JYU DISSERTATIONS 807

Antti Kykkänen

Geodesic X-ray Transforms in Non-smooth Riemannian Geometries

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

JYU DISSERTATIONS 807

Antti Kykkänen

Geodesic X-ray Transforms in Non-smooth Riemannian Geometries

Esitetään Jyväskylän yliopiston matemaattis-luonnontieteellisen tiedekunnan suostumuksella julkisesti tarkastettavaksi yliopiston Mattilanniemen auditoriossa MaD259, elokuun 13. päivänä 2024 kello 12.

Academic dissertation to be publicly discussed, by permission of the Faculty of Mathematics and Science of the University of Jyväskylä, in Mattilanniemi, auditorium MaD259, on August 13, 2024 at 12 o'clock noon.

JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

JYVÄSKYLÄ 2024

Editors
Joonas Ilmavirta
Department of Mathematics and Statistics, University of Jyväskylä
Ville Korkiakangas
Open Science Centre, University of Jyväskylä

ISBN 978-952-86-0233-0 (PDF)
URN:ISBN:978-952-86-0233-0
ISSN 2489-9003

Permanent link to this publication: http://urn.fi/URN:ISBN:978-952-86-0233-0

Foreword

First and foremost I want to thank my supervisor Joonas Ilmavirta. I am truly grateful for the advice, lessons, patience and time you have dedicated to my supervision in the recent years. Your commitment to supervision cannot be overvalued. Thank you for your guidance into the world of inverse problems and thank you for showing me that real mathematics can be meaningful in science at large. One of my favourite parts of being a PhD student has been travelling and networking. I thank you for your efforts in making such experiences possible.

I thank the Department of Mathematics and Statistics at the University of Jyväskylä for the welcoming and supportive work environment during 2022–2024. I want to give special thanks to all members of the inverse problems research group at Jyväskylä for all the fun we have had with mathematics and non-mathematics on and off campus.

I thank my collaborators, Kelvin Lam, Maarten de Hoop and Rafe Mazzeo for many fruitful and productive discussions. You have taught me a lot. I wish to thank Maarten de Hoop for your hospitality during my visit at Rice University in May 2023 and October 2023. The inspiring discussions we had during the visits greatly motivated me in finishing my dissertation.

I wish to thank Lauri Oksanen who has agreed to be my opponent at the defence of my dissertation. I wish to thank the pre-examiners Gabriel Paternain and Katya Krupchyk for their valuable comments and encouragement.

There is no academic life without life outside it. I thank my friends and my family who have supported me in the times of my studies. I would not have been successful in my pursuit of the PhD degree without you.

Jyväskylä, June 2024
Department of Mathematics and Statistics
University of Jyväskylä
Antti Kykkänen

List of included articles

This dissertation consists of an introduction and the following four articles:

[A] **Pestov identities and X-ray tomography on manifolds of low regularity**
J. Ilmavirta, A. Kykkänen,
Inverse Problems and Imaging, 17(6):1301–1328 (December 2023).

[B] **Tensor tomography on negatively curved manifolds of low regularity**
J. Ilmavirta, A. Kykkänen,
Journal of Geometric Analysis, 34:147 (March 2024).

[C] **Microlocal analysis of the X-ray transform in non-smooth geometry**
J. Ilmavirta, A. Kykkänen and K. Lam,
Preprint (September 2023), arXiv: 2309.12702.

[D] **Geometric inverse problems on gas giants**
M. V. de Hoop, J. Ilmavirta, A. Kykkänen and R. Mazzeo,
Preprint (March 2024), arXiv:2403.05475.

The author of this dissertation has actively taken part in the research of the joint articles [A], [B] and [C].

The author has participated actively in the development of the joint article [D]. In particular, the author contributed development of the geometry in [D, Section 2] and is mainly responsible for proof of injectivity of the X-ray transform in [D, Section 3].

Abstract

Is a function uniquely determined by its integrals over geodesics of a Riemannian manifold? This question — known as geodesic X-ray tomography — is a geometric generalization of the classical problem of recovering a function from its integrals along lines encountered in medical applications of X-ray tomography. The geometric question naturally arises from a various geometric inverse problems such as boundary rigidity and spectral rigidity.

This thesis studies geodesic X-tomography problems in non-smooth Riemannian geometries. The central objects of interest — known as geodesic X-ray transforms — are various integral transforms encoding the integrals of a function or a tensor field over the geodesics. We encounter two different types of non-smooth geometries: globally non-smooth Riemannian metrics and Riemannian metrics singular at the boundary of the manifold. The thesis contains four articles recording results on X-ray transforms and the geometries themselves.

We prove that the geodesic X-ray transform of Lipschitz scalar functions is injective on simple Riemannian manifolds with $C^{1,1}$ regular metrics. We prove that the X-ray transforms of $C^{1,1}$ smooth 1-forms and tensor fields of higher rank are solenoidally injective on simple Riemannian manifolds of non-positive sectional curvature with $C^{1,1}$ regular metrics. These results are based on energy methods and the use of the so called Pestov identity. In addition to injectivity results, we produce a redefinition of simplicity that is compatible with non-smooth geometry, and prove that the redefinition is equivalent to any standard definition of simplicity for C^∞ smooth Riemannian metrics.

We supplement the injectivity results by considering the normal operator of the X-ray transform in non-smooth geometry. Based on non-smooth microlocal analysis of the normal operator we prove that the geodesic X-ray transform is injective on L^2 when the Riemannian metric is simple but only finitely differentiable. The number of derivatives needed depends explicitly on the dimension of the manifold.

Riemannian metrics that are C^∞ smooth in the interior of a manifold with boundary but have a conformal blow up of a specific strength at the boundary are called gas giant metrics. Such Riemannian metrics are different from but relatives of asymptotically hyperbolic metrics, and arise naturally in the study of wave propagation in gas giant planets. The specific type of singularity is related to the fact that unlike on terrestrial planets the density of a gas giant planet goes to zero at the surface. The specific blow up rate comes from a polytropic model. We prove and apply Pestov identities in gas giant geometry to show that the X-ray transform on a gas giant is injective. We develop the differential geometry of gas giant metrics with an emphasis on the geometry of geodesics, and study the basic analytic properties of the Laplace–Beltrami operator associated to a gas giant metric.

The introduction part of the thesis contains an overview of the X-ray tomography in Riemannian geometry and the geometric preliminaries behind it. An overview of the included articles is also provided.

Tiivistelmä

Määräytyykö tuntematon funktio Riemannin monitolla yksikäsitteisesti integraaleistaan kaikkien geodeesien yli? Tämä kysymys, joka tunnetaan geodeettisenä röntgentomografiana, on geometrinen yleistys klassiselle lääketieteellisen kuvantamisen röntgentomografiaongelmalle, jossa halutaan löytää tuntematon funktio, kun tunnetaan sen integraalit suoria pitkin. Ongelman geometrinen yleistys tulee vastaan monien geometristen inversio-ongelmien, kuten reunajäykkyyden ja spektraalijäykkyyden, tutkimuksessa.

Tässä tutkielmassa tarkastellaan geodeettistä röntgentomografiaa epäsileässä Riemannin geometriassa. Keskeisiä tutkimuksen kohteita, jotka tunnetaan geodeettisinä röntgenmuunnoksina, ovat erinäiset integraalimuunnokset, jotka paketoivat yhteen funktion tai tensorikentän integraalit geodeesien yli. Tutkielmassa kohdataan kahden tyyppistä epäsileää geometriaa; globaalisti epäsileitä Riemannin metriikoita ja Riemannin metriikoita, jotka ovat singulaarisia moniston reunalla. Tutkielma koostuu neljästä artikkelista, joissa on tuloksia liittyen sekä röntgenmuunnoksiin, että itse epäsileisiin geometrioihin.

Tutkielmassa todistetaan, että Lipschitz-funktioiden geodeettinen röntgenmuunnos on injektiivinen yksinkertaisilla Riemannin monistoilla, kun Riemannin metriikka on $C^{1,1}$-säännöllinen. Todistetaan myös, että $C^{1,1}$-säännöllisten 1-muotojen ja korkeamman asteen tensorikenttien röntgenmuunnos on solenoidisesti injektiivinen epäpositiivisesti kaarevilla yksinkertaisilla Riemannin monistoilla, kun metriikka on $C^{1,1}$-säännöllinen. Nämä tulokset perustuvat energiametodeihin ja Pestov-identiteettien käyttöön. Injektiivisyystuloksien lisäksi annetaan määritelmä moniston yksinkertaisuudelle, joka on yhteensopiva matalan säännöllisyyden kanssa, ja osoitetaan, että uusi määritelmä on yhtäpitävä tavallisten määritelmien kanssa, kun metrinen säännöllisyys on C^∞.

Näitä injektiivisyystuloksia täydennetään tarkastelemalla röntgenmuunnoksen normaalioperaattoria epäsileässä geometriassa. Normaalioperaattorin epäsileään mikrolokaaliin analyysiin perustuen osoitetaan, että geodeettinen röntgenmuunnos on injektiivinen L^2-funktioilla, kun Riemannin metriikka on yksinkertainen, mutta äärellisen monta kertaan derivoituva. Tarvittujen derivaattojen lukumäärä riippuu eksplisiittisesti moniston dimensiosta.

Riemannin metriikoita, jotka ovat C^∞-sileitä moniston sisällä, mutta jotka konformisesti räjähtävät tiettyä tahtia moniston reunalla, kutsutaan kaasujättimetriikoiksi. Tällaiset Riemannin metriikat ovat sukua asymptoottisesti hyperbolisille metriikoille, mutta eroavat kuitenkin käytökseltään. Kaasujättimetriikat liittyvät luonnollisesti aaltoliikkeeseen kaasujättiplaneetoilla. Kaasujättien tiheys lähestyy nollaa planeetan pinnalla toisin kuin kiviplaneetoilla, joka määrää singulariteetin erityisen tyypin. Räjähdystahdin määrää polytrooppinen tilayhtälö. Tutkielmassa todistetaan ja sovelletaan Pestov-identiteettejä osoittamaan, että kaasujättien röntgenmuunnos on injektiivinen. Lisäksi tutkielmassa kehitetään kaasujättimetriikoiden differentiaaligeometriaa ja erityisesti geodeesien geometriaa sekä tutkitaan kaasujättimetriikoiden Laplace–Beltrami-operaattorin analyyttisiä ominainaisuuksia.

Tutkielman johdanto-osiosta löytyy yleiskatsaus röntgentomografiaan Riemannin geometriassa ja sen taustalta löytyviin geometrisiin esitietoihin. Johdanto-osio sisältää myös yleiskatsauksen tutkielman artikkeleihin.

Contents

1. Introduction

1.1. **Inverse problems.** Imagine a system whose behaviour depends on a set of parameters. A forward problem asks to determine the behaviour of the system given the parameters. An inverse problem asks to deduce in reverse. Given the behaviour of the system determine the set of parameters causing the experienced behaviour. Inverse problems are the mathematics of indirect measurement. Typical questions of interest are uniqueness, stability and existence of an algorithm.

- *Uniqueness*: Do the measurements determine the parameters uniquely?
- *Stability*: Do the unknown parameters depend on the measurements continuously?
- *Algorithms*: Is there an explicit algorithm for determining the parameters from the measurements?

The following are some classical examples of inverse problems in Riemannian geometry.

- *Geodesic X-ray tomography*: Determine a function or a tensor field on Riemannian manifold from the knowledge of its integrals over all maximal geodesics.
- *Boundary rigidity*: Determine the Riemannian metric inside a manifold with boundary from the knowledge of the Riemannian distances between boundary points.
- *Spectral rigidity*: Determine the Riemannian metric in a manifold without a boundary from the knowledge of the spectrum of the Laplace-Beltrami operator.
- *The Calderón problem*: Determine the electrical conductivity inside a manifold with boundary by making current and voltage measurements on the boundary.

All of the above example geometric inverse problems are related. There are variations of each of the last three problems where it is possible to use the measurements to recover integrals of a function over the geodesics of the manifold. Hence in a certain way geodesic X-ray transform is in the background of the other problems. This thesis focus on geodesic X-ray transforms in various non-smooth Riemannian geometries.

1.2. **Geodesic X-ray transforms on manifolds.** The starting point of research in geodesic integral transforms in Riemannian geometry is the work of Mukhometov [Muk75, Muk77, Muk78]. The transform of interest takes a function on Riemannian manifold and produces a function on the space of geodesics by integrating the given function along the geodesics. More formally, the space of geodesics on a compact Riemannian manifold with a smooth boundary is the inward pointing part $\partial_{\text{in}}(SM)$ of the unit sphere bundle SM over M at the boundary. The geodesic

X-ray transform of a function f is then defined by the formula

$$If(x,v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\,dt$$

for all $(x,v) \in \partial_{\text{in}}(SM)$. Here $\gamma_{x,v}$ is the unique unit speed geodesic in M corresponding to the initial conditions $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$, and $\tau(x,v)$ denotes the first time the geodesic $\gamma_{x,v}$ exists the manifold. Mukhometov introduced a so called Pestov identity in the study of such integral transforms and was able to prove that the transform I is injective under certain geometric assumptions.

There are various generalizations of the operator I. Instead of a scalar function, one can consider transforms of tensor fields [AR97, PS87, PSU13, PSU15, KMM19, BLP24]. The transform can be changed by adding an attenuation or a Higgs field [AMU18, HMS18, PSU12, SU11, GPSU16, BP23], or a magnetic field [Ain13, DPSU07]. The non-Abelian versions of the problem do not study invertibility of any explicit linear integral transform, but the data is encoded in the solution operator of a matrix ODE along the orbits of the geodesic flow [FU01, PS22, MNP21].

While the variations of the transform itself are many, the most relevant changes to the set-up, from the perspective of this thesis, are variations of the geometry. Geodesic X-ray transforms are well understood on Anosov manifolds which are closed counterparts of the smooth simple manifolds with a boundary [CS98, DS03, PSU14a, SU00, GL19].

A direction of generalization is to study which of the X-ray tomography results remain true in non-smooth geometry and how low the regularity assumptions of geometry can be pushed. A considerable amount of the work in this thesis is devoted to X-ray tomography in non-smooth low regularity geometry.

The first article of thesis studies X-ray transforms in $C^{1,1}$ smooth geometry. The main result of article [A] is that the X-ray transform of Lipschitz functions is injective in simple $C^{1,1}$ geometry (see Section 3.1.3). Article [A] also addresses the definition simple Riemannian geometry in low regularity context which is discussed in Section 2.1. One of the main results of article [A] is a definition of simple Riemannian manifold compatible with non-smooth Riemannian metrics yet equivalent to any standard definition of simplicity for C^∞-smooth Riemannian metrics.

Article [B] extends the work of article [A]. X-ray transforms of tensor fields of higher order are studied. The solenoidal injectivity results are obtained in article [B] discussed in Section 3.1.3.

Article [C] takes another approach to X-ray tomography on Riemannian manifolds. The article continues in non-smooth geometry and shows that the normal operator of the X-ray transform is an elliptic pseudodifferential operator in a low regularity pseudodifferential calculus. An application, which shows that the X-ray transform of L^2-functions is injective in certain non-smooth geometries. The results of article [C] are discussed in Section 3.2.2.

Normal operator and its microlocal analysis have been utilized in smooth Riemannian settings with far reaching applications [SU04, SU05, PU05]. It is generally understood that on 2-dimensional Riemannian manifolds ellipticity of the normal operator, stable invertibility of the X-ray transform and simplicity of the geometry are equivalent. See [MSU15] for instability in non-simple Riemannian geometry.

Prior to articles [A, B, C] the only known result in X-ray tomography on Riemannian manifolds of low regularity is [dHI17], where the authors considered spherically symmetric Riemannian metrics of $C^{1,1}$ smoothness that satisfy the so called Herglotz condition. Injectivity results for a large class of operators including the X-ray transform were obtained. The spherically symmetric geometric variation has a long history going back to [Her05, WZ07]. A tensorial variant of the problem has also been studied [Sha97].

The fourth article of the thesis is concerned with non-smooth geometry but instead of having globally uniformly non-smooth geometry article [D] studies Riemannian metrics that are singular at the boundary in a special way. The article provides the first insights into geometrization of wave propagation phenomena specific to gas giant planets such as Jupiter. The density of matter goes to zero at the boundary of a gas giant unlike on terrestrial planets such as the Earth. Modelling a gas giant, for example, as a polytrope leads to a power type singularity in the Riemannian metric at the boundary whereas the geometry of terrestrial planets in non-singular everywhere. The geometry of gas giant planets is discussed in Section 2.2 and a result on injectivity of the X-ray transform on gas giants is presented in Section 3.1.4.

The geometry of gas giant developed in article [D] has some close relatives which have a longer history. Geodesic X-ray tomography has been studied on non-compact Cartan-Hadamard manifolds [Leh16, LRS18]. Particularly close to the geometry of gas giants is asymptotically hyperbolic geometry, the difference being the strength of the singularity at the boundary. There are multiple articles on X-ray tomography in asymptotically hyperbolic geometry [Lef20, GGSU19, EG22, Ept22, Gre23] and the X-ray transform has been studied in asymptotically conical geometries [JV24, GLT20, VZ22] which are also Riemannian geometries with a special singularity.

There are some other geometric generalizations of geodesic X-ray tomography. For example one can add an obstacle in the manifold and study the geometry where geodesics reflect off the obstacle. Injectivity of the X-ray transform is known in some geometries with reflecting obstacles [IS16, IP22]. X-ray transforms have been studied in Finsler geometry which is a direct generalization of Riemannian geometry [AD18, IM23]. One of the motivations for Finsler geometries and X-ray tomography comes from the recent articles showing that the geometry of elastic wave propagation naturally leads to a certain class of Finsler manifolds [dHILS21, dHILS19].

More detailed history and a versatile introduction to the methods of geodesic X-ray tomography can be found in [Sha94, PSU14b, IM19, PSU23].

2. Non-smooth and singular Riemannian metrics

Simple manifolds are a class of Riemannian manifolds with boundary that arises naturally in the study of geodesic X-ray tomography. The known counter examples to injectivity of the X-ray transform all violate some properties that a simple Riemannian manifold would satisfy (see [PSU23]). In this section, we discuss the work conducted in articles [A] and [D] related to simplicity of non-smooth and conformally compact Riemannian metrics and the basic properties of such geometries. Before diving into the geometric results, we consider a well-known example of a non-simple manifold and a counter example to injectivity of the X-ray transform.

Example 1 (Large spherical cap [PSU23, Example 2.5.5]). *A large spherical cap M is constructed by removing a small spherical cap from the unit sphere $S^2 \subseteq \mathbb{R}^3$ making M larger than a hemisphere. More rigorously, we let*

$$M = \{\, (x, y, z) \in S^2 \,:\, z \leq 3/4 \,\}.$$

The large spherical cap M is a smooth manifold with boundary

$$\partial M = \{\, (x, y, z) \in S^2 \,:\, z = 3/4 \,\}$$

and comes equipped with the restriction of the round metric on S^2 to M. The geodesic segments of M are the arcs of the great circles in S^2.

Consider a smooth function $f \colon S^2 \to \mathbb{R}$ compactly supported in

$$\tilde{M} = \{\, (x, y, z) \in S^2 \,:\, |z| \leq 1/2 \,\} \subseteq M$$

and $f \equiv 1$ at the equator $\{z = 0\}$. Suppose that the function f is odd meaning that $f(-w) = -f(w)$ for all $w \in S^2$. Then, since f is supported in $\tilde{M}$, the integrals of f over the maximally extended geodesics of M starting from the boundary are equal to the integrals of f over the entire great circles. Thus, since f is odd, all integrals of f over maximal geodesics of M are zero. We have constructed a non-zero smooth function on M with vanishing X-ray transform showing that the X-ray transform of the large spherical cap is not injective.

In fact, the kernel of the X-ray transform on the large spherical cap M consists precisely of the odd functions on S^2 that are supported in M. This particular instance of the X-ray transform is closely related to the Funk transform (see e.g. [Hel11] *for details on the Funk transform).*

According to Example 1 the X-ray transform of functions is not injective on any Riemannian manifold with a smooth boundary. For this reason is we want to restrict the set of geometries under consideration. It is generally understood that simple manifolds are a reasonable class of manifolds where X-ray transforms behave well.

2.1. **Non-smooth simple manifolds.** Simple manifolds have many equivalent definitions, even when considering smooth Riemannian metrics. A collection of equivalent defining conditions is given in [PSU23, Chapter 3]. Out of the many possible definitions arise three geometric conditions that define simplicity completely:

(1) The boundary is strictly convex.
(2) There are no conjugate points.
(3) The manifold is non-trapping.

Strict convexity of the boundary in item (1) is defined in terms of the second fundamental form of ∂M. The second fundamental form is the quadratic form defined by

$$\mathrm{II}_x(v, w) = -\langle \nabla_v \nu(x), w \rangle_{g(x)}$$

for $v, w \in T_x\partial M$ and $x \in M$, where $\nu(x)$ is the inward unit normal at x. The boundary ∂M is *strictly convex* if II_x is positive definite for all $x \in \partial M$.

Intuitively speaking, absence of conjugate points in item (2) means that there are no geodesic segments starting from a common initial point converging back a common end point. More rigorously, two points $\gamma(a)$ and $\gamma(b)$ are said to be *conjugate along the geodesic segment* $\gamma\colon [a, b] \to M$ if there is a non-trivial Jacobi field $J\colon [a, b] \to TM$ along γ vanishing at a and b. By a Jacobi field we mean a smooth vector field along γ satisfying the Jacobi equation

$$D_t^2 J + R(J, \dot\gamma)\dot\gamma = 0$$

where D_t is the covariant derivative along γ and R is the Riemann curvature tensor.

A useful tool in analysing the presence of conjugate points is the so called index form I_γ along a geodesic segment $\gamma\colon [a, b] \to M$. The index form is a quadratic form along the geodesic defined by

$$I_\gamma(V, W) = \int_a^b \langle D_t V, D_t W\rangle - \langle R(V, \dot\gamma)\dot\gamma, W\rangle \, dt \tag{2.1}$$

for all vector fields V and W along γ. A classic result on the index form says that I_γ is

- positive definite, if there are no conjugate points along the segment γ,
- positive semidefinite, if the end points are conjugate along γ, and
- indefinite, if an interior point is conjugate to another point along γ.

A manifold M is *non-trapping* if all geodesics exit the manifold in a finite time, which means that for any maximally extended geodesics γ we have

$$\inf\{\, t > 0 \,:\, \gamma(t) \in \partial M \,\} < \infty.$$

We declare the following as the definition of a simple manifold, since it makes the comparison to the low regularity definition more straightforward. Definition 2 is equivalent to any other standard definition.

Definition 2. *Let (M, g) be a smooth Riemannian manifold with a smooth boundary. The manifold (M, g) is called* simple *if the following hold:*

A1: The boundary ∂M is strictly convex in the sense of the second fundamental form.
A2: Any two points of M can be joined by a unique geodesics in the interior of M whose length depends smoothly on its end points.

Next, we will describe the geometric set up of articles [A], [B] and [C]. Particularly, we discuss the work in article [A] on simple manifolds with non-smooth geometry. For the rest of the section M will be a smooth manifold with a smooth boundary. For $k \in \mathbb{N}$ and $\alpha \in [0, 1]$, we say that g is a $C^{k,\alpha}$ Riemannian metric on M and write $g \in C^{k,\alpha}(M)$, if g is a symmetric and positive definite 2-tensor field of class $C^{k,\alpha}$ in the sense of the smooth structure of M.

We restrict our attention to the case $k + \alpha \geq 2$, since unique solvability of the geodesic equation fails for $C^{1,\alpha}$ Riemannian metrics, when $\alpha < 1$ (see [SS18]). Then the set of maximally extended unit speed g-geodesics of M is naturally identified with the set of inwards pointing directions over the boundary, which we denote by $\partial_{\text{in}}(SM)$. The X-ray transform of a function is defined as in 1.2.

Since sufficiently smooth functions (or more generally tensor fields) can be integrated along 1-dimensional submanifolds, the X-ray transform is abstractly well-defined if we know the set of geodesics of the manifold. However, to access analytic tools to study the transform in Section 3, we need a more concrete realization of the set of geodesics. Hence we want the geodesic equation to have unique solutions.

One of the main themes of article [A] is to find a redefinition of a simple manifold that is better suited for Riemannian metrics of low regularity. In the article we provide a new definition of simplicity that addresses the following issues faced with low regularity simplicity.

- We want to describe the absence of conjugate points without using Jacobi fields and the Jacobi equation directly. We take an approach via an integrated global index form. In our new definition, we use a quadratic form Q which is roughly defined by $Q = \int I_\gamma \, d\gamma$ where I_γ is the along a maximal geodesic γ. That is Q measures the integrated contribution of the index forms along maximal geodesics.
- We want to express convexity of the boundary ∂M via a definition without derivatives. The second fundamental form involves derivatives of the unit normal vector, and for a C^k boundary the unit normal is C^{k-1}. Instead of using the second fundamental form, our definition approaches convexity via properties of the exit time function τ.

The definition of a simple manifold with non-smooth geometry is formulated for Riemannian metrics $g \in C^{1,1}(M)$, which is the natural lower bound on regularity we aim in the inverse problem. The definition can also be used for Riemannian metrics $g \in C^{k,\alpha}(M)$ with $k + \alpha \geq 2$ as it stands.

In the definition all norms and inner products denote the natural L^2-norms and L^2-inner products. We let X be the geodesic vector field, we say that W is a section of N if $W \colon SM \to TM$ is a function so that $W(x, v) \in \{v\}^\perp \subseteq T_xM$ and $H^1_0(N, X)$ is the space of such section with $W, XW \in L^2(N)$ and $W|_{\partial(SM)} = 0$.

Definition 3 ([A, Definition 5])**.** *Let M be the closed Euclidean unit ball in $\mathbb{R}^n$. A Riemannian metric $g \in C^{1,1}(M)$ is called a* simple $C^{1,1}$ metric *and the pair (M, g) a* simple $C^{1,1}$ manifold *if the following hold:*

B1: There is $\varepsilon > 0$ so that

$$Q(W) \coloneqq \|XW\|^2 - (RW, W) \geq \varepsilon \|W\|^2$$

for all $W \in H^1_0(N, X)$. We say that the quadratic form Q is the global index form *of the manifold (M, g).*

B2: Any two points of M can be joined by a unique geodesic in the interior of M whose length depends continuously on its end points.

B3: The squared exit time function τ^2 is Lipschitz on SM.

One of the main theorems of article [A] is the following. It proves that the classes of simple $C^{1,1}$ manifolds (see Definition 3) and smooth simple manifolds (see Definition 2) are the same when the Riemannian metric is assumed to be C^∞ smooth. Therefore the new definition is "correct", and can be added to the long list of previously known equivalent definitions for a simple manifold in smooth geometry.

Theorem 4 ([A, Theorem 2])**.** *In smooth geometry, Definitions 2 and 3 are equivalent in the following sense.*

(1) If (M, g) is a smooth simple manifold and $g \in C^\infty(M)$, then M is diffeomorphic to the closed Euclidean unit ball in $\mathbb{R}^n$ and (M, g) is a simple $C^{1,1}$ manifold (see Definition 3).

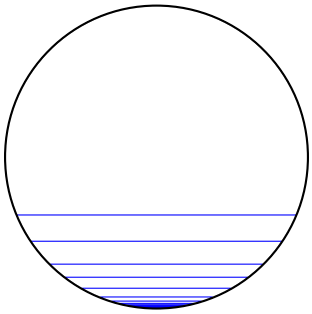

FIGURE 1. Geodesic lines approaching a boundary point horizontally.

(2) If (M, g) is a simple $C^{1,1}$ manifold and $g \in C^\infty(M)$ then (M, g) is a smooth simple manifold (see Definition 2).

In the proof of Theorem 4, we can always reduce to the case where M is the close unit ball. A simple $C^{1,1}$ manifold M is equal to the closed unit ball by definition. If (M, g) is a smooth simple manifold in the sense of Definition 2 and $g \in C^\infty(M)$, then M is necessarily diffeomorphic to the closed unit ball (see [PSU23, Proposition 3.8.5]).

Figure 2 below shows the correspondences between the conditions in Definitions 2 and 3, and serves as a proof plan. Since condition A1 trivially implies condition B2, and since prior to work of article [A] it was shown in [PSU15] that simple manifolds satisfy condition B1, we are left to prove the equivalence of conditions A1 and B3 and that B1 and B2 together imply A2.

Equivalence of conditions A1 and B3 means that the boundary of the manifold is strictly convex if and only if the squared exit time function τ^2 is Lipschitz continuous. The proof of this equivalence is based on the observation that the second fundamental form plays a role in asymptotic behavior of the exit time function. The idea for the condition that τ^2 should be Lipschitz comes from an elementary observation in Euclidean geometry which turns out to work in greater generality.

Example 5. *Consider the closed Euclidean unit disk in the plane $\mathbb{R}^2$, and equip the disk with the Euclidean metric. We use polar coordinates (r, φ) on the disc and a unit tangent vector $v_\theta = (\sin\theta, \cos\theta)$ can be identified angle $\theta \in [0, 2\pi)$. In these coordinates*

$$\tau(r, \varphi, \theta) = -r(x_\varphi \cdot v_\theta) + \sqrt{r^2((x_\varphi \cdot v_\theta)^2 - 1) + 1}.$$

where $x_\varphi = (\sin\varphi, \cos\varphi)$. The exit time function τ is differentiable in the interior of the disk and

$$\partial_r \tau(r, \varphi, \theta) = -x_\varphi \cdot v_\theta + \frac{(x_\varphi \cdot v_\theta) - 1}{\sqrt{r^2((x_\varphi \cdot v_\theta)^2 - 1) + 1}} r.$$

Consider horizontal lines through the geodesic starting at an interior point, where $r < 1$ and consider the limit of $\partial_r \tau(r, \varphi, \theta)$ when $r \to 1$ (see Figure 1). More formally this corresponds to taking $\varphi = 3\pi/2$ and $\theta = 0$ which yields

$$\partial_r \tau(r, 3\pi/2, 0) = -\frac{r}{\sqrt{1 - r^2}}.$$

We see that the derivative blows up on the limit $r \to 1$. The squared exit time $\tau^2(r, \varphi, \theta)$ is however smooth up to the boundary. For example, $\partial_r \tau^2(r, 3\pi/2, 0) = -r$ has a nice limit when $r \to 1$.

The final part of the proof of Theorem 4 is to show that conditions B2 and B1 together imply condition A2. Heuristically the quadratic form Q is defined by integrating together the index forms of maximal geodesics. Intuitively speaking we have $Q = \int_\gamma I_\gamma \, d\gamma$ where I_γ is the index form of γ. Using this idea, we prove that the estimate

$$Q(W) \geq \varepsilon \|W\|^2$$

for all $W \in H^1_0(N, X)$ localizes to a maximal geodesic γ_0 to give the estimate

$$I_{\gamma_0}(V, V) \geq \varepsilon \|V\|^2_{L^2(\gamma_0)}$$

for all non-trivial normal vector fields V along γ_0, where $\|V\|_{L^2(\gamma)}$ denotes the L^2-norm of the vector field V. This estimate says that the index form is positive definite, proving that there are no conjugate points along γ_0 as claimed in condition A2.

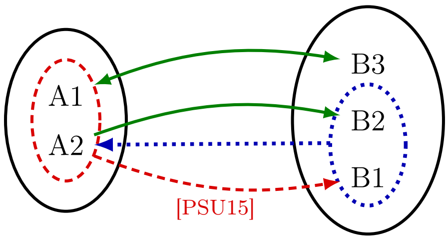

FIGURE 2. Illustration of the proof of theorem 4. The arrows represent implications except the one double headed arrow, which represents equivalence. The green (solid) arrows connect one condition to another. The red (dashed) and the blue (dotted) arrows indicate that one condition follows from the two conditions circled with the same color (style).

A smooth Riemannian manifold (M, g) with a smooth boundary is said to be *α-controlled* (see [PSU15]) for some $\alpha \in \mathbb{R}$ if

$$Q(W) \geq \alpha \|XW\|^2 \tag{2.2}$$

for all $W \in C^\infty_0(N)$. It is natural to ask, if condition B1 in Definition 3 is related to estimate (2.2) in the low regularity frame work. More specifically, are all simple $C^{1,1}$ manifolds *α-controlled* for some $\alpha > 0$ in the sense that estimate (2.2) holds for all $W \in H^1_0(N, X)$?

We proceed in the spirit of [PSU15, Lemma 11.2], but in low regularity geometry. Suppose that (M, g) is a simple $C^{1,1}$ manifold, and let $0 < \delta < 1$. Then

$$Q(W) = (1 - \delta)Q(W) + \delta Q(W) \geq (1 - \delta)\varepsilon \|W\|^2 + \delta Q(W)$$

by condition B1. Since the Riemann curvature tensor R is an L^∞ tensor field, we have $-(RW, W) \geq -C\|W\|^2$ for some $C > 0$. By choosing δ close enough to zero, it follows that

$$Q(W) \geq (1 - \delta)\varepsilon \|W\|^2 + \delta \|XW\|^2 - C\delta \|W\|^2 \geq \alpha(\|W\|^2 + \|XW\|^2) \tag{2.3}$$

for some $\alpha > 0$. This short argument shows that all simple $C^{1,1}$ manifolds are α-controlled for some positive α, and that even a stronger estimate (2.3) holds for all $W \in H^1_0(N, X)$.

2.2. **Riemannian metrics singular at the boundary.** In section 2.1, we discussed geometry that is non-smooth in a uniform way. However, uniform low regularity is not the only form of non-smoothness a Riemannian metric can have. In this section, we consider Riemannian metrics that are singular in a particular way. Also, we will see how these singular Riemannian metrics are related to wave propagation in gas giant planets, and discuss the basic properties of such gas giant geometries in accordance with the geometric results obtained in article [D].

We start from basic principles. Let M be a smooth manifold with a smooth boundary. A smooth function $x\colon M \to \mathbb{R}$ is a *boundary defining function* if $M = \{x > 0\}$, $\partial M = \{x = 0\}$ and $dx \neq 0$ at ∂M. Essentially, a boundary defining function is a measure of distance to the boundary. Let $\bar{g}$ be a Riemannian metric on M smooth up to the boundary. For any positive real number $\alpha > 0$ and any boundary defining function x on M we have a Riemannian metric $g = x^{-\alpha}\bar{g}$ in the interior of M. Such Riemannian metrics are singular at the boundary ∂M, and we call them *conformally compact*.

Example 6. *A Riemannian metric g in the interior of a smooth manifold M with smooth boundary is called* asymptotically hyperbolic *if $\bar{g} = x^2 g$ extends to a smooth Riemannian metric up to the boundary and $|dx|_{\bar{g}} = 1$ at ∂M. Asymptotically hyperbolic metrics correspond to conformally compact metrics for $\alpha = 2$.*

Asymptotically hyperbolic geometry is reasonably well understood (see e.g. [Maz88]*). For example, the geodesics approach the boundary normally and are parametrized by their second order deviation from normality. Maximally extended geodesics have infinite length. An asymptotically hyperbolic manifold has infinite Riemannian volume and the sectional curvatures are asymptotic to -1 at the boundary.*

If g is an asymptotically hyperbolic metric then the conformal class of $\bar{g}|_{T\partial M}$ is called the conformal infinity of g. Given any metric h in the conformal infinity there is a unique boundary defining function x so that $|dx|_{x^2 g} = 1$ near ∂M and $x^2 g|_{T\partial M} = h$. It follows that near ∂M the metric g can be written in the form

$$g = \frac{dx^2 + h_x}{x^2}$$

for a smooth family of Riemannian metrics h_x on the boundary with $h_0 = h$. The existence of such a normal form is the basis of asymptotic analysis in asymptotically hyperbolic geometry (cf. Proposition 8).

2.2.1. *Gas giant geometry.* In article [D] we study Riemannian metrics of the form $g = x^{-\alpha}\bar{g}$, where x is a boundary defining function, $\bar{g}$ is a Riemannian metric smooth up to the boundary and $\alpha \in (0, 2)$. In particular, the better understood cases $\alpha = 0$ and $\alpha = 2$ are excluded. This section covers basics of such geometries as developed in the article.

Such Riemannian metrics arise naturally when considering the dynamics of wave propagation in gas giant planets such as Jupiter. Unlike on a terrestrial planet, such as the Earth, on a gas giant the density of matter approaches zero at the surface of the planet. Using a polytropic model for a gas giant (see [Hor04]) one can compute that the sounds speed on a gas giant goes to zero at the surface at a rate asymptotic to the square root of the distance. This suggests that we should model wave propagation in gas giant with a Riemannian metric of the form $g = x^{-1}\bar{g}$, but we generalize a bit allowing blow up rates $\alpha \in (0, 2)$. For a detailed exposition on the polytropic model and hydrodynamics of gas giants see [D, Sections 1.2 and 5].

Definition 7. *Let M be a smooth manifold with a smooth boundary. An α-*gas giant metric *on M is a Riemannian metric of the form $g = x^{-\alpha}\bar{g}$ in the interior M° of M, where x is a boundary defining function, $\bar{g}$ is a Riemannian metric on M smooth up to the boundary and $\alpha \in (0, 2)$. The pair (M, g), where g is a gas giant metric is called an α-*gas giant.

One of the main themes of article [D] is studying the basic geometric properties of gas giant metrics and their geodesics. Next, we highlight some of the results in this direction obtained in the article.

Most of the analysis on a gas giant (M, g) happens near the boundary. The following proposition shows that there is a collar neighbourhood of the boundary and an associated coordinate system which brings the metric tensor to a particularly nice form and makes the analysis manageable. The form obtained for the metric is called Graham-Lee normal form after [GL91], where the authors showed the existence of an analogous coordinate system for asymptotically hyperbolic metrics.

Proposition 8 ([D, Proposition 2]). *Let g be an α-gas giant metric on M. Then there is a well-defined Riemannian metric h_0 on ∂M, and an associated boundary defining function x on M so that*

$$g(x, y, dx, dy) = \frac{dx^2 + h(x, y, dy)}{x^\alpha}, \tag{2.4}$$

where $h(0, y, dy) = h_0(y, dy)$ for all $y \in \partial M$.

The proof of Proposition 8 is based on ideas already introduced in [GL91]. If ϕ is a diffeomorphism from a neighbourhood of $\{0\} \times \partial M$ in $[0, \infty) \times \partial M$ to a neighbourhood of ∂M in M so that $\phi^* g$ is in the form (2.4) then the boundary defining function $\tilde{x} = x \circ \phi^{-1}$ solves the eikonal equation

$$\left| \frac{d\tilde{x}}{\tilde{x}^{\alpha/2}} \right|_g^2 = 1.$$

Then the objective is to solve the eikonal equation. In the asymptotically hyperbolic case the corresponding eikonal equation can be reduced to a non-singular form by a clever change of variables as is shown in [GL91]. This technique is not available in the case of gas giant metrics and we are forced to work with singular coefficients as is the case for general edge metric in [GK12]. Fortunately, we can use the existence results from [GK12] to prove existence also in our case.

Article [D] describes some basic geometric properties of gas giant geometries. As an immediate application for the normal form found in Proposition 8, we see that gas giants have infinite Riemannian volume unless the blow up in the metric is weak enough, and we can find the blow up rates of the volume. It was stated in [D, Proposition 3], that if (M, g) is an α-gas giant, then

- $\mathrm{Vol}_g(M) < \infty$ if and only if $\alpha < 2/n$,
- $\mathrm{Vol}_g(\{x \geq \varepsilon\})$ is asymptotic to $C\varepsilon^{1-\frac{n\alpha}{2}}$ when $\alpha > 2/n$, and
- $\mathrm{Vol}_g(\{x \geq \varepsilon\})$ is asymptotic to $-C\log(\varepsilon)$ when $\alpha = 2/n$.

These facts follow from the simple observation that the volume form of a gas giant metric g is $x^{-n\alpha/2}\, dx\, dV_h$ in the coordinates of Proposition 8, where dV_h is the volume form of the metric $h(x, \cdot\,)$ on ∂M.

In addition, the normal form is used to prove that the level sets $\{x = \varepsilon\}$ in M are stricly convex for small $\varepsilon > 0$ (see [D, Proposition 4]). This fact is a simple computation of the second fundamental forms of the level sets in the local coordinates provided by Proposition 8.

The blow up rates of sectional curvatures were computed in article [D]. It was shown that the blow up rates are determined by the geometry instrinsically in the sense that they can be derived solely from interior knowledge of the metric (see [D, Proposition 1]). More rigorously, suppose that g is an α-gas giant metric in the interior of a compact smooth manifold M with smooth boundary. Then there is a smoothly varying orthonormal basis of sections for TM such that the sectional curvatures for 2-planes spanned by pairs of these basis vectors are asymptotic to

$$(2.5) \qquad -\frac{2\alpha}{(2-\alpha)^2} d_g(\,\cdot\,, \partial M)^{-2}, \quad \text{or} \quad -\frac{\alpha^2}{(2-\alpha)^2} d_g(\,\cdot\,, \partial M)^{-2}$$

depending on the generator pairs. Interestingly the strength of the blow up is independent of α, but α shows up in the coefficients. The blow up rates can be determined by a direct coordinate computation. The proof reflects the fact that these formulas are not valid for $\alpha = 2$, which shows up in the last step were we use the fact that the function x is related to the distance s to the boundary by the formula $s = (1 - \alpha/2)x^{1-\alpha/2}$.

2.2.2. *Geodesics of a gas giant.* So far we have described the general geometry of a gas giant. Next, we study the geodesics of an α-gas giant (M, g). Let $z = (x, y)$ be coordinates near the boundary of M as in Proposition 8 and denote by $\zeta = (\xi, \eta)$ the corresponding coordinates for covectors. We use Hamiltonian formalism and consider the bicharacteristic curves in T^*M for the Hamiltonian

$$H(x, y, \xi, \eta) = \frac{1}{2}x^\alpha \xi^2 + \frac{1}{2}x^\alpha h^{ij}(x, y)\eta_i\eta_j.$$

The equations of motion read

$$(2.6) \qquad \dot{x} = x^\alpha \xi, \quad \dot{y}^i = x^\alpha h^{ij}(x, y)\eta_j, \quad \dot{\xi} = -\alpha x^{-1} H(x, y, \xi, \eta) - \frac{1}{2}x^\alpha \partial_x h^{ij}(x, y)\eta_i\eta_j,$$

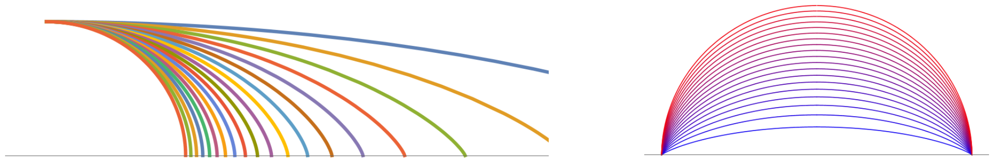

FIGURE 3. Geodesic on a gas giants near a flattened boundary depicted in two different ways. One curve in each picture corresponds to one value of α. On the left-hand side, geodesics start from a distance ε into a direction parallel to the boundary. On the right-hand side, geodesics connect two near by boundary points.

and

$$\dot{\eta}_i = -\frac{1}{2}x^\alpha \partial_{y^i} h^{kj}(x,y)\eta_k\eta_j.$$

We often write the bicharacteristics as $(z(t), \zeta(t))$ where $z(t) = (x(t), y(t))$ and $\zeta(t) = (\xi(t), \eta(t))$ where $x(t)$, $y(t)$, $\xi(t)$ and $\eta(t)$ correspond to coordinates (x, y) and (ξ, η).

The bicharacteristic curves have the following crucial property, which is unique to gas giant geometries among conformally compact geometries described by $g = x^\alpha \bar{g}$ and $\alpha > 0$. Let $\gamma(t) = (z(t), \zeta(t))$ be a bicharacteristic with $x(0) < \varepsilon$ and $\xi(0) \leq 0$ for $\varepsilon > 0$ small enough. Then $z(t)$ converges to a unique point $(0, \bar{y}) \in \partial M$ and $\eta(t)$ converges to a unique covector $\bar{\eta}$ at the boundary. In particular, the convergence happens in a finite time $T > 0$ (see [D, Lemma 6]). This means that gas giants are in a sense locally non-trapping. Any bicharacteristic starting close enough to the boundary and towards it, reaches the boundary in a finite time.

The local non-trapping property described above is not surprising. A heuristic explanation is as follows. Look at a radial geodesic $\gamma(t) = (x(t), 0)$ near the boundary, where $x(0) < \varepsilon$. It can be shown that x is strictly decreasing in t. Then if $\gamma(t)$ exists on the interval $[0, T]$ we have

$$T = -\int_0^T \frac{\dot{x}}{x^{\frac{\alpha}{2}}}\, dt = -\left(1 - \frac{\alpha}{2}\right)^{-1} \left(x(T)^{1-\frac{\alpha}{2}} - x(0)^{1-\frac{\alpha}{2}}\right).$$

Since $x(T) \geq 0$, this can be turned into the estimate

$$T \leq \left(1 - \frac{\alpha}{2}\right)^{-1} x(0)^{1-\frac{\alpha}{2}} < \left(1 - \frac{\alpha}{2}\right)^{-1} \varepsilon^{1-\frac{\alpha}{2}}.$$

Note that the upper bound is even independent of $\gamma(t)$ and only depends on ε and α.

A feature that gas giants share with asymptotically hyperbolic geometries is that geodesics hit the boundary normally. If $\gamma(t) = (z(t), \zeta(t))$ is a bicharacteristic, then it follows from (2.6) asymptotics computed in [D, Lemma 6] that

$$\frac{\partial y^i}{\partial x} = \frac{h^{ij}\eta_j}{\xi} \to 0$$

which means that $z(t)$ hits the boundary in the direction of ∂_x.

Due to natural non-trappingness of a gas giant near the boundary there are two notions of distance between a pair of boundary points. We can measure the distance between boundary points x and y in the sense of g and in the sense of the boundary metric h_0 (see Proposition 8). There is a power law type relation between the distances $d_g(x, y)$ and $d_{h_0}(x, y)$. This leads to a way to compute the Hausdorff dimension of the metric space $(\partial M, d_g)$ and consequently the Hausdorff dimension of (M, d_g).

Proposition 9 ([D, Proposition 15]). *If the dimension of M is n, the Hausdorff dimension of an α-gas giant (M, g) is*

$$\max\left\{n, \frac{2}{2-\alpha}(n-1)\right\}.$$

The proof of Proposition 9 is based on an observation that the metrics d_g and $d_{h_0}^{1-\alpha/2}$ are bi-Lipschitz equivalent on ∂M. Therefore to compute the Hausdorff dimension of $(\partial M, d_g)$ it is enough to compute the dimension of $(\partial M, d_{h_0}^{1-\alpha/2})$, which can be found in terms of the Hausdorff dimension of $(\partial M, d_{h_0})$. This leads to the result since the Hausdorff dimension (M, d_g) is the maximum of the Hausdorff dimensions of $(\partial M, d_g)$ and (M°, d_g). The ultimate fact that allows the entire sequence of deductions is the power law type relation between the metrics d_g and d_{h_0} on the boundary, which can be shown by simple computations with the geodesic equations.

2.2.3. *A travel time problem on a gas giant.* Lastly, we consider a travel time problem on a gas giant. For $z \in \partial M$ we define the function $r(z)\colon \partial M \to \mathbb{R}$ by $r(z)(y) = d_g(z, y)$ where $y \in \partial M$. The travel time data of M is the image of $r(M)$. The information of the maps $r(z)$ is conveniently packed into a single map $r\colon M \to C(\partial M)$ where $z \mapsto r(z)$ for all $z \in M$. In other words, the travel time data encodes the Riemannian distances from any point in the manifold to all boundary points. The objective is to show that such data determines the geometry to the degree allowed by coordinate invariance.

In the following, a *simple gas giant metric* is gas giant metric that is globally non-trapping and does not have conjugate points. We do not make assumptions on convexity of the boundary, since the boundary of a gas giant is strictly convex in the sense of [D, Proposition 4].

Theorem 10 ([D, Theorem 16]). *For $i = 1, 2$ let g_i be simple α_i-gas giant metrics on M for some $\alpha_i \in (0, 2)$. If $r_1(M) = r_2(M)$ then $\alpha_1 = \alpha_2$ and g_1 is isometric to g_2 by a diffeomorphism that is the identity on ∂M.*

The corresponding result is known in standard Riemannian geometry, and a proof can be found in [KKL01]. It was proved in [ILS23] that the recovery of g is stable, where the authors also simplified the proof of uniqueness by using the Myers-Steenrod theorem (see [MS39, Pal57]). The proof of Theorem 10 in article [D] proceeds along the lines of [ILS23] extending the method to more general families of Riemannian metrics. We are also able to recover the blow rate α, which is based on the observation that the sectional curvature blow rates can be recovered from interior knowledge of a gas giant metric alone, and α appears in the coefficients (see (2.5)).

3. Approaches to X-ray tomography in non-smooth and singular geometries

This section outlines results in geodesic X-ray tomography and tensorial X-ray tomography in non-smooth and gas giant geometries. The results we presented were obtained in articles [A], [B], [C] and [D]. Section 3.1 concerns injectivity results proved using the so called Pestov identities and is related to articles [A], [B] and [D]. Section 3.2 concerns an injectivity result proved using the so called normal operator of the X-ray tranform and is related to article [C]. We give short introductions to both methods before discussing the results of the articles.

There are two fundamental questions in X-ray tomography on Riemannian manifolds with a boundary. The first is *geodesic X-ray tomography.*

Question 11. *Is a function on a Riemannian manifold with boundary uniquely determined by its integrals over all maximal geodesics of the manifold?*

For transforms tensor fields the natural uniqueness question is different. Let p be a smooth 1-form vanishing on the boundary ∂M and consider its symmetrized covariant derivative $f = \sigma\nabla p$. For any maximal geodesic γ in M the integral of f over γ is

$$If(\gamma) = 2\int_0^{l(\gamma)} \nabla_i p_j(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt = 2\int_0^{l(\gamma)} \partial_t(p_j(\gamma(t))\dot\gamma^j(t))\,dt.$$

Thus, since p vanishes on the boundary, the X-ray transform of f is zero. The X-ray transform of symmetric tensor fields of rank m is not injective for $m \geq 1$, since *potential tensor fields* i.e. tensor fields of the form $\sigma\nabla p$ where p is an symmetric $(m-1)$-tensor field with $p|_{\partial M} = 0$ are in the kernel. The natural question is whether potential tensor field are the only obstruction to injectivity. Any sufficiently smooth symmetric m-tensor field can be uniquely expressed as

$$f = f^s + \sigma\nabla p$$

where f^s is the *solenoidal part* and $p|_{\partial M} = 0$. For this reason, we say that the X-ray transform of tensor fields is *solenoidally injective*, if its kernel consists solely of potential tensor fields.

The second fundamental question is *tensorial geodesic X-ray tomography.*

Question 12. *Is a symmetric solenoidal tensor field on a Riemannian manifold with boundary uniquely determined by its integrals over all maximal geodesics of the manifold?*

3.1. **Pestov identities.** In this section we discuss the so called Pestov identity method used to study a transport equation in SM related to the geodesic X-ray transform of functions and tensor fields. We begin with an overview of the technique in standard Riemannian geometry. In sections 3.1.3 and 3.1.4 we highlight the features of the method in non-smooth Riemannian geometries and gas giant geometries, and we discuss the results obtained in articles [A], [B] and [D] on the X-ray transforms in such geometries.

3.1.1. *A transport equation on the unit sphere bundle.* Let (M, g) smooth Riemannian manifold with a smooth boundary. For a smooth function $f \in C^\infty(SM)$ consider the function $u^f : SM \to \mathbb{R}$ defined by

$$u^f(x, v) = \int_0^{\tau(x,v)} f(\phi_t(x, v))\, dt$$

for all $(x, v) \in SM$, where ϕ_t denotes the geodesic flow. The function u^f is called the integral function of f and denoted simply by u in this section. It is clear that $If = u|_{\partial_{\text{in}}(SM)}$ and by the fundamental theorem of calculus $Xu = -f$. The equation is called the *transport equation in SM.* Vanishing of the X-ray transform of f is equivalent to the boundary condition $u|_{\partial_{\text{in}}(SM)} = 0$.

For now consider a sufficiently smooth function f on M and identify f with its pullback $\pi^* f$ that is a function on SM. Suppose that the transport equation is uniquely solvable i.e. if u is any sufficiently smooth function on SM with $Xu = -f$ and $u|_{\partial_{\text{in}}(SM)} = 0$, then $u = 0$ on SM. Then if f is a function with vanishing X-ray transform it follows that the integral function u of f satisfies the transport equation with zero boundary values. Thus $u = 0$ in SM by unique solvability, and more over $f = -Xu = 0$ proving that the X-ray transform is injective.

Conversely, if the X-ray transform is injective, then the transport problem is uniquely solvable. To see this let f be a sufficiently smooth function with $If = 0$ and consider the problem $Xu = -f$ and $u|_{\partial_{\text{in}}(SM)} = 0$. Since I is injective, we have $Xu = -f = 0$. This means that the solution u is invariant under the geodesic flow, and therefore we must have $u = 0$ in SM since $u|_{\partial_{\text{in}}(SM)} = 0$. We have reformulated injectivity of the X-ray transform of functions as an equivalent problem: If $Xu = -f$ and $u|_{\partial_{\text{in}}(SM)} = 0$ does it follow that $u = 0$ in SM?

Then consider a sufficiently smooth symmetric tensor field f on M of rank $m \geq 1$. Such a tensor field can be identified with a function λf on SM defined by the formula

$$\lambda f(x, v) = f_{i_1 \cdots i_m}(x) v^{i_1} \cdots v^{i_m}. \tag{3.1}$$

It is easily verified that $X(\lambda f) = \lambda(\sigma\nabla f)$.

Suppose that the X-ray transform of symmetric m-tensor fields is solenoidally injective. Then if f is a symmetric m-tensor field with vanishing X-ray transform, there is a symmetric $(m-1)$-tensor field p so that $f = \sigma\nabla p$ and $p|_{\partial M} = 0$. Then if we let $u = -\lambda p$ we find that $Xu = -f$ and $u|_{\partial_{\text{in}}(SM)} = 0$.

Conversely, assume that for all sufficiently smooth solutions u to the transport problem $Xu = -f$ and $u|_{\partial_{\text{in}}(SM)} = 0$ there is a symmetric $(m-1)$-tensor field p so that $u = -\lambda p$. Then if f has vanishing X-ray transform, we know that the integral function u of f satisfies $Xu = -f$ and $u|_{\partial_{\text{in}}(SM)} = 0$. Thus there is a symmetric $(m-1)$-tensor field p so that $u = -\lambda p$. It follows that $f = -Xu = \sigma\nabla p$ and $p|_{\partial M} = 0$.

In the case of tensorial X-ray transform, solenoidal injectivity is a question of proving that the integral function of a tensor field is necessarily induced by a tensor field via the identification (3.1). Then it is easily seen from the transport equation that the kernel of the tensorial ray transform consists of potential fields.

We established connections between the X-ray transform of a function or a tensor field and the transport equation in SM. Such a transport equation can be studied using the so called *Pestov identity method*, which we describe in the next section.

3.1.2. *A Pestov identity is smooth simple geometry.* In this section we outline a proof of injectivity of the geodesic X-ray transform using Pestov identities. This constitutes of deriving so called energy estimates for the transport equation $Xu = -f$ in SM. The study of the transport equation $Xu = -f$ outlined in section 3.1.1 begun in the works of Mukhometov (see [Muk75, Muk77, Muk78]), and has been extended to many different geometric set ups (see Section 1.2).

Let (M, g) be a smooth Riemannian manifold with a smooth boundary. If $u \in C^\infty(SM)$ vanishes on the boundary $\partial(SM)$, then

$$\|\overset{\mathtt{v}}{\nabla} Xu\|^2 = Q(\overset{\mathtt{v}}{\nabla} u) + (n-1)\|Xu\|^2, \tag{3.2}$$

where Q is a quadratic form on defined by the formula

$$Q(W) = \|XW\|^2 - (RW, W)$$

for all $W \in C^\infty(N)$ and $\overset{\mathtt{v}}{\nabla} u$ is the vertical part of the gradient of u with respect to the Sasaki metric on SM. This is known as the *Pestov identity.* A proof using commutator formulas for the operators X, $\overset{\mathtt{v}}{\nabla}$ and $\overset{\mathtt{h}}{\nabla}$ can be found in [PSU15].

Suppose that the manifold (M, g) is simple. We assume simplicity since it is well-known that the X-ray transform in not injective in some non-simple geometries (see Example 1). Let $f \in C^\infty(M)$ be a smooth function, and consider consider the problem $Xu = -f$ and $u|_{\partial_{\text{in}}(SM)} = 0$. We can apply the Pestov identity (3.2) to any smooth solution u of the problem to obtain

$$\|\overset{\mathtt{v}}{\nabla} f\|^2 = Q(\overset{\mathtt{v}}{\nabla} u) + (n-1)\|f\|^2. \tag{3.3}$$

The function f is a function on M interpreted as a function on SM via the pullback of the bundle map $\pi\colon SM \to M$. Thus, in particular, f is independent of direction in SM, which means that $\overset{\mathtt{v}}{\nabla} f = 0$.

To derive more information from (3.3), we use simplicity of the geometry. We take a closer look at the quadratic form Q. Let γ be a geodesic of M and let W be a smooth section of N that vanishes on $\partial(SM)$. The assignment $W_\gamma(t) := W(\gamma(t))$ defines a normal vector field W_γ along the geodesic γ that vanishes at the end points of γ. Immediately from definitions we see that $XW = D_t W_\gamma$, where D_t is the covariant derivative along γ. Then by Santaló's formula

$$Q(W) = \int_{\partial_{\text{in}}(SM)} I_\gamma(W_\gamma, W_\gamma)\, d\gamma,$$

where I_γ is the index form along γ defined in (2.1). We assumed that the manifold M is simple, and thus there are no conjugate points, which gives that $Q(W) \geq C\|W\|^2$.

Combining the estimate for $Q(W)$ and the vanishing of the vertical gradient $\overset{\mathtt{v}}{\nabla} f$, equation (3.3) reduces to $0 \geq (n-1)\|f\|^2$. Thus $f = 0$ and u flow invariant with $u|_{\partial_{\text{in}}(SM)} = 0$. We have shown that if $Xu = -f$ and $u|_{\partial_{\text{in}}(SM)} = 0$ then $u = 0$ in SM. This is equivalent to injectivity of the X-ray transform on the space of smooth functions as explained in section 3.1.1.

A vital step in the proof is to show that the integral function u^f defined in Section 3.1.1 is smooth enough for the Pestov identity. We need to verify that any smooth solution u to the transport problem exists. For general $f \in C^\infty(M)$ the integral function u^f is not C^∞ smooth in SM not even in simple geometry. The exit time function τ is not smooth in the region $\partial_0(SM)$, which is an issue. The remedying fact is that $u|_{\partial_{\text{in}}(SM)} = 0$, or equivalently that the X-ray transform of f vanishes. We have omit the details of this step here and refer the reader to [PSU23] instead.

Let f be a smooth 1-form in M with vanishing X-ray transform. In this case, we can still prove that f has to be a potential directly from the Pestov identity. The crucial observation is that

$$\|\overset{\mathrm{v}}{\nabla} f\|^2 = (n-1)\|f\|^2.$$

Then identity (3.3) yields $Q(\overset{\mathrm{v}}{\nabla} u) = 0$, which by positive definiteness of Q gives $\overset{\mathrm{v}}{\nabla} u = 0$. Therefore there is a function p on M so that $u = -\pi^* p$ where $\pi\colon SM \to M$ is the bundle map, which is merely a restatement of the fact that u is independent of direction. It follows that $f = -Xu = dp = \sigma\nabla p$ and $p|_{\partial M} = 0$ confirming that the X-ray transform of 1-forms is solenoidally injective.

Let $m \geq 2$ and consider a symmetric m-tensor field f in M with vanishing X-ray transform. We additionally assume that (M, g) has non-positive sectional curvature. In general solenoidal injectivity of tensor fields of order 2 or greater is an open question in simple geometry, but partial results are known (see Section 1.2).

We will recall some facts originating from Fourier analysis on the sphere $S^{n-1} \subseteq \mathbb{R}^n$. It is well-known that a function $u \in C^\infty(SM)$ can be uniquely decomposed as an L^2-convergent and orthogonal series

$$u = \sum_{k=0}^{\infty} u_k$$

where

$$u_k \in \Omega_k \coloneqq \{\, w \in C^\infty(SM) \,:\, \overset{\mathrm{v}}{\Delta} w = k(k-n+2)w \,\}$$

and $\overset{\mathrm{v}}{\Delta}$ is the vertical gradient. It is also true that a function u on SM is induced by a symmetric m-tensor field via identification (3.1) if and only if $u_k = 0$ for all $k > m$ and $u_k = 0$ for all $k \equiv m \pmod 2$.

The geodesic vector field X decomposes as a sum of operators X_+ and X_- with the property that $X_\pm\colon \Omega_k \to \Omega_{m\pm1}$ continuously. Then it can be shown that

$$\|X_+ u\|^2 = \sum_{k=0}^{\infty} \|X_+ u_k\|^2$$

from which we deduce that $\|X_+ u_k\| \to 0$ sufficiently fast as $k \to \infty$. This together with the Pestov identity and non-positivity of sectional curvature can be turned into estimates proving that the functions u_k have to vanish identically for $k \geq m$. For details we refer the reader to [PSU15, IP22, B].

It is a straightforward computation that $u_k = 0$ for all $k \equiv m \pmod 2$. We have shown that the integral function u must be induced by a symmetric $(m-1)$-tensor field p in the sense that $u = -\lambda p$ and since $u|_{\partial_{\mathrm{in}}(SM)} = 0$ it holds that $p|_{\partial M} = 0$. Therefore $f = -Xu = \sigma\nabla p$ and we have proved that the tensorial X-ray transform is solenoidally injective.

Next, we move onto results obtained in articles [A], [B] and [D] about injectivity of the X-ray transform in non-smooth and gas giant geometries.

3.1.3. *A Pestov identity in non-smooth simple geometry.* Theorem 1 of article [A] states that the X-ray transform of Lipschitz functions is injective in simple $C^{1,1}$ geometry. Theorem 1 of article [B] states that the X-ray transform of $C^{1,1}$ tensor fields solenoidally injective in simple $C^{1,1}$ geometry with almost everywhere non-positive sectional curvature. In this section, we explain the features of low regularity geometry in the Pestov method.

The following lemma is gives a Pestov identity in simple $C^{1,1}$ geometry. The form of the identity is the same as in smooth simple geometry, but the regularity assumptions of u have been modified.

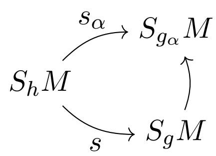

FIGURE 4. The various bundles and radial diffeomorphisms of the proof of Lemma 13.

Lemma 13 ([A, Lemma 9]). *Let (M,g) be a simple $C^{1,1}$ manifold. For all $u \in \mathrm{Lip}_0(SM)$ with $Xu \in H^1(SM)$ and $\overset{\mathrm{v}}{\nabla} u \in H^1(N,X)$ we have*

$$\|\overset{\mathrm{v}}{\nabla} Xu\|^2 = Q(\overset{\mathrm{v}}{\nabla} u) + (n-1)\|Xu\|^2. \tag{3.4}$$

The Riemannian curvature tensor of a $C^{1,1}$ Riemannian metric is interpreted as an L^∞ tensor field in identity (3.4).

The identity cannot be proved as in smooth geometry for the simple reason that the commutator formulas used in the standard proof are not classically well-defined in $C^{1,1}$ geometry. The proof of Lemma 13 uses smooth approximations g_α of the $C^{1,1}$ metric g and a smooth reference metric h. There is a familiar Pestov identity on each of the manifolds (M, g_α) and we translate these identities on the smooth Riemannian manifold (M,h). For the translations we use radial diffeomorphisms $s_\alpha\colon S_hM \to S_{g_\alpha}M$. Then we prove that the translated Pestov identities have a well-defined limit which translates to the claimed identity via another radial diffeomorphism $s\colon S_hM \to S_gM$. The various bundles and diffeomorphisms are depicted in Figure 4.

One of the main themes of article [A] is proving that the integral function u of a Lipschitz function f on SM with $f|_{\partial SM} = 0$ is again a Lipschitz function. This result allows us to prove that u has all necessary regularity for the Pestov identity to be applied.

When f is a Lipschitz function on M we prove that $f|_{\partial M} = 0$ given that f is in the kernel of the X-ray transform. Thus, in regularity consideration of u, it suffices to assume that $f|_{\partial M} = 0$. Vanishing of f at the boundary coupled with the fact that the squared exit time τ^2 is Lipschitz on a simple $C^{1,1}$ manifold is used to prove that the integral function u is also Lipschitz.

Boundary determination is more delicate when f is a tensor field. In Lemma 2 of article [B] we prove that tensor fields in the kernel of the transform are potential fields at the boundary. This is proved by an explicit local construction yielding better regularity for the potential field at the boundary than previous results (cf. [SU05]).

With regularity of the integral function and a Pestov identity the proof of injectivity proceeds as the proof in the smooth case with only minor modifications (cf. section 3.1.2). Instead of arguing that the quadratic form Q is positive definite, the definiteness is given since (M,g) is simple $C^{1,1}$ manifold. We arrive at the theorems.

Theorem 14 ([A, Theorem 1]). *Let (M,g) be a simple $C^{1,1}$ manifold. If a Lipschitz function f integrates to zero over all maximal geodesics of M then $f=0$.*

Theorem 15 ([B, Theorem 1]). *Let (M,g) be a simple $C^{1,1}$ manifold. Assume that M has almost everywhere non-positive sectional curvature. Let $m \geq 1$ be an integer.*

- *If $p \in C^{1,1}(M)$ then the X-ray transform of $\sigma\nabla p$ vanishes.*
- *If the X-ray transform of a symmetric m-tensor field $f \in C^{1,1}(M)$ vanishes, then there is a symmetric $(m-1)$-tensor field $p \in \mathrm{Lip}(M)$ so that $\sigma\nabla p = f$ almost everywhere in M.*

The apparent asymmetry in the regularity of p between the claims in Theorem 15 is most likely an artifact of our proof techniques. This asymmetry is discussed at length in Section 1.2. of article [B].

