Resurgent perturbation theory

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Author:

Joona Marjamäki

Supervisor:

Robert van Leeuwen

UNIVERSITY OF JYVÄSKYLÄ
DEPARTMENT OF PHYSICS

Abstract

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The aim of this thesis was to study the theory of resurgence and how it relates to perturbation theory. Resurgence theory is exponentially accurate asymptotic theory and it gives us tools to resum divergent series via the Borel resummation procedure. The essence of resurgence is in the singularities of the Borel transformed asymptotic series which correspond to exponentially small factors that ordinary perturbation theory misses, the non-perturbative effects. As an application of resurgence in quantum mechanics the exact WKB method was studied. The standard WKB method is known as an approximation method but via resurgence it can be shown to be an exact method. Using the exact WKB we can find the exact quantization conditions for spectral problems geometrically using Stokes graphs.

Keywords: Asymptotics, Resurgence theory, non-perturbative effects, Exact WKB

Tiivistelmä

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Resurgenttinen häiriöteoria
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Tämän tutkielman tavoitteena oli tutkia resurgenssiteoriaa ja miten se kytkeytyy häiriöteoriaan. Resurgenssiteoria on eksponentiaalisen tarkka asymptoottinen teoria ja sen avulla voimme uudelleensummata hajoavia sarjoja Borel summauksen avulla. Resurgenssiteorian ydin on Borel muunnetun hajoavan sarjan singulariteeteissä joihin kytkeytyy eksponentiaalisen pienet tekijät, joita ei voida selvittää tavallisen häiriöteorian avulla, niin kutsutut ei-häiriöteoreettiset ilmiöt. Resurgenssiteorian sovelluksena kvanttimekaniikkaan tutkittiin eksaktia WKB menetelmää. Tavallinen WKB menetelmä tiedetään olevan approksimatiivinen menetelmä, mutta resurgenssiteorian avulla siitä saadaan tarkka menetelmä. Eksaktin WKB menetelmää voidaan käyttää eksaktien kvantisaatioehtojen selvittämiseksi geometrisesti käyttäen Stokesin graafeja.

Avainsanat: Asymptotiikka, Resurgenssi teoria, ei-häiriöteoreettiset ilmiöt, Eksakti WKB

Contents

1 Introduction

In physics the problems that are exactly solvable are usually only found in textbooks so one has to resort to different approximation schemes to find a solution to the problem at hand. One such a method is perturbation theory where the observable to be calculated is expanded as a power series in a small variable

$$\mathcal{O} = \sum_n a_n \lambda^n \tag{1.1}$$

where the 0th order term is given by an exactly solvable system and it can lead to extremely accurate results such as the QED electron anomalous magnetic moment [1]. However, in many cases the perturbation series is divergent and has a vanishing radius of convergence. For instance in quantum field theories the perturbation series are generally divergent [2, 3] or the WKB series in semiclassical quantum mechanics [4]. These divergent perturbation series are called asymptotic series.

Naturally a question arises how can these asymptotic series give us any meaningful information if the series diverges? Classical asymtotics answers this by showing that even though the series is asymptotic it can give accurate approximations when optimally truncated. Furthermore it has been found that the asymptotic perturbative expansion contains information about the exact answer in the form of non-pertubative information. This connection between perturbative and non-perturbative physics is formalized by the theory resurgence.

Resurgence arose as a need to create a exponentially accurate asympototic theory [5] and was developed by Jean Écalle [6]. The main aim of the resurgent theory is to formalize a resummation method for divergent series, that is we can assign a value to a divergent series. The framework for this is Borel analysis: given a asymptotic series φ with coefficients of at most factorial growth, $a_n \sim A^{-n} n!$, one can improve the convergence by the means of Borel transform and get a convergent series with finite radius convergence on a Riemann surface called the Borel plane defined by the singularities of the Borel transform $\mathcal{B}\varphi$. If this new function fulfills certain conditions about analytic continuation and growth at infinity, the directional Laplace transform gives an analytic function with the divergent series being its asymptotic expansion. This is Borel-Laplace transform is know as the Borel resummation

$$\mathcal{S}^{\theta}\varphi(\lambda) = \int_0^{\infty e^{i\theta}} d\zeta\, e^{-\lambda\zeta}\mathcal{B}[\varphi](\zeta) \sim \varphi(\lambda) \tag{1.2}$$

If $\mathcal{B}\varphi$ has a singularity along the direction θ, also known as the Stokes line, the Borel resummation becomes ambiguous depending on which side the integration path avoids the singularity. This is the well known Stokes phenomena of asymptotic expansions and is entirely encoded by the singularities of the Borel transform. The difference between these two resummations, the discontiuity of the Borel sum along θ is described by the Stokes automorphism $\mathfrak{S}_{\theta}$

$$\mathcal{S}^{\theta^-} = \mathcal{S}^{\theta^+} \circ \mathfrak{S}_{\theta} \tag{1.3}$$

and the discontinuity is exponential, that is, non-perturbative in nature.

In quantum mechanics one of the main goals of resurgence is derive the exact quantization condition in spectral problems using the exact WKB method. The WKB method, named after Wentzel, Kramers and Brillouin [7–9] is usually known as an approximation method to high energy-states but via resurgence it can be shown to exact, applying everywhere in the spectrum. The exact WKB was pioneered by Voros using the Écalle theory in [10] and was further developed in [11, 12].

In the exact WKB method the WKB series are made into analytic resurgent functions via the Borel resummation method and the connection problem is solved geometrically using Stokes graphs which describe the Borel summability of the WKB wave functions and the Stokes phenomena. Due to the divergence of the WKB series at the turning points a certain normalization convention has to be defined. Inside a Stokes region WKB wave functions normalized at different turning points are connected by the Voros symbols V_{γ}. Solving the connection problem leads to the exact quantization condition given by the Voros symbols

$$f(V_{\gamma_1}, V_{\gamma_2}, V_{\gamma_3}, \dots) = 0 \tag{1.4}$$

The above quantization condition is exact because it includes Voros symbols corresponging to the non-perturbative tunneling regions.

The discontinuity of the Borel sum of Voros symbols is encoded by the Delabere-

Dillinger-Pham (DDP) formula [11]

$$\mathfrak{S}V_{\gamma_i} = \prod_{j=2} V_{\gamma_i}\left(1 + V_{\gamma_j}\right)^{(\gamma_j,\gamma_i)} \tag{1.5}$$

The discontinuity is non-perturbative and determined by its perturbative expansion.

In this thesis we first give a theoretical background on asymptotics, Borel resummation, resurgence theory and the exact WKB method. After which we study the resurgent perturbation theory via the quantum resurgence of the Airy-type Schrödinger equation first deriving the asymptotics expansions via the method of steepest descents focusing the Riemann surface structure, then studying the Stokes phenomena and finally the resurgence. The Airy-type Schrödinger equation is an important example understanding the exact WKB as it forms the building blocks of the Stokes graphs of more complicated potentials. After this we focus on the exact quantization condition of quantum mechanics and derive these for a few cases using the exact WKB framework. Finally we discuss the conclusions and future outlook of resurgence.

2 General theory

The resurgence theory is a theory for resummation of divergent asymptotic series and before we delve into resurgence, we have to understand what asymptotic series, or expansions, are. In this chapter we introduce most definitions, theorems and results used in asymptotics and resurgence theory. We'll use an example to illustrate the definitions and to understand them better. Let's start with the example:

Example 2.1. Consider the following first order differential equation, also known as the Euler equation[1],

$$\varphi'(z) - a\varphi(z) = -\frac{1}{z} \tag{2.1}$$

with $a \in \mathbb{C} \setminus \{0\}$ and let's say that we want to study the behavior of the solutions at the irregular singular point$z = \infty$. From the theory of ordinary differential equations we know that the general solution is

$$\phi(z) = Ce^{az} + e^{az} \int_z^\infty dt\, \frac{e^{-at}}{t} \tag{2.2}$$

Because the homogenous solution is non-analytic at $z = \infty$ the constant must be $C = 0$. Thus

$$\phi(z) = e^{az} \int_z^\infty dt\, \frac{e^{-at}}{t} = e^{az} E_1(az) \quad \text{as} \quad z \to \infty \tag{2.3}$$

where $E_1(z)$ is the exponential integral and can be expanded as [13]

$$E_1(z) = -\gamma - \log z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!n} \tag{2.4}$$

where γ is the Euler-Mascheroni constant, $\gamma = \lim_m \left(\sum_{n=1}^m 1/n - \ln m\right) = 0.5772\ldots$.

[1] The name is due to Euler studying the properties of a divergent series which is a solution of the equation with $z = a = 1$.

Then

$$\phi(z) = e^{az}\left(-\gamma - \log az - \sum_{n=1}^{\infty}\frac{(-1)^n(az)^n}{n!n}\right) \tag{2.5}$$

The convergence of the above expansion becomes very slow as z grows larger. To get an approximation accurate to three significant digits (with $a = 1$) for $z = 2$ we need 9 terms in the series, for $z = 5$, 19 terms and for $z = 10$, 36 terms. So for large values of z this expansion is very inefficient.

On the other hand we could have tried to solve the equation by making a series ansatz of the form

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^{-n} \tag{2.6}$$

which leads to the solution

$$\varphi(z) = \sum_{n=0}^{\infty}\frac{(-1)^n n!}{(az)^{n+1}} \tag{2.7}$$

This series, known as the Euler series, has a radius of convergence of 0, which means that it diverges for all $z \in \mathbb{C}$. How can a series which diverges everywhere be of any use to us? Turns out that these divergent series, which are also called asymptotic series (see definition 2.6), are of great use and play an important role in physics. For instance, asymptotic series gives an approximation to our solution and often even just the first terms gives an accurate approximation.

Now the error of the approximation given by the first term of the series, $1/az$ (again with $a = 1$), is 0.596 for $z = 1$, 0.0104 for $z = 5$, 0.00156 for $z = 10$, 0.000409 for $z = 20$ and $1.94 \cdot 10^{-6}$ for $z = 100$. As one can see, even the first term of the asympotic series gives more and more accurate approximation z grows.

One difference to the convergent expansion is that we cannot keep adding terms to inrease accuracy since series diverges. However there exists methods to obtain more accurate approximations as discussed in [14],[15].

Is there a connection between our two solutions? It turns out that the series solution (2.7) is an asymptotic expansion of the integral solution (2.3) and we write this as

$$\phi(z) = \int_z^{\infty} dt\,\frac{e^{-a(t-z)}}{t} \sim \varphi(z) = \sum_{n=0}^{\infty}\frac{(-1)^n n!}{(az)^{n+1}} \quad \text{as} \quad z \to \infty \tag{2.8}$$

Now that we have an example what an asymptotic series looks like and why they are useful we will define them properly.

2.1 Asymptotics

We'll start by giving the basic definitons of asymptotics in the complex plane, and then discuss the Stokes phenomena that arises for the asymptotic expansions in the complex plane. More information can be found in the appendix and for example in [4, 16].

Definition 2.2 (Asymptotic to)**.** Let $x_0 \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ be functions, such that $g(x) \neq 0$ in a neighbourhood of x_0, except possibly at x_0. Then we say that

$$f(x) \sim g(x) \quad \text{as} \quad x \to x_0, \tag{2.9}$$

if the relative error between f and g goes to zero as $x \to x_0$, that is,

$$f(x) - g(x) = o(g(x)) \quad \text{as} \quad x \to x_0, \tag{2.10}$$

where

$$f(x) = o(g(x)) \quad \text{as} \quad x \to x_0, \tag{2.11}$$

if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{2.12}$$

On the real line there is at most two possible directions along which we approach x_0 so we can easily distinguish the asymptotic behaviour whether we approach from below or above x_0. On the other hand, in the complex plane the limit $z \to z_0$ includes all the possible paths along which $z \to z_0$, so we can't distinguish these two limits anymore. This leads to nonuniqueness of the asympotic behaviour. Another cause for nonuniqueness is z approaching infinity as this includes (among all other possible paths) $z \to \pm\infty$ along the real line.

A solution for this problem is to restrict along which paths z can approach z_0. This restriction is called the sectorial domain:

Definition 2.3 (Sectorial domain)**.** A sectorial domain is a simply connected domain $\mathcal{D}(\theta_1,\theta_2)$ between the rays $z_0 + e^{i\theta_1}[0,\infty)$ and $z_0 + e^{i\theta_2}[0,\infty)$, $\theta_1 < \theta_2$. The angle $\theta_2 - \theta_1$ is called the opening of $\mathcal{D}$. An example is shown in figure 1.

Now we can define the asymptotic relation in the complex plane:

Definition 2.4 (Asymptotic relation in the complex plane)**.** Let $z_0 \in \mathbb{C}$ and $f, g : \mathbb{C} \to \mathbb{C}$ be complex functions, such that $g(z) \neq 0$ in a neighbourhood of z_0, except possibly at z_0. Then we say that

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0, \tag{2.13}$$

if

$$f(z) - g(z) = o(g(z)) \quad \text{as} \quad z \to z_0, \tag{2.14}$$

such that $z \to z_0$ along paths that lie in the sector of validity $\mathcal{D}(\theta_1, \theta_2)$ that depend on the functions f and g.

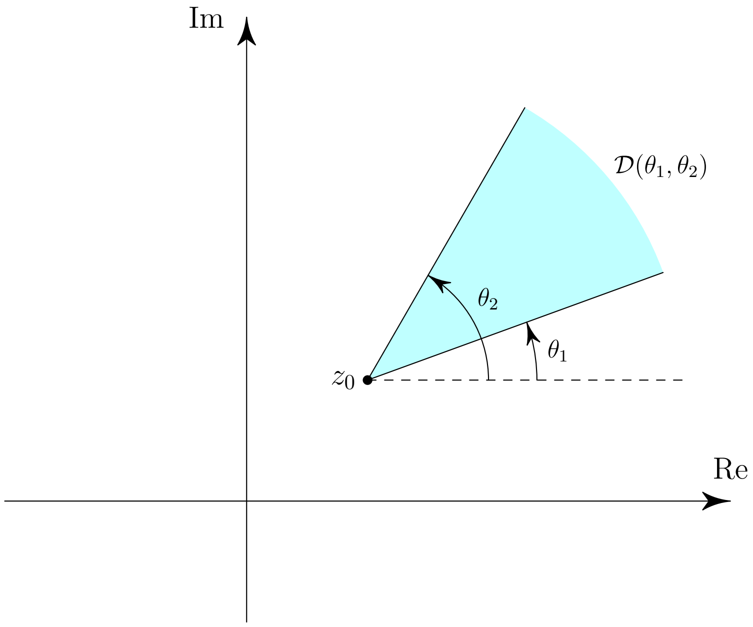

Figure 1. The sector of validity $\mathcal{D}(\theta_1, \theta_2)$, between angles θ_1 and θ_2, where the asymptotic relation is valid. The paths along which $z \to z_0$ must lie in this region.

Now that we know what it means for two function to be asymptotic, we can define the asymptotic series. Let's consider the formal series

$$\varphi(z) = \sum_{n=0}^{\infty} a_n \phi_n(z), \tag{2.15}$$

where $a_n \in \mathbb{C}$ and $\phi_n(z)$ are complex functions. Notice that symbol $\varphi(z)$ representing the formal series is not a function in general since the series is usually divergent.

Definition 2.5 (Asymptotic sequence)**.** A sequence $(\phi_n(z))_n$ of complex functions with a limit point z_0 in the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is an asymptotic sequence if there exists neighbourhood U of z_0 such that $\phi_n(z) \neq 0,\ z \in U \setminus \{z_0\}$ and for all n

$$\lim_{z \to z_0} \frac{\phi_{n+1}}{\phi_n} = 0. \tag{2.16}$$

Definition 2.6 (Asymptotic expansion in the complex plane)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function and $(\phi_n)_n$ be an asymptotic sequence for $z \to z_0$. The formal series

$$\varphi(z) = \sum_{n=0}^{N} a_n \phi_n(z), \tag{2.17}$$

is an asymptotic expansion, in the sense of Poincaré, of $f(z)$ to N as $z \to z_0$ in the sectorial domain $\mathcal{D}(\theta_1,\theta_2)$, if for all N

$$\lim_{z \to z_0} \frac{f(z) - \sum_{n=0}^{N} a_n \phi_n(z)}{\phi_N(z)} = 0 \tag{2.18}$$

or equivalently, if there exists constant $K_N > 0$ such that for all N

$$\left| f(z) - \sum_{n=0}^{N-1} a_n \phi_n(z) \right| \leq K_N \left| \phi_N(z) \right|, \tag{2.19}$$

and we write

$$f(z) \sim \sum_{n=0}^{N} a_n \phi_n(z), \quad z \to z_0. \tag{2.20}$$

Furthermore, the asymptotic expansion of the function $f(z)$ is unique in the sectorial domain.

Allthough the asymptotic expansion of $f(z)$ is unique, the asymptotic expansion is not unique to the function $f(z)$: An asymptotic expansion is asymptotic to a class of functions differing by exponentially small terms. Notice that asymptoticity is differs from the concept of convergence such that we are interested in what happens to the partial sum $\sum_n^N a_n \phi_n$ when $z \to z_0$ compared to when $N \to \infty$ and that asymptocity is a relative property of the expansion coefficients and the class of functions which it is asymptotic to. Asymptoticity of an expansion doesn't mean anything without knowing the function it is asymptotic to.

In our example 2.1 we had the series

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(az)^{n+1}} \tag{2.21}$$

with $a_n = (-1)^n n!/a^{n+1}$ and $\phi_n(z) = z^{-n-1}$. The sequence $(z^{-n-1})_n$ is an asymptotic sequence for $z \to \infty$ since

$$\lim_{z\to\infty} \frac{z^{-(n+1)-1}}{z^{-n-1}} = \lim_{z\to\infty} z^{-1} = 0 \tag{2.22}$$

Furthermore, the series is an asymptotic expansion of the function

$$\phi(z) = \int_z^{\infty} dt\, \frac{e^{-a(t-z)}}{t} = \int_0^{\infty} d\zeta\, \frac{e^{-z\zeta}}{\zeta + a} \tag{2.23}$$

as $z \to \infty$ in the sectorial domain $\mathcal{D}_\delta = \{z \in \mathbb{C} : |\arg z| < \pi/2 - \delta,\ \delta \in (0,\pi/2)\}$, shown in figure 2, where we made the change of variables $\zeta = a(t-z)/z$.

Let $S_{N-1} = \sum_{n=0}^{N-1} (-1)^n \zeta^n / a^{n+1}$. Using the relation

$$\zeta S_{N-1} = 1 - a S_{N-1} + (-1)^{N-1} \frac{\zeta^N}{a^n} \tag{2.24}$$

we can write $1/(\zeta + a)$ as

$$\frac{1}{z+a} = S_{N-1} + \frac{(-1)^N}{a^N} \frac{\zeta^N}{a+u} \tag{2.25}$$

and substituting this into the equation (2.3) we get

$$\begin{aligned}
\phi(z) &= \int_0^{\infty} d\zeta\, \frac{e^{-z\zeta}}{(\zeta + a)} = \int_0^{\infty} d\zeta\, e^{-z\zeta} \left(S_{N-1} + \frac{(-1)^N}{a^N} \frac{\zeta^N}{a+u} \right) \\
&= \sum_{n=0}^{N-1} \frac{(-1)^n}{a^{n+1}} \int_0^{\infty} d\zeta\, e^{-z\zeta} \zeta^n + \frac{(-1)^N}{a^N} \int_0^{\infty} d\zeta\, \frac{e^{-z\zeta}\zeta^N}{\zeta + a} \\
&= \sum_{n=0}^{N-1} \frac{(-1)^n}{a^{n+1}} \frac{1}{z^{n+1}} \Gamma(n+1) + \frac{(-1)^N}{a^N} \int_0^{\infty} d\zeta\, \frac{e^{-z\zeta}\zeta^N}{\zeta + a} \\
&= \sum_{n=0}^{N-1} \frac{(-1)^n n!}{(az)^{n+1}} + \frac{(-1)^N}{a^N} \int_0^{\infty} d\zeta\, \frac{e^{-z\zeta}\zeta^N}{\zeta + a}
\end{aligned} \tag{2.26}$$

Then

$$\left|\phi(z) - \sum_{n=0}^{N-1} \frac{(-1)^n n!}{(az)^{n+1}}\right| = \left|\frac{(-1)^N}{a^N} \int_0^\infty d\zeta\, \frac{e^{-z\zeta}\zeta^N}{\zeta + a}\right|$$
$$\leq \frac{1}{|a|^N} \int_0^\infty d\zeta\, \frac{e^{-\zeta \operatorname{Re} z}\zeta^N}{|\zeta + a|} \tag{2.27}$$

Assume $\operatorname{Re} a > 0$. Then $|\zeta + a| \geq |\operatorname{Re} a|$. Since $z \in \mathcal{D}_\delta$, $\sin\delta \leq \operatorname{Re} z / |z|$. Thus

$$\begin{aligned}
&\leq \frac{|a|^{-N}}{|\operatorname{Re} a|} \int_0^\infty d\zeta\, e^{-\zeta |z| \sin\delta} \zeta^N \\
&\leq \frac{|a|^{-N}}{|\operatorname{Re} a|} \frac{1}{|z|^{N+1} (\sin\delta)^{N+1}} \int_0^\infty d\omega\, e^{-\omega} \omega^N \\
&\leq \frac{|a|^{-N}}{|\operatorname{Re} a|} \frac{\Gamma(N+1)}{|z|^{N+1} (\sin\delta)^{N+1}} \\
&=: K_N |z|^{-N-1}
\end{aligned} \tag{2.28}$$

Thus

$$\phi(z) = \int_0^\infty d\zeta\, \frac{e^{-z\zeta}}{(\zeta + a)} \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(az)^{n+1}} \quad \text{as} \quad z \to \infty \tag{2.29}$$

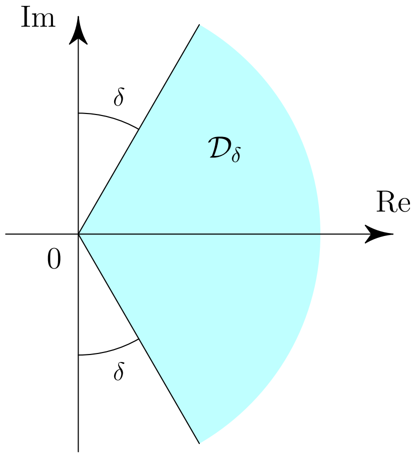

Figure 2. The sectorial domain $\mathcal{D}_\delta$, where $|\arg z| < \frac{\pi}{2} - \delta$.

2.1.1 Stokes phenomenon

As noted in the previous section, to have unique asymptotic behaviour in the complex plane, we had to restrict our domain to sectorial domains, where the asymptotic relation was valid. This means for the same analytic function different asymptotic expansion hold in different sectors, also known as the Stokes phenomenon. These different sectors are called Stokes regions and the lines separating the regions are called either Stokes lines or anti-Stokes lines depending on the behaviour of the expansion along these lines.

Assume that the asymptotic relation $f(z) \sim g(z)$ as $z \to z_0$ holds in some sectorial domain $\mathcal{D}(\theta_1,\theta_2)$. By our definition 2.4, this means that $f(z) - g(z)$ is small compared to $g(z)$ and in $f(z) = g(z) + (f(z) - g(z))$ we neglect the latter smaller term, which is called subdominant or recessive compared with with dominant terms $f(z)$ and $g(z)$. As we approach the boundary of the sector, the subdominant term $f - g$ becomes less subdominant and at the boundary $f - g$ and g are equal in magnitude, they are purely oscillatory. This line at boundary where the subdominant and dominant terms become equal in magnitude is called the anti-Stokes line. Inside the sector, there is line where the terms are most unequal, they are purely real. This line is called the Stokes line. Some references as [4] define the Stokes and anti-Stokes lines oppositely, but our definitions is the one used in the resurgence framework.

When we cross the anti-Stokes line, the subdominant term becomes dominant and dominant term becomes subdominant. This exchange of roles is the Stokes phenomenon and was first discovered by G.G. Stokes in 1864 [17].

Another equivalent definitions of the Stokes phenomena, Stokes lines and anti-Stokes lines are given within the Borel resummation method, exact WKB method and steepest descent method [18], [19], [20].

In our example the solution for the Euler equation was

$$\begin{aligned}\phi(z) &= Ce^{az} + \int_0^\infty d\zeta \, \frac{e^{-z\zeta}}{\zeta + a} \\ &=: \phi_1(z) + \phi_2(z) \end{aligned} \tag{2.30}$$

and we wanted physical solutions, that is, solutions that don't blow up at infinity. Here ϕ_1 is the dominant solution and ϕ_2 subdominant. When the argument of z becomes $\pm\pi/2$, they become oscillatory. We have found the anti-Stokes line, the imaginary axis. After crossing the imaginary axis ϕ_1 becomes subdominant and ϕ_2 dominant and at the negative real axis they become most subdominant and dominant respectively. This is the Stokes line. Because the solution ϕ_2 tends to infinity on the left half-plane the physical solution is

$$\phi(z) = Ce^{az} \quad \text{as} \quad z \to \infty, \ \arg z \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \tag{2.31}$$

2.2 Borel summation

In perturbation theory we expand our physical observable in a small parameter,

$$\mathcal{O} = \sum_n a_n \lambda^n \tag{2.32}$$

where for example $\lambda = \hbar$ in WKB, coupling constant $\lambda = g$ in perturbative quantum field theory. If the perturbation series is divergent asymptotic expansion with zero radius of convergence how can we calculate the value of the observable? For a series with coefficients growing not faster than $n!$ the Borel summation gives a method of resummation of the divergent series.

2.2.1 Borel transformation

In regards to asympotics we want to work at infinity rather than at the origin, so we set $z = 1/\lambda$ and consider the space of formal power series without a constant term $z^{-1}\mathbb{C}[z^{-1}]$

$$z^{-1}\mathbb{C}[z^{-1}] := \left\{ \varphi(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} = \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \dots \,\Big|\, a_n \in \mathbb{C} \right\} \tag{2.33}$$

This space becomes an algebra when equipped with the Cauchy product, see [21], and is called the formal or multiplicative model.

The Borel analysis of divergent formal series starts with improving the convergence of the divergent series by the means of the formal Borel transform.

Definition 2.7 (Borel transform)**.** The formal Borel transform is a linear map $\mathcal{B} : z^{-1}\mathbb{C}[z^{-1}] \to \mathbb{C}[\zeta]$ defined by

$$\mathcal{B} : \varphi(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} \mapsto \hat{\varphi}(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n \tag{2.34}$$

where $\mathbb{C}[\zeta]$ is the set of formal Borel transforms and it is a convention to use the indeterminate ζ and the "hat" notation.

The space $\mathbb{C}[\zeta]$ of formal Borel transforms forms a convolutive algebra when equipped with the convolution product

$$(\hat{\varphi} * \hat{\psi})(\zeta) = \mathcal{B}\varphi * \mathcal{B}\psi \tag{2.35}$$

and is called the convolutive model.

Because the Borel transform was defined on $z^{-1}\mathbb{C}[z^{-1}]$ where the Cauchy product doesn't have a unit element, the convolutive algebra has no unit element. This can be remedied by constructing an extended convolutive algebra on the space $\mathbb{C} \times \mathbb{C}[\zeta]$ by defining $\delta := (1{,}0) \in \mathbb{C} \times \mathbb{C}[\zeta]$ and denote the extended algebra by $\mathbb{C}\delta \oplus \mathbb{C}[\zeta]$. Then we can define the extended Borel transform

$$\mathcal{B} : \mathbb{C}[z^{-1}] \to \mathbb{C}\delta \oplus \mathbb{C}[\zeta], \quad \mathcal{B}(1) = \delta \tag{2.36}$$

and the unital convolutive algebra $\mathbb{C}\delta \oplus \mathbb{C}[\zeta]$ with

$$\mathcal{B}(\psi) = \mathcal{B}(c + \varphi) = c\delta + \hat{\varphi} \tag{2.37}$$

where $\psi = c + \varphi \in \mathbb{C} \oplus z^{-1}\mathbb{C}[z^{-1}]$. The complex number c is called the residual coefficient.

