q-1}\Big)j^{q-1}.$$

We can see that f_j is a viscosity solution to (1.1) in $\mathbb{R}^{n+1}$. Define

$$\begin{aligned} m_j := \inf_{\Theta\cap\partial G^j} f_j &= j\frac{q-1}{q}\Big(\frac{1}{j^{\frac{1}{s}}}\Big)^{\frac{q}{q-1}} - \Big(n + \frac{p-q}{q-1}\Big)j^{q-1}\frac{\theta}{j^{\frac{l}{s}}} \\ &= \frac{q-1}{q}j^{1-\frac{q}{s(q-1)}} - \Big(n + \frac{p-q}{q-1}\Big)\theta j^{q-1-\frac{l}{s}}. \end{aligned}$$

We want $m_j \to \infty$ as $j \to \infty$ to construct a barrier family. Note that the coefficient of the second term is always positive so we only need to take care of the exponents. We must have

$$1 - \frac{q}{s(q-1)} > 0 \quad \text{and} \quad 1 - \frac{q}{s(q-1)} > q - 1 - \frac{l}{s}$$

i.e.

$$s > \frac{q}{q-1} \quad \text{and} \quad l > s(q-2) + \frac{q}{(q-1)}. \tag{7.4}$$

The latter condition gives us the two cases depending on q. If $1 < q < 2$, the first term on the right-hand side is negative and we have

$$l > \frac{q}{q-1} \quad \text{for all} \quad s > \frac{q}{q-1}.$$

If $q > 2$, the first term on the right-hand side is positive and we have $l > q$ if we choose s sufficiently close to $\frac{q}{q-1}$. But we see that there exists a s that is suitable for both of these cases.

Now we define

$$h_j = \begin{cases} \min\{f_j, m_j\} & \text{in } G^j \\ m_j & \text{in } \Theta_0 \setminus G^j. \end{cases}$$

Now using the pasting lemma (Lemma 4.3) for $f_j\,|_{G^j}$ and m_j, we have that h_j is a positive viscosity supersolution to (1.1) in Θ_0 provided we choose j large enough. Conditions (a) and (b) of the definition of a barrier Definition 5.3 are clearly satisfied.

Now for these s and l satisfying (7.4), we have $m_j \to \infty$ and $|\xi| \to 0$ for all $\xi \in G_j$ as $j \to \infty$ and thus for any k, we are able to find some large $j(k)$ so condition (c) is satisfied. The family of functions h_j is thus a barrier family at $(0,0)$. Thus by Theorem 5.5, the point $(0,0)$ is regular with respect to Θ_0.

Regularity with respect to Θ follows by picking an open ball B containing $(0,0)$, using Corollary 5.7 for $\Theta_0 \cap B$ and then Proposition 5.8. $\square$

Finally, if the bottom of the set is flat, we get regularity for all of these points from the following useful lemma. It turns out that the earliest points time-wise are always regular.

Lemma 7.3. *Let $\xi_0 = (x_0, t_0) \in \partial\Theta$. If $\xi_0 \notin \partial\Theta_-$ for*

$$\Theta_- = \{(x,t) \in \Theta \mid t < t_0\},$$

then ξ_0 is regular with respect to Θ. In particular, this holds if $\Theta_- = \emptyset$.

Proof. Let

$$f_j(x,t) = j\frac{q-1}{q}|x-x_0|^{\frac{q}{q-1}} + \Big(n + \frac{p-q}{q-1}\Big)j^{q-1}(t-t_0).$$

For any $(x,t) \notin \partial\Theta_-$, we can use Lemma 5.4 to conclude

$$\Delta_p^q f_j(x,t) = j^{q-1}\Big(n + \frac{p-q}{q-1}\Big) = \partial_t f_j(x,t),$$

which implies that f_j are positive viscosity solutions to (1.1) in $\mathbb{R}^{n+1}$ for all j. They also clearly satisfy condition (b) of Definition 5.3 and lastly for any $k = 1, 2, \dots,$ $r = \frac{1}{k}$ and $\xi \in \Theta \setminus B_r(\xi_0)$, we can ensure

$$\liminf_{\Theta \ni \zeta \to \xi} f_j(\zeta) \geq j\frac{q-1}{q}r^{\frac{q}{q-1}} + \Big(n + \frac{p-q}{q-1}\Big)j^{q-1}r \geq k$$

by picking a large j, which implies condition (c). Thus f_j form a barrier family in Θ at a point ξ_0 and thus by Theorem 5.5, the point ξ_0 is regular with respect to Θ. $\square$

There remain many other geometric conditions known for the p-parabolic equation that could be expanded for equation (1.1) in the future.