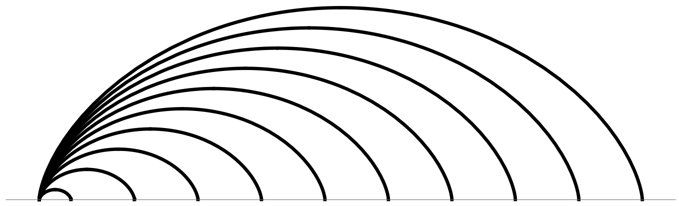

FIGURE 5. The fundamentally behind Lemma 17 is a construction of a sequence of geodesics converging to a given boundary point.

3.1.4. *A Pestov identity in gas giant geometry.* Theorem 17 of article [D] states that on a simple α-gas giant the X-ray transform is injective on $C^\infty(\bar{M})$. In this section we explain the steps needed to prove the theorem and particularly the features of gas giant geometry in play. The proof of the theorem is again based on a Pestov identity.

We provide a Pestov identity on an α-gas giant. The form of the identity is still familiar, but the regularity assumptions are modified to reflect expectations for the regularity of the integral function. In Lemma 16 we denote the unit cosphere bundle over the interior by S^*M° and $\nabla_G u$ is the gradient of u with respect to the Sasaki metric G.

Lemma 16 ([D, Lemma 21]). *Let (M, g) be a simple α-gas giant. For all $u \in x^\infty C^\infty(S^*M^\circ)$ satisfying $Xu = -f$ as well as $\overset{\mathtt{v}}{\nabla} Xu, X\overset{\mathtt{v}}{\nabla} u \in L^2(N)$ and $\nabla_G u \in x^\infty L^\infty(S^*M; TS^*M)$ it holds that*

$$\|\overset{\mathtt{v}}{\nabla} Xu\|^2 = Q(\overset{\mathtt{v}}{\nabla} u) + (n-1)\|Xu\|^2. \tag{3.5}$$

Note that this identity is formulated on the unit cosphere bundle S^*M and not on the unit sphere bundle SM. The structure is still essentially the same (see [D, Section 3.1]).

The proof of Lemma 16 begins by considering a truncated manifold $M_\varepsilon = \{x \geq \varepsilon\} \subseteq M$. For small $\varepsilon > 0$, the truncated manifold is a standard simple manifold with a smooth boundary, and it is known that the Pestov identity

$$\|\overset{\mathtt{v}}{\nabla} Xu\|^2_{L^2(N_\varepsilon)} = Q_\varepsilon(\overset{\mathtt{v}}{\nabla} u) + (n-1)\|Xu\|^2_{L^2(S^*M_\varepsilon)} + B_\varepsilon(u)$$

holds for all sufficiently smooth functions u on M_ε. There is a boundary term $B_\varepsilon(u)$ involved, which we need to take into account, since the integral function considered later might not vanish on the boundary ∂M_ε. The boundary term is

$$B_\varepsilon(u) = \int_{\partial(S^*M_\varepsilon)} \left(\langle \overset{\mathtt{v}}{\nabla} u, \overset{\mathtt{h}}{\nabla} u \rangle + (n-1)uXu \right) d\sigma_\varepsilon \tag{3.6}$$

where $d\sigma_\varepsilon$ is the volume form of the restriction of the Sasaki metric on S^*M_ε to $\partial S^*M_\varepsilon$. There are two points of interest when studying the limit of (3.6) when $\varepsilon \to 0$. We need to combat the blow up of the volume and the blow up of curvature as $\varepsilon \to 0$. Since the blow ups are polynomial in ε, we can use the assumptions that u and its gradient $\nabla_G u$ vanish faster than any polynomial at the boundary to prove that (3.6) has the limit (3.5). The reasonability of these vanishing assumptions is discussed next in Lemma 17 and in the paragraph after.

Lemma 17 ([D, Lemma 20]). *Let g be a simple α-gas giant metric on M, and let $f \in C^\infty(\bar{M})$. If the integrals of f over all maximal geodesic in M vanish, then $f \in x^\infty C^\infty(M)$.*

The fundamental construction behind the proof of Lemma 17 is proving the existence of a sequence of geodesics converging to a boundary point so that the lengths of the geodesics converge to 0 (see Figure 5). To construct such a sequence, pick a sequence (x_k) points along a smooth boundary curve converging to a fixed boundary point x. Then we can use the observation that the metrics $d_g(x, y)$ and $d_h(x, y)^{1-\alpha/2}$ are bi-Lipschitz equivalent (see Section 2.2.2) to prove that the geodesics γ_k connecting x to x_k have the desired properties.

Lemma 17 can be used to prove that the integral function u^f of a smooth function $f \in C^\infty(\bar{M})$ with vanishing X-ray transform also has to vanish to an infinite order at the boundary. Confirming that the gradient $\nabla_G u^f$ vanishes to infinite order at the boundary we use growth estimates for Jacobi fields and the fact that derivatives of u^f can be computed using these fields. Once we have the details on regularity of u^f, we can proceed as in section 3.1.2 with slight modifications. The injectivity result we obtain is the following.

Theorem 18 ([D, Theorem 17])**.** *Let (M, g) be a simple α-gas giant. If a function f smooth up to the boundary ∂M has vanishing X-ray transform, then $f = 0$.*

3.2. **The normal operator.** In this section, we introduce another method widely used in the study of geodesic X-ray tomography. The basic idea is that by combining the X-ray transform of functions with its adjoint we obtain an operator mapping functions on the manifold to functions on the manifold. The expectation is that the normal operator defined in this way is more manageable than the X-ray transform. The normal operator is often studied using techniques from microlocal analysis.

We provide a short introduction to the normal operator of the X-ray transform in Section 3.2.1 and discuss results obtained in article [C] related to the normal operator in non-smooth geometry in Section 3.2.2

3.2.1. *The normal operator in smooth simple geometry.* Let (M, g) be a simple Riemannian manifold with a smooth boundary. Define the function $\mu\colon \partial SM \to \mathbb{R}$ by $\mu(x, v) = \langle \nu(x), v\rangle$, where $\nu(x)$ is the inward unit normal vector at $x \in \partial M$. It is well-known that the X-ray transform of functions is a bounded linear operator

$$I\colon L^2(M) \to L^2_\mu(\partial_{\text{in}}(SM)),$$

where $L^2_\mu(\partial_{\text{in}}(SM))$ is the L^2 space on $\partial_{\text{in}}(SM)$ with the natural measure weighted by μ. The adjoint I^* of I can be computed using the Santaló's formula and

$$I^*h(x) = \int_{S_xM} h(\phi_{-\tau(x,-v)}(x, v))\, dS_x$$

for all $h \in L^2_\mu(\partial_{\text{in}}(SM))$. Composing I with its adjoint I^* gives a bounded linear operator

$$N\colon L^2(M) \to L^2(M)$$

known as the *normal operator*. Unraveling the formulas we find that

$$Nf(x) = 2\int_{S_xM}\int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt\, dS_x \tag{3.7}$$

for all $f \in L^2(M)$.

The usefulness of N is due to to the fact that N is an elliptic pseudodifferential operator of order -1 in M° with principal symbol $c_n\, |\xi|^{-1}_{g(x)}$ where c_n is a dimensional constant. This was proved in [PU05], where the authors use a clever change of variables to write the operator in the form

$$Nf(x) = 2\int_M \frac{a(x, y)}{d_g(x, y)^{n-1}} f(y)\, dV_g$$

where $a(x, y) = \det(d\exp_x|_{\exp_x^{-1}(y)})^{-1}$. The Schwartz kernel

$$K(x, y) = \frac{2a(x, y)\sqrt{\det(g(x))}}{d_g(x, y)^{n-1}}$$

of N has a singularity of type $|x - y|^{-n+1}$ proving that the operator is pseudodifferential and to the leading order behaves like the operator corresponding to the Schwartz kernel

$$\tilde{K}(x, y) = \frac{2\sqrt{\det(g(x))}}{d_g(x, y)^{n-1}}.$$

The symbol of the operator with kernel $\tilde{K}$ can be computed by taking the Fourier transform of $\tilde{K}(x, x-z)$ (see [Ste93]) which is explicitly computable yielding $c_n \left|\xi\right|_{g(x)}^{-1}$ as the principal symbol of N.

Ellipticity of N gives us the first application to X-ray tomography. Suppose that $f \in L^2(M)$ is compactly supported in the interior of M and $If = 0$. Then $Nf = 0$ and since N is elliptic there is a parametrix P inverting N up to smoothing terms. In symbols, we get

$$0 = PN = f + Rf$$

where R is a smoothing operator. Thus $f = -Rf$ is C^∞ smooth. The kernel of I on L^2 functions of compact support in M° consists of smooth functions only. Injectivity of I on smooth functions proves injectivity on L^2 functions of compact support in M°.

Another useful tool for elliptic pseudodifferential operators is the so called *pseudodifferential property*, which states that elliptic pseudodifferential operators do not create new singularities (in the sense of the wave front set) or destroy old ones. Therefore ellipticity of N means that the sharp features of f can be recovered from the X-ray transform If.

For deeper implications of elliptic pseudodifferentiality of N see [SU04, PU05, SU05].

3.2.2. *The normal operator in non-smooth simple geometry.* In this section we discuss the main results obtained in article [C]. The first result (Theorem 19) is an elliptic regularity result for an operator N defined by the same formula (3.7) as the normal operator in smooth geometry. Our operator N agrees with the with the normal operator of the X-ray transform on $L^2(M)$ in non-smooth geometry.

Theorem 19 ([C, Theorem 1])**.** *Let (M, g) be a simple manifold of dimension $n \geq 2$, where $g \in C^k(M)$ for some $k \geq 7 + n/2$. Then if $f \in H^s_c(M)$ for some $s > -k + 6 + n/2$ and $Nf = 0$ we have $f \in H^r_c(M)$ for all $s < r < k - 6 - n/2$.*

In the statement of Theorem 19, the space $H^s_c(M)$ consists of compactly supported functions in $H^s(M)$, and similarly $H^{-s}_c(M)$ is the subspace of compactly supported distributions in the continuous dual of $H^s(M)$.

The proof of Theorem 19 is based on a parametrix construction for operator N. The operator is not an elliptic pseudodifferential operator in the standard sense, since its Schwartz kernel is non-smooth even off the diagonal. Instead, we prove that N is an elliptic pseudodifferential operator in a non-smooth calculus due to [Mar96] and construct the parametrix there. The limits of regularity indices in Theorem 19 stem from continuous Sobolev mapping properties of pseudodifferential operator in the non-smooth calculus.

The second main result of article [C] is an application of Theorem 19 to X-ray tomography.

Theorem 20 ([C, Theorem 3])**.** *Let (M, g) be a simple manifold of dimension $n \geq 2$, where the Riemannian metric $g \in C^{8+n}(M)$. Then the X-ray transform of (M, g) is injective on $L^2(M)$.*

The proof uses continuous mapping properties for the error term in the parametrix construction and a Sobolev embedding theorem to prove that any L^2 function in the kernel of the X-ray transform is infact Lipschitz continuous. Then we can use [A, Theorem 1] to prove injectivity of the X-ray transform.

References

[AD18] Y. M. Assylbekov and N. S. Dairbekov. The X-ray transform on a general family of curves on Finsler surfaces. *J. Geom. Anal.*, 28(2):1428–1455, 2018.

[Ain13] G. Ainsworth. The attenuated magnetic ray transform on surfaces. *Inverse Probl. Imaging*, 7(1):27–46, 2013.

[AMU18] Y. M. Assylbekov, F. Monard, and G. Uhlmann. Inversion formulas and range characterizations for the attenuated geodesic ray transform. *J. Math. Pures Appl. (9)*, 111:161–190, 2018.

[AR97] Yu. E. Anikonov and V. G. Romanov. On uniqueness of determination of a form of first degree by its integrals along geodesics. *J. Inverse Ill-Posed Probl.*, 5(6):487–490, 1997.

[BLP24] J. Bohr, T. Lefeuvre, and G. P. Paternain. Invariant distributions and the transport twistor space of closed surfaces, 2024. Preprint: arXiv:2301.07391.

[BP23] J. Bohr and G. P. Paternain. The transport Oka-Grauert principle for simple surfaces. *J. Éc. polytech. Math.*, 10:727–769, 2023.

[CS98] C. B. Croke and V. A. Sharafutdinov. Spectral rigidity of a compact negatively curved manifold. *Topology*, 37(6):1265–1273, 1998.

[dHI17] M. V. de Hoop and J. Ilmavirta. Abel transforms with low regularity with applications to X-ray tomography on spherically symmetric manifolds. *Inverse Problems*, 33(12):124003, November 2017. Special issue "100 Years of the Radon Transform".

[dHILS19] M. V. de Hoop, J. Ilmavirta, M. Lassas, and T. Saksala. Determination of a compact Finsler manifold from its boundary distance map and an inverse problem in elasticity. *Communications in Analysis and Geometry*, January 2019. To appear.

[dHILS21] M. V. de Hoop, J. Ilmavirta, M. Lassas, and T. Saksala. A foliated and reversible Finsler manifold is determined by its broken scattering relation. *Pure and Applied Analysis*, 4(4):789–811, December 2021.

[DPSU07] N. S. Dairbekov, G. P. Paternain, P. Stefanov, and G. Uhlmann. The boundary rigidity problem in the presence of a magnetic field. *Adv. Math.*, 216(2):535–609, 2007.

[DS03] N. S. Dairbekov and V. A. Sharafutdinov. Some problems of integral geometry on Anosov manifolds. *Ergodic Theory Dynam. Systems*, 23(1):59–74, 2003.

[EG22] N. Eptaminitakis and C. R. Graham. Local X-ray transform on asymptotically hyperbolic manifolds via projective compactification. *New Zealand J. Math.*, 52:733–763, 2021 [2021–2022].

[Ept22] N. Eptaminitakis. Stability estimates for the X-ray transform on simple asymptotically hyperbolic manifolds. *Pure Appl. Anal.*, 4(3):487–516, 2022.

[FU01] D. Finch and G. Uhlmann. The X-ray transform for a non-abelian connection in two dimensions. volume 17, pages 695–701. 2001. Special issue to celebrate Pierre Sabatier's 65th birthday (Montpellier, 2000).

[GGSU19] C. R. Graham, C. Guillarmou, P. Stefanov, and G. Uhlmann. X-ray transform and boundary rigidity for asymptotically hyperbolic manifolds. *Ann. Inst. Fourier (Grenoble)*, 69(7):2857–2919, 2019.

[GK12] C. R. Graham and J. M. Kantor. Normal Form for Edge Metrics, 2012. Preprint: arXiv:1207.1343.

[GL91] C. R. Graham and J. M. Lee. Einstein metrics with prescribed conformal infinity on the ball. *Adv. Math.*, 87(2):186–225, 1991.

[GL19] C. Guillarmou and T. Lefeuvre. The marked length spectrum of Anosov manifolds. *Ann. of Math. (2)*, 190(1):321–344, 2019.

[GLT20] C. Guillarmou, M. Lassas, and L. Tzou. X-ray Transform in Asymptotically Conic Spaces. *International Mathematics Research Notices*, 2022(5):3918–3976, 11 2020.

[GPSU16] C. Guillarmou, G. P. Paternain, M. Salo, and G. Uhlmann. The X-ray transform for connections in negative curvature. *Comm. Math. Phys.*, 343(1):83–127, 2016.

[Gre23] H. Grebnev. The Non-Abelian X-Ray Transform on Asymptotically Hyperbolic Spaces, 2023. Preprint: arXiv:2312.09408.

[Hel11] Sigurdur Helgason. *Integral geometry and Radon transforms*. Springer, New York, 2011.

[Her05] G. Herglotz. Über die Elastizität der Erde bei Berücksichtigung ihrer variablen Dichte. *Zeitschr. für Math. Phys*, 52(3):275—-299, 1905.

[HMS18] S. Holman, F. Monard, and P. Stefanov. The attenuated geodesic x-ray transform. *Inverse Problems*, 34(6):064003, 26, 2018.

[Hor04] G. P. Horedt. *Polytropes: Applications in Astrophysics and Related Fields*, volume 1 of *Astrophysics and Space Science Library*. Springer Dordrecht, 2004.

[ILS23] J. Ilmavirta, B. Liu, and T. Saksala. Three travel time inverse problems on simple Riemannian manifolds. *Proc. Amer. Math. Soc.*, 151(10):4513–4525, 2023.

[IM19] J. Ilmavirta and F. Monard. Integral geometry on manifolds with boundary and applications. In Ronny Ramlau and Otmar Scherzer, editors, *The Radon Transform: The First 100 Years and Beyond*. de Gruyter, April 2019.

[IM23] J. Ilmavirta and K. Mönkkönen. The geodesic ray transform on spherically symmetric reversible Finsler manifolds. *The Journal of Geometric Analysis*, 33, February 2023.

[IP22] J. Ilmavirta and G. P. Paternain. Broken ray tensor tomography with one reflecting obstacle. *Comm. Anal. Geom.*, 30(6):1269–1300, 2022.

[IS16] J. Ilmavirta and M. Salo. Broken ray transform on a Riemann surface with a convex obstacle. *Communications in Analysis and Geometry*, 24(2):379–408, June 2016.

[JV24] Q. Jia and A. Vasy. The tensorial X-ray transform on asymptotically conic spaces. *Inverse Problems and Imaging*, 2024.

[KKL01] A. Katchalov, Y. Kurylev, and M. Lassas. *Inverse boundary spectral problems*, volume 123 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2001.

[KMM19] V. P. Krishnan, R. K. Mishra, and F. Monard. On solenoidal-injective and injective ray transforms of tensor fields on surfaces. *J. Inverse Ill-Posed Probl.*, 27(4):527–538, 2019.

[Lef20] T. Lefeuvre. Boundary rigidity of negatively-curved asymptotically hyperbolic surfaces. *Comment. Math. Helv.*, 95(1):129–166, 2020.

[Leh16] J. Lehtonen. The geodesic ray transform on two-dimensional Cartan-Hadamard manifolds, 2016. Preprint: arXiv:1612.04800.

[LRS18] J. Lehtonen, J. Railo, and M. Salo. Tensor tomography on Cartan-Hadamard manifolds. *Inverse Problems*, 34(4):044004, 27, 2018.

[Mar96] J. Marschall. Nonregular pseudo-differential operators. *Z. Anal. Anwendungen*, 15(1):109–148, 1996.

[Maz88] R. Mazzeo. The Hodge cohomology of a conformally compact metric. *J. Differential Geom.*, 28(2):309–339, 1988.

[MNP21] F. Monard, R. Nickl, and G. P. Paternain. Consistent inversion of noisy non-Abelian X-ray transforms. *Comm. Pure Appl. Math.*, 74(5):1045–1099, 2021.

[MS39] S. B. Myers and N. E. Steenrod. The group of isometries of a Riemannian manifold. *Ann. of Math. (2)*, 40(2):400–416, 1939.

[MSU15] F. Monard, P. Stefanov, and G. Uhlmann. The geodesic ray transform on Riemannian surfaces with conjugate points. *Comm. Math. Phys.*, 337(3):1491–1513, 2015.

[Muk75] R. G. Mukhometov. Inverse kinematic problem of seismic on the plane. *Math. Problems of Geophysics, Akad. Nauk. SSSR, Sibirsk. Otdel., Vychisl. Tsentr, Novosibirsk*, 6:243—-252, 1975.

[Muk77] R. G. Mukhometov. The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry. *Dokl. Akad. Nauk SSSR*, 232(1):32–35, 1977.

[Muk78] R. G. Mukhometov. A problem of integral geometry on the plane. In *Methods of functional analysis in problems of mathematical physics (Russian)*, pages 30–37, 180. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1978.

[Pal57] R. S. Palais. On the differentiability of isometries. *Proc. Amer. Math. Soc.*, 8:805–807, 1957.

[PS87] L. N. Pestov and V. A. Sharafutdinov. Integral geometry of tensor fields on a manifold of negative curvature. *Dokl. Akad. Nauk SSSR*, 295(6):1318–1320, 1987.

[PS22] G. P. Paternain and M. Salo. The non-Abelian X-ray transform on surfaces, 2022. Preprint: arXiv:2006.02257.

[PSU12] G. P. Paternain, M. Salo, and G. Uhlmann. The attenuated ray transform for connections and Higgs fields. *Geom. Funct. Anal.*, 22(5):1460–1489, 2012.

[PSU13] G. P. Paternain, M. Salo, and G. Uhlmann. Tensor tomography on surfaces. *Invent. Math.*, 193(1):229–247, 2013.

[PSU14a] G. P. Paternain, M. Salo, and G. Uhlmann. Spectral rigidity and invariant distributions on Anosov surfaces. *J. Differential Geom.*, 98(1):147–181, 2014.

[PSU14b] G. P. Paternain, M. Salo, and G. Uhlmann. Tensor tomography: progress and challenges. *Chinese Ann. Math. Ser. B*, 35(3):399–428, 2014.

[PSU15] G. P. Paternain, M. Salo, and G. Uhlmann. Invariant distributions, Beurling transforms and tensor tomography in higher dimensions. *Math. Ann.*, 363(1-2):305–362, 2015.

[PSU23] G. P. Paternain, M. Salo, and G. Uhlmann. *Geometric inverse problems—with emphasis on two dimensions*, volume 204 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2023. With a foreword by András Vasy.

[PU05] L. Pestov and G. Uhlmann. Two dimensional compact simple Riemannian manifolds are boundary distance rigid. *Ann. of Math. (2)*, 161(2):1093–1110, 2005.

[Sha94] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

[Sha97] V. A. Sharafutdinov. Integral geometry of a tensor field on a surface of revolution. *Sibirsk. Mat. Zh.*, 38(3):697–714, iv, 1997.

[SS18] C. Sämann and R. Steinbauer. On geodesics in low regularity. *J. Phys. Conf. Ser.*, 968:012010, 14, 2018.

[Ste93] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[SU00] V. A. Sharafutdinov and G. Uhlmann. On deformation boundary rigidity and spectral rigidity of Riemannian surfaces with no focal points. *J. Differential Geom.*, 56(1):93–110, 2000.

[SU04] P. Stefanov and G. Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity. *Duke Math. J.*, 123(3):445–467, 2004.

[SU05] P. Stefanov and G. Uhlmann. Boundary rigidity and stability for generic simple metrics. *J. Amer. Math. Soc.*, 18(4):975–1003, 2005.

[SU11] M. Salo and G. Uhlmann. The attenuated ray transform on simple surfaces. *J. Differential Geom.*, 88(1):161–187, 2011.

[VZ22] A. Vasy and E. Zachos. The X-ray transform on asymptotically conic spaces, 2022. Preprint: arXiv:2204.11706.

[WZ07] E. Wiechert and K. Zoeppritz. Uber Erdbebenwellen. *Nachr. Königl. Ges. Wiss. Göttingen*, 4:415—-549, 1907.

Included articles

[A]

Pestov identities and X-ray tomography on manifolds of low regularity

Joonas Ilmavirta and Antti Kykkänen

First published in *Inverse Problems and Imaging* Volume 17 no. 6 (2023)
https://doi.org/10.3934/ipi.2023017

Inverse Problems and Imaging

doi:10.3934/ipi.2023017

PESTOV IDENTITIES AND X-RAY TOMOGRAPHY ON MANIFOLDS OF LOW REGULARITY

Joonas Ilmavirta[*1] and Antti Kykkänen[1,2]

[1]Department of Mathematics and Statistics, University of Jyväskylä, Finland

[2]Department of Mathematics and Statistics, University of Helsinki, Finland

(Communicated by Matti Lassas)

Abstract. We prove that the geodesic X-ray transform is injective on scalar functions and (solenoidally) on one-forms on simple Riemannian manifolds (M, g) with $g \in C^{1,1}$. In addition to a proof, we produce a redefinition of simplicity that is compatible with rough geometry. This $C^{1,1}$-regularity is optimal on the Hölder scale. The bulk of the article is devoted to setting up a calculus of differential and curvature operators on the unit sphere bundle atop this non-smooth structure.

1. **Introduction.**

How regular does a Riemannian metric have to be for the geodesic X-ray transform to be injective? It is well known (see e.g. [25, 26, 36, 3]) that on a smooth simple Riemannian manifold this injectivity property holds. If the regularity is too low, the question itself falls apart: If the Riemannian metric is $C^{1,\alpha}$ for $\alpha < 1$, then the geodesic equation can fail to have unique solutions [13, 38]. Therefore it is indeed in a sense optimal on the Hölder scale when we prove that on a $C^{1,1}$-smooth simple Riemannian manifold the geodesic X-ray transform is injective on scalars and one-forms, the latter one up to natural gauge.

The geodesic X-ray transform is ubiquitous in the theory of geometric inverse problems. It appears either directly or through linearization in many imaging problems of anisotropic and inhomogeneous media. Most inverse problems have been studied in smooth geometry but the nature is not smooth. The irregularities of the structure of the Earth range from individual rocks (zero-dimensional, small) to interfaces like the core–mantle boundary (two-dimensional, global scale). Irregularity across various scales and dimensions are most conveniently captured in a single geometric structure of minimal regularity assumptions. Specific kinds of irregularities can well be analyzed further, but we restrict our attention to a uniform and global but low regularity.

We prove this injectivity result by using a Pestov identity, an approach that can well be called classical (cf. [25, 26, 36, 3, 32, 16, 42, 34]). What requires care is keeping track of regularity. The manifold does not have natural structure beyond $C^{1,1}$, so regularity beyond is both useless and inaccessible. The natural differential operators on the manifold and its unit sphere bundle are not smooth,

2020 *Mathematics Subject Classification.* Primary: 44A12, 53C22, 53C65; Secondary: 58J32.

Key words and phrases. Geodesic X-ray tomography, Pestov identity, non-smooth geometry, integral geometry, inverse problems.

*Corresponding author: Joonas Ilmavirta.

and only a couple of derivatives of any kind can be taken at all. The various commutators that appear in the calculations have to be interpreted in a suitable way, so that $[A, B]$ exists reasonably even when the products AB and BA do not. We employ two methods around these obstacles: approximation by smooth structures and careful analysis in the non-smooth geometry.

We say that a function is in the class $C^{1,1}$ if it is continuously differentiable and the derivative is Lipschitz, and we define in definition 1.5 what a $C^{1,1}$ simple Riemannian metric is. Throughout the article our manifolds are assumed to be connected and to have dimension $n \geq 2$.

Theorem 1.1. *Let (M, g) be a simple $C^{1,1}$ manifold in the sense of definition 1.5.*

1. *If f is a Lipschitz function on M that integrates to zero over all maximal geodesics of M, then $f = 0$.*
2. *Let h be a Lipschitz 1-form on M that vanishes on the boundary ∂M. Then h integrates to zero over all maximal geodesics of M if and only if there is a scalar function $p \in C^{1,1}(M)$ vanishing on the boundary ∂M so that $h = \mathrm{d}p$.*

We have to redefine simplicity to be tractable in our rough setup, and we regard this new definition as one of our main results. To verify that our redefinition is a valid one, we prove that it agrees with the classical definition when the metric is smooth. The classical definition of a smooth simple manifold implies the existence of global coordinates, but in the $C^{1,1}$ case we assume the coordinates in the definition — in light of the following theorem the coordinate assumption is not superfluous.

Theorem 1.2. *In smooth geometry definitions 1.4 and 1.5 are equivalent in the following sense:*

1. *If M is a simple C^∞ Riemannian manifold (see definition 1.4), then it is diffeomorphic to a closed ball in $\mathbb{R}^n$ and it is a simple $C^{1,1}$ Riemannian manifold (see definition 1.5).*
2. *If M is a simple $C^{1,1}$ Riemannian manifold (see definition 1.5) and its metric tensor is C^∞-smooth, then M is a smooth simple Riemannian manifold (see definition 1.4).*

Remark 1.3. The assumption $h|_{\partial M} = 0$ in claim 2 of theorem 1.1 is probably not necessary. Not assuming this is fine in smooth geometry but leads to technical difficulties in our rough setup. This added assumption is the only way in which our results fail to correspond to the classical smooth results.

1.1. Related results.

Geodesic X-ray transforms have been studied a lot on smooth manifolds equipped with C^∞-smooth Riemannian metrics. Injectivity of the transform is reasonably well understood both on manifolds with a boundary and on closed manifolds. On manifolds with boundary one integrates over maximal geodesics between two boundary points, whereas on closed manifolds one integrates over periodic geodesics.

After Mukhometov's introduction of the Pestov identity for scalar tomography [24, 25, 26], the method has been applied to 1-forms and higher order tensor fields [3, 35, 30, 33] on many simple manifolds. When one passes from simple manifolds with boundary to closed Anosov manifolds, the Pestov identity remains the same but the other tools around it change somewhat [6, 8, 31, 33, 43]. Cartan–Hadamard manifolds are a non-compact analogue of simple manifolds, and the familiar Pestov identity works well [20, 21]. Other variations of the problem change the Pestov identity, but a variant remains true and useful: In the presence of reflecting rays a boundary term on the reflector is added [18, 17], an attenuation or a Higgs field [37, 29, 12] and magnetic flows [7, 1, 22] add a term to the geodesic vector field, non-abelian versions of the problem remove the concept of a line integral entirely [11, 28, 23], and on Finsler surfaces a number of new terms are needed to account for non-Riemannian geometry [4]. On pseudo-Riemannian manifolds a Pestov identity useful for the light ray transform only seems to exist in product geometry of at least $2 + 2$ dimensions [15].

Pestov identities are not the only tool in the box for studying ray transforms on manifolds. For the variety of other methods we refer the reader to the review [16].

Inverse problems in integral geometry have been mostly studied on manifolds whose Riemannian metric is smooth or otherwise substantially above our $C^{1,1}$ in regularity. Injectivity of the scalar X-ray transform is known on spherically symmetric manifolds of regularity $C^{1,1}$ satisfying the so-called Herglotz condition when the conformal factor of the metric is in $C^{1,1}$ [9].

Some geometric inverse problems outside integral geometry have been solved in low regularity. A manifold with a metric tensor in a suitable Zygmund class is determined by its boundary spectral data [2], interior spectral data [5] or by its boundary distance function [19].

1.2. **Preliminaries.** In this subsection we will set up enough language to be able to state our definitions and give our proofs on a higher level. For a similar framework in the traditional smooth setting, see e.g. [33]. We will cover the foundations in more detail in section 4 before embarking on the detailed proofs of our key lemmas.

The Riemannian manifold (M, g), where g is $C^{1,1}$ regular, comes equipped with the unit sphere bundle $\pi\colon SM \to M$. The geodesic flow is a dynamical system on SM and its generator X is called the geodesic vector field. Properties and coordinate representations of X will be given later.

We will make frequent use of the bundle N over SM defined next. If π^*TM is the pullback of TM over SM, then N is the subbundle of π^*TM with fibers $N_{(x,v)} = \{v\}^\perp \subseteq T_xM$. It is well known (see [27]) that the tangent bundle TSM of SM has an orthogonal splitting

$$TSM = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V} \tag{1}$$

with respect to the so-called Sasaki metric, where $\mathcal{H}$ and $\mathcal{V}$ are called horizontal and vertical subbundles respectively. Roughly speaking, $\mathcal{H}_{(x,v)}$ corresponds to derivatives on SM in the base without components in the direction of v and $\mathcal{V}_{(x,v)}$ corresponds to derivatives on a fiber S_xM. It is natural to identify $\mathcal{H}_{(x,v)} = N_{(x,v)}$ and $\mathcal{V}_{(x,v)} = N_{(x,v)}$.

Given $z \in SM$, let γ_z be the unique geodesic corresponding to the initial condition z. We define the geodesic flow to be the collection of (partially defined) maps $\phi_t\colon SM \to SM$, $\phi_t(z) = (\gamma_z(t), \dot\gamma_z(t))$, where t goes through the values for which the right side is defined on SM. For any $z \in SM$ the geodesic γ_z is defined on a maximal interval $[\tau_-(z), \tau_+(z)]$. The travel time function $\tau\colon SM \to \mathbb{R}$ describes the first time a geodesic exists the manifold and it is defined by $\tau(z) = \tau_+(z)$ for $z \in SM$. Clearly $\gamma_z(\tau(z)) \in \partial M$ for any $z \in SM$.

A function f on M can be identified with the function π^*f on SM. If h is a 1-form on M, then it can be considered as a function $\tilde h\colon SM \to \mathbb{R}$ through the identification $\tilde h(x, v) = h_x(v)$ for $(x, v) \in SM$. Since $h_x\colon T_xM \to \mathbb{R}$ is linear, $\tilde h$ uniquely corresponds to h. The integral function $u^f : SM \to \mathbb{R}$ of $f \in \mathrm{Lip}(SM)$ is defined by

$$u^f(x,v) \coloneqq \int_0^{\tau(x,v)} f(\phi_t(x,v))\,\mathrm{d}t \tag{2}$$

for all $(x,v) \in SM$.

The lift of a unit speed curve $\gamma\colon I \to M$ is $\tilde{\gamma}\colon I \to SM$ given by $\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$. The curve γ is a geodesic if and only if the lift satisfies $\dot{\tilde{\gamma}}(t) = X(\tilde{\gamma}(t))$. The geodesic vector field X acts naturally on scalar fields by differentiation, and on sections V of N it acts by

$$XV(z) = D_t V(\phi_t(z))|_{t=0},$$

where D_t is the covariant derivative along the curve $t \mapsto \gamma_z(t)$. This operator maps indeed sections of N to sections of N.

According to (1) the gradient of a C^1 function u on SM we can be written as

$$\nabla_{SM} u = (Xu)X + \overset{\mathtt{h}}{\nabla} u + \overset{\mathtt{v}}{\nabla} u.$$

This gives rise to two new differential operators $\overset{\mathtt{v}}{\nabla}$ and $\overset{\mathtt{h}}{\nabla}$, called, respectively, the vertical and the horizontal gradient. Both $\overset{\mathtt{v}}{\nabla} u$ and $\overset{\mathtt{h}}{\nabla} u$ are naturally interpreted as sections of N; see [34] for details. There are natural L^2 spaces for functions on the sphere bundle as well as for the sections of the bundle N. These will be denoted $L^2(SM)$ and $L^2(N)$ and we will often label the corresponding inner products explicitly. Formal adjoints of $\overset{\mathtt{v}}{\nabla}$ and $\overset{\mathtt{h}}{\nabla}$ with respect to appropriate L^2 inner products are the vertical and horizontal divergences $-\overset{\mathtt{v}}{\mathrm{div}}$ and $-\overset{\mathtt{h}}{\mathrm{div}}$ respectively. The mapping properties of the operators in $C^{1,1}$ regular metric setting are

$$\begin{aligned} X\colon C^1(SM) &\to C(SM) \\ X\colon C^1(N) &\to C(N), \\ \overset{\mathtt{v}}{\nabla}, \overset{\mathtt{h}}{\nabla}\colon C^1(SM) &\to C(N), \quad \text{and} \\ \overset{\mathtt{v}}{\mathrm{div}}, \overset{\mathtt{h}}{\mathrm{div}}\colon C^1(N) &\to C(SM). \end{aligned}$$

These mapping properties are easily verified by inspecting the explicit formulas in local coordinates; see section 4.

We will deal with Sobolev spaces $H^1_{(0)}(SM)$ and $H^1_{(0)}(N)$ defined as completions of $C^1_{(0)}$ regular functions or sections in the relevant norms (see section 4), where the optional subscript 0 indicates zero boundary values. Similarly, we denote by $\mathrm{Lip}_0(M)$ and $\mathrm{Lip}_0(SM)$ the spaces of Lipschitz functions zero boundary values. As the last function space we introduce a Sobolev space $H^1_{(0)}(N, X)$, which only gives control over the operator X operating on sections of N. From definitions of various Sobolev norms it will be clear that all differential operators are bounded $H^1 \to L^2$ and thus extend to operators between Sobolev spaces.

Finally, there is a special quadratic form Q appearing in the Pestov identity. To define it, we use the Riemannian curvature tensor $R\colon L^\infty(N) \to L^\infty(N)$ acting on sections of N by

$$R(x,v)V(x,v) = R(V(x,v), v)v.$$

In order to verify the mapping property of R, observe that the second partial derivatives of $g \in C^{1,1} = W^{2,\infty}$ are in L^∞. We define Q by letting

$$Q(W) = \|XW\|^2_{L^2(N)} - (RW, W)_{L^2(N)}\,.$$

for all $W \in H^1(N, X)$.

To conclude the preliminaries we recall in definition 1.4 the traditional definition of a simple Riemannian manifold (cf. [33]). In what follows a manifold satisfying conditions A1 and A2 is called *simple C^∞ manifold*. In definition 1.5 we redefine the notion of simplicity on manifolds equipped with non-smooth Riemannian metrics.

Definition 1.4 (Simple C^∞ manifold)**.** Let (M, g) be a compact smooth Riemannian manifold with a smooth boundary. The manifold (M, g) is called *simple C^∞ Riemannian manifold*, if the following hold:

- A1: The boundary ∂M is strictly convex in the sense of the second fundamental form.
- A2: Any two points on M can be joined by a unique geodesic in the interior of M, and its length depends smoothly on its end points.

Definition 1.5 (Simple $C^{1,1}$ manifold)**.** Let $M \subseteq \mathbb{R}^n$ be the closed unit ball and g a $C^{1,1}$ regular Riemannian metric on M. We say that (M, g) is a *simple $C^{1,1}$ Riemannian manifold* if the following hold:

- B1: There is $\varepsilon > 0$ so that $Q(W) \geq \varepsilon \|W\|^2_{L^2(N)}$ for all $W \in H^1_0(N, X)$.
- B2: Any two points of M can be joined by a unique geodesic in the interior of M, whose length depends continuously on its end points.
- B3: The function τ^2 is Lipschitz on SM.

Remark 1.6. In definition 1.5 the assumption that M is the closed unit ball is not restrictive — any simple C^∞ Riemannian manifold is diffeomorphic to a closed ball in a Euclidean space. In the absence of conjugate points the exponential map $\exp_x$, related to an interior point $x \in \text{int}(M)$, maps its maximal domain D_x diffeomorphically to M and D_x is itself diffeomorphic to the closed unit ball in $\mathbb{R}^n$ (see [34]). We use global coordinates on a simple $C^{1,1}$ Riemannian manifold and we have decided to include their existence in the definition.

Remark 1.7. If one is to define a rough simple manifold as the limit of smooth simple manifolds, the simplicity needs to be quantified. The example of a hemisphere as the limit of expanding polar caps shows that the smooth limit of smooth simple manifolds can be a smooth but non-simple manifold. The limit procedure can introduce conjugate points and failure of strict convexity on the boundary. An example of quantified simplicity can be found in [10], but we do not take this limit route in our definition here.

2. **Proof of theorem 1.1.** This section contains the proof of theorem 1.1. The proofs of the necessary lemmas are postponed to section 5. More detailed definitions of function spaces and operators can be found from section 4.

We will freely identify a scalar function f and a one-form h on M with scalar functions on SM as described above. Interpreting f and h as functions on SM we can apply formula (2) to both.

Lemma 2.1 (Regularity of integral functions)**.** *Let (M, g) be a simple $C^{1,1}$ manifold.*

1. *Let f be a Lipschitz function on M that integrates to zero over all maximal geodesics of M and let u^f be the integral function of f defined by (2). Then $u^f \in \mathrm{Lip}_0(SM)$, $Xu^f \in H^1(SM)$ and $\overset{\mathrm{v}}{\nabla} u^f \in H^1_0(N, X)$.*
2. *Let h be a Lipschitz 1-form on M that integrates to zero over all maximal geodesics of M and vanishes on the boundary ∂M. If u^h is the integral function of h defined by (2), then $u^h \in \mathrm{Lip}_0(SM)$, $Xu^h \in H^1(SM)$ and $\overset{\mathrm{v}}{\nabla} u^h \in H^1_0(N, X)$.*

Lemma 2.2 (Pestov identity)**.** *Let (M, g) be a simple $C^{1,1}$ manifold and let $u \in \mathrm{Lip}_0(SM)$ be such that $Xu \in H^1(SM)$ and $\overset{\mathrm{v}}{\nabla} u \in H^1(N, X)$. Then*

$$\left\| \overset{\mathrm{v}}{\nabla} Xu \right\|^2_{L^2(N)} = Q\left(\overset{\mathrm{v}}{\nabla} u\right) + (n-1)\left\| Xu \right\|^2_{L^2(SM)}. \tag{3}$$

Lemma 2.1 provides enough regularity to apply the Pestov identity (3) to the integral functions u^f and u^h because we will see in remark 4.3 that $\mathrm{Lip}(SM) \subseteq H^1(SM)$ even if the metric tensor in only in $C^{1,1}$. The following lemma shows that certain norms of the integral function u^h of a 1-form cancel in the identity.

Lemma 2.3. *Let (M, g) be a simple $C^{1,1}$ manifold and let h be a Lipschitz 1-form on M. Then*

$$\left\| \overset{\mathrm{v}}{\nabla} h \right\|^2_{L^2(N)} = (n-1)\left\| h \right\|^2_{L^2(SM)}.$$

We are ready to prove theorem 1.1.

Proof of theorem 1.1. 1. The integral function u^f of $f \in \mathrm{Lip}(M)$ satisfies $Xu^f \in H^1(SM)$ and $\overset{\mathrm{v}}{\nabla} u^f \in H^1(N, X)$ by lemma 2.1. Thus we can apply the Pestov identity of lemma 2.2 to u^f. By the fundamental theorem of calculus $Xu^f = -f$ and thus $\overset{\mathrm{v}}{\nabla} Xu^f = 0$, since f does not depend on the direction $v \in S_xM$. By $C^{1,1}$ simplicity (definition 1.5) of (M, g), the quadratic form Q is non-negative. Thus the Pestov identity reduces to

$$0 \geq (n-1)\left\| Xu^f \right\|^2_{L^2(SM)}.$$

Hence $f = -Xu^f = 0$ in $L^2(SM)$ as claimed.

2. If $h = \mathrm{d}p$ for some scalar function $p \in C^{1,1}(M)$ with $p|_{\partial M} = 0$, then by the fundamental theorem of calculus h integrates to zero over all maximal geodesics of M.

Let h be a Lipschitz 1-form on M that integrates to zero over all maximal geodesic of M and vanishes on the boundary ∂M. We will show that $h = \mathrm{d}p$ for some function $p \in C^{1,1}(M)$ vanishing on ∂M. Lemma 2.1 allows us to apply the Pestov identity to the integral function u^h of h. Due to lemma 2.3, the identity reduces to

$$Q\left(\overset{\mathrm{v}}{\nabla} u^h\right) = 0.$$

Since the manifold is simple $C^{1,1}$, this can only happen if $\overset{\mathrm{v}}{\nabla} u^h = 0$. The function u^h is Lipschitz and independent of the direction $v \in S_xM$ on each fiber and therefore there is a Lipschitz scalar function p on M so that $u^h = -\pi^* p$ on SM. Additionally, $p|_{\partial M} = u^h|_{\partial(SM)} = 0$, since h integrates to zero over all maximal geodesics of M. Since $Xu^h = -h$, we have shown that $\mathrm{d}p = h$ in the weak sense. Because h is

Lipschitz-continuous by assumption, we have that $\mathrm{d}p$ is Lipschitz and thus $p \in C^{1,1}$ and the proof is complete. □

3. **Proof of theorem 1.2.** In this section we prove that in the smooth setting definition 1.5 of $C^{1,1}$ simplicity is equivalent to definition 1.4 of C^∞ simplicity. Proofs of lemmas 3.1 and 3.2 are given in section 6. Theorem 1.2 readily follows from lemmas 3.1 and 3.2.

Lemma 3.1. *Let (M, g) be a simple $C^{1,1}$ manifold with C^∞-smooth Riemannian metric g. Then there are no conjugate points in M, not even on the boundary.*

Lemma 3.2. *Let M be a compact Riemannian manifold with smooth boundary and a C^∞-smooth Riemannian metric g. Suppose that (M, g) is non-trapping. Then ∂M is strictly convex in the sense of the second fundamental form if and only if $\tau^2 \in \mathrm{Lip}(SM)$.*

Proof of theorem 1.2. By remark 1.6 each simple C^∞ Riemannian manifold is diffeomorphic to the closed unit ball $\overline{B}$ in $\mathbb{R}^n$. Thus we may assume that $M = \overline{B}$ and let g be a C^∞-smooth Riemannian metric on M. It suffices to show that (M, g) satisfies conditions A1–A2 in definition 1.4 if and only if it satisfies conditions B1–B3 in definition 1.5. We have illustrated these implications in figure 1.

By lemma 3.2 conditions A1 and B3 are equivalent. By lemma 3.1 the condition B1 implies that there are no conjugate points on M. Thus we can promote the continuous dependence in B2 to smooth dependence A2. Therefore simple $C^{1,1}$ manifolds satisfy both conditions A1 and A2 of C^∞ simplicity. Conversely, simple C^∞ manifolds satisfy B1 (see [33, Lemma 11.2]) and clearly B2 is strictly weaker than A2. □

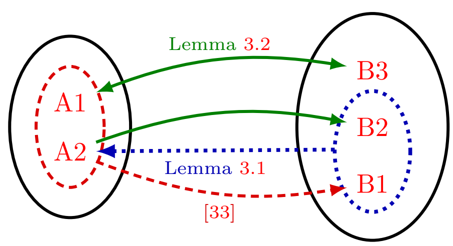

Figure 1. Illustration of the proof of theorem 1.2. The arrows represent implications except the one double headed arrow, which represents equivalence. The green (solid) arrows connect one condition to another. The red (dashed) and the blue (dotted) arrows indicate that one condition follows from the two conditions circled with the same color (style).

4. **Bundles, function spaces and operators.** This section complements the preliminaries in subsection 1.2. The main focus is on a detailed description of structures, functions spaces and operators build on a compact Riemannian manifold (M, g) with a $C^{1,1}$ regular Riemannian metric.

4.1. **Function spaces on smooth manifolds.** Let M be a compact smooth manifold with a smooth boundary. The space of smooth functions on M is denoted $C^\infty(M)$ and the space of differentiable functions with Lipschitz derivatives is denoted $C^{1,1}(M)$. We let $C^{1,1}(T^2M)$ denote the space of 2-tensor fields on M, whose component functions are in $C^{1,1}(M)$.

If h is a smooth Riemannian metric on M, then $L^2_h(M)$ and $L^\infty_h(M)$ will respectively denote spaces of square integrable and essentially bounded functions on M, where the Riemannian volume form of h is used as the measure. Similarly, $W^{1,p}_h(M)$ and $W^{2,p}_h(M)$ will respectively denote Sobolev spaces with p-integrable covariant derivatives of the first order and of the second order. Norms of the covariant derivatives on the tangent spaces are always defined by the metric h.

4.2. **Structures in low regularity.** Let (M, g) be a compact Riemannian manifold with a smooth boundary. We assume that $g \in C^{1,1}(T^2M)$. The unit sphere bundle $SM = \{v \in TM : |v| = 1\}$ is a submanifold of TM, but not in general a smooth one. Despite the non-smoothness of $SM \subseteq TM$ as a submanifold, it can be equipped with an induced smooth structure: SM is naturally homeomorphic to the quotient space $(TM \setminus 0)/\sim$, where $v \sim \lambda v$ for all $\lambda > 0$ and $v \in T_xM$. Metric structures like the Sasaki metric are still non-smooth, so this smooth structure is of little use. We will only see SM as a submanifold of TM.

For $k \in \{0, 1\}$ a function $u\colon SM \to \mathbb{R}$ is said to be in $C^k(SM)$ if u is k times continuously differentiable — for $k \geq 2$ this concept is undefined in our setting. As a C^1 submanifold of TM the sphere bundle has enough regularity to define both $C(SM)$ and $C^1(SM)$. The subset $C^k_0(SM)$ of $C^k(SM)$ consists of functions vanishing on

$$\partial(SM) = \{\, (x, v) \in SM \,:\, x \in \partial M \,\}.$$

The set of Lipschitz functions on SM is denoted by $\mathrm{Lip}(SM)$. We denote the inward unit normal vector field to the boundary ∂M by ν. The boundary $\partial(SM)$ is divided into parts pointing inwards and outwards, respectively denoted by

$$\partial_{\mathrm{in}}(SM) \coloneqq \{\, (x, v) \in \partial(SM) \,:\, \langle v, \nu(x)\rangle \geq 0 \,\}$$

and

$$\partial_{\mathrm{out}}(SM) \coloneqq \{\, (x, v) \in \partial(SM) \,:\, \langle v, \nu(x)\rangle \leq 0 \,\}.$$

Their intersection consists of tangential directions

$$\partial_0(SM) \coloneqq \partial_{\mathrm{in}}(SM) \cap \partial_{\mathrm{out}}(SM).$$

Many differential operators considered in this article operate on sections of the bundle N. To describe C^k spaces of sections of N, recall that N is the subbundle of π^*TM with fibers $N_{(x,v)} = \{v\}^\perp \subseteq T_xM$. A section V of the bundle N is a section of the bundle π^*TM with the property that $\langle V(x, v), v\rangle_{g(x)} = 0$ for all $(x, v) \in SM$. We say that such a section is in $C^k(N)$ for $k \in \{0, 1\}$ if the corresponding section of π^*TM is k times continuously differentiable. Differentiability of a section W of π^*TM is well defined since W is a certain function between two differentiable manifolds SM and TM. The subspace $C^k_0(N) \subseteq C^k(N)$ consists of sections V of N that vanish on $\partial(SM)$.

Let (x, v) be a local coordinate system on TM and let ∂_{x^j} and ∂_{v^k} be corresponding coordinate vector fields. We introduce new vector fields $\delta_{x^j} = \partial_{x^j} - {\Gamma^l}_{jk} v^k \partial_{v^l}$ on TM, where ${\Gamma^l}_{jk}$ are the Christoffel symbols of the metric g. As the metric tensor in our results is of regularity $C^{1,1}$, it follows that the Christoffel symbols and thus the vector fields δ_{x^j} are only Lipschitz.

4.3. **Differential operators.** Next we define differential operators on SM and N. The basic coordinate derivatives of a function $u \in C^1(SM)$ are defined by

$$\delta_j u \coloneqq \delta_{x^j}(u \circ r)|_{SM} \quad \text{and} \quad \partial_k u \coloneqq \partial_{v^k}(u \circ r)|_{SM},$$

where $r\colon TM \setminus 0 \to SM$ is the radial function $r(x,v) = (x, v\,|v|^{-1}_{g(x)})$. We denote $\delta^j \coloneqq g^{jk}\delta_k$ and $\partial^j \coloneqq g^{jk}\partial_k$. We use the basic derivatives to define operators in local coordinates.

The geodesic vector field X is a differential operator that acts both on functions on SM and on sections of the bundle N. The actions on a scalar function u and on a section V are defined by

$$Xu = v^j \delta_j u \quad \text{and} \quad XV = (XV^j)\partial_{x^j} + \Gamma^l{}_{jk} v^j V^k \partial_{x^l}. \tag{4}$$

Vertical and horizontal gradients are differential operators defined respectively by

$$\overset{\mathtt{v}}{\nabla} u = (\partial^j u)\partial_{x^j} \quad \text{and} \quad \overset{\mathtt{h}}{\nabla} u = (\delta^j u - (Xu)v^j)\partial_{x^j}.$$

Coordinate formulas indicate that $\overset{\mathtt{v}}{\nabla}$ is the gradient in v and $\overset{\mathtt{h}}{\nabla}$ is the gradient in x with the direction of v being projected out. The adjoint operators of $\overset{\mathtt{v}}{\nabla}$ and $\overset{\mathtt{h}}{\nabla}$ are the vertical and the horizontal divergences

$$\overset{\mathtt{v}}{\operatorname{div}} V = \partial_j V^j \quad \text{and} \quad \overset{\mathtt{h}}{\operatorname{div}} V = (\delta_j + \Gamma^i{}_{ji})V^j.$$

The Riemannian curvature tensor R of the metric g has an action on sections of N defined by

$$RV = R^l{}_{ijk} V^i v^j v^k \partial_{x^l}.$$

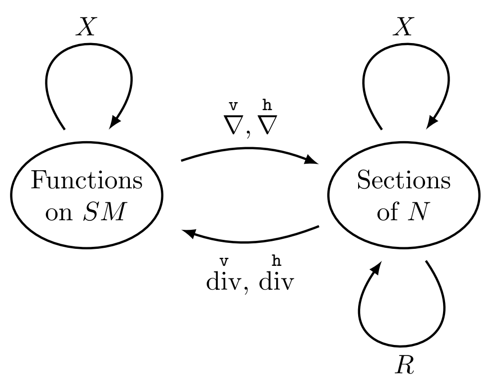

FIGURE 2. Interplay of the operators defined in subsection 4.3. The gradients map functions on SM to sections of N. The divergences map sections of N back to function on SM. The geodesic vector field maps functions to functions and sections to sections. The curvature operator acts only on sections and produces sections.

4.4. **Integration and Sobolev spaces.** A simple $C^{1,1}$ manifold M is orientable, so the Riemannian volume form on it can be defined in local coordinates as

$$\mathrm{d}V_g(x) \coloneqq |\det(g(x))|^{1/2}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

For any $x \in M$ the pair $(S_xM, g(x))$ is a Riemannian manifold. Let $\mathrm{d}S_x$ be the associated Riemannian volume form on S_xM. We use $\mathrm{d}V_g$ and $\mathrm{d}S_x$ to define the volume form $\mathrm{d}\Sigma_g$ on SM, given in local coordinates by

$$\mathrm{d}\Sigma_g(x,v) = \mathrm{d}S_x(v) \wedge \mathrm{d}V_g(x).$$

The form $\mathrm{d}\Sigma_g$ is natural as it coincides with the Riemannian volume form of the Sasaki metric on SM. Since $\mathrm{d}V_g$ has as much regularity as g, so does $\mathrm{d}\Sigma_g$.

The L^2-norm of a scalar function u on SM is denoted by $\|u\|_{L^2(SM)}$ and the L^2-norm of a section V of N is denoted by $\|V\|_{L^2(N)}$. The L^2-norms are induced by the inner products

$$(u,w)_{L^2(SM)} := \int_{SM} uw \,\mathrm{d}\Sigma_g$$

and

$$(V,W)_{L^2(N)} := \int_{SM} g_{ij} V^i W^j \,\mathrm{d}\Sigma_g.$$

We define the $\|\cdot\|_{H^1(SM)}$-norm of a function $u \in C^1(SM)$ by

$$\|u\|^2_{H^1(SM)} = \|u\|^2_{L^2(SM)} + \|Xu\|^2_{L^2(SM)} + \left\|\overset{\mathtt{v}}{\nabla} u\right\|^2_{L^2(N)} + \left\|\overset{\mathtt{h}}{\nabla} u\right\|^2_{L^2(N)}.$$

The Sobolev space $H^1(SM)$ is defined to be the completion of the subset of $C^1(SM)$ that consists of functions with finite $H^1(SM)$-norm. We denote by $H^1_0(SM)$ the closure of $C^1_0(SM)$ in $H^1(SM)$.

Sobolev spaces for sections of N are defined in an analogous fashion. For a section $V \in C^1(N)$ we define the two Sobolev norms

$$\|V\|^2_{H^1(N)} = \|V\|^2_{L^2(N)} + \|XV\|^2_{L^2(N)} + \left\|\overset{\mathtt{v}}{\mathrm{div}} V\right\|^2_{L^2(SM)} + \left\|\overset{\mathtt{h}}{\mathrm{div}} V\right\|^2_{L^2(SM)}$$

and

$$\|V\|^2_{H^1(N,X)} = \|V\|^2_{L^2(N)} + \|XV\|^2_{L^2(N)}.$$

The corresponding Sobolev spaces (the completions of $C^1(N)$ under these norms) are denoted by $H^1(N)$ and $H^1(N,X)$, and the Sobolev spaces of sections vanishing on the boundary $\partial(SM)$ are denoted by $H^1_0(N)$ and $H^1_0(N,X)$.

Remark 4.1. Contrary to what one might expect, the norm on $H^1(N)$ defined above does not contain derivatives in all possible directions, as it only includes divergences in the vertical and horizontal directions. We will use these norms only to estimate from above, so this omission of derivatives makes no difference.

In the case where g is a C^∞-smooth Riemannian metric, we introduce one more Sobolev space, $K^2(SM)$. The defining norm on the dense subspace $C^2(N)$ is

$$\|u\|^2_{K^2(SM)} = \|u\|^2_{H^1(SM)} + \|Xu\|^2_{H^1(SM)} + \left\|\overset{\mathtt{v}}{\nabla} u\right\|^2_{H^1(N,X)}.$$

Remark 4.2. It is important to realize that we cannot define Sobolev spaces using smooth test functions as in the smooth case. The reason is two-fold. First, the natural structure of SM as an submanifold TM is not regular enough to define the function class $C^\infty(SM)$. Second, the differential operators themselves are not smooth. Applying any of the differential operators immediately drops regularity to that of the coefficients, and they involve the metric tensor.

4.5. **Differential operators on Sobolev spaces.** It is clear from the definitions that all of our differential operators are bounded $H^1 \to L^2$. Thus all classically defined operators extend to operators between Sobolev spaces. We therefore have the continuous operators

$$\begin{aligned} X\colon H^1(SM) &\to L^2(SM), \\ X\colon H^1(N) &\to L^2(N), \\ \overset{\mathtt{v}}{\nabla}, \overset{\mathtt{h}}{\nabla}\colon H^1(SM) &\to L^2(N), \quad \text{and} \\ \overset{\mathtt{v}}{\mathrm{div}}, \overset{\mathtt{h}}{\mathrm{div}}\colon H^1(N) &\to L^2(SM). \end{aligned}$$

Basic integration by parts holds for the extended operators: If $u, w \in H^1(SM)$ and $V, W \in H^1(N)$ and w and W vanish on the boundary, then

$$\begin{aligned} (Xu, w)_{L^2(SM)} &= -(u, Xw)_{L^2(SM)}, \\ (XV, W)_{L^2(N)} &= -(V, XW)_{L^2(N)}, \\ \left(\overset{\mathtt{v}}{\nabla} u, W\right)_{L^2(N)} &= -\left(u, \overset{\mathtt{v}}{\mathrm{div}} W\right)_{L^2(SM)}, \quad \text{and} \\ \left(\overset{\mathtt{h}}{\nabla} u, W\right)_{L^2(N)} &= -\left(u, \overset{\mathtt{h}}{\mathrm{div}} W\right)_{L^2(SM)}. \end{aligned}$$

We can use the space $C^1_0(SM)$ as test functions and $C^1_0(N)$ as test sections.

4.6. **Switching between different unit sphere bundles.** Suppose we have two Riemannian metrics $g, h \in C^{1,1}(T^2M)$ on the manifold M. Let S_gM and S_hM denote the corresponding unit sphere bundles. There is a natural radial $C^{1,1}$-diffeomorphism

$$s\colon S_gM \to S_hM, \quad s(x, v) = (x, v\,|v|_h^{-1}).$$

In section 5 we will have three Riemannian metrics $g \in C^{1,1}(T^2M)$ and $\overset{\alpha}{g}, h \in C^\infty(T^2M)$ with certain roles. In this case we will denote the corresponding radial $C^{1,1}$-diffeomorphisms by

$$\overset{\alpha}{s}\colon S_{\overset{\alpha}{g}}M \to S_hM, \quad \overset{\alpha}{r}\colon S_gM \to S_{\overset{\alpha}{g}}M \quad \text{and} \quad s\colon S_gM \to S_hM.$$

In section 5 we frequently use the convention that the bundles related to $\overset{\alpha}{g}$ are denoted $\overset{\alpha}{S}M := S_{\overset{\alpha}{g}}M$ and $\overset{\alpha}{N} := N_{\overset{\alpha}{g}}$, the operators related to $\overset{\alpha}{g}$ are decorated with α on top or as a subscript, the bundles and the operators related to h are decorated with subscripts h, and the bundles and the operators related to the metric g are written without decorations.

Remark 4.3. We can switch between sphere bundles and corresponding Sobolev spaces using pullbacks along radial functions. If u is a scalar function on SM, then $(s^{-1})^*u$ is a scalar function on S_hM. To see that pullback behaves well on the Sobolev scale, note that the $H^1(SM)$-norm controls all possible derivatives on SM since $TSM = \mathbb{R}X \oplus \mathcal{V} \oplus \mathcal{H}$. Thus the $H^1(SM)$-norm is equivalent to

$$\|u\| = \|u\|_{L^2(SM)} + \|\mathrm{d}_{SM} u\|_{L^2(T^*SM)} \tag{5}$$

with the norm of the differential interpreted with respect to any Riemannian metric on SM. With the norm (5) we see that regularity on Sobolev scale is preserved, since $(s^{-1})^*(\mathrm{d}_{SM} u) = \mathrm{d}_{S_hM}(u \circ s^{-1})$ by standard properties of the pullback.

Remark 4.3 allows us to prove continuous Sobolev embeddings between Sobolev spaces of low regularity metrics. We present one example that will be useful to us later. Let $g \in C^{1,1}(T^2M)$ and $h \in C^\infty(T^2M)$ be two Riemannian metrics on M. If $u \in \mathrm{Lip}(SM)$, then $(s^{-1})^*u \in \mathrm{Lip}(S_hM)$. Since h is C^∞-smooth, we have $(s^{-1})^*u \in H^1(S_hM)$ Then since $\|u\|_{H^1(SM)} \lesssim \left\|(s^{-1})^*u\right\|_{H^1(S_hM)}$ by remark 4.3, we see that $u \in H^1(SM)$. We have shown that the inclusion $\mathrm{Lip}(SM) \subseteq H^1(SM)$ holds even when the metric tensor is only $C^{1,1}$.

5. **Lemmas in low regularity.**

5.1. **The Pestov identity.** In this subsection (M,g) is a simple $C^{1,1}$ Riemannian manifold. We prove a variant of the commutator formula $[X, \overset{\mathtt{v}}{\nabla}] = -\overset{\mathtt{h}}{\nabla}$ and the Pestov identity on (M,g). First, we show that both results are valid for Sobolev functions on a manifold equipped with a C^∞-smooth Riemannian metric. Then we show that only $C^{1,1}$ regularity of the Riemannian metric is needed. The main focus of the subsection is on proving the Pestov identity of lemma 2.2.

Lemma 5.1. *Let (M,h) be a compact smooth manifold with a smooth boundary, where h is a C^∞-smooth Riemannian metric. The commutator formula $[X, \overset{\mathtt{v}}{\nabla}] = -\overset{\mathtt{h}}{\nabla}$ holds in the H^1 sense on (M,h): For $u \in H^1_0(S_hM)$ and $V \in C^1(N_h)$, we have*

$$\left(\overset{\mathtt{h}}{\nabla}_h u, V\right)_{L^2(N_h)} = \left(\overset{\mathtt{v}}{\nabla}_h u, X_h V\right)_{L^2(N_h)} - \left(X_h u, \overset{\mathtt{v}}{\mathrm{div}}_h V\right)_{L^2(S_hM)}.$$

Proof. Let $u \in H^1_0(S_hM)$ and $V \in C^\infty(N_h)$. Since V is smooth, by [33, Lemma 2.1.] we have

$$X_h \overset{\mathtt{v}}{\mathrm{div}}_h V - \overset{\mathtt{v}}{\mathrm{div}}_h X_h V = -\overset{\mathtt{h}}{\mathrm{div}}_h V.$$

Thus

$$\begin{aligned}\left(\overset{\mathtt{h}}{\nabla}_h u, V\right)_{L^2(N_h)} &= -\left(u, \overset{\mathtt{v}}{\mathrm{div}}_h X_h V\right)_{L^2(S_hM)} + \left(u, X \overset{\mathtt{v}}{\mathrm{div}}_h V\right)_{L^2(S_hM)} \\ &= \left(\overset{\mathtt{v}}{\nabla}_h u, X_h V\right)_{L^2(N_h)} - \left(X_h u, \overset{\mathtt{v}}{\mathrm{div}}_h V\right)_{L^2(S_hM)},\end{aligned}$$

where the last equality holds since $u \in H^1_0(S_hM)$, and since $X_hV \in C^\infty(N_h)$ and $\overset{\mathtt{v}}{\mathrm{div}}_h V \in C^\infty(S_hM)$. The same identity holds for $V \in C^1(N_h)$ by approximation, since only first order derivatives appear in the statement. □

Lemma 5.2. *Let (M,h) be a compact smooth manifold with a smooth boundary, where h is a C^∞-smooth Riemannian metric. Suppose that $u \in K^2(S_hM)$ vanishes on the boundary $\partial(S_hM)$. Then*

$$\left\|\overset{\mathtt{v}}{\nabla}_h X_h u\right\|^2_{L^2(N_h)} = Q_h\left(\overset{\mathtt{v}}{\nabla}_h u\right) + (n-1)\left\|X_h u\right\|^2_{L^2(S_hM)}, \tag{6}$$

where Q_h is the quadratic form defined for $W \in H^1_0(N_h, X_h)$ by

$$Q_h(W) = \|X_h W\|^2_{L^2(N_h)} - (R_h W, W)_{L^2(N_h)}.$$

Proof. Since $u \in K^2(S_hM)$ and u vanishes on the boundary $\partial(S_hM)$, there is a sequence $\big(\overset{\beta}{u}\big)_{\beta\in\mathbb{N}}$ of smooth functions on S_hM vanishing on $\partial(S_hM)$ so that $\overset{\beta}{u} \to u$ in $K^2(S_hM)$. We see that

$$\left\|\overset{\mathrm{v}}{\nabla}_h X_h \overset{\beta}{u} - \overset{\mathrm{v}}{\nabla}_h X_h u\right\|^2_{L^2(N_h)} \le \left\|X_h \overset{\beta}{u} - X_h u\right\|^2_{H^1(S_hM)} \le \left\|\overset{\beta}{u} - u\right\|^2_{K^2(S_hM)} \tag{7}$$

and

$$\left\|X_h \overset{\beta}{u} - X_h u\right\|^2_{L^2(N_h)} \le \left\|\overset{\beta}{u} - u\right\|^2_{H^1(S_hM)} \le \left\|\overset{\beta}{u} - u\right\|^2_{K^2(S_hM)}. \tag{8}$$

Therefore $\overset{\mathrm{v}}{\nabla}_h X_h \overset{\beta}{u} \to \overset{\mathrm{v}}{\nabla}_h X_h u$ in $L^2(N_h)$ and $X_h \overset{\beta}{u} \to X_h u$ in $L^2(S_hM)$ as $\beta \to \infty$. Additionally, since the curvature operator R of the metric h continuously maps $L^\infty(N_h) \to L^\infty(N_h)$, we have

$$\left\|\overset{\mathrm{v}}{\nabla}_h \overset{\beta}{u} - \overset{\mathrm{v}}{\nabla}_h u\right\|^2_{L^2(N_h)} \le \left\|\overset{\beta}{u} - u\right\|^2_{H^1(S_hM)} \le \left\|\overset{\beta}{u} - u\right\|^2_{K^2(S_hM)} \tag{9}$$

and

$$\left\|R_h \overset{\mathrm{v}}{\nabla}_h \overset{\beta}{u} - R_h \overset{\mathrm{v}}{\nabla}_h u\right\|^2_{L^2(N_h)} \lesssim \left\|\overset{\beta}{u} - u\right\|^2_{H^1(S_hM)} \le \left\|\overset{\beta}{u} - u\right\|^2_{K^2(S_hM)}. \tag{10}$$

Thus $Q_h\left(\overset{\mathrm{v}}{\nabla}_h \overset{\beta}{u}\right) \to Q_h\left(\overset{\mathrm{v}}{\nabla}_h u\right)$ as $\beta \to \infty$. By the Pestov identity for smooth functions and metrics (see [33, Remark 2.3.]) we have

$$\left\|\overset{\mathrm{v}}{\nabla}_h X_h \overset{\beta}{u}\right\|^2_{L^2(N_h)} = Q_h\left(\overset{\mathrm{v}}{\nabla}_h \overset{\beta}{u}\right) + (n-1)\left\|X_h \overset{\beta}{u}\right\|^2_{L^2(S_hM)}. \tag{11}$$

We now let $\beta \to \infty$ in (11). By our estimates (7), (8), (9), and (10) we end up with the claimed identity (6). ☐

The rest of this section focuses on showing that we can replace the C^∞-smooth Riemannian metric h in lemmas 5.1 and 5.2 by a $C^{1,1}$ regular Riemannian metric. Let (M, g) be a $C^{1,1}$ simple Riemannian manifold. Next, we construct approximations of g by C^∞-smooth Riemannian metrics $\overset{\alpha}{g}$.