For convergent Borel transforms the convolution product is defined by the convolution integral:

Let $\hat{\varphi}, \hat{\psi} \in \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$ and let $R < \min\{\rho_1, \rho_2\}$, where ρ_1 and ρ_2 are the convergence radii of the analytic germs $\hat{\varphi}$ and $\hat{\psi}$ respectively. Then the formula

$$(\hat{\varphi} * \hat{\psi})(\zeta) = \int_0^\zeta du\, \hat{\varphi}(u)\hat{\psi}(\zeta - u). \tag{2.38}$$

defines a holomorphic function in the disk $B(0{,}R)$ with a radius of convergence $\geq R$.

The Borel transform has the following properties:

(i)
$$\mathcal{B}(z^{-\alpha-1}) = \frac{\zeta^\alpha}{\Gamma(\alpha+1)}, \ \alpha > -1 \tag{2.39}$$

(ii)

$$\mathcal{B}\left(\frac{d}{dz}\varphi\right) = -\zeta\hat{\varphi}(\zeta) \tag{2.40}$$

(iii)

$$\mathcal{B}(\varphi(z+c)) = e^{-c\zeta}\hat{\varphi}(\zeta),\ c \in \mathbb{C} \tag{2.41}$$

(iv)

$$\mathcal{B}(z^{-1}\varphi) = \int_0^\zeta du\,\hat{\varphi}(u) \tag{2.42}$$

(v)

$$\mathcal{B}(z\varphi) = \frac{d}{d\zeta}\varphi(\zeta) \tag{2.43}$$

Example 2.8. The Borel trasform of the Euler series (2.7) is

$$\hat{\varphi}(\zeta) = \mathcal{B}(\varphi(z)) = \mathcal{B}\left(\sum_{n=0}^{\infty}\frac{(-1)^n n!}{(az)^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}\zeta^n = \frac{1}{\zeta+a} \tag{2.44}$$

A particularly interesting subset of formal power series is a type of series called Gevrey series and especially Gevrey order 1 series.

Definition 2.9 (Gevrey series of order $\frac{1}{s}$)**.** A formal series $\sum_{n=0}^{\infty} a_n z^{-n-1}$ is said to be of Gevrey order $\frac{1}{s}$, or $\frac{1}{s}$-Gevrey, if there exists constants $\alpha > 0$ and $A > 0$ such that for all n the coefficients a_n satisfy

$$|a_n| \leq \alpha A^n (n!)^s. \tag{2.45}$$

The constants α and A do not depend on n. The set of all $\frac{1}{s}$-Gevrey series is denoted by $\mathbb{C}[z^{-1}]_{1/s}$. For 1-Gevrey series we also use the notation $\mathbb{C}_G[z^{-1}]$.

Example 2.10. The Euler series is a 1-Gevrey series:

$$|a_n| = \left|\frac{(-1)^n n!}{a^{n+1}}\right| = \frac{n!}{a^{n+1}} := \alpha A^n n! \tag{2.46}$$

with $\alpha = A = a^{-1}$.

The connection between 1-Gevrey series and the Borel tranform is that usually our formal series is divergent but its coefficients a_n don't grow faster than the factorial $n!$. In the Borel transform we divide the coefficients of the formal series by the factorial

$n!$ so that we can improve its convergence and get a convergent series in the Borel ζ-plane.

Proposition 2.11. *A formal power series $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ is 1-Gevrey series if and only if its Borel transform has a positive radius of convergence, $\hat{\varphi} = \mathcal{B}\varphi \in \mathbb{C}\{\zeta\}$.*

Proposition 2.12. *A formal power series $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ is convergent, i.e. $\varphi \in z^{-1}\mathbb{C}\{z^{-1}\}$, if and only if its Borel transform $\hat{\varphi} = \mathcal{B}\varphi$ defines an entire function of exponential order.*

From the above results we can see that if we have a divergent series, its Borel transform cannot be an entire function, it has singularities in the Borel plane. Thus if a divergent series $\varphi \in \mathbb{C}[z^{-1}]$ is 1-Gevrey series, its Borel transform $\hat{\varphi} = \mathcal{B}\varphi \in \mathbb{C}\{\zeta\}$ defines an analytic germ at the origin in the Borel plane (ζ-plane) with a radius of convergence defined by the nearest singularity. This gives us the notion that analytical continuation can be used in the Borel-plane in hopes of getting a suitable analytic extension of the original divergent series when transformed back into to the original complex plane. This transformation is achieved by the Laplace transform: Using the gamma function

$$n! = \Gamma(n+1) = \int_0^\infty dt\, e^{-t} t^n = z^{n+1} \int_0^\infty d\zeta\, e^{-z\zeta} \zeta^n \tag{2.47}$$

we can write the formal series as

$$\begin{aligned}
\varphi(z) &= \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} = \sum_{n=0}^{\infty} a_n \int_0^\infty d\zeta\, e^{-z\zeta} \frac{\zeta^n}{n!} \\
&= \int_0^\infty d\zeta\, e^{-z\zeta} \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n \\
&= \mathcal{L}(\mathcal{B}\varphi)(z) \qquad (2.48)
\end{aligned}$$

From the above equation we see that the Borel transform is an inverse of the Laplace transform, $\mathcal{B} = \mathcal{L}^{-1}$. The equality to the ordinary integral repsesentation of the inverse Laplace transform can be seen as follows:

$$
\begin{aligned}
2\pi i \mathcal{L}^{-1}\varphi(z) &= \int_{\sigma-i\infty}^{\sigma+i\infty} dz\, e^{z\zeta}\varphi(z) = \int_{\sigma-i\infty}^{\sigma+i\infty} dz\, e^{z\zeta}\varphi(z) + \underbrace{\int_{C_R} dz\, e^{z\zeta}\varphi(z)}_{=0,\ R\to\infty} \\
&= \lim_{R\to\infty}\left(\int_{\sigma-iR}^{\sigma+iR} + \int_{C_R}\right) dz\, e^{z\zeta}\varphi(z) = \lim_{R\to\infty}\sum_{n=0}^{\infty} a_n \left(\int_{\sigma-iR}^{\sigma+iR} + \int_{C_R}\right) dz\, \frac{e^{z\zeta}}{z^{n+1}} \\
&= \lim_{R\to\infty}\sum_{n=0}^{\infty} a_n \cdot 2\pi i\, \mathrm{Res}(e^{z\zeta}/z^{n+1}, 0) \\
&= 2\pi i \sum_{n=0}^{\infty} a_n \frac{1}{n!} \lim_{z\to 0} \frac{d^n}{dz^n}\left(z^{n+1}\frac{e^{z\zeta}}{z^{n+1}}\right) \\
&= 2\pi i \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n
\end{aligned} \tag{2.49}
$$

where $C_R(t) = \sigma + Re^{it},\ \pi/2 \leq t \leq 3\pi/2$, and the integral over C_R goes to 0 as R goes to ∞ since $\mathrm{Re}(\zeta(\sigma + re^{it})) = \zeta\sigma + \zeta R\cos t$ and $\cos t < 0$ for all $\pi/2 \leq t \leq 3\pi/2$. And thus

$$
\mathcal{L}^{-1}\varphi = \mathcal{B}\varphi. \tag{2.50}
$$

2.2.2 Directional Laplace Transform

If we our Borel transform has singularities along the positive real line we can't use the regular Laplace transform. But we can define a Laplace transform along a ray from the origin other than the positive real line where the Borel transform is analytic.

Definition 2.13 (Directional Laplace transform)**.** The Laplace transform in the direction θ is defined as a linear map $\mathcal{L}^\theta$:

$$
(\mathcal{L}^\theta\hat{\varphi})(z) = \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta}\hat{\varphi}(\zeta) \tag{2.51}
$$

where the path of integration is a ray from the origin, $\zeta = te^{i\theta},\ t \in [0,\infty)$. If $\hat{\varphi}$ is analytic and exponentially bounded, $|\hat{\varphi}(\zeta)| \leq Ae^{c_0\zeta}$, A,$c_0 > 0$, along the ray $e^{i\theta}\mathbb{R}^+$ the integral converges and $\mathcal{L}^\theta\hat{\varphi}$ is analytic on the half-plane $\Pi^\theta_{c_0} := \{z \in \mathbb{C} : \mathrm{Re}(ze^{i\theta}) > c_0\}$.

The usual properties of the Laplace transform also apply to the directional Laplace transform.

The directional Laplace transform maps exponentially bounded analytic functions along a ray in the Borel ζ-plane to analytic functions on a half-plane in the original complex z-plane. If the exponentially bounded function $\hat{\varphi}$ is a Borel transform $\mathcal{B}\varphi$ of a 1-Gevrey series φ, then the directional Laplace transform $\mathcal{L}^\theta\hat{\varphi}$ admits the series φ as an asymptotic expansion and we write

$$(\mathcal{L}^\theta\hat{\varphi})(z) \sim \varphi(z) \quad \text{as} \quad z \to \infty \tag{2.52}$$

Proof can be found in [21].

2.2.3 Borel summation

Now we can define a composite operator $\mathcal{S}^\theta := \mathcal{L}^\theta \circ \mathcal{B}$ which maps formal power series $\varphi \in \mathbb{C}[z^{-1}]$ to analytic functions $\bar{\varphi} := \mathcal{S}^\theta\varphi = (\mathcal{L}^\theta \circ \mathcal{B})\varphi = \mathcal{L}^\theta\hat{\varphi}$ on the half-plane $\Pi^\theta_{c_0}$ and $\mathcal{S}^\theta\varphi$ is called the Borel-Laplace summation or the Borel sum of the formal power series φ. The Borel-Laplace summation gives us a tool for resummation of the divergent 1-Gevrey series.

Definition 2.14 (Borel-Laplace summation)**.** Let φ 1-Gevrey series with a non-zero residual coefficient. Then we define the linear map $\mathcal{S}^\theta$ as

$$\mathcal{S}^\theta\varphi(z) = \mathcal{L}^\theta(\mathcal{B}\varphi)(z) = \mathcal{L}^\theta(c\delta + \hat{\varphi})(z) = c + \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta}\hat{\varphi}(\zeta), \tag{2.53}$$

and the series φ is called Borel summable in the direction θ. The Borel summation is an algebra homomorphism.

If a Borel transform of a 1-Gevrey series is exponentially bounded, then replacing $(\mathcal{L}^\theta\hat{\varphi})(z)$ by $(\mathcal{S}^\theta\varphi)(z)$ in (2.52) we get that

$$(\mathcal{S}^\theta\varphi)(z) \sim \varphi(z) \quad \text{as} \quad z \to \infty \tag{2.54}$$

Example 2.15. We found before that the Euler series (2.7)

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(az)^{n+1}} \tag{2.55}$$

is a 1-Gevrey series and its Borel transform is

$$\hat{\varphi}(\zeta) = \frac{1}{\zeta + a} \tag{2.56}$$

$$\begin{tikzcd}
\begin{matrix}\text{Formal Series} \\ \varphi(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}\end{matrix} \arrow[rr, "\text{Borel Transform}"] & & \begin{matrix}\text{Convergent Series} \\ \mathcal{B}\left[\varphi\right](\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n\end{matrix} \arrow[dl, "\text{Laplace Transform}"] \\
& \begin{matrix}\text{Analytic function} \\ \mathcal{S}^{\theta}\varphi(z) = \int_0^{\infty e^{i\theta}} d\zeta e^{-z\zeta}\mathcal{B}\left[\varphi\right](\zeta)\end{matrix} \arrow[ul, "\text{Asymptotic Expansion}"] &
\end{tikzcd}$$

Figure 3. Borel summation graph

The Borel transform $\hat{\varphi}(\zeta)$ is analytic in the disc $D = \{\zeta \in \mathbb{C} : |\zeta| < |a|\}$ and it can be analytically continued along any direction $\theta \neq \arg(-a)$ and it is exponentially bounded with $c_0 = 0$. Then for all directions $\theta \neq \arg(-a)$ the Borel sum of the Euler series is

$$(\mathcal{S}^{\theta}\varphi)(z) = \int_0^{\infty e^{i\theta}} d\zeta \, \frac{e^{-z\zeta}}{\zeta + a} \tag{2.57}$$

and

$$\int_0^{\infty e^{i\theta}} d\zeta \, \frac{e^{-z\zeta}}{\zeta + a} \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(az)^{n+1}} \quad \text{as} \quad z \to \infty \tag{2.58}$$

For $a > 0$ and $\theta = 0$ this is the same result as in example 2.1.

2.2.4 Lateral Borel sum

In the last example the Borel sum was well-defined for all directions but $\arg(-a)$ because the function $\hat{\varphi}(\zeta) = 1/(\zeta + a)$ has a pole at $\zeta = -a$. To understand what happens on the singular direction we deform our path of integration to go around the singularity.

Definition 2.16 (Lateral Borel sum)**.** Let φ be a formal series with residual coefficient

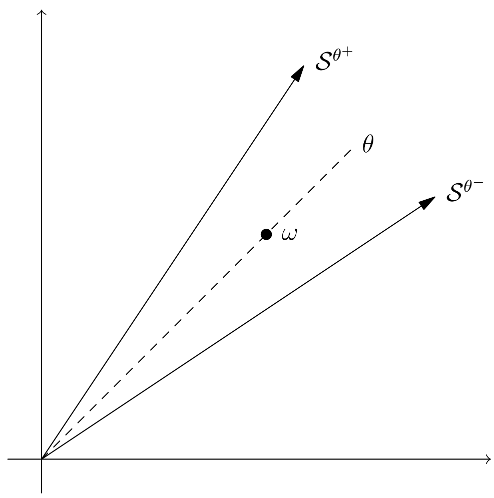

Figure 4. The lateral Borel sums in the direction of θ

c. Then we define the right ($\theta^+ > \theta$) and the left ($\theta^- < \theta$) lateral Borel sum as

$$\mathcal{S}^{\theta^+}\varphi(z) = c + \int_0^{\infty e^{i\theta^+}} d\zeta\, e^{-z\zeta}\hat{\varphi}(\zeta) \tag{2.59}$$

$$\mathcal{S}^{\theta^-}\varphi(z) = c + \int_0^{\infty e^{i\theta^-}} d\zeta\, e^{-z\zeta}\hat{\varphi}(\zeta) \tag{2.60}$$

where $\theta^{\pm} = \theta \pm \varepsilon$ for some small $\varepsilon > 0$.

The name right and left comes from the singularity being seen on the right or on the left when integrating along the direction θ^+ or θ^-.

We know by the monodromy theorem that analytic continuations of $\hat{\varphi}$ along two different paths are equal only if the paths are homotopic. Because of the singularity ω the paths corresponging to the lateral Borel sums aren't homotopic and thus lateral Borel sums do not give rise to the same analytic continuation.

Example 2.17. For simplicity assume that $a > 0$. Then the singular direction of the Borel transform of the Euler series is $\theta = \pi$ and the lateral Borel sums are

$$\mathcal{S}^{\pi^+}\varphi(z) = \int_0^{-\infty - i\varepsilon} d\zeta\, \frac{e^{-z\zeta}}{\zeta + a} \tag{2.61}$$

$$\mathcal{S}^{\pi^-}\varphi(z) = \int_0^{-\infty + i\varepsilon} d\zeta\, \frac{e^{-z\zeta}}{\zeta + a} \tag{2.62}$$

Now the difference between the lateral Borel sums is

$$
\begin{aligned}
(\mathcal{S}^{\theta^+} - \mathcal{S}^{\theta^-})\varphi(z) &= \int_0^{\infty e^{i\theta^+}} d\zeta\, e^{-z\zeta} \frac{e^{-z\zeta}}{\zeta + a} - \int_0^{\infty e^{i\theta^-}} d\zeta\, \frac{e^{-z\zeta}}{\zeta + a} \\
&= \int_{\mathcal{H}} d\zeta\, \frac{e^{-z\zeta}}{\zeta + a} \\
&= \int_{C_\varepsilon} d\zeta\, \frac{e^{-z\zeta}}{\zeta + a} \\
&= -2\pi i e^{az}
\end{aligned} \tag{2.63}
$$

where $\mathcal{H}$ is the Hankel contour coming from ∞ to $\zeta = -a$ along the negative real axis, going around the singlularity $\zeta = -1$ clockwise along C_ε, the circular path of radius ε around $\zeta = -1$, and going back to ∞ along the negative real axis as shown in the figure 5. On the third line we used fact that the integrals along the negative real axis cancel out.

Remembering the general solution of the Euler equation (2.2) we notice that the difference between the lateral Borel sums is exactly the homogenous solution. Due to the properties of the Borel summation the lateral Borel sums are themselves solutions of the Euler equation so their linear combination should also be a solution.

Because the lateral Borel sums differ in the singular direction there is a discontinuity in the Borel summation when crossing the singular direction. This shouldn't come as a surprise beacuse of the Stokes phenomena and due to this the singular direction is the Stokes line. The region divided by the Stokes line is the Stokes region.

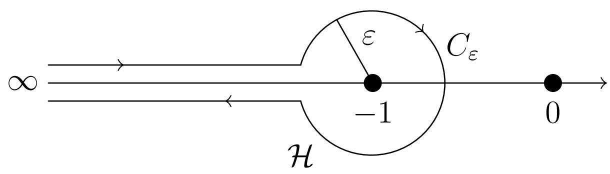

Figure 5. The Hankel contour around the point $\zeta = -a$.

2.3 Resurgence theory

As we saw before the Borel transform of a divergent formal series has a finite positive radius of convergence. This meant that there exist singularities in the Borel plane and that the Borel transform defines an analytic germ at the origin with the radius of convergence defined by the nearest singularity. And for the Borel summation to be well defined the Borel transform had to be analytically continuable along the direction of integration. In the Euler equation example the Borel transform had only one singularity and it could analytically continued to $\mathbb{C} \setminus \{a\}$. But what if there are infinitely many singularities such that the Borel transform cannot be analytically continued along any direction? Well then the Laplace integral isn't defined and the Borel sum doesn't make sense. This leads to having some restriction on the number of singularities that the Borel transform has. The analytic germs which satisfy these conditions on the Borel plane are called resurgent functions and the formal series corresponding to these are resurgent formal series.

Important fact to note is that the Borel plane is a Riemann surface (see Appendix C) and that there might be infinitely many singularites. This is nicely illustrated by the following example:

Consider a formal series whose Borel transform is

$$\hat{\varphi}(\zeta) = \frac{1}{\zeta} \operatorname{Log}(\zeta + 1) \tag{2.64}$$

where $\operatorname{Log} z$ is the principal branch of the logarithm. Now the Borel plane is the Riemann surface of the logarithm. Clearly $\zeta = -1$ is a logarithmic singularity and $\zeta = 0$ is a removable singlarity on the principal sheet. If we analytically continue $\hat{\varphi}$ such that that path γ goes once anticlockwise around $\zeta = -1$ the continuation is

$$\hat{\varphi}_\gamma(\zeta) = \frac{1}{\zeta} \left(\operatorname{Log}(\zeta + 1) + 2\pi i\right) = \frac{2\pi i}{\zeta} + \frac{1}{\zeta} \operatorname{Log}(\zeta + 1) \tag{2.65}$$

Now the removable singularity $\zeta = 0$ becomes a pole on an other Riemann sheet.

2.3.1 Simple resurgent functions

As noted above, the Borel plane is a, possibly infinite sheeted, Riemann surface. In its most abstract definition, a Riemann surface is just a two dimensional manifold with a complex structure (see Appendix C). Now an other way to define the Riemann surface if we know the positions of all of the singularities is to define it as an universal covering of the complex plane with the singularities removed.

Definition 2.18 (Riemann surface)**.** Let $\Gamma \subset \mathbb{C}$ be a possibly infinite disrete subset. Let $\hat{\Sigma}$ be the set of equivalence classes of paths $\gamma : [0,1] \to \mathbb{C} \setminus \Gamma$ with fixed endpoint homotopy. Then let $\pi : \hat{\Sigma} \to \mathbb{C} \setminus \Gamma$ be the covering map such that $\pi(\zeta) = \gamma(1)$, where γ is a representative of the equivalence class $\zeta = [\gamma]$. Then $\hat{\Sigma}$ is a Riemann surface by pulling back by π the complex structure of $\mathbb{C} \setminus \Gamma$.

Now we can define the restriction on the singularities of the Borel transform in the Borel plane.

Definition 2.19 (Endlessly continuable)**.** An analytic germ at the origin $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ is endlessly continuable on the Riemann surface $\hat{\Sigma}$ if for every $R > 0$ there exists finite set $\Gamma_R(\hat{\varphi}) \in \mathbb{C}$ of singularities of $\hat{\varphi}$ such that $\hat{\varphi}$ can be analytically continued along all paths γ whose length is less than R avoiding the singularities $\Gamma_R(\hat{\varphi})$.

Definition 2.20 (Resurgent function)**.** An endlessly continuable function $\hat{\varphi}$ on the Riemann surface $\hat{\Sigma}$ is called a resurgent function. The formal series whose Borel transform is a resurgent function is called a formal resurgent series. The space of resurgent fuctions is denoted by $\hat{\mathcal{R}}$ and the space of formal resurgent series by $\hat{\mathcal{R}}$.

Sometimes the Laplace transform of a endlessly continuable function is called a resurgent function [22].

The space of resurgent functions is an algebra when equipped with the convolution product and it is stable under the convolution. For more details see [21], [23].

For now we will focus on a class of resurgent functions with a certain type of singularities. We restrict the singularities to be either poles or logarithmic singularities and call these simple singularities.

Definition 2.21 (Simple Singularity)**.** An analytic function $\hat{\varphi}$ in a open disk $B \subset \mathbb{C}$ is said to have a simple singlularity at ω, ω in closure of B, if there exists $\alpha \in \mathbb{C}$ and

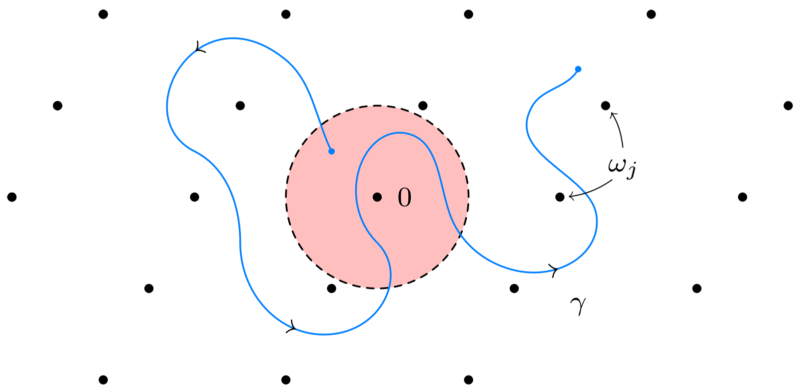

Figure 6. The path γ of analytic continuation of endlessly continuable germ $\hat{\varphi}$

two analytic germs at the origin $\hat{\Phi}, R \in \mathbb{C}\{\zeta\}$ such that

$$\hat{\varphi}(\zeta) = \frac{\alpha}{2\pi i(\zeta-\omega)} + \frac{1}{2\pi i}\hat{\Phi}(\zeta-\omega)\log(\zeta-\omega) + R(\zeta-\omega) \tag{2.66}$$

for all $\zeta \in D$ close enough to ω, i.e. $|\zeta-\omega|$ is small enough. α is the residuum of $\hat{\varphi}$, $\hat{\Phi}$ the variation or minor of the singularity of $\hat{\varphi}$ at ω and R the analytic terms close to ω. The mapping σ_ω

$$\sigma_\omega\hat{\varphi} = \alpha\delta + \hat{\Phi} \in \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\} \tag{2.67}$$

describes the singularity of $\hat{\varphi}$ at ω. Similarly in the formal model we define

$$\sigma_\omega\varphi = \alpha + \Phi \in \mathbb{C} \oplus \mathbb{C}_G[z^{-1}] \tag{2.68}$$

where $\Phi = \mathcal{B}^{-1}\hat{\Phi}$.

Definition 2.22 (Simple resurgent function)**.** A simple resurgent function is any $\hat{\varphi} \in \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$ endlessly continuable on the Riemann surface $\hat{\mathcal{R}}$ such that for any $\omega \in \Gamma$ and for each γ starting from the origin, avoiding all the singularities and ending in the disk D, $D \cap \Gamma = \{\omega\}$, the the branch of analytic continuation of $\hat{\varphi}$ has a simple singularity at ω. The space of simple resurgent is denoted by $\hat{\mathcal{R}}^S$

The space of simple resurgent functions $\hat{\mathcal{R}}^S$ is a subalgebra of the convolution algebra and the algebra of resurgent functions. This means that it is stable under convolution and the convolution product of simple resurgent functions yields new singularities which are either poles or logarithmic singularities.

Example 2.23. (i) The Euler series is a simple resurgent series and its Borel transform a simple resurgent function since

$$\hat{\varphi}(z) = \frac{1}{\zeta + a} = \frac{2\pi i}{2\pi i(\zeta + a)} =: \frac{\alpha}{2\pi i(\zeta - \omega)} \tag{2.69}$$

with $\alpha = 2\pi i$ and $\omega = -a$.

(ii) The function $\hat{\varphi}(\zeta) = \zeta^{-1}\,\mathrm{Log}(\zeta + 1)$ is a simple resurgent function:

$$\hat{\varphi}_\gamma(\zeta) = \frac{1}{\zeta}\left(\mathrm{Log}(\zeta + 1) + 2\pi i N\right) = \frac{2\pi i N}{\zeta} + \frac{1}{\zeta}\,\mathrm{Log}(\zeta + 1) \tag{2.70}$$

where $N \in \mathbb{Z}$ indexes the Riemann sheets. The singular behaviour is

$$\alpha + \hat{\Phi}(\zeta) = (2\pi i)^2 N + \frac{2\pi i}{\zeta - 1} \tag{2.71}$$

which means that principal sheet we have logarithmic sigularity on all the other sheet a pole and a logarithmic singularity.

Example 2.24. Let $\hat{\varphi}(\zeta) = \dfrac{1}{\zeta - \omega_1}$ and $\hat{\psi}(\zeta) = \dfrac{1}{\zeta - \omega_2}$. Then the convolution product is

$$(\hat{\varphi} * \hat{\psi})(\zeta) = \int_0^\zeta du\,\hat{\varphi}(u)\hat{\psi}(\zeta - u) = \int_0^\zeta du\,\frac{1}{u - \omega_1}\frac{1}{\zeta - u - \omega_2} \tag{2.72}$$

Using partial fraction decomposition we get

$$\begin{aligned}(\hat{\varphi} * \hat{\psi})(\zeta) &= \frac{1}{\zeta - (\omega_1 + \omega_2)}\left(\int_0^\zeta du\,\frac{1}{u - \omega_1} + \int_0^\zeta du\,\frac{1}{u - \omega_2}\right)\\ &= \frac{1}{\zeta - (\omega_1 + \omega_2)}\left[\log\left(1 - \frac{\zeta}{\omega_1}\right) + \log\left(1 - \frac{\zeta}{\omega_2}\right)\right]\end{aligned} \tag{2.73}$$

(i) if $\omega_1 = \omega_2 =: \omega$, then we can write the convolution as

$$\begin{aligned}(\hat{\varphi} * \hat{\psi})(\zeta) &= \frac{2}{\zeta - 2\omega}\left(\log(\zeta - \omega) - \log\omega + Mi\pi\right)\\ &= \frac{2\pi i M}{\zeta - 2\omega} + \frac{2}{\zeta - 2\omega}\log\left(1 + \frac{\zeta - 2\omega}{\omega}\right)\end{aligned} \tag{2.74}$$

On the other hand we could have written this as

$$
\begin{aligned}
(\hat{\varphi} * \hat{\psi})(\zeta) &= \frac{2}{(\zeta-\omega)-\omega}\log(\zeta-\omega) + R(\zeta-\omega) \\
&= 2\hat{\varphi}(\zeta-\omega)\log(\zeta-\omega) + R(\zeta-\omega)
\end{aligned} \tag{2.75}
$$

From these equations we see that the convolution is a simple resurgent function with a pole at $\zeta = 2\omega$ and a logarithmic singularity at $\zeta = \omega$.