8. Superparabolic

Our definition of barriers differs from what was used by Björn, Björn, Gianazza, and Parviainen in their paper, and we will prove in this final section that the definitions coincide. The definition is otherwise the same but they assume the function satisfies the comparison principle in arbitrary time cylinders instead of directly defining them to be viscosity supersolutions. This type of function has various names in the literature for different equations. For the usual p-parabolic equation these are sometimes called p-superparabolic in the literature and generalized supersolutions for the normalized equation in [BG14]. We will just use the term superparabolic for simplicity.

Definition 8.1. *A function $u : \Theta \to (-\infty, \infty]$ is superparabolic to equation (1.1) in Θ if*

(i) *u is lower semicontinuous,*
(ii) *u is finite in a dense subset of Θ,*

(iii) u *satisfies the following comparison principle on each space-time cylinder* Ω_{t_1,t_2}*: If* $v \in C(\overline{\Omega}_{t_1,t_2})$ *is a viscosity solution to* (1.1) *in* Ω_{t_1,t_2} *satisfying* $v \le u$ *on* $\partial_p \Omega_{t_1,t_2}$*, then* $v \le u$ *in* Ω_{t_1,t_2}*.*

Superparabolic functions defined in this way for p-parabolic and normalized p-parabolic equations are the same as the corresponding viscosity solutions as shown in [BG14] and [JLM01]. We will prove this same result for equation (1.1).

Lemma 8.2. *In a given domain, the viscosity supersolutions and superparabolic functions to* (1.1) *are the same.*

Proof. A viscosity supersolution is clearly superparabolic because conditions (i) and (ii) already match and supersolution satisfies the comparison principle Theorem 2.3.

Let $\Theta \subset \mathbb{R}^{n+1}$ and u be a superparabolic to (1.1) in Θ. We assume thriving for a contradiction that u is not a viscosity supersolution in Θ. Then we must have a point $(x_0, t_0) \in \Theta$ and at least one admissible $\varphi \in C^2(\Theta)$ that touches u at (x_0, t_0) from below but we have one of the following cases.

Case 1: $\partial_t \varphi(x_0, t_0) - \Delta_p^q \varphi(x_0, t_0) < 0$ and $\nabla \varphi(x_0, t_0) \neq 0$. Because the inequality is strict, we necessarily have that for small $\rho > 0$ the function φ is a classical subsolution to (1.1) inside a small cylinder Q_ρ. By the definition of touching from below, it is possible for us to choose ρ so small that we can pick $\delta > 0$ so that

$$\varphi + \delta \le u \text{ on } \partial_p Q_\rho.$$

Let v be a viscosity solution to the Dirichlet problem (2.3) with $\varphi + \delta$ as boundary values on the set Q_ρ. This exists by Theorem 2.4. Because $\varphi + \delta$ is a subsolution in this set, we can use the comparison principle Theorem 2.3 to deduce $\varphi + \delta \le v$ in Q_ρ. This combined with condition (iii) from u being superparabolic gives us

$$\varphi(x,t) + \delta \le v(x,t) \le u(x,t) \text{ for all } (x,t) \in Q_\rho.$$

But this is a contradiction as this implies $\varphi(x_0, t_0) + \delta \le u(x_0, t_0) = \varphi(x_0, t_0)$.

Case 2: $\partial_t \varphi(x_0, t_0) < 0$ and $\nabla \varphi(x_0, t_0) = 0$. Because φ is admissible, we have by Definition 2.1, that for some $\rho > 0$, $f \in \mathcal{F}(F)$ and $g(x) = f(|x|)$, we have

$$|\varphi(x,t) - \varphi(x_0,t_0) - \partial_t \varphi(x_0,t_0)(t-t_0)| \le g(x-x_0) + \sigma(t-t_0) \tag{8.1}$$

for all $(x,t) \in B_\rho(x_0) \times (t_0 - \rho, t_0 + \rho)$. Define

$$\phi(x,t) = u(x_0,t_0) + \partial_t \varphi(x_0,t_0)(t-t_0) - g(x-x_0) - \sigma(t-t_0)$$

which is an admissible test function touching u at (x_0, t_0) from below because

$$\begin{aligned} |\phi(x,t) - \phi(x_0,t_0) - \partial_t \phi(x_0,t_0)(t-t_0)| &= |\phi(x,t) - u(x_0,t_0) - \partial_t \varphi(x_0,t_0)(t-t_0)| \\ &\le g(x-x_0) + \sigma(t-t_0). \end{aligned}$$