Let $(x^1, \dots, x^n)$ be the usual Cartesian coordinates on the Euclidean closed ball $M \subset \mathbb{R}^n$ and extend all components $g_{ij} \in C^{1,1}(M)$ of g to functions $\overline{g}_{ij} \in C^{1,1}(\mathbb{R}^n)$. Such extensions exist since $C^{1,1} = W^{2,\infty}$ and the boundary of M is smooth (see [41, Chapter 6, Theorem 5]). Let us then choose a non-negative compactly supported smooth function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ with unit integral and define a sequence of standard mollifiers $\overset{\alpha}{\varphi}(x) = \alpha^n \varphi(\alpha x)$ for $\alpha \in \mathbb{N}$. Then we define

$$\overset{\alpha}{g}_{ij} := (\overset{\alpha}{\varphi} * \overline{g}_{ij})|_M \in C^\infty(M). \tag{12}$$

Lemma 5.3. *Let (M, g) be a simple $C^{1,1}$ manifold. Let h be a smooth reference metric on M. There exists a sequence $(\overset{\alpha}{g})_{\alpha\in\mathbb{N}}$ of C^∞-smooth metrics on M such that*

1. $\overset{\alpha}{g}_{ij} \to g_{ij}$ *in* $W^{2,2}_h(M)$ *and in* $W^{1,\infty}_h(M)$,
2. $\overset{\alpha}{g}^{ij} \to g^{ij}$ *in* $W^{1,2}_h(M)$ *and in* $L^\infty_h(M)$,
3. $\overset{\alpha}{\Gamma}{}^i{}_{jk} \to \Gamma^i{}_{jk}$ *in* $W^{1,1}_h(M)$ *and in* $L^\infty_h(M)$,
4. $\overset{\alpha}{R}{}^i{}_{jkl} \to R^i{}_{jkl}$ *in* $L^1_h(M)$.

Proof. For each $\alpha \in \mathbb{N}$ let $\overset{\alpha}{g} \coloneqq \overset{\alpha}{g}_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j \in C^\infty(T^2M)$, where $\overset{\alpha}{g}_{ij}$ are as in (12). We will show that a subsequence of the sequence $(\overset{\alpha}{g})_{\alpha\in\mathbb{N}}$ consists of smooth Riemannian metrics and satisfies conditions 1–4.

We see that for large α each $\overset{\alpha}{g}$ is a C^∞-smooth Riemannian metric. Smoothness follows standard properties of the mollifiers $\overset{\alpha}{\varphi}$. Each $\overset{\alpha}{g}$ is symmetric by construction. For large α each $\overset{\alpha}{g}$ is positive definite since this is an open condition and pointwise convergence $\overset{\alpha}{g}_{ij} \to g_{ij}$ follows from continuity and item 1.

1. Because $\overline{g}_{ij}$ is compactly supported and in both spaces $W^{2,2}(\mathbb{R}^n)$ and $W^{2,\infty}(\mathbb{R}^n)$, the convolution converges $\overset{\alpha}{\varphi} * \overline{g}_{ij} \to \overline{g}_{ij}$ in both spaces $W^{2,2}(\mathbb{R}^n)$ and $W^{1,\infty}(\mathbb{R}^n)$. This implies convergence in the corresponding function spaces over the subdomain $M \subset \mathbb{R}^n$.

2. Let us denote the adjugate of a matrix A by $\mathrm{adj}(A)$; we interpret rank two tensor fields as matrix-valued functions on $M \subset \mathbb{R}^n$. By item 1 we have

$$\det(\overset{\alpha}{g}) \to \det(g) \quad \text{and} \quad \mathrm{adj}(\overset{\alpha}{g})^{ij} \to \mathrm{adj}(g)^{ij} \tag{13}$$

in $L^\infty_h(M)$. Thus for sufficiently large α the matrices $\overset{\alpha}{g}$ are uniformly invertible in the sense that

$$\left\|\det(\overset{\alpha}{g})^{-1}\right\|_{L^\infty_h(M)} \leq C.$$

Since

$$\overset{\alpha}{g}{}^{ij}(x) = \det(\overset{\alpha}{g}(x))^{-1}\,\mathrm{adj}(\overset{\alpha}{g}(x))^{ij},$$

we have that $\overset{\alpha}{g}{}^{ij} \to g^{ij}$ in $L^\infty_h(M)$. Derivatives of the inverse satisfy $\partial_k \overset{\alpha}{g}{}^{ij} = -\overset{\alpha}{g}{}^{il}(\partial_k \overset{\alpha}{g}_{lm})\overset{\alpha}{g}{}^{mj}$, which implies convergence of the derivatives in $L^2_h(M)$.

3. This follows from

$$\overset{\alpha}{\Gamma}{}^i{}_{jk} = \frac{1}{2}\overset{\alpha}{g}{}^{il}\left(\partial_j \overset{\alpha}{g}_{kl} + \partial_k \overset{\alpha}{g}_{jl} - \partial_l \overset{\alpha}{g}_{jk}\right)$$

and items 1 and 2.

4. This follows from

$$\overset{\alpha}{R}_{ijk}{}^l = \partial_i \overset{\alpha}{\Gamma}{}^l{}_{jk} - \partial_j \overset{\alpha}{\Gamma}{}^m{}_{jk} + \overset{\alpha}{\Gamma}{}^m{}_{jk}\overset{\alpha}{\Gamma}{}^l{}_{im} - \overset{\alpha}{\Gamma}{}^m{}_{ik}\overset{\alpha}{\Gamma}{}^l{}_{jm}$$

and item 3. ∎

Next we prove the Pestov identity for $C^{1,1}$ regular metrics. In the context of lemma 2.2 the manifold (M,g) is simple $C^{1,1}$, the Riemannian metric g is $C^{1,1}$ regular, the function u is in $\mathrm{Lip}_0(SM)$ and satisfies $Xu \in H^1(SM)$ and $\overset{\mathrm{v}}{\nabla} u \in H^1(N,X)$.

Proof of lemma 2.2. Choose a smooth reference Riemannian metric h on M and let $(\overset{\alpha}{g})_{\alpha\in\mathbb{N}}$ be a sequence of smooth metrics approximating g as in lemma 5.3. For each $\alpha \in \mathbb{N}$ denote $\overset{\alpha}{u} \coloneqq u \circ \overset{\alpha}{r}$. Then by remark 4.3 we have $\overset{\alpha}{u} \in H^1_0(\overset{\alpha}{S}M)$, $\overset{\alpha}{X}\overset{\alpha}{u} \in H^1(\overset{\alpha}{S}M)$ and $\overset{\mathrm{v}}{\nabla}_\alpha \overset{\alpha}{u} \in H^1(\overset{\alpha}{N}, \overset{\alpha}{X})$, which implies that $\overset{\alpha}{u} \in K^2(\overset{\alpha}{S}M)$ and $\overset{\alpha}{u}|_{\partial(\overset{\alpha}{S}M)} = 0$. For each α an application of lemma 5.2 gives

$$\left\|\overset{\mathrm{v}}{\nabla}_\alpha \overset{\alpha}{X}\overset{\alpha}{u}\right\|^2_{L^2(\overset{\alpha}{N})} = Q_\alpha\left(\overset{\mathrm{v}}{\nabla}_\alpha \overset{\alpha}{u}\right) + (n-1)\left\|\overset{\alpha}{X}\overset{\alpha}{u}\right\|^2_{L^2(\overset{\alpha}{S}M)}. \tag{14}$$

We will show that

$$\lim_{\alpha\to\infty}\left\|\overset{\mathrm{v}}{\nabla}_\alpha \overset{\alpha}{X}\overset{\alpha}{u}\right\|^2_{L^2(\overset{\alpha}{N})} = \left\|\overset{\mathrm{v}}{\nabla} X u\right\|^2_{L^2(N)}. \tag{15}$$

Similar arguments can be used to deduce that

$$\lim_{\alpha\to\infty} Q_\alpha\left(\overset{\mathrm{v}}{\nabla}_\alpha \overset{\alpha}{u}\right) = Q\left(\overset{\mathrm{v}}{\nabla} u\right) \tag{16}$$

and

$$\lim_{\alpha\to\infty} \left\|\overset{\alpha}{X}\overset{\alpha}{u}\right\|^2_{L^2(\overset{\alpha}{S}M)} = \|Xu\|^2_{L^2(SM)}. \tag{17}$$

Then letting $\alpha\to\infty$ in equation (14) proves the claim of the theorem. Since the arguments showing equations (16) and (17) are analogous to what is presented below, we omit them. (The fact that components of the curvature tensor only converge in L^1 and not in L^∞ is where the assumption $u\in \mathrm{Lip}(SM)$ is useful.) Coordinate formulas required to show equations (16) and (17) are given in appendix A.

For any L^p convergence to make sense, we fix S_hM to be our common reference bundle for objects to be integrated on. First we study how the $L^2(\overset{\alpha}{N})$ norm on the left-hand side of (15) transforms under $\overset{\alpha}{s}$. Let $\tilde{u} := \overset{\alpha}{u}\circ \overset{\alpha}{s}^{-1}$ and fix $\hat{z}\in S_hM$ and $z\in \overset{\alpha}{S}M$ such that $\overset{\alpha}{s}(z)=\hat{z}$. By basic properties of pushforwards we have

$$\left(\left((\overset{\alpha}{s}_*\overset{\alpha}{X})\tilde{u}\right)\circ\overset{\alpha}{s}\right)(z) = (\overset{\alpha}{s}_*\overset{\alpha}{X})_{\hat{z}}\tilde{u} = \overset{\alpha}{X}_z(\tilde{u}\circ\overset{\alpha}{s}) = \overset{\alpha}{X}_z\overset{\alpha}{u}.$$

Thus

$$(\overset{\alpha}{s}_*\overset{\alpha}{\partial}{}^j)_{\hat{z}}\left((\overset{\alpha}{s}_*\overset{\alpha}{X})\tilde{u}\right) = \overset{\alpha}{\partial}{}^j_z\left((\overset{\alpha}{s}_*\overset{\alpha}{X})\tilde{u}\circ\overset{\alpha}{s}\right) = \overset{\alpha}{\partial}{}^j_z\left(\overset{\alpha}{X}\overset{\alpha}{u}\right).$$

Since $\pi(z)=\pi(\hat{z})\in M$ we get

$$\begin{aligned}
\left\|\overset{\mathrm{v}}{\nabla}_\alpha\overset{\alpha}{X}\overset{\alpha}{u}\right\|^2_{L^2(\overset{\alpha}{N})} &= \int_{z\in\overset{\alpha}{S}M} \overset{\alpha}{g}_{ij}(\pi(z))\left(\overset{\alpha}{\partial}{}^i_z(\overset{\alpha}{X}\overset{\alpha}{u})\right)\left(\overset{\alpha}{\partial}{}^j_z(\overset{\alpha}{X}\overset{\alpha}{u})\right)\,\mathrm{d}\overset{\alpha}{\Sigma}(z)\\
&= \int_{\hat{z}\in S_hM} \overset{\alpha}{g}_{ij}(\pi(\hat{z}))\left(\overset{\alpha}{\partial}{}^i_{\overset{\alpha}{s}^{-1}(\hat{z})}(\overset{\alpha}{X}\overset{\alpha}{u})\right)\\
&\qquad\times\left(\overset{\alpha}{\partial}{}^j_{\overset{\alpha}{s}^{-1}(\hat{z})}(\overset{\alpha}{X}\overset{\alpha}{u})\right)\left|\det\left(\mathrm{d}\overset{\alpha}{s}{}^{-1}_{\hat{z}}\right)\right|\,\mathrm{d}\Sigma_h(\hat{z})\\
&= \int_{\hat{z}\in S_hM} \overset{\alpha}{g}_{ij}(\pi(\hat{z}))\left((\overset{\alpha}{s}_*\overset{\alpha}{\partial}{}^i)_{\hat{z}}\left((\overset{\alpha}{s}_*\overset{\alpha}{X})\tilde{u}\right)\right)\\
&\qquad\times\left((\overset{\alpha}{s}_*\overset{\alpha}{\partial}{}^j)_{\hat{z}}\left((\overset{\alpha}{s}_*\overset{\alpha}{X})\tilde{u}\right)\right)\left|\det\left(\mathrm{d}\overset{\alpha}{s}{}^{-1}_{\hat{z}}\right)\right|\,\mathrm{d}\Sigma_h(\hat{z}).
\end{aligned}$$

An analogous formula holds for the right-hand side of equation (15). Note that $\tilde{u} = u\circ s^{-1}$. Thus we see that to prove equation (15) we need to prove the following two items.

(i) $(\overset{\alpha}{s}_*\overset{\alpha}{\partial}{}^j)\left((\overset{\alpha}{s}_*\overset{\alpha}{X})\tilde{u}\right)\to(s_*\partial^j)\left((s_*X)\tilde{u}\right)$ in $L^\infty(S_hM)$.

(ii) $\det\left(\mathrm{d}\overset{\alpha}{s}{}^{-1}\right)\to\det\left(\mathrm{d}s^{-1}\right)$ in $L^\infty(S_hM)$.

The push-forward $\overset{\alpha}{s}_*$ has a useful block matrix representation in the coordinates of the unit sphere bundles. Let $(x,\overset{\alpha}{w})\in\overset{\alpha}{S}M$ and $(x,v)\in S_hM$ correspond to each other through $\overset{\alpha}{s}(x,\overset{\alpha}{w})=(x,v)$. To $(x,\overset{\alpha}{w})$ and (x,v) we associate the coordinate vector fields $\partial_{x^1},\dots,\partial_{x^n},\partial_{\overset{\alpha}{w}{}^1},\dots,\partial_{\overset{\alpha}{w}{}^n}$ and $\partial_{x^1},\dots,\partial_{x^n},\partial_{v^1},\dots,\partial_{v^n}$ respectively. The matrix representation of $\overset{\alpha}{s}_*$ in a block form with respect to the bases $\partial_{x^1},\dots,\partial_{x^n},\partial_{\overset{\alpha}{w}{}^1},\dots,\partial_{\overset{\alpha}{w}{}^n}$ and $\partial_{x^1},\dots,\partial_{x^n},\partial_{v^1},\dots,\partial_{v^n}$ is

$$\overset{\alpha}{s}_* = \begin{pmatrix} I & 0\\ \partial_x v & \partial_{\overset{\alpha}{w}}v\end{pmatrix}.$$

To find coordinate expressions for $\overset{\alpha}{s}_*\overset{\alpha}{\partial}{}^j$ and $\overset{\alpha}{s}_*\overset{\alpha}{X}$, the vector fields $\overset{\alpha}{\partial}{}^j$ and $\overset{\alpha}{X}$ need to be expressed in the basis $\partial_{x^1},\dots,\partial_{x^n},\partial_{\overset{\alpha}{w}{}^1},\dots,\partial_{\overset{\alpha}{w}{}^n}$. As long as everything is only evaluated on $\overset{\alpha}{S}M$, we have $\overset{\alpha}{X}\overset{\alpha}{u} = \overset{\alpha}{w}^j \overset{\alpha}{\delta}_j \hat{u}$ for any $\hat{u}\colon TM \setminus 0 \to \mathbb{R}$ such that $\hat{u}|_{\overset{\alpha}{S}M} = \overset{\alpha}{u}$. Therefore, in local coordinates and as long as we are careful to only evaluate only $\overset{\alpha}{S}M$, we have

$$\overset{\alpha}{X} = \overset{\alpha}{w}^j \partial_{x^j} - \overset{\alpha}{\Gamma}{}^k{}_{jl} \overset{\alpha}{w}^j \overset{\alpha}{w}^l \partial_{\overset{\alpha}{w}^k}.$$

Similarly we get $\overset{\alpha}{\partial}{}^j = \overset{\alpha}{g}{}^{jl} \partial_{\overset{\alpha}{w}^l}$.

Coordinate formulas for the push-forwards vector fields can be found by multiplying with $\overset{\alpha}{s}_*$. This time only evaluating on S_hM, we have

$$\overset{\alpha}{s}_* \overset{\alpha}{X} = \overset{\alpha}{w}^j \partial_{x^j} + \left(\overset{\alpha}{w}^k \partial_{x^k} v^j - \overset{\alpha}{\Gamma}{}^k{}_{lm} \overset{\alpha}{w}^l \overset{\alpha}{w}^m (\partial_{\overset{\alpha}{w}^k} v^j) \right) \partial_{v^j}$$

and

$$\overset{\alpha}{s}_* \overset{\alpha}{\partial}{}^j = \overset{\alpha}{g}{}^{jl} (\partial_{\overset{\alpha}{w}^l} v^k) \partial_{v^k}.$$

From these expressions it is clear that convergence in item (i) comes down to three matters. In the base there are derivatives of components of $\overset{\alpha}{g}$ up to the second order and derivatives of components of $\overset{\alpha}{g}{}^{-1}$ up to the first order. Again, components of the metric $\overset{\alpha}{g}$ appear in the coefficients $\overset{\alpha}{w}^j$. The behaviour on the limit of all of these matters is controlled by lemma 5.3. We have concluded item (i).

To prove item (ii), we write the matrix $\mathrm{d}\overset{\alpha}{s}{}^{-1}$ in the block form

$$\mathrm{d}\overset{\alpha}{s}{}^{-1} = \begin{pmatrix} I & 0 \\ \partial_x \overset{\alpha}{w} & \partial_v \overset{\alpha}{w} \end{pmatrix}.$$

Clearly $\det\left(\mathrm{d}\overset{\alpha}{s}{}^{-1}\right) = \det\left(\partial_v \overset{\alpha}{w}\right)$. Therefore the behaviour as $\alpha \to \infty$ depends on sums and products of derivatives $\partial_{v^k}(v\,|v|_\alpha^{-1})$, which comes down to the metric $\overset{\alpha}{g}$. By lemma 5.3 we have $\det\left(\mathrm{d}\overset{\alpha}{s}{}^{-1}\right) \to \det\left(\mathrm{d}s^{-1}\right)$ in $L^\infty(S_hM)$.

We have shown both items (i) and (ii) and thus we have proved equation (15). This finishes the proof of the lemma. □

Lemma 5.4. *Let (M, g) be a simple $C^{1,1}$ manifold. The commutator formula $[X, \overset{\mathtt{v}}{\nabla}] = -\overset{\mathtt{h}}{\nabla}$ holds on (M, g) in the H^1 sense: For all $u \in H^1_0(SM)$ and $V \in C^1(N)$ we have*

$$\left(\overset{\mathtt{h}}{\nabla}u, V\right)_{L^2(N)} = \left(\overset{\mathtt{v}}{\nabla}u, XV\right)_{L^2(N)} - \left(Xu, \overset{\mathtt{v}}{\mathrm{div}}V\right)_{L^2(SM)}. \tag{18}$$

Proof. Let h be a smooth reference metric on M and choose a sequence $(\overset{\alpha}{g})_{\alpha\in\mathbb{N}}$ of smooth metrics approximating g as in lemma 5.3. For each $\alpha \in \mathbb{N}$ denote $\overset{\alpha}{u} := u \circ \overset{\alpha}{r}$ and $\overset{\alpha}{V} := V \circ \overset{\alpha}{r}$. Then by remark 4.3 we have $\overset{\alpha}{u} \in H^1_0(\overset{\alpha}{S}M)$ and $\overset{\alpha}{V} \in C^1(\overset{\alpha}{N})$. We apply lemma 5.1 to $\overset{\alpha}{u}$ and $\overset{\alpha}{V}$ to get

$$\left(\overset{\mathtt{h}}{\nabla}_\alpha \overset{\alpha}{u}, \overset{\alpha}{V}\right)_{L^2(\overset{\alpha}{N})} = \left(\overset{\mathtt{v}}{\nabla}_\alpha \overset{\alpha}{u}, \overset{\alpha}{X}\overset{\alpha}{V}\right)_{L^2(\overset{\alpha}{N})} - \left(\overset{\alpha}{X}\overset{\alpha}{u}, \overset{\mathtt{v}}{\mathrm{div}}_\alpha \overset{\alpha}{V}\right)_{L^2(\overset{\alpha}{S}M)}. \tag{19}$$

Letting $\alpha \to \infty$ in equation (19) proves the claimed identity (18), after we have shown that

$$\lim_{\alpha\to\infty} \left(\overset{\mathtt{h}}{\nabla}_\alpha \overset{\alpha}{u}, \overset{\alpha}{V}\right)_{L^2(\overset{\alpha}{N})} = \left(\overset{\mathtt{h}}{\nabla}u, V\right)_{L^2(N)}, \tag{20}$$

$$\lim_{\alpha\to\infty} \left(\overset{\mathtt{v}}{\nabla}_\alpha \overset{\alpha}{u}, \overset{\alpha}{X}\overset{\alpha}{V}\right)_{L^2(\overset{\alpha}{N})} = \left(\overset{\mathtt{v}}{\nabla}u, XV\right)_{L^2(N)} \tag{21}$$

and

$$\lim_{\alpha\to\infty}\left(\overset{\alpha}{X}\overset{\alpha}{u},\overset{\mathrm{v}}{\mathrm{div}}_\alpha\overset{\alpha}{V}\right)_{L^2(\overset{\alpha}{S}M)}=\left(Xu,\overset{\mathrm{v}}{\mathrm{div}}V\right)_{L^2(SM)}. \tag{22}$$

All formulas (20), (21) and (22) can be shown by arguments analogous to those used in proving the formula (15) in the proof of lemma 2.2. Thus we omit the details. Coordinate formulas needed to complete the proofs of the formulas are given in appendix A. ☐

5.2. **Regularity of the integral function.** Let (M,g) be a simple $C^{1,1}$ manifold. In this section we will prove lemma 2.1 concerning regularity properties of the integral functions of Lipschitz functions and one-forms. We prove a Lipschitz property for the geodesic flow in lemma 5.5. The Lipschitz property lets us prove that the integral functions are Lipschitz in lemma 5.6. To prove $H^1(N,X)$ regularity for the vertical gradients of the integral functions in lemma 5.7, we use the commutator formula from lemma 5.4.

On a compact manifold M and its unit sphere bundle SM all reasonable notions of distance are bi-Lipschitz equivalent. Since in this section M will be the Euclidean closed ball, we choose Euclidean distances.

Lemma 5.5. *Let (M,g) be a simple $C^{1,1}$ manifold. For $z\in SM$ let $[\tau_-(z),\tau_+(z)]$ be the maximal interval of existence of the geodesic γ_z. The geodesic flow ϕ_t is Lipschitz continuous in $z\in SM$: There is a uniform $L>0$ so that for all $z,\hat z\in SM$ and $t\in[\tau_-(z),\tau_+(z)]\cap[\tau_-(\hat z),\tau_+(\hat z)]$ we have*

$$d_{SM}(\phi_t(z),\phi_t(\hat z))\le L d_{SM}(z,\hat z).$$

Proof. Let $z,\hat z\in SM$. Note that both lifted geodesics $t\mapsto\phi_t(z)$ and $t\mapsto\phi_t(\hat z)$ satisfy the equation $X(\psi(t))=\dot\psi(t)$ on the interval $[\tau_-(z),\tau_+(z)]\cap[\tau_-(\hat z),\tau_+(\hat z)]$. Since the distance d_{SM} can be taken to be Euclidean and the Christoffel symbols of the metric $g\in C^{1,1}(T^2M)$ are Lipschitz continuous, we get by Grönwall's inequality that

$$d_{SM}(\phi_t(z),\phi_t(\hat z))\le e^{K|t-0|}d_{SM}(\phi_0(z),\phi_0(\hat z))=e^{K|t|}d_{SM}(z,\hat z),$$

for some $K>0$ independent of t, z and $\hat z$. Since M is simple $C^{1,1}$, the function τ is uniformly bounded on SM. Thus we find a constant $L>0$ independent of z and $\hat z$ such that $e^{K|t|}\le L$ uniformly for $t\in[\tau_-(z),\tau_+(z)]\cap[\tau_-(\hat z),\tau_+(\hat z)]$, which finishes the proof. ☐

Lemma 5.6. *Let (M,g) be a simple $C^{1,1}$ manifold. Let $f\in\mathrm{Lip}_0(SM)$ and let u^f be the integral function of f defined by (2). Then $u^f\in\mathrm{Lip}(SM)$.*

Proof. Let $z,\hat z\in SM$ be so that $\tau(\hat z)\le\tau(z)$. Then by a simple calculation

$$\begin{aligned}\left|u^f(z)-u^f(\hat z)\right|&\le(\tau(z)-\tau(\hat z))\sup_{t\in[\tau(\hat z),\tau(z)]}|f(\phi_t(z))|\\&\quad+\int_0^{\tau(\hat z)}|f(\phi_t(z))-f(\phi_t(\hat z))|\,\mathrm{d}t.\end{aligned} \tag{23}$$

We will show that both summands on the right-hand side of equation (23) are bounded by $Cd_{SM}(z,\hat z)$ for some constant $C>0$ independent of z and $\hat z$.

First, we treat the second term on the right-hand side of (23). Since by lemma 5.5 the geodesic flow ϕ_t and f both are Lipschitz, there is a constant $K>0$ independent of t, z and $\hat z$ so that

$$|f(\phi_t(z))-f(\phi_t(\hat z))|\le Kd_{SM}(z,\hat z).$$

Since the manifold M is simple $C^{1,1}$, there is a constant $L > 0$ independent of $\hat{z}$ so that $\tau(\hat{z}) \leq L$. It follows that

$$\int_0^{\tau(\hat{z})} |f(\phi_t(z)) - f(\phi_t(\hat{z}))| \, \mathrm{d}t \leq KLd_{SM}(z, \hat{z}), \tag{24}$$

which proves the desired bound for the second term.

Then we turn to the first term on the right-hand side of (23). Since f is Lipschitz and vanishes on the boundary $\partial(SM)$, for all $t \in [\tau(\hat{z}), \tau(z)]$ we have

$$\begin{aligned} |f(\phi_t(z))| &= \left|f(\phi_t(z)) - f(\phi_{\tau(z)}(z))\right| \\ &\leq \operatorname{Lip}(f) d_{SM}(\phi_t(z), \phi_{\tau(z)}(z)) \\ &\leq \operatorname{Lip}(f)(\tau(z) - t) \\ &\leq \operatorname{Lip}(f)(\tau(z) - \tau(\hat{z})) \\ &\leq \operatorname{Lip}(f)(\tau(z) + \tau(\hat{z})). \end{aligned} \tag{25}$$

The function τ^2 is Lipschitz since the manifold is simple $C^{1,1}$, and so

$$\begin{aligned} (\tau(z) - \tau(\hat{z})) \sup_{t \in [\tau(\hat{z}), \tau(z)]} |f(\phi_t(z))| &\leq \operatorname{Lip}(f)(\tau^2(z) - \tau^2(\hat{z})) \\ &\leq \operatorname{Lip}(f) \operatorname{Lip}(\tau^2) d_{SM}(z, \hat{z}), \end{aligned} \tag{26}$$

as desired.

Combining estimates (23), (24) and (26) yields a Lipschitz estimate for the integral function u^f. □

Lemma 5.7. *Let (M, g) be a simple $C^{1,1}$ manifold. Assume that $f \in \operatorname{Lip}_0(SM)$ integrates to zero over all maximal geodesics in M. Then $\overset{\mathtt{v}}{\nabla} u^f \in H^1(N, X)$, where u^f is the integral function of f defined by equation (2).*

Proof. The integral function u^f is in $\operatorname{Lip}(SM)$ by lemma 5.6 and $u^f|_{\partial(SM)} = 0$ since f integrates to zero over all maximal geodesics of M. Thus by remark 4.3 we have $u^f \in H^1_0(SM)$. Then an application of lemma 5.4 gives

$$\left(\overset{\mathtt{v}}{\nabla} u^f, XV\right)_{L^2(N)} = \left(\overset{\mathtt{h}}{\nabla} u^f, V\right)_{L^2(N)} - \left(Xu^f, \overset{\mathtt{v}}{\operatorname{div}} V\right)_{L^2(SM)} \tag{27}$$

for any $V \in C^1(N)$. Here $Xu^f \in H^1(SM)$, since $Xu^f = -f \in \operatorname{Lip}(SM)$. As $Xu^f = -f = 0$ at ∂SM, for any $V \in C^1(N)$ we can integrate by parts in (27) to get

$$\left(\overset{\mathtt{v}}{\nabla} u^f, XV\right)_{L^2(N)} = \left((\overset{\mathtt{h}}{\nabla} - \overset{\mathtt{v}}{\nabla} X) u^f, V\right)_{L^2(N)}.$$

Therefore $X\overset{\mathtt{v}}{\nabla} u^f = (\overset{\mathtt{v}}{\nabla} X - \overset{\mathtt{h}}{\nabla}) u^f \in L^2(N)$, which shows that $\overset{\mathtt{v}}{\nabla} u^f \in H^1(N, X)$. □

Lemma 5.8. *Let (M, g) be a simple $C^{1,1}$ manifold. Then for any $x \in \partial M$ and $v \in S_x(\partial M)$, there is a sequence of vectors $v_k \in S_x M$ so that $\tau(x, v_k) > 0$, $v_k \to v$ and $\tau(x, v_k) \to 0$ as $k \to \infty$.*

Proof. Let $x \in \partial M$ and $v \in S_x(\partial M)$. Choose a C^1 boundary curve σ defined on an interval I so that $\sigma(0) = x$ and $\dot{\sigma}(0) = v$. Choose a sequence (x_k) of boundary points on $\sigma(I)$ so that $x_k \to x$. For each k let $v_k \in S_x M$ be the initial velocity of the unique geodesic γ_k joining x to x_k in the interior of M — the geodesic γ_k exists by simplicity. Then $\tau(x, v_k) > 0$ for each k. Since the lengths of the geodesics depend continuously on their end points, we get $\tau(x, v_k) = l(\gamma_k) \to 0$ as $k \to \infty$.

It remains to verify that $v_k \to v$. The geodesic equation gives

$$\left|\ddot{\gamma}_k^i(t)\right| = \left|\sum_{j,l} \Gamma^i{}_{jl}(\gamma_k(t))\dot{\gamma}_k^j(t)\dot{\gamma}_k^l(t)\right| \le n^2 \sup_x \left|\Gamma^i{}_{jl}(x)\right| \sup_t \left|\dot{\gamma}_k(t)\right|^2,$$

where all norms are the Euclidean ones of the global coordinates and the suprema range over all coordinates. Therefore $|\ddot{\gamma}_k(t)|$ is uniformly bounded for all t and k, and so by Taylor approximation in the coordinates

$$x_k = \gamma_k(\tau_k) = x + \tau_k v_k + \mathcal{O}(\tau_k^2),$$

where the error term is uniform over k. Therefore (in local coordinates)

$$\begin{aligned} v &= \dot{\sigma}(0) \\ &= \lim_{k\to\infty} \frac{x_k - x}{\tau_k} \\ &= \lim_{k\to\infty} \frac{\tau_k v_k + \mathcal{O}(\tau_k^2)}{\tau_k} \\ &= \lim_{k\to\infty} v_k \end{aligned}$$

as claimed. □

Lemma 5.9. *Let (M, g) be a simple $C^{1,1}$ manifold. Suppose that $f \in \mathrm{Lip}(M)$ integrates to zero over all maximal geodesics of M. Then f vanishes on the boundary ∂M.*

Proof. Let $x \in \partial M$ be a boundary point. Suppose that $v \in S_x(\partial M)$. By lemma 5.8 there is a sequence of tangent vectors $v_k \in S_x M$ so that $\tau(x, v_k) > 0$, $\tau(x, v_k) \to 0$ and $v_k \to v$ when $k \to \infty$. Since integrals of f over all maximal geodesics vanish, the integral function u^f of f vanishes on the boundary $\partial(SM)$. As the lengths of the geodesics approach zero we get

$$f(x) = \lim_{k\to\infty} \frac{1}{\tau(x, v_k)} \int_0^{\tau(x,v_k)} f(\gamma_{x,v_k}(t))\,\mathrm{d}t = \lim_{k\to\infty} \frac{1}{\tau(x, v_k)} u^f(x, v_k) = 0$$

as claimed. □

If $f \in \mathrm{Lip}(SM)$, then the proof above only gives $f|_{\partial_0(SM)} = 0$, not $f|_{\partial(SM)} = 0$. This is true also in the smooth case, and this conclusion is optimal for general functions on SM. If a function on SM arises from a tensor field, then the natural boundary determination is more involved in low regularity and we shall not discuss it here; cf. remark 1.3.

Proof of lemma 2.1. 1. Let f be a Lipschitz function on M that integrates to zero over all maximal geodesics of M. Define the integral function u^f of f as in (2). We have $f \in \mathrm{Lip}_0(M)$ by lemma 5.9. Thus $u^f \in \mathrm{Lip}_0(SM)$ by lemma 5.6. We have $\overset{\mathrm{v}}{\nabla} u^f \in H_0^1(N, X)$ by lemma 5.7 since u^f vanishes on $\partial(SM)$. The last claim $Xu^f = -\pi^* f \in \mathrm{Lip}(SM) \subseteq H^1(SM)$ follows from the fundamental theorem of calculus.

2. Let h be a Lipschitz 1-form on M that integrates to zero over all maximal geodesics of M and vanishes on the boundary ∂M. Let u^h be the integral function of h defined by (2). Then $u^h \in \mathrm{Lip}_0(SM)$ by lemma 5.6. We see that $Xu^f \in H^1(SM)$ and $\overset{\mathrm{v}}{\nabla} u^h \in H_0^1(N, X)$ as in item 1. □

5.3. **The integral function in the Pestov identity.** This subsection concludes the proofs of the lemmas required to prove theorem 1.1. We verify that the integral function of a Lipschitz 1-form h on M behaves in the same way in the Pestov identity as it does in the smooth case. Recall that (M, g) is a simple $C^{1,1}$ Riemannian manifold and particularly g is a $C^{1,1}$ regular Riemannian metric on M.

Proof of lemma 2.3. Let h be a Lipschitz 1-form on M and denote by $\tilde{h}$ the associated function on SM. We will show that

$$\left\| \overset{\mathrm{v}}{\nabla} \tilde{h} \right\|^2_{L^2(N)} = (n-1) \left\| \tilde{h} \right\|^2_{L^2(SM)}. \tag{28}$$

The Lipschitz assumption guarantees that the left-hand side of (28) is well defined. Let ω stand for the $(n-1)$-dimensional measure of the unit sphere in $\mathbb{R}^n$. By [15, Lemma 4] we have

$$\int_{S_x M} \left| \tilde{h}(x, v) \right|^2 \, \mathrm{d}S_x = |h(x)|^2 \frac{\omega}{n}$$

and

$$\int_{S_x M} \left| \overset{\mathrm{v}}{\nabla} \tilde{h}(x, v) \right|^2 \, \mathrm{d}S_x = |h(x)|^2 \frac{\omega(n-1)}{n}$$

on every fiber $S_x M$ of the unit sphere bundle. We may integrate over x just as in [15, Lemma 4] despite having less regularity, and we find

$$\left\| \overset{\mathrm{v}}{\nabla} \tilde{h} \right\|^2_{L^2(N)} = (n-1) \int_M |h(x)|^2 \frac{\omega}{n} \, \mathrm{d}V_g = (n-1) \left\| \tilde{h} \right\|^2_{L^2(SM)}$$

as claimed. □

6. **Lemmas in smooth geometry.** This final section contains the proofs of the lemmas used to verify that the two definitions of simplicity (definitions 1.4 and 1.5) agree when the geometry is C^∞-smooth. We assume that $M \subseteq \mathbb{R}^n$ is the closed unit ball and we let g be a C^∞-smooth Riemannian metric on M.

We denote by I_γ the index form along a geodesic γ of M. Recall that if there are interior conjugate points along γ, then I_γ is indefinite and if the end points of γ are conjugate to each other along γ, then there is a normal vector field $V \not\equiv 0$ along γ so that $I_\gamma(V) = 0$. If V is a normal vector field along γ vanishing at the end points of γ, we abbreviate $I_\gamma(V) \coloneqq I_\gamma(V, V)$.

Proof of lemma 3.1. Let (M, g) be a simple $C^{1,1}$ manifold and assume that the Riemannian metric g is C^∞-smooth. Let $\gamma_0 \colon [a, b] \to M$ be a maximal geodesic in M and let $V \not\equiv 0$ be a normal vector field along γ_0 vanishing at the end points of γ_0. We will show that $I_{\gamma_0}(V) > 0$, proving that there cannot be conjugate points along γ even at its end points.

Let $(\gamma_0(0), \dot{\gamma}_0(0)) \eqqcolon z_0 \in \partial_{\mathrm{in}}(SM)$ be the initial data of a geodesic γ_0 and let $\tilde{\gamma}_0$ be the lift to the sphere bundle. The pullback bundle $\tilde{\gamma}_0^* N$ consists precisely of all normal vector fields along γ_0. Particularly, V is a section of $\tilde{\gamma}_0^* N$ vanishing at the end points, so by lemma B.1 (in appendix B) there is a smooth section $\tilde{V}$ of N vanishing on the boundary and satisfying $\tilde{V}|_{\tilde{\gamma}_0} = V$. We may assume that $\tilde{V}$ is supported in a small neighborhood of $\tilde{\gamma}_0$.

Choose for each $k \in \mathbb{N}$ a smooth function $a_k \colon \partial_{\mathrm{in}}(SM) \to \mathbb{R}$ so that $a_k^2 \to \delta_{z_0}$ in the weak sense and $\int_{\partial_{\mathrm{in}}(SM)} a_k^2 \mathrm{d}\mu = 1$, where $\mathrm{d}\mu(x, v) = \langle \nu(x), v \rangle \, \mathrm{d}\Sigma_g(x, v)$. Since we are working locally around z_0, it is enough to find such a sequence of functions in Euclidean space and we see that a sequence of square roots of positive standard mollifiers will suffice.

For each $k \in \mathbb{N}$ let $W_k \in C^\infty(N)$ be a section such that $W_k(\phi_t(z)) = a_k(z)\tilde{V}(\phi_t(z))$ for all $z \in \partial_{\text{in}}(SM)$ and $t \in [0, \tau(z)]$. By Santaló's formula (see [40, Lemma 3.3.2]) it follows that as $k \to \infty$ we have

$$\begin{aligned}
Q(W_k) &= \int_{z\in\partial_{\text{in}}(SM)} I_{\gamma_z}(W_k|_{\tilde{\gamma}_z})\,\mathrm{d}\mu(z) \\
&= \int_{z\in\partial_{\text{in}}(SM)} a_k^2(z) I_{\gamma_z}(\tilde{V}|_{\tilde{\gamma}_z})\,\mathrm{d}\mu(z) \\
&\to \int_{z\in\partial_{\text{in}}(SM)} \delta_{z_0}(z) I_{\gamma_z}(\tilde{V}|_{\tilde{\gamma}_z})\,\mathrm{d}\mu(z) \\
&= I_{\gamma_0}(\tilde{V}|_{\tilde{\gamma}_0}) = I_{\gamma_0}(V).
\end{aligned}$$

Here we have written the distribution δ_{z_0} as a function on SM to simplify notation. Similarly as $k \to \infty$ we get

$$\begin{aligned}
\|W_k\|^2_{L^2(N)} &= \int_{z\in\partial_{\text{in}}(SM)} \int_0^{\tau(z)} |W_k|_{\tilde{\gamma}_z}|^2\,\mathrm{d}t\,\mathrm{d}\mu(z) \\
&= \int_{z\in\partial_{\text{in}}(SM)} a_k^2(z) \left(\int_0^{\tau(z)} \left|\tilde{V}|_{\tilde{\gamma}_z}\right|^2 \mathrm{d}t\right) \mathrm{d}\mu(z) \\
&\to \int_{z\in\partial_{\text{in}}(SM)} \delta_{z_0}(z) \left(\int_0^{\tau(z)} \left|\tilde{V}|_{\tilde{\gamma}_z}\right|^2 \mathrm{d}t\right) \mathrm{d}\mu(z) \\
&= \int_0^{\tau(z_0)} \left|\tilde{V}|_{\tilde{\gamma}_0}\right|^2 \mathrm{d}t = \int_0^{\tau(z_0)} |V|^2\,\mathrm{d}t.
\end{aligned}$$

By $C^{1,1}$ simplicity of (M, g) and zero boundary values of W_k there is $\varepsilon > 0$ so that $Q(W_k) \geq \varepsilon \|W_k\|^2_{L^2(N)}$ for all k. We conclude that

$$I_{\gamma_0}(V) \geq \varepsilon \int_0^{\tau(z_0)} |V|^2\,\mathrm{d}t > 0,$$

which proves that there cannot be conjugate points along γ_0 even at its end points. □

Proof of lemma 3.2. Let (M, g) be a compact smooth Riemannian manifold with a smooth boundary. We assume that the Riemannian metric g is C^∞-smooth.

First, we will prove that strict convexity implies Lipschitz continuity of τ^2. As the boundary is strictly convex, all geodesics starting in the interior $\text{int}(SM)$ meet the boundary transversally. The implicit function theorem implies that τ is smooth in $\text{int}(SM)$. As $\tau\colon SM \to \mathbb{R}$ is continuous on all of SM, it suffices to show that the gradient of τ^2 (in Sasaki or any other Riemannian metric on SM) is uniformly bounded in the interior.

Let $z \in SM$ be an interior point and let $s \mapsto z_s$ be a smooth curve of interior points, where $s \in (-\varepsilon, \varepsilon)$ and $z_0 = z$. Choose $s \mapsto z_s$ to have unit speed with respect to the Sasaki metric related to the C^∞-smooth metric g. The implicit function theorem gives an explicit formula for the differential $\mathrm{d}\tau$ of τ. Applying the implicit function theorem to $\rho(\gamma_{z_s}(t))$ yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\tau(z_s) = -\frac{\left\langle \frac{\mathrm{d}}{\mathrm{d}s}\gamma_{z_s}(t), \nu(\gamma_{z_s}(t))\right\rangle}{\langle \dot\gamma_{z_s}(t), \nu(\gamma_{z_s}(t))\rangle}\bigg|_{t=\tau(z_s)}, \tag{29}$$

where ρ is a boundary defining function. To prove that $\mathrm{d}(\tau^2) = 2\tau\mathrm{d}\tau$ is uniformly bounded in the interior, we will show that

$$\tau(z_s)\frac{\mathrm{d}}{\mathrm{d}s}\tau(z_s) \tag{30}$$

is bounded by some absolute constant near $s = 0$. Boundedness of (30) will follow after we have shown that[1]

$$\tau(z) \lesssim |\langle\dot\gamma_z(\tau(z)), \nu(\gamma_z(\tau(z)))\rangle| \tag{31}$$

for all $z \in \mathrm{int}(SM)$, since by growth estimates for Jacobi fields and $|\dot z_0| = 1$ we have

$$\left|\left\langle \frac{\mathrm{d}}{\mathrm{d}s}\gamma_{z_s}(\tau(z_s)), \nu(\gamma_{z_s}(\tau(z_s)))\right\rangle\right| \le C,$$

where C is a constant depending only on curvature bounds and diameter. Since the right-hand side of (31) is constant along the geodesic γ_z, it is enough to prove boundedness for $z \in \partial_{\mathrm{in}}(SM)$.

Outside any neighbourhood of the compact set $\partial_0(SM)$, the right-hand side of (31) is uniformly bounded from below by a positive constant and $\tau(z)$ is also uniformly bounded from above. Thus if we can prove that there is a neighbourhood of the set $\partial_0(SM)$ where (31) holds, it will hold everywhere on $\partial(SM)$.

Take any $x \in \partial M$ and an inward pointing vector $v \in S_xM$. Let

$$\hat x := \gamma_{x,v}(\tau(x,v)) \quad \text{and} \quad \hat v := -\dot\gamma_{x,v}(\tau(x,v)).$$

Let ν be the inward unit normal vector at the boundary. We decompose the vector $\hat v$ as $\hat v^\perp\nu + \hat v^\parallel$, where $\hat v^\perp > 0$ and $\hat v^\parallel$ is parallel to the boundary. It follows from [14, Lemma 12] that as $\hat v^\perp \to 0$, we have

$$\tau(\hat x, \hat v) = 2v^\perp S(\hat v^\parallel, \hat v^\parallel)^{-1} + \mathcal{O}((\hat v^\perp)^2),$$

where S is the second fundamental form of ∂M and the error term is locally uniform. As the boundary is strictly convex, the second fundamental form is bounded uniformly from below by $c > 0$. Thus as $\hat v^\perp \to 0$ we get

$$\tau(\hat x, \hat v) \le 3c^{-1}\hat v^\perp = 3c^{-1}\langle\nu(\hat x), \hat v\rangle. \tag{32}$$

Therefore, since $\tau(x,v) = \tau(\hat x, \hat v)$, as $v^\perp \to 0$ we get

$$\tau(x,v) = \tau(\hat x,\hat v) \lesssim |\langle\nu(\hat x), \hat v\rangle| = |\langle\nu(\gamma_{x,v}(\tau(x,v))), \dot\gamma_{x,v}(\tau(x,v))\rangle|.$$

This shows that (31) holds in a neighbourhood of the tangential point $(x, v^\parallel)$. Thus estimate (31) holds in a neighbourhood of $\partial_0(SM)$.

Next we turn to the opposite statement. We assume that τ^2 is Lipschitz. If the boundary were not to be strictly convex everywhere, there would be a $v \in S_x(\partial M)$ so that $S_x(v,v) \le 0$.

As τ^2 is Lipschitz, the function τ itself is Hölder-continuous. Because the continuous function τ vanishes on $\partial_{\mathrm{out}}SM \setminus \partial_0(SM)$ (the geodesics stop immediately), we have

$$\tau|_{\partial_0(SM)} = 0 \tag{33}$$

[1] This estimate follows from [39, Lemma 4.1.2], but we reprove it here. Our method of proof is different and may be of interest to some readers.

as well.

We use boundary normal coordinates near the base point $x \in \partial M$. We construct a family $(\gamma_h)_{h\in[0,1]}$ of geodesics as follows. Parallel translate the vector v for time h along the geodesic starting normally inwards from x. Call this vector $v_h \in T_{x_h}M$. Let γ_h be the geodesic with the initial data $\dot\gamma_h(0) = v_h$. The geodesic γ_0 (with initial direction $v_0 = v$ at $x_0 = x$) starts at the boundary and may, depending on the convexity of the boundary, be only defined at $t = 0$.

As in [14, Eq. (2)] we extend the second fundamental form in the boundary normal coordinates near x. Denote $S_h(t) \coloneqq S_{\gamma_h(t)}(\dot\gamma_h(t), \dot\gamma_h(t))$. Since $S_x(v, v) \le 0$ we have

$$S_h(t) = S_0(0) + \mathcal{O}(h) + \mathcal{O}(|t|) \le C(h + |t|),$$

for some $C > 0$ when h and $|t|$ are small. If $z_h(t)$ is the distance from $\gamma_h(t)$ to the boundary, we have $z_h(0) = h$ and $\dot z_h(0) = 0$. By writing the geodesic equation in boundary normal coordinates (as in [14, Eq. (8)]) we find that

$$\ddot z_h(t) = -S_h(t) \ge -C(h + |t|). \tag{34}$$

The total length τ_h of the geodesic γ_h can be divided into forward and backward parts, denoted respectively by τ_h^+ and τ_h^-. We want to find estimates for τ_h^+ and τ_h^- from below.

Let us first consider the case of positive time, $t > 0$. Integrating the estimate (34) leads to

$$z_h(t) = h + \int_0^t \int_0^s \ddot z_h(r)\mathrm{d}r\mathrm{d}s \ge h - \frac{C}{2}ht^2 - \frac{C}{6}t^3 =: \hat z_h(t)$$

for all $t > 0$. If we choose $A \coloneqq \min\left(\sqrt{\frac{2}{3C}}, \sqrt[3]{\frac{2}{C}}\right)$ and $\hat\tau_h^+ \coloneqq Ah^{1/3}$, then for all $t \in [0, \hat\tau_h^+]$ we have

$$\hat z_h(t) \ge h\left[1 - \frac{1}{2}Ch^{2/3}A^2 - \frac{1}{6}CA^3\right] \ge \frac{h}{3}.$$

Therefore $z_h(t) \ge \hat z_h(t) > 0$ for $t \in [0, \hat\tau_h^+]$. This shows that $\tau_h^+ \ge \hat\tau_h^+$.

The case of negative time can be reduced to previous case by substituting $t = -s$, $s > 0$ and similarly we get $\tau_h^- \ge Ah^{1/3}$. Equation (33) implies $\tau_0 = 0$, and this together with the Lipschitz continuity of τ^2 implies that there is $B > 0$ so that $\tau_h^2 \le Bh$. As $0 < h \ll 1$, this gives us

$$Bh \ge \tau_h^2 = (\tau_h^+ + \tau_h^-)^2 \ge 4A^2h^{2/3},$$

which is impossible for small h. This is a contradiction so the boundary has to be strictly convex. □

Acknowledgments. Both authors were supported by the Academy of Finland (JI by grants 332890 and 351665, AK by 336254). We thank Matti Lassas for discussions and the anonymous referees for useful remarks.

Appendix A. **Coordinate formulas and norms.** We have collected here the remaining formulas from proofs of lemmas 2.2 and 5.4. In the context of the proof of lemma 2.2 following formulas hold. The Q_α-term in identity (14) is

$$Q_\alpha\left(\overset{\mathrm{v}}{\nabla}_\alpha \overset{\alpha}{u}\right) = \left\|\overset{\alpha}{X}\overset{\mathrm{v}}{\nabla}_\alpha \overset{\alpha}{u}\right\|_{L^2(\overset{\alpha}{N})} - \left(\overset{\alpha}{R}\overset{\mathrm{v}}{\nabla}_\alpha \overset{\alpha}{u}, \overset{\mathrm{v}}{\nabla}_\alpha \overset{\alpha}{u}\right)_{L^2(\overset{\alpha}{N})}.$$

For L^2 quantities in identity (14) we have

$$\begin{aligned}\left\|\mathring{X}\overset{\mathtt{v}}{\nabla}_\alpha\mathring{u}\right\|^2_{L^2(\mathring{N})} =&\int_{S_hM}\mathring{g}_{ij}\left(\mathring{w}^k\left(\mathring{s}_*\mathring{\delta}_k\right)\left((\mathring{s}_*\mathring{\partial}^i)\tilde{u}\right)+\mathring{\Gamma}^i{}_{lk}\mathring{w}^l(\mathring{s}_*\mathring{\partial}^k)\tilde{u}\right)\\&\times\left(\mathring{w}^k\left(\mathring{s}_*\mathring{\delta}_k\right)\left((\mathring{s}_*\mathring{\partial}^j)\tilde{u}\right)+\mathring{\Gamma}^j{}_{lk}\mathring{w}^l(\mathring{s}_*\mathring{\partial}^k)\tilde{u}\right)\\&\times\left|\det\left(\mathrm{d}\mathring{s}^{-1}\right)\right|\,\mathrm{d}\Sigma_h\end{aligned}\tag{35}$$

and

$$\left\|\mathring{X}\mathring{u}\right\|^2_{L^2(\mathring{S}M)}=\int_{S_hM}\left|\left(\mathring{s}_*\mathring{X}\right)\tilde{u}\right|^2\left|\det\left(\mathrm{d}\mathring{s}^{-1}\right)\right|\,\mathrm{d}\Sigma_h\tag{36}$$

and

$$\begin{aligned}\left(\mathring{R}\overset{\mathtt{v}}{\nabla}_\alpha\mathring{u},\overset{\mathtt{v}}{\nabla}_\alpha\mathring{u}\right)_{L^2(\mathring{N})}=&\int_{S_hM}\mathring{g}_{ij}\left(\mathring{R}^i{}_{jkl}\left((\mathring{s}_*\mathring{\partial}^j)\tilde{u}\right)\mathring{w}^k\mathring{w}^l\right)\\&\times\left((\mathring{s}_*\mathring{\partial}^j)\tilde{u}\right)\left|\det\left(\mathrm{d}\mathring{s}^{-1}\right)\right|\,\mathrm{d}\Sigma_h.\end{aligned}\tag{37}$$

For the vector fields $\mathring{s}_*\mathring{\delta}_k$, $\mathring{s}_*\mathring{X}$ and $\mathring{s}_*\mathring{\partial}^j$ appearing in formulas (35), (36) and (37) we have coordinate formulas

$$\begin{aligned}\mathring{s}_*\mathring{\delta}_k=&\partial_{x^k}+(\partial_{x^k}v^j)\partial_{v^j}-\mathring{\Gamma}^i{}_{kj}\mathring{w}^j(\partial_{\mathring{w}^i}v^l)\partial_{v^l},\\ \mathring{s}_*\mathring{X}=&\mathring{w}^j\partial_{x^j}+\left(\mathring{w}^k\partial_{x^k}v^j-\mathring{\Gamma}^k{}_{lm}\mathring{w}^l\mathring{w}^m(\partial_{\mathring{w}^k}v^j)\right)\partial_{v^j}\quad\text{and}\\ \mathring{s}_*\mathring{\partial}^j=&\mathring{g}^{jl}(\partial_{\mathring{w}^l}v^k)\partial_{v^k}.\end{aligned}$$

In the context of the proof of lemma 5.4 the following formulas hold. For the L^2 inner products in equation (19) we have

$$\left(\overset{\mathtt{h}}{\nabla}_\alpha\mathring{u},\mathring{V}\right)_{L^2(\mathring{N})}=\int_{S_hM}\mathring{g}_{ij}\left((\mathring{s}_*\mathring{\delta}^i)\tilde{u}+\mathring{w}^i(\mathring{s}_*\mathring{X})\tilde{u}\right)\tilde{V}^j\left|\det\left(\mathrm{d}\mathring{s}^{-1}\right)\right|\,\mathrm{d}\Sigma_h\tag{38}$$

and

$$\begin{aligned}\left(\overset{\mathtt{v}}{\nabla}_\alpha\mathring{u},\mathring{X}\mathring{V}\right)_{L^2(\mathring{N})}=&\int_{S_hM}\mathring{g}_{ij}\left((\mathring{s}_*\mathring{\partial}^i)\tilde{u}\right)\left((\mathring{s}_*\mathring{X})\tilde{V}^j+\mathring{\Gamma}^j{}_{lk}\mathring{w}^l\tilde{V}^k\right)\\&\times\left|\det\left(\mathrm{d}\mathring{s}^{-1}\right)\right|\,\mathrm{d}\Sigma_h\end{aligned}\tag{39}$$

and

$$\left(\mathring{X}\mathring{u},\overset{\mathtt{v}}{\mathrm{div}}_\alpha\mathring{V}\right)_{L^2(\mathring{S}M)}=\int_{S_hM}\left((\mathring{s}_*\mathring{X})\tilde{u}\right)\left((\mathring{s}_*\mathring{\partial}_j)\tilde{V}^j\right)\left|\det\left(\mathrm{d}\mathring{s}^{-1}\right)\right|\,\mathrm{d}\Sigma_h.\tag{40}$$

New vector fields $\mathring{s}_*\mathring{\partial}_j$ and $\mathring{s}_*\mathring{\delta}^k$ appear in equations (38), (39) and (40). For them we have the coordinate formulas

$$\mathring{s}_*\mathring{\partial}_j=(\partial_{\mathring{w}^j}v^k)\partial_{v^k},$$

and

$$\mathring{s}_*\mathring{\delta}^k=\mathring{g}^{kl}\partial_{x^l}+(\mathring{g}^{kl}(\partial_{x^l}v^j))\partial_{v^j}-\mathring{g}^{kl}\mathring{\Gamma}^i{}_{lm}\mathring{w}^m(\partial_{\mathring{w}^i}v^j)\partial_{v^j}.$$

Appendix B. **Smooth extension from a curve.** This appendix is devoted to the proof of the following lemma. We will comment on some of the definitions and give examples after the statement. In this appendix everything is smooth and all manifolds and bundles have finite dimension.

Lemma B.1. *Let M be a smooth manifold with boundary and $\pi\colon B \to M$ a bundle over it whose fiber is a closed manifold. Let $\Pi\colon E \to B$ be a vector bundle over B.*

Let $\sigma\colon [a, b] \to B$ be a smooth curve without self-intersections so that the end points $\pi(\sigma(a))$ and $\pi(\sigma(b))$ are on ∂M and $\pi(\sigma(t)) \in \operatorname{int}(M)$ for all $t \in (a, b)$. Suppose the exit directions $\dot\sigma(a)$ and $\dot\sigma(b)$ are not tangent to the boundary $\partial B \coloneqq \pi^{-1}(\partial M)$.

Let V be a smooth section of the pullback bundle σ^*E so that $V(a) = V(b) = 0$. Then there is a smooth section W of E so that $W|_{\partial B} = 0$ and $W(\sigma(t)) = V(t)$ for all $t \in [a, b]$.

The fiber of the bundle B is a smooth and compact manifold of any finite dimension, including zero. The result is valid in the trivial case where the fiber is a singleton and $B = M$. If E is the trivial line bundle $B \times \mathbb{R}$, then sections of it are merely scalar functions $B \to \mathbb{R}$. Therefore the lemma covers extensions of scalar functions from smooth curves γ on M but also much more. The result will only be applied in the case $B = SM$ and $E = N$, but we record it in more generality as it adds no cost.

As $\sigma\colon [a, b] \to B$ is an injective smooth map, a section of the pullback bundle σ^*E is simply a smooth map $W\colon [a, b] \to E$ so that $\Pi(W(t)) = \sigma(t)$ for all $t \in [a, b]$.

Proof of lemma B.1. Denote the projected curve by $\gamma \coloneqq \pi \circ \sigma\colon [a, b] \to M$. The assumption that $\dot\sigma(a)$ and $\dot\sigma(b)$ are not tangential to ∂B implies that the end directions $\dot\gamma(a)$ and $\dot\gamma(b)$ on the base are not tangential to ∂M.

The point $x = \gamma(a)$ has a neighborhood $\omega_1 \subset M$ where we may choose local coordinates $\phi\colon \omega_1 \to \mathbb{R}^n$ so that $\phi(\partial M \cap \omega_1) = \{x_n = 0\}$ and for all interior points $y \in M \setminus \partial M$ we have $\phi_n(y) > 0$. In these coordinates the initial direction satisfies $\dot\gamma_n(a) > 0$, and so the map

$$\theta\colon [a, a + \varepsilon) \ni t \mapsto \gamma_n(t) \in [0, h)$$

is a diffeomorphism for some choice of $\varepsilon, h > 0$.

We may shrink ω_1 so that $\phi(\omega_1) \subset \mathbb{R}^{n-1} \times [0, h)$ and the bundle B is locally trivial: $B \supset \pi^{-1}(\omega_1) \approx \omega_1 \times F$, where F is a closed manifold (the typical fiber of B). Denote $y = \sigma(a) \in B_x = F$. There is a neighborhood $U \ni y$ in F so that the bundle E is trivial over $\omega_1 \times U =: \Omega_1 \subset \pi^{-1}(\omega_1) \subset B$ (with the product in the sense of the local trivialization of B) in the sense that $\Pi^{-1}(\Omega_1) \approx \Omega_1 \times \mathbb{R}^K$, where $K \in \mathbb{N}$ is the dimension of the fiber of E. In these coordinates the section V of σ^*E may be written as a smooth function $[a, b] \to \mathbb{R}^K$, and we denote the component functions as $V_k\colon [a, b] \to \mathbb{R}$. By the non-intersecting property of σ we may assume the neighborhoods $\omega_1 \subset M$ and $\Omega_1 \subset B$ to be so small that the curve σ does not return to Ω_1 after leaving it.

We define a function $W_1\colon \Omega_1 \to \mathbb{R}^K$ by letting its components be

$$W_1(z)_k = V_k(\theta^{-1}(\pi(z)_n)). \tag{41}$$

This defines a section W_1 of the bundle E in a neighborhood of the point $(x, y) \in B$. By construction $W_1(z) = 0$ when $z \in \partial B$, as that corresponds to the set where $\pi(z)_n = 0$ and we have $V(a) = 0$. This section W_1 satisfies the required restriction property where it is defined: Whenever $t \in [a,b]$ satisfies $\sigma(t) \in \Omega_1$, we have $W_1(\sigma(t)) = V(t)$.

Similarly, there is a neighborhood Ω_2 of $(\gamma(b), \sigma(b)) \in B$ and a local section $W_2 \colon \Omega_2 \to E$ with the same property: Whenever $t \in [a,b]$ satisfies $\sigma(t) \in \Omega_2$, we have $W_2(\sigma(t)) = V(t)$.

In addition to satisfying the restriction property, both of these local sections W_1 and W_2 of E vanish on the boundary ∂B when defined there. The point of the construction in (41) is to ensure that the local extension vanishes on the boundary.

For any $t \in (a,b)$ it is easy to provide local extensions as σ has no self-intersections and there are no boundary conditions to worry about. Using compactness of $\sigma([a,b])$ to pass to a finite subcover, we find sets $\Omega_3, \dots, \Omega_J \subset B \setminus \partial B$ and local sections $W_j \colon \Omega_j \to E$ of E so that $W_j(\sigma(t)) = V(t)$ whenever $\sigma(t) \in \Omega_j$ and $\sigma([a,b]) \subset \bigcup_{j=1}^J \Omega_j$.

We also let $\Omega_0 = B \setminus \sigma([a,b])$ and let $W_0 \colon \Omega_0 \to E$ be the zero section. The vector field W_0 has the same boundary conditions and restriction properties as the other W_js but for trivial reasons.

The sets $\Omega_0, \dots, \Omega_J$ are an open cover of the smooth manifold B with boundary ∂B. Let the functions $\psi_0, \dots, \psi_J \in C_c^\infty(B)$ be a partition of unity subordinate to this cover in the sense that each ψ_j is supported in Ω_j and $\sum_{j=0}^J \psi_j(z) = 1$ for all $z \in B$. The functions $B \to E$ defined by $\psi_j(z)W_j(z)$ are smooth (interpreted to be zero outside Ω_j where W_j is defined) and the global smooth section $W \colon B \to E$ given by

$$W(z) = \sum_{j=0}^J \psi_j(z) W_j(z)$$

is quickly verified to have all the required properties. □

REFERENCES

[1] G. Ainsworth, The attenuated magnetic ray transform on surfaces, *Inverse Probl. Imaging*, **7** (2013), 27-46.

[2] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas and M. Taylor, Boundary regularity for the Ricci equation, geometric convergence, and Gel'fand's inverse boundary problem, *Invent. Math.*, **158** (2004), 261-321.

[3] Y. Anikonov and V. Romanov, On uniqueness of determination of a form of first degree by its integrals along geodesics, *J. Inverse Ill-Posed Probl.*, **5** (1997), 487-490.

[4] Y. Assylbekov and N. Dairbekov, The X-ray transform on a general family of curves on Finsler surfaces, *J. Geom. Anal.*, **28** (2018), 1428-1455.

[5] R. Bosi, Y. Kurylev and M. Lassas, Reconstruction and stability in Gel'fand's inverse interior spectral problem, *Anal. PDE*, **15** (2022), 273-326.

[6] C. Croke and V. Sharafutdinov, Spectral rigidity of a compact negatively curved manifold, *Topology*, **37** (1998), 1265-1273.