(ii) if $\omega_1 \neq \omega_2$, then $\hat{\varphi} * \hat{\psi}$ is simple resurgent function at $\zeta = \omega_1$ and at $\zeta = \omega_2$:

$$
\begin{aligned}
(\hat{\varphi} * \hat{\psi})(\zeta) &= \frac{1}{\zeta-(\omega_1+\omega_2)}\left[\log\left(1-\frac{\zeta}{\omega_1}\right) + \log\left(1-\frac{\zeta}{\omega_2}\right)\right] \\
&= \frac{1}{\zeta-\omega_2-\omega_1}\log(\zeta-\omega_1) \\
&\quad + \frac{M\pi i - \log(\omega_1)}{\zeta-(\omega_1+\omega_2)}\log\left(1-\frac{\zeta}{\omega_2}\right) \\
&= \frac{1}{2\pi i}\hat{\Phi}_{\omega_1}(\zeta-\omega_1)\log(\zeta-\omega_1) + R_{\omega_1}(\zeta-\omega_1)
\end{aligned} \tag{2.76}
$$

where $R(\zeta)$ is analytic in the disk $\{|\zeta+\omega_1| < |\omega_2|\}$. Notice that

$$
\hat{\Phi}_{\omega_1}(\zeta) = \frac{2\pi i}{\zeta-\omega_2} = 2\pi i\hat{\psi}(\zeta). \tag{2.77}
$$

Similarly for $\zeta = \omega_2$

$$
\hat{\Phi}_{\omega_2}(\zeta) = \frac{2\pi i}{\zeta-\omega_1} = 2\pi i\hat{\varphi}(\zeta). \tag{2.78}
$$

Then the convolution can be written as

$$
(\hat{\varphi} * \hat{\psi})(\zeta) = \hat{\psi}(\zeta-\omega_1)\log(\zeta-\omega_1) + R_{\omega_1}(\zeta-\omega_1) \tag{2.79}
$$

$$
(\hat{\varphi} * \hat{\psi})(\zeta) = \hat{\varphi}(\zeta-\omega_2)\log(\zeta-\omega_2) + R_{\omega_2}(\zeta-\omega_2) \tag{2.80}
$$

For $\zeta = \omega_1 + \omega_2$ we get that the convolution is a simple resurgent function with a pole at $\zeta = \omega_1 + \omega_2$: If $\arg\omega_1 \neq \arg\omega_2$, taking any analytic continuation of $\hat{\varphi} * \hat{\psi}$ along a path γ avoiding ω_1 and ω_2 and taking a path $\eta = \gamma * [0, \omega_1 + \omega_2]$

we get by Cauchy's integral formula

$$\begin{aligned}(\hat{\varphi} * \hat{\psi})_\gamma(\zeta = \omega_1 + \omega_2) &= \frac{1}{\zeta - (\omega_1 + \omega_2)} \left(\int_\gamma \frac{du}{u - \omega_1} - \int_\gamma \frac{du}{u + \omega_1} \right) \\ &= \frac{1}{\zeta - (\omega_1 + \omega_2)} \left(\int_\eta \frac{du}{u - \omega_1} + \int_\eta \frac{du}{u - \omega_2} \right) \\ &= \frac{2\pi i}{\zeta - (\omega_1 + \omega_2)} \big(n(\eta, \omega_1) + n(\eta, \omega_2) \big) \\ &= \frac{2\pi i N}{\zeta - (\omega_1 + \omega_2)} \end{aligned} \tag{2.81}$$

where N is the sum of winding numbers of ω_1 and ω_2 along η. With $N = 0$ $\zeta = \omega_1 + \omega_2$ is a removable singularity. If $\arg \omega_1 = \arg \omega_2 =: \theta$ and $|\omega_1| \neq |\omega_2|$, then with $\omega_1 = r_1 e^{i\theta}$ and $\omega_2 = r_2 e^{i\theta}$, $r_1 \neq r_2$ taking a path γ going from 0 to $\omega_1 + \omega_2$ avoiding ω_1 and ω_2 by anti-clockwise half-circles we get

$$\begin{aligned}(\hat{\varphi} * \hat{\psi})_\gamma(\omega_1 + \omega_2) &= \frac{1}{\zeta - (\omega_1 + \omega_2)} \left(\int_\gamma \frac{du}{u - \omega_1} + \int_\gamma \frac{du}{u - \omega_2} \right) \\ &= \frac{1}{\zeta - (\omega_1 + \omega_2)} \left(\int_{\gamma_{\omega_1}} \frac{du}{u} + \int_{\gamma_{\omega_2}} \frac{du}{u} \right) \\ &= \frac{1}{\zeta - (\omega_1 + \omega_2)} \left(\int_{-r_1}^{-1} + \int_{C_1} + \int_1^{r_2} + \int_{-r_2}^{-1} + \int_{C_1} + \int_1^{r_1} \right) \\ &= \frac{1}{\zeta - (\omega_1 + \omega_2)} \left(-\ln r_1 + \pi i + \ln r_2 - \ln r_2 + \pi i + \ln r_1 \right) \\ &= \frac{2\pi i}{\zeta - (\omega_1 + \omega_2)} \end{aligned} \tag{2.82}$$

where γ_{ω_1} is a path from $-r_1$ to -1, going around origin to 1 and then going to r_2. Similarly for γ_{ω_2}.

Looking again at the simple resurgent structure of the convolution in the equations (2.79) and (2.80)

$$(\hat{\varphi} * \hat{\psi})(\zeta) = \hat{\psi}(\zeta - \omega_1) \log(\zeta - \omega_1) + R_{\omega_1}(\zeta - \omega_1) \tag{2.83}$$

$$(\hat{\varphi} * \hat{\psi})(\zeta) = \hat{\varphi}(\zeta - \omega_2) \log(\zeta - \omega_2) + R_{\omega_2}(\zeta - \omega_2) \tag{2.84}$$

we can see the origin of the name resurgence. At the singularity of the resurgent function $\hat{\varphi}(\zeta) = 1/(\zeta - \omega_1)$ we find the resurgence or reappearance of the resurgent function $\hat{\psi}(\zeta) = 1/(\zeta - \omega_2)$ and at the singularity of $\hat{\psi}$ we find the resurgence of $\hat{\varphi}$.

2.3.2 Stokes automorphism

In section 2.2.4 we noticed that along the singular, or Stokes, direction the Borel sum is discontinuous and the lateral Borel sums are different functions. There is an ambiguity in the resummation. How can we can calculate the observable if the resummation is ambiguous? Turns out, that the lateral Borel sums are conncted via an automorphism called the Stokes automorphism $\mathfrak{S}_\theta$:

$$\mathcal{S}^{\theta^-} = \mathcal{S}^{\theta^+} \circ \mathfrak{S}_\theta \tag{2.85}$$

The Stokes automorphism describes the full discontinuity across the Stokes line.

To understand how the Stokes automorphism behaves we consider a simple resurgent function

$$\hat{\varphi}(\zeta) = \frac{\alpha}{2\pi i(\zeta - \omega)} + \frac{1}{2\pi i}\hat{\Phi}(\zeta - \omega)\log(\zeta - \omega) + R(\zeta - \omega) \tag{2.86}$$

with a singularity at $\zeta = \omega$ and which is holomorphic along the ray $|\omega|\, e^{i\theta}[1, \infty)$, $\theta = \arg\omega$. The logarithmic branch cut is along the aforementioned ray and the argument has values in $(\theta, \theta + 2\pi]$ Then deforming the paths of integrations of the lateral Borel sums into a Hankel contour, going around ω clockwise as in figure 7, we get

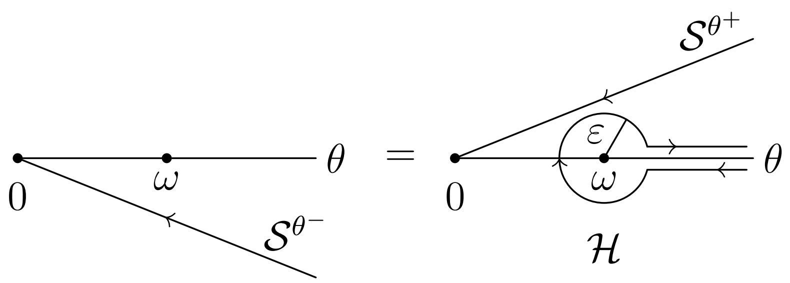

Figure 7. The the contour of $\mathcal{S}^{\theta^-}$ along the direction θ^- deformed into contour of $\mathcal{S}^{\theta^+}$ and the Hankel contour.

$$\begin{aligned}\left(\mathcal{S}^{\theta^+} - \mathcal{S}^{\theta^-}\right)\varphi(z) &= \int_{\mathcal{H}} d\zeta\, e^{-z\zeta}\hat{\varphi}(\zeta) \\ &= \frac{\alpha}{2\pi i}\int_{\mathcal{H}} d\zeta\, \frac{e^{-z\zeta}}{\zeta - \omega} + \frac{1}{2\pi i}\int_{\mathcal{H}} d\zeta\, e^{-z\zeta}\hat{\Phi}(\zeta - \omega)\log(\zeta - \omega)\end{aligned} \tag{2.87}$$

The first integral is

$$\begin{aligned}\int_{\mathcal{H}} d\zeta\, \frac{e^{-z\zeta}}{\zeta-\omega} &= \int_{\infty e^{i\theta}}^{\omega+\varepsilon e^{i\theta}} d\zeta\, \frac{e^{-z\zeta}}{\zeta-\omega} + \int_{C_\varepsilon} d\zeta\, \frac{e^{-z\zeta}}{\zeta-\omega} + \int_{\omega+\varepsilon e^{i\theta}}^{\infty e^{i\theta}} d\zeta\, \frac{e^{-z\zeta}}{\zeta-\omega} \\ &= \int_{C_\varepsilon} d\zeta\, \frac{e^{-z\zeta}}{\zeta-\omega} \\ &= -2\pi i e^{-z\omega} \end{aligned} \tag{2.88}$$

where $C_\varepsilon(t) = \omega + \varepsilon e^{-it},\ \theta \leq t \leq \theta + 2\pi$. The second integral is

$$\begin{aligned} &= \int_{\mathcal{H}} d\zeta\, e^{-z\zeta} \hat{\Phi}(\zeta-\omega) \log(\zeta-\omega) \\ &= \left(\int_{C_\varepsilon} + \int_{\omega+\varepsilon e^{i\theta+i\varepsilon}}^{\infty e^{i\theta+i\varepsilon}} - \int_{\omega+\varepsilon e^{i\theta-i\varepsilon}}^{\infty e^{i\theta-i\varepsilon}} \right) d\zeta\, e^{-z\zeta} \hat{\Phi}(\zeta-\omega) \log(\zeta-\omega) \\ &= e^{-z\omega} \left(\int_{C'_\varepsilon} + \int_{\varepsilon e^{i\theta+i\varepsilon}}^{\infty e^{i\theta+i\varepsilon}} - \int_{\varepsilon e^{i\theta-i\varepsilon}}^{\infty e^{i\theta-i\varepsilon}} \right) d\zeta\, e^{-z\zeta} \hat{\Phi}(\zeta) \log\zeta \end{aligned} \tag{2.89}$$

Now $C'_\varepsilon(t) = \varepsilon e^{-it},\ \theta + \varepsilon \leq t \leq \theta + 2\pi - \varepsilon$, and since $\hat{\Phi}$ is analytic germ at the origin it defines a holomorphic function around the origin and hence is continuous, we can choose ε small enough that $\hat{\Phi}$ is bounded in a closed disk which includes C'_ε. Then

$$\begin{aligned} \left| \int_{C'_\varepsilon} d\zeta\, e^{-z\zeta} \hat{\Phi}(\zeta) \log(\zeta) \right| &\leq \int_{C'_\varepsilon} |d\zeta|\, e^{-\operatorname{Re}(z\zeta)} \left| \hat{\Phi}(\zeta) \log\zeta \right| \\ &\leq \varepsilon \int_{\theta+\varepsilon}^{\theta+2\pi-\varepsilon} dt\, e^{-\varepsilon \cos t \operatorname{Re}(z)} \left| \hat{\Phi}(\varepsilon e^{-it}) \right| (\ln\varepsilon + t) \\ &\longrightarrow 0 \quad \varepsilon \to 0 \end{aligned} \tag{2.90}$$

since $\varepsilon \ln\varepsilon \to 0,\ \varepsilon \to 0$. Because of the logarithmic branch cut along the direction θ, in the limit $\varepsilon \to 0$ we get $\log(\zeta + \varepsilon e^{i\theta+i\varepsilon}) \to \log(\zeta) - 2\pi i$. Then the integral in (2.89) is equal to

$$-2\pi i e^{-z\omega} \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta} \hat{\Phi}(\zeta) \tag{2.91}$$

Thus the difference between the lateral Borel sums is

$$\begin{aligned}\left(\mathcal{S}^{\theta^+} - \mathcal{S}^{\theta^-}\right)\varphi(z) &= -\alpha e^{-z\omega} - e^{-z\omega}\int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta}\hat{\Phi}(\zeta) \\ &= -e^{-z\omega}\int_0^{\infty e^{i\theta^+}} d\zeta\, e^{-z\zeta}\left(\alpha\delta + \hat{\Phi}(\zeta)\right) \\ &= -e^{-z\omega}\mathcal{L}^{\theta^+}\left(\alpha\delta + \hat{\Phi}(\zeta)\right) \\ &= -e^{-z\omega}\mathcal{S}^{\theta^+}\left(\alpha + \Phi(z)\right) \end{aligned} \tag{2.92}$$

where $\Phi(z) = \mathcal{B}^{-1}(\hat{\Phi}(\zeta))$ and we used the fact that integrals along θ and θ^+ are equal due to Cauchy's theorem . We can write the above equation as

$$\begin{aligned}\mathcal{S}^{\theta^-}\varphi(z) &= \mathcal{S}^{\theta^+}\varphi(z) + e^{-z\omega}\mathcal{S}^{\theta^+}(\alpha + \Phi) \\ &= \mathcal{S}^{\theta^+}\circ\left(\operatorname{Id} + e^{-z\omega}\sigma_\omega\right)\varphi(z)\end{aligned} \tag{2.93}$$

and

$$\mathcal{S}^{\theta^-} = \mathcal{S}^{\theta^+}\circ\left(\operatorname{Id} + e^{-z\omega}\sigma_\omega\right) \tag{2.94}$$

As we can see there is a discontinuity along the singular direction θ between the lateral Borel sums. This discontinuity is described by the Stokes automorphism $\mathfrak{S}_\theta$

$$\mathfrak{S}_\theta = \operatorname{Id} + e^{-z\omega}\sigma_\omega \tag{2.95}$$

It is an automorphism since the lateral Borel sums are homomorphisms.

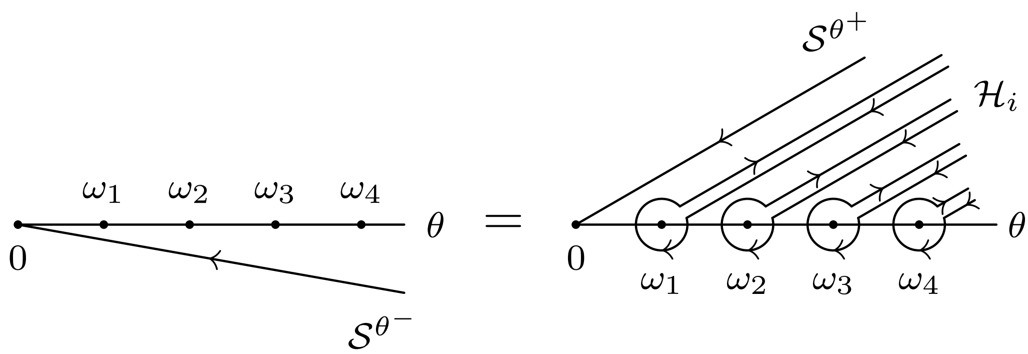

Figure 8. The contour of $\mathcal{S}^{\theta^-}$ deformed into contour of $\mathcal{S}^{\theta^+}$ and the Hankel contours $\mathcal{H}_i$ around the points ω_i. Note that the contours are actually in the direction θ since the functions $\hat{\Phi}_{\omega_i}$ are holomorphic along the ray $|\omega_i|\, e^{i\theta}[1,\infty)$.

If our simple resurgent function $\hat{\varphi}$ is not holomorphic along the direction θ and it has multiple simple singularities on the ray along θ the above calculation isn't valid anymore. (For example the convolution of $\hat{\varphi} = 1/(\zeta+1)$ with itself $\hat{\varphi} * \hat{\varphi} = 2/(\zeta+2)\log(1+\zeta)$ has two singularities along the negative real line: logarithmic singularity at $\zeta = -1$ and a pole at $\zeta = -2$). An easy fix is to deform the path along θ^- into a concatenation of path along θ^+ and Hankel contours coming from infinity in the direction θ^+, going around the singularities anti-clockwise and going back to infinity along θ^+ as shown in the figure 8: Choosing the branch of the logarithm to be along the direction θ^+ we get

$$(\mathcal{S}^{\theta^+} - \mathcal{S}^{\theta^-})\varphi = \sum_{\omega_i \in \Gamma_\theta} \int_{\mathcal{H}_{\omega_i}} d\zeta\, e^{-z\zeta} \hat{\varphi}(\zeta) \tag{2.96}$$

where Γ_θ is the set of singularities along θ and $\mathcal{H}_{\omega_i}$ is the Hankel contour along the singularity ω_i. We can calculate the integrals along the Hankel contours using the same arguments as before and get

$$\begin{aligned} \mathcal{S}^{\theta^-}\varphi &= \mathcal{S}^{\theta^+}\varphi + \sum_{\omega_i \in \Gamma_\theta} e^{-z\omega_i} \mathcal{S}^{\theta^+}\left(\alpha_{\omega_i} + \Phi_{\omega_i}\right) \\ &= \mathcal{S}^{\theta^+} \circ \left(\mathrm{Id} + \sum_{\omega_i \in \Gamma_\theta} e^{-z\omega_i} \sigma_{\omega_i}\right) \varphi(z) \end{aligned} \tag{2.97}$$

and as an operator

$$\mathcal{S}^{\theta^-} = \mathcal{S}^{\theta^+} \circ \left(\mathrm{Id} + \sum_{\omega_i \in \Gamma_\theta} e^{-z\omega_i} \sigma_{\omega_i}\right) \tag{2.98}$$

Now we define the Stokes automorphism in the general case as

Definition 2.25 (Stokes automorphism)**.** The automorphism $\mathfrak{S}_\theta : \mathcal{R}^S \to \mathcal{R}^S$,

$$\mathfrak{S}_\theta = \mathrm{Id} + \sum_{\omega_i \in \Gamma_\theta} e^{-z\omega_i} \sigma_{\omega_i} \tag{2.99}$$

$$\mathcal{S}^{\theta^-} = \mathcal{S}^{\theta^+} \circ \mathfrak{S}_\theta \tag{2.100}$$

is called the Stokes automorphism and it describes the discontinuity between the lateral Borel sums along a singular direction.

2.4 Exact WKB

2.4.1 WKB solutions of the Schrödinger equation

Starting with the Schrödinger equation

$$\left(-\hbar^2 \frac{d^2}{dz^2} + V(z)\right)\psi(z) = E\psi(z) \tag{2.101}$$

we can write it in the form

$$\left(\frac{d^2}{dz^2} + \frac{1}{\hbar^2}p^2(z)\right)\psi(z) = 0 \tag{2.102}$$

where we defined the momentum $p(z) = (E - V(z))^{1/2}$. A common choice in the literature is to use $\tilde{p}(z) = -ip(z)$. We'll keep the "classical momentum" notation. The domain of the Schrödinger equation is the two sheeted Riemann surface Σ of the momentum, it is the two-fold covering of $\mathbb{C} \setminus P$, where P is the set of turning points and poles of $p(z)$.

Making an ansatz of the form

$$\psi(z, \hbar) = \exp\left(\int^z du\, Y(u,\hbar)\right) \tag{2.103}$$

where Y is a formal series of powers of $\hbar$:

$$Y(z,\hbar) = \sum_{n=-1}^{\infty} \hbar^n Y_n(z) = \hbar^{-1}Y_{-1}(z) + Y_0(z) + \hbar Y_1 + \cdots \tag{2.104}$$

translates the Scrödinger equation (2.102) into a Riccati equation

$$\frac{dY}{dz} + Y^2 = -\frac{1}{\hbar^2}p^2(z). \tag{2.105}$$

Inserting the series to (2.105) and comparing the powers of $\hbar$ we get the recursion relation

$$\begin{cases} Y_{-1} = \pm ip(z) \\ 2Y_{-1}Y_{n+1} + \displaystyle\sum_{j+k=n} Y_j Y_k + \frac{dY_n}{dz} = 0 \end{cases} \tag{2.106}$$

If the momentum was also a function of $\hbar$, i.e.

$$p(z,\hbar) = \sum_{n=0}^{\infty} \hbar^n p_n(z) \tag{2.107}$$

the recursion relation is

$$\begin{cases} Y_{-1} = \pm i p(z) \\ 2Y_{-1}Y_{n+1} + \displaystyle\sum_{j+k=n} Y_j Y_k + \frac{dY_n}{dz} = p_{n+2}(z) \end{cases} \tag{2.108}$$

Since we have to possible determinations of the momentum p on the Riemann surface Σ defined by it and by the above recursion relation the coefficients Y_n are odd in p, we have two formal solutions

$$Y^{\pm}(z,\hbar) = \sum_{n=-1}^{\infty} \hbar^n Y_n^{\pm}(z) = \pm\hbar^{-1}p(z) + \cdots \tag{2.109}$$

of the Riccati equation (2.105).

Writing $Y^{\pm}$ as a sum of its even and odd powers of $\hbar$

$$Y = Y_{even} + Y_{odd} \tag{2.110}$$

and plugging into (2.105) and comparing odd powers of $\hbar$ we get

$$Y'_{odd} + 2Y_{even}Y_{odd} = 0 \tag{2.111}$$

from which we find

$$Y_{even} = -\frac{1}{2}\frac{Y'_{odd}}{Y_{odd}} = -\frac{1}{2}\frac{d}{dz}\log Y_{odd} \tag{2.112}$$

Thus the formal solution can be written in terms of only odd powers:

$$\begin{aligned} Y(z,\hbar) &= -\frac{1}{2}\frac{d}{dz}\log Y_{odd} + Y_{odd} \\ &=: -\frac{1}{2}\frac{d}{dz}\log Q + Q \end{aligned} \tag{2.113}$$

where we defined Q as odd powers of Y:

$$Q = Y_{odd} = \hbar^{-1}Y_{-1} + \hbar Y_1 + \hbar^3 Y_3 + \cdots =: \sum_{n=0}^{\infty} \hbar^{2n-1} Q_{2n-1} \tag{2.114}$$

Thus our ansatz (2.103) becomes

$$\psi^{\pm}(z,\hbar) = \frac{1}{\sqrt{Q}} \exp\left(\pm \int^{z} du\, Q(u,\hbar)\right). \tag{2.115}$$

These solutions are known as the *formal WKB solutions* of the Schrödinger equation (2.102)

Definition 2.26 (Formal WKB solutions)**.** The formal WKB soltutions of the Schrödinger equation are

$$\psi^{\pm}(z,\hbar) = \frac{1}{\sqrt{Q}} \exp\left(\pm \int^{z} du\, Q(u,\hbar)\right). \tag{2.116}$$

where

$$Q(z,\hbar) = \sum_{n=0}^{\infty} \hbar^{2n-1} Q_{2n-1}(z) \tag{2.117}$$

$$Q_{-1}(z) = Y_{-1} = ip(z) \tag{2.118}$$

is an asymptotic series in $\hbar$ and the functions Q_{2n-1} are multi-valued and holomorphic on the Riemann surface Σ. Furthermore, defining the action integral as

$$S(z) = i \int^{z} du\, p(u) \tag{2.119}$$

we can write the WKB solutions as

$$\psi^{\pm}(z,\hbar) = \frac{1}{\sqrt{Q}} \exp\left(\pm \frac{1}{\hbar} S(z)\right) \exp\left(\pm \int^{z} du \left(Q(u,\hbar) - \frac{i}{\hbar} p(z)\right)\right) \tag{2.120}$$

Definition 2.27 (Well normalized formal WKB solutions)**.** The formal WKB solutions (2.116) are called well normalized in the Riemann surface Σ. Around a turning point a the coefficients of $Q(z,\hbar)$ have a singularity. The WKB solution

$$\psi^{\pm}(z,\hbar) = \frac{1}{\sqrt{Q}} \exp\left(\pm \int_{a}^{z} du\, Q(u,\hbar)\right). \tag{2.121}$$

is well normalized at a turning point if the integral is defined as

$$\int_{a}^{z} du\, Q(u,\hbar) = \frac{1}{2} \int_{\gamma} du\, Q(u,\hbar) \tag{2.122}$$

where γ is a path around the turning point as shown in figure **??**

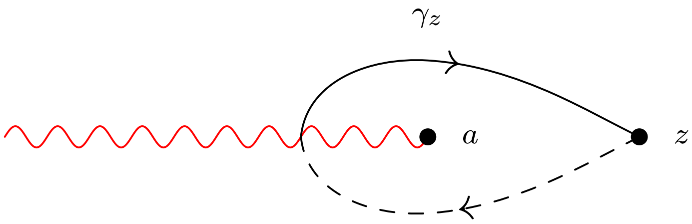

Figure 9. Normalization at a turning point.

This is due to the fact that when crossing to the other Riemann sheet, the momentum changes sign. Around a pole $= \infty$ the normalization is

$$\psi^{\pm}(z,\hbar) = \frac{1}{\sqrt{Q}} \exp\left(\pm\frac{i}{\hbar}\int_a^z du\, p(u)\right) \exp\left(\pm\int_\infty^z du \left(Q(u,\hbar) - \frac{i}{\hbar}p(z)\right)\right) \quad (2.123)$$

since $Q - i\hbar^{-1}p$ is integrable at infinity.

The formal WKB solutions (2.116) can be expanded as a formal series in $\hbar$

$$\psi^{\pm}(z,\hbar) = \exp\left(\pm\frac{1}{\hbar}S(z)\right)\sum_{n=0}^{\infty}\hbar^{n+\frac{1}{2}}\psi_n^{\pm}(z) \quad (2.124)$$

an in the leading order

$$\psi^{\pm}(z,\hbar) = \frac{\hbar^{1/2}}{p(z)^{1/2}} \exp\left(\pm\frac{i}{\hbar}\int^z du\, p(u)\right) \quad (2.125)$$

This is the well-known WKB approximation [4].

2.4.2 Stokes lines and graphs

Definition 2.28 (Stokes lines and Stokes regions)**.** Let a be a turning point of the Schrödinger equation. Then the Stokes lines of the Schrödinger equation are curves defined by the equation

$$\mathrm{Im}\int_a^z du\, ip(u) = 0 \quad (2.126)$$

The regions divided by the Stokes lines are the Stokes regions.

The Stokes lines can be classified to types. An unbounded Stokes line connects a turning point and ∞ and bounded Stokes lines connect two turning points as shown in figure 10.