By definition of $\mathcal{F}(F)$, g satisfies

$$\lim_{\substack{x\to 0\\ x\neq 0}} \Delta_p^q g(x-x_0) = 0. \tag{8.2}$$

Because we assumed that $\partial_t\varphi(x_0,t_0) < 0$ and $\sigma'(t-t_0) > 0$ in some small punctured neighborhood of t_0 by definition of Σ, a direct calculation using (8.2) yields

$$\partial_t\phi(x,t) - \Delta_p^q\phi(x,t) = \partial_t\varphi(x_0,t_0) - \sigma'(t-t_0) - \Delta_p^q g(x-x_0) < 0$$

in some small punctured neighborhood V^* of (x_0,t_0). This means that ϕ is a classical subsolution to (1.1) in V^*. Because we have $\partial_t\phi(x_0,t_0) < 0$, we can conclude by [OS97, Lemma 4.1] that ϕ is a viscosity subsolution to (1.1) in $V = V^* \cup \{(x_0,t_0)\}$.

The rest is similar to Case 1. By the definition of touching from below, it is possible for us to choose $\tilde\rho$ so small that we can pick $\delta > 0$ so that

$$\phi + \delta \leq u \text{ on } \partial_p Q_{\tilde\rho}$$

and $Q_{\tilde\rho} \subset V$. Let v be a viscosity solution to the Dirichlet problem (2.3) with $\phi+\delta$ as boundary values on the set $Q_{\tilde\rho}$. This exists by Theorem 2.4. Because $\phi+\delta$ is a viscosity subsolution in this set, we can use the comparison principle Theorem 2.3 to deduce $\phi+\delta \leq v$ in $Q_{\tilde\rho}$. This combined with condition (iii) from u being superparabolic gives us

$$\phi(x,t) + \delta \leq v(x,t) \leq u(x,t) \text{ for all } (x,t) \in Q_{\tilde\rho}.$$

But this is a contradiction as this implies $\phi(x_0,t_0) + \delta \leq u(x_0,t_0) = \phi(x_0,t_0)$.

Both cases lead to a contradiction and thus u is a viscosity supersolution. □

References

[AR20] A. Attouchi and E. Ruosteenoja. Gradient regularity for a singular parabolic equation in non-divergence form. *Discrete and Continuous Dynamical Systems*, 40(10):5955–5972, 2020.

[AS22] P. D. S. Andrade and M. S. Santos. Improved regularity for the parabolic normalized p-Laplace equation. *Calculus of Variations and Partial Differential Equations*, 61(5):196, 2022.

[Att20] A. Attouchi. Local regularity for quasi-linear parabolic equations in non-divergence form. *Nonlinear Analysis, Theory, Methods and Applications*, 199:1–32, 2020.

[BBG17] A. Björn, J. Björn, and U. Gianazza. The Petrovskiĭ criterion and barriers for degenerate and singular p-parabolic equations. *Mathematische Annalen*, 368:885–904, 2017.

[BBGP15] A. Björn, J. Björn, U. Gianazza, and M. Parviainen. Boundary regularity for degenerate and singular parabolic equations. *Calculus of Variations and Partial Differential Equations*, 52(3):797–827, 2015.

[BBGS18] A. Björn, J. Björn, U. Gianazza, and J. Siljander. Boundary Regularity for the Porous Medium Equation. *Archive for Rational Mechanics and Analysis*, 230(2):493–538, 2018.

[BBP19] A. Björn, J. Björn, and M. Parviainen. The tusk condition and Petrovskiĭ criterion for the normalized p-parabolic equation. *Journal of the London Mathematical Society*, 100(2):623–643, 2019.

[BG14] A. Banerjee and N. Garofalo. On the Dirichlet boundary value problem for the normalized p-laplacian evolution. *Communications on Pure and Applied Analysis*, 14:1–21, 2014.