[7] N. Dairbekov, G. P. Paternain, P. Stefanov and G. Uhlmann, The boundary rigidity problem in the presence of a magnetic field, *Adv. Math.*, **216** (2007), 535-609.

[8] N. Dairbekov and V. Sharafutdinov, Some problems of integral geometry on Anosov manifolds, *Ergodic Theory Dynam. Systems*, **23** (2003), 59-74.

[9] M. V. de Hoop, J. Ilmavirta and V. Katsnelson, Spectral rigidity for spherically symmetric manifolds with boundary, *J. Math. Pures Appl.*, **160** (2022), 54-98.

[10] M. V. de Hoop, J. Ilmavirta, M. Lassas and T. Saksala, Stable reconstruction of simple Riemannian manifolds from unknown interior sources, preprint, 2021, `arXiv:2102.11799`.

[11] D. Finch and G. Uhlmann, The x-ray transform for a non-Abelian connection in two dimensions, *Inverse Probl.*, **17** (2001), 695-701.

[12] C. Guillarmou, G. P. Paternain, M. Salo and G. Uhlmann, The X-ray transform for connections in negative curvature, *Comm. Math. Phys.*, **343** (2016), 83-127.
[13] P. Hartman, On the local uniqueness of geodesics, *Amer. J. Math.*, **72** (1950), 723-730.
[14] J. Ilmavirta, Boundary reconstruction for the broken ray transform, *Ann. Fenn. Math.*, **39** (2014), 485-502.
[15] J. Ilmavirta, X-ray transforms in pseudo-Riemannian geometry, *J. Geom. Anal.*, **28** (2018), 606-626.
[16] J. Ilmavirta and F. Monard, Integral geometry on manifolds with boundary and applications, *The Radon Transform: The First 100 Years and Beyond*, de Gruyter, **22** (2019), 43-113.
[17] J. Ilmavirta and G. P. Paternain, Broken ray tensor tomography with one reflecting obstacle, *Commun. Anal. Geom.*.
[18] J. Ilmavirta and M. Salo, Broken ray transform on a Riemann surface with a convex obstacle, *Commun. Anal. Geom.*, **24** (2016), 379-408.
[19] A. Katsuda, Y. Kurylev and M. Lassas, Stability of boundary distance representation and reconstruction of Riemannian manifolds, *Inverse Probl. Imaging*, **1** (2007), 135-157.
[20] J. Lehtonen, The geodesic ray transform on two-dimensional Cartan-Hadamard manifolds, preprint, 2016, `arXiv:1612.04800`.
[21] J. Lehtonen, J. Railo and M. Salo, Tensor tomography on Cartan-Hadamard manifolds, *Inverse Probl.*, **34** (2018), 044004.
[22] W. Merry and G. P. Paternain, *Inverse Problems in Geometry and Dynamics*, lecture notes, 2011.
[23] F. Monard, R. Nickl and G. P. Paternain, Consistent inversion of noisy non-Abelian X-ray transforms, *Comm. Pure Appl. Math.*, **74** (2021), 1045-1099.
[24] R. G. Mukhometov, Inverse kinematic problem of seismic on the plane, *Math. Problems of Geophysics*, **6** (1975), 243-252.
[25] R. G. Mukhometov, The reconstruction problem of a two-dimensional Riemannian metric, *Dokl. Akad. Nauk SSSR*, **232** (1977), 32-35.
[26] R. G. Mukhometov, On one problem of reconstruction of Riemannian metric, *Siberian Math. J.*, **22** (1981), 119-135.
[27] G. P. Paternain, *Geodesic Flows*, Birkhäuser Basel, Boston, 1999.
[28] G. P. Paternain and M. Salo, The non-Abelian X-ray transform on surfaces, preprint, 2020, `arXiv:2006.02257`.
[29] G. P. Paternain, M. Salo and G. Uhlmann, The attenuated ray transform for connections and Higgs fields, *Geom. Funct. Anal.*, **22** (2012), 1460-1489.
[30] G. P. Paternain, M. Salo and G. Uhlmann, Tensor tomography on surfaces, *Invent. Math.*, **193** (2013), 229-247.
[31] G. P. Paternain, M. Salo and G. Uhlmann, Spectral rigidity and invariant distributions on Anosov surfaces, *J. Differ. Geom.*, **98** (2014), 147-181.
[32] G. P. Paternain, M. Salo and G. Uhlmann, Tensor tomography: Progress and challenges, *Chin. Ann. Math. Ser. B*, **35** (2014), 399-428.
[33] G. P. Paternain, M. Salo and G. Uhlmann, Invariant distributions, Beurling transforms and tensor tomography in higher dimensions, *Math. Ann.*, **363** (2015), 305-362.
[34] G. P. Paternain, M. Salo and G. Uhlmann, *Geometric Inverse Problems—with Emphasis on Two Dimensions*, Cambridge University Press, Cambridge, 2023.
[35] L. Pestov and V. Sharafutdinov, Integral geometry of tensor fields on manifold of negative curvature, *Sib. Math. J.*, **29** (1988), 427-441.
[36] V. Romanov, *Inverse Problems of Mathematical Physics*, De Gruyter, Berlin, Boston, 1986.
[37] M. Salo and G. Uhlmann, The attenuated ray transform on simple surfaces, *J. Differ. Geom.*, **88** (2011), 161-187.
[38] C. Sämann and R. Steinbauer, On geodesics in low regularity, *J. Phys. Conf. Ser.*, **968** (2018), 012010.
[39] V. Sharafutdinov, *Integral Geometry of Tensor Fields*, Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.
[40] V. Sharafutdinov, *Ray Transform on Riemannian Manifolds: Eight Lectures on Integral Geometry*, lecture notes, 1999.
[41] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.
[42] G. Uhlmann, Inverse problems: Seeing the unseen, *Bull. Math. Sci.*, **4** (2014), 209-279.

[43] G. Uhlmann and V. Sharafutdinov, On deformation boundary rigidity and spectral rigidity of Riemannian surfaces with no focal pointss, *J. Differ. Geom.*, **56** (2000), 93-110.

Received January 2022; revised January 2023; early access April 2023.

[B]

Tensor tomography on negatively curved manifolds of low regularity

Joonas Ilmavirta and Antti Kykkänen

First published in *Journal of Geometric Analysis* Volume 34 no. 147 (2024)
https://doi.org/10.1007/s12220-024-01588-8

The Journal of Geometric Analysis (2024) 34:147
https://doi.org/10.1007/s12220-024-01588-8

Tensor Tomography on Negatively Curved Manifolds of Low Regularity

Joonas Ilmavirta[1] · Antti Kykkänen[1]

Received: 19 May 2023 / Accepted: 13 February 2024

Abstract

We prove solenoidal injectivity for the geodesic X-ray transform of tensor fields on simple Riemannian manifolds with $C^{1,1}$ metrics and non-positive sectional curvature. The proof of the result rests on Pestov energy estimates for a transport equation on the non-smooth unit sphere bundle of the manifold. Our low regularity setting requires keeping track of regularity and making use of many functions on the sphere bundle having more vertical than horizontal regularity. Some of the methods, such as boundary determination up to gauge and regularity estimates for the integral function, have to be changed substantially from the smooth proof. The natural differential operators such as covariant derivatives are not smooth.

Keywords Geodesic X-ray tomography · Non-smooth geometry · Tensor tomography · Integral geometry · Inverse problems

Mathematics Subject Classification 44A12 · 53C22 · 53C65 · 58C99

1 Introduction

What are the minimal smoothness assumptions on a Riemannian metric under which the geodesic X-ray transform of tensor fields on the Riemannian manifold is solenoidally injective? Solenoidal injectivity on smooth simple manifolds with negative curvature was proved in [44]. Since [44], many solenoidal injectivity results have been shown under different variations of the geometric setup. Solenoidal injectivity is known for all real analytic simple Riemannian metrics [51] and for all smooth simple

✉ Antti Kykkänen
antti.k.kykkanen@jyu.fi

Joonas Ilmavirta
joonas.ilmavirta@jyu.fi

1 Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), Jyväskylä, FI 40014, Finland

Published online: 27 March 2024

Riemannian metrics with certain bounds on their terminator values [43]. The study of the X-ray transform on manifolds with Riemannian metrics of low regularity was started recently [18], where the authors prove that the X-ray transform of scalar functions is injective on all simple manifolds with $C^{1,1}$ Riemannian metrics. We extend this result and prove that the X-ray transform of tensor fields of any order is solenoidally injective for all simple $C^{1,1}$ Riemannian metrics with almost everywhere non-positive sectional curvature.

X-ray tomography problems of 2-tensor fields naturally arise as linearized problems of travel time tomography or boundary rigidity [49]. The travel time problem arises in applications, such as seismological imaging, where one asks whether the sound speed in a medium can uniquely be determined from the knowledge of the arrival times of waves on the boundary. Because of the geophysical nature of such problems, it is relevant to ask how well the studied model corresponds to the real world. From this point of view, the smoothness assumption of the model manifold is merely a mathematical convenience, which is why we have set out to relax such assumptions.

Our main objective is to optimize the regularity assumptions imposed on the Riemannian metric g of the manifold. We focus on global and uniform non-smoothness (as opposed to, say, interfaces with jump discontinuities), and as in [18] the natural optimality to aim at remains $C^{1,1}$. If g is only assumed to be in the Hölder space $C^{1,\alpha}$ for $\alpha < 1$, the geodesic equation fails to have unique solutions [15, 47] and the X-ray transform itself becomes ill defined. In this sense,our result is optimal on the Hölder scale, as we provide a solenoidal injectivity result (theorem 1) for the class of simple $C^{1,1}$ Riemannian metrics with almost everywhere non-positive sectional curvature.

The non-positivity assumption on the curvature is likely unnecessary — milder assumptions on top of simplicity could suffice. Even in the smooth case relaxing the curvature assumption causes technical difficulties and solenoidal injectivity for all simple Riemannian metrics is not understood. Since our setting is complicated enough as it is, we decided not to include manifolds with possible positive curvature.

A popular method for proving injectivity results relies on interplay between the X-ray transform and a transport equation. In the smooth case, the transport equation is studied using the so-called Pestov identity and energy estimates derived from it (see e.g. [16, 36, 42] and references therein).

We employ a similar approach in our non-smooth setting. Our proof is structurally the same as those in smooth geometry, so the main content of this article is to ensure that everything is well defined and behaved in our non-smooth setting: the unit sphere bundle and operators on it, commutator formulas, function spaces, Santaló's formula, and others.

1.1 Main Results

We record as our main result the following kernel description for the geodesic X-ray transform of tensor fields. In the literature of the geodesic X-ray transform, similar results are often called solenoidal injectivity results. Throughout the article, M will be a compact and connected smooth manifold with a smooth boundary ∂M. The dimension of M will always be $n \geq 2$. The manifold M comes equipped with a $C^{1,1}$ regular Riemannian metric g. That is, the metric g is continuously differentiable and the derivative is Lipschitz.

We define what it means for (M, g) to be simple in Sect. 2.1. Simple $C^{1,1}$ manifolds have global coordinates by definition, but for smooth simple manifolds, this is a consequence of the definitions. When $g \in C^{\infty}$,the definition of $C^{1,1}$ simplicity is equivalent to the classical definition [18, Theorem 2] and thus assuming existence of global coordinates is not superfluous. We say that g has almost everywhere non-positive sectional curvature if for almost all $x \in M$ we have $\langle R(w, v)v, w\rangle_{g(x)} \leq 0$ where $v, w \in T_x M$ are orthogonal. The curvature tensor R is well defined by the familiar formula almost everywhere in M. The X-ray transform of tensor fields is defined in section 2.1.4.

Theorem 1 *Let (M, g) be a simple $C^{1,1}$ manifold (see Sect. 2.1) with almost everywhere non-positive sectional curvature. Let $m \geq 1$ be an integer.*

(1) *If $p \in C^{1,1}(M)$ is a symmetric $(m-1)$-tensor field vanishing on ∂M, then the X-ray transform $I(\sigma \nabla p)$ of its symmetrized covariant derivative vanishes.*
(2) *If the X-ray transform If of a symmetric m-tensor field $f \in C^{1,1}(M)$ vanishes, there is a symmetric $(m-1)$-tensor field $p \in \mathrm{Lip}(M)$ vanishing on ∂M so that $f = \sigma \nabla p$ almost everywhere on M.*

1.2 Regularity Discussion

Claims 1 and 2 in theorem 1 are not symmetric. The difference is in the regularity of the potential p and we believe this is only a consequence of our proof techniques.

There are two notions of smoothness of any given order of a tensor field: regularity with respect to the smooth structure and existence of high-order covariant derivatives. The covariant concept of smoothness is more natural on a Riemannian manifold. For a typical tensor field f that is C^{∞} smooth in the sense of the smooth structure, the covariant derivative ∇f is typically only Lipschitz when $g \in C^{1,1}$. The metric tensor g and its tensor powers are examples of non-vanishing and non-smooth (in the sense of the smooth structure) tensor fields for which covariant derivatives of all orders are well defined. Thus neither of the two notions of smoothness implies the other in general. The two notions of $C^{1,1}$ and less regular Hölder spaces of tensor fields agree, but they disagree for higher regularity. Therefore there are, for example, two different spaces $C^{2,1}$ and we do not use such confusing spaces at all.

We focus on optimizing the regularity of the Riemannian metric g, but we did not pursue optimizing regularity of the tensor fields f or p, the boundary ∂M or the integral function u^f of f (see equation (3)).

It is important for our key regularity result (lemma 3 below) that the boundary values of the tensor field are determined by the data to the extent allowed by gauge freedom. A boundary determination result for 2-tensor fields in the smooth case, where g is C^{∞}, can be found in [51, Lemma 4.1]. Their result is based on clever analysis of equation $2 f_{ij} = p_{i;j} + p_{j;i}$ in boundary normal coordinates. Although the argument in [51] works nicely in the smooth case, it does not give the desired result if g is only $C^{1,1}$ and f is $C^{1,1}$. The immediate conclusion of their argument in the non-smooth case would be that p has derivatives in some directions and is Lipschitz continuous, whereas in lemma 2,we find a p in the class $C^{1,1}$. The other difficulty in adapting similar arguments to the non-smooth case is the regularity of boundary normal coordinates.

To avoid these issues,we prove a boundary determination result (lemma 2) by a more explicit approach. Our construction gives a potential $p \in C^{1,1}(M)$ satisfying $\sigma\nabla p|_{\partial M} = f|_{\partial M}$ when $f \in C^{1,1}(M)$. The cost of our method compared to the method of [51] is losing control of the 1-jets in any neighbourhood of the boundary, but leading order boundary determination suffices for our needs.

We lose a derivative in the regularity of p twice in our argument:

(1) We lose a derivative of p in the boundary determination result. Even if the tensor field $f \in C^{l,1}(M)$ and the Riemannian metric $g \in C^{k,1}(M)$ are assumed to have any (finite) amounts of derivatives, we only get $p \in C^{\min(k,l),1}(M)$. Particularly, p is only $C^{1,1}$, when g and f are $C^{1,1}$. To our knowledge, our boundary determination result is optimal in the literature for differentiability of the potential p with properties $\sigma\nabla p = f$ and $p = 0$ on the boundary.
One might expect $f|_{\partial M} = \sigma\nabla p|_{\partial M}$, where $f \in C^{1,1}(M)$ and $p \in C^{2,1}(M)$. The space $C^{2,1}(M)$ is problematic as described above. In order to improve the regularity of p, one needs to make sense of higher regularity and prove a suitable ellipticity result, but we will not explore this avenue.

(2) Secondly, we lose a derivative of p in the transition of regularity from the spherical harmonic components of f to the spherical harmonic components of the integral function $u{:=}u^f$ of f (see Sect. 2.1). Consider the smooth case, where $g \in C^\infty$, and let $f = f_m + f_{m-2} + f_{m-4} + \cdots$ and $u = u_{m-1} + u_{m-3} + u_{m-5} + \cdots$ be the spherical harmonic decompositions of f and u. The geodesic vector field X on the unit sphere bundle of M splits into the two operators X_+ and X_- in each spherical harmonic degree (see Sect. 2.1). Projecting the transport equation $Xu = -f$ into each spherical harmonic degree gives $X_+u_{m-1} = -f_m$ and $X_+u_{k-1} = -f_k - X_-u_{k+1}$ for $k \leq m-2$ with $k \equiv m \pmod 2$. The operator X_+ is known to be an elliptic pseudodifferential operator of order one (see,e.g. [43]) and thus by elliptic regularity, we see that each u_k has one more derivative than the corresponding component f_{k+1}. This argument shows that u has one more derivative than f, proving that p is $C^{1,1}$ when f is Lipschitz.
However, when $g \in C^{1,1}(M)$,the phase space SM is not equipped with a smooth structure and the meaning of ellipticity and its implications,such as existence of a parametrix, become less clear. The exact formulation and application of ellipticity in the present low regularity setting would be a considerable task and would still not give fully matching regularities in the two parts of theorem 1. Therefore,we take a simpler route and do not pursue a fully symmetric version of our main theorem.

1.3 Related Results

The study of the X-ray transform via the transport equation and Pestov identity approach begun with the work of Mukhometov [30, 31, 33], where injectivity results for the transform of scalar functions were proved. Since Mukhometov's seminal articles, the Pestov identity method has been applied to the case of 1-forms in [2] and to higher-order tensors in [40, 43]. Besides manifolds with boundaries, Pestov identities are useful in the study of integral data of functions and tensor fields over closed curves on closed Anosov manifolds [7, 8, 41, 43, 48]. The method is even applicable in non-compact geometries. For results on Cartan–Hadamard manifolds, see [26, 27]. There are plenty of other geometrical variations of the problem, which have been studied employing a Pestov identity. These include reflecting obstacles inside the manifold [20, 21], attenuations and Higgs fields [13, 39, 46], manifolds with magnetic flows [1, 10, 22, 23, 28], and non-Abelian variations [12, 29, 35, 37]. The Pestov identity approach has been studied in more general geometries than Riemannian. For results in Finsler geometry, see [3, 19] and for pseudo-Riemannian geometry, [17].

Only few injectivity results exist outside smooth geometry, whether Riemannian or not. Injectivity of the scalar X-ray transform is known spherically symmetric $C^{1,1}$ regular manifolds satisfying the Herglotz condition, when the conformal factor of the metric is $C^{1,1}$ [11]. The scalar (and 1-form) X-ray transform is (solenoidally) injective on simple $C^{1,1}$ manifolds [18]. The proof of injectivity in [18] is based on a Pestov identity.

The boundary rigidity problem is a geometrization of the travel time tomography problem and its linearization is the X-ray tomography problem of 2-tensor fields. For results in boundary rigidity, see [4–6, 14, 24, 32, 34, 45, 50, 52]. For a comprehensive survey on results in travel time tomography and tensor tomography, see [16, 49].

2 Proof of the Main Theorem

2.1 Basic Definitions and Notation

In this subsection,we present enough terminology and notation to state and prove our main theorem. The preliminaries of the non-smooth setting are complemented in Sect. 3.

Throughout the article, M will be a compact and connected smooth manifold with a smooth boundary ∂M. The manifold M is equipped with a $C^{1,1}$ regular Riemannian metric g.

2.1.1 Bundles

The tangent bundle TM of M has a subbundle SM called the unit sphere bundle, which consists of the unit vectors in TM. As the level set $F^{-1}(1)$ of the $C^{1,1}$ map $F\colon TM \to \mathbb{R}$ defined by $F(x, v) = g_x(v, v)$, the unit sphere bundle is a $C^{1,1}$ submanifold[1] of TM. The boundary

$$\partial(SM) := \{ (x, v) \in SM : x \in \partial M \} \tag{1}$$

[1] It is easily verified by inspecting the vertical component that the differential $\mathrm{d}F$ is non-zero when $F = 1$. The smooth regular level set theorem [25] can easily be adapted to our case.

of SM is divided into inwards and outwards pointing parts $\partial_{\text{in}}(SM)$ and $\partial_{\text{out}}(SM)$ with respect to the inner product $\langle\cdot,\cdot\rangle_g$ and a unit normal vector field ν to the boundary ∂M. The subset of $\partial(SM)$ consisting of the vectors v such that $\langle v,\nu\rangle_g = 0$ is denoted by $\partial_0(SM)$ and it is disjoint from $\partial_{\text{in}}(SM)$ and $\partial_{\text{out}}(SM)$.

Let $\pi\colon SM \to M$ be the standard projection and let $\pi^*(TM)$ be the pullback of TM over SM. We denote by N the subbundle of $\pi^*(TM)$ with the fibre $N_{(x,v)}$ being the g-orthogonal complement of v in T_xM.

2.1.2 Horizontal–Vertical Decomposition

The tangent bundle $T(SM)$ of SM has an orthogonal splitting $T(SM) = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}$ with respect to the so-called Sasaki metric, where $\mathcal{H}$ and $\mathcal{V}$ are the horizontal and vertical subbundles,respectively, and X is the geodesic vector field on SM. We denote $\mathbb{R}X \oplus \mathcal{H}$ by $\overline{\mathcal{H}}$ and call it the total horizontal subbundle. Elements of $\mathcal{H}$ and $\mathcal{V}$ are, respectively, referred to as horizontal and vertical derivatives or vectors on SM. The summands $\mathcal{H}$ and $\mathcal{V}$ are each naturally identified with a copy of the bundle N. The horizontal–vertical geometry is essentially the same as the smooth one (see [38]) and works fine when $g \in C^{1,1}$.

2.1.3 Geodesic Flow

Since the Christoffel symbols of a $C^{1,1}$ metric are Lipschitz, there is a unique unit speed geodesic γ_z corresponding to a given initial condition $z \in SM$ by standard ODE theory. We define the geodesic flow on the unit sphere bundle to be the collection of (partially defined) maps $\phi_t\colon SM \to SM$, $\phi_t(z) = (\gamma_z(t), \dot\gamma_z(t))$, where t goes through all real numbers so that the right-hand side is defined. The infinitesimal generator X of the flow is called the geodesic vector field on SM. For any $z \in SM$, the geodesic γ_z is defined on a maximal interval of existence $[-\tau_-(z), \tau_+(z)]$, where $\tau_-(z)$ and $\tau_+(z)$ are positive. We call $\tau(z){:=}\tau_+(z)$ the travel time function on SM. The geodesic vector field X acts naturally on functions by differentiation and on sections W of the bundle N, it acts by

$$XW(z) = D_t W(\phi_t(z))|_{t=0}, \tag{2}$$

where D_t is the covariant derivative along the curve $t \mapsto \phi_t(z)$. The result XW of the action (2) is again a section of N.

2.1.4 The X-Ray Transform

Any symmetric m-tensor field f on M can be considered as a function on the unit sphere bundle. Given $(x, v) \in SM$,we let $f(x,v){:=}f_x(v,\dots,v)$. In lemma 7 and proposition 11 and their proofs, we denote the induced maps by $\lambda_x f\colon S_xM \to \mathbb{R}$ and $\lambda f\colon SM \to \mathbb{R}$ with $\lambda f(x,v) = \lambda_x f(v)$. Otherwise, we freely identify f with λf since there is no danger of confusion.

The integral function $u^f : SM \to \mathbb{R}$ of a continuous symmetric m-tensor field f is defined by

$$u^f(x, v) := \int_0^{\tau(x,v)} \lambda f(\phi_t(x, v))\, \mathrm{d}t \tag{3}$$

for all $(x, v) \in SM$. The X-ray transform of f is the restriction of the integral function to the inward pointing part of the boundary $\partial(SM)$, so we may declare $If := u^f|_{\partial_{\text{in}}(SM)}$.

2.1.5 Differentiability

We exclude the rank of the tensor field from our notations for function spaces. For tensor fields,the derivatives are covariant. We use the subscript 0 to indicate zero boundary values. Thus, for example, $f \in C_0^{1,\alpha}(M)$ for a tensor field f means that $f|_{\partial M} = 0$ and ∇f is α-Hölder. We use two kinds of functions on the sphere bundle SM, scalars (e.g. $C^1(SM)$) and sections of the bundle N (e.g. $C^1(N)$) defined in subsection 2.1.1.

We define $C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM)$ as the subset of $C(SM)$ consisting of functions with k many α-Hölder horizontal derivatives and l many β-Hölder vertical derivatives as well as any combination of k horizontal and l vertical derivatives, which are assumed to be ω-Hölder for $\omega := \min(\alpha, \beta)$. We let

$$C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{\infty}(SM) := \bigcap_{l=0}^{\infty} C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,1}(SM). \tag{4}$$

According to the splitting $T(SM) = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}$, the gradient of a C^1 function u on SM can be written as

$$\nabla u = ((Xu)X, \overset{\mathrm{h}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u). \tag{5}$$

This gives rise to two new differential operators; the vertical gradient $\overset{\mathrm{h}}{\nabla}$ and the horizontal gradient $\overset{\mathrm{v}}{\nabla}$. Both $\overset{\mathrm{h}}{\nabla} u$ and $\overset{\mathrm{v}}{\nabla} u$ are naturally identified with sections of the bundle N. The horizontal and vertical divergences are the L^2 adjoints of the corresponding gradients. The L^2 adjoint of X is $-X$. The vertical Laplacian on the sphere bundle is $\overset{\mathrm{v}}{\Delta} := -\overset{\mathrm{v}}{\mathrm{div}} \overset{\mathrm{v}}{\nabla}$; see [43, Appendix A] for details on the differential operators.

2.1.6 Curvature

By Rademacher's theorem, a Lipschitz continuous scalar function on a Euclidean domain is differentiable almost everywhere and the derivative is in L^∞. Using local coordinates and studying the individual components show that the Riemann curvature tensor $R_{ijkl}(x)$ corresponding to a Riemannian metric $g \in C^{1,1}$ has all components well defined for almost all $x \in M$. Thus we may interpret the curvature tensor R as an L^∞ tensor field. The curvature tensor $R\colon L^\infty(N) \to L^\infty(N)$ acts on sections of the bundle N by $R(x,v)W(x,v){:=}R(W(x,v),v)v$ producing again L^∞ sections of the bundle N.

We say that the sectional curvature of the manifold M is almost everywhere non-positive, if for almost all $x \in M$,it holds that $\langle R(w,v)v, w\rangle_{g(x)} \leq 0$ for all linearly independent $v, w \in T_x M$.

2.1.7 Sobolev Spaces

There are natural L^2 spaces for functions on the sphere bundle as well as for sections of the bundle N, which we will denote by $L^2(SM)$ and $L^2(N)$. We define the Sobolev spaces $H^1(SM)$ and $H^1(N, X)$,respectively, defined as completions of $C^1(SM)$ and $C^1(N)$ with respect to the norms

$$\begin{aligned} \|u\|^2_{H^1(SM)} &{:=} \|u\|^2_{L^2(SM)} + \|Xu\|^2_{L^2(SM)} + \left\|\overset{\mathrm{h}}{\nabla} u\right\|^2_{L^2(SM)} + \left\|\overset{\mathrm{v}}{\nabla} u\right\|^2_{L^2(SM)}, \quad \text{and} \\ \|W\|^2_{H^1(N,X)} &{:=} \|W\|^2_{L^2(N)} + \|XW\|^2_{L^2(N)}. \end{aligned} \tag{6}$$

We denote zero boundary values by a subindex 0. For example, $H^1_0(SM)$ is the subspace of $H^1(SM)$ with zero boundary values.

2.1.8 Spherical Harmonics

Given $x \in M$, the unit sphere $S_x M$ has the Laplace–Beltrami operator $\overset{\mathrm{v}}{\Delta}_x {:=} -g^{ij}(x)\partial_{v^i}\partial_{v^j}$. Letting $x \in M$ vary we get a second-order operator $\overset{\mathrm{v}}{\Delta} = -\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{v}}{\nabla}$ on the unit sphere bundle called the vertical Laplacian, where $-\overset{\mathrm{v}}{\mathrm{div}}$ is the formal L^2-adjoint of $\overset{\mathrm{v}}{\nabla}$.

Let $S^{n-1} \subseteq \mathbb{R}^n$ be the Euclidean unit sphere. It is well known that any function $f \in L^2(S^{n-1})$ can be decomposed as an L^2-convergent series $f = \sum_{k=0}^\infty f_k$, where f_k are eigenfunctions of the spherical Laplacian on S^{n-1} corresponding to the eigenvalues $k(k+n-2)$. Similarly, any function $u \in L^2(SM)$ can be decomposed as an $L^2(SM)$-convergent series $u = \sum_{k=0}^\infty u_k$, where $\overset{\mathrm{v}}{\Delta} u_k = k(k+n-2)u_k$ for all $k \in \mathbb{N}$. We call u_k the kth spherical harmonic component of u. For $k \in \{0, 1\}$, $k, l \in \mathbb{N}$ and $\alpha, \beta \in [0, 1]$ we let

$$\Omega^{k,\alpha}_{\mathrm{h}}\Omega^{l,\beta}_{\mathrm{v}}(m){:=}\{\, u \in C^{k,\alpha}_{\mathrm{h}} C^{l,\beta}_{\mathrm{v}}(SM) : \overset{\mathrm{v}}{\Delta} u = m(m+n-2)u \,\} \tag{7}$$

and

$$\Omega^{k,\alpha}_{\mathrm{h}}\Omega^{\infty}_{\mathrm{v}}(m){:=}\bigcap_{l\in\mathbb{N}} \Omega^{k,\alpha}_{\mathrm{h}}\Omega^{l,1}_{\mathrm{v}}(m). \tag{8}$$

Furthermore, we denote

$$\Lambda^k_{\mathrm{h}}\Lambda^l_{\mathrm{v}}(m) = \{ u \in H^k_{\mathrm{h}} H^l_{\mathrm{v}}(SM) : \overset{\mathrm{v}}{\Delta} u = m(m+n-2)u \}. \tag{9}$$

For all $m \in \mathbb{N}$,there are operators $X_\pm \colon \Omega^1_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m) \to \Omega^0_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m \pm 1)$ with the convention that $\Omega^0_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(-1) = 0$ so that $X = X_+ + X_-$. These mapping properties and validity of this decomposition in low regularity are addressed in proposition 12.

2.1.9 Simple $C^{1,1}$ Manifolds

The global index form Q of the manifold (M, g) (not of a single geodesic) is the quadratic form defined for $W \in H^1_0(N, X)$ by

$$Q(W) := \|XW\|^2_{L^2(N)} - (RW, W)_{L^2(N)}\,. \tag{10}$$

It was proved in [18, Lemma 11] that there are no conjugate points on a Riemannian manifold (M, g), $g \in C^\infty$, if the global index form Q of (M, g) is positive definite.

We conclude this subsection by recalling a definition of a simple manifold in the case $g \in C^{1,1}$. Our definition is equivalent to the definition of traditional simple manifold when $g \in C^\infty$ [18]. Let $M \subseteq \mathbb{R}^n$ be the closed Euclidean unit ball and let g be a $C^{1,1}$ regular Riemannian metric on M. We say that (M, g) is *a simple $C^{1,1}$ Riemannian manifold* if the following hold:

- A1: There is $\varepsilon > 0$ so that $Q(W) \geq \varepsilon \|W\|^2_{L^2(N)}$ for all $W \in H^1_0(N, X)$.
- A2: Any two points of M can be joined by a unique geodesic in the interior of M, whose length depends continuously on its end points.
- A3: The squared travel time function τ^2 (see 2.1.3) is Lipschitz on SM.

2.2 Proof of the Theorem

In this subsection, we prove our main result, theorem 1. We state the lemmas required for the proof of 1, and the proofs of the lemmas are postponed to sections 4, 5, and 6 .

Lemma 2 (Boundary determination) *Let (M, g) be a simple $C^{1,1}$ manifold. If $f \in C^{1,1}(M)$ is a symmetric m-tensor field with $If = 0$, then there is a symmetric $(m-1)$-tensor field $p \in C^{1,1}(M)$ so that $f|_{\partial M} = \sigma\nabla p|_{\partial M}$ and $p|_{\partial M} = 0$.*

Lemma 3 (Regularity of spherical harmonic components) *Let (M, g) be a simple $C^{1,1}$ manifold. Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field on M with $If = 0$ and let $u := u^f$ be the integral function of f defined by (3). If the spherical harmonic decomposition of u is $u = \sum_{k=0}^\infty u_k$, then $u_k \in \Omega^{0,1}_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(k)$ and $u_k|_{\partial(SM)} = 0$ for all $k \in \mathbb{N}$.*

Lemma 4 *Let (M, g) be a simple $C^{1,1}$ manifold. Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field on M with $If = 0$ and let $u := u^f$ be the integral function of f defined by (3). Then $X_+ u \in L^2(SM)$.*

Lemma 4 follows immediately from lemmas 3 and 17 .

Recall that n is the dimension of M. For natural numbers k and l,we define the two constants

$$C(n,k):=\frac{2k+n-1}{2k+n-3} \quad \text{and} \quad B(n,l,k):=\prod_{p=1}^{l} C(n,k+2p). \tag{11}$$

Lemma 5 *Let (M,g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field with $If = 0$ and denote by $u:=u^f$ the integral function of f defined by (3). If the spherical harmonic decomposition of u is $u = \sum_{k=0}^{\infty} u_k$, then for all $k \geq m$ and $l \in \mathbb{N}$,we have*

$$\|X_+u_k\|^2_{L^2(SM)} \leq B(n,l,k)\, \|X_+u_{k+2l}\|^2_{L^2(SM)}\,. \tag{12}$$

Lemma 6 (Injectivity of X_+) *Let (M,g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. Suppose that $u \in \Omega^{0,1}_{\mathrm{h}}\Omega^{\infty}_{\mathrm{v}}(k)$ and $u|_{\partial(SM)} = 0$. Then $X_+u = 0$ implies that $u = 0$.*

Lemma 7 *Let (M,g) be a simple $C^{1,1}$ manifold. Let $f \in \mathrm{Lip}(M)$ be a symmetric m-tensor field. Suppose that p is a symmetric $(m-1)$-tensor field and $u = -\lambda p$ is a Lipschitz function in SM so that $Xu = -\lambda f$ everywhere in SM. Then $\sigma\nabla p = f$ almost everywhere in M.*

Proof of theorem 1 Item 1: Suppose that $p \in C^{1,1}(M)$ is a symmetric $(m-1)$-tensor field vanishing on $\partial(M)$. Then using the fundamental theorem of calculus along each geodesic gives $If = I(\sigma\nabla p) = 0$ (see [36, Lemma 6.4.2]), which proves item 1.

Item 2: Suppose that the X-ray transform of a symmetric m-tensor field $f \in C^{1,1}(M)$ vanishes. We will prove that there is a symmetric $(m-1)$-tensor field p vanishing on ∂M so that $f = \sigma\nabla p$.

By boundary determination in lemma 2, there is a symmetric $(m-1)$-tensor field $p_0 \in C^{1,1}(M)$ so that $p_0|_{\partial M} = 0$ and $f|_{\partial M} = \sigma\nabla p_0|_{\partial M}$. Let $\hat{f}:=f-\sigma\nabla p_0$. Then $\hat{f} \in \mathrm{Lip}_0(M)$ is a symmetric m-tensor field on M and $I\hat{f} = If = 0$.

Let $u = \sum_{k=0}^{\infty} u_k$ be the spherical harmonic decomposition of $u:=u^{\hat{f}}$. Then $u_k \in \Omega^{0,1}_{\mathrm{h}}\Omega^{\infty}_{\mathrm{v}}(k)$ by lemma 3. First, we prove that $u_k = 0$ for all k for which $k \equiv m \pmod 2$.

Since for all $(x,v) \in SM$ it holds that $\hat{f}(x,-v) = (-1)^m \hat{f}(x,v)$, we have

$$\begin{aligned} u(x,-v) &= \int_0^{\tau_+(x,-v)} \hat{f}(\gamma_{x,-v}(t), \dot{\gamma}_{x,-v}(t))\, \mathrm{d}t \\ &= (-1)^m \int_{-\tau_-(x,v)}^{0} \hat{f}(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))\, \mathrm{d}t. \end{aligned} \tag{13}$$

Therefore,

$$u(x,-v) + (-1)^m u(x,v) = (-1)^m I\hat{f}(\phi_{-\tau_-(x,v)}(x,v)) = 0. \tag{14}$$

This shows that $u(x, -v) = (-1)^{m+1}u(x, v)$ for all $(x, v) \in SM$ and thus $u_k = 0$ whenever $k \equiv m \pmod 2$. Next, we will show that $u_k = 0$ for all $k \geq m$.

Let $m_0 \geq m$ and suppose that $A_1 := \|X_+ u_{m_0}\|^2_{L^2(SM)} > 0$. For all $l \in \mathbb{N}$, lemma 5 yields the estimate

$$B(n, l, m_0)^{-1} \|X_+ u_{m_0}\|^2_{L^2(SM)} \leq \|X_+ u_{m_0+2l}\|^2_{L^2(SM)}. \tag{15}$$

By an elementary estimate (see [20, Lemma 13]), there is a constant $A_2 > 0$ only depending on m_0 and n so that

$$B(n, l, m_0)^{-1} \geq \left(1 + \frac{4l}{2m_0 + n - 3}\right)^{-1/2} \geq A_2 l^{-1/2}. \tag{16}$$

Thus the estimate (15) gives

$$\sum_{l=1}^{\infty} \|X_+ u_{m_0+2l}\|^2_{L^2(SM)} \geq A_1 A_2 \sum_{l=1}^{\infty} l^{-1/2} = \infty. \tag{17}$$

On the other hand, $X_+ u \in L^2(SM)$ by lemma 4. Hence orthogonality implies that

$$\sum_{l=1}^{\infty} \|X_+ u_{m_0+2l}\|^2_{L^2(SM)} \leq \sum_{k=0}^{\infty} \|X_+ u_k\|^2_{L^2(SM)} \leq \|X_+ u\|^2_{L^2(SM)} < \infty. \tag{18}$$

This contradiction proves that $\|X_+ u_k\|^2_{L^2(SM)} = 0$ for all $k \geq m$. Since additionally $u_k|_{\partial(SM)} = 0$ for $k \geq m$, lemma 6 says $u_k = 0$ for all $k \geq m$.

We have shown $u_k = 0$ for $k \geq m$ and $u_k = 0$ for $k \equiv m \pmod 2$. Thus $-u \in \mathrm{Lip}_0(SM)$ is identified with a symmetric $(m-1)$-tensor field $p_1 \in \mathrm{Lip}_0(M)$. As u solves the transport equation $Xu = -\hat{f}$ everywhere on SM we have $\sigma \nabla p_1 = \hat{f}$ almost everywhere on M by lemma 7. Thus we conclude that $f = \sigma \nabla p$ almost everywhere in M, where $p := p_0 + p_1 \in \mathrm{Lip}(M)$ is a symmetric $(m-1)$-tensor field with $p|_{\partial M} = 0$. □

3 Preliminaries

In this article, we consider compact and connected smooth manifolds with smooth boundaries. We assume that such a manifold M comes equipped with a symmetric and positive definite 2-tensor field g so that its component functions g_{jk} are $C^{1,1}$-functions on M. In this case, we refer to g as a $C^{1,1}$ *Riemannian metric* and to (M, g) as a *(non-smooth) Riemannian manifold*.

3.1 Spaces of Tensor Fields

Since g is a $C^{1,1}$ Riemannian metric, componentwise differentiability and existence of covariant derivatives are not the same. Even if the components of a tensor field f in any local coordinates are C^k functions for $k \geq 2$ (which is possible since M is assumed to have a smooth structure), the covariant derivative ∇f falls into $\mathrm{Lip}(M)$. Since most of our considerations are related to the metric structure and componentwise differentiability is not compatible with the covariant derivative, the correct definition of a $C^{1,1}$ tensor field is by covariant differentiability. However, with covariant differentiability, we are restricted to $C^{1,1}(M)$ and higher regularity does not exist on the Hölder scale.

The space $L^2(M)$ of L^2-tensor fields of order m on M is defined to be the completion of the space of continuous m-tensor fields with respect to the norm induced by the inner product

$$(f, h)_{L^2(M)} := \int_M g^{j_1 k_1} \cdots g^{j_m k_m} f_{j_1 \cdots j_m} h_{k_1 \cdots k_m} \, \mathrm{d}V_g. \tag{19}$$

Here $\mathrm{d}V_g$ is the Riemannian volume form of M. The space $H^1(M)$ of H^1-tensor fields of order m on M is defined to be the closure of the space of continuously differentiable m-tensor fields with respect to the norm

$$\|f\|^2_{H^1(M)} := \|f\|^2_{L^2(M)} + \|\nabla f\|^2_{L^2(M)} \,. \tag{20}$$

Let $p \in [1, \infty)$. The spaces $L^p(M)$ and $W^{1,p}(M)$ of L^p- and $W^{1,p}$-tensor fields of order m are defined analogously to the spaces $L^2(M)$ and $H^1(M)$.

We could give definitions of the spaces $H^2(M)$ and $W^{2,p}(M)$ for tensor fields of any order similar to the definitions of spaces $H^1(M)$ and $W^{1,p}(M)$. Again, since g is only a $C^{1,1}$ regular Riemannian metric, there are no spaces $H^3(M)$ and $W^{3,p}(M)$ compatible with the geometry. A compatible space should be defined using covariant derivatives in the norms, which would force the spaces $W^{k,p}(M)$ trivial, when $k \geq 3$.

If $f \in C^1(M)$ is a symmetric m-tensor field on M, its symmetrized covariant derivative is $\sigma\nabla f$. The symmetrization σ is defined for all m-tensor fields h on M by

$$(\sigma h)_{j_1 \cdots j_m} := \frac{1}{m!} \sum_{\pi} h_{j_{\pi(1)} \cdots j_{\pi(m)}} \tag{21}$$

where the summation is over all permutations π of $\{1, \ldots, m\}$. Note that since $\|\sigma\nabla f\|_{L^2} \leq \|\nabla f\|_{L^2}$, the symmetrized covariant derivative is bounded between Sobolev spaces.

The trace of a symmetric m-tensor field f on M is denoted by $\mathrm{tr}_g(f)$. In local coordinates, $\mathrm{tr}_g(f)_{i_1 \cdots i_{m-2}} = g^{jk} f_{jki_1 \cdots i_{m-2}}$. A symmetric m-tensor field is called trace-free, if its trace is zero.

3.2 Vertical and Horizontal Differentiability

Let M be a compact smooth manifold with a smooth boundary and let g be a $C^{1,1}$ Riemannian metric on M. Let $k \in \mathbb{N}$ and $\alpha \in [0,1]$ be so that $k+\alpha \le 2$. For $l \in \mathbb{N}$ and $\beta \in [0,1]$, the set $C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM)$ consists of all functions $u \in C(SM)$ with

$$H_1 \cdots H_k u \in C^{0,\alpha}(SM) \quad \text{and} \quad V_1 \cdots V_l u \in C^{0,\beta}(SM) \tag{22}$$

for any k vector fields $H_1, \ldots, H_k \in \overline{\mathcal{H}}$ and any l vector fields $V_1, \ldots, V_l \in \mathcal{V}$. Additionally, we require that for any $k+l$ vector fields $Z_1, \ldots, Z_{k+l} \in T(SM)$ out of which exactly k are in $\overline{\mathcal{H}}$ and exactly l are in $\mathcal{V}$,we have

$$Z_1 \cdots Z_{k+l} u \in C^{0,\omega}(SM), \quad \text{where} \quad \omega := \min(\alpha, \beta). \tag{23}$$

We let

$$C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{\infty}(SM) := \bigcap_{l \in \mathbb{N}} C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,1}(SM). \tag{24}$$

Remark 8 In the definition of $C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM)$,the vertical differentiability indices l and β can surpass the smoothness of charts of SM. It is not necessary for SM to have C^∞ smooth charts, since vertical vector fields operate on a fixed fibre and for a fixed point x in M, the scaling $s(x,v) = (x, v\,|v|_g^{-1})$ is smooth on $T_x M \backslash 0$. The slit tangent space $T_x M \setminus 0$ has a smooth structure even if M does not.

Remark 9 Any commutator $[H, V] = HV - VH$, where $H \in \overline{\mathcal{H}}$ and $V \in \mathcal{V}$, can be defined classically on the space $C_{\mathrm{h}}^1 C_{\mathrm{v}}^1(SM)$, since for any $u \in C_{\mathrm{h}}^1 C_{\mathrm{v}}^1(SM)$, the derivatives HVu and VHu are in $C(SM)$.

The set $C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(N)$ consists of all continuous sections W of the bundle N with W^j in $C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM)$ when $W = W^j \partial_{x^j}$. A section W of the bundle N is continuous, if it is continuous as a map $SM \to TM$.

As one might expect, vertical operators preserve horizontal differentiability and horizontal operators preserve vertical differentiability. That is

$$X : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM) \to C_{\mathrm{h}}^{k-1,\alpha} C_{\mathrm{v}}^{l,\beta}(SM), \tag{25}$$

$$X : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(N) \to C_{\mathrm{h}}^{k-1,\alpha} C_{\mathrm{v}}^{l,\beta}(N), \tag{26}$$

$$\overset{\mathrm{v}}{\nabla} : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM) \to C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l-1,\beta}(N), \tag{27}$$

$$\overset{\mathrm{v}}{\mathrm{div}} : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(N) \to C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l-1,\beta}(SM), \tag{28}$$

$$\overset{\mathrm{h}}{\nabla} : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM) \to C_{\mathrm{h}}^{k-1,\alpha} C_{\mathrm{v}}^{l,\beta}(N), \quad \text{and} \tag{29}$$

$$\overset{\mathrm{h}}{\mathrm{div}} : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(N) \to C_{\mathrm{h}}^{k-1,\alpha} C_{\mathrm{v}}^{l,\beta}(SM). \tag{30}$$

3.3 Sobolev Spaces of Different Vertical and Horizontal Indices

Standard Sobolev spaces on SM are defined in Sect. 2.1.7. Here, we define Sobolev spaces for scalar functions on SM of different vertical and horizontal indices. If $k, l \in \{0, 1\}$ and u is a scalar function in $C^k_{\mathrm{h}} C^l_{\mathrm{v}}(SM)$, we define the $H^k_{\mathrm{h}} H^l_{\mathrm{v}}(SM)$-norm of u to be

$$\|u\|^2_{H^k_{\mathrm{h}} H^l_{\mathrm{v}}(SM)} := \|u\|^2_{L^2(SM)} + k\,\|Xu\|^2_{L^2(SM)} + k\left\|\overset{\mathrm{h}}{\nabla} u\right\|^2_{L^2(N)} + l\left\|\overset{\mathrm{v}}{\nabla} u\right\|^2_{L^2(N)}. \quad (31)$$

The Sobolev space $H^k_{\mathrm{h}} H^l_{\mathrm{v}}(SM)$ for $k, l \in \{0, 1\}$ is defined to be the completion of $C^k_{\mathrm{h}} C^l_{\mathrm{v}}(SM)$ with respect to the norm $\|\cdot\|_{H^k_{\mathrm{h}} H^l_{\mathrm{v}}(SM)}$.

Similarly, we define spaces $H^0_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$ and $H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$ to be the completions of $C^0_{\mathrm{h}} C^2_{\mathrm{v}}(SM)$ and of $C^1_{\mathrm{h}} C^2_{\mathrm{v}}(SM)$ with respect to the norms

$$\|u\|^2_{H^0_{\mathrm{h}} H^2_{\mathrm{v}}(SM)} := \|u\|^2_{L^2(SM)} + \left\|\overset{\mathrm{v}}{\Delta} u\right\|^2_{L^2(SM)}, \quad \text{and} \quad (32)$$

$$\|u\|^2_{H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)} := \|u\|^2_{H^1_{\mathrm{h}} H^1_{\mathrm{v}}(SM)} + \|u\|^2_{H^0_{\mathrm{h}} H^2_{\mathrm{v}}(SM)} \quad (33)$$

$$+ \left\|X\overset{\mathrm{v}}{\Delta} u\right\|^2_{L^2(SM)} + \left\|\overset{\mathrm{v}}{\Delta} Xu\right\|^2_{L^2(SM)}. \quad (34)$$

Note that the norm on $H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$ does not cover all possible combinations of a horizontal derivative and two vertical derivatives (e.g. $\overset{\mathrm{v}}{\mathrm{div}} X \overset{\mathrm{v}}{\nabla}$). This is intentional, since the missing combinations will not be needed.

Proposition 10 *Let M be a compact smooth manifold with a smooth boundary and let g be a $C^{1,1}$ Riemannian metric on M. The following commutator formulas hold on $H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$:*

$$[X, \overset{\mathrm{v}}{\nabla}] = -\overset{\mathrm{h}}{\nabla}, \quad (35)$$

$$\overset{\mathrm{h}}{\mathrm{div}} \overset{\mathrm{v}}{\nabla} - \overset{\mathrm{v}}{\mathrm{div}} \overset{\mathrm{h}}{\nabla} = (n-1)X, \quad (36)$$

$$[X, \overset{\mathrm{v}}{\Delta}] = 2\overset{\mathrm{v}}{\mathrm{div}} \overset{\mathrm{h}}{\nabla} + (n-1)X. \quad (37)$$

The following commutator formula holds on $H^1_{\mathrm{h}} H^1_{\mathrm{v}}(N)$:

$$[X, \overset{\mathrm{v}}{\mathrm{div}}] = -\overset{\mathrm{h}}{\mathrm{div}}. \quad (38)$$

Proof Formulas (35), (36) and (37) on $C^1_{\mathrm{h}} C^2_{\mathrm{v}}(SM)$ and (38) on $C^1_{\mathrm{h}} C^1_{\mathrm{v}}(N)$ can be proved by a computation similar to [43, Appendix], since the computations use one horizontal derivative and two vertical for (35), (36) and (37) and one horizontal and one vertical derivative for (38). The same formulas hold on $H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$ and $H^1_{\mathrm{h}} H^1_{\mathrm{v}}(N)$ by approximation. □

3.4 Vertical Fourier Analysis

In this subsection, we recall the identification of trace-free symmetric tensor fields and spherical harmonics (the vertical Fourier modes). We state and prove proposition 11 in order to emphasize what changes in these well known results when applied to a case of non-smooth Riemannian metrics. More details in the case of C^∞-smooth Riemannian metrics can be found for example in [36] and [9].

Proposition 11 *Let M be a compact smooth manifold with a smooth boundary and let g be a $C^{1,1}$ Riemannian metric on M. Let $k \in \{0, 1\}$ and $\alpha \in [0, 1]$. The map $\lambda \colon f \mapsto \lambda f$ is defines a linear isomorphism from the space of symmetric trace-free m-tensor fields in $C^{k,\alpha}(M)$ to the space $\Omega^{k,\alpha}_{\mathrm{h}} \Omega^{\infty}_{\mathrm{v}}(m)$. There is a constant $C_{m,n} > 0$ so that for all symmetric trace-free m-tensor fields $f \in C^0(M)$, we have*

$$\|\lambda f\|_{L^2(SM)} = C_{m,n} \|f\|_{L^2(M)} . \tag{39}$$

Furthermore, there are positive constants $c, C > 0$ so that for any two m-tensor fields f and h in $C^0(M)$, we have

$$c\,(\lambda f, \lambda h)_{L^2(SM)} \leq (f, h)_{L^2(M)} \leq C\,(\lambda f, \lambda h)_{L^2(SM)} . \tag{40}$$

Proof As in the smooth case [9, Lemma 2.5.],the map λ_x isomorphically maps trace-free m-tensors to spherical harmonics $S_x M$ of degree m. Since the dependence on x is of the form $\lambda f(x, v) = f_{j_1 \ldots j_m}(x) v^{j_1} \cdots v^{j_m}$, the map λ maps on trace-free m-tensor fields in $C^{k,\alpha}(M)$ into $\Omega^{k,\alpha}_{\mathrm{h}} \Omega^{\infty}_{\mathrm{v}}(m)$.

For any symmetric and trace-free m-tensor fields $f, h \in C^0(M)$, a fibrewise calculation [9, Lemma 2.4.] shows that for all $x \in M$,we have

$$\int_{S_x M} (\lambda_x f)(\lambda_x h)\, \mathrm{d}S_x = C_{m,n} \langle f, h \rangle_{g(x)} \tag{41}$$

for some $C_{m,n} > 0$. Since the computation is fibrewise, it remains valid when $g \in C^{1,1}$. Integrating equation (41) over M gives

$$(\lambda f, \lambda h)_{L^2(SM)} = C_{m,n} (f, h)_{L^2(M)} , \tag{42}$$

which proves (39). Furthermore, the last claim (40) follows from (41), since any symmetric m-tensor field can be decomposed into a sum of symmetric trace-free tensor fields of orders less than or equal to m [36]. □

3.5 Decomposition of the Geodesic Vector Field

In this subsection,we recall the fact that the geodesic vector field maps from spherical harmonic degree m to spherical harmonic degrees $m-1$ and $m+1$. This mapping property induces a decomposition of X into operators X_+ and X_-. See [36, Section 6.6.] for details of the decomposition when $g \in C^\infty$. We record in proposition 12 what changes in the decomposition, when the Riemannian metric g is only $C^{1,1}$-smooth.

Proposition 12 *Let M be a compact smooth manifold with a smooth boundary and let g be a $C^{1,1}$ Riemannian metric on M. The geodesic vector field maps*

$$X\colon \Omega^1_\mathrm{h}\Omega^\infty_\mathrm{v}(m) \to \Omega^0_\mathrm{h}\Omega^\infty_\mathrm{v}(m-1) \oplus \Omega^0_\mathrm{h}\Omega^\infty_\mathrm{v}(m+1). \tag{43}$$

Therefore X decomposes into operators X_+ and X_- in each spherical harmonic degree so that

$$X_\pm\colon \Omega^1_\mathrm{h}\Omega^\infty_\mathrm{v}(m) \to \Omega^0_\mathrm{h}\Omega^\infty_\mathrm{v}(m\pm 1). \tag{44}$$

Proof Let $u \in \Omega^1_\mathrm{h}\Omega^\infty_\mathrm{v}(m)$ and pick a point $x \in M$. Then $Xu(x,v) = v^j \delta_j u(x,v)$ for all $v \in S_xM$, where v^j is a spherical harmonic of degree 1 on S_xM and $\delta_j u(x,\cdot)$ is a spherical harmonic of degree m on S_xM. Since any product of spherical harmonics of degrees 1 and m is a sum of spherical harmonics of degrees $m-1$ and $m+1$ we see that

$$X\colon \Omega^1_\mathrm{h}\Omega^\infty_\mathrm{v}(m) \to \Omega^0_\mathrm{h}\Omega^\infty_\mathrm{v}(m-1) \oplus \Omega^0_\mathrm{h}\Omega^\infty_\mathrm{v}(m+1). \tag{45}$$

Here the spherical harmonic components of Xu have one horizontal derivative less than u since $X \in \overline{\mathcal{H}}$. □

Remark 13 Since X maps continuously with respect to the H^1- and L^2-norms the mapping properties from proposition 12 carry over to the Sobolev space. In other words

$$\begin{aligned} X\colon \Lambda^1_\mathrm{h}\Lambda^2_\mathrm{v}(m) &\to \Lambda^0_\mathrm{h}\Lambda^2_\mathrm{v}(m-1) \oplus \Lambda^0_\mathrm{h}\Lambda^2_\mathrm{v}(m+1), \quad \text{and} \\ X_\pm\colon \Lambda^1_\mathrm{h}\Lambda^2_\mathrm{v}(m) &\to \Lambda^0_\mathrm{h}\Lambda^2_\mathrm{v}(m\pm 1). \end{aligned} \tag{46}$$

As stated above, proposition 12 gives degreewise defined operators X_- and X_+ acting on $\Lambda^1_\mathrm{h}\Lambda^2_\mathrm{v}(SM)$. If $u \in H^1_\mathrm{h}H^2_\mathrm{v}(SM)$ and $u = \sum_{k=0}^\infty u_k$ is the spherical harmonic decomposition of u, we define

$$X_\pm u = \sum_{k=0}^\infty X_\pm u_k. \tag{47}$$

We prove in lemma 17 that the series in (47) converges (absolutely) in $L^2(SM)$.

The following lemma 14 is a low regularity version of [43, Lemma 3.3.], the only difference being the regularity of u.

Lemma 14 *Let M be a compact smooth manifold with a smooth boundary and let g be a $C^{1,1}$ Riemannian metric on M. If $u \in \Lambda^1_{\mathrm{h}}\Lambda^2_{\mathrm{v}}(m)$ then*

$$[X_+, \overset{\mathrm{v}}{\Delta}]u = -(2m+n-1)X_+u, \quad \textit{and} \tag{48}$$

$$[X_-, \overset{\mathrm{v}}{\Delta}]u = (2m+n-3)X_-u. \tag{49}$$

Proof By density, it is enough to prove the claimed formulas for $u \in \Omega^1_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m)$. By eigenvalue property of u and by the mapping property of X_+, we have

$$X_+\overset{\mathrm{v}}{\Delta}u = m(m+n-2)X_+u. \tag{50}$$

Similarly, by the eigenvalue property of X_+u, we have

$$\overset{\mathrm{v}}{\Delta}X_+u = (m+1)((m+1)+n-2)X_+u. \tag{51}$$

Subtracting (50) from (51) shows that

$$[X_+, \overset{\mathrm{v}}{\Delta}]u = -(2m+n-1)X_+u. \tag{52}$$

The identity (49) can be proved similarly. □

4 Boundary Determination and Regularity Lemmas

This section is devoted to the study of the integral function u^f of a tensor field f with vanishing X-ray transform. We prove a vital boundary determination result (lemma 2) that allows us to prove that u^f is a Lipschitz function on SM in subsection 4.2. In subsection 4.3, we exploit the particular form of the identification of trace-free tensor fields and spherical harmonics to prove our main regularity lemma 3.

4.1 Boundary Determination

The boundary determination lemma 2 is proved in two parts. In lemma 15,we give an explicit local construction. In more detail, we prove that if If vanishes for some tensor field f, then in local coordinates near any boundary point, we construct a tensor field p so that the symmetrized covariant derivative of p equals f when restricted to the boundary. We prove that lemma 2 follows from the local construction by a partition of unity argument.

Lemma 15 *Let (M, g) be a simple $C^{1,1}$ manifold and suppose that $f \in C^{1,1}(M)$ is a symmetric m-tensor field on M so that in $If = 0$. For each $x \in \partial M$, there is a neighbourhood $W \subseteq M$ of x and a symmetric $(m-1)$-tensor field $p \in C^{1,1}(W)$ so that $p|_{W\cap\partial M} = 0$ and $\sigma\nabla p|_{W\cap\partial M} = f|_{W\cap\partial M}$.*

Proof Let $x_0 \in \partial M$ be a boundary point. Choose a neighbourhood $W \subseteq M$ of x_0, where we have C^∞ coordinates $\phi\colon W \to \mathbb{R}^n$ so that

$$\phi(W \cap \partial M) = \{x^n = 0\} \quad \text{and} \quad \phi(W \cap M^{\text{int}}) = \{x^n > 0\}. \tag{53}$$

The smooth coordinate function ϕ exists, since M is a smooth manifold with a smooth boundary. Denote $\hat{x}{:=}(x^1, \dots, x^{n-1})$ so that $x = (\hat{x}, x^n)$.

In these coordinates,the required tensor field p can be defined in the following way. Given $l \in \{0, \dots, m-1\}$ and $j_1, \dots, j_l \in \{1, \dots, n-1\}$,we let the component of p corresponding to the indices $j_1 \cdots j_l n \cdots n$ be

$$p_{j_1 \cdots j_l n \cdots n}(\hat{x}, x^n){:=}\frac{m}{m-l} x^n f_{j_1 \cdots j_l n \cdots n}(\hat{x}, 0). \tag{54}$$

Here the index n appears $m-1-l$ times in $p_{j_1 \cdots j_l n \cdots n}$ and $m-l$ times in $f_{j_1 \cdots j_l n \cdots n}$. We can insist that p is symmetric by requiring

$$p_{j_1 \cdots j_{m-1}}(\hat{x}, x^n) = p_{j_{\pi(1)} \cdots j_{\pi(m-1)}}(\hat{x}, x^n), \tag{55}$$

where π is any permutation of $\{1, \dots, m-1\}$ so that $j_{\pi(1)} \leq \cdots \leq j_{\pi(m-1)}$. This causes no contradictions, since f is symmetric. Clearly, it holds that $p|_{x^n=0} = 0$ and $p \in C^{1,1}(M)$ since $f \in C^{1,1}(M)$.

It remains to show that $\sigma \nabla p|_{x^n=0} = f|_{x^n=0}$, which follows from two claims:

(1) We prove $f_{j_1 \cdots j_m}(\hat{x}, 0) = 0$ in the coordinates in W when $j_1, \dots, j_m \in \{1, \dots, n-1\}$.
(2) We verify that $(\sigma \nabla p)_{j_1 \dots j_m}|_{x^n=0} = f_{j_1 \dots j_m}|_{x^n=0}$ in the coordinates in W.

Both claims are proved in appendix A. The idea is that item 1 follows from the fact $If = 0$, and item 2 can then be verified by a straightforward computation in the coordinates in W. □

Proof of lemma 2 Let $f \in C^{1,1}(M)$ be a symmetric m-tensor field with $If = 0$. We construct a symmetric $(m-1)$-tensor field $p \in C^{1,1}(M)$ so that $p|_{\partial M} = 0$ and $\sigma \nabla p|_{\partial M} = f|_{\partial M}$.

For each $x \in \partial M$ pick a neighbourhood $W_x \subseteq M$ of x and a symmetric $(m-1)$-tensor field $p_x \in C^{1,1}(W_x)$. Such neighbourhoods W_x and tensor fields p_x exist by lemma 15. Since ∂M is compact, there is a finite subcover $\{W_{x_i}\}_{i=1}^k$ of the open cover $\{W_x\}_{x \in \partial M}$ of ∂M. Denote $W_i{:=}W_{x_i}$ and $p_i{:=}p_{x_i}$. We add $W_0{:=}M \setminus \partial M$ to get a finite open cover of M. Choose a partition of unity $\{\psi_i\}_{i=1}^n \cup \{\psi_0\}$ subordinate to $\{W_i\}_{i=1}^n \cup \{W_0\}$. We let the tensor field p_0 corresponding to W_0 to be identically zero. The products $\psi_i p_i$ are $C^{1,1}$ tensor fields in neighbourhoods W_i and we can extend them by zero outside W_i to get $C^{1,1}$ tensor fields on M since each $W_i \setminus \operatorname{supp} \psi_i$ is open. We define an $(m-1)$-tensor field p by

$$p(x) = \sum_{i=0}^{n} \psi_i(x) p_i(x). \tag{56}$$

Since $\psi_i p_i$ are zero outside $\operatorname{supp}\psi_i$ and $p_i|_{\partial M\cap \operatorname{supp}\psi_i} = 0$ by construction, we see that $p|_{\partial M} = 0$. The final step is to check that $\sigma\nabla p = f$ on the boundary ∂M. By the product rule,we have $\nabla(\psi_i p_i) = \nabla\psi_i \otimes p_i + \psi_i(\nabla p_i)$ for all i. Since symmetrization commutes with multiplication by a scalar function and ψ_i is a scalar, we have

$$\sigma\nabla p = \sum_{i=0}^{n}[\sigma((\nabla\psi_i)\otimes p_i) + \psi_i\sigma(\nabla p_i)]. \tag{57}$$

Since symmetrization and tensor product commute with pointwise evaluations,we have $\sigma((\nabla\psi_i)\otimes p_i)|_{\partial M} = 0$. Since $\psi_i = 0$ in $M\setminus \operatorname{supp}\psi_i$ we have $\sigma\nabla\psi_i = 0$ in the same open set $M\setminus \operatorname{supp}\psi_i$. Together with $p_i = 0$ on $\partial M \cap \operatorname{supp}\psi_i \subseteq \partial M \cap W_i$, vanishing of the covariant derivative $\sigma\nabla\psi_i$ in $M \setminus \operatorname{supp}\psi_i$ implies

$$\begin{aligned}\sigma\nabla p|_{\partial M} &= \sum_{i=0}^{n}(\psi_i(\sigma\nabla p_i))|_{\partial M} = \sum_{i=0}^{n}\psi_i(\sigma\nabla p_i|_{\partial M\cap W_i})\\ &= \sum_{i=0}^{n}\psi_i(f|_{\partial M\cap W_i}) = \sum_{i=0}^{n}(\psi_i f)|_{\partial M} = f|_{\partial M}.\end{aligned} \tag{58}$$

Thus p has the desired properties. □

4.2 Regularity of the Integral Function

Let (M, g) be a simple $C^{1,1}$ manifold and let $f \in C^{1,1}(M)$ be a symmetric m-tensor field with $If = 0$. Since the main objective is to prove that there is a symmetric $(m-1)$-tensor field p on M so that $\sigma\nabla p = f$ and by lemma 2, we can find a tensor field $p \in C^{1,1}(M)$ with this property on the boundary ∂M, we can move to studying tensor fields $f \in \mathrm{Lip}_0(M)$ vanishing on the boundary. The following lemma is a special case of [18, Lemma 21]. We record it for the convenience of the reader.

Lemma 16 *Let (M, g) be a simple $C^{1,1}$ manifold. Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field on M and let $u:=u^f$ be the integral function of f defined by* (3). *Then $u \in$* $\mathrm{Lip}(SM)$.

Proof Since f is in $\mathrm{Lip}_0(M)$ the corresponding function on the sphere bundle is in $\mathrm{Lip}_0(SM)$. It was shown in [18, Lemma 21] that the integral function of a function in $\mathrm{Lip}_0(SM)$ is again a Lipschitz function on SM. □

Next we prove lemma 7 which states that if a Lipschitz function u on SM arising from of tensor field $-p$ satisfies the transport equation $Xu = -f$, then $\sigma\nabla p = f$ holds pointwise almost everywhere.