The Stokes regions cannot be bounded so the bounded Stokes lines cannot be cyclically connected

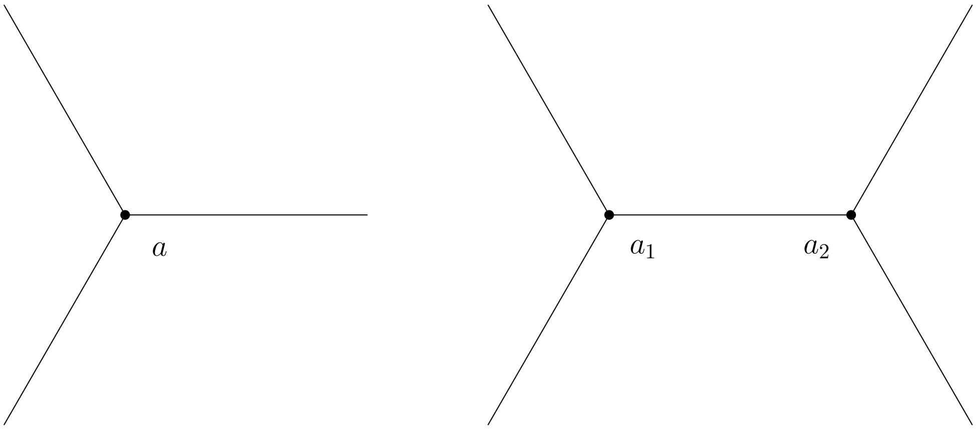

Figure 10. Examples of unbounded and bounded Stokes lines.

The formal WKB solutions (2.116) are Borel summable in the Stokes regions [§2.8 19], [Theorem 5.12 24]

2.4.3 Borel summation of formal WKB solutions

From the properties (2.39) and (2.41) of the Borel transform 2.7 we can define Borel transform for formal series with non-integer powers and exponential factors as

$$\mathcal{B} : e^{Az} \sum_{n=0}^{\infty} \frac{a_n}{z^{n+\alpha}} \mapsto \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+\alpha)} (\zeta + A)^{n+\alpha-1} \tag{2.127}$$

where $\alpha \notin \mathbb{Z}_-$

Then the Borel transform of the formal WKB solutions is

$$\hat{\psi}^{\pm}(z,\zeta) = \mathcal{B}(\psi^{\pm}(z,\hbar))(\zeta) = \sum_{n=0}^{\infty} \frac{\psi_n^{\pm}(z)}{\Gamma(n+\frac{1}{2})} (\zeta \pm S(z))^{n-\frac{1}{2}} \tag{2.128}$$

And the Borel sum

$$\mathcal{S}^{\theta}\psi^{\pm}(z,\hbar) = \int_{S(z)}^{\infty e^{i\theta}} d\zeta \, e^{-\zeta/\hbar} \hat{\psi}^{\pm}(z,\zeta) \tag{2.129}$$

The formal WKB symbols are Borel summable in the Stokes regions 2.28. This is proved for example in [19] and [24].

Resurgence properties of formal WKB solutions not fully proved, [19],[24],[22]

Definition 2.29 (Stokes graph)**.** A Stokes graph of the Schrödinger equation is a graph in the Riemann surface Σ of the momentum whose vertices are turning points and poles and edges are the Stokes lines.

Classically forbidden regions are correspond to the Stokes lines: the tunneling regions are bounded Stokes lines and Borel resummation is discontinous along this direction

2.4.4 Airy-type Stokes graph

Locally around every non-degenerate turning point the potential looks like Airy like potential $V(z) \sim \pm cz$. Thus the Airy-type Schrödinger equation

$$\psi''(z) - \frac{1}{\hbar^2} z\psi(z) = 0 \tag{2.130}$$

is an important case and its Stokes graphs can be used as a building blocks of the Stokes graph of the Schrödinger equation with general potential.

The Airy momentum function is

$$p_{Ai}(z) = -iz^{1/2} \tag{2.131}$$

with a turning point at $z = 0$. The Stokes lines are given by the equation (2.126)

$$0 = \operatorname{Im} \int_0^z du\, i(-iu^{1/2}) = -\operatorname{Im} \int_0^z du\, u^{1/2} = -\operatorname{Im} \frac{2}{3} z^{3/2} \tag{2.132}$$

Thus the Stokes line are given by

$$\arg z = \frac{2k\pi}{3} \tag{2.133}$$

and shown in figure 11.

The connection formula of the WKB wave fuctions, the Borel sums of the WKB solutions, across the Stokes line labeled + from the region I to the region II is

$$\begin{cases} \mathcal{S}^{0-}\psi^+_{a,\mathrm{I}} = \mathcal{S}^{0+}\psi^+_{a,\mathrm{II}} + i\mathcal{S}^{0+}\psi^-_{a,\mathrm{II}} \\ \mathcal{S}^{0-}\psi^-_{a,\mathrm{I}} = \mathcal{S}^{0+}\psi^-_{a,\mathrm{II}} \end{cases} \tag{2.134}$$

and across the Stokes line labeled − from the region II to the region III is

$$\begin{cases} \mathcal{S}^{\frac{2\pi}{3}-}\psi^+_{a,\mathrm{II}} = \mathcal{S}^{\frac{2\pi}{3}+}\psi^+_{a,\mathrm{III}} \\ \mathcal{S}^{\frac{2\pi}{3}-}\psi^-_{a,\mathrm{I}} = \mathcal{S}^{\frac{2\pi}{3}+}\psi^-_{a,\mathrm{II}} + i\mathcal{S}^{\frac{2\pi}{3}+}\psi^+_{a,\mathrm{III}} \end{cases} \tag{2.135}$$

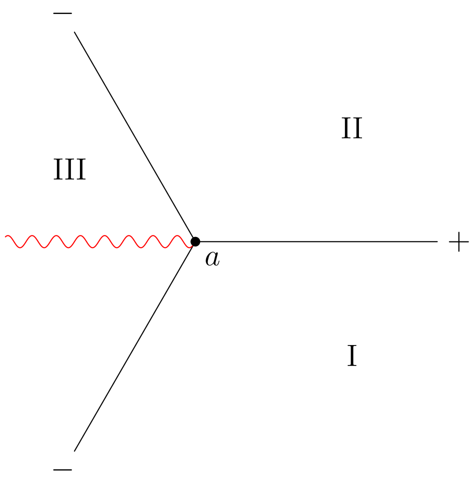

Figure 11. The Airy-type Stokes graph with the turning point a. The label $\pm$ denotes where the WKB solution $\psi^{\pm}$ is dominant and the red line is a branch cut.

In matrix form we can write these as

$$\begin{pmatrix} \psi^{+}_{a,\mathrm{I}} \\ \psi^{-}_{a,\mathrm{I}} \end{pmatrix} = M_{+} \begin{pmatrix} \psi^{+}_{a,\mathrm{II}} \\ \psi^{-}_{a,\mathrm{II}} \end{pmatrix} \tag{2.136}$$

and

$$\begin{pmatrix} \psi^{+}_{a,\mathrm{I}} \\ \psi^{-}_{a,\mathrm{I}} \end{pmatrix} = M_{-} \begin{pmatrix} \psi^{+}_{a,\mathrm{II}} \\ \psi^{-}_{a,\mathrm{II}} \end{pmatrix} \tag{2.137}$$

where we defined the matrices

$$M_{+} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_{-} = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \tag{2.138}$$

Across the branch cut we have

$$\begin{pmatrix} \psi^{+}_{a,\mathrm{III}} \\ \psi^{-}_{a,\mathrm{III}} \end{pmatrix} = M_{b} \begin{pmatrix} \psi^{+}_{a,\mathrm{IV}} \\ \psi^{-}_{a,\mathrm{IV}} \end{pmatrix} \tag{2.139}$$

where

$$M_b = M_-^{-1} M_+^{-1} M_-^{-1} = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{2.140}$$

The above connection formulas are proved in for example [10] and [25] in the exact WKB formalism and in the section 3.4 using the steepest descent method.

2.4.5 Voros symbols

On the Riemann surface Σ of the momentum p we define a cycle γ to be a path such that the analytic continuation of momentum along this path gives the same determination of the momentum. On the two sheeted Riemann surface this means that if γ goes through the branch cut, it has cross it again to give the same determination. Let λ be a path between points z_0 and z in the Riemann surface Σ not crossing the branch cut. Then the WKB solution can be written as

$$\psi(z,\hbar) = \frac{1}{\sqrt{Q}} \exp\Big(\int_\lambda du\, Q(u,\hbar)\Big) \tag{2.141}$$

Then analytic continuation of the WKB solution along a cycle γ is

$$\begin{aligned}
\psi_\gamma &= Q_\gamma^{-1/2} \exp\Big(\int_{\lambda+\gamma} du\, Q(u,\hbar)\Big) \\
&= (-i)^{n(\gamma)} Q^{-1/2} \exp\Big(\int_\lambda du\, Q(u,\hbar)\Big) \exp\Big(\oint_\gamma du\, Q(u,\hbar)\Big) \\
&= (-i)^{n(\gamma)} \psi \exp\Big(\oint_\gamma du\, \big(Q(u,\hbar) - i\hbar^{-1} p(u)\big)\Big) \exp\Big(\frac{i}{\hbar} \oint_\gamma du\, p(u)\Big)
\end{aligned} \tag{2.142}$$

where $n(\gamma)$ is the index of the cycle γ, that is the number of times in encircles the turning points. The multiplier $(-i)^{n(\gamma)}$ comes from the fact that Q is odd in p so $Q^{1/2}$ has four fold monodromy. For a cycle that goes around the turning point twice we have $(-i)^{n(\gamma)} = -1$.

In the equation (2.142) we define the exponential terms as the Voros symbols.

Definition 2.30 (Voros symbols)**.** Let γ be a cycle in the Riemann surface Σ of the momentum p. Then the Voros symbols for the cycle γ are the formal series

$$V_\gamma = \exp\Big(\oint_\gamma du\, Q(u,\hbar)\Big) \tag{2.143}$$

and

$$a_\gamma = \exp\Big(\oint_\gamma du\, \big(Q(u,\hbar) - i\hbar^{-1} p(u)\big)\Big) \tag{2.144}$$

The integral

$$S_\gamma = i \oint_\gamma du\, p(u) \tag{2.145}$$

is called the action integral. The Voros symbols are connected by

$$V_\gamma = a_\gamma e^{S_\gamma/\hbar} \tag{2.146}$$

Now consider two WKB solutions normalized at different turning points a_1 and a_2, see Definition 2.27. The wave functions are connected by the the Voros symbol V_γ. The Voros symbol changes normalization between the turning points:

$$\begin{aligned}
\psi^{\pm}_{a_1}(z,\hbar) &= \frac{1}{\sqrt{Q}} \exp\left(\pm \int_{a_1}^{z} du\, Q(u,\hbar)\right) = \frac{1}{\sqrt{Q}} \exp\left(\pm \left(\int_{a_1}^{a_2} + \int_{a_2}^{z}\right) du\, Q(u,\hbar)\right) \\
&= \frac{1}{\sqrt{Q}} \exp\left(\pm \frac{1}{2}\left(\int_{\gamma_{12}} + \int_{\gamma_2}\right) du\, Q(u,\hbar)\right) \\
&= \exp\left(\pm \frac{1}{2} \int_{\gamma_{12}} du\, Q(u,\hbar)\right) \frac{1}{\sqrt{Q}} \exp\left(\pm \frac{1}{2} \int_{\gamma_2} du\, Q(u,\hbar)\right) \\
&= \exp\left(\pm \frac{1}{2} \int_{\gamma_{12}} du\, Q(u,\hbar)\right) \frac{1}{\sqrt{Q}} \exp\left(\pm \int_{a_2}^{z} du\, Q(u,\hbar)\right) \\
&= V_{\gamma_{12}}^{\pm 1/2} \psi^{\pm}_{a_2}(z,\hbar)
\end{aligned} \tag{2.147}$$

where the paths γ_1,γ_2 and γ_{12} are shown in figure 12 . This can be written in the

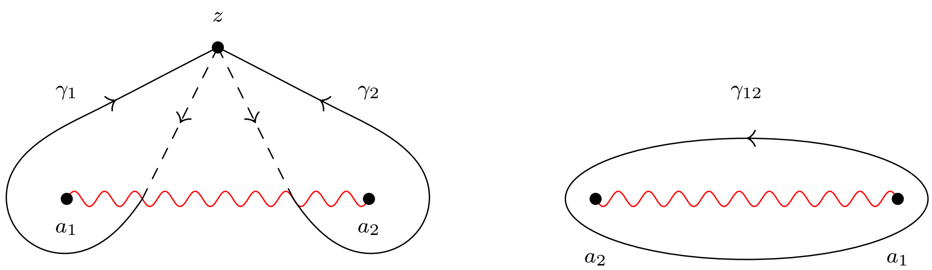

Figure 12. The contours γ_1, γ_2 and γ_{12} which define the normalization at turning points and the Voros symbol $V_{\gamma_{12}}$.

matrix form as

$$\begin{pmatrix} \psi^{+}_{a_1} \\ \psi^{-}_{a_1} \end{pmatrix} = N_{a_1 a_2} \begin{pmatrix} \psi^{+}_{a_2} \\ \psi^{-}_{a_2} \end{pmatrix} \tag{2.148}$$

where

$$N_{a_1 a_2} = \begin{pmatrix} V_{\gamma_{12}}^{1/2} & 0 \\ 0 & V_{\gamma_{12}}^{-1/2} \end{pmatrix} \tag{2.149}$$

is the change of normalization matrix.

2.4.6 DDP formula

The Voros symbols are Borel summable in the Stokes regions but if the path of integration intersects a bounded Stokes line the Borel summability breaks and there is a discontinuity between the lateral Borel sums. This means that a bounded Stokes line causes Stokes phenomena of the Voros symbols. This is due to the changing of Stokes graphs depending on $\arg \hbar$, as shown in figure 37 in the case of the double well potential.

This discontinuity between the lateral Borel sums of the Voros symbols is described by the Delabaere-Dillinger-Pham (DDP) formula [22]. The DDP formula is resurgence relation: the Voros symbols of a classically allowed region is related to the Voros symbols of a classically forbidden tunneling region and the discontinuity is non-perturbative.

Before stating the DDP formula we define an intersection form (γ,β) between two paths γ and β on the Riemann surface Σ of the momentum. It is normalized such that (x-axis,y-axis) = +1 and the rules are shown in figure 13.

Theorem 2.31 (DDP formula)**.** *The lateral Borel sums of the Voros symbols for a cycle γ are related by the DDP formula*

$$\mathcal{S}^{\theta^-}\left[V_\gamma\right] = \mathcal{S}^{\theta^+}\left[V_\gamma\right]\left(1+\mathcal{S}^{\theta^+}\left[V_{\gamma_0}\right]\right)^{-(\gamma_0,\gamma)} \tag{2.150}$$

$$\mathcal{S}^{\theta^-}\left[a_\gamma\right] = \mathcal{S}^{\theta^+}\left[a_\gamma\right]\left(1+\mathcal{S}^{\theta^+}\left[a_{\gamma_0}\right]\right)^{-(\gamma_0,\gamma)} \tag{2.151}$$

where $(\cdot,\cdot)$ is the intersection form of the cycles γ_0 and γ and γ_0 a cycle surrounding the turning points.

Thd DDP formula is originally proved in [11], and a more recent proof can be found in [19].

Using the Stokes automorphism (2.85) the DDP formula can be written as

$$\mathfrak{S}_\theta\left[a_\gamma\right] = a_\gamma\left(1+a_{\gamma_0}\right)^{-(\gamma_0,\gamma)} \tag{2.152}$$

$$\mathfrak{S}_\theta\left[V_\gamma\right] = V_\gamma\left(1+V_{\gamma_0}\right)^{-(\gamma_0,\gamma)} \tag{2.153}$$

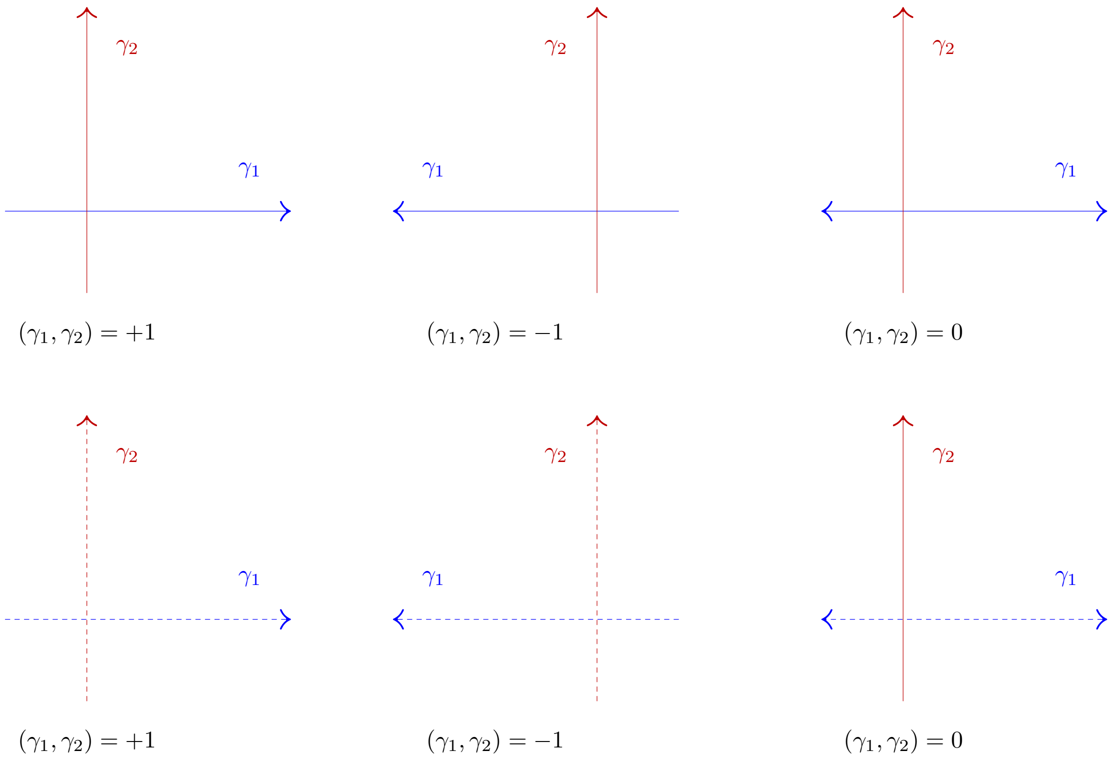

Figure 13. The rules for finding the intersection number (γ_1,γ_2). The first path γ_1 is drawn horizontally and the second path γ_2 vertically.

3 Quantum Resurgence of the Airy equation

The work on this chapter was inspired by work of Jidoumou [18].

Consider the Airy Schrödinger equation

$$\psi''(q) - \frac{1}{\hbar^2} q\psi(q) = 0 \tag{3.1}$$

where q is complex variable. The well known integral representation solutions of the Airy equation [13] translate to

$$\begin{aligned} \mathcal{A}_k(q,\hbar) &= -\frac{1}{2\pi i}\int_{\gamma_k} du\, e^{-\frac{1}{\hbar}(\frac{1}{3}u^3 - qu)} \\ &= -\frac{1}{2\pi i}\int_{\gamma_k} du\, e^{-\frac{1}{\hbar}s(u,q)} \end{aligned} \tag{3.2}$$

where the path γ_k are shown in figure 14 and where we defined the function

$$s(u) := s(u,q) = \frac{1}{3}u^3 - qu \tag{3.3}$$

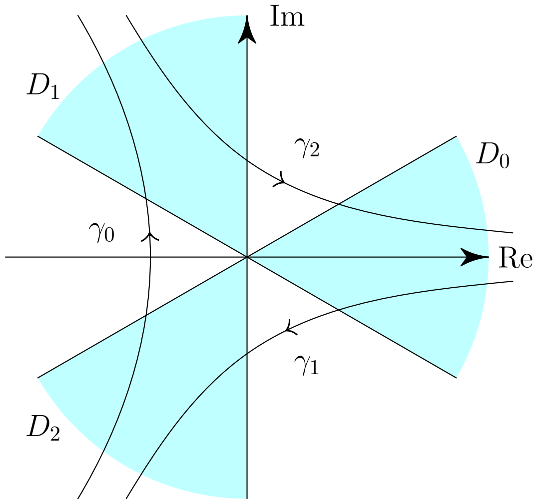

Figure 14. The contours γ_0, γ_1 and γ_2 which define the solutions of the Airy Schrödinger equation

The two linearly independent solutions of the Schrödinger equation are chosen to be $A := \mathcal{A}_0$ and $B := i\mathcal{A}_2 - \mathcal{A}_1$ so the general solution is

$$\psi(q,\hbar) = c_1 A(q,\hbar) + c_2 B(q,\hbar) \tag{3.4}$$

The asymptotic expansions of the Airy equation are well know but to illustrate the Riemann surface structure and the Borel summation procedure we will derive the expansions using the steepest descent method. For a more detailed introduction see for example [4].

The steepest contours are those along which $\exp(-s(u)/\hbar)$ changes most rapidly as $|1/\hbar| \to \infty$ or equivalently those along which $\operatorname{Im} s(u)$ is constant, i.e. constant phase contours. Thus the steepest contours are given by the equation

$$\operatorname{Im} s(u) = \text{constant} \tag{3.5}$$

and some of them are shown in figure 15. The saddle points of $s(u)$ are $\pm q^{1/2}$ since

$$s'(u) = u^2 - q = 0 \iff u = \pm q^{1/2}. \tag{3.6}$$

For now taking $q > 0$ real, we have

$$s(q^{1/2}) = -\frac{2}{3}q^{3/2} \quad \text{and} \quad s(-q^{1/2}) = \frac{2}{3}q^{3/2} \tag{3.7}$$

so

$$-\frac{1}{\hbar}\operatorname{Re} s(q^{1/2}) = \frac{2}{3}q^{3/2}\frac{1}{\hbar} \quad \text{and} \quad \frac{1}{\hbar}\operatorname{Re} s(-q^{\frac{1}{2}}) = -\frac{2}{3}q^{3/2}\frac{1}{\hbar}. \tag{3.8}$$

Thus the $+q^{1/2}$ is the dominant saddle and $-q^{1/2}$ is the recessive saddle and we denote them as

$$u_d := +q^{1/2} \quad \text{and} \quad u_r := -q^{1/2}. \tag{3.9}$$

From figure 15 one can see that the contours γ_k can only be deformed to steepest descent contours that go through the saddle points. The steepest descent contours are shown in figure 16 and denoted by C_k.

To find the full asymptotic expansion we start with a change of variables

$$\zeta = s(u,q) = \frac{1}{3}u^3 - qu, \quad du = \frac{du}{d\zeta}d\zeta \tag{3.10}$$

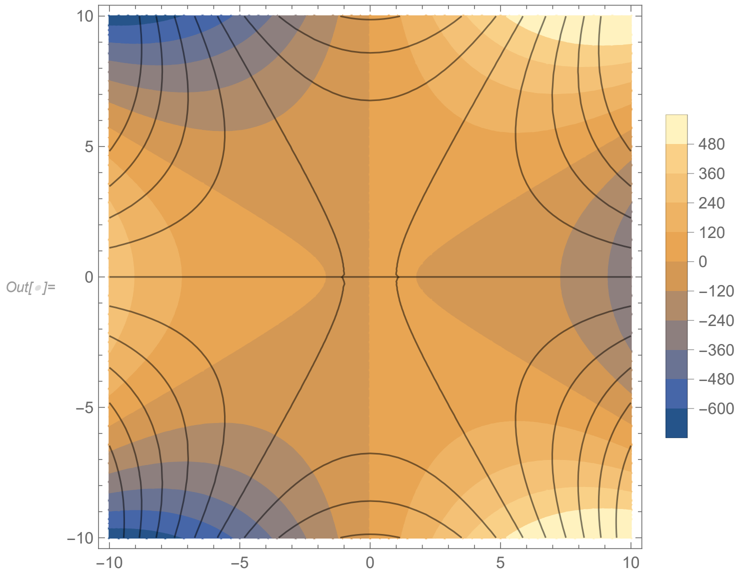

Figure 15. Contour plot of $s(u)$ with $q = 1$: the black lines are some of the steepest contours, and the color shading gives the level curves such that the darker colors indicate steepest descent contours and the lighter colors indicate steepest ascent contours.

and define the function

$$\hat{\mathcal{A}}(\zeta) = -\frac{1}{2\pi i}\frac{du}{d\zeta}, \tag{3.11}$$

and the paths $\Gamma_k = s(C_k)$ which are shown in the figure 17. Then the solutions to the Schrödinger equation can be written as

$$\mathcal{A}_k(q,\hbar) = -\frac{1}{2\pi i}\int_{\Gamma_k} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}(\zeta) \tag{3.12}$$

which is almost in the familiar form of the Laplace transform. The use notation hat-$\mathcal{A}$ and variable ζ implies that the function $\hat{\mathcal{A}}$ is defined on the Borel ζ-plane. Next we'll show that the function $\hat{\mathcal{A}}$ is multi-valued in the complex-plane but well-defined on Riemann surface of $u(\zeta)$.

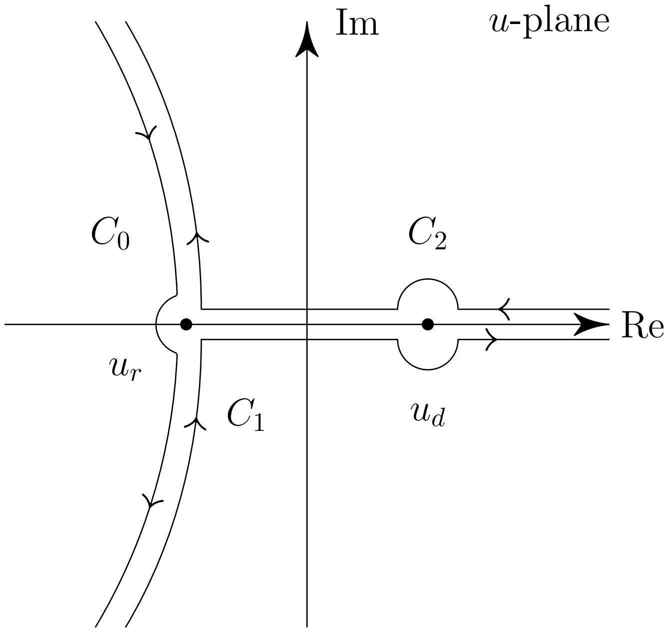

Figure 16. The steepest descent contours C_k deformed from γ_k in the u-plane. In reality the contours go through the saddle points, but the above contours are drawn for clarity.

3.1 Riemann surface structure of the multi-valued function $u(\zeta)$

The inverse of the function ζ defined by the change of variables (3.10), $u(\zeta)$ is a multi-valued function with branch points

$$\zeta_d := s(u_d) = -\frac{2}{3}q^{3/2} \quad \text{and} \quad \zeta_r := s(u_r) = \frac{2}{3}q^{3/2} \tag{3.13}$$

and the Riemann surface Σ_u of $u(\zeta)$ is locally around the branch points the Riemann surface of a square root. This can be seen as follows:

The Taylor expansions of ζ around the saddle points u_d and u_r are

$$\zeta = -\frac{2}{3}q^{3/2} + q^{1/2}(u - q^{1/2})^2 + \frac{1}{3}(u - q^{1/2})^3 \tag{3.14}$$

and

$$\zeta = \frac{2}{3}q^{3/2} - q^{1/2}(u + q^{1/2})^2 + \frac{1}{3}(u + q^{1/2})^3 \tag{3.15}$$

From these it is easy to see that

$$u(\zeta) - q^{1/2} \sim \pm q^{-1/4}(\zeta + \frac{2}{3}q^{3/2})^{1/2} \tag{3.16}$$

and

$$u(\zeta) + q^{1/2} \sim \pm i q^{-1/4}(\zeta - \frac{2}{3}q^{3/2})^{1/2} \tag{3.17}$$

From the above equations one can see that the change of variables $\zeta = s(u)$ introduces branch cuts to the ζ plane because of the square root terms, as shown in figure 17. The points $\zeta_d = -(2/3)q^{3/2}$ and $\zeta_r = (2/3)q^{3/2}$ are branch points of $u(\zeta)$ and therefore also branch points of $\hat{\mathcal{A}}(\zeta)$. The Riemann surface Σ_u is two sheeted Riemann surface: the multi-valued function $u(\zeta)$ has 3 different determinations but the domains of these determinations overlap such that a 2 sheeted Riemann surface is formed. This can be seen after explicitly giving the paths $\Gamma_k = s(C_k)$.