[CIL92] M. G. Crandall, H. Ishii, and P. L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27(1):1–67, 1992.

[DPZZ20] H. Dong, F. Peng, Y. R.-Y. Zhang, and Y. Zhou. Hessian estimates for equations involving p-Laplacian via a fundamental inequality. *Advances in Mathematics*, 370:107212, 40, 2020.

[EG82] L. C. Evans and R. F. Gariepy. Wiener's criterion for the heat equation. *Archive for Rational Mechanics and Analysis*, 78(4):293–314, 1982.

[Gig06] Y. Giga. *Surface Evolution Equations: A Level Set Approach*. Monographs in Mathematics. Birkhäuser Basel, 2006.

[GLM86] S. Granlund, P. Lindqvist, and O. Martio. Note on the PWB-method in the nonlinear case. *Pacific Journal of Mathematics*, 125(2):381–395, 1986.

[HKM06] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear Potential Theory of Degenerate Elliptic Equations*. Dover Publications, 2006.

[HL19] F. A. Høeg and P. Lindqvist. Regularity of solutions of the parabolic normalized p-Laplace equation. *Advances in Nonlinear Analysis*, 9(1):7–15, 2019.

[IJS19] C. Imbert, T. Jin, and L. Silvestre. Hölder gradient estimates for a class of singular or degenerate parabolic equations. *Advances in Nonlinear Analysis*, 8(1):845–867, 2019.

[JLM01] P. Juutinen, P. Lindqvist, and J. J. Manfredi. On the Equivalence of Viscosity Solutions and Weak Solutions for a Quasi-Linear Equation. *SIAM Journal on Mathematical Analysis*, 33(3):699–717, 2001.

[JS17] T. Jin and L. Silvestre. Hölder gradient estimates for parabolic homogeneous p-Laplacian equations. *J. Math. Pures Appl.*, 108(1):63–87, 2017.

[KL96] T. Kilpeläinen and P. Lindqvist. On the Dirichlet Boundary Value Problem for a Degenerate Parabolic Equation. *SIAM Journal on Mathematical Analysis*, 27(3):661–683, 1996.

[Koi04] S. Koike. *A Beginner's Guide to the Theory of Viscosity Solutions*. MSJ Memoirs, Vol. 13. Mathematical Society of Japan, 2004.

[KPS23] T. Kurkinen, M. Parviainen, and J. Siltakoski. Elliptic Harnack's inequality for a singular nonlinear parabolic equation in non-divergence form. *Bulletin of the London Mathematical Society*, 55(1):470–489, 2023.

[KS23] T. Kurkinen and J. Siltakoski. Intrinsic Harnack's inequality for a general nonlinear parabolic equation in non-divergence form. *To Appear in Potential Analysis*, 2023.

[Leb24] H. Lebesgue. On the Dirichlet Boundary Value Problem for a Degenerate Parabolic Equation. *Comptes Rendus de l'Académie des Sciences*, 178:349–354, 1924.

[MPR10] J. J. Manfredi, M. Parviainen, and J. D. Rossi. An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. *SIAM Journal on Mathematical Analysis*, 42(5):2058–2081, 2010.

[OS97] M. Ohnuma and K. Sato. Singular degenerate parabolic equations with applications. *Communications in Partial Differential Equations*, 22(3-4):381–441, 1997.

[Pet34] I. Petrovskiĭ. Über die Lösungen der ersten Randwertaugabe der Wärmerleitungsgleichung. *Uchenye Zapiski Moskovskogo Gosudarstvennogo Universiteta*, 2:55–59, 1934.

[Pet35] I. Petrovskiĭ. Zur ersten Randwertaufgabe der Wärmeleitungsgleichung. *Compositio Mathematica*, 1:383–419, 1935.

[Poi90] H. Poincaré. Sur les Equations aux Dérivées Partielles de la Physique Mathématique. *American Journal of Mathematics*, 12(3):211–294, 1890.

[PV20] M. Parviainen and J. L. Vázquez. Equivalence between radial solutions of different parabolic gradient-diffusion equations and applications. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 21(Special Issue):303–359, 2020.

[Wie24] N. Wiener. The Dirichlet Problem. *Journal of Mathematics and Physics*, 3:127–146, 1924.

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