Proof of lemma 7 Let $f \in \mathrm{Lip}(M)$ is a symmetric m-tensor field. Suppose that $p \in \mathrm{Lip}(M)$ is a symmetric m-tensor field so that the Lipschitz function $u := -\lambda p$ solves the transport equation $Xu = -f$ everywhere in SM. We prove that $\sigma\nabla p = f$ almost everywhere on SM by proving that

$$(\sigma\nabla p - f, \eta)_{L^2(M)} = 0 \tag{59}$$

for all symmetric m-tensor fields $\eta \in C_0^1(M)$. Since by proposition 11, there are positive constants $c, C > 0$ so that

$$c\,(\lambda h_1, \lambda h_2)_{L^2(SM)} \leq (h_1, h_2)_{L^2(M)} \leq C\,(\lambda h_1, \lambda h_2)_{L^2(SM)} \tag{60}$$

for all symmetric m-tensor fields $h_1, h_2 \in \mathrm{Lip}(M)$ it is enough to prove that

$$(\lambda\sigma\nabla p - \lambda f, \lambda\eta)_{L^2(SM)} = 0. \tag{61}$$

Consider a maximal geodesic γ of M so that $\gamma(0) = x \in \partial M$ and $\dot\gamma(0) = v \in \partial_{\text{in}}(SM)$. We denote $z{:=}(x, v)$ and write $\eta{:=}\lambda\eta$ and $f{:=}\lambda f$. Furthermore, we denote $\theta(t){:=}\phi_t(z)$ and $\eta(t){:=}\eta(\theta(t))$. Then we have

$$\int_0^{\tau(z)} (\lambda\sigma\nabla p)(\theta(t))\eta(t)\,\mathrm{d}t = \int_0^{\tau(z)} (\nabla p)_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))\eta(t)\,\mathrm{d}t. \tag{62}$$

Since γ is a geodesic, it satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$. Therefore,the Leibniz rule implies

$$\begin{aligned}\int_0^{\tau(z)} (\nabla p)_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))\eta(t)\,\mathrm{d}t &= \int_0^{\tau(z)} \partial_t(p_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t)))\eta(t)\,\mathrm{d}t \\ &= -\int_0^{\tau(z)} p_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))\partial_t\eta(t)\,\mathrm{d}t.\end{aligned} \tag{63}$$

By assumption $u(\theta(t)) = -p_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))$ for all $t \in [0, \tau(z)]$ and thus

$$\begin{aligned}-\int_0^{\tau(z)} p_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))\partial_t\eta(t)\,\mathrm{d}t &= \int_0^{\tau(z)} u(\theta(t))\partial_t\eta(t)\,\mathrm{d}t \\ &= -\int_0^{\tau(z)} \partial_t u(\theta(t))\eta(t)\,\mathrm{d}t \\ &= \int_0^{\tau(z)} f(\theta(t))\eta(t)\,\mathrm{d}t,\end{aligned} \tag{64}$$

where the last equality holds since $Xu = -f$ and X is the infinitesimal generator of the geodesic flow ϕ_t. Together, equations (62), (63) and (64) show that

$$\int_0^{\tau(z)} (\lambda\sigma\nabla p)(\theta(t))\eta(t)\,\mathrm{d}t = \int_0^{\tau(z)} f(\theta(t))\eta(t)\,\mathrm{d}t. \tag{65}$$

We integrate (65) over $\partial_{\text{in}}(SM)$ and use Santaló's formula (lemma 24) to see that

$$\begin{aligned}\int_{SM} (\lambda\sigma\nabla p)\eta \,\mathrm{d}\Sigma_g &= \int_{\partial_{\text{in}}(SM)} \int_0^{\tau(z)} (\lambda\sigma\nabla p)(\theta(t))\eta(t)\,\mathrm{d}t\,\mu \mathrm{d}\Sigma_{j^*g} \\ &= \int_{\partial_{\text{in}}(SM)} \int_0^{\tau(z)} f(\theta(t))\eta(t)\,\mathrm{d}t\,\mu \mathrm{d}\Sigma_{j^*g} \\ &= \int_{SM} f\eta\,\mathrm{d}\Sigma_g.\end{aligned} \tag{66}$$

Equation (61) follows immediately from (66), which finishes the proof. □

4.3 Regularity of the Spherical Harmonic Components

In this subsection,we use the special form of spherical harmonics and the identification of trace-free tensor fields and spherical harmonics to prove lemma 3. Also, we prove that the degreewise definition of operators $X_\pm$ acting on functions on SM is reasonable by proving that series in (47) converge absolutely in $L^2(SM)$.

Proof of lemma 3 Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field with vanishing X-ray transform and let $u:=u^f$ be the integral function of f defined by (3). The integral function u is in $\mathrm{Lip}(SM)$ by lemma 16. We prove that the spherical harmonic components u_k of u are in $\Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$ and that $u_k|_{\partial(SM)} = 0$.

For a fixed $x \in M$,the fibre $S_x M$ is isometric to the Euclidean unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ via the map

$$s_x : S_x M \to S^{n-1}, \quad s_x(v) = g(x)^{1/2} v, \tag{67}$$

where $g(x)^{1/2}$ is the unique square root of a positive definite matrix $g(x)$. Since u is in $\mathrm{Lip}(SM)$, its restriction $u_x:=u(x,\cdot)$ to $S_x M$ is in $\mathrm{Lip}(S_x M)$. Thus the functions $\tilde{u}_x$ on S^{n-1} corresponding to u_x via s_x has a decomposition

$$\tilde{u}_x = \sum_{k=0}^{\infty} (\tilde{u}_x, \phi_k)_{L^2(S^{n-1})}\, \phi_k, \tag{68}$$

where ϕ_k is the eigenfunction of the Laplacian on S^{n-1} corresponding to the eigenvalue $k(k+n-2)$. Tracing back through s_x,we find a $L^2(S_x M)$ convergent decomposition

$$u_x = \sum_{k=0}^{\infty} (u_x, \psi_k)_{L^2(S_x M)}\, \psi_k, \tag{69}$$

where $\psi_k(v) = \phi_k(s_x^{-1}(v))$. On the level of the bundle SM, we denote $\psi_k(x,v):= \phi_k(s_x^{-1}(v))$, and thus get the formula $u_k = (u,\psi_k)_{L^2(S_x M)}\,\psi_k$. Here ψ_k is in $C^{1,1}(SM)$, since ϕ_k is in $C^\infty(S^{n-1})$ and the map $(x,v) \mapsto s_x(v)$ is in $C^{1,1}(SM)$. This proves that $u_k \in \mathrm{Lip}(SM)$. We note that by lemma 11 for all k,there is a symmetric and trace-free k-tensor field $h_k \in \mathrm{Lip}(M)$ so that $u_k(x,v) = (h_k)_{j_1\cdots j_k}(x)v^{j_1}\cdots v^{j_k}$. This proves that $u_k \in \Omega^{0,1}_{\mathrm{h}}\Omega^{\infty}_{\mathrm{v}}(k)$ for all k, since u_k is polynomial in v.

Finally, we prove that $u_k|_{\partial(SM)} = 0$. Since the X-ray transform of f is zero, the restriction of u on the boundary $\partial(SM)$ is zero. Thus for any $x \in \partial M$ we have

$$0 = \|u(x,\cdot)\|^2_{L^2(S_xM)} = \sum_{k=0}^{\infty} \|u_k(x,\cdot)\|^2_{L^2(S_xM)}\,. \tag{70}$$

Therefore, since $u_k(x,\cdot) \in C^\infty(S_xM)$, we have $u_k(x,\cdot) = 0$ pointwise on S_xM for all k, which implies that $u_k|_{\partial(SM)} = 0$ for all k. □

Lemma 17 *Let (M,g) be a simple $C^{1,1}$ manifold. Given $u \in H^1_{\mathrm{h}}H^2_{\mathrm{v}}(SM)$, if $u = \sum_{k=0}^{\infty} u_k$ is the spherical harmonic decomposition of u, then the series $\sum_{k=0}^{\infty} X_\pm u_k$ converge absolutely in $L^2(SM)$. Here we use the convention that $X_- u_0 = 0$.*

Proof We prove convergence of both of series $\sum_{k=0}^{\infty} X_\pm u_k$ at once by proving that

$$\sum_{k=0}^{\infty} \|X_+u_k\|^2_{L^2(SM)} + \sum_{k=1}^{\infty} \|X_-u_k\|^2_{L^2(SM)} \le \|u\|^2_{H^1_{\mathrm{h}}H^0_{\mathrm{v}}(SM)}\,. \tag{71}$$

The proof of (71) is identical to the proofs of [43, Lemma 4.4] and [26, Lemma 5.1], where the authors proved that

$$\|X_+u\|^2_{L^2(SM)} + \|X_-u\|^2_{L^2(SM)} \le \|Xu\|^2_{L^2(SM)} + \left\|\overset{\mathrm{h}}{\nabla} u\right\|_{L^2(SM)}. \tag{72}$$

The major difference to the results in [43] and [26] is that we work in non-smooth geometry instead of a smooth geometry, so the tools in the proof have changed. For completeness, we repeat the arguments in appendix B to document the fact that all steps go through in lower regularity with suitably chosen function spaces. □

Remark 18 For $u \in H^1_{\mathrm{h}}H^2_{\mathrm{v}}(SM)$,we defined $X_\pm u$ to be the series $\sum_{k=0}^{\infty} X_\pm u_k$, when $u = \sum_{k=0}^{\infty} u_k$ is the spherical harmonic decomposition of u. By lemma 17 both X_+u and X_-u are well- defined functions in $L^2(SM)$ and by orthogonality

$$\|X_\pm u\|^2_{L^2(SM)} = \sum_{k=0}^{\infty} \|X_\pm u_k\|^2_{L^2(SM)}\,. \tag{73}$$

5 Energy Estimates and a Santaló Formula

In this section, we show that the L^2-estimate in lemma 5 follows from the Pestov identity, and we establish the Santaló's formula in low regularity in lemma 24.

5.1 Pestov Energy Identity

Let (M, g) be a simple $C^{1,1}$ manifold. Recall that the global index form Q of (M, g) is defined by

$$Q(W) := \|XW\|_{L^2(N)} - (RW, W)_{L^2(N)} \tag{74}$$

for $W \in H_0^1(N, X)$.

Lemma 19 (Pestov identity) *Let (M, g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. If $u \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$ and $u|_{\partial(SM)} = 0$, then*

$$\left\| \overset{\mathrm{v}}{\nabla} Xu \right\|_{L^2(N)}^2 = Q\left(\overset{\mathrm{v}}{\nabla} u\right) + (n-1)\|Xu\|_{L^2(SM)}^2. \tag{75}$$

Proof Since $u \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$, we have $u \in \mathrm{Lip}_0(SM)$, $\overset{\mathrm{v}}{\nabla} Xu \in L^2(N)$ and $X\overset{\mathrm{v}}{\nabla} u \in L^2(N)$. It was proved in [18, Lemma 9] that the Pestov identity (75) holds for this class of functions on simple $C^{1,1}$ manifolds. □

When $g \in C^\infty$, the estimate in Lemma 20 was derived in [20, Section 6]. We present a proof compatible with low regularity employing the Pestov identity in Lemma 19.

Lemma 20 *Let (M, g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. If $u \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$ and $u|_{\partial(SM)} = 0$, then*

$$\left(Xu, [X, \overset{\mathrm{v}}{\Delta}]u\right)_{L^2(SM)} \leq 0. \tag{76}$$

Proof Since the sectional curvature of (M, g) is almost everywhere non-positive, $Q(W) \geq \|XW\|^2$ for all $W \in H_0^1(N, X)$ and we have

$$\left\| \overset{\mathrm{v}}{\nabla} Xu \right\|_{L^2(N)}^2 \geq \left\| X\overset{\mathrm{v}}{\nabla} u \right\|_{L^2(N)}^2 + (n-1)\|Xu\|_{L^2(SM)}^2 \tag{77}$$

by the Pestov identity (lemma 19). On the other hand, using commutator formulas from proposition 10,we see that

$$\begin{aligned} \left\| X\overset{\mathrm{v}}{\nabla} u \right\|^2 &= \left\| \overset{\mathrm{v}}{\nabla} Xu - \overset{\mathrm{h}}{\nabla} u \right\|^2 \\ &= \left\| \overset{\mathrm{v}}{\nabla} Xu \right\|^2 - 2\left(\overset{\mathrm{v}}{\nabla} Xu, \overset{\mathrm{h}}{\nabla} u\right) + \left\| \overset{\mathrm{h}}{\nabla} u \right\|^2 \\ &= \left\| \overset{\mathrm{v}}{\nabla} Xu \right\|^2 + \left(Xu, 2\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{h}}{\nabla} u\right) + \left\| \overset{\mathrm{h}}{\nabla} u \right\|^2. \end{aligned} \tag{78}$$

Combining estimate (77) and equation (78) and applying the commutator formula (37),we get

$$0 \geq \left(Xu, 2\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{h}}{\nabla}u \right) + \left\| \overset{\mathrm{h}}{\nabla}u \right\|^2 + (n-1) \|Xu\|^2$$
$$\geq \left(Xu, 2\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{h}}{\nabla}u + (n-1)Xu \right) \tag{79}$$
$$= \left(Xu, [X, \overset{\mathrm{v}}{\Delta}]u \right)$$

as claimed. □

Lemma 21 *Let (M, g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. Suppose that $f \in \mathrm{Lip}_0(M)$ is a symmetric m-tensor field on M with vanishing X-ray transform If. Let $u{:=}u^f$ be the integral function of f defined by (3). If $k \geq m$ or $k \equiv m \pmod 2$, we have*

$$\|X_+u_k\|^2_{L^2(SM)} = \|X_-u_{k+2}\|^2_{L^2(SM)}. \tag{80}$$

Proof Since $f \in \mathrm{Lip}_0(M)$ and the X-ray transform of f vanishes, we have $u \in \mathrm{Lip}_0(SM)$ by lemma 16. By the fundamental theorem of calculus u solves $Xu = -f$. Projecting this transport equation onto spherical harmonic degree $k+1$ gives

$$-f_{k+1} = X_+u_k + X_-u_{k+2}. \tag{81}$$

If $k \geq m$ or $k \equiv m \pmod 2$, then $f_{k+1} = 0$ and the claim (80) follows by taking L^2-norms. □

Recall that the constants $C(n, k)$ and $B(n, l, k)$ in lemma 5 are

$$C(n,k){:=}\frac{2k+n-1}{2k+n-3} \quad \text{and} \quad B(n,l,k){:=}\prod_{p=1}^{l} C(n, k+2p). \tag{82}$$

Lemma 22 *Let (M, g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. Suppose that $f \in \mathrm{Lip}_0(M)$ is a symmetric m-tensor field with $If = 0$. Let $u{:=}u^f$ be integral function of f defined by (3). If $2k + n - 3 > 0$, we have*

$$\|X_-u_k\|^2_{L^2(SM)} \leq C(n,k) \|X_+u_k\|^2_{L^2(SM)}, \tag{83}$$

where u_k are the spherical harmonic components of u.

Proof Let $2k+n-3>0$. Since $u_k \in \Omega^{0,1}_{\mathrm{h}}\Omega^{\infty}_{\mathrm{v}}(k)$ by lemma 3, we can use lemma 20, which together with commutator formulas in 14 gives

$$
\begin{aligned}
(2k+n-1)\,\|X_+u_k\|^2 &\geq (2k+n-1)\,\|X_+u_k\|^2 + \left(Xu_k, [X, \overset{\mathrm{v}}{\Delta}]u_k\right) \\
&= (2k+n-1)\,\|X_+u_k\|^2 + \left(X_+u_k, [X_+, \overset{\mathrm{v}}{\Delta}]u_k\right) \\
&\quad + \left(X_-u_k, [X_-, \overset{\mathrm{v}}{\Delta}]u_k\right) \\
&= (2k+n-3)\,\|X_-u_k\|^2 .
\end{aligned}
\tag{84}
$$

Dividing by $2k+n-3>0$ proves the claimed estimate (83). □

Proof of lemma 5 Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field so that $If = 0$ and denote by $u{:=}u^f$ its integral function defined by (3). Let $k \geq m$. By lemma 3,we have $u \in \Omega^{0,1}_{\mathrm{h}}\Omega^{\infty}_{\mathrm{v}}(k)$ and thus lemmas 21 and 22,we get

$$
\|X_+u_k\|^2_{L^2(SM)} = \|X_-u_{k+2}\|^2_{L^2(SM)} \leq C(n,k+2)\,\|X_+u_{k+2}\|^2_{L^2(SM)} . \tag{85}
$$

Iterating lemmas 21 and 22 a total of $l \in \mathbb{N}$ times yields

$$
\|X_+u_k\|^2 \leq \|X_+u_{k+2l}\|^2 \prod_{p=1}^{l} C(n,k+2p) = B(n,l,k)\,\|X_+u_{k+2l}\|^2 \tag{86}
$$

as claimed. □

5.2 Santaló's Formula

The proof of Santaló's formula on a smooth simple manifolds (M,g) is based on the so-called Liouville's theorem and can be found e.g. in [36]. We give a similar proof of the formula on a simple $C^{1,1}$ manifold based on the following formulation of Liouville's theorem.

Lemma 23 *Let (M,g) be a simple $C^{1,1}$ manifold. Denote by L_X the Lie derivative into the direction of the geodesic vector field X on SM. Then for any $u \in \mathrm{Lip}(SM)$ it holds that*

$$
\int_{SM} u L_X(\mathrm{d}\Sigma_g) = 0. \tag{87}
$$

The proof of lemma 23 is based on smooth approximation of the Riemannian metric g and can be found in Appendix C.

If ν is the inner unit normal vector field to ∂M, let $\mu(x,v){:=}\langle \nu(x), v\rangle_{g(x)}$ for all $(x,v) \in SM$. If ω is a differential k-form on SM, then denote by $i_X\omega$ the contraction of ω with the geodesic vector field X. That is, for any vector fields $Y_1, \dots, Y_{k-1}$ on SM, we define $i_X\omega$ by letting $i_X\omega(Y_1, \dots, Y_{k-1}) = \omega(X, Y_1, \dots, Y_{k-1})$.

Lemma 24 (Santaló's formula) *Let (M, g) be a simple $C^{1,1}$ manifold. For any function $f \in \mathrm{Lip}_0(SM)$ the integral of f over SM with respect to $\mathrm{d}\Sigma_g$ can be written as*

$$\int_{SM} f\, d\Sigma_g = \int_{\partial_{\mathrm{in}} SM} \int_0^{\tau(z)} f(\phi_t(z))\, \mathrm{d}t\, \mu(z) \mathrm{d}\Sigma_{j^*g}. \tag{88}$$

Here $j\colon \partial(SM) \to SM$ is the inclusion map and j^*g is the Riemannian metric of ∂M induced by the inclusion j.

Proof Let $f \in \mathrm{Lip}_0(SM)$ and consider its integral function $u{:=}u^f$. The integral function satisfies $Xu = -f$ and $u \in \mathrm{Lip}(SM)$ by lemma 16. By Cartan's formula, we have

$$\int_{SM} L_X(u\, \mathrm{d}\Sigma) = \int_{SM} i_X d(u\, \mathrm{d}\Sigma) + \int_{SM} d(i_X u\, \mathrm{d}\Sigma), \tag{89}$$

where d is the exterior derivative. Since $u\, \mathrm{d}\Sigma$ is a volume form, the first term on the right in (89) vanishes. By Stoke's theorem

$$\int_{SM} d(i_X u\, \mathrm{d}\Sigma_g) = \int_{\partial(SM)} j^*(u i_X \mathrm{d}\Sigma_g). \tag{90}$$

As in the smooth case ([36, Proposition 3.6.6.]), we compute that

$$\begin{aligned}\int_{\partial(SM)} j^*(u i_X \mathrm{d}\Sigma_g) &= \int_{SM} (j^*u)(j^* i_X \mathrm{d}\Sigma_g)\\ &= \int_{SM} (j^*u)\, \langle X, \nu\rangle\, \mathrm{d}\Sigma_{j^*g}\\ &= \int_{SM} (j^*u)\mu\, \mathrm{d}\Sigma_{j^*g}.\end{aligned} \tag{91}$$

Finally, since j^*u is merely a restriction to the boundary, we invoke the definition of u and lemma 23 to see that

$$\begin{aligned}\int_{SM} f\, \mathrm{d}\Sigma_g &= \int_{SM} L_X(u)\, \mathrm{d}\Sigma\\ &= \int_{SM} L_X(u\, \mathrm{d}\Sigma) - \int_{SM} u L_X(\mathrm{d}\Sigma)\\ &= \int_{SM} L_X(u\, \mathrm{d}\Sigma) = \int_{\partial(SM)} (j^*u)\mu\, \mathrm{d}\Sigma_{j^*g}\\ &= \int_{\partial(SM)} \int_0^{\tau(z)} f(\phi_t(z))\, \mathrm{d}t \mu\, \mathrm{d}\Sigma_{j^*g}.\end{aligned} \tag{92}$$

Since $\tau(z) = 0$ for $z \notin \partial_{\mathrm{in}}(SM)$, the claim (88) follows at once from (92). □

6 Friedrich's Inequalities

In this section, we prove that L^2-norms of scalar functions on SM and sections of the bundle N are bounded above by constant multiples of L^2-norms of their derivatives along the geodesic flow. We call these estimates Friedrich's inequalities on SM. We apply the inequalities to prove lemma 6.

Lemma 25 *Let (M, g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. Let d be the diameter of M. Then*

$$d^2 \|Xu\|^2_{L^2(SM)} \geq \|u\|^2_{L^2(SM)} \quad \textit{and} \quad d^2 \|XW\|^2_{L^2(N)} \geq \|W\|^2_{L^2(N)} \tag{93}$$

for any $u \in H^1_0(SM)$ and $W \in H^1_0(N, X)$.

Proof First, we prove the inequality for functions. By density it is enough to consider the case $u \in C^1_0(SM)$. By Santaló's formula (lemma 24), we can write

$$\|Xu\|^2_{L^2(SM)} = \int_{\partial_{\mathrm{in}}(SM)} \int_0^{\tau(z)} |Xu(\phi_t(z)|^2 \,\mathrm{d}t\, \mu \mathrm{d}\Sigma_{j^*g}, \tag{94}$$

where $j\colon \partial(SM) \to SM$ is the inclusion. Let us denote $u_z(t) := u(\phi_t(z))$. Then $u_z \in H^1_0([0, \tau(z)])$ and we have

$$Xu(\phi_t(z)) = \frac{\mathrm{d}}{\mathrm{d}s} u(\phi_{t+s}(z))\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} u_z(t+s)\Big|_{s=0} = \dot{u}_z(t). \tag{95}$$

By the usual Friedrich's inequality of $H^1_0([0, \tau(z)])$, we see that

$$d^2 \int_0^{\tau(z)} |\dot{u}_z(t)|^2 \,\mathrm{d}t \geq \tau(z)^2 \int_0^{\tau(z)} |\dot{u}_z(t)|^2 \,\mathrm{d}t \geq \int_0^{\tau(z)} |u_z(t)|^2 \,\mathrm{d}t. \tag{96}$$

Combining equation (95) with inequality (96), we get

$$\begin{aligned} d^2 \|Xu\|^2_{L^2(SM)} &\geq d^2 \int_{\partial_{\mathrm{in}}(SM)} \int_0^{\tau(z)} |\dot{u}_z(t)|^2 \,\mathrm{d}t\, \mu \mathrm{d}\Sigma_{j^*g} \\ &\geq \int_{\partial_{\mathrm{in}}(SM)} \int_0^{\tau(z)} |u_z(t)|^2 \,\mathrm{d}t\, \mu \mathrm{d}\Sigma_{j^*g} \\ &= \|u\|^2_{L^2(SM)}, \end{aligned} \tag{97}$$

which is the claimed inequality for functions.

Next, we prove the inequality for sections of the bundle N. Let $W \in H^1_0(N, X)$. In this case, Santaló's formulas (lemma 24) gives

$$\|XW\|^2_{L^2(SM)} = \int_{\partial_{\text{in}}(SM)} \int_0^{\tau(z)} |XW(\phi_t(z))|^2_g \, \mathrm{d}t \, \mu(z) \, \mathrm{d}\Sigma_{\partial(SM)}. \tag{98}$$

We let $W_z(t) := W(\phi_t(z))$. Then $W_z(t)$ is a H^1_0 vector field along γ_z and it holds that $XW(\phi_t(z)) = D_t W_z(t)$. Choose a parallel frame $(E_1, \ldots, E_n)$ along γ_z. Then we have $D_t W_z = \dot{W}^i_z E_i$, when $W_z = W^i_z E_i$. Since W_z is a H^1_0 vector field along γ_z we have $W^i_z \in H^1_0([0, \tau(z)])$ for all i. Thus we read from equation (96) that

$$d^2 \int_0^{\tau(z)} \left|\dot{W}^i_z\right|^2 \mathrm{d}t \geq \int_0^{\tau(z)} \left|W^i_z\right|^2 \mathrm{d}t. \tag{99}$$

From equations (98) and (99) we see that

$$\begin{aligned} d^2 \|XW\|^2_{L^2(N)} &= d^2 \int_{\partial_{\text{in}}(SM)} \int_0^{\tau(z)} |D_t W_z(t)|^2_g \, \mathrm{d}t \, \mu(z) \, \mathrm{d}\Sigma_{\partial(SM)} \\ &= d^2 \sum_{i=1}^n \int_{\partial_{\text{in}}(SM)} \int_0^{\tau(z)} \left|\dot{W}^i_z(t)\right|^2 \mathrm{d}t \, \mu(z) \, \mathrm{d}\Sigma_{\partial(SM)} \\ &\geq \sum_{i=1}^n \int_{\partial_{\text{in}}(SM)} \int_0^{\tau(z)} \left|W^i_z(t)\right|^2 \mathrm{d}t \, \mu(z) \, \mathrm{d}\Sigma_{\partial(SM)} \\ &= \|W\|^2_{L^2(N)}, \end{aligned} \tag{100}$$

which is the second claimed inequality. □

Proof of lemma 6 Let $u \in \Omega^{0,1}_{\mathrm{h}} \Omega^{\infty}_{\mathrm{v}}(k)$ be so that $u|_{\partial(SM)} = 0$ and $X_+ u = 0$. By lemma 14,we have

$$\begin{aligned} (2k+n-3) \|X_- u\|^2 &= -(2k+n-1) \|X_+ u\|^2 + (2k+n-3) \|X_- u\|^2 \\ &= \left([X_+, \overset{\mathrm{v}}{\Delta}]u, X_+ u\right) + \left([X_-, \overset{\mathrm{v}}{\Delta}]u, X_- u\right) \\ &= \left([X_+, \overset{\mathrm{v}}{\Delta}]u, Xu\right) + \left([X_-, \overset{\mathrm{v}}{\Delta}]u, Xu\right) \\ &= \left([X, \overset{\mathrm{v}}{\Delta}]u, Xu\right). \end{aligned} \tag{101}$$

The last inner product in (101) is non-positive by lemma 20. Thus $X_- u = 0$ almost everywhere on SM. Let d be the diameter of M. Lemma 25 then provides

$$\|u\|^2_{L^2(SM)} \leq d^2 \|Xu\|^2_{L^2(SM)} = d^2(\|X_+ u\|^2_{L^2(SM)} + \|X_- u\|^2_{L^2(SM)}) = 0. \tag{102}$$

Thus $u = 0$ almost everywhere on SM, but since u is continuous, we have shown that $u = 0$ everywhere on SM. □

Even though we do not need the result, we next show for completeness that there are no conjugate points in the sense of the global index form Q when the sectional curvature is non-positive.

Proposition 26 *Let M be the closed Euclidean unit ball in $\mathbb{R}^n$. Suppose that M comes equipped with a $C^{1,1}$ Riemannian metric g so that the sectional curvature of (M, g) is almost everywhere non-positive. Then there is $\varepsilon > 0$ so that $Q(W) \geq \varepsilon \|W\|^2_{L^2(N)}$ for all $W \in H^1_0(N, X)$.*

Proof Since the sectional curvature is almost everywhere non-positive,

$$(RW, W)_{L^2(N)} = \int_{(x,v)\in SM} \langle R(W(x, v), v)v, W(x, v)\rangle_g \, \mathrm{d}\Sigma_g \leq 0 \tag{103}$$

for all $W \in H^1_0(N, X)$, since $W(x, v)$ and v are always orthogonal. Thus $Q(W) \geq \|XW\|^2_{L^2(N)}$ for all $W \in H^1_0(N, X)$. Then it follows from lemma 25 that for all $W \in H^1_0(N, X)$,we have

$$Q(W) \geq \|XW\|^2_{L^2(N)} \geq \frac{1}{d^2} \|W\|^2_{L^2(N)}. \tag{104}$$

We take $\varepsilon = 1/d^2$ which finishes the proof. □

Acknowledgements Both authors we supported by the Academy of Finland (JI by grant 351665, AK by grant 351656). AK was supported by the Finnish Academy of Science and Letters. This work was supported by the Research Council of Finland (Flagship of Advanced Mathematics for Sensing Imaging and Modelling grant 359208 and Centre of Excellence of Inverse Modelling and Imaging 353092). We thank the anonymous referees for many valuable comments and suggestions.

Funding Open Access funding provided by University of Jyväskylä (JYU).

Appendix A: Completion of the Proof of Boundary Determination

We complete the details in the proof of lemma 15 by proving items 1 and 2. Recall that we work in local coordinates $\phi\colon W \to \mathbb{R}^n$ so that

$$\phi(W \cap \partial M) = \{x^n = 0\}, \quad \text{and} \quad \phi(W \cap M^{\text{int}}) = \{x^n > 0\}. \tag{105}$$

We denote $\hat{x} = (x^1, \dots, x^{n-1})$. The local tensor field p is defined in these coordinates by

$$p_{j_1 \cdots j_l n \cdots n}(\hat{x}, x^n) = \frac{m}{m-l} x^n f_{j_1 \cdots j_l n \cdots n}(\hat{x}, 0), \tag{106}$$

where n appears $m-1-l$ times in $p_{j_1 \cdots j_l n \cdots n}$ and $m-l$ times in $f_{j_1 \cdots j_l n \cdots n}$.

First we prove item 1. We begin by proving that $f_x(v, \dots, v) = 0$ for all $v \in S_x(W \cap \partial M)$ and $x \in W \cap \partial M$. Given $v \in S_x(W \cap \partial M)$,we choose a sequence (v_k) of vectors $v_k \in S_x(W \cap \partial M)$ so that $\tau(x, v_k) > 0$, and $\tau(x, v_k) \to 0$ and $v_k \to v$ when $k \to \infty$. Such a sequence of vectors exists by $C^{1,1}$ simplicity as proved in [18, Lemma 23]. Since the lengths of the geodesics corresponding to (x, v_k) become arbitrarily short and $If = 0$, we find that

$$\begin{aligned} f_x(v, \dots, v) &= \lim_{k \to \infty} \frac{1}{\tau(x, v_k)} \int_0^{\tau(x, v_k)} f(\phi_t(x, v_k))\, \mathrm{d}t \\ &= \lim_{k \to \infty} \frac{If(x, v_k)}{\tau(x, v_k)} \\ &= 0. \end{aligned} \tag{107}$$

We have shown that $f_x(v, \dots, v) = 0$ for all $v \in S_x(W \cap \partial M)$. Next, we prove that $f_{j_1 \cdots j_m}(\hat{x}, 0) = 0$ in $W \cap \partial M$ for all $j_1, \dots, j_m \in \{1, \dots, n-1\}$.

Let $\iota \colon \partial M \to M$ be the inclusion map. The pullback $\iota^* f$ is an m-tensor field on ∂M. Since $f_x(v, \dots, v) = 0$ for all $v \in S_x(W \cap \partial M)$ we have $(\iota^* f)_x(v, \dots, v) = 0$ for all $v \in S_x(W \cap \partial M)$. Then a fibrewise computation [9, Lemma 2.4] shows that

$$0 = \int_{W \cap \partial M} (\iota^* f)_x(v, \dots, v)^2 \, \mathrm{d}S_x = C_{m,n-1} \left| \iota^* f \right|^2_{g(x)} \tag{108}$$

for all $x \in W \cap \partial M$. We have shown that $\iota^* f|_{W \cap \partial M} = 0$ which written in the coordinates in W gives $f_{j_1 \cdots j_m}(\hat{x}, 0) = 0$ for all $j_1, \dots, j_m \in \{1, \dots, n-1\}$. We have proved item 1.

We proceed to proving item 2. Let $l \in \{0, \dots, m-1\}$ and $j_1, \dots, j_l \in \{1, \dots, n-1\}$. To compute the restriction to boundary of the component functions of $\sigma \nabla p$, we first compute $\nabla_n p_{j_1 \cdots j_l n \cdots n}(\hat{x}, 0)$ and $\nabla_{j_s} p_{j_1 \cdots \widehat{j_s} \cdots j_l n \cdots n}(\hat{x}, 0)$. We have

$$\begin{aligned} \nabla_n p_{j_1 \cdots j_l n \cdots n} &= \partial_n p_{j_1 \cdots j_l n \cdots n} \\ &\quad - \sum_{s=1}^{l} \Gamma^k_{n j_s} p_{j_1 \cdots k \cdots j_l n \cdots n} - \sum_{s=l+1}^{m-1} \Gamma^k_{nn} p_{j_1 \cdots j_l n \cdots k \cdots n}. \end{aligned} \tag{109}$$

Thus by the construction of p,we find that

$$\nabla_n p_{j_1\cdots j_l n\cdots n}(\hat{x}, x^n) = \frac{m}{m-l} f_{j_1\cdots j_l n\cdots nn}(\hat{x}, 0)$$
$$-\frac{m}{m-l} x^n \sum_{s=1}^{l} \Gamma^k_{nj_s} f_{j_1\cdots k\cdots j_l n\cdots nn}(\hat{x}, 0) - \frac{m}{m-l} x^n \sum_{s=l+1}^{m-1} \Gamma^k_{nn} f_{j_1\cdots j_l n\cdots k\cdots nn}(\hat{x}, 0). \tag{110}$$

On the boundary $\{x^n = 0\}$, equation (110) reduces to

$$\nabla_n p_{j_1\cdots j_l n\cdots n}(\hat{x}, 0) = \frac{m}{m-l} f_{j_1\cdots j_l n\cdots nn}(\hat{x}, 0). \tag{111}$$

As in equation (109), we have

$$\begin{aligned} \nabla_{j_s} p_{j_1\cdots \widehat{j_s}\cdots j_l n\cdots n} &= \partial_{j_s} p_{j_1\cdots \widehat{j_s}\cdots j_l n\cdots n} \\ &\quad - \sum_{r=1}^{l-1} \Gamma^k_{j_s j_r} p_{j_1\cdots k\cdots j_l n\cdots n} \\ &\quad - \sum_{r=l}^{m-1} \Gamma^k_{j_s j_r} p_{j_1\cdots j_l n\cdots k\cdots n}. \end{aligned} \tag{112}$$

By the construction of p, equation (112) gives

$$\begin{aligned} \nabla_{j_s} p_{j_1\cdots \widehat{j_s}\cdots j_l n\cdots n}(\hat{x}, x^n) &= \frac{m}{m-l} x^n \partial_{j_s} f_{j_1\cdots \widehat{j_s}\cdots j_l n\cdots nn}(\hat{x}, 0) \\ &\quad - \frac{m}{m-l} x^n \sum_{r=1}^{l-1} \Gamma^k_{j_s j_r} f_{j_1\cdots k\cdots j_l n\cdots nn}(\hat{x}, 0) \\ &\quad - \frac{m}{m-l} x^n \sum_{r=l}^{m-1} \Gamma^k_{j_s n} f_{j_1\cdots j_l n\cdots k\cdots nn}(\hat{x}, 0). \end{aligned} \tag{113}$$

Therefore, on the boundary $\{x^n = 0\}$,we get

$$\nabla_{j_s} p_{j_1\cdots \widehat{j_s}\cdots j_l n\cdots n}(\hat{x}, 0) = 0. \tag{114}$$

Now we are ready to compute $(\sigma\nabla p)_{j_1\ldots j_l n\cdots n}$, when $l \in \{0, \ldots, m-1\}$. Denote $j_{l+1} = \cdots = j_m = n$. There are $(m-l)(m-1)!$ permutations π of $\{1, \ldots, m\}$ so that $j_{\pi(1)} = n$, when no restrictions are set on the remaining indices $j_{\pi(2)}, \ldots, j_{\pi(m)}$. Thus using symmetry of p we find that

$$\begin{aligned}(\sigma\nabla p)_{j_1\cdots j_l n\cdots n} &= \frac{(m-l)(m-1)!}{m!}\nabla_n p_{j_1\cdots j_l n\cdots n} + \frac{(m-1)!}{m!}\sum_{s=1}^{l}\nabla_{j_s} p_{j_1\cdots\widehat{j_s}\cdots j_l n\cdots n} \\ &= \frac{m-l}{m}\nabla_n p_{j_1\cdots j_l n\cdots n} + \frac{1}{m}\sum_{s=1}^{l}\nabla_{j_s} p_{j_1\cdots\widehat{j_s}\cdots j_l n\cdots n}.\end{aligned} \tag{115}$$

Evaluating (115) on the boundary $\{x^n = 0\}$ and substituting (111) and (114) results in

$$(\sigma\nabla p)_{j_1\ldots j_l n\cdots n}(\hat{x}, 0) = f_{j_1\cdots j_l n\cdots n}(\hat{x}, 0). \tag{116}$$

The last step is to prove that

$$(\sigma\nabla p)_{j_1\cdots j_m}(\hat{x}, 0) = f_{j_1\cdots j_m}(\hat{x}, 0) \tag{117}$$

when $j_1, \ldots, j_m \in \{1, \ldots, n-1\}$. By the definition of the symmetrized covariant derivative,

$$(\sigma\nabla p)_{j_1\cdots j_m} = \frac{1}{m!}\sum_{\pi}\nabla_{j_{\pi(1)}} p_{j_{\pi(2)}\cdots j_{\pi(m)}} \tag{118}$$

where the summation is over all permutations π of $\{1, \ldots, m\}$. Since $j_{\pi(k)} < n$ for all $k \in \{1, \ldots, m\}$, we can compute as in (113) to see that

$$\nabla_{j_{\pi(1)}} p_{j_{\pi(2)}\cdots j_{\pi(m)}}|_{x^n=0} = 0 \tag{119}$$

for all permutations π of $\{1, \ldots, m\}$. Thus

$$(\sigma\nabla p)_{j_1\cdots j_m}|_{x^n=0} = 0 = f_{j_1\cdots j_m}|_{x^n=0}. \tag{120}$$

We have finally used item 1 of the proof, where we proved that $f_{j_1\cdots j_m}(\hat{x}, 0) = 0$ for all $j_1, \ldots, j_m \in \{1, \ldots, n-1\}$. This concludes the proof item 2 and thus the proof of lemma 15 is completed.

Appendix B: A Regularity Computation

The following calculation completes the proof of lemma 3. It is based on the proofs of [43, Lemma 4.4] and [26, Lemma 5.1].

Let $u \in H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$ and let $w_k \in \Omega^1_{\mathrm{h}} \Omega^\infty_{\mathrm{v}}(k)$ be so that $w_k|_{\partial(SM)} = 0$. Then $\overset{\mathrm{h}}{\nabla} u \in H^1_{\mathrm{h}} H^1_{\mathrm{v}}(SM)$ and thus

$$\left(\overset{\mathrm{h}}{\nabla} u, \overset{\mathrm{v}}{\nabla} w_k\right)_{L^2(N)} = -\left(\overset{\mathrm{v}}{\mathrm{div}} \overset{\mathrm{h}}{\nabla} u, w_k\right)_{L^2(N)}. \tag{121}$$

Using proposition 10,the right side can rewritten as

$$-\left(\overset{\mathrm{v}}{\mathrm{div}} \overset{\mathrm{h}}{\nabla} u, w_k\right) = -\frac{1}{2}\left([X, \overset{\mathrm{v}}{\Delta}] u, w_k\right) + \frac{n-1}{2}(Xu, w_k). \tag{122}$$

If $u_k \in \Lambda^1_{\mathrm{h}} \Lambda^2_{\mathrm{v}}(k)$ are the spherical harmonic components of u, then by orthogonality and lemma 14 we have

$$\begin{aligned}\left([X, \overset{\mathrm{v}}{\Delta}] u, w_k\right) &= \left([X_+, \overset{\mathrm{v}}{\Delta}] u_{k-1} + [X_-, \overset{\mathrm{v}}{\Delta}] u_{k+1}, w_k\right) \\ &= \left(-\frac{2k+n-3}{2} X_+ u_{k-1} + \frac{2k+n-1}{2} X_- u_{k+1}, w_k\right).\end{aligned} \tag{123}$$

Together, equations (121), (122) and (123) show that

$$\left(\overset{\mathrm{h}}{\nabla} u, \overset{\mathrm{v}}{\nabla} w_k\right) = ((k+n-2) X_+ u_{k-1} - k X_- u_{k+1}, w_k). \tag{124}$$

Then we let $w \in C^1_{\mathrm{h}} C^2_{\mathrm{v}}(SM)$ so that $w|_{\partial(SM)} = 0$. If we decompose w into spherical harmonics w_k, then $w_k \in \Omega^1_{\mathrm{h}} \Omega^\infty_{\mathrm{v}}(k)$. We sum equation (124) over $k \in \mathbb{N}$ and use $k(k+n-2) w_k = \overset{\mathrm{v}}{\Delta} w_k$ to get

$$\begin{aligned}\left(\overset{\mathrm{h}}{\nabla} u, \overset{\mathrm{v}}{\nabla} w\right) &= \sum_{k=0}^{\infty} ((k+n-2) X_+ u_{k-1} + k X_- u_{k+1}, w_k) \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{k} X_+ u_{k-1} + \frac{1}{k+n-2} X_- u_{k+1}, \overset{\mathrm{v}}{\Delta} w_k\right) \\ &= \left(\sum_{k=0}^{\infty} \overset{\mathrm{v}}{\nabla}\left[\frac{1}{k} X_+ u_{k-1} + \frac{1}{k+n-2} X_- u_{k+1}\right], \overset{\mathrm{v}}{\nabla} w_k\right).\end{aligned} \tag{125}$$

Thus there is $W(u) \in H^0_{\mathrm{h}} H^1_{\mathrm{v}}(N)$ so that $\overset{\mathrm{v}}{\mathrm{div}}(W(u)) = 0$ and

$$\overset{\mathrm{h}}{\nabla} u = \sum_{k=0}^{\infty} \overset{\mathrm{v}}{\nabla}\left[\frac{1}{k} X_+ u_{k-1} + \frac{1}{k+n-2} X_- u_{k+1}\right] + W(u). \tag{126}$$

It follows from the eigenvalue property that

$$\left\| \overset{\mathrm{v}}{\nabla} u_k \right\|_{L^2(N)}^2 = k(k+n-2) \left\| u_k \right\|_{L^2(SM)}^2 . \tag{127}$$

Thus equation (126) yields

$$\begin{aligned}
\left\| \overset{\mathrm{h}}{\nabla} u \right\|^2 &= \sum_{k=0}^{\infty} k(k+n-2) \left\| \frac{1}{k} X_+ u_{k-1} + \frac{1}{k+n-2} X_- u_{k+1} \right\|^2 + \| W(u) \|^2 \\
&= \sum_{k=0}^{\infty} \Bigg(\frac{k+n-2}{k} \| X_+ u_{k-1} \|^2 - 2 \left(X_+ u_{k-1}, X_- u_{k+1} \right) \\
&\quad + \frac{k}{k+n-2} \| X_- u_{k+1} \|^2 \Bigg) + \| W(u) \|^2 .
\end{aligned} \tag{128}$$

Again, by orthogonality, we have

$$\begin{aligned}
\| X u \|^2 &= \sum_{k=0}^{\infty} \| X_+ u_{k-1} + X_- u_{k+1} \|^2 \\
&= \sum_{k=0}^{\infty} \left(\| X_+ u_{k-1} \|^2 + 2 \left(X_+ u_{k-1}, X_- u_{k+1} \right) + \| X_- u_{k+1} \|^2 \right)
\end{aligned} \tag{129}$$

We add equations (128) and (129) to get

$$\begin{aligned}
\| u \|_{H^1_{\mathrm{h}} H^0_{\mathrm{v}}(SM)}^2 &= \| X u \|^2 + \left\| \overset{\mathrm{h}}{\nabla} u \right\|^2 \\
&= \sum_{k=0}^{\infty} \left(1 + \frac{k+n-2}{k} \right) \| X_+ u_{k-1} \|^2 \\
&\quad + \sum_{k=0}^{\infty} \left(1 + \frac{k}{k+n-2} \right) \| X_- u_{k+1} \|^2 + \| W(u) \|^2 \\
&\geq \sum_{k=0}^{\infty} \| X_+ u_{k-1} \|^2 + \sum_{k=0}^{\infty} \| X_- u_{k+1} \|^2 ,
\end{aligned} \tag{130}$$

This is estimate (71).

Appendix C: Proof of Liouville's Theorem

This appendix is devoted to the proof of lemma 23. We let M be a compact smooth manifold with a smooth boundary. Suppose that we are given two $C^{1,1}$ Riemannian metrics g and h on M. Let the corresponding unit sphere bundles be $S_g M$ and $S_h M$.

There is a natural radial $C^{1,1}$-diffeomorphism $(x, v) \mapsto (x, v\,|v|_h^{-1})$ from S_gM to S_hM, the inverse map from S_hM to S_gM being $(x, w) \mapsto (x, w\,|w|_g^{-1})$.

In the proof of lemma 23, we use three types of Riemannian metrics on M. We will have a $C^{1,1}$ Riemannian metric g and two types of smooth Riemannian metrics h and $\overset{\alpha}{g}$. We denote the corresponding radial diffeomorphisms by

$$\overset{\alpha}{s}\colon S_hM \to \overset{\alpha}{S}M, \quad s\colon S_hM \to S_gM, \quad \text{and} \quad \overset{\alpha}{r}\colon \overset{\alpha}{S}M \to S_gM. \tag{131}$$

In the proof of lemma 23, we will use the convention that the unit sphere bundle related $\overset{\alpha}{g}$ is denoted $\overset{\alpha}{S}M{:=}S_{\overset{\alpha}{g}}M$, the operators and differential forms related to $\overset{\alpha}{g}$ are decorated with α on top or as a subscript, the sphere bundle, operators and differential forms related to h are decorated with subscripts h and the bundles and the operators related to the metric g are written without decorations.

Proof of lemma 23 The proof is based on smooth approximations of the Riemannian metric g. Let h be a smooth fixed reference Riemannian metric on M. Let $(\overset{\alpha}{g})$ be a sequence of smooth Riemannian metrics on M so that

$$\overset{\alpha}{g}_{jk} \to g_{jk} \quad \text{in } W_h^{1,\infty}(M) \quad \text{and} \quad \overset{\alpha}{\Gamma}{}^i{}_{jk} \to \Gamma^i{}_{jk} \quad \text{in } L_h^\infty(M). \tag{132}$$

Existence of such sequence was proved in [18, Lemma 18]. Let $u \in \mathrm{Lip}(SM)$ and denote $\overset{\alpha}{u}{:=}\overset{\alpha}{r}{}^*u$ and $\tilde{u}{:=}s^*u$. We note that $\tilde{u} = \overset{\alpha}{s}{}^*\overset{\alpha}{u}$. We will prove that

$$\lim_{\alpha\to\infty} \int_{\overset{\alpha}{S}M} \overset{\alpha}{u} L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma}) = \int_{SM} u L_X(\mathrm{d}\Sigma). \tag{133}$$

Establishing equation (133) proves the claim, since by Liouville's theorem [36, Lemma 3.6.4.], we have

$$L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma}) = 0 \tag{134}$$

for all $\alpha \in \mathbb{N}$ and thus the limit integral in equation (133) is zero.

Recall that $\tilde{u} = s^*u = \overset{\alpha}{s}{}^*\overset{\alpha}{u}$. Thus by basic properties of pullback,it is enough prove that

$$\lim_{\alpha\to\infty} \int_{S_hM} \tilde{u}\overset{\alpha}{s}{}^*(L_{\overset{\alpha}{X}}\mathrm{d}\overset{\alpha}{\Sigma}) = \int_{S_hM} \tilde{u}s^*(L_X\mathrm{d}\Sigma) \tag{135}$$

The manifold M is the Euclidean unit ball in $\mathbb{R}^n$ and we let $(x^1, \dots, x^n)$ be usual Cartesian coordinates on M. We consider coordinates $(x^1, \dots, x^n, w^1, \dots, w^n)$ on S_hM and corresponding coordinates

$$(x^1, \dots, x^n, \overset{\alpha}{v}{}^1, \dots, \overset{\alpha}{v}{}^n) \quad \text{on } \overset{\alpha}{S}M \quad \text{and} \quad (x^1, \dots, x^n, v^1, \dots, v^n) \quad \text{on } SM$$

so that $\overset{\alpha}{s}(x, w) = (x, \overset{\alpha}{v})$ and $s(x, w) = (x, v)$. We associate to (x, w) the coordinate vector fields $\partial_{x^1}, \dots, \partial_{x^n}, \partial_{w^1}, \dots, \partial_{w^n}$ and similarly to $(x, \overset{\alpha}{v})$ we associate

$\partial_{x^1}, \dots, \partial_{x^n}, \partial_{\overset{\alpha}{v}^1}, \dots, \partial_{\overset{\alpha}{v}^n}$ and to (x, v) we associate $\partial_{x^1}, \dots, \partial_{x^n}, \partial_{v^1}, \dots, \partial_{v^n}$. We let

$$
\begin{aligned}
&\mathrm{d}x^1, \dots, \mathrm{d}x^n, \mathrm{d}w^1, \dots, \mathrm{d}w^n, \\
&\mathrm{d}x^1, \dots, \mathrm{d}x^n, \mathrm{d}\overset{\alpha}{v}^1, \dots, \mathrm{d}\overset{\alpha}{v}^n, \quad \text{and} \\
&\mathrm{d}x^1, \dots, \mathrm{d}x^n, \mathrm{d}v^1, \dots, \mathrm{d}v^n
\end{aligned}
\tag{136}
$$

be the dual basis one-forms characterized by

$$
\begin{aligned}
&\mathrm{d}x^j(\partial_{x^k}) = \delta^j_k, \quad \mathrm{d}x^j(\partial_{w^k}) = 0, \quad \mathrm{d}w^j(\partial_{x^k}) = 0, \quad \mathrm{d}w^j(\partial_{w^k}) = \delta^j_k, \\
&\mathrm{d}x^j(\partial_{x^k}) = \delta^j_k, \quad \mathrm{d}x^j(\partial_{\overset{\alpha}{v}^k}) = 0, \quad \mathrm{d}\overset{\alpha}{v}^j(\partial_{x^k}) = 0, \quad \mathrm{d}\overset{\alpha}{v}^j(\partial_{\overset{\alpha}{v}^k}) = \delta^j_k, \\
&\mathrm{d}x^j(\partial_{x^k}) = \delta^j_k, \quad \mathrm{d}x^j(\partial_{v^k}) = 0, \quad \mathrm{d}v^j(\partial_{x^k}) = 0, \quad \mathrm{d}v^j(\partial_{v^k}) = \delta^j_k.
\end{aligned}
\tag{137}
$$

Next, we will write the integrals in equation (135) in coordinates on $S_h M$ and we will argue that equation (135) follows from (132). We will derive a local coordinate formula for $L_X(\mathrm{d}\Sigma)$. A similar formula for $L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma})$ can be derived analogously. Then we will compute how the coordinate presentations transform under the pullbacks s^* and $\overset{\alpha}{s}{}^*$.

We denote by $|g|$ the determinant of g. Since $\mathrm{d}\Sigma$ is a volume form (differential form of the highest order), Cartan's formula implies that

$$
L_X(\mathrm{d}\Sigma) = d(i_X \mathrm{d}\Sigma). \tag{138}
$$

Since

$$
i_X \mathrm{d}x^i = \mathrm{d}x^i(X) = \mathrm{d}x^i(v^j \partial_{x^j} - \Gamma^l{}_{jk} v^j v^k \partial_{v^l}) = v^i \tag{139}
$$

and

$$
i_X \mathrm{d}v^i = \mathrm{d}v^i(X) = \mathrm{d}v^i(v^j \partial_{x^j} - \Gamma^l{}_{jk} v^j v^k \partial_{v^l}) = -\Gamma^i{}_{jk} v^j v^k \tag{140}
$$

we see that

$$
\begin{aligned}
i_X \mathrm{d}\Sigma &= \sum_{i=1}^n v^i \, |g| \, \mathrm{d}x^1 \wedge \dots \wedge \widehat{\mathrm{d}x^i} \wedge \dots \wedge \mathrm{d}x^n \wedge \mathrm{d}v^1 \wedge \dots \wedge \mathrm{d}v^n \\
&\quad + \sum_{i=1}^n (-\Gamma^i{}_{jk} v^j v^k \, |g|) \, \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n \wedge \mathrm{d}v^1 \wedge \dots \wedge \widehat{\mathrm{d}v^i} \wedge \dots \wedge \mathrm{d}v^n,
\end{aligned}
\tag{141}
$$

where $\widehat{\mathrm{d}x^i}$ and $\widehat{\mathrm{d}v^i}$ indicate that one-forms $\mathrm{d}x^i$ and $\mathrm{d}v^i$ are omitted from the wedge product. From (141),it follows that

$$\begin{aligned} d(i_X \mathrm{d}\Sigma) &= \sum_{i=1}^{n}(-1)^{i-1}\partial_{x^i}(v^i\,|g|)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}v^1\wedge\cdots\wedge\mathrm{d}v^n \\ &\quad+\sum_{i=1}^{n}(-1)^{n+i-1}\partial_{v^i}(-\Gamma^i{}_{jk}v^jv^k\,|g|)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}v^1\wedge\cdots\wedge\mathrm{d}v^n \\ &= \sum_{i=1}^{n}(-1)^{i-1}(\partial_{x^i}(v^i\,|g|)+(-1)^{n+1}\partial_{v^i}(\Gamma^i{}_{jk}v^jv^k\,|g|)) \\ &\qquad\times\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}v^1\wedge\cdots\wedge\mathrm{d}v^n. \end{aligned} \tag{142}$$

Similarly, we see that

$$\begin{aligned} L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma}) &= \sum_{i=1}^{n}(-1)^{i-1}\partial_{x^i}(\overset{\alpha}{v}{}^i\,|\overset{\alpha}{g}|)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}\overset{\alpha}{v}{}^1\wedge\cdots\wedge\mathrm{d}\overset{\alpha}{v}{}^n \\ &\quad+\sum_{i=1}^{n}(-1)^{n+i-1}\partial_{\overset{\alpha}{v}{}^i}(-\overset{\alpha}{\Gamma}{}^i{}_{jk}\overset{\alpha}{v}{}^j\overset{\alpha}{v}{}^k\,|\overset{\alpha}{g}|)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}\overset{\alpha}{v}{}^1\wedge\cdots\wedge\mathrm{d}\overset{\alpha}{v}{}^n \\ &= \sum_{i=1}^{n}(-1)^{i-1}(\partial_{x^i}(\overset{\alpha}{v}{}^i\,|\overset{\alpha}{g}|)+(-1)^{n+1}\partial_{\overset{\alpha}{v}{}^i}(\overset{\alpha}{\Gamma}{}^i{}_{jk}\overset{\alpha}{v}{}^j\overset{\alpha}{v}{}^k\,|\overset{\alpha}{g}|)) \\ &\qquad\times\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}\overset{\alpha}{v}{}^1\wedge\cdots\wedge\mathrm{d}\overset{\alpha}{v}{}^n. \end{aligned} \tag{143}$$

Next, we pullback formulas (142) and (143) onto S_hM. We can compute

$$s^*\mathrm{d}v^j = \mathrm{d}(s^*v^j) = \mathrm{d}(w^j\,|w|_g^{-1}) = |w|_g^{-1}\,\mathrm{d}w^j + w^j\mathrm{d}(|w|_g^{-1}). \tag{144}$$

If we write

$$\mathrm{d}(|w|_g^{-1}) = \mu_i\mathrm{d}x^i + \lambda_i\mathrm{d}w^i, \tag{145}$$

then

$$\mu_k = \mu_i\mathrm{d}x^i(\partial_{x^k}) = \mathrm{d}(|w|_g^{-1})(\partial_{x^k}) = \partial_{x^k}\,|w|_g^{-1} \quad \text{and} \quad \lambda_k = \partial_{w^k}\,|w|_g^{-1}. \tag{146}$$

Thus

$$s^*\mathrm{d}v^j = w^j(\partial_{x^k}\,|w|_g^{-1})\mathrm{d}x^k + (|w|_g^{-1}\,\delta_k^j + w^j\partial_{w^k}\,|w|_g^{-1})\mathrm{d}w^k. \tag{147}$$

Similarly, we get

$$s^*\mathrm{d}\overset{\alpha}{v}{}^j = w^j(\partial_{x^k}\,|w|_\alpha^{-1})\mathrm{d}x^k + (|w|_\alpha^{-1}\,\delta_k^j + w^j\partial_{w^k}\,|w|_\alpha^{-1})\mathrm{d}w^k. \tag{148}$$

Since s and $\overset{\alpha}{s}$ act identically on the base point x, we have

$$s^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \quad \text{and} \quad \overset{\alpha}{s}{}^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{149}$$

Using the fact that a wedge product vanishes whenever repetition appears,we get

$$\begin{aligned} &s^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}v^n) \\ &\quad = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge (|w|_g^{-1}\,\delta_k^j + w^1(\partial_{w^k}|w|_g^{-1}))\mathrm{d}w^k \wedge \cdots \\ &\qquad \cdots \wedge (|w|_g^{-1}\,\delta_k^n + w^n(\partial_{w^k}|w|_g^{-1}))\mathrm{d}w^k \\ &\quad = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \bigwedge_{j=1}^{n}(|w|_g^{-1}\,\delta_k^j + w^j(\partial_{w^k}|w|_g^{-1}))\mathrm{d}w^k. \end{aligned} \tag{150}$$

By a similar computation

$$\begin{aligned} &s^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}v^n) \\ &\quad = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \bigwedge_{j=1}^{n}(|w|_g^{-1}\,\delta_k^j + w^j(\partial_{w^k}|w|_\alpha^{-1}))\mathrm{d}w^k. \end{aligned} \tag{151}$$

To complete formulas for the pullback of (142) and (143) we use the facts that $s^* = s_*^{-1}$ and $\overset{\alpha}{s}{}^* = \overset{\alpha}{s}{}_*^{-1}$ to compute

$$s^*\partial_{x^i} = \partial_{x^i} + (\partial_{x^i} w^j)\partial_{w^j} \quad \text{and} \quad \overset{\alpha}{s}{}^*\partial_{x^i} = \partial_{x^i} + (\partial_{x^i} w^j)\partial_{w^j} \tag{152}$$

as well as

$$s^*\partial_{v^i} = (\partial_{v^j} w^j)\partial_{w^j} \quad \text{and} \quad \overset{\alpha}{s}{}^*\partial_{\overset{\alpha}{v}{}^i} = (\partial_{\overset{\alpha}{v}{}^j} w^j)\partial_{w^j}. \tag{153}$$

Thus we get

$$s^*(\partial_{x^i} v^i\,|g|) = \partial_{x^i}(w^i\,|w|_g^{-1}\,|g|) + (\partial_{x^i} w^j)(\partial_{w^j}(w^i\,|w|_g^{-1}\,|g|)), \tag{154}$$

$$\overset{\alpha}{s}{}^*(\partial_{x^i} v^i\,|\overset{\alpha}{g}|) = \partial_{x^i}(w^i\,|w|_\alpha^{-1}\,|\overset{\alpha}{g}|) + (\partial_{x^i} w^j)(\partial_{w^j}(w^i\,|w|_\alpha^{-1}\,|\overset{\alpha}{g}|)), \tag{155}$$

and

$$s^*\partial_{v^i}(\Gamma^i{}_{jk}v^j v^k\,|g|) = \Gamma^i{}_{jk}\,|g|\,(\partial_{v^i} w^l)\partial_{w^l}(w^i\,|w|_g^{-1}\,w^k\,|w|_g^{-1}), \tag{156}$$

$$\overset{\alpha}{s}{}^*\partial_{\overset{\alpha}{v}{}^i}(\overset{\alpha}{\Gamma}{}^i{}_{jk}\overset{\alpha}{v}{}^j\overset{\alpha}{v}{}^k\,|\overset{\alpha}{g}|) = \overset{\alpha}{\Gamma}{}^i{}_{jk}\,|\overset{\alpha}{g}|\,(\partial_{\overset{\alpha}{v}{}^i} w^l)\partial_{w^l}(w^i\,|w|_\alpha^{-1}\,w^k\,|w|_\alpha^{-1}). \tag{157}$$

The formulas we get for the pullbacks of $L_X(\mathrm{d}\Sigma)$ along s and of $L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma})$ along $\overset{\alpha}{s}$ are

$$\begin{aligned} s^* L_X(\mathrm{d}\Sigma) = \sum_{i=1}^{n} (-1)^{i-1} \Big(& \partial_{x^i}(w^i\, |w|_g^{-1}\, |g|) + (\partial_{x^k} w^j)(\partial_{w^j}(w^k\, |w|_g^{-1}\, |g|)) \\ & + (-1)^{n+1} \Gamma^i{}_{jk}\, |g|\, (\partial_{v^m} w^l) \partial_{w^l}(w^m\, |w|_g^{-1}\, w^k\, |w|_g^{-1}) \Big) \\ & \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \bigwedge_{j=1}^{n} (|w|_g^{-1}\, \delta_k^j + w^j(\partial_{w^k} |w|_g^{-1}))\mathrm{d}w^k \end{aligned} \tag{158}$$

and

$$\begin{aligned} \overset{\alpha}{s}{}^* L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma}) = \sum_{i=1}^{n} (-1)^{i-1} \Big(& \partial_{x^i}(w^i\, |w|_\alpha^{-1}\, |\overset{\alpha}{g}|) + (\partial_{x^k} w^j)(\partial_{w^j}(w^k\, |w|_\alpha^{-1}\, |\overset{\alpha}{g}|)) \\ & + (-1)^{n+1} \overset{\alpha}{\Gamma}{}^i{}_{jk}\, |\overset{\alpha}{g}|\, (\partial_{\overset{\alpha}{v}{}^m} w^l) \partial_{w^l}(w^m\, |w|_\alpha^{-1}\, w^k\, |w|_\alpha^{-1}) \Big) \\ & \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \bigwedge_{j=1}^{n} (|w|_\alpha^{-1}\, \delta_k^j + w^j(\partial_{w^k} |w|_\alpha^{-1}))\mathrm{d}w^k. \end{aligned} \tag{159}$$

From formulas (158) and (159) we see that can conclude the equation (135) if the following holds:

$$\begin{aligned} & \partial_{x^i}(w^i\, |w|_\alpha^{-1}\, |\overset{\alpha}{g}|) \prod_{j\in S} (|w|_\alpha^{-1}\, \delta_k^j) \prod_{j\in S'} (w^j(\partial_{w^k} |w|_\alpha^{-1})) \\ & \quad \to \partial_{x^i}(w^i\, |w|_\alpha^{-1}\, |g|) \prod_{j\in S} (|w|_g^{-1}\, \delta_k^j) \prod_{j\in S'} (w^j(\partial_{w^k} |w|_g^{-1})), \end{aligned} \tag{160}$$

$$\begin{aligned} & (\partial_{x^i} w^j)(\partial_{w^j}(w^k\, |w|_\alpha^{-1}\, |\overset{\alpha}{g}|)) \prod_{j\in S} (|w|_\alpha^{-1}\, \delta_k^j) \prod_{j\in S'} (w^j(\partial_{w^k} |w|_\alpha^{-1})) \\ & \quad \to (\partial_{x^i} w^j)(\partial_{w^j}(w^k\, |w|_g^{-1}\, |g|)) \prod_{j\in S} (|w|_g^{-1}\, \delta_k^j) \prod_{j\in S'} (w^j(\partial_{w^k} |w|_g^{-1})), \end{aligned} \tag{161}$$

$$\begin{aligned} & \overset{\alpha}{\Gamma}{}^i{}_{jk}\, |\overset{\alpha}{g}|\, (\partial_{\overset{\alpha}{v}{}^m} w^l)(\partial_{w^l}(w^m\, |w|_\alpha^{-1}\, w^l\, |w|_\alpha^{-1})) \prod_{j\in S} (|w|_\alpha^{-1}\, \delta_k^j) \prod_{j\in S'} (w^j(\partial_{w^k} |w|_\alpha^{-1})) \\ & \to \Gamma^i{}_{jk}\, |g|\, (\partial_{v^m} w^l)(\partial_{w^l}(w^m\, |w|_g^{-1}\, w^l\, |w|_g^{-1})) \prod_{j\in S} (|w|_g^{-1}\, \delta_k^j) \prod_{j\in S'} (w^j(\partial_{w^k} |w|_g^{-1})) \end{aligned} \tag{162}$$

in $L^1(S_hM)$, where S and S' are any subsets of $\{1,\ldots,n\}$. We chose the approximating sequence $\left(\overset{\alpha}{g}\right)$ so that

$$\overset{\alpha}{g}_{jk} \to g_{jk} \quad \text{in } W_h^{1,\infty}(M) \quad \text{and} \quad \overset{\alpha}{\Gamma}{}^i{}_{jk} \to \Gamma^i{}_{jk} \quad \text{in } L_h^\infty(M). \tag{163}$$