Now because the paths of steepest descent are also paths of constant phase, that is

$$\operatorname{Im} s(u) = \text{constant } u \in |C_k|\,, \tag{3.18}$$

the paths C_k will transform to paths Γ_k determined by

$$\operatorname{Im}\zeta = \text{constant}. \tag{3.19}$$

Since along the steepest descent paths $\operatorname{Re} s(u) \to \infty$ as $|u| \to \infty$ we have

$$\operatorname{Re}\zeta \to \infty \quad \text{as} \quad |\zeta| \to \infty. \tag{3.20}$$

From these conditions we get that the paths Γ_k come from infinity along the $\operatorname{Re}\zeta$-axis, turn around at the point ζ_r or ζ_d and tend back to infinity along the $\operatorname{Re}\zeta$-axis, as shown in figure 17.

Now that we know how the contours Γ_k look like we can find what the three determinations of $u(\zeta)$ are:

i) $u_0(\zeta)$: defined by Γ_0, holomorphic in the cut plane $\mathbb{C} \setminus [\zeta_r,\infty)$

ii) $u_1(\zeta)$: defined by Γ_2, holomorphic in the cut plane $\mathbb{C} \setminus [\zeta_d,\infty)$, whose extension from the left is singular at the point ζ_r and extension from the right is holomorphic at the point ζ_r. This follows from the path C_1 coming from infinity, going through the saddle point u_r first, then u_d and finally going back to infinity it doesn't see the point u_r again.

iii) $u_2(\zeta)$: defined by Γ_2, holomorphic in the cut plane $\mathbb{C} \setminus [\zeta_d,\infty)$, whose extension from the left is holomorphic at the point ζ_r and extension from the right is singular at the point ζ_r. Similarly this follows from the path C_2 coming from infinity, going through the saddle point u_d first, then u_r where it's singular and then going to infinity.

Now using these determinations of $u(\zeta)$ we can construct the two-sheeted Riemann surface Σ_u, see figures 17, 18 and 19.

When the path C_1 comes from infinity, it first goes through the saddle point u_r and then to the saddle point u_d, at same time when the path Γ_1 comes from from infinity it sees the branch point ζ_r on the left (from the right) and continues to the branch point ζ_d. From u_d C_1 continues to infinity. On the other hand Γ_1 goes through the branch cut to the second sheet. Why it has to go to another sheet and not remain on on first sheet? Because as C_1 continues to infinity from u_d it never sees the saddle point u_r. The same must be true for Γ_1 and the branch point ζ_r, it doesn't see ζ_r again so Γ_1 must go to infinity on a another sheet.

In the case of Γ_2 it comes from infinity on the second sheet because it doesn't see the branch point ζ_r and crosses to the first sheet at ζ_d. On the first sheet Γ_2 sees the branch point ζ_r on right (from the left) and continues to infinity along the first sheet.

Another way to visualize that the Riemann surface will be two-sheeted and not three-sheeted as it would be in the case is to look at the graph of the function $\zeta = \frac{1}{3}u^3 - qu$ shown in figure 20. From the graph we can see that locally around the points u_r and u_d the graph looks like a parabola, which corresponds to square root like behavior of $u(\zeta)$ shown in equations (3.16) and (3.17), and that one branch around ζ_r agrees with one branch around ζ_d. In the case where $q \to 0$ we'd get the three-sheeted Riemann surface.

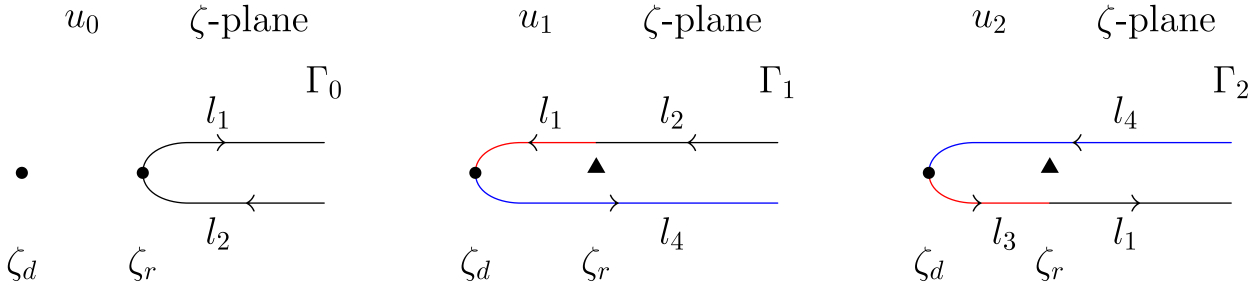

Figure 17. The paths Γ_k and the determinations of $u_k(\zeta)$. The colors correspond to different parts of contours C_k. The "nail" indicates the direction where singularity is seen.

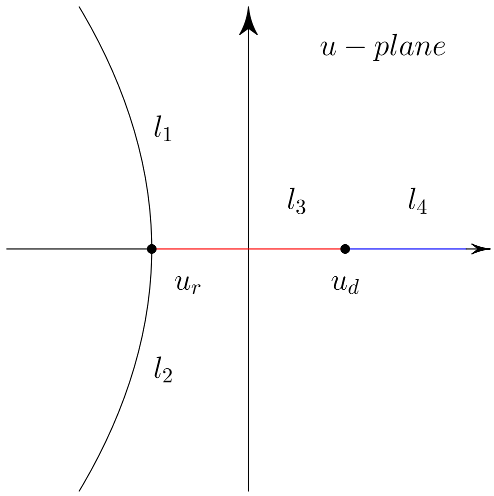

Figure 18. The steepest descent contour divided to different parts l_k to construct the Riemann surface.

3.2 Asymptotic expansion of the Airy functions $\mathcal{A}_k(q,\hbar)$

In the equations (3.16) and (3.17) we obtained the first and second terms in the expansion of $u(\zeta)$. To obtain the full expansion we can use the Lagrange inversion theorem [26]: for a power series of the form

$$w = f(z) = f(a) + \sum_{k=m}^{\infty} b_k(z-a)^k, \;\; m \geq 2, \;\; bm \neq 0 \tag{3.21}$$

with a as a critical point of f, the series of the inverse function $z(w)$ is the Puiseux series with a algebraic branch point at $b = f(a)$ of order $m-1$:

$$z(w) = a + \sum_{k=1}^{\infty} a_k(w-b)^{k/m} \tag{3.22}$$

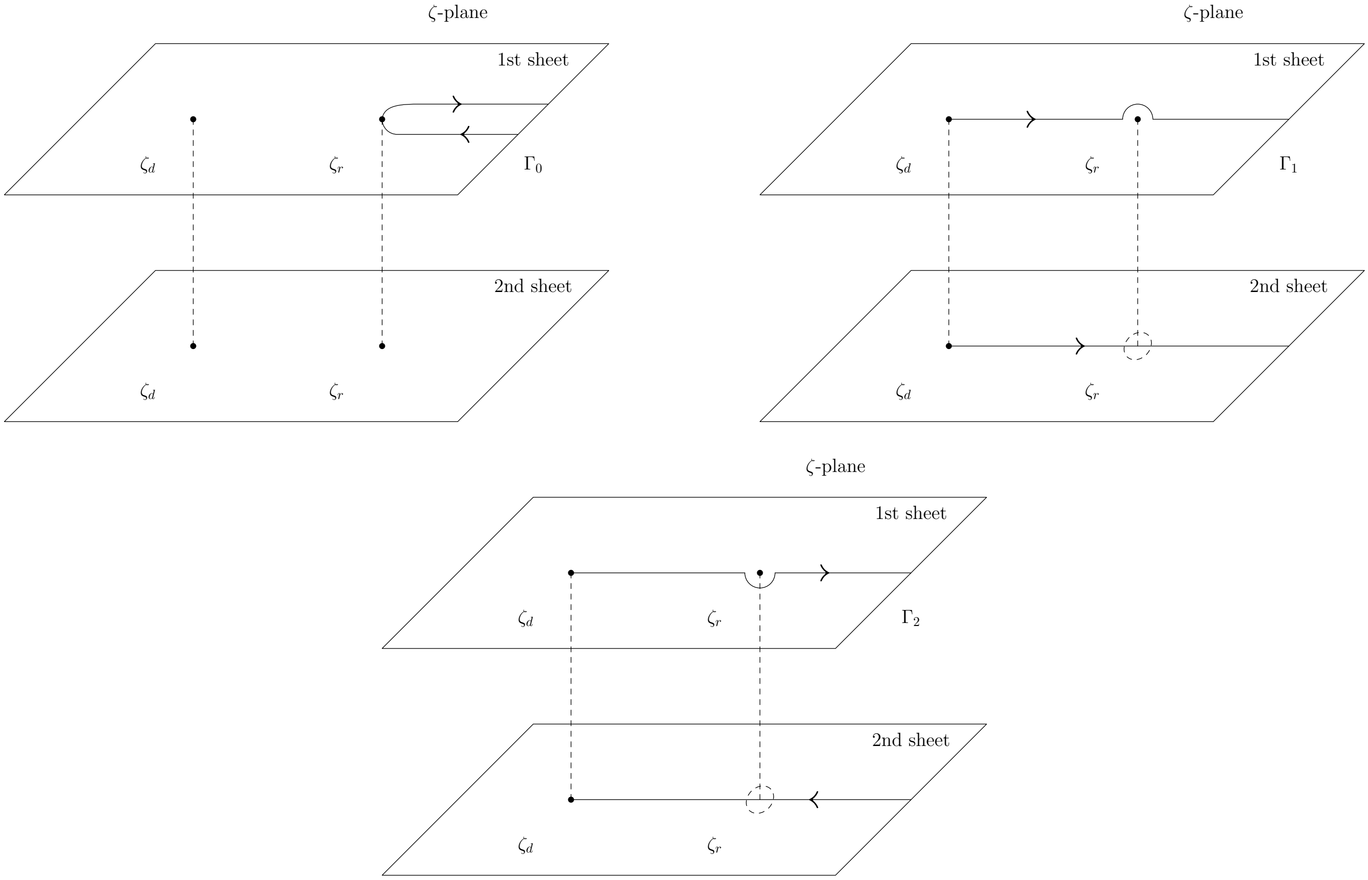

Figure 19. Another visualization of the paths Γ_k on different sheets of the Riemann surface

where the coefficients are given by

$$a_k = \frac{1}{k!} \lim_{\zeta \to a} \frac{d^{k-1}}{d\zeta^{k-1}} \left[\left(\frac{\zeta - a}{(f(\zeta) - b)^{1/m}} \right)^k \right] \tag{3.23}$$

Now because our series expansions of $\zeta = s(u)$ are expanded at the saddle points, the inverse series are given by the Puiseux series

$$u(\zeta) = u_{d/r} + \sum_{k=2}^{\infty} a_k (\zeta - \zeta_{d/r})^{k/2} \tag{3.24}$$

where the coefficients are given by

$$a_k = \frac{1}{k!} \lim_{u \to u_{d/r}} \frac{d^{k-1}}{du^{k-1}} \left[\left(\frac{u - u_{d/r}}{(s(u) - \zeta_{d/r})^{1/2}} \right)^k \right] \tag{3.25}$$

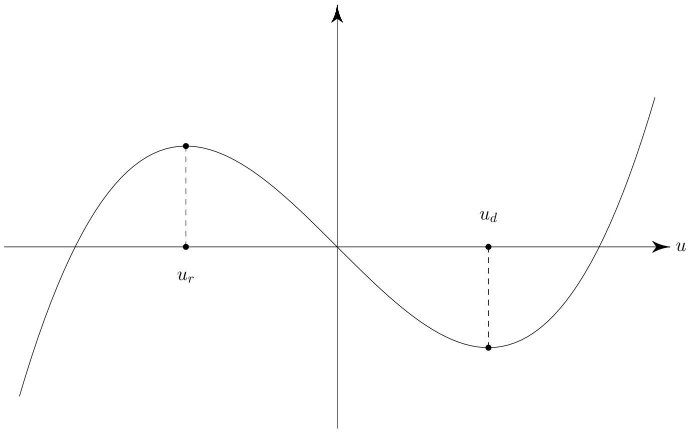

Figure 20. The graph of $\zeta = \frac{1}{3}u^3 - qu$.

The Taylor expansions of $\zeta = s(u)$ were

$$s(u) = \zeta_{d/r} + u_{d/r}(u - u_{d/r})^2 + \frac{1}{3}(u - u_{d/r})^3 \tag{3.26}$$

so the function inside the square brackets in the Puiseux coefficients can be written as

$$\frac{(u - u_{d/r})^k}{\left(u_{d/r}(u - u_{d/r})^2 + \frac{1}{3}(u - u_{d/r})^3\right)^{k/2}} = \left(u_{d/r} + \frac{1}{3}(u - u_{d/r})\right)^{-k/2} \tag{3.27}$$

Denoting

$$g(u) = u_{d/r} + \frac{1}{3}(u - u_{d/r}) \text{ with } g'(u) = 1/3 \text{ and } \lim_{u \to u_{d/r}} g(u) = u_{d/r} \tag{3.28}$$

the Puiseux coefficients can be written as

$$a_k = \frac{1}{k!} \lim_{u \to u_{d/r}} \frac{d^{k-1}}{du^{k-1}} \left(g(u)^{-k/2}\right) \tag{3.29}$$

The derivatives give

$$\begin{aligned}
&\frac{d^{k-1}}{du^{k-1}}\left(g(u)^{-k/2}\right)\\
&= \frac{d^{k-2}}{du^{k-2}}\left(-\frac{1}{3}\frac{k}{2}g(u)^{-k/2-1}\right)\\
&= \frac{d^{k-3}}{du^{k-3}}\left((-1)^2\frac{1}{3^2}\left(\frac{k}{2}+1\right)\frac{k}{2}g(u)^{-k/2-2}\right)\\
&= \dots\\
&= \frac{d^{k-k}}{du^{k-k}}\left((-1)^{k-1}\frac{1}{3^{k-1}}\left(\frac{k}{2}+k-2\right)\left(\frac{k}{2}+k-3\right)\cdots\left(\frac{k}{2}+1\right)\frac{k}{2}g(u)^{-k/2-(k-1)}\right)\\
&= (-1)^{k-1}\frac{1}{3^{k-1}}\left(\frac{k}{2}+k-2\right)\left(\frac{k}{2}+k-3\right)\cdots\left(\frac{k}{2}+1\right)\frac{k}{2}q^{-k/4-(k-1)/2} \qquad (3.30)
\end{aligned}$$

When k is even, $k = 2m$, $m \in \mathbb{N}$, we have

$$\begin{aligned}
&\left(\frac{k}{2}+k-2\right)\left(\frac{k}{2}+k-3\right)\cdots\left(\frac{k}{2}+1\right)\frac{k}{2}\\
&= (m+2m-2)(m+2m-3)\cdots(m+2)(m+1)m\\
&= \frac{\Gamma(m+2m-1)}{\Gamma(m)}\\
&= \frac{\Gamma(3m-1)}{\Gamma(m)} \qquad (3.31)
\end{aligned}$$

where we used the formula

$$x^{(n)} = x(x+1)(x+2)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)} \qquad (3.32)$$

Thus for even values of k the coefficients are

$$a_{2m} = -\frac{1}{(2m)!3^{2m-1}}\frac{\Gamma(3m-1)}{\Gamma(m)}q^{-(6m+2)/4} \qquad (3.33)$$

where we used $(-1)^{k-1} = (-1)^{2m-1} = -1$.

For odd k, $k = 2m+1$

$$
\begin{aligned}
&\left(\frac{k}{2}+k-2\right)\left(\frac{k}{2}+k-3\right)\cdots\left(\frac{k}{2}+1\right)\frac{k}{2} \\
&= (3m-1/2)((3m-1/2)-1)\cdots((3m-1/2)-(2m-2))((3m-1/2)-(2m-1)) \\
&= \frac{\Gamma((3m-1/2)+1)}{\Gamma((3m-1/2)-(2m-1))} \\
&= \frac{\Gamma(3m+1/2)}{\Gamma(m+1/2)}
\end{aligned} \tag{3.34}
$$

where we used the formula

$$\frac{\Gamma(z+1)}{\Gamma(z-k)} = z(z-1)(z-2)\cdots(z-k) \tag{3.35}$$

Thus for odd values of k the coefficients are

$$a_{2m+1} = \frac{1}{(2m+1)!3^{2m}}\frac{\Gamma(3m+1/2)}{\Gamma(m+1/2)}q^{-(6m+1)/4} \tag{3.36}$$

Thus the expansion of the function $u(\zeta)$ around the point ζ_d is

$$u(\zeta) = u_d \pm \sum_{m=0}^{\infty} a_{2m+1}(\zeta-\zeta_d)^{m+1/2} + \sum_{m=1}^{\infty} a_{2m}(\zeta-\zeta_d)^m \tag{3.37}$$

where the the $\pm$ refer to the two different branches of the square root around the point ζ_d. The series converges in the disk $D(\zeta_d, r)$ where r is the distance between the branch points, $r = |\zeta_r - \zeta_d| = \frac{4}{3}q^{3/2}$.

Since the saddle point $-q^{1/2}$ is analytic continuation of $q^{1/2}$ by $e^{2\pi i}$:

$$-q^{1/2} = e^{\pi i}q^{1/2} = (e^{2\pi i}q)^{1/2}, \tag{3.38}$$

we can find the expansion around ζ_r by the analytic continuation

$$q \to e^{2\pi i}q \tag{3.39}$$

Now

$$q^{3/2} \to e^{3\pi i}q^{3/2} = -q^{3/2} \tag{3.40}$$

so

$$\zeta_d \to \zeta_r, \tag{3.41}$$

and

$$q^{(-6m+1)/4} \to e^{i(-6m+1)\pi/2} q^{(-6m+1)/4} = i(-1)^m q^{(-6m+1)/4} \tag{3.42}$$

$$q^{(-6m+2)/4} \to e^{i(-6m+2)\pi/2} q^{(-6m+1)/4} = (-1)^{m+1} q^{(-6m+1)/4} \tag{3.43}$$

so

$$a_{2m+1} \to i(-1)^m a_{2m+1} \tag{3.44}$$

$$a_{2m} \to (-1)^{m+1} a_{2m} \tag{3.45}$$

Thus the expansion around ζ_r is given by

$$u(\zeta) = u_r \pm i \sum_{m=0}^{\infty} (-1)^m a_{2m+1} (\zeta - \zeta_r)^{m+1/2} + \sum_{m=1}^{\infty} (-1)^{m+1} a_{2m} (\zeta - \zeta_r)^m \tag{3.46}$$

with disk of convergence $D(\zeta_d, r)$, $r = |\zeta_r - \zeta_d| = \frac{4}{3} q^{3/2}$.

Since the function $u(\zeta)$ has three determinations defined on Riemann surface $\hat{\mathcal{R}}_u$, it follows from equation (3.11) that function $\hat{\mathcal{A}}(\zeta)$ also has also three determinations

$$\hat{\mathcal{A}}_k(\zeta) = -\frac{1}{2\pi i} \frac{du_k}{d\zeta} \tag{3.47}$$

Then the Airy functions can be written as

$$\mathcal{A}_k(q,\hbar) = \int_{\Gamma_k} d\zeta \, e^{-\zeta/\hbar} \hat{\mathcal{A}}_k(\zeta) \tag{3.48}$$

where the paths G_k are shown in figure 21.

In the following we will denote by $\hat{\mathcal{A}}_k^+$ the extension from the right and by $\hat{\mathcal{A}}_k^-$

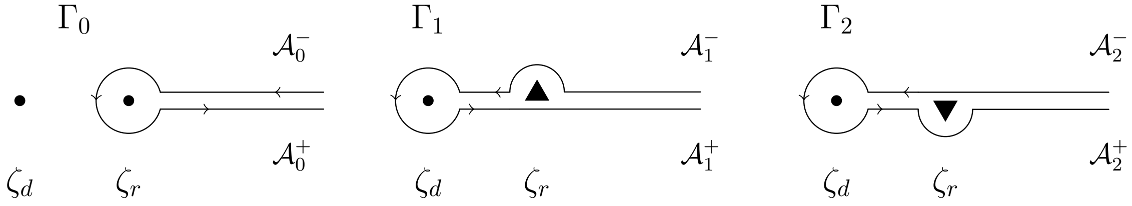

Figure 21. The paths Γ_k in the ζ-plane corresponding to the Airy functions $\mathcal{A}_k$ and the extension extensions of $\hat{\mathcal{A}}_k$ from the left and from the right.

the extension from the left of $\hat{\mathcal{A}}$ as shown in figure 21. Now

$$\begin{aligned}
\mathcal{A}_0(q,\hbar) &= \int_{\Gamma_0} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_0(\zeta) \\
&= \int_{\infty}^{\zeta_r} d\ \ e^{-\zeta/\hbar}\hat{\mathcal{A}}_0^-(\zeta) + \oint_{\zeta_{r,\varepsilon}} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_0(\zeta) + \int_{\zeta_r}^{\infty} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_0^+(\zeta) \\
&= -\int_{\zeta_r}^{\infty} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_0^-(\zeta) + \int_{\zeta_r}^{\infty} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_0^+(\zeta) \\
&= \int_{\zeta_r}^{\infty} d\zeta\, e^{-\zeta/\hbar}(\hat{\mathcal{A}}_0^+ - \hat{\mathcal{A}}_0^-)(\zeta)
\end{aligned} \tag{3.49}$$

where the second integral was calculated using equations (3.46) and (3.47):

$$\begin{aligned}
&\oint_{\zeta_{r,\varepsilon}} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_0(\zeta) \\
&= -\frac{1}{2\pi i}\oint_{\varepsilon} d\zeta\, e^{-\zeta/\hbar}\left(\frac{\pm i q^{-1/4}}{2(\zeta-\zeta_r)^{1/2}} + \text{ holomorphic }\right) \\
&= \mp\frac{i q^{-1/4}}{2\cdot 2\pi i}\int_0^{2\pi} dt\, \frac{i\varepsilon e^{-x\varepsilon e^{it}}}{\sqrt{\varepsilon}e^{it/2}} \\
&= \mp\frac{i q^{-1/4}}{4\pi i}\cdot i\sqrt{\varepsilon}\int_0^{2\pi} dt\, e^{-x\varepsilon e^{i\theta}-it/2} \\
&\longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0
\end{aligned} \tag{3.50}$$

where $\oint_{\zeta_{r,\varepsilon}}$ means that we integrate along a contour $t \in [0, 2\pi] \to \zeta_r + \varepsilon e^{it}$. Similar calculation shows that

$$\oint_{\zeta_{d,\varepsilon}} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_{1,2}(\zeta) = 0 \tag{3.51}$$

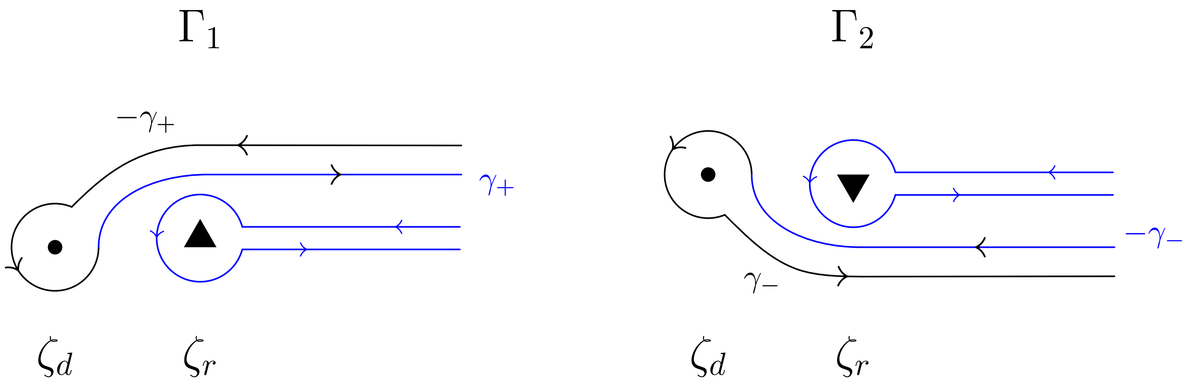

Figure 22. The deformations of paths Γ_1 and Γ_2. The blue color indicates that the path lies on the second Riemann sheet. Note that the singularity at ζ_r is seen in the first sheet.

Deforming the path Γ_1 and Γ_2 as shown in figure 22 we get

$$\begin{aligned}
\mathcal{A}_1(q,\hbar) &= \int_{\Gamma_1} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1(\zeta) \\
&= \int_{-\gamma^+} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^-(\zeta) + \oint_{\zeta_d,\varepsilon} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1(\zeta) + \int_{\gamma^+} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^+(\zeta) + \\
&\quad + \int_{\infty}^{\zeta_r} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^+(\zeta) + \oint_{\zeta_r,\varepsilon} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^+(\zeta) + \int_{\zeta_r}^{\infty} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^+(\zeta) \\
&= -\int_{\gamma^+} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^-(\zeta) + \int_{\gamma^+} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^+(\zeta) \\
&= \int_{\gamma^+} d\zeta\, e^{-\zeta/\hbar} (\hat{\mathcal{A}}_1^+ - \hat{\mathcal{A}}_1^-)(\zeta)
\end{aligned} \tag{3.52}$$

where the path γ^+ is the path from ζ_d to infinity avoiding ζ_r from the left as shown in figure 23.

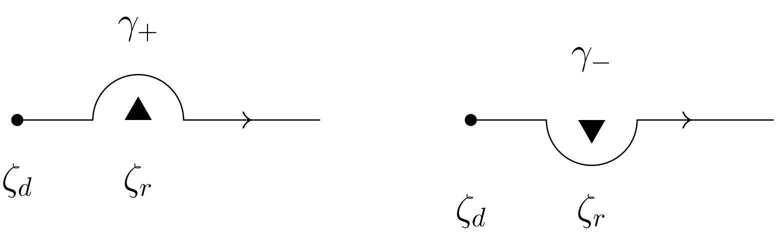

Figure 23. The path γ^+ circumvents ζ_r above and the path γ^- below.

Similarly for A_2 we get

$$\begin{aligned}
\mathcal{A}_2(q,\hbar) &= \int_{\Gamma_2} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_2(\zeta) \\
&= \int_{\zeta_r}^{\infty} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_2^-(\zeta) + \oint_{\zeta_r,\varepsilon} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_2(\zeta) + \int_{\infty}^{\zeta_r} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_2^-(\zeta) \\
&\quad + \int_{-\gamma^-} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_2^-(\zeta) + \oint_{\zeta_d,\varepsilon} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_{1,2}(\zeta) + \int_{\gamma^-} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_2^+(\zeta) + \\
&= -\int_{\gamma^-} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_2^-(\zeta) + \int_{\gamma^+} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_2^+(\zeta) \\
&= \int_{\gamma^-} d\zeta\, e^{-\zeta/\hbar}(\hat{\mathcal{A}}_2^+ - \hat{\mathcal{A}}_2^-)(\zeta) \qquad (3.53)
\end{aligned}$$

where the path γ^- is shown in figure 23.