From (163),we see that

$$\begin{aligned}
\partial_{x^i}(w^i\,|w|_\alpha^{-1}\,|\overset{\alpha}{g}|) &\to \partial_{x^i}(w^i\,|w|_g^{-1}\,|g|),\\
|w|_\alpha^{-1}\,\delta_k^j &\to |w|_g^{-1}\,\delta_k^j,\\
w^j(\partial_{w^k}\,|w|_\alpha^{-1})) &\to w^j(\partial_{w^k}\,|w|_g^{-1})),\\
\partial_{w^j}(w^k\,|w|_\alpha^{-1}\,|\overset{\alpha}{g}| &\to \partial_{w^j}(w^k\,|w|_g^{-1}\,|g|,\\
\overset{\alpha}{\Gamma}{}^i{}_{jk}\,|\overset{\alpha}{g}| &\to \Gamma^i{}_{jk}\,|g|,\\
\partial_{\overset{\alpha}{v}{}^m}\,w^l &\to \partial_{v^m}\,w^l,\\
\partial_{w^l}(w^m\,|w|_\alpha^{-1}\,w^l\,|w|_\alpha^{-1}) &\to \partial_{w^l}(w^m\,|w|_g^{-1}\,w^l\,|w|_g^{-1})
\end{aligned} \tag{164}$$

in $L^\infty(S_hM)$. Thus we can take products and we conclude that (160), (161) and (162) hold, which finishes the proof. □

References

1. Ainsworth, G.: The attenuated magnetic ray transform on surfaces. Inverse Probl. Imaging **7**(1), 27–46 (2013)
2. Anikonov, Y.E., Romanov, V.G.: On uniqueness of determination of a form of first degree by its integrals along geodesics. J. Inverse Ill-Posed Probl. **5**(6), 487–490 (1997)
3. Assylbekov, Y.M., Dairbekov, N.S.: The X-ray transform on a general family of curves on Finsler surfaces. J. Geom. Anal. **28**(2), 1428–1455 (2018)
4. Burago, D., Ivanov, S.: Boundary rigidity and filling volume minimality of metrics close to a flat one. Ann. Math. **2**(171), 1183–1211 (2010)
5. Croke, C.B.: Rigidity for surfaces of nonpositive curvature. Comment. Math. Helv. **65**(1), 150–169 (1990)
6. Croke, C.B.: Rigidity and the distance between boundary points. J. Differential Geom. **33**(2), 445–464 (1991)
7. Croke, C.B., Sharafutdinov, V.A.: Spectral rigidity of a compact negatively curved manifold. Topology **37**(6), 1265–1273 (1998)
8. Dairbekov, N.S., Sharafutdinov, V.A.: Some problems of integral geometry on Anosov manifolds. Ergodic Theory Dynam. Syst. **23**(1), 59–74 (2003)
9. Dairbekov, N.S., Sharafutdinov, V.A.: Conformal Killing symmetric tensor fields on Riemannian manifolds. Mat. Tr. **13**(1), 85–145 (2010)
10. Dairbekov, N.S., Paternain, G.P., Stefanov, P., Uhlmann, G.: The boundary rigidity problem in the presence of a magnetic field. Adv. Math. **216**(2), 535–609 (2007)
11. de Hoop, M.V., Ilmavirta, J.: Abel transforms with low regularity with applications to x-ray tomography on spherically symmetric manifolds. Inverse Problems **33**(12), 124003,36 (2017)
12. Finch, D., Uhlmann, G.: The x-ray transform for a non-abelian connection in two dimensions. Inverse Problems **17**(4), 695 (2001)
13. Guillarmou, C., Paternain, G.P., Salo, M., Uhlmann, G.: The X-ray transform for connections in negative curvature. Comm. Math. Phys. **343**(1), 83–127 (2016)

14. Guillarmou, C., Mazzucchelli, M., Tzou, L.: Boundary and lens rigidity for non-convex manifolds. Amer. J. Math. **143**(2), 533–575 (2021)
15. Hartman, P.: On the local uniqueness of geodesics. Amer. J. Math. **72**, 723–730 (1950)
16. Ilmavirta, J., Monard, F.: Integral geometry on manifolds with boundary and applications. In The Radon transform—the first 100 years and beyond, volume 22 of Radon Ser. Comput. Appl. Math., pages 43–113. Walter de Gruyter, Berlin, [2019] (2019)
17. Ilmavirta, J.: X-ray transforms in pseudo-Riemannian geometry. J. Geom. Anal. **28**(1), 606–626 (2018)
18. Ilmavirta, J., Kykkänen, A.: Pestov identities and X-ray tomography on manifolds of low regularity. Inverse Probl. Imaging **17**(6), 1301–1328 (2023)
19. Ilmavirta, J., Mönkkönen, K.: The geodesic ray transform on spherically symmetric reversible Finsler manifolds. J. Geom. Anal. **33**(4), 27 (2023)
20. Ilmavirta, J., Paternain, G.P.: Broken ray tensor tomography with one reflecting obstacle. Comm. Anal. Geom. **30**(6), 1269–1300 (2022)
21. Ilmavirta, J., Salo, M.: Broken ray transform on a Riemann surface with a convex obstacle. Comm. Anal. Geom. **24**(2), 379–408 (2016)
22. Jollivet, A.: On inverse scattering in electromagnetic field in classical relativistic mechanics at high energies. Asymptot. Anal. **55**(1–2), 103–123 (2007)
23. Jollivet, A.: On inverse problems in electromagnetic field in classical mechanics at fixed energy. J. Geom. Anal. **17**(2), 275–319 (2007)
24. Lassas, M., Sharafutdinov, V., Uhlmann, G.: Semiglobal boundary rigidity for Riemannian metrics. Math. Ann. **325**(4), 767–793 (2003)
25. Lee, J.M.: Introduction to smooth manifolds, volume 218 of Graduate Texts in Mathematics, 2nd edn. Springer, New York (2013)
26. Lehtonen, J., Railo, J., Salo, M.: Tensor tomography on Cartan-Hadamard manifolds. Inverse Problems **34**(4), 044004, 27 (2018)
27. Lehtonen, J.: The geodesic ray transform on two-dimensional Cartan-Hadamard manifolds (2016). arXiv:1612.04800 [math.DG]
28. Merry, W., Paternain, G.: Lecture notes: Inverse problems in geometry and dynamics. https://www.dpmms.cam.ac.uk/~gpp24/ipgd(3).pdf (2011)
29. Monard, F., Nickl, R., Paternain, G.P.: Consistent inversion of noisy non-Abelian X-ray transforms. Comm. Pure Appl. Math. **74**(5), 1045–1099 (2021)
30. Muhometov, R.G.: The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry. Dokl. Akad. Nauk SSSR **232**(1), 32–35 (1977)
31. Muhometov, R.G.: On a problem of reconstructing Riemannian metrics. Sibirsk. Mat. Zh. **22**(3), 119–135 (1981)
32. Muhometov, R.G., Romanov, V.G.: On the problem of finding an isotropic Riemannian metric in an n-dimensional space. Dokl. Akad. Nauk SSSR **243**(1), 41–44 (1978)
33. Mukhometov, R.G.: Inverse kinematic problem of seismic on the plane. Mathematical Problems of Geophysics, Akad. Nauk. SSSR, Sibirsk. Otdel., Vychisl. Tsentr, Novosibirsk **6**, 243–252 (1975)
34. Mukhometov, R.G.: The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry (Russian). Dokl. Akad. Nauk SSSR **232**(1), 32–35 (1977)
35. Novikov, R.G.: Non-Abelian Radon transform and its applications. In The Radon transform—the first 100 years and beyond, volume 22 of Radon Ser. Comput. Appl. Math., pages 115–127. Walter de Gruyter, Berlin [2019] (2019)
36. Paternain, G.P., Salo, M., Uhlmann, G.: Geometric inverse problems—with emphasis on two dimensions, volume 204 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge. With a foreword by András Vasy (2023)
37. Paternain, G.P., Salo, M.: The non-Abelian X-ray transform on surfaces, to appear in J. Differ. Geom (2022)
38. Paternain, G.P.: Geodesic flows. Progress in Mathematics, vol. 180. Birkhäuser Boston Inc, Boston, MA (1999)
39. Paternain, G.P., Salo, M., Uhlmann, G.: The attenuated ray transform for connections and Higgs fields. Geom. Funct. Anal. **22**(5), 1460–1489 (2012)
40. Paternain, G.P., Salo, M., Uhlmann, G.: Tensor tomography on surfaces. Invent. Math. **193**(1), 229–247 (2013)
41. Paternain, G.P., Salo, M., Uhlmann, G.: Spectral rigidity and invariant distributions on Anosov surfaces. J. Differential Geom. **98**(1), 147–181 (2014)

42. Paternain, G.P., Salo, M., Uhlmann, G.: Tensor tomography: progress and challenges. Chinese Ann. Math. Ser. B **35**(3), 399–428 (2014)
43. Paternain, G.P., Salo, M., Uhlmann, G.: Invariant distributions, Beurling transforms and tensor tomography in higher dimensions. Math. Ann. **363**(1–2), 305–362 (2015)
44. Pestov, L.N., Sharafutdinov, V.A.: Integral geometry of tensor fields on a manifold of negative curvature. Sibirsk. Mat. Zh., 29(3):114–130 (1988)
45. Pestov, L., Uhlmann, G.: Two dimensional compact simple Riemannian manifolds are boundary distance rigid. Ann. Math. (2) **161**(2), 1093–1110 (2005)
46. Salo, M., Uhlmann, G.: The attenuated ray transform on simple surfaces. J. Differential Geom. **88**(1), 161–187 (2011)
47. Sämann, C., Steinbauer, R.: On geodesics in low regularity. J. Phys. Conf. Ser. **968**(012010), 14 (2018)
48. Sharafutdinov, V., Uhlmann, G.: On deformation boundary rigidity and spectral rigidity of Riemannian surfaces with no focal points. J. Differential Geom. **56**(1), 93–110 (2000)
49. Stefanov, P., Uhlmann, G., Vasy, A., Zhou, H.: Travel time tomography. Acta Math. Sin. (Engl. Ser.), **35**(6), 1085–1114 (2019)
50. Stefanov, P., Uhlmann, G.: Rigidity for metrics with the same lengths of geodesics. Math. Res. Lett. **5**(1–2), 83–96 (1998)
51. Stefanov, P., Uhlmann, G.: Boundary rigidity and stability for generic simple metrics. J. Amer. Math. Soc. **18**(4), 975–1003 (2005)
52. Uhlmann, G.: Inverse problems: seeing the unseen. Bull. Math. Sci. **4**(2), 209–279 (2014)

[C]

Microlocal analysis of the X-ray transform in non-smooth geometry

Joonas Ilmavirta, Antti Kykkänen and Kelvin Lam

Preprint (September 2023)
https://arxiv.org/abs/2309.12702

MICROLOCAL ANALYSIS OF THE X-RAY TRANSFORM IN NON-SMOOTH GEOMETRY

JOONAS ILMAVIRTA

Department of Mathematics and Statistics, University of Jyväskylä

ANTTI KYKKÄNEN

Department of Mathematics and Statistics, University of Jyväskylä

KELVIN LAM

Department of Mathematics, University of Washington

Abstract. We prove that the geodesic X-ray transform is injective on L^2 when the Riemannian metric is simple but the metric tensor is only finitely differentiable. The number of derivatives needed depends explicitly on dimension, and in dimension 2 we assume $g \in C^{10}$. Our proof is based on microlocal analysis of the normal operator: we establish ellipticity and a smoothing property in a suitable sense and then use a recent injectivity result on Lipschitz functions. When the metric tensor is C^k, the Schwartz kernel is not smooth but C^{k-2} off the diagonal, which makes standard smooth microlocal analysis inapplicable.

1. Introduction

We show that on a simple Riemannian manifold (M, g) where $g \in C^k$ for a finite and explicit k the geodesic X-ray transform is injective on L^2 (Theorem 3). We do this using a typical two-step approach, first showing that a function in the kernel of the transform is smoother than assumed a priori and then showing that injectivity holds for smooth functions. Both of the two steps of the proof have to be adapted to low regularity. The "smooth" injectivity (on Lipschitz functions) was established in [3], so it remains to prove that a function in the kernel of the X-ray transform has to be Lipschitz.

E-mail addresses: `joonas.ilmavirta@jyu.fi`, `antti.k.kykkanen@jyu.fi`, `klam0008@uw.edu`.

Date: September 25, 2023.

This regularity result (Theorem 1) is based on microlocal analysis of the normal operator. This normal operator is not a pseudodifferential operator in the usual sense because the "smooth" off-diagonal part of the Schwartz kernel is only C^{k-2}. Also, when the metric tensor is not infinitely differentiable, the Sobolev scale of H^s spaces only makes sense for a bounded range of indices s in both the positive and the negative direction. These two issues mean that the concepts of ellipticity, smoothing, and a parametrix need careful treatment.

1.1. **Main results.** We consider two operators: The X-ray transform I and its normal operator N. These are defined separately, and we only prove that $N = I^*I$ when acting on L^2 functions. Precise definitions of the operators and spaces we employ are given in section 2 below.

We prove two main theorems. Theorem 1 concerns functions in the kernel of the operator N and proves that they have, a priori, improved regularity. Theorem 3 can be compared to a recent result in [3]. We prove that the X-ray transform is injective on $L^2(M)$ while requiring more metric regularity whereas [3, Theorem 1] proves that the X-ray transform is injective only on Lipschitz functions.

Theorem 1. *Let (M, g) be a simple manifold, $n := \dim M \geq 2$ and $g \in C^k(M)$ for some $k \geq 7 + \frac{n}{2}$. Then if $f \in H^s_c(M)$ for some $s > -k + 6 + \frac{n}{2}$ and $Nf = 0$, we have $f \in H^r_c(M)$ for all $s < r < k - 6 - \frac{n}{2}$.*

Theorem 1 can be applied to geodesic X-ray tomography in low metric regularity assuming that the X-ray transform I acts on $L^2(M)$, since then $N = I^*I$ is in fact the normal operator for the X-ray transform I.

Proposition 2. *Let (M, g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 2$. Then $I^*I = N$ on $L^2(M)$.*

Theorem 3. *Let (M, g) be a simple manifold, $n := \dim M \geq 2$ and $g \in C^k(M)$ for some $k \geq 8+n$. Then the X-ray transform I is injective on $L^2(M)$.*

The proofs of the theorems rely on microlocal tools. We study the so-called normal operator $N = I^*I$ related to the X-ray transform I. We prove that N is a non-smooth elliptic operator and construct a principal parametrix with an error term smoothing of order $\tau \in (0, 1)$. The construction and its implications use a non-smooth microlocal calculus developed in [7] and, in particular, we use the non-smooth symbol and operator classes, continuous Sobolev mapping properties and a commutator theorem there introduced. The details are recalled in section 2.

1.2. **Related results.** The geodesic X-ray transform on a Riemannian manifold has been studied in a variety of contexts and with a variety of tools [11, 13, 5, 10]. The current article focus on the aspect of not studying the X-ray transform directly but via the related normal operator. This approach has seen plenty of applications in C^∞-smooth metric regularity.

In [12] it was proved that the normal operator on a simple Riemannian manifold is an elliptic pseudodifferential operator in the interior of the manifold — a result that is essential in their proof that all two dimensional simple Riemannian manifolds are boundary rigid. The normal operator has also played a role in later developments in boundary rigidity [14, 15]. Microlocal methods in relation to the normal operator are useful in geometries permitting conjugate points [16, 17, 2]. More recently, there has been interest in isomorphic mapping properties of the normal operator and its variants between suitably weighted function spaces [8, 9].

Microlocal analysis of the normal operator in the X-ray tomography is in non-smooth geometries virtually unexplored. However, injectivity for the X-ray transform of Lipschitz scalar and $C^{1,1}$ tensor fields on simple $C^{1,1}$ manifolds was proved in two recent articles [3, 4], and injectivity is known for the scalar transform on spherically symmetric $C^{1,1}$ manifolds satisfying the Herglotz condition [1].

The current article uses non-smooth microlocal methods. As references on pseudodifferential operators with symbols non-smooth in both variables we mention [6, 7] and as references to paradifferential methods we mention [19].

1.3. **Acknowledgements.** JI was supported by the Research Council of Finland (grant 351665). AK was supported by the Research Council of Finland (grant 351656) and by the Finnish Academy of Science and Letters. KL was supported by NSF. We thank John M. Lee, Gabriel P. Paternain, Mikko Salo, Hart F. Smith, and Gunther Uhlmann for discussions.

2. Preliminaries

In this section we introduce the geometric set-up, the function spaces, and the operators used throughout the article. We also recall the parts of the non-smooth calculus and theorems from [7] that are required for the proofs of our main results.

2.1. Simple manifolds.

In this section we recall the geometric set-up in which we study geodesic X-ray transforms. Since the Riemannian metrics we consider are not C^∞-smooth, we include the following definition for clarity.

Definition 4. Let k be an integer so that $k \geq 2$. Let M be a compact smooth manifold with a smooth boundary and equip M with a C^k smooth Riemannian metric g. We say that (M, g) is simple if M is C^k-diffeomorphic to the closed Euclidean unit ball in $\mathbb{R}^n$ and the following hold:

(1) The boundary ∂M is strictly convex in the sense of the second fundamental form.
(2) The manifold is non-trapping, i.e., all geodesics hit the boundary in a finite time.
(3) There are no conjugate points in M.

When the Riemannian metric g is C^∞-smooth, definition 4 is equivalent to any standard definition of a simple manifold.

Remark 5. Our analysis of the non-smooth operators is carried out on the closed Euclidean unit ball, which allows us to use smooth global coordinates on our manifold. This allows us to use smooth functions on the manifold without having to worry about limitations on regularity indices. However, we have to interpret our results in the original manifold via a C^k-diffeomorphism which restrict the meaningful range of any regularity indices (Hölder or Sobolev) to $[-k, k]$ in the up coming sections.

To prove Theorem 3 we will use [3, Theorem 1]. There the authors use a slightly different notion is simplicity, but definition 4 is equivalent to their definition for Riemannian metrics $g \in C^k(M)$ when $k \geq 10$ which holds in the case of Theorem 3. The proof of equivalence of definitions in [3, Theorem 2] carries over to our simple Riemannian metrics $g \in C^k(M)$ for $k \geq 10$ by the arguments given in [3] and since we assume that M is C^k-diffeomorphic to the closed unit ball in $\mathbb{R}^n$.

Since the conditions defining a simple manifold (M, g) with $g \in C^k(M)$ are open, there is a small open extension $M \subseteq U \subseteq \mathbb{R}^n$ and an extension $\tilde{g}$ of g so that $(\overline{U}, \tilde{g})$ is a simple manifold with $\tilde{g} \in C^k(\overline{U})$. For details on the existence of simple extension we refer the reader to [11, Proposition 3.8.7].

2.2. **Function spaces.** Our definition of a simple manifold includes global coordinates. Therefore no partitions of unity are needed and the definitions of some operators and function spaces are somewhat simplified.

Let (M, g) be a simple manifold where $g \in C^k(M)$ for some $k \geq 2$. Since M is C^k-diffeomorphic to the closed Euclidean unit ball $\overline{B} \subseteq \mathbb{R}^n$ we take $M = \overline{B}$ from now on and all computations are to be interpreted via a C^k-diffeomorphism as explained in remark 5.

We use smooth global coordinates $(x^1, \dots, x^n)$ in the definitions of our functions spaces. We use the Riemannian volume for dVol_g to define $L^2(M)$ in the standard way i.e. $L^2(M) = L^2(M, \mathrm{dVol}_g)$.

For $s > 0$ we denote by $H^s_c(M)$ the space of compactly supported functions in $H^s(M)$. For $s > 0$ we let $H^{-s}(M)$ be the continuous dual of $H^s(M)$ and $H^{-s}_c(M)$ be the subspace of compactly supported distributions.

Similarly, we define the Zygmund space $C^r_*(M)$ to be the space of continuous functions f on M whose zero extension to $\mathbb{R}^n$ is in $C^r_*(\mathbb{R}^n)$ and the norm of a such functions is its $C^r_*(\mathbb{R}^n)$-norm.

2.3. **Geodesic X-ray transforms.** Let (M, g) be a simple manifold where $g \in C^k(M)$ for some $k \geq 2$. For a given unit vector $v \in T_xM$ there is a unique geodesic $\gamma_{x,v}$ corresponding to the initial conditions $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$. Since the manifold is non-trapping, the geodesic $\gamma_{x,v}$ is defined on a maximal interval of existence $[-\tau_-(x, v), \tau_+(x, v)]$ where $\tau_\pm(x, v) \geq 0$ and we abbreviate $\tau := \tau_+$.

The X-ray transform If of a function $f \in L^2(M)$ is defined for all inwards pointing unit vectors $(x, v) \in \partial_{\text{in}} SM$ by the formula

$$If(x, v) := \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, \mathrm{d}t. \tag{1}$$

The backprojection I^*h of a function h on $L^2(\partial_{\text{in}}(SM))$ is defined for all $x \in M$ by the formula

$$I^*h(x) := \int_{S_xM} h(\phi_{-\tau(x,-v)}(x, v))\, \mathrm{d}S_x(v). \tag{2}$$

Finally, we define the operator N which we will call the normal operator and which will be the main focus of our study. The normal operator is defined on $L^2(M)$ by the formula

$$Nf(x) = 2 \int_{S_xM} \int_0^{\tau(x,v)} f(\gamma_{x,v}(t)))\, \mathrm{d}t\, \mathrm{d}S_x(v). \tag{3}$$

We will prove in proposition 2 that N agrees with the composition I^*I on $L^2(M)$, justifying calling it the normal operator.

2.4. **Non-smooth operators and symbols.** In this section we recall the basics of a non-smooth pseudodifferential calculus introduced in [7]. We rerecord the results that are relevant to the current work for the convenience of the reader.

Let $m \in \mathbb{R}$ and $r, L \in \mathbb{N}$ be given. Multi-indices in $\mathbb{N}^n$ are denoted by α and β. For all $\rho, \delta \in [0, 1]$ the symbol class $S^m_{\rho\delta}(r, L)$ consists of continuous functions $p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying the estimates

$$\left|\partial_\xi^\alpha p(x, \xi)\right| \leq C_\alpha (1 + |\xi|)^{m - \rho|\alpha|} \tag{4}$$

and

$$\left\|\partial_\xi^\alpha p(\,\cdot\,, \xi)\right\|_{C^r_*} \leq C_{\alpha r} (1 + |\xi|)^{m + r\delta - \rho|\alpha|} \tag{5}$$

for all $|\alpha| \leq L$.

Given a symbol $p \in S^m_{\rho\delta}(r, L)$ the corresponding operator $\mathrm{Op}(p)$ is defined by its action

$$\mathrm{Op}(p)f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi)\, \mathrm{d}\xi \tag{6}$$

on functions f in $L^2(\mathbb{R}^n)$. The identity operator Id is the operator corresponding to the constant symbol 1.

We end the preliminaries by isolating two useful results on the operators of class $\Psi^m(r, L)$. For the proofs of the lemmas we refer the reader to [7].

Lemma 6 ([7] Theorem 2.1.). *Let $p \in S^m_{\rho\delta}(r, L)$ and consider the operator $P \coloneqq \mathrm{Op}(p)$. Suppose that $\rho, \delta \in [0, 1]$ and $r, L > 0$ satisfy*

$$\delta \leq \rho, \quad L > \frac{n}{2}, \quad r > \frac{1-\rho}{1-\delta}\frac{n}{2}. \tag{7}$$

Then the operator $P\colon H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is bounded when

$$(1-\rho)\frac{n}{2} - (1-\delta)r < s < r. \tag{8}$$

Lemma 7 ([7] Theorem 3.5.). *Let $p \in S^{m_1}_{\rho_1\delta_1}(r, L)$ and $q \in S^{m_2}_{\delta_2\rho_2}(r, L + \frac{n}{2} + 1)$ and suppose that $\delta_1 < \rho_2$ and $L > \frac{n}{2}$. Denote the corresponding operators by $P \coloneqq \mathrm{Op}(p)$ and $Q \coloneqq \mathrm{Op}(q)$. Let $\tau \in (0, 1]$ be such that $0 < \tau < r$. Define*

$$\delta \coloneqq \max\{\delta_1 + (\rho_1 - \delta_2)\tau, \delta_2\} \quad \text{and} \quad \rho \coloneqq \min\{\rho_1, \rho_2\}. \tag{9}$$

Assume that $\delta \leq \rho$ and in the case $\rho < 1$ suppose in addition that $r > \frac{1-\rho}{1-\delta}\frac{n}{2} + \tau$. Then the commutator

$$QP - \mathrm{Op}(qp)\colon H^{s+m_1+m_2-(\rho_1-\delta_2)\tau}(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \tag{10}$$

is bounded when

$$\max\{-m_2, 0\} + (1-\rho)\frac{n}{2} - (1-\delta)(r-\tau) < s < r - \max\{m_2, 0\}. \tag{11}$$

3. Parametrix construction for the normal operator

This section provides a detailed analysis of the operator N culminating in a leading order parametrix construction in the non-smooth symbol calculus presented in section 2.4. The parametrix construction is the main tool used in the proofs of our main theorems.

3.1. **The Schwartz kernel and the symbol.** The objective of this section is to study the operator N as a non-smooth elliptic pseudodifferential operator. We begin from the Schwartz kernel of the operator and analyse its symbol by dissecting it into manageable parts. The end result containing the principal part of the symbol is presented in corollary 16.

Lemma 8. *Let (M, g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 2$. Let $a(x, y) = \det(\mathrm{d}\exp_x|_{\exp_x^{-1}(y)})^{-1}$. Then for all $f \in L^2(M)$ we have*

$$Nf(x) = 2\int_M a(x, y) d_g(x, y)^{1-n} f(y)\, \mathrm{dVol}_g(y). \tag{12}$$

Proof. The same formula is derived in [11, Lemma 8.1.10] when $g \in C^\infty(M)$. The computation works when $g \in C^k(M)$ with $k \geq 2$. □

The Schwartz kernel of the operator N is

$$K(x,y) = 2a(x,y)d_g(x,y)^{1-n} \tag{13}$$

on $M \times M$. We will construct leading order parametrices for operators on $\mathbb{R}^n$ related to the Schwartz kernels of the form

$$\tilde{K}(x,y) := \psi(x)2a(x,y)d_g(x,y)^{1-n}\det(g(y))^{\frac{1}{2}}\phi(y) \tag{14}$$

where ψ and ϕ are suitable cut-off functions in $\mathbb{R}^n$.

Consider $\Omega \subseteq M$ and consider $f \in H^s_c(M)$ so that $\operatorname{supp} f \subseteq \Omega$. We can choose a cut-off function $\phi \in C^\infty_c(M)$ so that $\phi f = f$ on M. Then if $\psi \in C^\infty_c(M)$ is to that $\psi = 1$ on Ω we have for all $x \in \Omega$ that

$$\begin{aligned} Nf(x) &= \int_{\mathbb{R}^n} \psi(x)K(x,y)\det(g(y))^{\frac{1}{2}}\phi(y)f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \tilde{K}(x,y)f(y)\,\mathrm{d}y. \end{aligned} \tag{15}$$

We let $\tilde{N}$ be the operator corresponding to the kernel $\tilde{K}$. Then $Nf(x) = \tilde{N}f(x)$ on Ω which shows that it is enough to only consider operators with kernel of the form (14). For the details see the proof of Theorem 1 in section 4. From now on we let $N = \tilde{N}$ to avoid cluttered notation and we keep the cut-off functions ψ and ϕ fixed for the remainder of this section.

We will prove that $N \in \Psi^{-1}(k-s, s-4)$ for all $s \in \mathbb{N}$ with $4 \leq s \leq k$. This is accomplished by studying the operator in the global coordinates of the Euclidean unit ball and by computing the symbol of the operator. By [11, Lemma 8.1.12] we can write in the coordinates that

$$\tilde{K}(x,y) = \psi(x)\frac{2a(x,y)\det(g(y))^{1/2}}{[G_{jk}(x,y)(x-y)^j(x-y)^k]^{\frac{n-1}{2}}}\phi(y) \tag{16}$$

for some functions G_{jk} with $G_{jk}(x,x) = g_{jk}(x)$.

Lemma 9. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 3$. Then $\tilde{K} \in C^{k-2}(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta)$ where $\Delta := \{(x,x) : x \in \mathbb{R}^n\}$ is the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. The kernel $\tilde{K}$ can be expressed in the form

$$\tilde{K}(x,y) = \psi(x)2a(x,y)d_g(x,y)^{1-n}\det(g(y))^{\frac{1}{2}}\phi(y). \tag{17}$$

By standard ODE theory the geodesic flow has C^{k-1} smooth initial value dependence when $g \in C^k(M)$, and thus the exponential function is also C^k. It follows that $a \in C^{k-2}(M \times M)$. In addition, since $d_g(x, \exp_x(v)) = |v|_g$ for $(x,v) \in TM$ it follows that $d_g(x,y) \in C^{k-1}(M \times M \setminus \Delta)$. Finally, since the determinant term in (17) is C^k we see that $\tilde{K}$ is C^{k-2} off diagonal as claimed. □

By denoting

$$k(x,z) \coloneqq \tilde{K}(x, x-z) \tag{18}$$

and letting

$$a(x,\xi) \coloneqq \int_{\mathbb{R}^n} e^{-iz\cdot\xi} k(x,z)\,\mathrm{d}z \tag{19}$$

the normal operator on $L^2(M)$ can be brought to the form

$$Nf(x) = \int_{R^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi. \tag{20}$$

The following lemma is a finite regularity adaptation of the classical result [18, Chapter VI.7.4].

Lemma 10. *Let $m < 0$ and suppose that $\kappa \in C_c^l(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ where $l \in \mathbb{N}$ satisfies estimates*

$$\left|\partial_x^\alpha \partial_z^\beta \kappa(x,z)\right| \le C_{\alpha\beta} \left|z\right|^{-m-n-|\beta|}, \quad z \neq 0, \tag{21}$$

for $|\alpha| + |\beta| \le l$. Then the function on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$b(x,\xi) \coloneqq \int_{\mathbb{R}^n} e^{-iz\cdot\xi} \kappa(x,z)\,\mathrm{d}z \tag{22}$$

is a symbol in the class $S^m(l-s, s-2)$ for all $s \in \mathbb{N}$ with $2 \le s \le l$.

Proof. Since by assumption

$$\left|\partial_z^\beta \kappa(x,z)\right| \le C_\beta \left|z\right|^{-m-n-|\beta|}, \quad z \neq 0, \tag{23}$$

holds for all $|\beta| \le l$ and since κ is compactly supported, it can be shown by using [18, VI 4.5.] as in [18, VI 7.4.] that b is a continuous function on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$\left|\partial_\xi^\beta b(x,\xi)\right| \le C_\beta (1 + |\xi|)^{m-|\beta|} \tag{24}$$

for all $|\beta| \le l-2$, which is the first estimate we set out to prove.

Then let $s \in [2, l]$ be an integer. Since κ is compactly supported we have

$$\partial_x^\alpha b(x,\xi) = \int_{\mathbb{R}^n} e^{-iz\cdot\xi} \partial_x^\alpha \kappa(x,z)\,\mathrm{d}z. \tag{25}$$

Let us denote $\kappa_\alpha(x,z) = \partial_x^\alpha \kappa(x,z)$. Then it holds that

$$\left|\partial_z^\beta \kappa_\alpha(x,z)\right| \le C_{\alpha\beta} \left|z\right|^{-m-n-|\beta|}, \quad z \neq 0, \tag{26}$$

for all $|\beta| \le l - |\alpha|$. Therefore by a similar application of [18, VI 4.5] we have

$$\left|\partial_\xi^\beta \partial_x^\alpha b(x,\xi)\right| \le C_{\alpha\beta} (1 + |\xi|)^{m-|\beta|} \tag{27}$$

for all $|\alpha| + |\beta| \le l-2$. Then it follows that

$$\left\|\partial_\xi^\beta b(\,\cdot\,,\xi)\right\|_{C_*^{l-s}} \le \left\|\partial_\xi^\beta b(\,\cdot\,,\xi)\right\|_{C^{l-s}} \le C_{\alpha\beta} (1 + |\xi|)^{m-|\beta|}, \tag{28}$$

which uses compactness of the support of κ again. By estimates (24) and (28) we have shown $b \in S^m(l-s, s-2)$ for all integers $s \in [2, l]$ as claimed. □

Lemma 11. *Let (M, g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 5$. Then the function a defined by (19) belongs to $S^{-1}(k-s, s-4)$ for all $s \in [4, k]$ with $4 \leq s \leq k$.*

Proof. We write the kernel in (18) in the form

$$k(x,z) = |z|^{1-n}\,\psi(x)\frac{2a(x,x-z)\det(g(x-z))^{\frac{1}{2}}}{[G_{jk}(x,x-z)\frac{z^j}{|z|}\frac{z^k}{|z|}]^{\frac{n-1}{2}}}\phi(x-z) \tag{29}$$

and denote

$$k_0(x,z) = \psi(x)\frac{2a(x,x-z)\det(g(x-z))^{\frac{1}{2}}}{[G_{jk}(x,x-z)\frac{z^j}{|z|}\frac{z^k}{|z|}]^{\frac{n-1}{2}}}\phi(x-z). \tag{30}$$

Then $k_0(x,z)$ is C^{k-2} for $z \neq 0$ and its derivatives $\partial_x^\alpha \partial_z^\beta k_0(x,z)$, $z \neq 0$, are bounded for all $|\alpha| + |\beta| \leq k-2$ since $k_0(x,z)$ is compactly supported. Thus k satisfies estimates

$$\left|\partial_x^\alpha \partial_z^\beta k(x,z)\right| \leq C_{\alpha\beta}\,|z|^{1-n-|\beta|}, \quad z \neq 0, \tag{31}$$

for all $|\alpha| + |\beta| \leq k-4$ and the claim follows from lemma 10. □

Remark 12. The symbol $a(x,\xi)$ is smooth in ξ but our argument does not prove that $a(x,\xi)$ satisfies the estimates of the class $S^m(k, L)$ for all orders L of ξ-derivatives. Thus we cannot use paradifferential calculus to study N.

Lemma 11 shows that $N \in \Psi^{-1}(k-s, s-4)$ for all $s \in \mathbb{N}$ with $4 \leq s \leq k$ when the Riemannian metric is in $C^k(M)$ when $k \geq 5$. The rest of this section is devoted to computing the principal symbol of the normal operator. We start by writing the kernel k as

$$k(x,z) = |z|^{1-n}\,h\left(x, z, \frac{z}{|z|}\right) \tag{32}$$

where h is a function on $\mathbb{R}^n \times [0,\infty) \times S^{n-1}$ defined by

$$h(x,r,\omega) = \psi(x)\frac{2a(x,x-r\omega)\det(g(x-r\omega))^{\frac{1}{2}}}{[G_{jk}(x,x-r\omega)\omega^j\omega^k]^{\frac{n-1}{2}}}\phi(x-r\omega). \tag{33}$$

Since $G_{jk}(x,x-r\omega)\omega^j\omega^k$ is non-vanishing we see that $h \in C^{k-2}(\mathbb{R}^n \times [0,\infty) \times S^{n-1})$. By the Fundamental theorem of calculus

$$h(x,r,\omega) = h(x,0,\omega) + r\int_0^1 \partial_r h(x,rt,\omega)\,\mathrm{d}t \tag{34}$$

and we can decompose $k(x,z) = k_{-1}(x,z) + r(x,z)$ where

$$k_{-1}(x,z) := |z|^{1-n}\,h\left(x, 0, \frac{z}{|z|}\right) \tag{35}$$

and

$$r(x,z) \coloneqq |z|^{2-n} \int_0^1 \partial_r h\left(x, |z|\, t, \frac{z}{|z|}\right) \mathrm{d}t. \tag{36}$$

Since $k(x,z)$ is compactly supported in z we can choose a cut-off function $\chi(z)$ so that $0 \le \chi \le 1$ and $\chi = 1$ near the origin so that

$$k(x,z) = \chi(z)k(x,z) = \chi(z)k_{-1}(x,z) + \chi(z)r(x,z). \tag{37}$$

Now the full symbol of N is decomposed as

$$\begin{aligned} a(x,\xi) &= \mathcal{F}(\chi(\,\cdot\,)k_{-1}(x,\,\cdot\,))(\xi) + \mathcal{F}(\chi(\,\cdot\,)r(x,\,\cdot\,))(\xi) \\ &=: a_{-1}(x,\xi) + c(x,\xi). \end{aligned} \tag{38}$$

Lemma 13. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \ge 5$. Then $a_{-1} \in S^{-1}(k-s, s-4)$ for all $s \in \mathbb{N}$ with $4 \le s \le k$ and $c \in S^{-2}(k-s, s-5)$ for all $s \in \mathbb{N}$ with $5 \le s \le k$.*

Proof. Since $h \in C^{k-2}(\mathbb{R}^n \times [0,\infty) \times S^{n-1})$ is compactly supported in x and r and S^{n-1} is compact, we can extend h to a compactly supported function on $\mathbb{R}^n \times [0,\infty) \times S^{n-1}$. Thus $\partial_x^\alpha \partial_r^l \partial_\omega^\beta h(x,r,\omega)$ is continuous and compactly supported for all $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^{n-1}$ and $l \in \mathbb{N}$ for which we have $|\alpha| + l + |\beta| \le k-2$.

First, we prove the claim about the Fourier transform c of the remainder. Since derivatives of h are continuous and compactly supported, a simple computation using the chain rule shows that

$$\left|\partial_x^\alpha \partial_{z^j} \partial_r h\left(x, |z|\, t, \frac{z}{|z|}\right)\right| \le C\, |z|^{-1} \tag{39}$$

near $z = 0$ and for all $t \in [0,1]$ when $|\alpha| + 2 \le k-2$. Therefore by iteration

$$\left|\partial_x^\alpha \partial_z^\beta \partial_r h\left(x, |z|\, t, \frac{z}{|z|}\right)\right| \le C_{\alpha\beta}\, |z|^{-|\beta|} \tag{40}$$

near $z = 0$ when $|\alpha| + |\beta| + 1 \le k-2$. The above estimate applied to the remainder term $r(x,z)$ yields

$$\left|\partial_x^\alpha \partial_z^\beta \int_0^1 \partial_r h\left(x, |z|\, t, \frac{z}{|z|}\right) \mathrm{d}t\right| \le C_{\alpha\beta}\, |z|^{-|\beta|} \tag{41}$$

near $z = 0$ when $|\alpha| + |\beta| + 1 \le k-2$, which implies that

$$\left|\partial_x^\alpha \partial_z^\beta (\chi(z) r(x,z))\right| \le C_{\alpha\beta}\, |z|^{2-n-|\beta|} \tag{42}$$

for all z and $|\alpha| + |\beta| \le k-3$ since the cut-off $\chi(z)$ implies that only have to derive the estimate near $z = 0$. It follows from lemma 10 that $c \in S^{-2}(k-s, s-5)$ for all $s \in \mathbb{N}$ with $5 \le s \le k$.

By a similar computation we see that k_{-1} satisfies estimates

$$\left|\partial_x^\alpha \partial_z^\beta (\chi(z) k_{-1}(x,z))\right| \le C_{\alpha\beta}\, |z|^{1-n-|\beta|} \tag{43}$$

for all z and $|\alpha| + |\beta| \le k-2$. Thus, again, by lemma 10 we have $a_{-1} \in S^{-1}(k-s, s-4)$ for all $s \in \mathbb{N}$ with $4 \le s \le k$, which finishes the proof. ∎

In the next section we construct a leading order parametrix for N. To this end we need to find a more explicit representation for a_{-1}. We write $\chi(z)k_{-1}(x,z) = k_{-1}(x,z) - (1-\chi(z))k_{-1}(x,z)$ and analyze the Fourier transforms of the parts separately.

Lemma 14. *For a dimensional constant C it holds that*

$$\int_{\mathbb{R}^n} e^{-iz\cdot\xi} k_{-1}(x,z)\,\mathrm{d}z = C\psi(x)\,|\xi|_{g(x)}^{-1}\,\phi(x). \tag{44}$$

Proof. The Fourier transform of

$$k_{-1}(x,z) = \psi(x)\frac{2\det(g(x))^{1/2}}{(g_{jk}(x)z^j z^k)^{\frac{n-1}{2}}}\phi(x) \tag{45}$$

in z is computed in [11, Chapter 8.1]. The only difference is regularity in x, which does not affect the computation. □

Lemma 15. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 3$. Let $b(x,\xi) := \mathcal{F}((1-\chi(\,\cdot\,))k_{-1}(x,\,\cdot\,))(\xi)$. Then $b \in C_x^{k-2}C_\xi^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$. Moreover, b has a singularity of type $|\xi|_g^{-1}$ at the origin, and satisfies $|b(x,\xi)| \leq C\,|\xi|^{2-k}$ when $|\xi|$ is large enough.*

Proof. The fact that $b(x,\xi)$ has a singularity of type $|\xi|_{g(x)}^{-1}$ at the origin follows from the fact that $b(x,\xi) = a(x,\xi) - C\psi(x)\,|\xi|_{g(x)}^{-1}\,\phi(x)$ near $\xi = 0$ and $a \in C_x^{k-2}C_\xi^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

Next, we prove the claim about the decay of b away from $\xi = 0$. Since $(1-\chi(z))k_{-1}(x,z) = 0$ for z near the origin and since

$$(1-\chi(z))k_{-1}(x,z) = (1-\chi(z))\psi(x)\frac{2\det(g(x))^{1/2}}{(g_{jk}(x)z^j z^k)^{\frac{n-1}{2}}}\phi(x) \tag{46}$$

for $z \neq 0$, we know that $(1-\chi)k_{-1}$ is in $C^{k-2}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ and compactly supported in x. As in the proof of lemma 13 we can use boundedness of the derivatives of $h(x,0,z\,|z|^{-1})$ to prove that

$$|\partial_z^\alpha k_{-1}(x,z)| = \left|\partial_z^\alpha\left((1-\chi(z))\,|z|^{1-n}\,h\left(x,0,\frac{z}{|z|}\right)\right)\right| \leq C_\alpha\,|z|^{-n-1} \tag{47}$$

for $2 \leq |\alpha| \leq k-2$ which proves that for a fixed x we have $\partial_z^\alpha k_{-1}(x,z) \in L^1(\mathbb{R}^n)$. Therefore by the Riemann–Lebesgue lemma we conclude that

$$\begin{aligned} &|\xi^\alpha \mathcal{F}((1-\chi(\,\cdot\,))k_{-1}(x,\,\cdot\,))(\xi)| \\ &\quad= |\mathcal{F}(\partial_z^\alpha((1-\chi(\,\cdot\,))k_{-1}(x,\,\cdot\,)))(\xi)| \\ &\quad\to 0 \end{aligned} \tag{48}$$

for all $2 \leq |\alpha| \leq k-2$ as $|\xi| \to \infty$. Thus since 48 holds for all $2 \leq |\alpha| \leq k-2$ we have $|b(x,\xi)| \leq C\,|\xi|^{2-k}$ for $|\xi|$ large enough. □

Lemmas 13, 14 and 15 together prove the following corollary.

Corollary 16. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 5$. Then $N \in \Psi^{-1}(k-s, s-4)$ for all $s \in \mathbb{N}$ with $4 \leq s \leq k$. The principal symbol of N is*

$$a_{-1}(x,\xi) = C\psi(x)\left|\xi\right|_g^{-1}\phi(x) - b(x,\xi) \in S^{-1}(k-s, s-4) \tag{49}$$

where b is as in lemma 15 and $s \in \mathbb{N}$ with $4 \leq s \leq k$, in particular this shows that N is elliptic of order -1 in the sense of principal symbol.

The function a_{-1} is a function on the whole cotangent bundle and thus b has to have a singularity of type $\left|\xi\right|_{g(x)}^{-1}$ at $\xi = 0$ to cancel out the singularity in $C\psi(x)\left|\xi\right|_g^{-1}\phi(x)$.

3.2. **Parametrix construction.** In this section we construct a leading order parametrix for the normal operator. The construction is based on a commutator result in [7]. We define $p(x,\xi) \coloneqq C^{-1}\zeta(\xi)\left|\xi\right|_{g(x)}$ for some $\zeta \in C^\infty(\mathbb{R}^n)$ so that $0 \leq \zeta \leq 1$, $\zeta = 0$ near $\xi = 0$ and $\zeta = 1$ for large ξ, and where C is the same dimensional constant as in lemma 14. We will prove that the operator corresponding to the symbol p which is in $S^1(k-s, N)$ for all $s \in \mathbb{N}$ with $4 \leq s \leq k$ and $N \in \mathbb{N}$ provides the parametrix to the leading order.

Lemma 17. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 7 + \frac{n}{2}$. Let $P = \mathrm{Op}(p)$. If $\tau \in (0,1]$ is fixed then the operator*

$$PN - \mathrm{Op}(pa)\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{50}$$

is continuous when $-(1-\tau)(k-5-\frac{n}{2}-\tau) < t < k-6-\frac{n}{2}$.

Proof. Choose $s \in \mathbb{N}$ so that $s \in (4+\frac{n}{2}, k-1)$ which is possible since $k \geq 7+\frac{n}{2}$. Let $L \coloneqq s-4$ and let $r \coloneqq k-s$. Then $L > \frac{n}{2}$ and $r > 1 \geq \tau$. By lemma 11 we have $N \in \Psi^{-1}(r,L)$ and also it holds that $P \in \Psi^1(r, L+1+\frac{n}{2})$, which means that we are in the setting of lemma 7. For δ and ρ as the lemma it holds that

$$\delta = \tau, \quad \rho = 1 \quad \text{and} \quad \delta < \rho. \tag{51}$$

Thus since $m_1 = -1$ and $m_2 = 1$ in the lemma the commutator

$$PN - \mathrm{Op}(pa)\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{52}$$

is continuous for

$$\max\{-1,0\} - (1-\tau)(r-\tau) < t < r - \max\{1,0\} \tag{53}$$

which simplifies to

$$-(1-\tau)(k-s-\tau) < t < k-s-1. \tag{54}$$

To have a non-empty range of indices t we must have

$$k - s > \frac{1+(1-\tau)\tau}{2-\tau} \tag{55}$$

which is satisfied since $s < k-1$ and by an elementary computation it holds that $\frac{1+(1-\tau)\tau}{2-\tau} \leq 1$ for all $\tau \in (0,1]$.

Finally to conclude the proof we note that if

$$-(1-\tau)(k-5-\frac{n}{2}-\tau) < t < k-6-\frac{n}{2} \tag{56}$$

there is $s_t \in \mathbb{N}$ so that $s_t \in [4+\frac{n}{2}, k-1)$ and

$$-(1-\tau)(k-s_t-\tau) < t < k-s_t-1 \tag{57}$$

since $k \geq 7+\frac{n}{2}$ and thus the operator $PN - \mathrm{Op}(pa)\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$ is continuous as claimed. □

Lemma 18. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 7+\frac{n}{2}$. Then $\mathrm{Op}(pa_{-1}) = \mathrm{Id} + R_1$ where Id is an operator acting as the identity on elements in $H^{t+2-k}(\mathbb{R}^n)$ which are supported in the set where $\psi = 1 = \phi$ and the remainder*

$$R_1\colon H^{t+2-k}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{58}$$

is continuous when $-k+2 < t < k-2$.

Proof. By corollary 16 the principal symbol a_{-1} of N can be decomposed as

$$a_{-1}(x,\xi) = C\psi(x)\,|\xi|_{g(x)}^{-1}\,\phi(x) - b(x,\xi) \tag{59}$$

where $b(x,\xi)$ is in $C_x^{k-2}C_\xi^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ which is compactly supported in x and decays faster than $|\xi|^{2-k}$ in ξ. Therefore

$$\begin{aligned} p(x,\xi)a_{-1}(x,\xi) &= \zeta(\xi)\psi(x)\phi(x) - C^{-1}\zeta(\xi)b(x,\xi) \\ &= \psi(x)\phi(x) - (1-\zeta(\xi))\psi(x)\phi(x) - C^{-1}\zeta(\xi)b(x,\xi). \end{aligned} \tag{60}$$

Since $(1-\zeta(\xi))\psi(x)\phi(x)$ is smooth and compactly supported, it decays faster than $|\xi|^{-l}$ for any $l \in \mathbb{N}$. Since $\psi(x)\phi(x)$ equals to 1 on in the set where $\psi = 1 = \phi$ the corresponding operator acts as the identity on functions in $H^{t+2-k}(\mathbb{R}^n)$ which are supported in this set. Also, by lemma 15 the function $\zeta(\xi)b(x,\xi)$ decays faster than $|\xi|^{2-k}$. Therefore, since the support in x is compact and $b \in C_x^{k-2}C_\xi^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ and $\zeta(\xi) = 0$ near $\xi = 0$ it follows from the definitions that

$$\tilde{b}(x,\xi) \coloneqq -(1-\zeta(\xi))\psi(x)\phi(x) - C^{-1}\zeta(\xi)b(x,\xi) \tag{61}$$

is a symbol in the class $S^{2-k}(k-2, 1+\lfloor\frac{n}{2}\rfloor)$. Therefore by lemma 6 that $\mathrm{Op}(\tilde{b})\colon H^{t+2-k}(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$ for all $-k+2 < t < k-2$ since $1+\lfloor\frac{n}{2}\rfloor > \frac{n}{2}$, which proves the claim. □

Lemma 19. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 7+\frac{n}{2}$. Then the operator*

$$\mathrm{Op}(pc)\colon H^{t-1}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{62}$$

is continuous when $-k+6+\frac{n}{2} < t < k-6-\frac{n}{2}$.

Proof. Let $s \in \mathbb{N}$ be so that $s \in (5 + \frac{n}{2}, k)$. Since p is in $S^1(k - s, s-5)$ and c is in $S^{-2}(k-s, s-5)$ by lemma 13 the product pc is in $S^{-1}(k-s, s-5)$. Furthermore, since $s - 5 > \frac{n}{2}$ it follows from lemma 6 that $\operatorname{Op}(pc)$ continuously maps from $H^{t-1}(\mathbb{R}^n)$ to $H^t(\mathbb{R}^n)$ for all $-k+s < t < k-s$. To see that the continuous mapping property holds for all $-k+6+\frac{n}{2} < t < k-6-\frac{n}{2}$, we note that given any such t we can choose any $s_t \in \mathbb{N}$ so that $s_t \in (5+\frac{n}{2}, k-t)$ when $t \geq 0$ or $s_t \in (5+\frac{n}{2}, k+t)$ when $t < 0$ and it holds that $s_t \in \mathbb{N}$ with $s_t \in (5+\frac{n}{2}, k)$ and $-k+s_t < t < k-s_t$. This finishes the proof. □

Lemma 20. *Let (M, g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 7 + \frac{n}{2}$. Let $P = \operatorname{Op}(p)$. Then there is $\varepsilon > 0$ so that $PN = \operatorname{Id} + R$ where* Id *is an operator acting as the identity on elements in $H^{t-\tau}(\mathbb{R}^n)$ which are supported in the set where $\psi = 1 = \phi$ and the remainder*

$$R\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{63}$$

is continuous whenever $0 < \tau \leq \varepsilon$ and

$$-k+6+\frac{n}{2} < t < k-6-\frac{n}{2}. \tag{64}$$

Proof. By lemma 13 we may write

$$\begin{aligned} PN &= \operatorname{Op}(pa) + (PN - \operatorname{Op}(pa)) \\ &= \operatorname{Op}(pa_{-1}) + (PN - \operatorname{Op}(pa)) + \operatorname{Op}(pc). \end{aligned} \tag{65}$$

Let $\tau \in (0, 1]$. Then by lemma 17 we have that

$$PN - \operatorname{Op}(pa)\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{66}$$

is continuous for $-(1-\tau)(k-5-\frac{n}{2}-\tau) < t < k-6-\frac{n}{2}$. By lemmas 18 and 19 the operator $\operatorname{Op}(pa_{-1})$ is the identity up to an operator R_1 that is smoothing by 2 degrees and $\operatorname{Op}(pc)$ is smoothing by 1 degree, and therefore R_1 and $\operatorname{Op}(pc)$ are also smoothing by τ degrees. More precisely, $\operatorname{Op}(pa_{-1}) = \operatorname{Id} + R_1$, and we have that

$$R_1\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{67}$$

is continuous for $-k+2 < t < k-2$ and

$$\operatorname{Op}(pc)\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{68}$$

is continuous for $-k+6+\frac{n}{2} < t < k-6-\frac{n}{2}$. Letting R be the sum of the operators in (66), (67) and (68) we find that $PN = \operatorname{Id} + R$. Now suppose that τ is close enough to zero. Then the remainder continuously maps $H^{t-\tau}(\mathbb{R}^n)$ to $H^t(\mathbb{R}^n)$ for

$$-k+6+\frac{n}{2} < t < k-6-\frac{n}{2} \tag{69}$$

since $k-6-\frac{n}{2}$ is the smallest among the upper bound requirements and when τ is close to zero $-k+6+\frac{n}{2}$ is the largest of the lower bound. This proves the claimed identity and the mapping properties. □

4. Proofs of main theorems

In the last section we show that the parametrix construction in lemma 20 in combination with the recent result [3, Theorem 1] can be used to prove our main results.

Proof of theorem 1. Let $f \in H^s_c(M)$ for some $s > -k+6+\frac{n}{2}$ and assume that $Nf = 0$. Let $\operatorname{supp} f \subseteq \Omega$. There is a cut-off function $\phi \in C^\infty_c(M)$ so that $\phi f = f$ and moreover there is a cut-off $\psi \in C^\infty_c(M)$ with $\psi = 1$ on Ω so that $Nf(x) = (\psi N \phi) f(x) = 0$ for all $x \in M$. The operator $\psi N \phi$ has Schwartz kernel of the form (14) so by lemma 20 there is an operator P and $\varepsilon > 0$ so that $P(\psi N \phi) = \mathrm{Id} + R$ where Id acts as the identity on elements in $H^t(\mathbb{R}^n)$ with support in Ω and $R\colon H^t_c(M) \to H^{t+\tau}(\mathbb{R}^n)$ is continuous for $\tau \in (0, \varepsilon]$ and

$$-k+6+\frac{n}{2}-\tau < t < k-6-\frac{n}{2}-\tau. \tag{70}$$

We may choose τ so small that $s > -k+6+\frac{n}{2}-\tau$. Then $f \in H^s_c(M)$ and

$$\phi f = P(\psi N \phi) f - Rf = -Rf. \tag{71}$$

Thus $\phi f \in H^{t+\tau}(\mathbb{R}^n)$ and therefore $f \in H^{t+\tau}_c(M)$.

Then let $s < r < k-6-\frac{n}{2}$. By possibly choosing τ to be even smaller we may assume that there is $m \in \mathbb{N}$ so that $r < s+m\tau < k-6-\frac{n}{2}-\tau$. Then by iterating m times the argument in the previous paragraph we see that $f \in H^{s+m\tau}_c(M) \subseteq H^r_c(M)$ as claimed in the theorem. □

Proof of proposition 2. The composition of I and I^* was computed in [11, Lemma 8.1.5] for $g \in C^\infty(M)$. The same computation works for $g \in C^k(M)$ when $k \geq 2$. □

Proof of theorem 3. Let $(\tilde{M}, \tilde{g})$ be a simple extension of (M, g) and let $\tilde{I}$ be the X-ray transform of $(\tilde{M}, \tilde{g})$. Suppose that $f \in L^2(M)$ and $If = 0$. Then zero extension of f to $\tilde{M}$ still denoted by f satisfies $\tilde{I} f = 0$. Therefore $\tilde{N} f = \tilde{I}^* \tilde{I} f = 0$ by proposition 2 where $\tilde{N}$ and $\tilde{I}^*$ are the operators on $\tilde{M}$ defined by (2) and (1) with all objects replaced by corresponding objects of $(\tilde{M}, \tilde{g})$. Therefore by theorem 1 applied to the simple extension $(\tilde{M}, \tilde{g})$ implies that $f \in H^r_c(\tilde{M})$ for all $s < r < k-6+\frac{n}{2}$. Since $k \geq n+8$ there is some $r \in \mathbb{R}$ so that $\lceil 1+\frac{n}{2} \rceil < r < k-6+\frac{n}{2}$ and $f \in H^r_c(\tilde{M})$. Sobolev embedding yields

$$H^r_c(\tilde{M}) \subseteq W^{1,\infty}(\tilde{M}) = \mathrm{Lip}(\tilde{M}). \tag{72}$$

Thus $f \in \mathrm{Lip}(\tilde{M})$ and since f vanishes in $\tilde{M} \setminus M$ we have $f \in \mathrm{Lip}_0(M)$. We see that $f = 0$ since I is injective on $\mathrm{Lip}(M)$ by [3, Theorem 1.] which finishes the proof. □

References

[1] Maarten V. de Hoop and Joonas Ilmavirta. Abel transforms with low regularity with applications to x-ray tomography on spherically symmetric manifolds. *Inverse Problems*, 33(12):124003, 36, 2017.

[2] Sean Holman and Gunther Uhlmann. On the microlocal analysis of the geodesic X-ray transform with conjugate points. *J. Differential Geom.*, 108(3):459–494, 2018.

[3] Joonas Ilmavirta and Antti Kykkänen. Pestov identities and X-ray tomography on manifolds of low regularity. *Inverse Problems and Imaging*, December 2021. To appear.

[4] Joonas Ilmavirta and Antti Kykkänen. Tensor tomography on negatively curved manifolds of low regularity. March 2023.

[5] Joonas Ilmavirta and François Monard. Integral geometry on manifolds with boundary and applications. In *The Radon transform—the first 100 years and beyond*, volume 22 of *Radon Ser. Comput. Appl. Math.*, pages 43–113. Walter de Gruyter, Berlin, 2019.

[6] Ivan Ivec and Ivana Vojnović. Basic calculus of pseudodifferential operators with nonsmooth symbols. *J. Pseudo-Differ. Oper. Appl.*, 13(2):Paper No. 13, 34, 2022.

[7] J. Marschall. Nonregular pseudo-differential operators. *Z. Anal. Anwendungen*, 15(1):109–148, 1996.

[8] Rohit Kumar Mishra, François Monard, and Yuzhou Zou. The C^∞-isomorphism property for a class of singularly-weighted x-ray transforms. *Inverse Problems*, 39(2):Paper No. 024001, 24, 2023.

[9] François Monard, Richard Nickl, and Gabriel P. Paternain. Efficient nonparametric Bayesian inference for X-ray transforms. *Ann. Statist.*, 47(2):1113–1147, 2019.

[10] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Tensor tomography: progress and challenges. *Chinese Ann. Math. Ser. B*, 35(3):399–428, 2014.

[11] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. *Geometric inverse problems—with emphasis on two dimensions*, volume 204 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2023. With a foreword by András Vasy.

[12] Leonid Pestov and Gunther Uhlmann. Two dimensional compact simple Riemannian manifolds are boundary distance rigid. *Ann. of Math. (2)*, 161(2):1093–1110, 2005.

[13] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

[14] Plamen Stefanov and Gunther Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity. *Duke Math. J.*, 123(3):445–467, 2004.

[15] Plamen Stefanov and Gunther Uhlmann. Boundary rigidity and stability for generic simple metrics. *J. Amer. Math. Soc.*, 18(4):975–1003, 2005.

[16] Plamen Stefanov and Gunther Uhlmann. Integral geometry on tensor fields on a class of non-simple Riemannian manifolds. *Amer. J. Math.*, 130(1):239–268, 2008.

[17] Plamen Stefanov and Gunther Uhlmann. The geodesic X-ray transform with fold caustics. *Anal. PDE*, 5(2):219–260, 2012.

[18] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[19] Michael E. Taylor. *Partial differential equations. III*, volume 117 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1997. Nonlinear equations, Corrected reprint of the 1996 original.

[D]

Geometric inverse problems on gas giants

Maarten de Hoop, Joonas Ilmavirta, Antti Kykkänen and Rafe Mazzeo

Preprint (March 2024)
https://arxiv.org/abs/2403.05475

Department of Mathematics and Statistics, University of Jyväskylä
Email address: antti.k.kykkanen@jyu.fi

GEOMETRIC INVERSE PROBLEMS ON GAS GIANTS

MAARTEN V. DE HOOP, JOONAS ILMAVIRTA, ANTTI KYKKÄNEN, AND RAFE MAZZEO

ABSTRACT. On gas giant planets the speed of sound is isotropic and goes to zero at the surface. Geometrically, this corresponds to a Riemannian manifold whose metric tensor has a conformal blow-up near the boundary. The blow-up is tamer than in asymptotically hyperbolic geometry: the boundary is at a finite distance.

We study the differential geometry of such manifolds, especially the asymptotic behavior of geodesics near the boundary. We relate the geometry to the propagation of singularities of a hydrodynamic PDE and we give the basic properties of the Laplace–Beltrami operator. We solve two inverse problems, showing that the interior structure of a gas giant is uniquely determined by different types of boundary data.

1. INTRODUCTION

The study of propagation of acoustic waves on a gas giant planet leads to a Riemannian geometry that lies between asymptotically hyperbolic geometry and standard geometry with boundary. Some of the phenomena in this geometry are unlike those seen at either end; for example, constant curvature is not possible. We set out to study this geometry, the related analytic model, and inverse problems for determining the geometry from boundary measurements.

On a gas giant planet, unlike a rocky planet, the speed of sound goes to zero at the boundary. Geometrically, the power law decay rate of the speed of sound corresponds to a specific conformal blow-up rate of a Riemannian metric. This rate is slower than on asymptotically hyperbolic manifolds and the boundary is at a finite distance from interior points.

We study the basic geometry of gas giant Riemannian metrics, including properties of geodesics near the boundary (Propositions 10 and 11), the Hausdorff dimension of the boundary (Proposition 15), and discreteness of the spectrum of the Laplace–Beltrami operator (Proposition 29).

We solve two inverse problems for simple gas giants planets, proving that the metric is uniquely determined by its boundary distance data (Theorem 16) and that the geodesic X-ray transform is injective (Theorem 17).

A brief introduction to gas giant physics and how it leads to our geometric model is given in Section 1.2 below and a more detailed model is discussed in Section 5.

1.1. **Gas giant geometry.** Let M be an $(n+1)$-dimensional compact manifold with boundary. A metric g on M is called a gas giant metric of order $\alpha \in (0,2)$ if it can be written in the form

$$g = \frac{\overline{g}}{x^\alpha}$$

where $\overline{g}$ is a smooth non-degenerate metric on M, including up to its boundary. Observe that any such metric is incomplete. There are two limiting cases: when $\alpha \to 0$, g becomes the ordinary incomplete metric $\overline{g}$ on M, but when $\alpha \to 2$, then g converges to a complete asymptotically metric of a type often called conformally compact; cf. e.g. [Maz88]. We shall typically use a useful normal form. We may choose local coordinates (x, y) on M, where $x \geq 0$ and $x = 0$ but $dx \neq 0$ on ∂M and y restricts to a coordinate system on the boundary. There is an associated collar neighborhood of the boundary $\mathcal{U} \cong [0,1)_x \times \partial M$ and a smooth family of metrics h_x on ∂M such that

$$g = \frac{dx^2 + h_x}{x^\alpha}.$$

This is an analogue of the Graham-Lee normal form for conformally compact metrics. We establish this below in Section 2.1.

Our goals in this paper are to develop a number of facts about the geometry and analysis of this class of singular metrics. The first steps involve a series of calculations concerning the more elementary geometric considerations. We also consider the somewhat more subtle problem of understanding the asymptotics of escaping geodesics, and of the limiting dynamics of the geodesic flow. This leads to a first sort of inverse question: is there a way to characterize a gas giant metric intrinsically? More specifically, if (M°, g) is an open manifold with an incomplete metric, then is it possible to determine from this metric alone the compactification M, as a smooth manifold with boundary, the metric $\overline{g}$, the constant α and the boundary defining function x?

We consider some deeper inverse problems related to this class of metrics. In particular, we prove that the X-ray transform I_g on (M, g) is injective. In the final sections of this paper we also consider the Laplace–Beltrami operator Δ_g. We study its spectrum, mapping properties and whether it is essentially self-adjoint.

This paper is an initial foray into the analysis and geometry of gas giant metrics. Our aim here is to develop a number of fundamental results, either ab initio or as consequences of other related studies, which will then make it possible to consider some deeper inverse problems for this class of metrics. This paper splits into two not altogether distinct sections. In the first we develop a number of fundamental facts about the Riemannian geometry, including the behavior of geodesics, for gas giant metrics. Some properties are slightly simpler in the special case $\alpha = 1$ but we present all our results for all values $\alpha \in (0, 2)$. The second part of the paper studies various analytic properties of the scalar Laplace–Beltrami operators for such metrics. In between these two parts, we also prove some Pestov-type identities, which involve the vector field generating geodesic flow on the cosphere bundle, and use these to solve an inverse problem.

1.2. **Geometry from the equation of state.** As a leading order approximation, we take a gas giant planet to be a ball and assume all physical quantities to be invariant under rotations. Spherical symmetry is irrelevant for the geometric model introduced above, but it makes physics simpler.

Many celestial bodies are modelled to leading order as polytropes, a far more detailed discussion of which can be found in [Hor04]. The defining feature of a polytrope is the polytropic equation of state

$$p = K\rho^{1+1/n}$$

relating the pressure p and the density ρ via the polytropic constant K and the polytropic index n. The leading order approximation to a self-gravitating and spherically symmetric polytropic body can be written in terms of an auxiliary radial function $\theta(r)$ that satisfies $p(r) = p_0\theta(r)^{n+1}$ and $\rho(r) = \rho_0\theta(r)^n$. If the ambient dimension is d and the polytropic index satisfies $n > -1$, the function θ satisfies the Lane–Emden equation

$$\theta''(r) + (N-1)r^{-1}\theta'(r) + Cr^n = 0,$$

where $C > 0$. By rescaling the radial variable one can achieve $C = 1$.

At the surface of the body where $r = R$ we have $\theta(R) = 0$, and by virtue of being a positive (inside the body) solution to the second order Lane–Emden equation the function θ must satisfy $\theta'(R) < 0$.

The speed of sound can be computed as the (isentropic) derivative

$$c = \sqrt{\frac{\partial p}{\partial \rho}} = K'\rho^{1/2n} = K''p^{1/2(n+1)} = K'''\theta^{1/2}$$

for new constants K' and K'' and K'''. This means that the speed of sound is comparable to the square root of the distance to the surface, no matter the value of the polytropic index. For gas giants the polytropic index is usually taken to agree with the adiabatic index, which is $n = 5/3$ in the case of a monoatomic gas.

The polytropic model is only a leading order approximation and is not expected to hold perfectly. Bodies are also not perfectly rotationally symmetric due to rotation and inhomogeneities. Therefore we do not take the polytropic model as the truth, but as a guide to choosing a realistic mathematical model.

If e is the Euclidean Riemannian metric on a smooth domain $B \subset \mathbb{R}^n$, then the speed of sound $c(r)$ can be modeled by the conformally Euclidean Riemannian metric $g = c^{-2}e$. For a symmetric planet B would be a ball. If x is a boundary defining function for B (i.e. $x(z) > 0$ for $z \in B$, $x(z) = 0$ for $z \in \partial B$, and $\mathrm{d}x \neq 0$ at ∂B), the polytropic model suggests that $c(z) \approx x(z)^{1/2}$, and this is the simplest model for a gas giant. For a rocky planet the speed of sound has a non-zero limit at the boundary and so $c(z) \approx 1$.

Therefore we take for a general model a speed of sound $c(z) \approx x(z)^{\alpha/2}$. For a gas giant we expect the value of the parameter α to be 1 and for rocky planets 0. For realistic gaseous celestial bodies we may thus reasonably expect that α is close to 1. We thus allow $\alpha \in (0, 2)$. The extreme case $\alpha = 0$ corresponds physically to solid bodies and mathematically to manifolds with boundary, and the other extreme $\alpha = 2$ corresponds to asymptotically hyperbolic geometry but is far from all planetary models.

Therefore we say that a gas giant metric on a smooth manifold M with boundary is a Riemannian metric g on M° so that $g = x^{-\alpha}h$, where x is a boundary defining function for M and h is a well-defined Riemannian metric up to the boundary. The fact that h is neither zero nor infinite at ∂M implies a specific blow-up rate for g near the boundary. This conformal power-law blow-up is the key geometric feature of gas giant metrics. Both extremes $\alpha = 0$ and $\alpha = 2$ are quite well understood mathematically, but the intermediate cases $\alpha \in (0, 2)$ have been studied far less. The physically most relevant case $\alpha = 1$ does not appear to be geometrically substantially different from other values in the range we allow apart from some minor conveniences and inconveniences that are not important for the present paper.

For a more detailed physical model for the hydrodynamics of a gas giant planet, see Section 5 below.

Acknowledgements. MVdH was supported by the Simons Foundation under the MATH + X program, the National Science Foundation under grant DMS-2108175, and the corporate members of the Geo-Mathematical Imaging Group at Rice University. A significant part of the work of MVdH was carried out while he was an invited professor at Centre Sciences des Données at Ecole Normale Supérieure, Paris. JI and AK were supported by the Research Council of Finland (Flagship of Advanced Mathematics for Sensing Imaging and Modelling grant 359208; Centre of Excellence of Inverse Modelling and Imaging grant 353092; and other grants 351665, 351656, 358047). AK was supported by the Finnish Academy of Science and Letters.

2. The geometry of gas giant manifolds

We begin with an 'extrinsic' study of the metric $g = x^{-\alpha}\overline{g}$. Namely, we assume that the metric takes this form and proceed to study its various geometric properties.

2.1. Normal forms and asymptotic curvatures.

A first observation is that if a metric g is know to be a gas giant metric for some α, then this value can be determined from the intrinsic geometry of g.

Proposition 1. *Suppose that g is an α-gas giant metric on the interior of some manifold with boundary M. Then g is incomplete, and there is a smoothly varying orthonormal basis of sections for TM such that the sectional curvatures for 2-planes spanned by pairs of these basis vectors are asymptotic to*

$$-\frac{2\alpha}{(2-\alpha)^2}\mathrm{dist}\,(\cdot, \partial M)^{-2}, \quad \textit{and} \quad -\frac{\alpha^2}{(2-\alpha)^2}\mathrm{dist}\,(\cdot, \partial M)^{-2}.$$

Thus α can be recovered from these asymptotic sectional curvatures.

We prove this Proposition below, but before doing so, first describe a "normal form" for the metric near the boundary. This is modelled on a very useful normal form, due to Graham and Lee [GL91, Lemma 5.2], in the case when $\alpha = 2$, in which case the metric g is complete, and is called conformally compact. In that case, one can define $\overline{g} = x^2 g$ where x is any choice of boundary defining function, and by definition, $\overline{g}$ is a smooth non-degenerate metric up to the boundary. The restriction of $\overline{g}$ to ∂M is a metric on the boundary; however, replacing x by $x' = ax$ where a is any positive smooth function results in a new metric on ∂M conformal to the first one. In other words, only the conformal class of the metric is well-defined. The Graham–Lee theorem states that if h_0 is any representative of that conformal class, there is a unique boundary defining function x such that

$$g = \frac{dx^2 + h(x,y,dy)}{x^2}, \quad h(0,y,dy) = h_0.$$

Here h is a family of metrics on ∂M (pulled back to the level sets $x = \text{const.}$) depending smoothly on x, and y is any local coordinate system of the boundary. In particular, $-\log x$ is a distance function for the metric g.

In the gas-giant setting we can attempt to prove the same thing, but there is no longer "free data" (analogous to the choice of representative of the conformal class).

Proposition 2. *Let g be an α gas giant metric. Then there is a well-defined metric h_0 on ∂M, and an associated boundary defining function*

x on M such that

$$g = \frac{dx^2 + h(x, y, dy)}{x^\alpha}, \quad \textit{where} \qquad h(0, y, dy) = h_0.$$

Proof. First choose an arbitrary boundary defining function $\tilde{x}$. We modify it in two steps. In the first, we seek a new boundary defining function $\hat{x} = a\tilde{x}$ such that $|d\hat{x}/\hat{x}^{\alpha/2}|_g^2|_{\partial M} \equiv 1$. For this, we compute

$$\frac{d(a\tilde{x})}{(a\tilde{x})^{\alpha/2}} = a^{1-\alpha/2}\frac{d\tilde{x}}{\tilde{x}^\alpha/2} + \mathcal{O}(\tilde{x}),$$

hence we simply need choose a along ∂M so that $a^{2-\alpha}|d\tilde{x}|^2/\tilde{x}^\alpha \equiv 1$ there.

The metric h_0 on ∂M is then defined as the pullback of $\hat{x}^\alpha g$ to the boundary. The computation above shows that there is no leeway: the 1-jet of the boundary defining function, and hence this boundary metric, are completely fixed by the requirement that $|dx/x^\alpha|_g \equiv 1$.

We now make a further change, setting $x = e^\omega \hat{x}$, and study the equation $|dx/x^{\alpha/2}|_g^2 \equiv 1$, not just at the boundary but in the collar neighborhood of the boundary. Writing $\hat{g} = \hat{x}^\alpha g$, we can rewrite this as

$$e^{(2-\alpha)\omega}\frac{|d\hat{x} + \hat{x}d\omega|_g^2}{\hat{x}^\alpha} = e^{(2-\alpha)\omega}|d\hat{x} + \hat{x}d\omega|_{\hat{g}}^2 = 1.$$

Expanding and rearranging yields

$$|d\hat{x}|_{\hat{g}}^2 + 2\hat{x}\langle d\hat{x}, d\omega\rangle_{\hat{g}} + \hat{x}^2|d\omega|_{\hat{g}}^2 = e^{(\alpha-2)\omega}.$$

Using the normalization of $\hat{x}$ and writing $\langle d\hat{x}, d\omega\rangle_{\hat{g}} = \partial_{\hat{x}}\omega$, we recast this in the form

$$\hat{x}\partial_{\hat{x}}\omega = -\hat{x}^2|d\omega|^2 + (1 - |d\hat{x}|^2) + G(\omega)\omega, \tag{1}$$

where $G(\omega) = \omega^{-1}(e^{(\alpha-2)\omega} - 1)$ is a smooth function of ω (including where ω vanishes). It is important to note that $G(0) = \alpha - 2 < 0$.

This is a characteristic Hamilton–Jacobi equation. Fortunately the main result in [GK12] is an existence theorem for equations of precisely this form. That theorem applies to equations of the form

$$\hat{x}\partial_{\hat{x}}\omega = F(x, y, \omega, \partial_y\omega), \quad \omega(0, y) = \omega_0(y),$$

where $F(x, y, \omega, q)$ is smooth and satisfies

$$F(0, y, \omega_0, \partial_y\omega_0) = 0, \; F_\omega(0, y, \omega_0, \partial_y\omega_0) < 1, \quad F_q(0, y, \omega_0, \partial_y\omega_0) = 0.$$

The conclusion in [GK12] is that there exists a unique smooth solution in some small interval $0 \le \hat{x} < \hat{x}_0$. In the proof they observe that the stronger condition $F_\omega < 0$ at $(0, y, \omega_0, \partial_y\omega_0)$ implies that the solution is unique even amongst continuing solutions.

To apply this theorem to our setting, we impose the initial condition $\omega(0, y) = \omega_0 = 0$. We then write $\hat{x}\partial_{\hat{x}}\omega + \hat{x}^2|d\omega|_{\hat{g}}^2 = H(\hat{x}\partial_{\hat{x}}\omega, \hat{x}\partial_y\omega)$, where $H(q_1, q_2)$ satisfies $H_{q_1}(0,0) = 1$, $H_{q_2}(0,0) = 0$. Applying the implicit function theorem, we can thus rewrite (1) as

$$\hat{x}\partial_{\hat{x}}\omega = \mathcal{F}(\hat{x}, y, \omega, \partial_y \omega)$$

where the differential of $\mathcal{F}$ in its third argument at $\omega_0 = 0$ equals $G(0)$. Since $F_\omega G(0) = \alpha - 2 < 0$, the result of Graham and Kantor can be applied. In fact, even the stronger form, which gives uniqueness even amongst all continuous solutions, also holds. □

We now return to the assertion about curvature asymptotics.

Proof of Proposition 1. We first compute sectional curvatures for the warped product metric $x^{-\alpha}(dx^2 + h_0)$. The shortest way uses Cartan's method of moving frames, which we recall briefly. We choose a g-orthonormal family of 1-forms $\{\omega_i\}$ which span T_p^*M at each point. Thus, essentially by definition, $g = \sum \omega_i \otimes \omega_i$. A simple lemma states that there exist uniquely defined 1-forms ω_{ij} which are skew-symmetric in the indices, i.e., $\omega_{ji} = -\omega_{ij}$, such that

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j.$$

This is called Cartan's lemma, and the forms ω_{ij} encode the Levi-Civita connection. We then define 2-forms

$$\Omega_{ij} := d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}.$$

It is then not difficult to show (and this is explained in many sources) that

$$\Omega_{ij} = -\sum_{k,\ell} R_{ijk\ell}\omega_k \wedge \omega_\ell,$$

where $R_{ijk\ell}$ are the components of the Riemann curvature tensor in this basis at each point.

We apply this as follows. Let $\bar{\omega}_\beta$, $\beta = 1, \ldots, n-1$, denote a smoothly varying orthonormal basis of 1-forms on $(\partial M, h_0)$, and write

$$\omega_0 = \frac{dx}{x^{\alpha/2}}, \quad \omega_\beta = \frac{\bar{\omega}_\beta}{x^{\alpha/2}}.$$

A short calculation then shows that for $\beta, \gamma = 1, \ldots, n-1$,

$$\omega_{\beta\gamma} = \bar{\omega}_{\beta\gamma}, \qquad \omega_{\beta 0} = \frac{\alpha}{2} x^{(\alpha-2)/2}\omega_\beta.$$

Here $\bar{\omega}_{\beta\gamma}$ are the connection 1-forms for the metric h_0 on ∂M (extended to the neighborhood $\mathcal{U}$ by the product decomposition).

Finally we compute that

$$\Omega_{\beta\gamma} = \frac{\alpha^2}{4} x^{\alpha-2}\omega_\beta \wedge \omega_\gamma + \mathcal{O}(x^\alpha), \quad \text{and} \quad \Omega_{\beta 0} = \frac{\alpha}{2} x^{\alpha-2}\omega_\beta \wedge \omega_0 + \mathcal{O}(x^{\alpha-1}).$$

The estimate of the remainder term uses, for example, that $|\Omega_{\beta\gamma}|_g^2 = x^{2\alpha}|\Omega_{\beta\gamma}|_{\overline{g}}^2$. We conclude that the principal components of the curvature tensor (which agree with the corresponding sectional curvatures because of our use of orthonormal coframes) satisfy

$$R_{\beta 0\beta 0} \sim -\frac{\alpha}{2}x^{\alpha-2}, \quad R_{\beta\gamma\beta\gamma} \sim -\frac{\alpha^2}{4}x^{\alpha-2}.$$

The function x is related to the distance function s by $(1-\alpha/2)x^{1-\alpha/2} = s$, hence

$$R_{\beta 0\beta 0} \sim -\frac{-2\alpha}{(2-\alpha)^2}, \quad R_{\beta\gamma\beta\gamma} \sim -\frac{\alpha^2}{(2-\alpha)^2},$$

as claimed.

We have shown that any gas giant metric can be written in this simple warped product form up to remainders which are $\mathcal{O}(x)$. However, there is something mildly circular in that we used an initial knowledge of α in proving that normal form. To show that this is not a true issue, observe that we can carry out with only moderately more work the same computations as above if we only know that the metric g is a gas-giant metric for some parameter α, and have set $\overline{g} = x^\alpha g$ for an arbitrary boundary defining function x. The leading asymptotics then determine the value of α just as above. □

We list a few more basic properties.

Proposition 3. *If (M, g) is a gas giant metric, then* $\mathrm{Vol}\,(M, g) < \infty$ *if and only if $\alpha < 2/n$. If $\alpha > 2/n$, then* $\mathrm{Vol}\,(\{x \geq \varepsilon\}) \sim C\varepsilon^{1-n\alpha/2}$, *while if $\alpha = 2/n$, then* $\mathrm{Vol}\,(\{x \geq \varepsilon\}) \sim -C\log\varepsilon$.

Proof. In the special coordinates above, $dV_g = x^{-n\alpha/2}dx dV_h$, so the total volume is finite if $-n\alpha/2 > -1$, i.e., $\alpha < 2/n$. The other assertions are immediate. □

Proposition 4. *The second fundamental form of the level sets $\{x = \varepsilon\}$ are strictly convex.*

Proof. This is a standard computation, which is left to the reader. The conclusion is that

$$\nabla_{\partial_{y_i}}(-x\partial_x) = \frac{\alpha}{2}\partial_{y_i} + \mathcal{O}(x).$$

This shows that the second fundamental form of these level sets is, asymptotically, $\alpha/2$ times the identity, and in particular is positive definite. □

2.2. **Geodesics.** We now turn to a study of the geodesic flow on (M, g).

In the following we always use an adapted coordinate system (x, y) near the boundary, where x is the special boundary defining function obtained in Proposition 2 and y is any coordinate system on the boundary. We denote by (ξ, η) the associated covectors. We shall use the Hamiltonian formalism, namely we write the equations for the bicharacteristics for the Hamiltonian function

$$H(x, y, \xi, \eta) = \frac{1}{2}|(\xi, \eta)|^2_{g_{x,y}} = \frac{1}{2}(x^\alpha \xi^2 + x^\alpha h^{ij}(x, y)\eta_i\eta_j).$$

These bicharacteristics are curves in T^*M which project to the geodesics on M. These equations are:

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial \xi} = x^\alpha \xi, \quad \dot{y}_i = \frac{\partial H}{\partial \eta_i} = x^\alpha h^{ij}(x, y)\eta_j \\
\dot{\xi} &= -\frac{\partial H}{\partial x} = -\alpha x^{-1} H(x, y, \xi, \eta) - \frac{1}{2}x^\alpha \frac{\partial h^{ij}}{\partial x}\eta_i\eta_j, \\
\dot{\eta_i} &= -\frac{\partial H}{\partial y_i} = -\frac{1}{2}x^\alpha \frac{\partial h^{jk}(x, y)}{\partial y_i}\eta_j\eta_k.
\end{aligned}$$

We may as well restrict to geodesics of a fixed speed, and thus suppose that $H \equiv 1/2$ along the solution curves. This simplifies the first summand in the equation for $\dot{\xi}$ to being simply $-\alpha/2x$. We often write a bicharacteristic as $(z(t), \zeta(t))$, where $z(t) = (x(t), y(t))$ and $\zeta(t) = (\xi(t), \eta(t))$.

Before we begin to analyze this system, there are some preliminary observations. First, $x^\alpha(\xi^2 + h^{ij}\eta_i\eta_j) \equiv 2$ along each orbit, so from this it follows that if A^{ij} is any matrix which is uniformly bounded on M, e.g., one written in terms of partial derivatives of the h^{ij} with respect to any of the variables x or y_k, then

$$\left|x^\alpha A^{ij}\eta_i\eta_j\right| \leq C, \tag{2}$$

along each orbit, where C depends only on the norm of A. In the following, we use $\mathcal{O}(1)$, $\mathcal{O}(x^\alpha)$, etc., to denote quantities which are bounded by C, Cx^α, etc., where the constants C depend only on the metric and are independent of the orbit.

Lemma 5. *For $\varepsilon > 0$ small enough, if $\gamma(t) = (z(t), \zeta(t))$ is any bicharacteristic with $x(0) < \varepsilon$ and $\xi(0) = 0$, then $\xi(t) < 0$ for all $t \in \mathbb{R}$.*

Proof. The hypothesis is invariant with respect to replacing t by $-t$, so we prove the assertion for $t \geq 0$. First observe that, by (2),

$$\dot{\xi} = -\alpha x^{-1} + \mathcal{O}(1) < -\frac{1}{2}\alpha\varepsilon^{-1} < 0$$

if ε is sufficiently small. Again, the penultimate inequality here is independent of the trajectory.

This argument shows that if $\xi(0) = 0$, then $\xi(t) < 0$ for $t > 0$ sufficiently small, but in fact it shows that for any $t_0 > 0$, if $\xi(t_0) < 0$ and $x(t_0) < \varepsilon$, then $\xi(t)$ remains bounded above by a strictly negative constant. This proves that $\xi(t) < 0$ for all $t \geq 0$, and for any $t_0 > 0$, $\xi(t) \leq -c < 0$ for $t \geq t_0$. ∎

Lemma 6. *If $\gamma(t) = (z(t), \zeta(t))$ is any bicharacteristic with $x(0) < \varepsilon$, where ε is chosen as in Lemma 5, and $\xi(0) \leq 0$, then $z(t)$ converges to a unique point $(0, \bar{y}) \in \partial M$ at some finite time $T > 0$ and $\eta(t)$ converges to some $\bar{\eta}$ as $t \nearrow T$ as well.*

Proof. We have just shown that the function $x(t)$ is strictly monotone decreasing. Denote the maximal time of existence by $T \leq \infty$. There are a number of possibilities: either $T < \infty$ or $T = \infty$, and in each of these cases, either $x(t) \searrow x_0 > 0$ as $t \nearrow T$ or else $x(t) \searrow 0$. We aim to show that $T < \infty$ and $x(t) \searrow 0$.

Suppose first that $x(t) \searrow x_0 > 0$. If in addition $T < \infty$, then the system of equations remains non-degenerate and we could simply take a limit as $t \to T$ to define $\gamma(T)$ and then continue the solution for later times $t > T$. On the other hand, if $T = \infty$, then using that $\dot{\xi}(t) \leq -c < 0$ for $t \geq t_0$, we obtain $\xi(t) \to -\infty$, which would contradict that $x_0^\alpha \xi^2 < x^\alpha \xi^2 \leq 1$. Neither of these scenarios are possible, hence $x(t) \searrow 0$.

We next show that $\gamma(t)$ reaches $x = 0$ in finite time. Since x is monotone, we may use it as the independent parameter. Thus, writing $\xi = \xi(x)$, we have

$$\frac{d\xi}{dx} = \frac{-(\alpha/2)x^{-1} + \mathcal{O}(1)}{x^\alpha \xi} \Longrightarrow \frac{d}{dx}\xi(x)^2 = -\alpha x^{-\alpha-1} + \mathcal{O}(x^{-\alpha}).$$

Writing the final term as $x^{-\alpha}F$, where F is bounded, and integrating from x to 1, gives

$$\xi(1)^2 - \xi(x)^2 = (1 - x^{-\alpha}) + \int_x^1 s^{-\alpha} F(s)\, ds,$$

whence $\xi(x) = -x^{-\alpha/2}(1 + \mathcal{O}(x) + \mathcal{O}(x^\alpha))$. (The case $\alpha = 1$ is of course slightly different, but we omit the details.) Now insert this into the equation for $\dot{x}$ to get that

$$\begin{aligned} \frac{dx}{dt} = x^\alpha \xi = -\, x^{\alpha/2}(1 + \mathcal{O}(x) + \mathcal{O}(x^\alpha)) \Longrightarrow \\ x^{-\alpha/2}\dot{x} = -1 + \mathcal{O}(x) + \mathcal{O}(x^\alpha). \end{aligned} \tag{3}$$

Bounding the last two terms by $C(\varepsilon + \varepsilon^\alpha)$, and integrating from t_0 to t_1, we get

$$x(t_1)^{1-\alpha/2} = x(t_0)^{1-\alpha/2} - (1 - \alpha/2)(t_1 - t_0) + \mathcal{O}(\varepsilon + \varepsilon^\alpha)(t_1 - t_0).$$

As $t_1 \nearrow T$, $x(t_1)^{1-\alpha/2} \to 0$, which then shows that t_1 cannot become arbitrarily large. This proves that $T < \infty$.

We next observe that by the Hamiltonian constraint, $\dot{\eta}_i = \mathcal{O}(1)$, and and thus $\eta_i(t)$ converges to some limiting value $\bar{\eta}_i$ as $t \to T$ since T is finite. Using this, we also conclude that $y_i(t) \to \bar{y}_i$, and furthermore that

$$(y(t), \eta(t)) = (\bar{y}, \bar{\eta}) + \mathcal{O}(x^\alpha).$$

Note, however, that $\xi(t)$ is unbounded, and more specifically, $\xi(t) \sim -x(t)^{-\alpha/2} \to -\infty$ as $t \nearrow T$. □

We now improve these estimates by showing that along a fixed trajectory, the functions $x(t)$, $y(t)$, $\xi(t)$ and $\eta(t)$ have complete asymptotic expansions in powers of $\tau = T - t$ as $\tau \to 0$. This is achieved by an iteration argument and a careful examination of the methods used in the preceding proof. To simplify notation below, we use τ as a new independent variable, and for any function $f(\tau)$, denote $df/d\tau$ by f' (so $f' = -\dot{f}$). We proceed with the calculations, and summarize the outcomes of all of this at the end.

First, integrate $x^{-\alpha/2}x' = \mathcal{O}(1)$ from 0 to τ to get $x(\tau) = \mathcal{O}(\tau^{\frac{2}{2-\alpha}})$. Substituting this into (3) yields $x^{-\alpha/2}x' = 1 + \mathcal{O}(\tau^{2/(2-\alpha)} + \tau^{2\alpha/(2-\alpha)})$, which then implies that

$$x(\tau) = (1-\alpha/2)^{2/(2-\alpha)}\tau^{2/(2-\alpha)}(1 + \mathcal{O}(\tau^{2/(2-\alpha)} + \tau^{2\alpha/(2-\alpha)})). \quad (4)$$

This gives a leading asymptotic term for the function $x(\tau)$.

For the next step, observe that since we have already proved that y and η remain bounded, the equations of motion show that $(y', \eta') = \mathcal{O}(x^\alpha) = \mathcal{O}(\tau^{2\alpha/(2-\alpha)})$, so that

$$(y(\tau), \eta(\tau)) = (\bar{y}, \bar{\eta}) + \mathcal{O}(\tau^{(2+\alpha)/(2-\alpha)}). \quad (5)$$

Finally,

$$\xi(\tau) = -(1-\alpha/2)^{2/(2-\alpha)}\tau^{-\alpha/(2-\alpha)}(1 + \mathcal{O}(\tau^{2/(2-\alpha)} + \tau^{2\alpha/(2-\alpha)})). \quad (6)$$

The equations (4), (5) and (6) show that each of the functions $x(\tau), y(\tau), \xi(\tau), \eta(\tau)$ has a leading asymptotic term plus a lower order remainder as $\tau \to 0$. For many purposes this is sufficient. However, it is straightforward to set up an inductive scheme to prove the existence of complete polyhomogeneous expansions for these functions in powers of τ. (When $\alpha = 1$, these expansions also involve positive integer powers of $\log \tau$ as well.) This is done by iteratively substituting the partial expansions of these functions into the equations of motion and integrating from 0 to τ, which produces an expansion with one further term in the asymptotic plus an error term which vanishes even more quickly.

Since it will be very helpful below, we carry out the first step of this iteration. In the following, set

$$c_\alpha = (1-\alpha/2)^{2/(2-\alpha)},$$

and for simplicity, indicate higher order remainders by "...". Now, insert the expansions (4) and (5) into the equation for $y_i'(\tau)$ to get that

$$\begin{aligned} y_i'(\tau) &= x^\alpha h^{ij}(x,y)\eta_j \\ &= (c_\alpha \tau^{2/(2-\alpha)} + \dots)^\alpha h^{ij}(c_\alpha \tau^{2/(2-\alpha)} + \dots, \bar{y} + \dots)(\bar{\eta}_j + \dots) \\ &= c_\alpha^\alpha \tau^{2\alpha/(2-\alpha)} h^{ij}(0,\bar{y})\bar{\eta}_j + \dots, \end{aligned}$$

Note that $h^{ij}(0,\bar{y})\bar{\eta}_j = \bar{v}^i$ is the i^{th} coordinate of the vector $\bar{v}$ which is h_0-dual to $\bar{\eta}$ at $\bar{y}$. Thus

$$y(\tau) = \bar{y} + c'_\alpha \tau^{(2+\alpha)/(2-\alpha)}\bar{v} + \ldots,$$

where $c'_\alpha = \alpha^{-1}((2-\alpha)/2)^{(2+\alpha)/(2-\alpha)}$.

From this we immediately deduce the following.

Corollary 7. *Any geodesic $(x(t), y(t))$ which approaches the boundary does so along a curve asymptotic to*

$$y - \bar{y} = c''_\alpha x^{(2+\alpha)/2}\bar{v}$$

for some $\bar{v} \in T_{\bar{y}}\partial M$, where c''_α is a universal constant depending only on α.

Collecting all of the calculations, and proceeding as explained above, we have proved the following result.

Proposition 8. *Each trajectory $(z(t), \zeta(t))$ which remains in the region $\{x < \varepsilon\}$ for $t \geq 0$ reaches the boundary at $x = 0$ at some finite time T. The coordinate functions $(x(t), y(t), \xi(t), \eta(t))$ for a given trajectory admit complete asymptotic expansions in powers of $T - t$ (and when $\alpha = 1$, also $\log(T - \tau)$. In particular, $(y(t), \eta(t))$ converges to some fixed point $(\bar{y}, \bar{\eta})$ in the cotangent bundle of the boundary as $t \to T$.*

We now consider all points (z_0, ζ_0), $\zeta_0 \in T^*_{z_0}M$, with $0 < x_0 < \varepsilon$, where the forward trajectory $(z(t), \zeta(t))$ remains in the region $x < \varepsilon$ for all $t \geq 0$ and converges to $x = 0$. To simplify matters, assume that $\xi_0 = 0$, so that η_0 satisfies the Hamiltonian constraint $x_0^\alpha|\eta_0|^2_{h(x_0,y_0)} = 1$. Our goal is to determine the dependence of the exit time T and exit point $(\bar{y}, \bar{\eta})$ as functions of (x_0, y_0, η_0).

Lemma 9. *The function $T(x_0, y_0, \eta_0)$ is smooth when $x_0 > 0$ and has a complete asymptotic expansion in powers of x_0 as $x_0 \to 0$, with leading term $T \sim c x_0^{1-\alpha/2}$ for some $c > 0$.*

Proof. Strictly speaking, the analysis in the preceding proof assumes that $\xi(0) < 0$. We arrange this by first using that the one-parameter family of local diffeomorphisms Φ_t associated to this flow for some small time $t = \ell(x_0, y_0, \eta_0) > 0$ defines a smooth map $\Phi_{t(x_0)} \colon (x_0, y_0, \eta_0) \mapsto (x_1, y_1, \xi_1, \eta_1)$. We choose this function $\ell(x_0, y_0, \eta_0)$ so that $x_1 = x_0/2$. The "height" x_1 depends on all the variables (x_0, y_0, η_0) (and the function ℓ too), so the image of this map as η_0 varies but (x_0, y_0) remains fixed is a small $(n-2)$-sphere which is not of constant height, but along which ξ_1 is everywhere negative. By Lemma 5, the continuing trajectory converges to ∂M.

Now use that $x(\tau) = c\tau^{2/(2-\alpha)} + \ldots$, which implies, equivalently, that $\tau = c' x^{(1-\alpha/2)} + \ldots$. These functions (and the terms in the expansions) are smooth in all the remaining data. This shows that the time τ needed to move along this given trajectory from $(0, \bar{y})$ to (x_1, y_1) depends smoothly on (x_0, y_0, η_0) and is polyhomogeneous in x_0.

We have just shown that elapsed time $h(x_0, y_0, \eta_0)$ for the path to move from x_0 to $x_0/2$ is on the order of $c'' x_0^{1-\alpha/2}$ for some $c'' > 0$. This function is readily seen to be polyhomogeneous as $x_0 \to 0$, as is the concatenation with the map that gives the elapsed time for the trajectory to move from $x_0/2$ to the boundary. □

We next study the "endpoint mapping" from the set of initial conditions $\mathcal{S} := \{(x_0, y_0, \eta_0) \colon H(x_0, y_0, 0, \eta_0) = 1/2\}$ to the limiting covector on the boundary:

$$F \colon \mathcal{S} \longrightarrow T^*\partial M, \qquad F(x_0, y_0, \eta_0) = (\bar{y}, \bar{\eta}).$$

Of course, F is well-defined only when restricted to the set $\mathcal{S}_\varepsilon = \{(x_0, y_0, \eta_0) \in \mathcal{S} : 0 < x_0 < \varepsilon\}$ for some sufficiently small ε, and we henceforth fix such an ε and the restriction of F to this set. Note that both the Hamiltonian constraint set $\mathcal{S}_\varepsilon$ and $T^*\partial M$ are $(2n-2)$-dimensional. In the following, we systematically identify covectors $\zeta = (\xi, \eta)$ with vectors v using the metric g. However, for covectors $(\bar{y}, \bar{\eta}) \in T^*_{\bar{y}}\partial M$, we identify $\bar{\eta}$ with a vector $\bar{v} \in T_{\bar{y}}\partial M$ via the metric h_0.

Proposition 10. *The map $F : \mathcal{S}_\varepsilon \longrightarrow T^*\partial M$ is a diffeomorphism onto its image. Furthermore, it is smooth, in a precise sense to be made explicit during the course of the proof, in the limit as $x_0 \to 0$.*

Proof. First note that along geodesics starting on $\mathcal{S}_\varepsilon$, we have that $\xi = -x^{-\alpha/2}(1/2 - x^\alpha h^{ij}(x,y)\eta_i\eta_j)^{1/2}$. Inserting this into the equation for $\dot{x}$ yields that

$$\frac{dx}{dt} = -x^{\alpha/2}(1/2 - x^\alpha h^{ij}(x,y)\eta_i\eta_j)^{1/2} =: -x^{\alpha/2}K.$$

The quantity K is simply the second factor with the square root. As we have done before, let us shift to using x as the independent variable. We can then rewrite the equations for $\dot{y}_i$ and $\dot{\eta}_i$ as

$$\begin{aligned} \frac{dy_i}{dx} &= \frac{dy_i/dt}{dx/dt} = -x^{\alpha/2} h^{ij}(x,y)\eta_j, \\ \frac{d\eta_i}{dx} &= \frac{d\eta_i/dt}{dx/dt} = \frac{1}{2} x^{\alpha/2} \partial_{y_i} h^{pq}(x,y)\eta_p\eta_q. \end{aligned}$$

What we have done is to rewrite the equations for (y, η) as "self-contained" equations involving only the new independent variable x and (y, η). This system takes the form

$$\frac{d}{dx}\begin{bmatrix} y \\ \eta \end{bmatrix} = x^{\alpha/2} K(x,y,\eta) G(x,y,\eta), \quad \text{where } G(x,y,\eta) = \begin{bmatrix} -h^{ij}\eta_j \\ \frac{1}{2}\partial_{y_i} h^{pq}\eta_p\eta_q \end{bmatrix}.$$

Rewrite this as $x^{-\alpha/2}\frac{d}{dx}\begin{bmatrix} y \\ \eta \end{bmatrix} = K(x,y,\eta)G(x,y,\eta)$. This suggests that we reparametrize again, setting $u = x^{1+\alpha/2}/(1+\alpha/2)$ so that $\frac{d}{du} =$

$\frac{dx}{du}\frac{d}{dx} = x^{-\alpha/2}\frac{d}{dx}$. The system then becomes

$$\frac{d}{du}\begin{bmatrix} y \\ \eta \end{bmatrix} = K(x(u), y, \eta), G(x(u), y, \eta).$$

Finally, the endpoint map we are studying corresponds to the flow of this system between the two values $u_0 = x_0^{1+\alpha/2}/(1+\alpha/2)$ and $u_1 = 0$. Since $x(u) = (1+\alpha/2)^{2/(2+\alpha} u^{2/(2+\alpha)}$, the functions on the right are polyhomogeneous in u, but not smooth at $u = 0$. However, the lack of full regularity in the independent variable is not relevant in the key fact needed here, which is smooth dependence on initial conditions (u_0, y_0, η_0). This map $\mathcal{S}_\varepsilon \ni (x_0, y_0, \eta_0) \mapsto (y(0), \eta(0))$ is this thus smooth, and patently reversible, hence defines a diffeomorphism from the domain $\mathcal{S}_\varepsilon$ to its image, an open subset of $T^*\partial M$.

For the final statement, we employ a scaling argument to study this map as $x_0 \to 0$. Fix a point $(0, 0) \in \partial M$, and consider the family of dilations $\delta_\lambda : (x, y) \mapsto (\lambda x, \lambda y)$. The pullback of the fixed metric g with respect to δ_λ is

$$\delta_\lambda^*(x^{-\alpha}(dx^2 + h_{ij}(x, y)dy^i dy^j)) = \lambda^{2-\alpha} x^{-\alpha}(dx^2 + h_{ij}(\lambda x, \lambda y)dy^i dy^j),$$

and after normalizing, this has a limit:

$$\lim_{\lambda \to 0} \lambda^{\alpha-2}\delta_\lambda^* g = x^{-\alpha}(dx^2 + h_{ij}(0, 0)dy^i dy^j).$$

This last metric is defined on the entire half-space $\mathbb{R}^n_+ = \{(x, y) \in \mathbb{R}^n : x > 0\}$. The (co)geodesic flow of these dilated rescaled metrics are simply reparametrizations of the geodesics for the initial metric g.

We employ this as follows. To understand the behavior of $F(x_0, y_0, \eta_0)$ as $x_0 \to 0$, it suffices to consider the family of mappings $F_{x_0}(1, y_0, \eta_0)$ which are defined in the same way as F, but for the family of rescaled metric $x_0^{\alpha-2}\delta_{x_0}^* g$. These rescaled metrics converge smoothly as $x_0 \to 0$, and this implies easily that this family of mappings F_{x_0} also converge smoothly. □

With this analysis, we can now use the map F to understand further maps of interest.

Proposition 11. *Let y_1 and y_2 be two nearby points on ∂M. Then there exists a unique geodesic γ which connects y_1 to y_2.*

Proof. Given any (y_1, η_1) and a point (x_0, y_0) with y_0 sufficiently near to y_1, x_0 sufficiently small and $|\eta_1| \leq C$, there exists a unique trajectory $(x(t), y(t), \xi(t), \eta(t))$ with initial condition $(x_0, y_0, 0, \eta_0)$ for some η_0 satisfying $H(x_0, y_0, 0, \eta_0) = 1/2$ and such that $(y(t), \eta(t)) \to (y_1, \eta_1)$. Now follow this trajectory past (x_0, y_0). This continuation hits the boundary at some point $(\bar{y}, \bar{\eta}) = F(x_0, y_0, -\eta_0)$. The elapsed time for the entire trajectory is $T(x_0, y_0, \eta_0) + T(x_0, y_0, -\eta_0)$. This defines a smooth invertible map, which is the analogue of the scattering relation in this setting,

$$E\colon T^*\partial M \to T^*\partial M, \quad E(y_1,\eta_1) = (\bar{y},\bar{\eta}).$$

Now fix y_1 and project E off the $\bar{\eta}$ component. Since there are no conjugate points, this defines a diffeomorphism from a small punctured ball $B_c(0)\setminus\{0\}\subset T^*_{y_1}\partial M$ of non-zero covectors to its image, a punctured neighborhood of y_1 in ∂M. Thus to every y_2 in this neighborhood, there exists some η_1 such that $E(y_1,\eta_1(y_2)) = (y_2,\eta_2)$ for some η_2. This associates to the pair (y_1,y_2) first the covector (y_1,η_1) and then the apex of the corresponding geodesic $F^{-1}(y_1,\eta_1)$, and finally the entire geodesic. □

We have now shown that the interior distance function $d_g(y_1,y_2)$ is well-defined for (y_1,y_2) lying in a sufficiently small punctured neighborhood of the diagonal of $(\partial M)^2$. Let us reparametrize the space of such pairs with the new variables $(\bar{y},\bar{v})$; here $\bar{y}$ is defined as the midpoint of the h_0-geodesic connecting y_1 to y_2 and $\bar{v}\in T_{\bar{y}}\partial M$ is the tangent vector to that geodesic (in the direction from y_1 toward y_2) with length $d_g(y_1,y_2)/2$. Thus $y_1,y_2 = \exp^{h_0}_{\bar{y}}(\pm\bar{v})$. Write $\bar{v} = r\omega$ in spherical coordinates, so $r = d_g(y_1,y_2)/2 \geq 0$ and $\omega\in S^{n-2}$.

Corollary 12. *The interior distance function $d_g(y_1,y_2)$ is polyhomogeneous as $r\to 0$, with $d_g(y_1,y_2)\sim r^{1-\alpha/2}$.*

Proof. We have already shown that that there is a well-defined diffeomorphism which maps (y_1,y_2) to (x_0,y_0,η_0). We also analyzed that this map has a smooth limit as $y_2\to y_1$, i.e., for $r\to 0$. In particular, x_0 depends smoothly on r. Next, by Lemma 9, the elapsed times $T(x_0,y_0,\pm\eta_0)$ to descend either of the two halves of this geodesic toward y_1 and y_2 are polyhomogeneous as $x_0\to 0$. Finally, the lengths of these two half-geodesics $\gamma_\pm$ are computed using the usual formula

$$\ell(\gamma_\pm) = \int_0^{T(x_0,y_0,\pm\eta_0)} |\gamma_\pm'(t)|\,dt,$$

which is smooth in T, and hence polyhomogeneous in x_0 and thus also in r . Since $T\sim cx_0^{1-\alpha/2}$, this is the behavior of $d_g(y_1,y_2)$ as well. □

We note that it is possible to arrive at essentially the same conclusion, at least at the level of an estimate of order of growth but without the expansion, by a more elementary method.

Proposition 13. *Then there are uniform constants $0 < C_1 < C_2$ so that*

$$C_1 d_g(y_1,y_2) \leq d_{h_0}(y_1,y_2)^{1-\frac{\alpha}{2}} \leq C_2 d_g(y_1,y_2).$$

for all $y_1,y_2\in\partial M$.

Remark 14. Observe that this result shows in yet a different way that boundary measurements determine α.

Proof of proposition 13. It is convenient here to use coordinates (s, y) where

$$g = ds^2 + s^{-\beta}(1-\alpha/2)^{-\beta}h(s, y, dy) \quad \text{and} \quad h(0, y, dy) = h_0. \tag{7}$$

In fact, $s = x^{1-\alpha/2}/(1-\alpha/2)$ and $\beta = 2\alpha/(2-\alpha)$.

Suppose, as before, that $y_1, y_2 \in \partial M$ are sufficiently close to one another. We work locally near the boundary in the coordinates (s, y) so that the metric is of the form (7), and in particular the distance of a point (s, y) to the boundary is s. Let γ be the unique interior geodesic connecting these boundary points, and set $k = \max_t d_g(\partial M, \gamma(t))$. In the following, C_α denotes various positive constants depending on α but not y_1, y_2.

We now approximate γ by a "quasi-geodesic". Define curves γ_1, γ_2 by $\gamma_j(t) = (t, y_j)$, $j = 1, 2$, $0 \le t \le \varepsilon$, where ε is to be determined, and let γ_3 be the interior geodesic which connects $\gamma_1(\varepsilon)$ to $\gamma_2(\varepsilon)$. Denote by γ_ε the concatenation of these three curves; this connects y_1 to y_2. Hence $d_g(y_1, y_2) \le \min_\varepsilon \ell_g(\gamma_\varepsilon)$, the length with respect to g of this piecewise curve. Furthermore,

$$l_g(\gamma_\varepsilon) = l_g(\gamma_1) + l_g(\gamma_2) + l_g(\gamma_3) = 2\varepsilon + C_\alpha \varepsilon^{-\frac{\alpha}{2-\alpha}} d_h(y_1, y_2). \tag{8}$$

The right-hand side of (8) is minimized at $\varepsilon = C_\alpha d_h(y_1, y_2)^{1-\frac{\alpha}{2}}$. This gives

$$\begin{aligned} d_g(y_1, y_2) &\le C_\alpha d_{h_0}(y_1, y_2)^{1-\frac{\alpha}{2}} + C_\alpha d_{h_0}(y_1, y_2)(d_h(y_1, y_2)^{1-\frac{\alpha}{2}})^{-\frac{\alpha}{2-\alpha}} \\ &= C_\alpha d_{h_0}(y_1, y_2)^{1-\frac{\alpha}{2}}. \end{aligned}$$

Next, choose t_0 so that $k = d_g(\partial M, \gamma(t_0))$. Writing $\gamma(t) = (s(t), y(t))$, then

$$\ell_g(\gamma) \ge \int_0^{\ell_g(\gamma)} |\dot{s}(t)|\, dt = \int_0^{t_0} \dot{s}(t)\, dt - \int_{t_0}^{\ell_g(\gamma)} \dot{s}(t)\, dt = 2k. \tag{9}$$

On the other hand, since $s(t) \le s(t_0)$ on the entire geodesic,

$$\begin{aligned} \ell_g(\gamma) &\ge \int_0^{\ell_g(\gamma)} s(t)^{-\frac{\alpha}{2-\alpha}} |\dot{y}(t)|_{h_0}\, dt \ge s(t_0)^{-\frac{\alpha}{2-\alpha}} \int_0^{\ell_g(\gamma)} |\dot{y}(t)|_{h_0}\, dt \\ &\ge k^{-\frac{\alpha}{2-\alpha}} d_{h_0}(y_1, y_2). \end{aligned} \tag{10}$$

Combining (9) and (10), we obtain that $d_g(y_1, y_2) \ge C_\alpha d_{h_0}(y_1, y_2)^{1-\frac{\alpha}{2}}$, while combining (8) and (10) yields $k \ge C_\alpha d_g(y_1, y_2)$. Therefore all d_g, d_{h_0} and k are comparable with constants only depending on α as claimed. □

Proposition 15. *The Hausdorff dimension of* $(\partial M, d_g)$ *equals* $\frac{2}{2-\alpha}(n-1)$. *The Hausdorff dimension of* M *equipped with this same metric equals* $\max\{n, \frac{2}{2-\alpha}(n-1)\}$.

Proof. By Proposition 13, $(\partial M, d_g)$ and $(\partial M, d_{h_0}^{1-\frac{\alpha}{2}})$ are bi-Lipschitz equivalent, and hence have the same Hausdorff dimension. It is enough then to compute $\dim(\partial M, d_{h_0}^{1-\frac{\alpha}{2}})$. For simplicity, write this metric space as $(\partial M, d_\alpha)$.

It follows from the definition of Hausdorff measure that for all $\delta > 0$, $\mathcal{H}^\delta(\partial M, d_\alpha) = \mathcal{H}^{(1-\alpha/2)\delta}(\partial M, d_{h_0})$, hence

$$\dim_{\mathcal{H}}(\partial M, d_\alpha) = \frac{2}{2-\alpha}\dim_{\mathcal{H}}(\partial M, d_{h_0}) = \frac{2}{2-\alpha}(n-1).$$

This proves the first claim. As for the second, this follows since $M = \partial M \cup M^\circ$ and $\dim_{\mathcal{H}}(M^\circ, d_g) = n$. □

2.3. **A travel time inverse problem for a gas giant.**

As a first application of our study of geodesics on gas giants, we consider a preliminary inverse problem which asks whether the interior geometry of a gas giant can be recovered (modulo isometries) from knowledge of the Riemannian distances from interior points to the boundary. This simple application can be seen as a proof of concept of our gas giant geometry, leading to a proof as simple as that in the case $\alpha = 0$.

The corresponding result is known both for compact Riemannian manifolds with boundary [KKL01] and in the Finsler setting [dHILS19]. There is a more straightforward proof [ILS23] in the Riemannian case when the metric is simple using a version of the Myers–Steenrod theorem from [dHILS23]. The result here is related to this simpler version.

Theorem 16. *Let M be a compact manifold with boundary and, for $i = 1, 2$, suppose that g_i are simple α_i-gas giant metrics on M. Denote by $d_i\colon M \times M \to \mathbb{R}^+$ the associated Riemannian distance functions. Define the maps $r_i\colon M \to \mathcal{C}(\partial M)$, where $r_i(x)$ is the function which sends $\partial M \ni z \mapsto d_i(x, z)$.*

If the ranges of the two maps r_1 and r_2 are the same in $\mathcal{C}(\partial M)$, then $\alpha_1 = \alpha_2$ and g_1 is isometric to g_2 by a diffeomorphism which is the identity on ∂M.

Proof. First note that each map r_i is well-defined, i.e., $r_i(x)$ is indeed a continuous function on ∂M. For standard incomplete metrics, this follows immediately from the triangle inequality. In this setting, the same conclusion holds because, if $z, z' \in \partial M$ and d_{h_i} is the distance function associated to the metric h_i on ∂M, then using the analysis of the last section gives the continuity estimate $|d_i(x, z) - d_i(x, z')| \le d_{h_i}(z, z')^{1-\alpha_i/2}$. We also observe the unique continuous extension of d_i to the closed manifold with boundary is also well-defined and continuous.

Next, it is also straightforward to check that each r_i is injective. Indeed, if there were to exist two distinct points $x, x' \in M^\circ$ such that $r_i(x) = r_i(x')$, i.e., $d_i(x, z) = d_i(x', z)$ for all $z \in \partial M$, then consider the maximally extended geodesic γ which is length minimizing between any two of its points (this is where we use simplicity of the metrics) passing through x and x'. Suppose that one end of γ meets ∂M at a point z, with x' between x and z. Then clearly $r_i(x)(z) = r_i(x')(z) + d_i(x, x')$, so $r_i(x) \neq r_i(x')$ in $\mathcal{C}(\partial M)$. The extended maps from all of M are also injective.

Continuing this same line of reasoning, we claim that in fact, if $x, x' \in M^\circ$, then $\|r_i(x) - r_i(x')\|_\infty = d_i(x, x')$. This follows since, on the one hand, by the triangle inequality, $\|r(x) - r(x')\|_\infty \leq d(x, x')$, while on the other, choosing the minimizing geodesic γ as above, then $d(z, x) - d(z, x') = \pm d(x, x')$, whence $\|r(x) - r(x')\|_\infty \geq d_g(x, x')$.

As continuous injective maps from the compact Hausdorff space (M, d_i) to $\mathcal{C}(\partial M)$, each r_i is a homeomorphism onto the common image $r_1(M) = r_2(M)$. We may then define $\Psi := r_2^{-1} \circ r_1 \colon M \to M$. By construction, this is a bijective metric isometry.

If $x \in \partial M$, then $0 = r_1(x)(x) = r_2(\Psi(x))(x)$, so $x = \Psi(x)$, i.e., Ψ is the identity on the boundary. The fact that Ψ is a Riemannian isometry from (M°, g_1) to (M°, g_2) is then a consequence of the Myers–Steenrod theorem [MS39, Pal57]. $\square$

3. Geodesic X-ray tomography on a gas giant

This section studies the problem of unique reconstructibility of a function on a gas giant from the knowledge of its X-ray data i.e. integrals over all maximal geodesics. We prove that the X-ray data uniquely determines functions smooth up to the boundary.

The study of geodesic X-ray tomography in standard smooth Riemannian geometry originated in the work of Mukhometov [Muh77], who first proved the case $\alpha = 0$ of our theorem 17 below. For a comprehensive survey of the results, history and motivation of geodesic X-ray tomography, see [Sha94, IM19, PSU23]. The X-ray transform is known to be injective on Cartan–Hadamard manifolds (see [LRS18]) and in asymptotically hyperbolic geometry (see [GGSU19]). These results are the closest relatives to our Theorem 17.

Theorem 17. *Let M be a smooth manifold with boundary of dimension $n + 1 \geq 2$. Let $g = x^{-\alpha}\overline{g}$ be a gas giant metric on M for some $\alpha \in (0, 2)$ which is simple, i.e., non-trapping and free of conjugate points. Suppose that a function $f \in \mathcal{C}^\infty(\bar{M})$ has zero integral over all maximally extended g-geodesics. Then $f = 0$.*

The proof of this theorem is based on a Pestov identity method. We begin by recalling relevant terminology, and refer to [Pat99] for more details about the geometry of unit sphere bundles.

3.1. Pestov identity with boundary terms on a regular boundary.

Let (M, g) be any compact smooth Riemannian manifold with smooth boundary (in this subsection g is assumed to be smooth up to ∂M), and S^*M its unit cosphere bundle. This has the standard projection $\pi \colon S^*M \to M$, as well as a connection map $K \colon TS^*M \to TM$, defined by $K(\theta) = D_t c^\sharp(0)$; here c is any curve in S^*M with $c(0) = (x, \xi)$

and $\dot{c}(0) = \theta$, $c^\sharp(t) = c(t)^\sharp$ is the family of dual covectors, and D_t is the Levi-Civita connection along $\pi(c(t))$.

There is an orthogonal decomposition

$$TS^*M = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}, \qquad \mathcal{V} = \operatorname{Ker} \mathrm{d}\pi, \quad \mathcal{H} = \operatorname{Ker} K. \tag{11}$$

We denote by $N \to S^*M$ the bundle whose fibers are $N_{x,\xi} = \operatorname{Ker} \xi_x \subseteq T_xM$. The maps $\mathrm{d}\pi|_{\mathcal{H}} \colon \mathcal{H} \to N$ and $K|_{\mathcal{V}} \colon \mathcal{V} \to N$ are isomorphisms and we freely identify $\mathcal{H} \oplus \mathcal{V} = N \oplus N$. We define the Sasaki metric G on S^*M by

$$G(\theta, \theta') = g(\mathrm{d}\pi(\theta), \mathrm{d}\pi(\theta')) + g(K(\theta), K(\theta'))$$

for $\theta, \theta' \in T_{x,\xi}S^*M$. The splitting (11) of TS^*M is orthogonal with respect to G.

The G-gradient of a smooth function u on S^*M can be written as

$$\nabla_G u = (Xu, \nabla^{\mathcal{H}} u, \nabla^{\mathcal{V}} u)$$

where the horizontal and vertical gradients $\nabla^{\mathcal{H}} u$ and $\nabla^{\mathcal{H}} u$ are smooth sections of the bundle N and X is the Hamiltonian vector field on S^*M. The Riemannian curvature tensor maps sections W of N to sections of N by the action

$$RW(x, \xi) = R(W(x, \xi), \xi^\sharp)\xi^\sharp.$$

Let $\mathrm{d}\Sigma$ be the volume form of the Sasaki metric.

Now pull the volume form $\mathrm{d}\Sigma$ by the inclusion $\partial S^*M \to S^*M$ to get a natural volume form $\mathrm{d}\sigma$ on ∂S^*M. For all $u \in C^\infty(S^*M)$ define

$$B(u) := \int_{\partial S^*M} \langle \nabla^{\mathcal{V}} u, \nabla^{\mathcal{H}} u \rangle + nuXu \, d\sigma.$$

The following Pestov identity was proved in [GGSU19, p. 60].

Lemma 18. *With notation as above, then for all $u \in C^\infty(S^*M)$,*

$$\|\nabla^{\mathcal{V}} Xu\|^2 = \|X\nabla^{\mathcal{V}} u\|^2 - (R\nabla^{\mathcal{V}} u, \nabla^{\mathcal{V}} u) + n\|Xu\|^2 + B(u). \tag{12}$$

Remark 19. By approximation, the identity (12) continues to hold if $u \in C^1(S^*M)$ has $\nabla^{\mathcal{V}} Xu \in L^2(N)$ and $X\nabla^{\mathcal{V}} u \in L^2(N)$.

3.2. **Proof of theorem 17.** We return to the case where g is a simple gas giant metric and present the proof of theorem 17 using a collection of lemmas, the proofs of which appear in sections 3.3, 3.4 and 3.5.

Lemma 20. *Let g be a simple α-gas giant metric on M, and let $f \in \mathcal{C}^\infty(\bar{M})$. If f integrates to zero over all maximally extended g-geodesics of M then $f \in x^\infty \mathcal{C}^\infty(M)$.*

Lemma 21. *Let g be a simple α-gas giant metric on M, and suppose that $f \in x^\infty \mathcal{C}^\infty(M)$. Then there is a solution $u \in x^\infty \mathcal{C}^\infty(S^*M^\circ)$ to the transport equation $Xu = -f$ in S^*M° with $\nabla_G u \in x^\infty L^\infty(S^*M; TS^*M)$ and $X\nabla^{\mathcal{V}} u, \nabla^{\mathcal{V}} Xu \in L^2(N)$.*

Lemma 22. *Let* $u \in x^\infty C^\infty(S^*M^\circ)$ *satisfy* $Xu = -f$ *as well as* $\nabla^{\mathcal{V}} Xu, X\nabla^{\mathcal{V}} u \in L^2(N)$ *and* $\nabla_G u \in x^\infty L^\infty(S^*M; TS^*M)$*. Then*

$$\|\nabla^{\mathcal{V}} Xu\|^2 = \|X\nabla^{\mathcal{V}} u\|^2 - (R\nabla^{\mathcal{V}} u, \nabla^{\mathcal{V}} u) + n\|Xu\|^2.$$

In the following, $\mathcal{C}^\infty(N^\circ)$ denotes the space of sections of N which are smooth over the interior S^*M°.

Lemma 23. *Let* g *be a simple* α*-gas giant metric on* M*. Then*

$$Q(W) = \|XW\|^2 - (RW, W) \geq 0$$

for all $W \in x^\infty \mathcal{C}^\infty(N^\circ)$ *with* $W \in x^\infty L^\infty(S^*M; TS^*M)$*.*

Proof of theorem 17. Suppose that $f \in \mathcal{C}^\infty(M)$ integrates to zero over all maximally extended geodesics. Then $f \in x^\infty \mathcal{C}^\infty(M)$ by Lemma 20 and so by Lemma 21 there is a solution $u \in x^\infty \mathcal{C}^\infty(S^*M^\circ)$ to the transport equation $Xu = -f$ with $\nabla_G u \in x^\infty L^\infty(S^*M; TS^*M)$ and $\nabla^{\mathcal{V}} Xu, X\nabla^{\mathcal{V}} u \in L^2(N)$. Apply the Pestov identity in Lemma 22 to u to get

$$\|\nabla^{\mathcal{V}} f\|^2 = Q(\nabla^{\mathcal{V}} u) + n\|f\|^2. \tag{13}$$

Since f is the lift of a function on M to S^*M, $\nabla^{\mathcal{V}} f \equiv 0$. In addition, $Q(\nabla^{\mathcal{V}} u) \geq 0$. Thus by Lemma 23, the Pestov identity (13) reduces to $0 \geq n\|f\|^2$, so $f \equiv 0$ as claimed. □

3.3. **Boundary determination.** In this section we prove that a function smooth up to the boundary of M is uniquely determined to any order at the boundary by its integrals over all maximal g-geodesics in M. We first prove an auxiliary result about geodesics converging to a given boundary point.

Lemma 24. *Let* g *be a simple* α*-gas giant metric on* M*. For any* $\bar{y} \in \partial M$*, there exists a sequence* $\zeta_k \in S^*_{\bar{y}}M$ *such that the lengths* $l_g(\gamma_k)$ *of the bicharacteristics* $\gamma_k(t) = (z_k(t), \zeta_k(t))$ *with* $\gamma_k(0) = (\bar{y}, \zeta_k)$ *are positive for all* k *and converge to zero as* $k \to \infty$*.*

Proof. Choose a smooth boundary curve $c\colon (-\varepsilon, \varepsilon) \to \partial M$ with $c(0) = \bar{y}$, and set $\bar{y}_k := c(1/k)$. By simplicity, there is a unique unit speed bicharacteristic $\gamma_k(t) = (z_k(t), \zeta_k(t))$ with $z_k(0) = \bar{y}$, $z_k(\tau_k) = \bar{y}_k$; here τ_k is the exit time of z_k (this is finite by lemma 6). We then let $\zeta_k \zeta_k(0)$. Since $\bar{y}_k \neq \bar{y}$, each $l_g(\gamma_k)$ has positive length. Moreover, by Proposition 13 we have

$$l_g(\gamma_k) = d_g^M(\bar{y}_k, \bar{y}) \leq C d_h^{\partial M}(\bar{y}_k, \bar{y})^{1-\alpha/2},$$

so the lengths converge to zero, as needed. □

We can now prove the following boundary determination lemma. We use arguments similar to the proof of [LSU03, Theorem 2.1]. The only step in its proof where simplicity of the metric is needed is when Lemma 24 is invoked.

Proof of lemma 20. We prove by induction on ℓ that for all $\bar{y} \in \partial M$ and every $\ell \geq 0$, $(\partial_x^\ell f)(0, \bar{y}) = 0$.

When $\ell = 0$, choose a sequence $\zeta_k \in S^*_{\bar{y}} M$ so that that the corresponding bicharacteristics $\gamma_k(t) = (z_k(t), \zeta_k(t))$ have lengths $l_g(\gamma_k)$ tending to zero, as in Lemma 24. By hypothesis,

$$\frac{1}{\tau_k} \int_0^{\tau_k} f(z_k(t))\, dt = 0,$$

where τ_k is the length of the geodesic z_k. Since f is smooth, there exist $t_k \in (0, \tau_k)$ such that $f(z_k(t_k)) = 0$. Clearly $t_k < \tau_k \to 0$. Thus

$$f(0, \bar{y}) = \lim_{k \to \infty} f(z_k(t_k)) = 0,$$

as claimed.

Now assume, for any $\ell > 0$, that $\partial_x^j f(x, y)|_{(0, \bar{y})} = 0$ for all $0 \leq j < \ell$. We prove that $(\partial_x^\ell f)(0, \bar{y}) = 0$ by assuming the contrary, that $(\partial_x^\ell f)(0, \bar{y}) \neq 0$ and arriving at a contradiction.

Assume that $(\partial_x^\ell f)(0, \bar{y}) > 0$. Since f is smooth, $(\partial_x^\ell f)(x, y) > 0$ for all (x, y) in some neighborhood $\mathcal{U}$ of $(0, \bar{y})$. Taking the Taylor expansion of f at any $(0, y)$ and using the inductive hypothesis, we have that

$$f(x, y) = x^\ell \partial_x^\ell f(0, y) + \mathcal{O}(x^{\ell+1}).$$

By the positivity of the ℓ^{th} derivatives, there is a smaller neighbourhood $\bar{y} \in \mathcal{U}' \subseteq \mathcal{U}$ such that $f(x, y) > 0$ in $\mathcal{U}'$. Since $l_g(\gamma_k) \to 0$, the entire geodesic z_k lies in $\mathcal{U}'$ when k is large. Hence the integral of f over z_k cannot vanish, a contradiction.

This proves that f vanishes to order ℓ along ∂M, and since this is true for all $\ell > 0$, we are done. □

As a corollary of this Lemma, we prove that the transport equation $Xu = -f$ admits a solution which is smooth in M°, and that this solution vanishes to all orders at ∂M if $f \in \mathcal{C}^\infty(M)$ is in the kernel of the X-ray transform.

Given $f \in \mathcal{C}^\infty(\bar{M})$, we define u^f to be the function on S^*M defined by the formula[1]

$$u^f(z, \zeta) = \int_0^{\tau(z,\zeta)} f(\phi_t(z, \zeta))\, dt;$$

here f is identified with its pullback $\pi^* f$, and $\phi_t(z, \zeta)$ is the cogeodesic flow.

Corollary 25. *Let g be a simple α-gas giant metric on M, and let $f \in \mathcal{C}^\infty(\bar{M})$. If the integral of f over all maximal geodesics in M is zero, then u^f solves the transport equation $Xu = -f$ in S^*M°, and $u^f \in x^\infty \mathcal{C}^\infty(S^*M^\circ)$.*

[1] In this section, unlike above, we denote the exit time by τ to adhere with the common convection.

Proof. Since f, ϕ_t and τ are all smooth (see Lemma 9), clearly $u^f \in \mathcal{C}^\infty(S^*M^\circ)$.

We prove that $u^f(z,\zeta) = \mathcal{O}(x^\ell)$ for all $\ell > 0$, where the constant depend only on ℓ. It suffices to prove this at $(z_0,\zeta_0) \in S^*M^\circ$, so that $x_0 \in (0,\varepsilon)$ and $\xi_0 < 0$. For positive ξ_0 the claim follows from this one and vanishing integrals of f over maximal geodesics.

Let $\gamma(t) = (z(t),\zeta(t))$ be a bicharacteristic $\gamma(0) = (z_0,\zeta_0)$. We have already shown that $f(x,y) = \mathcal{O}(x^\ell)$ for any ℓ. Since $x(t)$ is strictly decreasing by lemma 5 and g is non-trapping, we have

$$\begin{aligned}\left|u^f(z_0,\zeta_0)\right| &\le \int_0^{\tau(z_0,\zeta_0)} |f(\phi_t(z_0,\zeta_0)|\, dt \le C_k \int_0^{\tau(z,\zeta)} x(t)^\ell\, dt \\ &\le \tilde{C}_k x(0)^\ell = \tilde{C}_k x_0^\ell\end{aligned}$$

for all $\ell > 0$, hence $u^f \in x^\infty\mathcal{C}^\infty(S^*M^\circ)$.

To prove that u^f solves $Xu = -f$ in S^*M°, we compute just as for the classical case of metrics smooth up to the boundary. The point is simply that X differentiates along the cogeodesic flow and u^f is defined by integration along the orbits of this flow. □

3.4. **Derivatives of the integral function.** We now prove lemma 21. This involves an estimate of the derivatives of u^f, where f has vanishing X-ray transform. The first step is to show that normal Jacobi fields cannot blow up at the boundary with respect to the metric $\overline{g} = x^\alpha g$.

Lemma 26. *Let $J(t)$ be a Jacobi field everywhere normal to a bicharacteristic curve $(z(t),\zeta(t))$ with $x(0) < \varepsilon$ and $\xi(0) \le 0$. Then $|J(t)|_{\overline{g}} \le C$ and $|D_t J(t)|_{\overline{g}} \le Cx(t)^{-1}$ for all $t \in [0,\tau(z(0),\zeta(0))]$.*

Proof. Choose any local coordinate system on M near the endpoint of the projected geodesic. The Jacobi equation takes the form

$$\ddot{J}^i + 2\Gamma^i_{jk}\dot{\gamma}^j\dot{J}^k + (\partial_k\Gamma^i_{jl})\dot{\gamma}^j\dot{\gamma}^l J^k = 0,$$

where $\ddot{J}$ and $\dot{J}$ denote the usual derivatives of the coordinates of J with respect to t. The Christoffel symbols of the actual metric g are

$$\begin{aligned}\Gamma^0_{00} &= -\frac{\alpha}{2}x^{-1}, \quad \Gamma^0_{i0} = 0, \quad \Gamma^m_{00} = 0 \\ \Gamma^0_{ij} &= -\frac{\alpha}{2}x^{-1}h_{ij} + \frac{1}{2}h_{ij} \\ \Gamma^m_{i0} &= -\frac{\alpha}{2}x^{-1}\delta^m_i + \frac{1}{2}h^{mk}\partial_x h_{ki} \\ \Gamma^i_{jk} &= \frac{1}{2}h^{mk}(\partial_j h_{ki} + \partial_i h_{kj} - \partial_k h_{ij}) := H^i_{jk},\end{aligned}$$

where H^i_{jk} is defined by this last equality. When $J(t)$ is normal to $z(t)$, the Jacobi equation reduces to

$$\begin{aligned}0 &= \ddot{J}^i + 2\Gamma^i_{0k}\dot{x}\dot{J}^k + 2\Gamma^i_{jk}\dot{y}^j\dot{J}^k + 2(\partial_k\Gamma^i_{0l})\dot{x}\dot{y}^l J^k + (\partial_k\Gamma^i_{jl})\dot{y}^j\dot{y}^l J^k \\ &= \ddot{J}^i - \alpha x^{-1}\dot{x}\dot{J}^i + (h^{il}\partial_x h_{lk}\dot{x} + 2H^i_{jk}\dot{y}^j)\dot{J}^k \\ &\quad + (\partial_k(h^{ip}\partial_x h_{pl})\dot{x}\dot{y}^l + (\partial H^i_{jl})\dot{y}^j\dot{y}^l)J^k.\end{aligned} \tag{14}$$

The coefficient of the third term on the right, involving $\dot{J}^k$, is bounded for $x \geq 0$, and since $\dot{y} = \mathcal{O}(x^\alpha)$, equation (14) becomes

$$\ddot{J}^i - \alpha x^{-1}\dot{x}\dot{J}^i + F^i_k\dot{J}^k + x^\alpha G^i_k J^k = 0 \tag{15}$$

for some bounded functions F^i_k and G^i_k.