From the figures 17 and 21 one sees that $\hat{\mathcal{A}}_2^+ = \hat{\mathcal{A}}_1^-$ and $\hat{\mathcal{A}}_2^- = \hat{\mathcal{A}}_1^+$, so

$$\hat{\mathcal{A}}_2^+ - \hat{\mathcal{A}}_2^- = -(\hat{\mathcal{A}}_1^+ - \hat{\mathcal{A}}_1^-) \qquad (3.54)$$

Then defining the functions

$$\hat{\mathcal{A}}_r := \hat{\mathcal{A}}_0^+ - \hat{\mathcal{A}}_0^- \qquad (3.55)$$

$$\hat{\mathcal{A}}_d := \hat{\mathcal{A}}_1^+ - \hat{\mathcal{A}}_1^- \qquad (3.56)$$

we get

$$\mathcal{A}_0(q,\hbar) = e^{-\zeta_r/\hbar}\int_0^{\infty} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_r(\zeta+\zeta_r) \qquad (3.57)$$

$$\mathcal{A}_1(q,\hbar) = e^{-\zeta_d/\hbar}\int_{\gamma^+} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_d(\zeta+\zeta_d) \qquad (3.58)$$

$$\mathcal{A}_2(q,\hbar) = -e^{-\zeta_d/\hbar}\int_{\gamma^-} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_d(\zeta+\zeta_d) \qquad (3.59)$$

where the γ^+ and γ^- are the paths shown in figure 23 but shifted such that they start from origin and the singularity is at $\zeta_r - \zeta_d$.

One then notices that right hand sides of above equations are in the from of Borel and lateral Borel sums:

$$\mathcal{A}_0(q,\hbar) = e^{-\zeta_r/\hbar}\mathcal{S}^0(\mathcal{B}^{-1}(\hat{\mathcal{A}}_r(\zeta+\zeta_r))) := e^{-\zeta_r/\hbar}\mathcal{S}^0\mathcal{A}_r(q,\hbar) \qquad (3.60)$$

$$\mathcal{A}_1(q,\hbar) = e^{-\zeta_d/\hbar}\mathcal{S}^{0^+}(\mathcal{B}^{-1}(\hat{\mathcal{A}}_d(\zeta+\zeta_d))) := e^{-\zeta_d/\hbar}\mathcal{S}^{0^+}\mathcal{A}_d(q,\hbar) \qquad (3.61)$$

$$\mathcal{A}_2(q,\hbar) = -e^{-\zeta_d/\hbar}\mathcal{S}^{0^-}(\mathcal{B}^{-1}(\hat{\mathcal{A}}_d(\zeta+\zeta_d))) := -e^{-\zeta_d/\hbar}\mathcal{S}^{0^-}\mathcal{A}_d(q,\hbar) \qquad (3.62)$$

Now using equations (3.46) and (3.47) we have

$$\begin{aligned}
\hat{\mathcal{A}}_0 &= -\frac{1}{2\pi i}\frac{du}{d\zeta} \\
&= -\frac{1}{2\pi i}\Bigg[\frac{\pm i}{(\zeta-\zeta_r)^{1/2}}\sum_{m=0}^{\infty}(-1)^m a_{2m+1}(m+1/2)(\zeta-\zeta_r)^m \\
&\quad + \sum_{m=1}^{\infty}(-1)^{m+1}a_{2m}m(\zeta-\zeta_r)^{m-1}\Bigg] \qquad (3.63)
\end{aligned}$$

The extension from the right of $u(\zeta)$ is given by negative imaginary singular part in (3.46) and the expansion from the left is given the positive imaginary part. Then we get

$$\begin{aligned}
\hat{\mathcal{A}}_r(\zeta) &= (\hat{\mathcal{A}}_0^+ - \hat{\mathcal{A}}_0^-)(\zeta) \\
&= -\frac{1}{2\pi i}\Big(-i-(+i)\Big)\sum_{m=0}^{\infty}(-1)^m a_{2m+1}(m+1/2)(\zeta-\zeta_r)^{m-1/2} \\
&= \frac{1}{2\pi}(\zeta-\zeta_r)^{-1/2}\sum_{m=0}^{\infty}(-1)^m a_{2m+1}(2m+1)(\zeta-\zeta_r)^m \qquad (3.64)
\end{aligned}$$

Because the above series converges, it inverse Borel transform is a divergent asymptotic series:

$$\begin{aligned}
\mathcal{A}_r(q,\hbar) &= \mathcal{B}^{-1}(\hat{\mathcal{A}}_r(\zeta+\zeta_r)) \\
&= \frac{1}{2\pi}\sum_{m=0}^{\infty}(-1)^m a_{2m+1}(2m+1)\Gamma(m+1/2)\hbar^{m+1/2} \qquad (3.65)
\end{aligned}$$

where the Borel transform was defined by

$$\hbar^{\nu+1} \longrightarrow \frac{\zeta^\nu}{\Gamma(\nu+1)}, \quad \nu \neq 0,-1,-2,\ldots \qquad (3.66)$$

Similarly for $\hat{\mathcal{A}}_1$ the extension from the right is given by negative singular part in (3.37) and from the left by the negative singular part. Then the discontinuity $\hat{\mathcal{A}}_d$ can be written as

$$\begin{aligned}\hat{\mathcal{A}}_d(\zeta) &= (\hat{\mathcal{A}}_1^+ - \hat{\mathcal{A}}_1^-)(\zeta)\\ &= -\frac{1}{2\pi i}\Big(1-(-1)\Big)\sum_{m=0}^{\infty} a_{2m+1}(m+1/2)(\zeta-\zeta_d)^{m-1/2}\\ &= \frac{i}{2\pi}(\zeta-\zeta_d)^{-1/2}\sum_{m=0}^{\infty} a_{2m+1}(2m+1)(\zeta-\zeta_d)^m \end{aligned} \tag{3.67}$$

$$\tag{3.68}$$

and the inverse Borel transform is

$$\begin{aligned}\mathcal{A}_d(q,\hbar) &= \mathcal{B}^{-1}(\hat{\mathcal{A}}_d(\zeta+\zeta_d))\\ &= \frac{i}{2\pi}\sum_{m=0}^{\infty} a_{2m+1}(2m+1)\Gamma(m+1/2)\hbar^{m+1/2}\end{aligned} \tag{3.69}$$

Using the the equation (3.36) for a_{2m+1} we have

$$a_{2m+1}(2m+1)\Gamma(m+1/2) = \frac{(6m)!\Gamma(1/2)}{3^{2m}(2m)!4^{3m}(3m)!}q^{-(6m+1)/4}, \tag{3.70}$$

where we used the Legendre multiplication formula

$$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \Gamma(1/2)\Gamma(2z). \tag{3.71}$$

Thus the formal series $\mathcal{A}_r(q,\hbar)$ and $\mathcal{A}_d(q,\hbar)$ can be written as

$$\mathcal{A}_r(q,\hbar) = \frac{1}{2\sqrt{\pi}}q^{-1/4}\hbar^{1/2}\left(1+\sum_{m=1}^{\infty}(-1)^m b_m \hbar^m\right) \tag{3.72}$$

and

$$\mathcal{A}_d(q,\hbar) = \frac{i}{2\sqrt{\pi}}q^{-1/4}\hbar^{1/2}\left(1+\sum_{m=1}^{\infty} b_m \hbar^m\right) \tag{3.73}$$

where

$$b_m = \frac{(6m)!}{3^{2m}(2m)!4^{3m}(3m)!}q^{-3m/2},\ m \geq 1. \tag{3.74}$$

It can be shown that the Borel series (3.72) and (3.73) can be analytically continued along the positive real axis and that the Laplace integrals converge, see for example Appendix 1 in[18]. Thus the Borel sums are asymptotic to the inverse

Borel transforms

$$\mathcal{S}^0 \mathcal{A}_r(q,\hbar) \sim \mathcal{A}_r(q,\hbar) \quad \text{as} \quad \hbar \to 0 \tag{3.75}$$

$$\mathcal{S}^{0^+} \mathcal{A}_d(q,\hbar) \sim \mathcal{A}_d(q,\hbar) \quad \text{as} \quad \hbar \to 0 \tag{3.76}$$

$$\mathcal{S}^{0^-} \mathcal{A}_d(q,\hbar) \sim \mathcal{A}_d(q,\hbar) \quad \text{as} \quad \hbar \to 0 \tag{3.77}$$

and the asymptotic expansions of the functions $\mathcal{A}_k(q,\hbar)$ are given by

$$\mathcal{A}_0(q,\hbar) \sim \frac{1}{2\sqrt{\pi}} q^{-1/4} \hbar^{1/2} \exp\left(-\frac{2}{3} q^{3/2} \frac{1}{\hbar}\right) \left(1 + \sum_{m=1}^{\infty} (-1)^m b_m \hbar^m\right), \tag{3.78}$$

$$\mathcal{A}_1(q,\hbar) \sim \frac{i}{2\sqrt{\pi}} q^{-1/4} \hbar^{1/2} \exp\left(\frac{2}{3} q^{3/2} \frac{1}{\hbar}\right) \left(1 + \sum_{m=1}^{\infty} b_m \hbar^m\right), \tag{3.79}$$

$$\mathcal{A}_2(q,\hbar) \sim -\frac{i}{2\sqrt{\pi}} q^{-1/4} \hbar^{1/2} \exp\left(\frac{2}{3} q^{3/2} \frac{1}{\hbar}\right) \left(1 + \sum_{m=1}^{\infty} b_m \hbar^m\right), \tag{3.80}$$

as $\hbar \to 0$. Then the asymptotic expansions of the solutions of the Airy equation are

$$A(q,\hbar) = \mathcal{A}_0(q,\hbar) \sim \frac{1}{2\sqrt{\pi}} q^{-1/4} \hbar^{1/2} \exp\left(-\frac{2}{3} q^{3/2} \frac{1}{\hbar}\right) \left(1 + \sum_{m=1}^{\infty} (-1)^m b_m \hbar^m\right) \tag{3.81}$$

and

$$\begin{aligned} B(q,\hbar) &= i\left(\mathcal{A}_2(q,\hbar) - \mathcal{A}_1(q,\hbar)\right) \\ &\sim -2i e^{-\zeta_d/\hbar} \mathcal{A}_d(q,\hbar) \\ &= \frac{1}{\sqrt{\pi}} q^{-1/4} \hbar^{1/2} \exp\left(\frac{2}{3} q^{3/2} \frac{1}{\hbar}\right) \left(1 + \sum_{m=1}^{\infty} b_m \hbar^m\right) \end{aligned} \tag{3.82}$$

3.3 Stokes phenomena of resurgent symbols $e^{-\zeta_r/\hbar} A_r$ and $e^{-\zeta_d/\hbar} A_d$

In the steepest descent method the Stokes phenomena occurs when the steepest descent contour goes through multiple saddle points which happens on the Stokes lines [20]. Transforming into the Borel plane the Stokes phenoma occurs when the point corresponging to the recessive saddle point lies on the cut from the point corresponding to the dominant saddle point [18].

Now the Stokes lines are defined as the directions in the complex q-plane which correspond to the case where the recessive point ζ_r lies on the cut from the dominant point ζ_d to infinity in the complex ζ-plane (Borel-plane). In the complex u-plane this happens when a steepest descent contour goes through a second saddle point. From equation (3.5) we see that this happens when

$$\operatorname{Im}\Big(s(u_d) - s(u_r)\Big) = 0 \tag{3.83}$$

or equivalently

$$\operatorname{Im}\Big(\zeta_d - \zeta_r\Big) = 0 \tag{3.84}$$

This gives

$$\operatorname{Im}(q^{3/2}) = 0 \iff \frac{3}{2}\arg q = k\pi \iff \arg q = \frac{2k\pi}{3} \tag{3.85}$$

Thus the Stokes lines of the Airy function are

$$L_0 = [0, \infty), \tag{3.86}$$

$$L_1 = e^{i2\pi/3}[0, \infty), \tag{3.87}$$

$$L_2 = e^{-i2\pi/3}[0, \infty) \tag{3.88}$$

as shown in figure 24 and are exactly the same as we found in 2.4.4

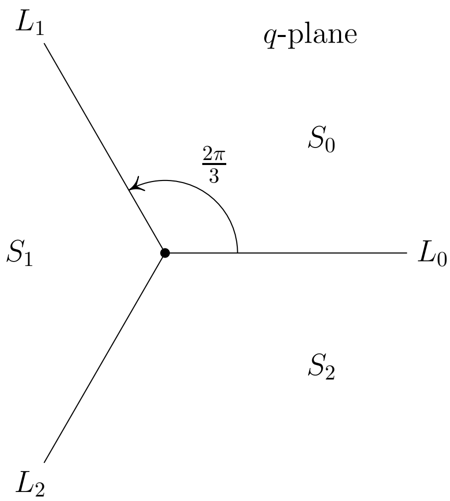

Figure 24. The Stokes lines L_k of the Airy equation and the Stokes regions S_k bounded by the Stokes lines in the comlex q-plane.

In deriving the equations (3.61) and (3.62):

$$\mathcal{A}_1(q,\hbar) = e^{-\zeta_d/\hbar}\mathcal{S}^{0^+}\mathcal{A}_d(q,\hbar)$$

$$\mathcal{A}_2(q,\hbar) = -e^{-\zeta_d/\hbar}\mathcal{S}^{0^-}\mathcal{A}_d(q,\hbar)$$

the deformation of the paths Γ_1 and Γ_2 used is not the only possible deformation. We can deform them by "pulling" the singular part through the cut between the branch points to the other sheet, as shown in figure 25, and get

$$\Gamma_1 = \gamma_r * \gamma_d^- \quad \text{and} \quad \Gamma_2 = \gamma_r * \gamma_d^+ \tag{3.89}$$

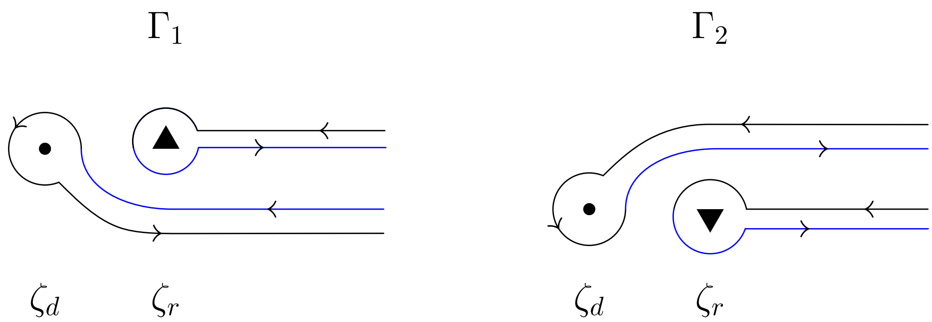

Figure 25. Another deformation of paths Γ_1 and Γ_2. The blue lines indicate that the path lies on the second Riemann sheet.

Then we have

$$\begin{aligned}
\mathcal{A}_1(q,\hbar) =& \int_{\Gamma_1} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1(\zeta) \\
=& \left(\int_{\gamma_d^-} + \int_{\gamma_r} \right) d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1(\zeta) \\
=& \int_{-\gamma^-} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^-(\zeta) + \int_{\zeta_d}^{\infty} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^+(\zeta) + \oint_{\zeta_{d,\varepsilon}} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1(\zeta) \qquad (3.90) \\
&+ \int_{\infty}^{\zeta_r} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^-(\zeta) + \oint_{\zeta_{d,\varepsilon}} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1(\zeta) + \int_{\zeta_r}^{\infty} d\zeta\, e^{-\zeta/\hbar} \hat{\mathcal{A}}_1^+(\zeta) \\
=& \int_{\gamma_-} d\zeta\, e^{-\zeta/\hbar} \Big(\hat{\mathcal{A}}_1^+ - \hat{\mathcal{A}}_1^- \Big)(\zeta) + \int_{\zeta_r}^{\infty} d\zeta\, e^{-\zeta/\hbar} \Big(\hat{\mathcal{A}}_1^+ - \hat{\mathcal{A}}_1^- \Big)(\zeta) \qquad (3.91)
\end{aligned}$$

where we used that $\hat{\mathcal{A}}_1$ is integrable at ζ_s and ζ_r, γ^- is the path shown in figure 23. From figure 17 we see that $\hat{\mathcal{A}}_1^-$ has the negative imaginary singular part in (3.46) (the parts l_2 have the same determination) and by crossing to the another sheet $\hat{\mathcal{A}}_1^+$

must have the positive imaginary singular part. Then as in (3.64)

$$
\begin{aligned}
(\hat{\mathcal{A}}_1^+ - \hat{\mathcal{A}}_1^-)(\zeta) &= -\frac{1}{2\pi i}\Big(i-(-i)\Big)\sum_{m=0}^{\infty}(-1)^m a_{2m+1}(m+1/2)(\zeta-\zeta_r)^{m-1/2} \\
&= -\frac{1}{2\pi}(\zeta-\zeta_r)^{-1/2}\sum_{m=0}^{\infty}(-1)^m a_{2m+1}(2m+1)(\zeta-\zeta_r)^m \\
&= -\hat{\mathcal{A}}_r(\zeta) \qquad (3.92) \\
& \qquad (3.93)
\end{aligned}
$$

Thus we get, after a change of variables $\zeta = \zeta + \zeta_d$ and $\zeta = \zeta + \zeta_r$

$$
\begin{aligned}
\mathcal{A}_1(q,\hbar) &= e^{-\zeta_d/\hbar}\int_{\gamma_-} d\zeta\, e^{-\zeta/\hbar}\hat{\mathcal{A}}_d(\zeta+\zeta_d) + e^{-\zeta_d/\hbar}\int_0^{\infty} d\zeta\, e^{-\zeta/\hbar}(-\mathcal{A}_r)(\zeta+\zeta_r) \\
&= e^{-\zeta_d/\hbar}\mathcal{S}^{0^-}A_d(q,\hbar) - e^{-\zeta_r/\hbar}\mathcal{S}^0\hat{\mathcal{A}}_r(q,\hbar) \qquad (3.94)
\end{aligned}
$$

Now a similar calculation for $\mathcal{A}_2$ gives

$$
\begin{aligned}
\mathcal{A}_2(q,\hbar) &= e^{-\zeta_d/\hbar}\int_{\gamma_+} d\zeta\, e^{-\zeta/\hbar}(\hat{\mathcal{A}}_2^+ - \hat{\mathcal{A}}_2^-)(\zeta+\zeta_d) + e^{-\zeta_d/\hbar}\int_{\zeta_r}^{\infty} d\zeta\, e^{-\zeta/\hbar}(\hat{\mathcal{A}}_2^+ - \hat{\mathcal{A}}_2^-)(\zeta) \\
&= e^{-\zeta_d/\hbar}\int_{\gamma_+} d\zeta\, e^{-\zeta/\hbar}(-\hat{\mathcal{A}}_d)(\zeta+\zeta_d) + e^{-\zeta_d/\hbar}\int_{\zeta_r}^{\infty} d\zeta\, e^{-\zeta/\hbar}(-\hat{\mathcal{A}}_r)(\zeta+\zeta_r) \\
&= -e^{-\zeta_d/\hbar}\mathcal{S}^{0^+}A_d(q,\hbar) - e^{-\zeta_r/\hbar}\mathcal{S}^0 A_r(q,\hbar) \qquad (3.95)
\end{aligned}
$$

where $\hat{\mathcal{A}}_2^+$ has the positive imaginary singular part in (3.46) and $\hat{\mathcal{A}}_2^-$ has the negative imaginary singular part, so at ζ_r

$$
(\hat{\mathcal{A}}_2^+ - \hat{\mathcal{A}}_2^-)(\zeta) = -\hat{\mathcal{A}}_r(\zeta) \qquad (3.96)
$$

Since $\hat{\mathcal{A}}_r(\zeta+\zeta_r)$ is holomorphic in a strip along $[0,\infty)$,

$$
\mathcal{S}^0\mathcal{A}_r(q,\hbar) = \mathcal{S}^{0^-}\mathcal{A}_r(q,\hbar) = \mathcal{S}^{0^+}\mathcal{A}_r(q,\hbar) \qquad (3.97)
$$

we get

$$
\begin{aligned}
\mathcal{A}_1(q,\hbar) &= \mathcal{S}^{0^-}\Big(e^{-\zeta_d/\hbar}A_d(q,\hbar) - e^{-\zeta_r/\hbar}A_r(q,\hbar)\Big) \qquad (3.98) \\
\mathcal{A}_2(q,\hbar) &= \mathcal{S}^{0^+}\Big(-e^{-\zeta_d/\hbar}A_d(q,\hbar) - e^{-\zeta_r/\hbar}A_r(q,\hbar)\Big) \qquad (3.99)
\end{aligned}
$$

Thus we have two different Borel sums of $\mathcal{A}_1$: (3.61) and (3.98) and two Borel sums of $\hat{\mathcal{A}}_2$: (3.62) and (3.99)

$$\mathcal{A}_1(q,\hbar) = \mathcal{S}^{0^+}\Big(e^{-\zeta_d/\hbar}\mathcal{A}_d(q,\hbar)\Big) \tag{3.100}$$

$$\mathcal{A}_1(q,\hbar) = \mathcal{S}^{0^-}\Big(e^{-\zeta_d/\hbar}A_d(q,\hbar) - e^{-\zeta_r/\hbar}A_r(q,\hbar)\Big) \tag{3.101}$$

$$\mathcal{A}_2(q,\hbar) = \mathcal{S}^{0^-}\Big(- e^{-\zeta_d/\hbar}\mathcal{A}_d(q,\hbar)\Big) \tag{3.102}$$

$$\mathcal{A}_2(q,\hbar) = \mathcal{S}^{0^+}\Big(- e^{-\zeta_d/\hbar}A_d(q,\hbar) - e^{-\zeta_r/\hbar}A_r(q,\hbar)\Big) \tag{3.103}$$

As mentioned the before, the Stokes phenomena is given be the discontinuity between the lateral Borel sums. As we can see from the above equations there is a discontinuity between the lateral Borel sums: when we cross the Stokes line L_0 from below the recessive point ζ_r (saddle u_r) starts to contribute to the asymptotic expansion. Why it didn't contribute before we crossed the Stokes line? Below the Stokes line L_0, the recessive point ζ_r is "hidden" on the second Riemann sheet as seen in figure 26. As the argument of q varies, the points ζ_d and z_r rotate in the complex ζ-plane and in the case $\arg q < 0$ the recessive point ζ_r lies on the second sheet of the Riemann surface defined by the branch point ζ_d.

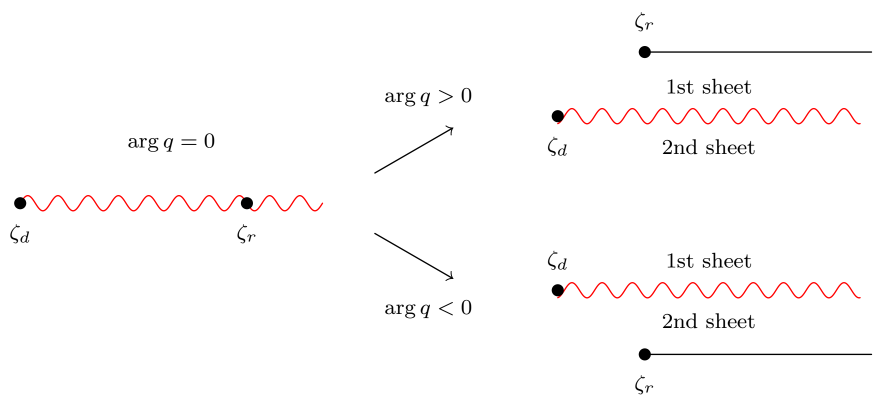

Figure 26. Rotation the points ζ_d and ζ_r when crossing the Stokes line L_0. The red lines are the branch cuts.

Another way to see this is to look at how the steepest descent contours in the u-plane behave as the argument of q varies. When $\arg q < 0$ the contour γ_1 corresponding to $\mathcal{A}_1$ in (3.2) can be deformed to the steepest descent contour going through the dominant saddle u_d (the intersection of the orange lines) as shown in figure 27, so there is no contribution from the recessive saddle u_r.

When $\arg q > 0$, 28, the deformation of the contour γ_1 into a steepest descent contour first goes through the recessive saddle u_r (the intersection of the blue lines) and then the dominant saddle u_d (intersection of the orange lines). Thus both saddles contribute to the asymptotic expansion. The transformation of these contours to the ζ-plane are shown in figure 29.

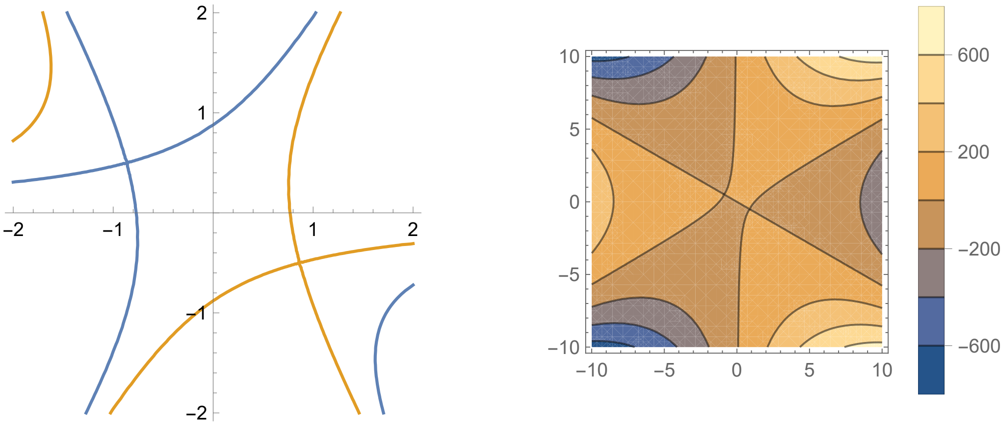

Figure 27. $\arg q = -\pi/3$ Left: the steepest descent contours on the u-plane. Blue lines cross at the recessive saddle $u_r = -q^{1/2}$ and orange lines cross the dominant saddle $u_d = q^{1/2}$. Right: Level curves.