Equation (15) can be reduced to a non-singular equation by rescaling $\dot{J}$. Define $W_1(t) = J(t)$ and $W_2(t) = x(t)^{-\alpha}\dot{J}(t)$, so that $\dot{W}_1 = x^\alpha W_2$. Substituting into (15) gives

$$0 = \alpha x^{\alpha-1}\dot{x}W^i_2 + x^\alpha\dot{W}^i_2 - \alpha x^{\alpha-1}\dot{x}W^i_2 + x^\alpha F^i_k W^k_2 + x^\alpha G^i_k W^k_1$$

which reduces to $\dot{W}^i_2 = -F^i_k W^K_2 - G^i_k W^k_1$. This shows that $W = (W_1, W_2)$ satisfies $\dot{W} = AW$ where

$$A = \begin{pmatrix} 0 & x^\alpha I \\ -G & -F \end{pmatrix}$$

and I is the identity matrix, and $F = (F^i_k)$ and $G = (G^i_k)$.

It suffice to prove boundedness of the Jacobi field in the Euclidean metric e with respect to the (x, y) coordinates. We compute

$$\partial_t |W(t)|^2_e = 2\dot{W}(t) \cdot W(t) = 2A(t)W(t) \cdot W(t).$$

Since A is continuous up to ∂M, and hence bounded, we get $\partial_t |W(t)|^2_e \leq C|W(t)|^2_e$. By Grönwall's inequality, $|W(t)|^2_e \leq C|W(0)|^2_e$. This proves that $|J(t)|^2_e \leq C$ and $|\dot{J}(t)| \leq Cx(t)^{2\alpha}$, and hence $|D_t J(t)|_e \leq Cx(t)^{-1}$, as claimed. $\square$

By Lemma 26, we can now estimate derivatives of u^f.

Lemma 27. *If $f \in x^\infty\mathcal{C}^\infty(M)$, then $\nabla_G u^f(z,\zeta) = \mathcal{O}(x^\ell)$ for any $\ell \geq 0$, where the constants are uniform in $(z,\zeta) \in S^*M^\circ$.*

Proof. It suffices to prove that $\partial_\theta u^f(z,\zeta) = \mathcal{O}(x^\ell)$ uniformly on S^*M°, where $x < \varepsilon$, $\xi \leq 0$ and $\theta \in T_{z,\zeta}S^*M^\circ$ with $\theta \perp X$. For convenience, identify f with its lift π^*f to S^*M°. Choose a smooth curve $c(s)$ in S^*M° with $c(0) = (z,\zeta)$ and $\dot{c}(0) = \theta$. Then

$$\begin{aligned}\partial_\theta u^f(z,\zeta) &= \frac{d}{ds}\int_0^{\tau(c(s))} f(\phi_t(c(s)))\,dt\bigg|_{s=0} \\ &= f(\phi_{\tau(z,\zeta)}(z,\zeta))\frac{d}{ds}\tau(c(s))\bigg|_{s=0} + \int_0^{\tau(x,\zeta)} \frac{d}{ds}f(\phi_t(c(s)))\bigg|_{s=0} dt.\end{aligned}$$

Since τ is smooth in S^*M° and f vanishes on ∂M, the first term on the right here vanishes. The second interior term is estimated using Jacobi fields.

Let $J_\theta(t)$ be the Jacobi field along the geodesic $\pi(\phi_t(z,\zeta))$ with initial conditions $J_\theta(0) = d\pi(\theta)$ and $D_t J_\theta(0) = K(\theta)$. In the splitting of TS^*M, the differential $d\phi_t(\theta)$ splits into $J_\theta(t) = d\pi(d\phi_t(\theta))$ and $D_t J_\theta(t) = K(d\phi_t(\theta))$. Thus

$$\begin{aligned}\frac{d}{ds} f(\phi_t(c(s)))\bigg|_{s=0} &= d(\pi^* f)(\partial_s \phi_t(c(s))|_{s=0}) \\ &= \pi^*(df)(J_\theta(t), D_t J_\theta(t)) \\ &= df(J_\theta(t)).\end{aligned}$$

Now, both f and df are $\mathcal{O}(x^\ell)$ for all ℓ. Since $\theta \perp X$, J_θ is normal to this geodesic, Lemma 26 implies that $|J_\theta(t)|_{\overline{g}}$ remains bounded. Thus the integrand in the second term is bounded by $Cx(t)^\ell$. Since x is strictly decreasing on $[0, \tau(z,\zeta)]$ (and the metric is non-trapping), this shows that $\partial_\theta u^f(z,\zeta) = \mathcal{O}(x^\ell)$ for all ℓ, as claimed. □

Proof of lemma 21. Let $f \in x^\infty \mathcal{C}^\infty(M)$ and set $u = u^f$. By Corollary 25 and Lemma 27, the integral function u satisfies $Xu = -f$ in S^*M° and $u \in x^\infty \mathcal{C}^\infty(S^*M^\circ)$ and $\nabla_G u \in x^\infty L^\infty(S^*M; TS^*M)$. It remains to prove that $\nabla^{\mathcal{V}} Xu, X\nabla^{\mathcal{V}} u \in L^2(N)$.

Since u solves the transport equation and the lift of f to S^*M depends only on x, we see that $\nabla^{\mathcal{V}} Xu = -\nabla^{\mathcal{V}} f = 0$, which is in $L^2(N)$. Now use the commutator formula $[X, \nabla^{\mathcal{V}}] = -\nabla^{\mathcal{H}}$, valid in S^*M°, (cf. [PSU15, Appendix A]) to see that

$$\|X\nabla^{\mathcal{V}} u\|_{L^2} = \|\nabla^{\mathcal{H}} u\|_{L^2} \le \|\nabla_G u\|_{L^2}. \tag{16}$$

Since $\nabla_G u \in x^\infty L^2(S^*M; TS^*M) \subset \nabla_G u \in L^2(S^*M; TS^*M)$, (16) gives that $X\nabla^{\mathcal{V}} u \in L^2(N)$. This proves all of the assertions. □

3.5. **Proof of the Pestov identity.** We complete this entire argument by proving Lemmas 22 and 23.

Proof of lemma 22. Let $u \in x^\infty \mathcal{C}^\infty(S^*M^\circ)$ be such that $\nabla^{\mathcal{V}} Xu, X\nabla^{\mathcal{V}} u \in L^2(N)$ and $\nabla_G u \in x^\infty L^\infty(S^*M; TS^*M)$. In adapted coordinates (x, y) near ∂M, consider the truncated manifold $M_\varepsilon := \{x \ge \varepsilon\}$. The restriction of g to this truncation is smooth and non-degenerate up to ∂M_ε. By Lemma 18, for any $w \in \mathcal{C}^\infty(S^*M_\varepsilon)$,

$$\|\nabla^{\mathcal{V}} Xw\|_\varepsilon^2 = \|X\nabla^{\mathcal{V}} w\|_\varepsilon^2 - (R\nabla^{\mathcal{V}} w, \nabla^{\mathcal{V}} w)_\varepsilon + (n-1)\|Xw\|_\varepsilon^2 + B_\varepsilon(w).$$

In particular, this holds for the restriction of u to S^*M_ε. We prove that the identity on all of S^*M by taking the limit $\varepsilon \to 0$.

First, since $\nabla_G u \in x^\infty L^\infty(S^*M; TS^*M)$, we see that

$$|B_\varepsilon(u)| \le C\varepsilon^\ell \mathrm{Vol}(\{x = \varepsilon\})$$

for large ℓ in the sense of the inherited volume form of the submanifold $\{x = \varepsilon\}$. The volume of (M, g) is finite when $\alpha < 2/n$; if $\alpha = 2/n$, the volume of M_ε is asymptotic to $-C\log(\varepsilon)$, while for $\alpha > 2/n$ it is asymptotic to $C\varepsilon^{1-n\alpha/2}$. Choose ℓ large, it is clear that $B_\varepsilon(u) \to 0$ as $\varepsilon \to 0$.

We next prove that the term involving curvature converges to the corresponding term in S^*M. The sectional curvatures of g are asymptotic to $C_\alpha s^{-2}$ where s is the distance to the boundary with respect to g; it is related to x by $s = (1-\alpha/2)x^{1-\alpha/2}$. The pointwise inner product $\langle R\nabla^{\mathcal{V}}u, \nabla^{\mathcal{V}}u\rangle$ is thus bounded by a multiple of $x^{-2+\alpha}|\nabla_G u|^2$. It follows that

$$(R\nabla^{\mathcal{V}}u, \nabla^{\mathcal{V}}u) - (R\nabla^{\mathcal{V}}u, \nabla^{\mathcal{V}}u)_\varepsilon \le C\int_{S^*M\setminus S^*M_\varepsilon} x^{\ell-2+\alpha}\, d\Sigma$$

for ℓ large. We then compute that

$$\begin{aligned}\int_{S^*M\setminus S^*M_\varepsilon} x^{\ell-2+\alpha}\, d\Sigma &= C\int_{M\setminus M_\varepsilon} x^{\ell-2+\alpha-\frac{n\alpha}{2}}\, dx dV_h \\ &\le C\varepsilon^{\ell-2+\alpha-\frac{n\alpha}{2}}\mathrm{Vol}_{\overline{g}}(M\setminus M_\varepsilon).\end{aligned}$$

Since ℓ can be chosen as large as desired, this last term vanishes as $\varepsilon \to 0$, proving that

$$(R\nabla^{\mathcal{V}}u, \nabla^{\mathcal{V}}u)_\varepsilon \to (R\nabla^{\mathcal{V}}u, \nabla^{\mathcal{V}}u).$$

Finally, since the pointwise norms $|X\nabla^{\mathcal{V}}u|$, $|\nabla^{\mathcal{V}}u|$ and $|Xu|$ are bounded by $|\nabla_G u|$, a similar computation shows that we can take limits in the remaining terms $\|X\nabla^{\mathcal{V}}u\|_\varepsilon^2$, $\|\nabla^{\mathcal{V}}Xu\|_\varepsilon^2$ and $\|Xu\|_\varepsilon^2$. ☐

Proof of lemma 23. We prove finally that

$$Q(W) = \|XW\|^2 - (RW, W) \ge 0$$

for all $W \in x^\infty\mathcal{C}^\infty(N^\circ) \cap x^\infty L^\infty(S^*M; TS^*M)$.

Choose $\chi \in \mathcal{C}^\infty([0,\infty))$ with $\chi = 1$ in $[2,\infty)$, $\chi = 0$ in $[0,1]$ and $0 \le \chi \le 1$ everywhere. We use the special adapted coordinates (x, y). For $(z,\zeta) \in S^*M$, write $\chi_\varepsilon(z,\zeta) = \chi(x/\varepsilon)$, and define $W_\varepsilon = \chi_\varepsilon W$. Then W_ε is smooth in the interior of S^*M and supported in $S^*M_\varepsilon = \{(z,\zeta) \in S^*M : x \ge \varepsilon\}$. We claim that $Q(W_\varepsilon) \to Q(W)$ as $\varepsilon \to 0$.

By the product rule,

$$\begin{aligned}Q(W_\varepsilon) = &\int_{S^*M_\varepsilon} |\chi_\varepsilon|^2\left(|XW|^2 - \langle RW, W\rangle\right)\, d\Sigma_\varepsilon \\ &+ \int_{S^*M_\varepsilon} |(X\chi_\varepsilon)W|^2 - 2\chi_\varepsilon(X\chi_\varepsilon)\langle RW, W\rangle\, d\Sigma_\varepsilon.\end{aligned} \tag{17}$$

The first term on the right converges to $Q(W)$ as $\varepsilon \to 0$ by dominated convergence. It suffices to prove that the second term also vanishes as $\varepsilon \to 0$.

The derivative $X\chi_\varepsilon$ is supported in $\{\varepsilon \le x \le 2\varepsilon\}$. In addition, for all $(z,\zeta) \in S^*M^\circ$

$$X\chi_\varepsilon(z,\zeta) = \frac{d}{dt}\chi(x(t)/\varepsilon)\Big|_{t=0} = \varepsilon^{-1}\dot{x}(0)\chi'(x/\varepsilon),$$

so $|X\chi_\varepsilon| \le C\varepsilon^{-1}$ and the integrand in the last term of (17) is bounded by a constant multiple of $C\varepsilon^{-1}(|W|^2 + \langle RW, W\rangle)$ in $\{\varepsilon \le x \le 2\varepsilon\}$ and vanishes elsewhere. Also, the sectional curvatures are asymptotic to $C_\alpha x^{-2+\alpha}$. Since $W \in x^\infty L^\infty(SM; TS^*M)$, we can bound $C\varepsilon^{-1}(|W|^2 + \langle RW, W\rangle)$ by a multiple of $\varepsilon^{\ell-3+\alpha}$ in $\{\varepsilon \le x \le 2\varepsilon\}$, and hence the second integral in (17) is bounded by

$$C\varepsilon^{\ell-3+\alpha}\mathrm{Vol}_G(\{\varepsilon \le x \le 2\varepsilon\}) = \tilde{C}\varepsilon^{\ell-3+\alpha}\mathrm{Vol}_g(\{\varepsilon \le x \le 2\varepsilon\}).$$

The volume grows no faster than a fixed power of ε, so choosing ℓ sufficiently large, we see that this term also vanishes in the limit. Thus $Q(W_\varepsilon) \to Q(W)$ as $\varepsilon \to 0$.

The final step is to note that since the W_ε are smooth and compactly supported in S^*M_ε, and since the truncated manifold M_ε is simple in the traditional sense, it follows from [PSU15, Lemma 11.2] that $Q(W_\varepsilon) \ge 0$ for all $\varepsilon > 0$. Thus its limit $Q(W)$ is also non-negative. □

4. The Laplacian of g

We now turn to the final major theme of this paper, which is to determine a few of the fundamental analytic properties of the scalar Laplace–Beltrami operator Δ_g associated to a gas giant metric. This operator degenerates at $x = 0$, hence is poorly behaved from the point of view of classical theory. However, as we explain here, it can be regarded as an operator with a "uniform degeneracy" as $x \to 0$, and as such, can be transformed to lie in a class of operators for which there is already an extensive theory. We describe this transformation of Δ_g into an "elliptic 0-differential operator", as studied in [Maz91] (and elsewhere). Quoting results from that theory, we study some of basic mapping and regularity properties of Δ_g.

We begin by deriving an expression for this operator in terms of the Laplacian of the metric $\overline{g}$. First observe that

$$g^{ij} = x^\alpha \overline{g}^{ij}, \qquad \det(g_{ij}) = x^{-\alpha n}\det(\overline{g}_{ij}).$$

For simplicity, write $\det g = \det(g_{ij})$ and $\det\overline{g} = \det(\overline{g}_{ij})$. Using the usual special adapted coordinates $z = (z_0, z') = (x, y)$, we compute

$$\Delta_g = x^{\alpha n/2}\frac{1}{\det\overline{g}}\,\partial_{z_i}\left(x^{\alpha(1-n/2)}(\det\overline{g})\,\overline{g}^{ij}\,\partial_{z_j}\right) = x^\alpha\Delta_{\overline{g}} + x^{\alpha-1}\alpha(1-\frac{n}{2})\,\partial_x$$

As noted earlier, this operator is clearly degenerate at $x = 0$.

We now set this into the context of the class of uniformly degenerate, or 0-, differential operators. Using coordinates (x, y) near ∂M, we recall that a differential operator L is called a 0-operator if it can be expressed locally as a linear combination of products of smooth vector fields, each of which vanish at ∂M. The space of all smooth vector fields vanishing at ∂M is denoted $\mathcal{V}_0(M)$, and called the space of 0 vector fields. It is generated over $\mathcal{C}^\infty(M)$ by the 'basis' vector fields $x\partial_x, x\partial_{y_1}, \dots, x\partial_{y_{n-1}}$, i.e.,

$$\mathcal{V}_0(M) = \mathrm{span}_{\mathcal{C}^\infty}\{x\partial_x, x\partial_{y_1}, \dots, x\partial_{y_{n-1}}\}.$$

Thus a 0-operator can be written in small neighborhoods as a finite sum of smooth multiples of products of these vector fields. In particular, for example, a 0-operator of order 2 is one which takes the form

$$L = \sum_{j+|\beta|\le 2} a_{j\beta}(x,y)(x\partial_x)^j(x\partial_y)^\beta.$$

For the present purposes, the key point is that Δ_g assumes this form after multiplication by the factor $x^{2-\alpha}$. (In carrying out some of the arguments below, it is occasionally more transparent to maintain symmetry of the operator by pre- and post-multiplying by $x^{1-\alpha/2}$, but we shall not get into this level of detail). To illustrate this, let x be a special boundary defining function, so that $\overline{g} = dx^2 + h$, where $h(x)$ is a smooth family of metrics on ∂M, pulled back to this collar neighborhood by the projection $(x,y) \mapsto y$. Then

$$\begin{aligned}\Delta_g \coloneqq& x^\alpha \Delta_{\overline{g}} - \alpha(n/2-1)x^{\alpha-1}\partial_x \\ &= x^\alpha(\partial_x^2 + q(x,y)\partial_x + \Delta_{h(x)}) - x^{\alpha-1}\alpha(n/2-1)\partial_x.\end{aligned}$$

where $q(x,y)$ is related to derivatives of $\det h$, but its precise expression is irrelevant since it is a higher order error term. This operator is symmetric with respect to the measure $x^{-\alpha n/2}dxdV_{h(x)}$. From this we see directly that $x^{2-\alpha}\Delta_g$ is a 0-differential operator.

Associated to a 0-differential operator is its 0-symbol, which is obtained by writing L as a sum of products of the generating vector fields, and then replacing each $x\partial_x$ by ξ and $x\partial_{y_i}$ by η_i, and then dropping all terms with homogeneity less than that of the degree of L. In particular, for $L = x^{2-\alpha}\Delta_g$,

$${}^0\sigma_2(L)(x,y;\xi,\eta) \coloneqq \xi^2 + |\eta|^2_{h(x)}.$$

This is not apparently an invariant definition, but (ξ,η) turn out to be natural linear variables on the fiber of a certain replacement for the cotangent bundle T^*M, and ${}^0\sigma_2(L)$ is a well-defined homogeneous polynomial of degree 2 on these linear fibers. In any case, a 0-operator is called 0-elliptic if this symbol is non-vanishing (or invertible, if a system) when $(\xi,\eta) \neq (0,0)$; this operator L is obviously 0-elliptic. As such, the calculus of 0-pseudodifferential operators offers analogues of all the familiar constructions in pseudodifferential theory. In particular, there is an elliptic parametrix construction for L, and the various properties of the parametrix G for L obtained through this construction lead to sharp mapping and regularity properties which are used below.

Now consider the densely defined unbounded operator

$$\Delta_g \colon L^2(M, dV_g) \longrightarrow L^2(M, dV_g). \tag{18}$$

This is symmetric on the core domain $\mathcal{C}_0^\infty(M^\circ)$ of smooth functions compactly supported in the interior of M, and one of the starting points for the analysis of the Laplacian is to determine whether this symmetric operator has a unique self-adjoint extension, or if boundary conditions need to be imposed to obtain a self-adjoint realization? Once that is accomplished, one may proceed to study the spectrum of any such self-adjoint extension.

We first recall some facts relevant to determining whether Δ_g is essentially self-adjoint. In the following, we translate some definitions from the development of the 0-calculus to the present setting (rather than working directly with the 0-operator $L = x^{2-\alpha}\Delta_g$ simply to avoid too many confusing changes of notation.

A fundamental invariant of Δ_g in this geometric setting is its pair of indicial roots, $\gamma_\pm$. These are the values γ such that solutions of $\Delta_g u$ grow or decay like x^γ. More formally, these are the exponents which yield approximate solutions in the sense that

$$\Delta_g x^\gamma = \mathcal{O}(x^{\gamma-1+\alpha})$$

rather than the expected rate $\mathcal{O}(x^{\gamma-2+\alpha})$. In other words, γ is an indicial root if there is some leading order cancellation. To calculate these, we compute

$$\Delta_g x^\gamma = (\gamma(\gamma-1) - \alpha(n/2-1)\gamma)\, x^{\gamma-2+\alpha} + \mathcal{O}(x^{\gamma-1+\alpha}),$$

and hence γ must satisfy $\gamma^2 - (\alpha(n/2-1)+1)\gamma = 0$, or finally

$$\begin{aligned} \gamma_\pm =& 0, \alpha(n/2-1)+1 \\ =& \frac{1}{2}(\alpha(n/2-1)+1) \pm \frac{1}{2}(\alpha(n/2-1)+1). \end{aligned}$$

This last expression is included to emphasize the symmetry of $\gamma_\pm$ around their average, which is useful below.

Next, observe that a function x^γ lies in $L^2(dV_g)$ near $x = 0$ if and only if

$$\gamma > \frac{1}{2}(n\alpha/2 - 1).$$

We call this threshold the "L^2 cutoff weight". It is most natural to let Δ_g act on the Sobolev spaces adapted to the 0-vector fields:

$$H_0^k(M, dV_g) = \{u : V_1 \dots V_\ell u \in L^2(dV_g) \quad \ell \le k, \quad \text{each } V_i \in \mathcal{V}_0(M)\},$$

and their weighted version $x^\mu H_0^k = \{u = x^\mu v : v \in H_0^k\}$. It clear from this definition that

$$\Delta_g \colon x^\mu H_0^2 \longrightarrow x^{\mu-2+\alpha} L^2$$

is bounded for every μ. In particular, $\Delta_g u \in L^2$ if $u \in x^\mu H_0^2$ where $\mu \geq 2-\alpha$. Since $\mathcal{C}_0^\infty(M^\circ)$ is dense in $x^{2-\alpha}H_0^2$, it is clear that the minimal domain, i.e., the minimal closed extension from the core domain, of (18) is contained in $x^{2-\alpha}H_0^2$. Using the parametrix for Δ_g alluded to above, it can be proved that this is an equality:

$$\mathcal{D}_{\min}(\Delta_g) = x^{2-\alpha} H_0^2(M, dV_g).$$

On the other hand, we also define the maximal domain $\mathcal{D}_{\max} = \{u \in L^2 : \Delta_g u \in L^2\}$.

Proposition 28. *The operator Δ_g is essentially self-adjoint on L^2, i.e.,*

$$\mathcal{D}_{\min} = \mathcal{D}_{\max}$$

if and only if $\alpha > 2/n$.

Proof. They key issue is whether either of the indicial roots $\gamma_\pm$ lie in the critical weight interval

$$\mu_- := \frac{1}{2}(n\alpha/2 - 1) \leq \mu \leq \frac{1}{2}(n\alpha/2 - 1) + 2 - \alpha =: \mu_+.$$

Notice that the midpoint of this critical interval is $\frac{1}{2}(n\alpha/2-1)+1-\alpha = \frac{1}{2}(\alpha(n/2-1)+1)$, which is precisely the same as the midpoint of the gap between the two indicial roots. The width of this weight interval is $2-\alpha$, whereas $\gamma_+ - \gamma_- = \alpha(n/2-1)+1$. We claim that

$$\gamma_- < \mu_- < \mu_+ < \gamma_+$$

precisely when $\alpha > 2/n$, which is verified by noting that $\alpha(n/2-1)+1 > 2-\alpha$ precisely then.

The relevance of whether the indicial roots are included in the critical weight interval is that, using the parametrix carefully, one can deduce that if $\gamma_\pm$ do not lie in this critical weight interval, then $u \in L^2$ and $\Delta_g u \in L^2$ imply that $u \in x^{2-\alpha}H_0^2 = \mathcal{D}_{\min}$. However, when $\alpha \leq 2/n$, then we can only deduce that

$$u(x,y) \sim \sum a_j(y) x^{\gamma_- + j} + \sum b_j(y) x^{\gamma_+ + j}.$$

This asymptotic expansion has some complicating features, such as that if $a_0 \not\equiv 0$, then the coefficients a_j, b_j may only have finite regularity (and will have negative Sobolev regularity for large j. Conversely, there exists a solution of $\Delta_g u = 0$ where u has an expansion of this type with any prescribed smooth leading coefficient $a_0(y)$. In any case, the upshot is that the maximal domain is far bigger than the minimal domain in this case. □

When $\alpha < 2/n$, there are many possible self-adjoint extensions. The most prominent, and the one we shall use below, is the Dirichlet extension. This corresponds to the choice of domain $\mathcal{D}_{\mathrm{Dir}}$ consisting of those $u \in L^2$ such that $\Delta_g u \in L^2$ and where the leading coefficient $a_0(y)$ in the expansion above vanishes. Other self-adjoint extensions correspond to other types of conditions on the pair of leading coefficient $(a_0(y), b_0(y))$. We do not detail these below, except mentioning the most standard other ones: the Neumann extension, where $b_0(y) \equiv 0$, and the family of Robin extensions, corresponding to conditions of the form $A(y)a_0(y) + B(y)b_0(y) \equiv 0$, where A, B are given smooth functions.

Proposition 29. *Let $\mathcal{D}$ be a domain of self-adjointness for Δ_g as above. Then $(\Delta_g, \mathcal{D})$ is a Fredholm operator on L^2 with discrete spectrum.*

Proof. The first step is to show that this operator is Fredholm. This follows from the existence of its parametrix. This is a 0-pseudodifferential operator of order -2 which maps L^2 onto $\mathcal{D}$ (possibly modulo compact errors), and which satisfies $G \circ \Delta_g = \mathrm{Id} - R_1$, $\Delta_g \circ G = \mathrm{Id} - R_2$, where R_1 and R_2 are compact operators on L^2 and on $\mathcal{D}$ (with its graph topology) respectively. As noted earlier, the construction of this parametrix is one of the standard consequences of 0-ellipticity; details are given in [Maz91]. When $\alpha < 2/n$, a slightly more intricate construction is needed which incorporates the choice of boundary conditions; this appears in [MV14].

The key point here is that the operator G is constructed as an element of the 0-pseudodifferential calculus. This means that its Schwartz kernel is a very well-understood object which, as a distribution on $M \times M$, has explicit asymptotic expansions at the boundary faces of this product, and a slightly more intricate, but equally explicit expansion near the corner of M^2. The precise details are omitted. The upshot, however, is that it then follows by general properties of such pseudodifferential operators proved in [Maz91] that G is bounded on L^2. Of course, as a pseudoinverse to Δ_g, its range must lie in $\mathcal{D}$. Since it is a (0-)pseudodifferential operator of order -2, it is clear that the elements in $G(L^2)$ have two derivatives in L^2, at least in the interior of M. However, slightly more is true, and the precise statement is that for any $f \in L^2$ and any two vector fields $V_1, V_2 \in \mathcal{V}_0(M)$, we must have that $V_1 V_2(Gf) \in x^\varepsilon L^2$ for some fixed $\varepsilon > 0$ which is independent of f. This is summarized by saying that $G\colon L^2 \to x^\varepsilon H_0^2$, where the range is a weighted 0-Sobolev space. We may then invoke the L^2 version of the Arzelà–Ascoli theorem, which may be used to prove that $x^\varepsilon H_0^2 \hookrightarrow L^2$ is a compact embedding. This shows that the domain of Δ_g is compactly contained within L^2, and hence that $(\Delta_g, \mathcal{D})$ has discrete spectrum.

We say a few more words about this parametrix construction, particularly when $\alpha > 2/n$. Write $\Delta_g = x^{\alpha/2-1} L x^{\alpha/2-1}$; as noted earlier, L is an elliptic 0-operator. The singular factor has been distributed on opposite sides of L to preserve symmetry. Let $\overline{G}$ be a parametrix for L as constructed in [Maz91]. Thus $\mathrm{Id} - L\overline{G} = R_1'$, $\mathrm{Id} - \overline{G}L = R_2'$, where R_1', R_2' are operators with smooth kernels on the interior of $M \times M$, and which admit classical expansions at all boundary faces of a certain resolution (or blow-up) of this product, with coefficients in these expansion smooth functions on the corresponding boundary faces. We then write $G = x^{1-\alpha/2}\overline{G}x^{1-\alpha/2}$, so that $\Delta_g \circ G = \mathrm{Id} - x^{\alpha/2-1}R_1 x^{\alpha/2-1} = I - R_1$, $G \circ \Delta_g = \mathrm{Id} - x^{\alpha/2-1}R_2' x^{\alpha/2-1} = \mathrm{Id} - R_2$. These remainder terms are much better, inasmuch as they have smooth Schwartz kernels which have polyhomogeneous expansions at the two boundary hypersurfaces of M^2, without need for the resolution (or blow-up) process.

If $\Delta_g u = f \in L^2$, then applying G, we get that $u = R_1 u + Gf = R_1 u + x^{1-\alpha/2}\overline{G}x^{1-\alpha/2}f$. The first term is polyhomogeneous on M, and decays at a fixed rate strictly greater than the L^2 cutoff. When $\alpha > 2/n$, the range of G lies in $x^{2-\alpha}H_0^2$. This range is identified with the domain of self-adjointness $\mathcal{D}$ (again, when $\alpha > 2/n$), hence, as described above, $\mathcal{D} \subset L^2$ is indeed compact. ☐

We now take up our final problem. Fix a domain $\mathcal{D} \subset L^2$ where $(\Delta_g, \mathcal{D})$ is self-adjoint. To be very concrete below, we assume that this is the Dirichlet extension henceforth. As just proved, the Dirichlet Laplacian has discrete spectrum $0 \le \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots$.

Next consider the truncated manifold $M_\varepsilon = \{p \in M : x(p) \ge \varepsilon\}$, where $x(p)$ is just the value of the boundary defining function x at p (we assume that x has been extended to be a smooth function on the interior of M which is strictly positive on M_ε). Then Δ_g restricts to an operator acting on $H^2(M_\varepsilon)$ functions which vanish at ∂M_ε. This operator has discrete spectrum as well, by classical elliptic theory, and we denote its eigenvalues by $0 < \lambda_0(\varepsilon) < \lambda_1(\varepsilon) \le \ldots$. By classical perturbation theory, each $\lambda_j(\varepsilon)$ can be regarded as a continuous, and piecewise smooth, function of ε. The basic question is whether the spectrum of $(\Delta_g, \mathcal{D}_{\mathrm{Dir}})$ on M_ε converges to the spectrum of $(\Delta_g, \mathcal{D})$ on M. While there are such statements that can be made about the entire spectrum at once, we consider here the variation of individual eigenvalues.

Proposition 30. *For each $j = 0, 1, 2, \ldots$, the function $\lambda_j(\varepsilon)$ converges to λ_j as $\varepsilon \to 0$. In fact, there exists a constant $C_j > 0$ such that*

$$|\lambda_j(\varepsilon) - \lambda_j| \le C_j \varepsilon^{\alpha(n/2-1)+1}$$

Proof. We will denote ∂_ε by dot and ∂_x by prime.

Let us focus on a particular eigenvalue $\lambda_j(\varepsilon)$. For simplicity, we first make the computations below assuming that this is a simple eigenvalue, staying away from eigenvalue crossings. Thus, dropping the index j, assume that $\Delta_g \phi = \lambda(\varepsilon)\phi$ on M_ε, with $\phi = 0$ on ∂M_ε. We shall construct a family of diffeomorphisms $F_\varepsilon \colon M \to M_\varepsilon$, with $F_0 = \mathrm{Id}$. Using these to pull back all the data on M_ε, we consider the family of metrics $g_\varepsilon = F_\varepsilon^* g$, the associated Laplace operators Δ_{g_ε}, and eigenfunctions ϕ_ε which are smooth functions on M vanishing at ∂M. Choosing these to have $L^2(M, dV_\varepsilon)$ norms equal to 1, the proof involves estimating the quantity

$$\dot{\lambda} = \int_M (\dot{\Delta}_{g_\varepsilon} \phi_\varepsilon)\phi_\varepsilon \, dV_\varepsilon.$$

There are two parts to this. In the first, we obtain the uniform estimate

$$|\phi_\varepsilon| \leq Cx(\varepsilon + x)^{\alpha(n/2-1)},$$

with a constant C independent of ε. Thus ϕ_ε vanishes only like x when $\varepsilon > 0$, but like $x^{\alpha(n/2-1)+1} = x^{\gamma_+}$ when $\varepsilon = 0$. In the second, we must compute $\dot{\Delta}$.

To get started, we define the diffeomorphisms F_ε. Using a specially adapted boundary defining function, define $\mathcal{U} = \{x < c\}$ for some small $c > 0$, and identify $\mathcal{U}$ with $[0, c) \times \partial M$. Define $F_\varepsilon(x) = x + \varepsilon\chi(x/\varepsilon)$, where $\chi(s)$ is a smooth monotone non-negative function which equal 1 for $s \leq 1$ and 0 for $s \geq 4$. We also require that $|\chi'(s)| \leq 1/2$ for all s. We then have that

$$g_\varepsilon = x_\varepsilon^{-\alpha}(dx_\varepsilon^2 + h(x_\varepsilon)).$$

Define

$$\frac{dx_\varepsilon}{dx} := J = 1 + \chi'(x/\varepsilon), \quad \text{and} \quad \dot{x_\varepsilon} = \frac{dx_\varepsilon}{d\varepsilon} = -(x/\varepsilon)\chi'(x/\varepsilon).$$

Then, since $\partial_{x_\varepsilon} = J^{-1}\partial_x$, we obtain

$$\Delta_\varepsilon = x_\varepsilon^\alpha J^{-2}\partial_x - x_\varepsilon^{\alpha-1}(\alpha(n/2-1)J^{-1} - x_\varepsilon J'/J^2)\partial_x + x_\varepsilon \Delta_h.$$

We have actually made the tacit assumption here that $\bar{g} = dx^2 + h$ with h independent of x. The extra terms which appear when h depends on x are lower order in all the computations below, so we can safely omit them.

Computing further, we arrive at the expression

$$\dot{\Delta} = x_\varepsilon^{\alpha-1} A\partial_x^2 + x_\varepsilon^{\alpha-2} B\partial_x + x_\varepsilon^{\alpha-1} C\Delta_h,$$

where A, B and C are expressions which are sums of terms, each a smooth bounded multiples involving the quantities $x_\varepsilon \dot{J}$, $\dot{x}_\varepsilon$, $x_\varepsilon J'$ and $(x_\varepsilon J')\dot{}$. The key point here is that each of these terms is uniformly bounded in ε, and supported in the region $\varepsilon \leq x \leq 4\varepsilon$.

Now, granting the uniform estimate on the ϕ_ε stated above, we see that $\dot{\Delta}\phi \sim \varepsilon^{\alpha-1+\alpha(n/2-1)-1}$, and as before, supported in $x \in [\varepsilon, 4\varepsilon]$.

Thus

$$\int (\dot{\Delta}\phi)\phi \, dV_\varepsilon \sim \int_\varepsilon^{4\varepsilon} \varepsilon^{\alpha-1+\alpha(n/2-1)-1+\alpha(n/2-1)+1-n\alpha/2} \, dx$$
$$= \int_\varepsilon^{4\varepsilon} \varepsilon^{\alpha(n/2-1)-1} \, dx = 4\varepsilon^{\alpha(n/2-1)},$$

as claimed.

It remains to verify the assertion about the uniform bound on ϕ_ε. We indicate the more elementary of the two arguments. First note that since the L^2 norm of ϕ_ε equals 1, and interior estimates bound $|\nabla\phi_\varepsilon$ on any subset $\{x \geq c > 0\}$, the functions ϕ_ε are uniformly bounded on any compact subset of the interior of M, and all vanish at the boundary. We obtain a uniform upper bound of the form $|\phi| \leq Cx(\varepsilon + x)^\beta$ for any $\beta < \alpha(n/2 - 1)$. Since we may take β arbitrarily close to this upper limit, this suffices to give the eigenvalue variation limit above with arbitrary small loss in the exponent.

Now suppose that there is no uniform constant C such that $|\phi_\varepsilon(x, y)| \leq Cx(\varepsilon+x)^\beta$. The bound is clearly true for $\varepsilon \geq \varepsilon_0 > 0$ and for $x \geq c > 0$, so there must exist sequences (x_j, y_j) and ε_j with $x_j \to 0$, $\varepsilon_j \to 0$, such that after multiplying by a sequence of factors $1/C_j \to 0$, and writing ϕ_j instead of ϕ_{ε_j}, we have

$$|\phi_j(x, y)| \leq x(\varepsilon_j + x)^\beta, \quad |\phi_j(x_j, y_j)| = x_j(\varepsilon_j + x_j)^\beta.$$

Now rescale, setting $s = x/x_j$, $w = (y - y_j)/x_j$, to write this as

$$|\phi_j(x, y)| \leq x_j s(\varepsilon_j + x_j s)^\beta.$$

We now separate into two cases. In the first, $\varepsilon_j \gg x_j$, so we rewrite the right hand side of this inequality as $x_j\varepsilon_j^\beta s(1+(x_j/\varepsilon_j)s)^\beta$. Replacing ϕ_j by $\tilde{\phi}_j = \phi_j/x_j\varepsilon_j^\beta$, we see that

$$|\tilde{\phi}_j(s, w)| \leq s(1 + (x_j/\varepsilon_j)s)^\beta,$$

with equality at $(1, 0)$. Taking a limit as $j \to \infty$, we conclude the existence of a limit ϕ_∞ which satisfies $|\phi_\infty| \leq s$ for $s \geq 0$ and all $w \in \mathbb{R}^{n-1}$. Each of the ϕ_j is smooth up to $\partial M_{\varepsilon_j}$, and there is a uniform bound on the tangential derivatives (this follows from the parametrix methods); this implies that ϕ_∞ is in fact constant in w, and so must satisfy the ODE $s^2\partial_s^2\phi_\infty - \alpha(n/2 - 1)s\partial_s\phi_\infty = 0$, whence $\phi_\infty(s) = Cs^{1+\alpha(n/2-1)}$. This contradicts the bound above.

The other case is when $x_j \geq C\varepsilon_j$ as $\varepsilon_j \to 0$. Now rewrite the right hand side of the inequality as $x_j^{1+\beta}s((\varepsilon_j/x_j) + s)^\beta$. Now normalize by dividing by $x_j^{1+\beta}$ to define $\tilde{\phi}_j$, and take a limit as before. This yields a function ϕ_∞ such tat

$$|\phi_\infty(s, w)| \leq s(c + s)^\beta,$$

with equality at $(1, 0)$, and where c is the limit of (some subsequence) of the ε_j/x_j. This constant is finite, and possibly 0. As before, ϕ_∞ is independent of w and must equal a constant times $s^{\alpha(n/2-1)+1}$ for all $s \geq 0$, which is inconsistent with this limiting bound as s gets large.

As noted earlier, there is a more sophisticated way to obtain a sharper bound, and in fact complete asymptotic expansions for ϕ_ε as both $x \to 0$ and $\varepsilon \to 0$. This requires a generalization of the parametrix machinery described above. This generalization allows one to treat not only degenerate operators such as Δ_g, but also families of *degnerating* operators Δ_{g_ε}. However, for simplicity we do not describe or develop this point of view here. What we have proved with this more elementary argument is the slightly weaker estimate that each eigenvalue $\lambda(\varepsilon)$ satisfies

$$|\dot{\lambda}(\varepsilon)| \leq C_\delta \varepsilon^{\alpha(n/2-1)+1-\delta}$$

for any $\delta > 0$.

We then return to the case of a degenerate eigenvalue λ. The eigenspace at $\varepsilon = 0$ has the orthonormal basis $\{\phi^1, \dots, \phi^m\}$ so that each ϕ^k is the limit of eigenfunctions ϕ^k_ε of Δ_{g_ε} as $\varepsilon \to 0$. Each eigenvalue λ^k satisfies the same estimate. □

5. Original equations for gas giants

Here, we present the extraction of the Laplace–Beltrami operator and acoustic wave operator from the system of equations describing the seismology on, and free oscillations of solar system gas giants. The system has been applied to studying the interiors of Saturn and Jupiter [DMF+21]. The original system is given explicitly for the displacement and contains implicitly the pressure; most of the work to extract the acoustic wave operator involves eliminating the displacement. Such an elimination appeared already in the study of inertial modes, that is, a reduction of the original system and invoking incompressibility leading to the Poincaré equation. Here, we follow the work of Prat, Lignières and Ballot [PMA+16].

5.1. **Acoustic-gravitational system of equations.** The displacement vector of a gas or liquid parcel between the unperturbed and perturbed flow is u. The unperturbed values of pressure (P), density (ρ) and gravitational potential (Φ) are denoted with a zero subscript. The incremental Lagrangian stress formulation in the acoustic limit gives the equation of motion

$$\begin{aligned}\rho_0 \partial_t^2 u + 2\rho_0 \Omega \times \partial_t u = \nabla(\kappa \nabla \cdot u) - \nabla(\rho_0 u \cdot \nabla(\Phi_0 + \Psi^s)) \\ + (\nabla \cdot (\rho_0 u))\nabla(\Phi_0 + \Psi^s) - \rho_0 \nabla \Phi', \quad (19)\end{aligned}$$

where the perturbed gravitational potential, Φ', solves

$$\nabla^2 \Phi' = -4\pi G \nabla \cdot (\rho_0 u) \quad (20)$$

and Ψ^s denotes the centrifugal potential,

$$\Psi^s = -\tfrac{1}{2}(\Omega^2 x^2 - (\Omega \cdot x)^2)$$

($|\Omega|$ signifying the rotation rate of the planet). We may introduce the solution operator, S, for (20) such that

$$\Phi' = S(\rho_0 u). \tag{21}$$

We will use the shorthand notation,

$$g_0' = -\nabla(\Phi_0 + \Psi^s). \tag{22}$$

A spherically symmetric manifold requires $\Omega = 0$ from well-posedness arguments.

5.2. **Brunt–Väisälä frequency.** We rewrite the first two terms on the right-hand side of (19),

$$\nabla(\kappa \nabla \cdot u) - \nabla(\rho_0 u \cdot \nabla(\Phi_0 + \Psi^s)) = \nabla[\kappa \rho_0^{-1} (\nabla \cdot (\rho_0 u) - \tilde{s} \cdot u)], \tag{23}$$

in which

$$\tilde{s} = \nabla \rho_0 - g_0' \frac{(\rho_0)^2}{\kappa}, \quad \kappa = P_0 \gamma;$$

$\tilde{s}$ is related to the Brunt–Väisälä frequency, N^2,

$$N^2 = \rho_0^{-1}(\tilde{s} \cdot g_0'). \tag{24}$$

In (23), $\kappa \rho_0^{-1} (\nabla \cdot (\rho_0 u) - \tilde{s} \cdot u)$ can be identified with the dynamic pressure, $-P$ say. We recognize the acoustic wave speed,

$$c^2 = \kappa \rho_0^{-1}.$$

Thus (19) takes the form

$$\partial_t^2(\rho_0 u) + 2\Omega \times \partial_t(\rho_0 u) = \nabla[c^2 (\nabla \cdot (\rho_0 u) - \rho_0^{-1} \tilde{s} \cdot (\rho_0 u))] + (\nabla \cdot (\rho_0 u)) g_0' - \rho_0 \nabla \Phi'. \tag{25}$$

In (25) we can substitute (21) to arrive at an equation for u containing a nonlocal contribution.

5.3. **Equivalent system of equations and Cowling approximation.** Writing $v = \partial_t u$ for the velocity, we obtain the following equivalent system of equations,

$$\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho_0 v) &= 0, \\
\partial_t(\rho_0 v) + 2\Omega \times (\rho_0 v) &= -\nabla P + \rho g_0' - \rho_0 \nabla \Phi', \\
\partial_t P + v \cdot \nabla P_0 &= c^2(\partial_t \rho + v \cdot \nabla \rho_0),
\end{aligned}$$

using that $\nabla P_0 = -\rho_0 g_0'$. Well-posedness of the system of equations implies that

$$\rho_0^{-1}(\tilde{s} \cdot (\rho_0 u)) g_0' = \frac{\rho_0^{-1} \tilde{s} \cdot g_0'}{|g_0'|^2} (g_0' \cdot (\rho_0 u)).$$

Upon inserting (24), the third equation takes the form

$$\partial_t P = c^2 \left(\partial_t \rho + \frac{N^2}{|g_0'|^2} (g_0' \cdot (\rho_0 v) \right).$$

In the Cowling approximation the term $-\rho_0 \nabla \Phi'$ is dropped and the system reduces to

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho_0 v) &= 0, \\ \partial_t (\rho_0 v) + 2\Omega \times (\rho_0 v) &= -\nabla P + \rho g_0', \\ \partial_t P &= c^2 \left(\partial_t \rho + \frac{N^2}{|g_0'|^2} (g_0' \cdot (\rho_0 v) \right). \end{aligned}$$

This is identical to the system appearing in Prat *et al.* [PMA+16].

5.4. **"Truncation": Consistent boundary condition.** The free-surface boundary condition is given by the vanishing of the dynamic pressure (perturbation). If ρ_0 and c would not vanish at the boundary, we thus get the boundary condition

$$(\kappa \nabla \cdot u + \rho_0 g_0' \cdot u)|_{\partial M_\varepsilon} = 0.$$

(For comparison, the first term corresponds with the Lagrangian pressure perturbation.) This corresponds to taking the boundary condition slightly below the boundary rather than exactly at it. We prove in proposition 30 that if the gas giant manifold is truncated just before the boundary, then the eigenvalues on this slightly smaller manifold M_ε converge to those of the true manifold M at a specific rate. This truncation has been widely used in computations [DMF+21].

We have already noted the possibility of imposing certain types of boundary conditions when $\alpha < 2/n$, and this one here falls neatly into that framework. In particular, we can prove, just as in Section 4, that the domain of the operator augmented by this boundary condition is compactly contained in L^2, so that its spectrum is discrete. Furthermore, the spectra of the associated truncated problems converge at an estimable rate to the spectrum of this degenerate operator.

5.5. **Equation for the pressure and geometry.** We introduce

$$\nabla_z = \hat{\Omega} \cdot \nabla, \quad \nabla_\parallel = (-\hat{g}_0') \cdot \nabla,$$

with unit vectors

$$\hat{\Omega} = \frac{\Omega}{|\Omega|}, \quad \hat{g}_0' = \frac{g_0'}{|g_0'|}$$

and

$$\nabla_\perp = \nabla + \hat{g}_0' \nabla_\parallel, \quad \Delta_\perp = \nabla \cdot (\nabla_\perp).$$

Furthermore, e_ϕ is the unit vector in the direction of $\Omega \times g_0'$, noting that $\hat{\Omega}, \hat{g}_0', e_\phi$ form a non-orthogonal basis.

Lemma 31 ([PMA+16]). *The time-Fourier-transformed pressure, $\hat{P}$, satisfies the equation*

$$
\begin{aligned}
&\Delta\hat{P} - \frac{4}{\tau^2}\,\Omega\cdot\nabla(\Omega\cdot\nabla\hat{P}) \\
&\quad - \frac{N^2}{(\tau^2-4|\Omega|^2)}\Bigg[\Delta\hat{P} - \frac{1}{|g_0'|^2}\,g_0'\cdot\nabla(g_0'\cdot\nabla\hat{P}) - \frac{4}{\tau^2}\Omega\cdot\nabla(\Omega\cdot\nabla\hat{P}) \\
&\quad - \frac{4(\Omega\cdot g_0')^2}{\tau^2|g_0'|^2}\,\Delta\hat{P} + \frac{4(\Omega\cdot g_0')}{\tau^2|g_0'|^2}\,(\Omega\cdot\nabla(g_0'\cdot\nabla\hat{P}) + g_0'\cdot\nabla(\Omega\cdot\nabla\hat{P}))\Bigg] \\
&\qquad + \frac{1}{\tau^2(\tau^2-4|\Omega|^2)}\mathscr{V}\cdot\nabla\hat{P} + \frac{1}{c^2\tau^4(\tau^2-4|\Omega|^2)}\,\mathscr{R} \\
&+ \frac{1}{c^2}\left[\tau^2\left(1-\frac{4}{\tau^2}|\Omega|^2\right) + c^2\widehat{\mathcal{M}}\,\nabla\cdot\left(\frac{(4\tau^{-2}(g_0'\cdot\Omega)\Omega - g_0')}{c^2\widehat{\mathcal{M}}}\right)\right]\widehat{P} = 0,
\end{aligned}
\tag{32}
$$

where $\mathscr{R}$ and $\mathscr{V}$ are given below. In coordinates relative to the above mentioned non-orthogonal basis, the terms with leading, second-order spatial derivatives take the form $\Delta\hat{P} - \tau^{-2}(4|\Omega|^2\,\nabla_z^2\hat{P} + N^2\Delta_\perp\hat{P})$; the leading, second-order term in τ is given by $c^{-2}\tau^2\hat{P}$. Thus one identifies, to leading order, the acoustic wave operator on the one hand and an equation like Poincaré's equation in the (axi)symmetric case [RN99] *on the other hand.*

For clarity, we summarize the proof of this lemma. To eliminate v from the system of equations, one takes $2\Omega\times$ and $2\Omega\cdot$ of (30) and applies a time derivative to the resulting equations. Upon taking another time derivative, and substituting the second resulting equation in the first, one obtains the equation

$$
\begin{aligned}
\mathcal{L}(\rho_0 v) &= -4(\Omega\cdot\nabla P)\,\Omega + 4\rho(\Omega\cdot g_0')\,\Omega \\
&\qquad - \nabla\partial_t^2 P + (\partial_t^2\rho)g_0' + 2\Omega\times\nabla\partial_t P - 2(\partial_t\rho)\Omega\times g_0',
\end{aligned}
\tag{33}
$$

where

$$\mathcal{L} = \partial_t^3 + 2|\Omega|^2\partial_t.$$

With this operator, (29) implies

$$\partial_t\mathcal{L}(\rho) = -\nabla\cdot\mathcal{L}(\rho_0 v) \tag{34}$$

and (31) implies

$$\partial_t\mathcal{L}(P) = c^2\partial_t\mathcal{L}(\rho) + \beta g_0'\cdot\mathcal{L}(\rho_0 v), \tag{35}$$

where

$$\beta = \frac{c^2N^2}{|g_0'|^2}.$$

Substituting (33)) into (35) gives

$$\partial_t\mathcal{L}(P) + \beta[4(\Omega\cdot\nabla P)(g_0'\cdot\Omega) + g_0'\cdot\nabla\partial_t^2 P \\ + 2(\Omega\times g_0')\cdot\nabla\partial_t P] = c^2\partial_t\mathcal{L}(\rho) + \beta[\rho(2\Omega\cdot g_0')^2 + (\partial_t^2\rho)|g_0'|^2]. \quad (36)$$

Using the definition of $\mathcal{L}$, one may extend the operator notation to $c^2\mathcal{M}(\rho)$ for the right-hand side of this equation, with

$$\mathcal{M} = \partial_t^4 + (4|\Omega|^2 + N^2)\partial_t^2 + \frac{4N^2(\Omega\cdot g_0')^2}{|g_0'|^2}.$$

Introducing the dual, τ, of $\mathrm{i}\partial_t$, one writes

$$\widehat{\partial_t\mathcal{L}} = \tau^2(\tau^2 - 4|\Omega|^2), \quad \widehat{\mathcal{M}} = \tau^4 - (4|\Omega|^2 + N^2)\tau^2 + \frac{4N^2(\Omega\cdot g_0')^2}{|g_0'|^2}$$

for the relevant symbols, noting that N^2 and g_0' are dependent on the coordinates. Equation (36) then gives

$$\hat\rho = \frac{\widehat{\partial_t\mathcal{L}}\,\hat P + \beta[4(g_0'\cdot\Omega)\,\Omega - \tau^2 g_0' - 2\mathrm{i}\tau(\Omega\times g_0')]\cdot\nabla\hat P}{c^2\widehat{\mathcal{M}}}. \quad (37)$$

Taking the divergence of (33) yields

$$\nabla\cdot\mathcal{L}(\rho_0\hat v) = \tau^2\Delta\hat P \\ - 4\Omega\cdot\nabla(\Omega\cdot\nabla\hat P) + \nabla\cdot[\hat\rho\,(4(g_0'\cdot\Omega)\Omega - \tau^2 g_0' + 2\mathrm{i}\tau(\Omega\times g_0'))]. \quad (38)$$

Here, it was used that Ω is a constant vector (signifying uniform rotation). One then considers (34) and substitutes (38) to obtain an equation for $\hat P$ upon using (37) on the left-hand side:

$$\tau^2\Delta\hat P - 4\Omega\cdot\nabla(\Omega\cdot\nabla\hat P) \\ + \nabla\cdot\Bigg[\frac{(\widehat{\partial_t\mathcal{L}}\,\hat P + \beta[4(g_0'\cdot\Omega)\Omega - \tau^2 g_0' - 2\mathrm{i}\tau(\Omega\times g_0')]\cdot\nabla\hat P)}{c^2\widehat{\mathcal{M}}} \\ \cdot(4(g_0'\cdot\Omega)\Omega - \tau^2 g_0' + 2\mathrm{i}\tau(\Omega\times g_0'))\Bigg] \\ + \frac{\widehat{\partial_t\mathcal{L}}\,(\widehat{\partial_t\mathcal{L}}\,\hat P + \beta[4(g_0'\cdot\Omega)\Omega - \tau^2 g_0' - 2\mathrm{i}\tau(\Omega\times g_0')]\cdot\nabla\hat P)}{c^2\widehat{\mathcal{M}}} = 0.$$

As g_0' derives from a potential (cf. (22)), it follows that

$$\nabla\cdot(\Omega\times g_0') = 0.$$

Then

$$\begin{aligned}
&\tau^2 \Delta \hat{P} - 4\Omega \cdot \nabla(\Omega \cdot \nabla \hat{P}) \\
&\quad + \nabla \cdot \left[\frac{(\widehat{\partial_t \mathcal{L}}\, \hat{P} + \beta[4(g_0' \cdot \Omega)\Omega - \tau^2 g_0'] \cdot \nabla \hat{P})}{c^2 \widehat{\mathcal{M}}} \, (4(g_0' \cdot \Omega)\Omega - \tau^2 g_0') \right] \\
&\qquad + \frac{\widehat{\partial_t \mathcal{L}}\, (\widehat{\partial_t \mathcal{L}}\, \hat{P} + \beta[4(g_0' \cdot \Omega)\Omega - \tau^2 g_0'] \cdot \nabla \hat{P})}{c^2 \widehat{\mathcal{M}}} + \frac{1}{c^2 \widehat{\mathcal{M}}} \mathscr{R} = 0, \quad (39)
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{R} = -2\mathrm{i}\tau c^2 \widehat{\mathcal{M}}\, \nabla \cdot &\left[\frac{\beta}{c^2 \widehat{\mathcal{M}}} ((\Omega \times g_0') \cdot \nabla \hat{P})\, (4(\Omega \cdot g_0')\, \Omega + 2\mathrm{i}\tau \Omega \times g_0' - \tau^2 g_0') \right] \\
&+ 2\mathrm{i}\tau c^2 (\Omega \times g_0') \cdot \nabla \left[\frac{\beta}{c^2 \widehat{\mathcal{M}}} (4(\Omega \cdot g_0')\, \Omega - \tau^2 g_0') \cdot \nabla \hat{P} \right] \\
&\qquad + 2\mathrm{i}\tau c^2 \widehat{\partial_t \mathcal{L}}\, (\Omega \times g_0') \cdot \left[\nabla \left(\frac{\hat{P}}{c^2 \widehat{\mathcal{M}}} \right) - \frac{\beta}{c^2 \widehat{\mathcal{M}}} \nabla \hat{P} \right]
\end{aligned}$$

represents the sum of terms containing $2\mathrm{i}\tau(\Omega \times g_0')\,\cdot$. It is noted that in the axisymmetric case,

$$(\Omega \times g_0') \cdot \nabla \left(\frac{1}{c^2 \widehat{\mathcal{M}}} \right) = 0,$$

and that in a polytropic model (see Subsection 1.2), β is a constant, which simplifies the computations. Equation (39) can be rewritten as

$$\begin{aligned}
&\widehat{\mathcal{M}}\, \Delta \hat{P} - 4(\tau^2 - (4|\Omega|^2 + N^2))\, \Omega \cdot \nabla(\Omega \cdot \nabla \hat{P}) \\
&+ \frac{\beta \tau^2}{c^2}\, g_0' \cdot \nabla(g_0' \cdot \nabla \hat{P}) - \frac{4\beta}{c^2}\, (\Omega \cdot g_0')\, [\Omega \cdot \nabla(g_0' \cdot \nabla \hat{P}) + g_0' \cdot \nabla(\Omega \cdot \nabla \hat{P})] \\
&+ \frac{4}{c^2 \tau^2} \Bigg\{ \left[(1+\beta)\widehat{\partial_t \mathcal{L}} + c^2 \widehat{\mathcal{M}}\, \nabla \cdot \left(\frac{\beta(4(g_0' \cdot \Omega)\Omega - \tau^2 g_0')}{c^2 \widehat{\mathcal{M}}} \right) \right] (g_0' \cdot \Omega) \\
&\qquad + \frac{\beta}{2} \Omega \cdot \nabla(g_0' \cdot (4(g_0' \cdot \Omega)\Omega - \tau^2 g_0')) \Bigg\}\, \Omega \cdot \nabla \hat{P} \\
&- \frac{1}{c^2} \left\{ (1+\beta)\widehat{\partial_t \mathcal{L}} + c^2 \widehat{\mathcal{M}}\, \nabla \cdot \left(\frac{\beta(4(g_0' \cdot \Omega)\Omega - \tau^2 g_0')}{c^2 \widehat{\mathcal{M}}} \right) \right\} g_0' \cdot \nabla \hat{P} \\
&+ \frac{\widehat{\partial_t \mathcal{L}}}{c^2 \tau^2} \left[\widehat{\partial_t \mathcal{L}} + c^2 \widehat{\mathcal{M}}\, \nabla \cdot \left(\frac{(4(g_0' \cdot \Omega)\Omega - \tau^2 g_0')}{c^2 \widehat{\mathcal{M}}} \right) \right] \hat{P} + \frac{1}{c^2 \tau^2} \mathscr{R} = 0.
\end{aligned}$$

The sum of the two terms containing factors in between braces allow the shorthand notation $\mathscr{V} \cdot \nabla \hat{P}$:

$$(\tau^2 - 4|\Omega|^2)\,\Delta\hat{P} - (\tau^2 - 4|\Omega|^2)\,\frac{4}{\tau^2}\,\Omega\cdot\nabla(\Omega\cdot\nabla\hat{P}) \\ - N^2\Bigg[\Delta\hat{P} - \frac{1}{|g_0'|^2}\,g_0'\cdot\nabla(g_0'\cdot\nabla\hat{P}) - \frac{4}{\tau^2}\Omega\cdot\nabla(\Omega\cdot\nabla\hat{P}) \\ -\frac{4(\Omega\cdot g_0')^2}{\tau^2|g_0'|^2}\,\Delta\hat{P} + \frac{4(\Omega\cdot g_0')}{\tau^2|g_0'|^2}\,(\Omega\cdot\nabla(g_0'\cdot\nabla\hat{P}) + g_0'\cdot\nabla(\Omega\cdot\nabla\hat{P}))\Bigg] + \frac{1}{\tau^2}\mathscr{V}\cdot\nabla\hat{P} \\ + \frac{\widehat{\partial_t\mathcal{L}}}{c^2\tau^4}\left[\widehat{\partial_t\mathcal{L}} + c^2\tau^2\widehat{\mathcal{M}}\,\nabla\cdot\left(\frac{(4\tau^{-2}(g_0'\cdot\Omega)\Omega - g_0')}{c^2\widehat{\mathcal{M}}}\right)\right]\hat{P} + \frac{1}{c^2\tau^4}\,\mathscr{R} = 0.$$

One then divides the equation by $(\tau^2 - 4|\Omega|^2)$; the factor in front of $\hat{P}$ then takes the form

$$\frac{1}{c^2}\left[\tau^2\left(1 - \frac{4}{\tau^2}|\Omega|^2\right) + c^2\widehat{\mathcal{M}}\,\nabla\cdot\left(\frac{(4\tau^{-2}(g_0'\cdot\Omega)\Omega - g_0')}{c^2\widehat{\mathcal{M}}}\right)\right].$$

This results in equation (32).

5.6. **Propagation of singularities.** The propagation of singularities depends only on the leading order part of the system of equations. Ignoring lower order terms, equation (25) reads

$$\partial_t^2(\rho_0 u) - \nabla[c^2\nabla\cdot(\rho_0 u)] = 0$$

and the principal symbol at $(t, x; \tau, \xi)$ is $\tau^2\mathrm{Id} - c^2(x)\xi\xi^T$. From the way the matrix $\xi\xi^T$ acts we may read that pressure singularities propagate but shear ones do not. Pressure waves ("polarized" along the momentum ξ) follow the geodesics of the isotropic sound speed c just as the solutions of the scalar wave equation for pressure $(\partial_t^2 - c^2\Delta)P = 0$ as extracted from the original system in Lemma 31. Therefore, if only the travel times of singularities are concerned, it suffices to model a gas planet with a scalar wave equation.

The parametrix construction outlined in Section 4 is a very flexible one. Although we have used it to analyze the simpler operator Δ_g studied in the rest of this paper, the operator appearing in (32) is a perturbation of such a Laplacian, for appropriately defined gas-giant metric g, with all extra terms being of lower order in the sense of this calculus of degenerate operators. In other words, it is possible, just as easily, to construct a parametrix for this operator in the 0-pseudodifferential calculus, and to derive the same sorts of conclusions as we have discussed for the Laplacian. Furthermore, the lower order terms here are all compact relative to the main part of this operator, hence do not affect the discreteness of the spectrum, but do cause the usual sorts of perturbations to the spectrum caused by any such compact perturbations.

References

[dHILS19] Maarten V. de Hoop, Joonas Ilmavirta, Matti Lassas, and Teemu Saksala. Determination of a compact Finsler manifold from its boundary distance map and an inverse problem in elasticity. *Communications in Analysis and Geometry*, January 2019. To appear.

[dHILS23] Maarten V. de Hoop, Joonas Ilmavirta, Matti Lassas, and Teemu Saksala. Stable reconstruction of simple Riemannian manifolds from unknown interior sources. *Inverse Problems*, 39(9):Paper No. 095002, 43, 2023.

[DMF$^+$21] Janosz W. Dewberry, Christopher R. Mankovich, Jim Fuller, Dong Lai, and Wenrui Xu. Constraining saturn's interior with ring seismology: Effects of differential rotation and stable stratification. *Planet. sci. j.*, 2, 2021.

[GGSU19] C. Robin Graham, Colin Guillarmou, Plamen Stefanov, and Gunther Uhlmann. X-ray transform and boundary rigidity for asymptotically hyperbolic manifolds. *Ann. Inst. Fourier (Grenoble)*, 69(7):2857–2919, 2019.

[GK12] C. Robin Graham and Joshua M. Kantor. Normal Form for Edge Metrics, 2012. arXiv:1207.1343.

[GL91] C. Robin Graham and John M. Lee. Einstein metrics with prescribed conformal infinity on the ball. *Adv. Math.*, 87(2):186–225, 1991.

[Hor04] G. P. Horedt. *Polytropes: Applications in Astrophysics and Related Fields*, volume 1 of *Astrophysics and Space Science Library*. Springer Dordrecht, 2004.

[ILS23] Joonas Ilmavirta, Boya Liu, and Teemu Saksala. Three travel time inverse problems on simple Riemannian manifolds. *Proc. Amer. Math. Soc.*, 151(10):4513–4525, 2023.

[IM19] Joonas Ilmavirta and François Monard. Integral geometry on manifolds with boundary and applications. In Ronny Ramlau and Otmar Scherzer, editors, *The Radon Transform: The First 100 Years and Beyond*. de Gruyter, April 2019.

[KKL01] Alexander Katchalov, Yaroslav Kurylev, and Matti Lassas. *Inverse boundary spectral problems*, volume 123 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2001.

[LRS18] Jere Lehtonen, Jesse Railo, and Mikko Salo. Tensor tomography on Cartan-Hadamard manifolds. *Inverse Problems*, 34(4):044004, 27, 2018.

[LSU03] Matti Lassas, Vladimir Sharafutdinov, and Gunther Uhlmann. Semiglobal boundary rigidity for Riemannian metrics. *Math. Ann.*, 325(4):767–793, 2003.

[Maz88] Rafe Mazzeo. The Hodge cohomology of a conformally compact metric. *J. Differential Geom.*, 28(2):309–339, 1988.

[Maz91] Rafe Mazzeo. Elliptic theory of differential edge operators. I. *Comm. Partial Differential Equations*, 16(10):1615–1664, 1991.

[MS39] S. B. Myers and N. E. Steenrod. The group of isometries of a Riemannian manifold. *Ann. of Math. (2)*, 40(2):400–416, 1939.

[Muh77] R. G. Muhometov. The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry. *Dokl. Akad. Nauk SSSR*, 232(1):32–35, 1977.

[MV14] Rafe Mazzeo and Boris Vertman. Elliptic theory of differential edge operators, ii: Boundary value problems. *Indiana University Mathematics Journal*, 63(6):1911–1955, 2014.

[Pal57] Richard S. Palais. On the differentiability of isometries. *Proc. Amer. Math. Soc.*, 8:805–807, 1957.

[Pat99] Gabriel P. Paternain. *Geodesic flows*, volume 180 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1999.

[PMA+16] Prat, V., Mathis, S., Augustson, K., Lignières, F., Ballot, J., Alvan, L., and Brun, A. S. Asymptotic theory of gravity modes in rotating stars - i. ray dynamics. *A&A*, 587:A110, 2016.

[PSU15] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Invariant distributions, Beurling transforms and tensor tomography in higher dimensions. *Math. Ann.*, 363(1-2):305–362, 2015.

[PSU23] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. *Geometric inverse problems—with emphasis on two dimensions*, volume 204 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2023. With a foreword by András Vasy.

[RN99] M. Rieutord and K. Noui. On the analogy between gravity modes and inertial modes in spherical geometry. *European Physical Journal B*, 9(4):731–738, 1999.

[Sha94] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

Simons Chair in Computational and Applied Mathematics and Earth Science, Rice University

Department of Mathematics and Statistics, University of Jyväskylä

Department of Mathematics and Statistics, University of Jyväskylä

Department of Mathematics, Stanford University