3.4 Crossing of the Stokes line

What happens to our full solution of the Airy Schrödinger equation (3.4) when we cross a Stokes line? Because the function $\hat{\mathcal{A}}_r$ (3.64) is Borel summable along the positive real axis it holds that

$$\mathcal{S}^0\left(e^{-\zeta_r/\hbar}\mathcal{A}_r(q,\hbar)\right) = \mathcal{S}^{0^-}\left(e^{-\zeta_r/\hbar}\mathcal{A}_r(q,\hbar)\right) = \mathcal{S}^{0^+}\left(e^{-\zeta_r/\hbar}\mathcal{A}_r(q,\hbar)\right) \tag{3.104}$$

so the function $A = \mathcal{A}_0$ doesn't change when crossing the Stokes line L_0. On the other hand, using the equations (3.100) - (3.103) the second linearly independent

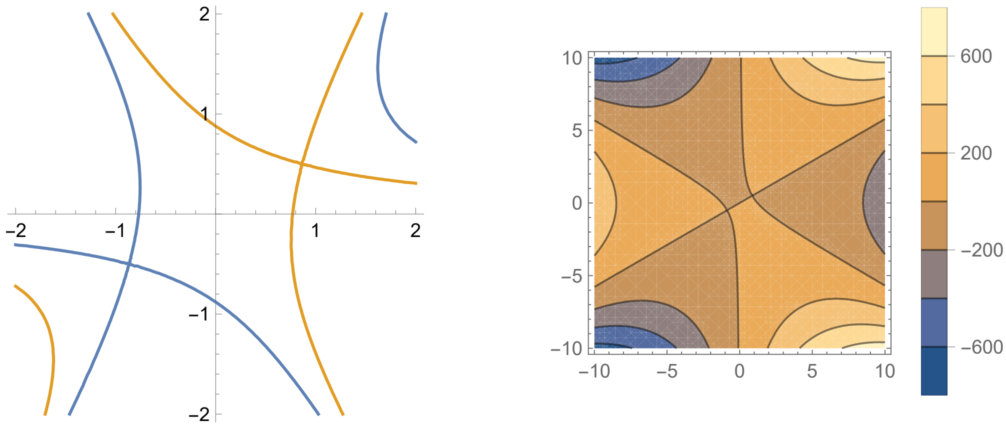

Figure 28. $\arg q = \pi/3$ Left: the steepest descent contours on the u-plane. Blue lines cross at the recessive saddle $u_r = -q^{1/2}$ and orange lines cross the dominant saddle $u_d = q^{1/2}$. Right: Level curves.

solution $B = i\mathcal{A}_1 - i\mathcal{A}_2$ can be written as

$$B_+(q,\hbar) = \mathcal{S}^{0^+}\left(2ie^{-\zeta_d/\hbar}\mathcal{A}_d(q,\hbar) + ie^{-\zeta_r/\hbar}\mathcal{A}_r(q,\hbar)\right) \tag{3.105}$$

$$B_-(q,\hbar) = \mathcal{S}^{0^-}\left(2ie^{-\zeta_d/\hbar}\mathcal{A}_d(q,\hbar) - ie^{-\zeta_r/\hbar}\mathcal{A}_r(q,\hbar)\right) \tag{3.106}$$

As we saw before, along the Stokes line L_0 the asymptotics of B was given by $B \sim -2ie^{-\zeta_d/\hbar}A_d$, so crossing the Stokes line introduces the term $\pm iA$. On the anti-Stokes line between the Stokes lines L_0 and L_1 the functions A and B change roles, A becomes dominant and B becomes recessive. Thus the crossing of the Stokes line L_1 yields $B \to B$ and $A \to A \pm iB$. And similarly for L_2. Thus the crossing of a Stokes line L_0 is given by

$$\begin{cases} B^{\mathrm{I}} = B^{\mathrm{II}} + iA^{\mathrm{II}} \\ A^{\mathrm{I}} = A^{\mathrm{II}} \end{cases} \tag{3.107}$$

and of L_1

$$\begin{cases} A^{\mathrm{II}} = A^{\mathrm{III}} + iB^{\mathrm{III}} \\ B^{\mathrm{II}} = B^{\mathrm{III}} \end{cases} \tag{3.108}$$

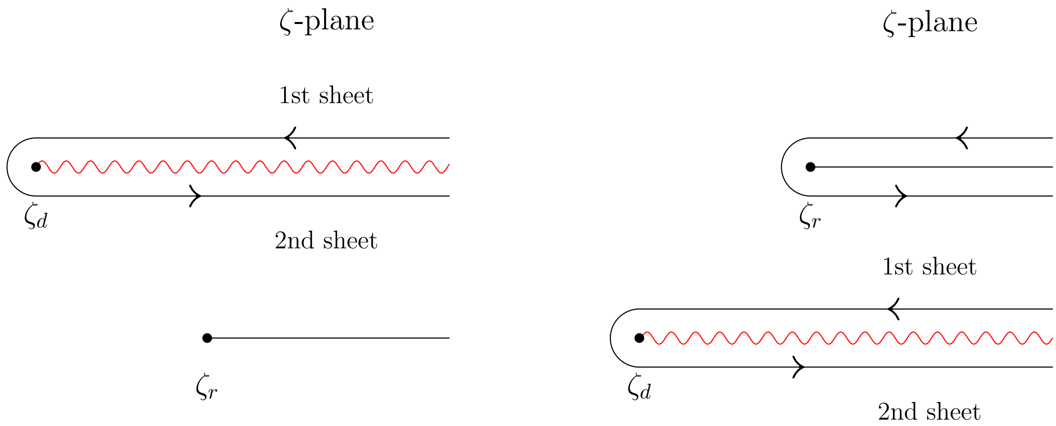

Figure 29. Transformations of the rotated steepest descent contours to the ζ-plane. Left: $\arg q < 0$. Right: $\arg q > 0$.

and of L_2

$$\begin{cases} A^{\mathrm{I}} = A^{\mathrm{IV}} + iB^{\mathrm{IV}} \\ B^{\mathrm{I}} = B^{\mathrm{IV}} \end{cases} \tag{3.109}$$

The above equations are the connection formula for the Airy-Schrödinger equation. These can also be written in matrix form:

$$M_0 = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \quad M_{1,2} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \tag{3.110}$$

where 0,1,2 refer to the corresponding Stokes line.

Crossing of the branch cut along the negative real axis is given by the difference on either side of the branch cut. Above the branch cut in the region III we have $A + iB$ and below in the region IV $A - iB$. This leads to the connection formula

$$M_b = \begin{cases} A^{\mathrm{III}} = -iB^{\mathrm{IV}} \\ B^{\mathrm{III}} = -iA^{\mathrm{IV}} \end{cases} \tag{3.111}$$

and in the matrix form

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{3.112}$$

The four matrices M_0,M_1,M_2 and M_b describe the monodromy around the turing

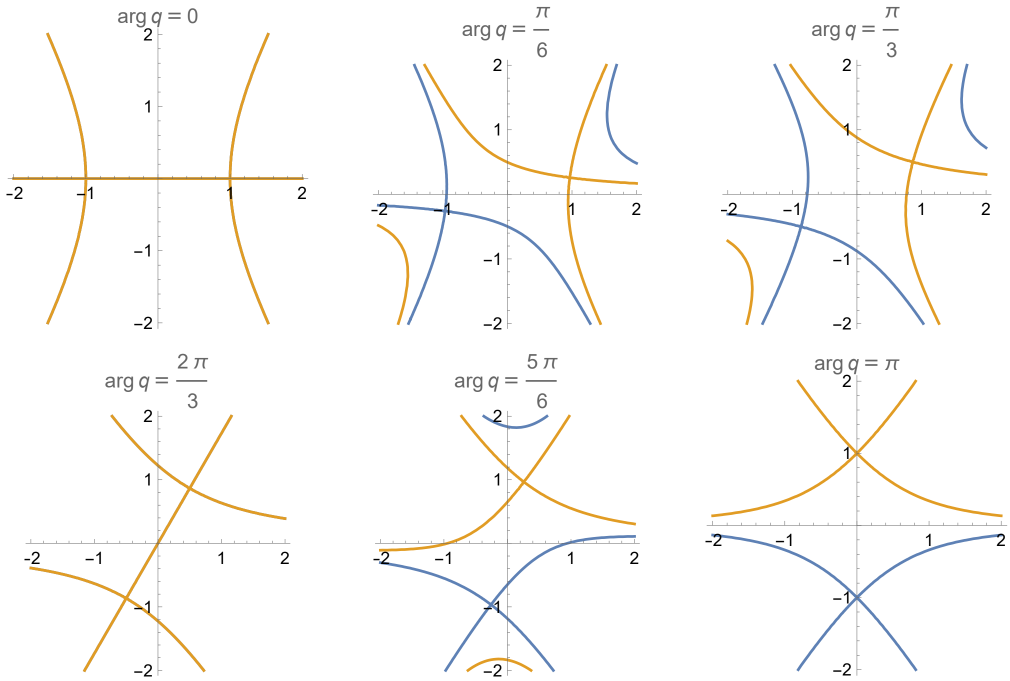

Figure 30. Behaviour of steepest descent contours on the u-plane when q rotates in the q-plane. Blue lines cross at the saddle $u_r = -q^{1/2}$ and the orange lines cross at the saddle $u_d = q^{1/2}$.

point such that

$$M_2 M_b M_1 M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = M_0^{-1} M_1^{-1} M_b^{-1} M_2^{-1} \tag{3.113}$$

3.5 Stokes regions

From figure 30 we can see that for the function

$$\mathcal{A}_0(q,\hbar) = -\frac{1}{2\pi i}\int_{\gamma_0} du\, e^{-xs(u,q)} \tag{3.114}$$

the path γ_0 can only be deformed to go through the saddle $-q^{1/2}$ when

$$-\frac{2\pi}{3} < \arg q < \frac{2\pi}{3}. \tag{3.115}$$

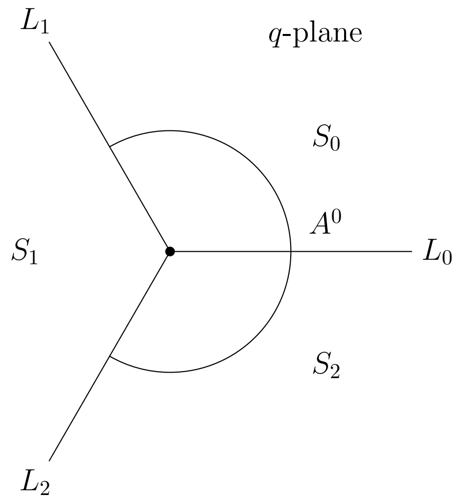

Figure 31. Extension A^0 of A_r in the sector $S_2 \cup S_0$

Thus the same asymptotic relation for $\mathcal{A}_0$ holds in this sector and we will denote it A^0 (figure 31):

$$A^0 := \text{extension of } e^{-\zeta_r/\hbar} A_r \text{ in the sector } |\arg q| < \frac{2\pi}{3}. \tag{3.116}$$

On the Riemann surface this means that the point ζ_d is "hidden" on the other sheet and doesn't contribute to the asymptotic expansion, that is, there is no Stokes phenomena for $\mathcal{A}_r$ along L_0. But when q crosses the Stokes line L_1 or L_2, ζ_d crosses the junction of the two sheets around ζ_r and reappears on the same sheet as ζ_r and starts to contribute to the asymptotic expansion. Now the roles of ζ_r and ζ_d have changed, ζ_r is the dominant term and ζ_d recessive. Thus on the edges of the sector, A^0 is dominant and on the bisector L_0 it is recessive.

After crossing the Stokes lines L_1 and L_2, ζ_r becomes the dominant term and ζ_d recessive, until along the anti-Stokes line $\arg q = \pi$ they become same order of magnitude and purely imaginary $(q^{3/2} \to -iq^{3/2})$. This leads to an oscillating behaviour rather than exponentially decaying/growing behaviour.

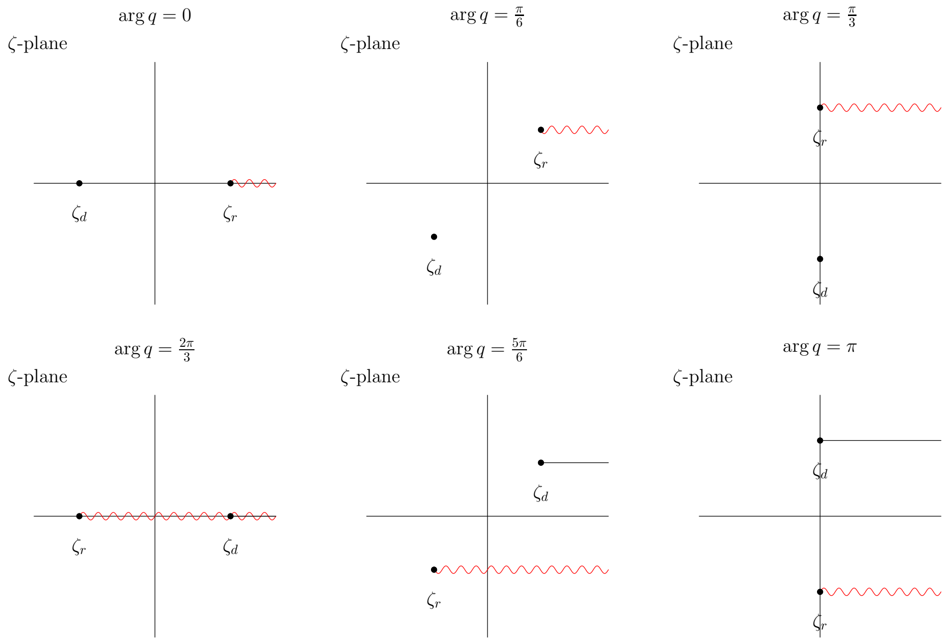

Figure 32. Behaviour of the points ζ_r and ζ_d as q rotates. At $\arg q = 2\pi/3$, ζ_d appears on the junction of two sheets around ζ_r and becomes "visible".

Thus the asymptotic expansion of $A_0(q,\hbar)$ is given by

$$\begin{aligned} A_0(q,\hbar) &\sim A^0(q,\hbar) = e^{-\zeta_r/\hbar} A_r(q,\hbar) \\ &= \frac{1}{2\sqrt{\pi}} q^{-1/4} \hbar^{1/2} \exp\left(-\frac{2}{3} q^{3/2} \frac{1}{\hbar}\right) \left(1 + \sum_{m=1}^{\infty} (-1)^m b_m(q) \hbar^m\right), \quad |\arg q| < \frac{2\pi}{3} \end{aligned} \tag{3.117}$$

and

$$A_0(q,\hbar) \sim e^{-\zeta_r/\hbar} A_r(q,\hbar) + e^{-\zeta_d/\hbar} A_d(q,\hbar), \quad \frac{2\pi}{3} < \arg q < \pi \text{ or } -\frac{2\pi}{3} < \arg q < -\pi \tag{3.118}$$

When finding the explicit expansion we have to use $q = |q|\, e^{i \arg q}$, so the form depends on the argument of q. On the anti-Stokes line $\arg q = \pi$ the expansion is given by (**??**):

$$\begin{aligned}
A_0(q,\hbar) \sim\ & \frac{e^{-i\pi/4}}{2\sqrt{\pi}}\,|q|^{-1/4}\,\hbar^{1/2}\exp\left(i\frac{2}{3}x\,|q|^{3/2}\right)\left(1+\sum_{m=1}^{\infty}(-i)^m b_m(|q|)\hbar^m\right)\\
&+\frac{e^{i\pi/4}}{2\sqrt{\pi}}\,|q|^{-1/4}\,\hbar^{1/2}\exp\left(-i\frac{2}{3}x\,|q|^{3/2}\right)\left(1+\sum_{m=1}^{\infty}i^m b_m(|q|)\hbar^m\right)\\
=\ & \frac{1}{2\sqrt{\pi}}\,|q|^{-1/4}\,\hbar^{1/2}\Bigg[\left(e^{i\left(\frac{2}{3}x|q|^{3/2}-\frac{\pi}{4}\right)}+e^{-i\left(\frac{2}{3}x|q|^{3/2}-\frac{\pi}{4}\right)}\right)\left(1+\sum_{m=2,4,6\ldots}b_m(|q|)\hbar^m\right)\\
&+i\left(e^{i\left(\frac{2}{3}x|q|^{3/2}-\frac{\pi}{4}\right)}-e^{-i\left(\frac{2}{3}x|q|^{3/2}-\frac{\pi}{4}\right)}\right)\sum_{m=1,3,5\ldots}b_m(|q|)\hbar^m\Bigg]\\
=\ & \frac{1}{\sqrt{\pi}}\,|q|^{-1/4}\,\hbar^{1/2}\Bigg[\cos\left(\frac{2}{3}x\,|q|^{3/2}-\frac{\pi}{4}\right)\sum_{m=0}^{\infty}b_{2m}(|q|)\hbar^{2m}\\
&+\sin\left(\frac{2}{3}x\,|q|^{3/2}-\frac{\pi}{4}\right)\sum_{m=0}^{\infty}b_{2m+1}(|q|)\hbar^{2m-1}\Bigg]
\end{aligned}\tag{3.119}$$

For the function

$$\mathcal{A}_1(q,\hbar)=-\frac{1}{2\pi i}\int_{\gamma_1}du\,e^{-xs(u,q)}\tag{3.120}$$

we see that in the Stokes region S_0 the path of steepest descent goes through both saddles, or equivalently ζ_d and ζ_r lie on the same Riemann sheet in the ζ-plane, thus both contributing to the asymptotic expansion. At $\arg q=\pi/3$, ζ_r becomes the dominant term and after q crosses the Stokes line L_1 at $\arg q=2\pi/3$ the point z_d, which is maximally subdominant on L_1, disappears on the other sheet since it crosses the cut between the two sheets around the dominant point ζ_r. It reappears after q crosses the Stokes line L_0 since rotation by $4\pi/3$ in the q-plane correspond to 2π rotation in the ζ-plane. Thus the the asymptotics of $\mathcal{A}_1(q,\hbar)$ are

$$\mathcal{A}_1(q,\hbar)\sim e^{-\zeta_d/\hbar}A_d(q,\hbar)-e^{-\zeta_r/\hbar}A_r(q,\hbar),\quad q\in S_0\tag{3.121}$$

and

$$\mathcal{A}_1(q,\hbar)\sim -e^{-\zeta_r/\hbar}A_r(q,\hbar),\quad \frac{2\pi}{3}<\arg q<2\pi.\tag{3.122}$$

and

$$\mathcal{A}_1(q,\hbar)\sim e^{-\zeta_d/\hbar}A_d(q,\hbar),\quad \frac{-2\pi}{3}<\arg q<0.\tag{3.123}$$

Similarly for the function $\mathcal{A}_2(q,\hbar)$ we have

$$\mathcal{A}_2(q,\hbar) \sim -e^{-\zeta_d/\hbar} A_d(q,\hbar) - e^{-\zeta_r/\hbar} A_r(q,\hbar), \quad q \in S_2 \tag{3.124}$$

and

$$\mathcal{A}_2(q,\hbar) \sim -e^{-\zeta_r/\hbar} A_r(q,\hbar), \quad -\frac{4\pi}{3} < \arg q < 0. \tag{3.125}$$

and

$$\mathcal{A}_2(q,\hbar) \sim e^{-\zeta_r/\hbar} A_r(q,\hbar), \quad 0 < \arg q < \frac{4\pi}{3}. \tag{3.126}$$

For the second solution of the Airy equation

$$B(q,\hbar) = i\mathcal{A}_2 - i\mathcal{A}_1 \tag{3.127}$$

we had the asymptotic expansion

$$\begin{aligned} B(q,\hbar) &\sim -2ie^{-\zeta_d/\hbar}\mathcal{A}_d \\ &= \frac{1}{\sqrt{\pi}} q^{-1/4}\hbar^{1/2} \exp\left(\frac{2}{3} q^{3/2}\hbar^{-1}\right)\left(1 + \sum_{m=1}^{\infty} b_m(q)\hbar^m\right) \end{aligned} \tag{3.128}$$

Since the subdominant term becomes of equal magnitude with the dominant term at the anti-Stokes lines $\arg q = \pm\pi/3$ the above relation is valid between these lines, that is, in the sector

$$|\arg q| < \frac{\pi}{3}. \tag{3.129}$$

The subdominant term has different coefficients depending on which direction the Stokes line L_0 is crossed.

3.6 Resurgence

From equation the equation (3.64)

$$\hat{\mathcal{A}}_r(\zeta + \zeta_r) = \frac{1}{2\pi} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(3m + 1/2)}{3^{2m}(2m)!\Gamma(m + 1/2)} q^{-(6m+1)/4} \zeta^{m-1/2} \tag{3.130}$$

the structure of simple resurgent function isn't obvious, but taking the inverse Borel transform as before we have

$$\begin{aligned} A_r(q,\hbar) &= \frac{1}{2\pi}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(3m+1/2)}{3^{2m}(2m)!}q^{-(6m+1)/4}\hbar^{m+1/2} \\ &= \frac{1}{2\sqrt{\pi}}q^{-1/4}\hbar^{1/2}\left(1+\sum_{m=1}^{\infty}\frac{(-1)^m\Gamma(3m+1/2)}{\sqrt{\pi}3^{2m}(2m)!}q^{-3m/2}\hbar^m\right) \end{aligned} \tag{3.131}$$

and define the formal series

$$\varphi_r(\hbar) := \sum_{m=1}^{\infty}\frac{(-1)^m\Gamma(3m+1/2)}{\sqrt{\pi}3^{2m}(2m)!}q^{-3m/2}\hbar^m \tag{3.132}$$

and denote the coefficients by c_m. Now using the Legendre multiplication formula

$$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \Gamma(1/2)\Gamma(2z). \tag{3.133}$$

we have

$$\Gamma(3m+1/2) = \frac{\Gamma(1/2)\Gamma(6m)}{2^{6m-1}\Gamma(3m)} \tag{3.134}$$

so

$$\begin{aligned} c_m &= \frac{(-1)^m\Gamma(6m)}{3^{2m}(2m)!2^{6m-1}\Gamma(3m)} \\ &= \frac{(-1)^m\Gamma(6m)}{3^{2m}4^{2m}m\Gamma(2m)\Gamma(3m)} \end{aligned} \tag{3.135}$$

where we used $(2m)! = \Gamma(2m+1) = 2m\Gamma(2m)$. Now using the Gauss multiplication formula

$$\prod_{k=0}^{n-1}\Gamma(z+k/n) = (2\pi)^{\frac{1}{2}(n-1)}n^{\frac{1}{2}-nz}\Gamma(nz) \tag{3.136}$$

we have

$$\begin{aligned} \Gamma(6m) &= (2\pi)^{-\frac{1}{2}(1-6)}6^{-\frac{1}{2}+6m}\prod_{k=0}^{6-1}\Gamma(m+k/6) \\ &= (2\pi)^{-\frac{5}{2}}6^{6m-\frac{1}{2}}\Gamma(m)\Gamma(m+1/6)\Gamma(m+1/3)\Gamma(m+1/2)\Gamma(m+2/3)\Gamma(m+5/6), \end{aligned} \tag{3.137}$$

$$\begin{aligned}\Gamma(3m) &= (2\pi)^{-\frac{1}{2}(1-3)}3^{3m-\frac{1}{2}}\prod_{k=0}^{3-1}\Gamma(m+k/3)\\ &= (2\pi)^{-1}3^{3m-\frac{1}{2}}\Gamma(m)\Gamma(m+1/3)\Gamma(m+2/3)\end{aligned} \tag{3.138}$$

and

$$\begin{aligned}\Gamma(2m) &= (2\pi)^{-\frac{1}{2}(1-2)}2^{2m-\frac{1}{2}}\prod_{k=0}^{2-1}\Gamma(m+k/2)\\ &= (2\pi)^{-\frac{1}{2}}2^{2m-\frac{1}{2}}\Gamma(m)\Gamma(m+1/2),\end{aligned} \tag{3.139}$$

so

$$\begin{aligned}c_m &= \frac{(-1)^m}{3^{2m}4^{2m}m}\cdot\frac{(2\pi)^{-\frac{5}{2}+1+\frac{1}{2}}6^{6m-\frac{1}{2}}}{3^{3m-\frac{1}{2}}2^{2m-\frac{1}{2}}}\cdot\frac{\Gamma(m+1/6)\Gamma(m+5/6)}{\Gamma(m)}\\ &= \frac{(-1)^m 6^{6m}}{2\pi 3^{2m+3m}4^{2m+m}m}\cdot\frac{\Gamma(m+1/6)\Gamma(m+5/6)}{\Gamma(m)}\\ &= \frac{(-1)^m}{2\pi}\left(\frac{3}{4}\right)^m\frac{\Gamma(m+1/6)\Gamma(m+5/6)}{m!}\end{aligned} \tag{3.140}$$

where we had

$$\frac{6^{6m}}{2\pi 3^{2m+3m}4^{2m+m}} = \left(\frac{6^6}{3^5\cdot 4^4}\right)^m = \left(\frac{3}{4}\right)^m. \tag{3.141}$$

Now

$$\begin{aligned}\varphi_r(\hbar) &= \sum_{m=1}^{\infty}\frac{(-1)^m}{2\pi}\left(\frac{3}{4}\right)^m\frac{\Gamma(m+1/6)\Gamma(m+5/6)}{m!}q^{-3m/2}\hbar^m\\ &= -\frac{3}{4}\cdot\frac{q^{-3/2}}{2\pi}\sum_{m=0}^{\infty}(-1)^m\left(\frac{3}{4}\right)^m\frac{\Gamma(m+11/6)\Gamma(m+7/6)}{(m+1)!}q^{-3m/2}\hbar^{m+1}.\end{aligned} \tag{3.142}$$

Taking the Borel transform we get

$$\begin{aligned}\hat{\varphi}_r(\zeta) = \mathcal{B}\varphi_r(x) &= -\frac{3}{4}\cdot\frac{q^{-3/2}}{2\pi}\sum_{m=0}^{\infty}\frac{\Gamma(m+11/6)\Gamma(m+7/6)}{\Gamma(m+2)(m)!}\left(-\frac{3}{4}q^{-3/2}\zeta\right)^m\\ &= -\frac{3}{4}\cdot\frac{q^{3/2}}{2\pi}\frac{\Gamma(11/6)\Gamma(7/6)}{\Gamma(2)}\frac{\Gamma(2)}{\Gamma(11/6)\Gamma(7/6)}\Bigg[\\ &\qquad\times\sum_{m=0}^{\infty}\frac{\Gamma(m+11/6)\Gamma(m+7/6)}{\Gamma(m+2)(m)!}\left(-\frac{3}{4}q^{-3/2}\zeta\right)^m\Bigg]\\ &= -\frac{5}{48}q^{-3/2}F(\tfrac{11}{6},\tfrac{7}{6},2;-\tfrac{3}{4}q^{-3/2}\zeta),\end{aligned} \tag{3.143}$$

where F is the Gauss hypereometric function

$$F(a,b,c\,;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(n+a)\Gamma(n+b)}{n!\Gamma(n+c)}z^n \tag{3.144}$$

and

$$\Gamma(11/6)\Gamma(7/6)=\frac{5}{6}\cdot\frac{1}{6}\Gamma(5/6)\Gamma(1-5/6)=\frac{5}{6}\cdot\frac{1}{6}\frac{\pi}{\sin\frac{5\pi}{6}}=\frac{5}{36}\cdot 2\pi \tag{3.145}$$

using the reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},\qquad z\notin\mathbb{Z}. \tag{3.146}$$

Because we have $c=2=\frac{11}{6}+\frac{7}{6}-1=a+b-1$, this leads to the Gauss hypergeometric function having logarithmic dependence [27] (Theorem 5.1). Now the hypergeometric function can be written as

$$\begin{aligned}
&F(a,b\,,a+b-1\,;z)\\
&=\frac{\Gamma(a+b-1)}{\Gamma(a-1)\Gamma(b-1)}F(a,b,\,2\,;1-z)\log(1-z)+\frac{1}{1-z}\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}\\
&+\frac{\Gamma(a+b-1)}{\Gamma(a-1)\Gamma(b-1)}\sum_{m=0}^{\infty}\frac{\Gamma(m+a)\Gamma(m+b)}{\Gamma(a)\Gamma(b)(m+1)!}\Big[\psi(m+a)+\psi(m+b)\\
&-\psi(m+2)-\psi(m+1)\Big]\frac{(1-z)^m}{m!}
\end{aligned} \tag{3.147}$$

where ψ is the logarithmic derivative of the gamma function, i.e. the digamma function

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}. \tag{3.148}$$

Now using equation (3.147) we get

$$\begin{aligned}\hat{\varphi}_r(\zeta) =& \frac{-5q^{-3/2}}{48}\Bigg(\frac{\Gamma(2)}{\Gamma(11/6)\Gamma(7/6)}\frac{1}{1+\frac{3}{4}q^{-3/2}\zeta}\\
&+\frac{\Gamma(2)}{\Gamma(5/6)\Gamma(1/6)}F(\tfrac{11}{6},\tfrac{7}{6},2;1+\tfrac{3}{4}q^{-3/2}\zeta)\log(1+\tfrac{3}{4}q^{-3/2}\zeta)\\
&+\frac{\Gamma(2)}{\Gamma(5/6)\Gamma(1/6)}\sum_{m=0}^{\infty}\frac{\Gamma(m+11/6)\Gamma(m+7/6)}{\Gamma(11/6)\Gamma(7/6)(m+1)!}\Big[\psi(m+11/6)+\psi(m+7/6)\\
&-\psi(m+2)-\psi(m+1)\Big]\frac{(1+\frac{3}{4}q^{-3/2}\zeta)^m}{m!}\Bigg)\\
=&-\frac{3q^{-3/2}}{2\pi\cdot 4(1+\frac{3}{4}q^{-3/2}\zeta)}\\
&-\frac{5q^{-3/2}}{2\pi\cdot 48}F(\tfrac{11}{6},\tfrac{7}{6},2;1+\tfrac{3}{4}q^{-3/2}\zeta)\Big(\log(\zeta+\tfrac{4}{3}q^{3/2})+\log(\tfrac{3}{4}q^{-3/2})\Big)\\
&-\frac{5q^{-3/2}}{2\pi\cdot 48}\sum_{m=0}^{\infty}\frac{\Gamma(m+11/6)\Gamma(m+7/6)}{\Gamma(11/6)\Gamma(7/6)(m+1)!}\Big[\psi(m+11/6)+\psi(m+7/6)\\
&-\psi(m+2)-\psi(m+1)\Big]\left(\frac{3}{4}\right)^m q^{-3m/2}\frac{(\zeta+\frac{4}{3}q^{3/2})^m}{m!}
\end{aligned}\tag{3.149}$$

$$\tag{3.150}$$

Since the hypergeometric function $F(a,b,c;z)$ is analytic when $\zeta \notin [1,\infty)$, we have that
$F(\frac{11}{6},\frac{7}{6},2;1+\frac{3}{4}q^{-3/2}\zeta)$ is analytic when $\zeta \notin [0,\infty)$ and the series in the last term converges for $\left|\zeta+\frac{4}{3}q^{3/2}\right| < \frac{4}{3}q^{3/2}$. Then denoting by $R(\zeta+\frac{4}{3}q^{3/2})$ the sum of the series term and the analytic term at $\zeta = -\frac{4}{3}q^{3/2}$ in the second term we have

$$\begin{aligned}\hat{\varphi}_r(\zeta) &= -\frac{3q^{-3/2}}{2\pi\cdot 4(1+\frac{3}{4}q^{-3/2}\zeta)}-\frac{5q^{-3/2}}{2\pi\cdot 48}F(\tfrac{11}{6},\tfrac{7}{6},2;1+\tfrac{3}{4}q^{-3/2}\zeta)\log(\zeta+\tfrac{4}{3}q^{3/2})\\
&+R(\zeta+\tfrac{4}{3}q^{3/2})\\
&=\frac{\alpha}{2\pi i(\zeta+\frac{4}{3}q^{3/2})}+\frac{1}{2\pi i}\hat{\Phi}_r(\zeta+\tfrac{4}{3}q^{3/2})\log(\zeta+\tfrac{4}{3}q^{3/2})+R(\zeta+\tfrac{4}{3}q^{3/2})
\end{aligned}\tag{3.151}$$

where the residuum is

$$\alpha = -i \tag{3.152}$$

and the minor is

$$\hat{\Phi}_r(\zeta) := \frac{-5iq^{3/2}}{48}F(\tfrac{11}{6},\tfrac{7}{6},2;\tfrac{3}{4}q^{-3/2}\zeta) \tag{3.153}$$

Then we have the resurgence relation

$$\begin{aligned}
\sigma_{-4/3q^{3/2}}\hat{\varphi}_r(\zeta) &= \alpha\delta + \hat{\Phi}_r \\
&= -i\delta - \frac{5iq^{3/2}}{48}F(\tfrac{11}{6},\tfrac{7}{6},2;\tfrac{3}{4}q^{-3/2}\zeta) \\
&= -i\left(\delta + \frac{5iq^{3/2}}{48}F(\tfrac{11}{6},\tfrac{7}{6},2;\tfrac{3}{4}q^{-3/2}\zeta)\right) \\
&= -i\left(\delta - \hat{\varphi}_r(-\zeta)\right). \qquad (3.154)
\end{aligned}$$

or equivalently in the formal model

$$\sigma_{-4/3q^{3/2}}\varphi_r(\hbar) = -i\left(1 + \varphi_r(-\hbar)\right). \qquad (3.155)$$

where we used

$$\mathcal{B}\varphi_r(-\hbar) = -\hat{\varphi}_r(-\zeta) \qquad (3.156)$$

Setting $\tilde{\varphi}_r(\hbar) := 1 + \varphi_r(\hbar)$ we get

$$\sigma_{-4/3q^{3/2}}\tilde{\varphi}_r(\hbar) = -i\tilde{\varphi}_r(-\hbar). \qquad (3.157)$$

Using equation (3.73):

$$\begin{aligned}
\mathcal{A}_d(q,\hbar) &= \frac{i}{2\sqrt{\pi}}q^{-1/4}\hbar^{1/2}\left(1 + \sum_{m=1}^{\infty} b_m\hbar^m\right) \\
&= \frac{i}{2\sqrt{\pi}}q^{-1/4}\hbar^{1/2}\left(1 + \varphi_d(\hbar)\right) \qquad (3.158)
\end{aligned}$$

we get the relation

$$\varphi_d(x) = \varphi(-x). \qquad (3.159)$$

Thus we get the resurgence relations

$$\sigma_{-4/3q^{3/2}}\tilde{\varphi}_r(\hbar) = -i\tilde{\varphi}_d(\hbar) \qquad (3.160)$$

and

$$\sigma_{4/3q^{3/2}}\tilde{\varphi}_d(\hbar) = -i\tilde{\varphi}_r(\hbar) \qquad (3.161)$$

As we can see the at the Borel singlularity of $\tilde{\varphi}_r$ we find the resurgence of $\tilde{\varphi}_d$ and vice versa. Furthermore the resurgence relations

$$\mathfrak{S}_\pi A_r = A_r + e^{4/3q^{3/2}/\hbar}\sigma_{-4/3q^{3/2}}A_r = A_r - ie^{4/3q^{3/2}/\hbar}A_d \tag{3.162}$$

and

$$\mathfrak{S}_0 A_d = A_d + e^{-4/3q^{3/2}/\hbar}\sigma_{4/3q^{3/2}}A_d = A_d - ie^{-4/3q^{3/2}/\hbar}A_r \tag{3.163}$$

which are exactly the same Stokes phenomena we found before.

The above resurgence relations to show that we would have only needed to know the asymptotic expansion of one solution to the Airy-Schrödinger equation and from its properties in the Borel plane we would find the asymptotic series corresponding to the other solution and the full Stokes phenomena across the Stokes lines. This shows the power of resurgence, one only needs to find one perturbative asymptotic expansion and its Borel analysis gives us the whole Stokes phenomena and the connection formulas.

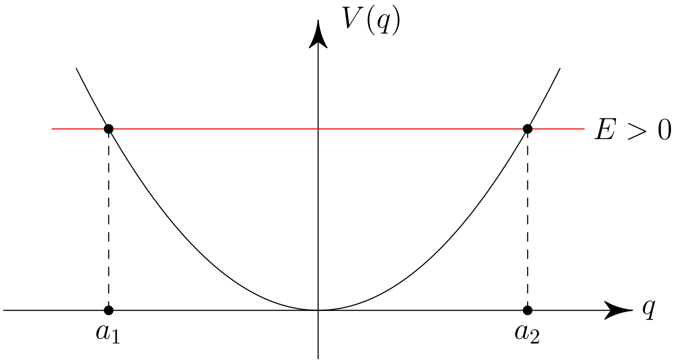

Figure 33. Harmonic oscillator potential with positive energy

4 Exact quantization condition

Resurgence and especially exact WKB is a great tool to solve the exact quantization conditions for spectral problems due to being able capture the non-pertubative tunneling effects to the energy spectrum. The exact quantization condition is given by the Voros symbols 2.30 which are resurgent functions in $\hbar$ and can be written implicitly as

$$f(V_{\gamma_1}, V_{\gamma_2}, V_{\gamma_3}, \dots) = 0 \tag{4.1}$$

4.1 Harmonic oscillator

In this example we study the harmonic oscillator and how the exact WKB formalism leads to the well known exact quantization condition. The harmonic oscillator potential is of the form $V(q) = q^2$ as shown in figure 33. Then the momentum is

$$p(q) = (E - V(q))^{1/2} = (E - q^2)^{1/2} \tag{4.2}$$

and the turning points are $a_1 = -\sqrt{E}$ and $a_2 = \sqrt{E}$. Now consider the case where the energy is postive, $E > 0$. Locally around the turning points the Stokes graph is the Airy type Stokes graph and the Stokes graph of the harmonic oscillator with positive energy is shown figure 34.

To derive the exact quantization condition we analytically continue the wave

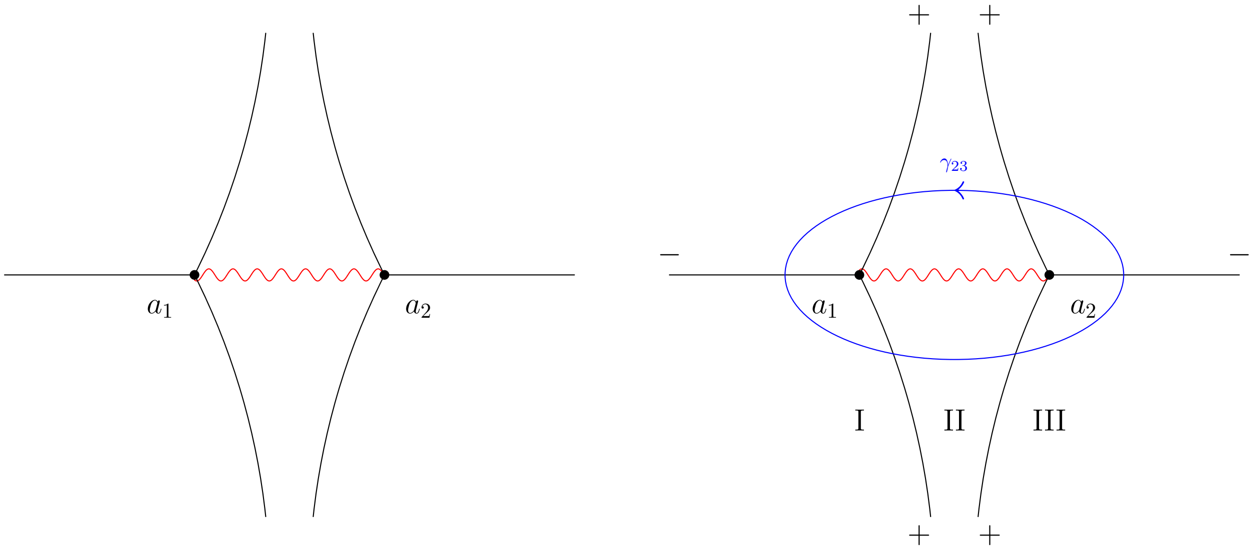

Figure 34. The Stokes graph of the harmonic oscillator with $E > 0$. The red line is a branch cut and blue cycle corresponds to the Voros symbol. The labeling of Stokes lines is given by the boundary condition.

function form the region I to the region III. The connection formula is then

$$\begin{aligned}
\begin{pmatrix}\psi^+_{a_1,\mathrm{I}}\\ \psi^-_{a_1,\mathrm{I}}\end{pmatrix} &= M_+\begin{pmatrix}\psi^+_{a_1,\mathrm{II}}\\ \psi^-_{a_1,\mathrm{II}}\end{pmatrix} = M_+N_{a_1a_2}\begin{pmatrix}\psi^+_{a_2,\mathrm{II}}\\ \psi^-_{a_2,\mathrm{II}}\end{pmatrix} = M_+N_{a_1a_2}M_+\begin{pmatrix}\psi^+_{a_2,\mathrm{III}}\\ \psi^-_{a_2,\mathrm{III}}\end{pmatrix}\\
&= M_+N_{a_1a_2}M_+N_{a_2a_1}\begin{pmatrix}\psi^+_{a_1,\mathrm{III}}\\ \psi^-_{a_1,\mathrm{III}}\end{pmatrix}\\
&= \begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}\begin{pmatrix}V^{1/2}_{\gamma_{12}} & i\\ 0 & V^{-1/2}_{\gamma_{12}}\end{pmatrix}\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}\begin{pmatrix}V^{-1/2}_{\gamma_{12}} & i\\ 0 & V^{1/2}_{\gamma_{12}}\end{pmatrix}\begin{pmatrix}\psi^+_{a_1,\mathrm{III}}\\ \psi^-_{a_1,\mathrm{III}}\end{pmatrix}\\
&= \begin{pmatrix}1 & i(1+V_{\gamma_{12}})\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi^+_{a_1,\mathrm{III}}\\ \psi^-_{a_1,\mathrm{III}}\end{pmatrix}\\
&= \begin{pmatrix}\psi^+_{a_1,\mathrm{III}} + i(1+V_{\gamma_{12}})\psi^-_{a_1,\mathrm{III}}\\ \psi^-_{a_1,\mathrm{III}}\end{pmatrix}
\end{aligned}\tag{4.3}$$

Now issuing the boundary condition that we have a physical solution in the region I we get that $\psi_{\mathrm{I}} \sim \psi^+_{a_1,\mathrm{I}}$ because ψ^- is exponentially growing and ψ^+ exponentially deacying in I. This leads to the labeling of the Stokes lines shown in figure **??** and that the wave function $\psi^-_{a_1,\mathrm{III}}$ is exponentially growing in the region III. To have a physical solution the the region III the coefficient of $\psi^-_{a_1,\mathrm{III}}$ needs to be zero. This gives the exact quantization condition

$$1 + V_{\gamma_{12}} = 0 \tag{4.4}$$

Because $-1 = e^{(2n+1)\pi}$ the quantization condition gives

$$\int_{\gamma_{12}} du\, Q(u,\hbar) = (2n+1)\pi = 2\pi i\left(n + \frac{1}{2}\right) \tag{4.5}$$

Now the integral on the LHS is

$$\int_{\gamma_{12}} du\, Q(u,\hbar) = \frac{i}{\hbar}\int_{\gamma_{12}} du\, p(u) + \int_{\gamma_{12}} du\left(\hbar Q_1 + \hbar^3 Q_3 + \cdots\right) \tag{4.6}$$

Using the recursion relation (2.106) we have

$$Q_{-1} = ip = i\sqrt{E - q^2} \tag{4.7}$$

$$Q_1 = -\frac{i(2E + 3q^2)}{8(E - q^2)^{5/2}} \tag{4.8}$$

Now

$$\int_{\gamma_{12}} du\, \hbar Q_1 = \frac{i\hbar}{8}\int_{-\gamma_{12}} du\, \frac{2E + 3u^2}{(E - u^2)^{5/2}} \tag{4.9}$$

The above integral can be calculated using residues at infinity: with a change of variables $u = 1/t$ we have

$$\mathrm{Res}(f(u), u = \infty) = -\,\mathrm{Res}\left(\frac{1}{t^2} f\left(\frac{1}{t}\right), t = 0\right) \tag{4.10}$$

Now

$$\frac{1}{t^2} f\left(\frac{1}{t}\right) = \frac{2Et^3 + 3t}{(Et^2 - 1)^{5/2}} = (2Et^3 + 3t)(-1)^{-5/2}\left(1 + \frac{5}{2}Et^2 + O(t^4)\right) \tag{4.11}$$

Since the above function is analytic at $t = 0$

$$\mathrm{Res}\left(\frac{1}{t^2} f\left(\frac{1}{t}\right), t = 0\right) = 0 \tag{4.12}$$

and thus

$$\int_{\gamma_{12}} du\, \hbar Q_1 = 0 \tag{4.13}$$

Similar calculations show that the integrals of the higher order terms also vanish. Thus

$$\int_{\gamma_{12}} du\, Q(u,\hbar) = \frac{i}{\hbar}\int_{\gamma_{12}} du\, p(u) = \frac{i}{\hbar}\int_{\gamma_{12}} du\, \sqrt{E-u^2} \tag{4.14}$$

Similarly the residues at infinity give

$$\mathrm{Res}(\sqrt{E-u^2},\infty) = -\,\mathrm{Res}\left(\frac{i}{t^3}\sqrt{1-Et^2},0\right) = i\frac{E}{2} \tag{4.15}$$

and

$$\int_{\gamma_{12}} du\, Q(u,\hbar) = \frac{i}{\hbar}2\pi i^2\frac{E}{2} = -\frac{\pi i}{\hbar}E \tag{4.16}$$

Now from the exact quantization condition (4.5) we get

$$E = -2\hbar\left(n+\frac{1}{2}\right) \tag{4.17}$$

Because $E > 0$ above is only valid for $n = -1, -2, \ldots,$ therefore

$$E_m = 2\hbar\left(m+\frac{1}{2}\right), \quad m = 0, 1, 2, \ldots \tag{4.18}$$

which is the familiar harmonic oscillator quantization.

4.2 The double well

In this example we move onto a little more complicated problem and consider the symmetric double well where the potential is of the form

$$V(q) = (q^2-b)^2, \quad b > 0 \tag{4.19}$$

and that the energy is positive, $E > 0$, as shown in figure 35. Now the turning points are non-degenerate and given by $q = \pm\sqrt{b\pm\sqrt{E}}$. the Stokes graph of the double well can again be constructed using the Airy-type Stokes graphs at the turning points

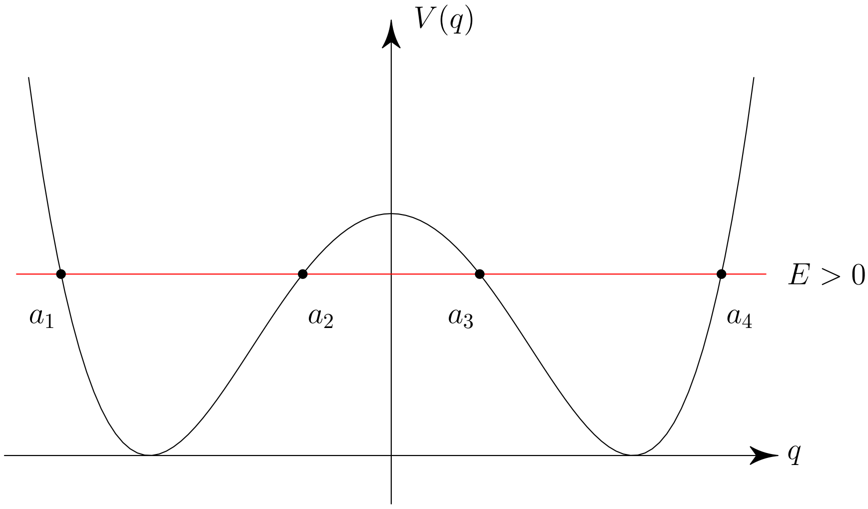

Figure 35. The symmetric double well potential with $E > 0$.

and it's shown in figure 36.

The Stokes line between the turning points a_2 and a_3 corresponding to the tunneling region is a bounded Stokes line. On a bounded Stokes line the Borel summability of the WKB solutions breaks and the connection formula across this Stokes line isn't valid anymore. Furthermore, the Voros symbols $V_{\gamma_{12}}$ and $V_{\gamma_{34}}$ are not Borel summable along the real axis, an imaginary ambiguity appears depending on $\arg\hbar$. The Stokes graph with $\arg\hbar > 0$ is shown in figure 37.

Let's consider the case $\arg\hbar > 0$ and analytic continuation of the wave function from the region I to III as shown in figure 37 with the boundary condition $\psi \sim \psi^-$ in I and the parity condition $\psi(0) = 0$ for an antisymmetric(odd) wave function and $\psi'(0) = 0$ for a symmetric(even) wave function. Then the connection formula is

$$\begin{aligned}\begin{pmatrix}\psi^+_{a_4,\mathrm{I}}\\ \psi^-_{a_4,\mathrm{I}}\end{pmatrix} &= M_- N_{a_4a_3} M_- N_{a_3a_4}\begin{pmatrix}\psi^+_{a_4,\mathrm{III}}\\ \psi^-_{a_4,\mathrm{III}}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ i(1+V_{\gamma_{34}}) & 1\end{pmatrix}\begin{pmatrix}\psi^+_{a_1,\mathrm{III}}\\ \psi^-_{a_1,\mathrm{III}}\end{pmatrix}\\ &= \begin{pmatrix}\psi^+_{a_4,\mathrm{III}}\\ i(1+V_{\gamma_{34}})\psi^+_{a_4,\mathrm{III}} + \psi^-_{a_4,\mathrm{III}}\end{pmatrix}\end{aligned} \tag{4.20}$$

Then the full solution is

$$\psi_{\mathrm{III}} = i(1+V_{\gamma_{34}})\psi^+_{a_4,\mathrm{III}} + \psi^-_{a_4,\mathrm{III}} \tag{4.21}$$

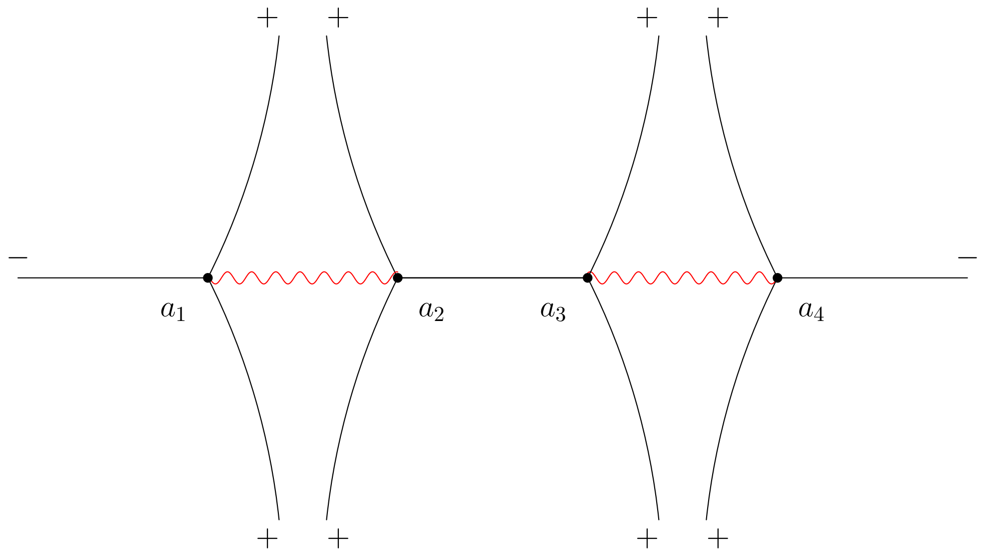

Figure 36. The Stokes graph of the symmetric double well potential with $E > 0$.

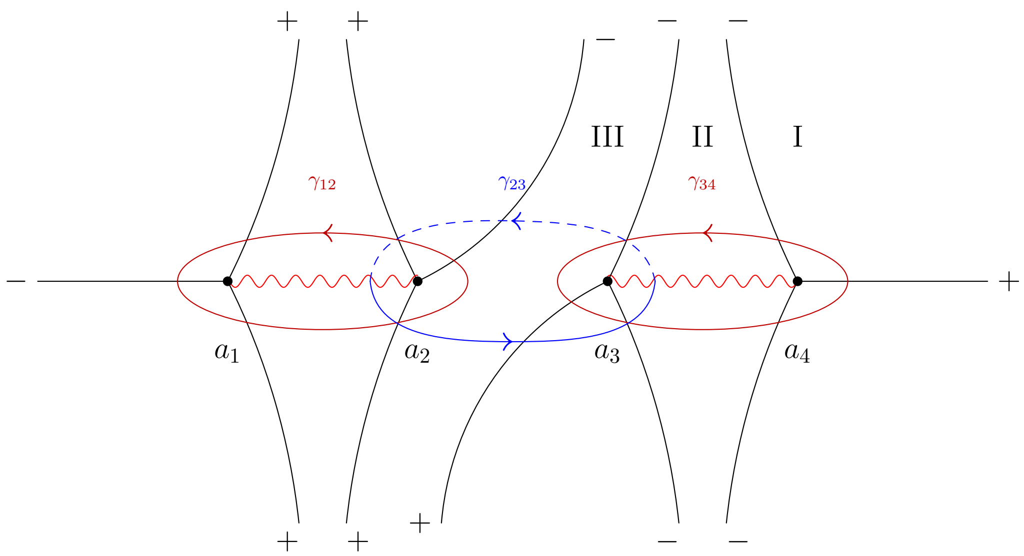

Figure 37. The Stokes graphs of the double well with $\arg \hbar > 0$ and the cycles corresponding to the Voros symbols. The labeling is given by the boundary condition $\psi_{\rm I} \sim \psi^{-}$.

In the antisymmetric case the parity condition gives

$$\begin{aligned}0 &= \psi_{\mathrm{III}}(0)\\ &= i(1+V_{\gamma_{34}})\exp\Big(\int_{a_4}^{0} du\, Q(u,\hbar)\Big) + \exp\Big(-\int_{a_4}^{0} du\, Q(u,\hbar)\Big)\end{aligned} \tag{4.22}$$

Since the momentum doesn't change its determination between a_2 and a_4 the integral in the exponential can be written as

$$\begin{aligned}\int_{a_4}^{0} du\, Q(u,\hbar) &= \Big(\int_{a_4}^{a_3} + \int_{a_3}^{0}\Big) du\, Q(u,\hbar)\\ &= \Big(\int_{a_4}^{a_3} + \frac{1}{2}\int_{a_3}^{a_2}\Big) du\, Q(u,\hbar)\\ &= \Big(\frac{1}{2}\oint_{\gamma_{43}} + \frac{1}{2}\frac{1}{2}\oint_{\gamma_{32}}\Big) du\, Q(u,\hbar)\\ &= -\Big(\frac{1}{2}\oint_{\gamma_{34}} + \frac{1}{4}\oint_{\gamma_{23}}\Big) du\, Q(u,\hbar)\end{aligned} \tag{4.23}$$

where γ_{34} and γ_{23} are the cycles surrounding the turning points a_3, a_4 and a_2, a_3 respectively.

Then the parity condition becomes

$$\begin{aligned}0 &= i(1+V_{\gamma_{34}}) + \exp\Big(-2\int_{a_4}^{0} du\, Q(u,\hbar)\Big)\\ &= i(1+V_{\gamma_{34}}) + \exp\Big(\Big(\oint_{\gamma_{43}} + \frac{1}{2}\oint_{\gamma_{32}}\Big) du\, Q(u,\hbar)\Big)\\ &= i(1+V_{\gamma_{34}}) + V_{\gamma_{34}} V_{\gamma_{23}}^{1/2}\\ \Longleftrightarrow\quad 0 &= 1 + V_{\gamma_{34}}^{-1} - iV_{\gamma_{23}}^{1/2}\\ &= 1 + V_{\gamma_{12}} - iV_{\gamma_{23}}^{1/2}\end{aligned} \tag{4.24}$$

where we used $V_{\gamma_{34}}^{-1} = V_{\gamma_{12}}$.

In the symmetric case $\psi'(0) = 0$ we have

$$\begin{aligned}\psi_{\mathrm{III}}'(0) = {}& i(1+V_{\gamma_{34}})\left[\frac{Q'(0)}{Q^{3/2}(0)} + Q^{1/2}(0)\right]\exp\Big(\int_{a_4}^{0} du\, Q(u,\hbar)\Big)\\ &+ \left[\frac{Q'(0)}{Q^{3/2}(0)} - Q^{1/2}(0)\right]\exp\Big(-\int_{a_4}^{0} du\, Q(u,\hbar)\Big)\end{aligned} \tag{4.25}$$

Since $p'(q) = -\frac{1}{2} - V'(q)\,(E - V(q))^{-1/2}$ and the derivative of the potential vanished at the origin $p'(0) = 0$. Thus $Q'_{-1}(0) = 0$ and using the recursion relation (2.106) we find that $Q'_{2n+1}(0) = 0$ for all $n = 0,1,2,\ldots$. Thus the parity condition reduces to

$$\begin{aligned} 0 &= i(1 + V_{\gamma_{34}}) - \exp\left(-2\int_{a_4}^{0} du\, Q(u,\hbar)\right) \\ &= i(1 + V_{\gamma_{34}}) - V_{\gamma_{34}} V_{\gamma_{23}}^{1/2} \\ \iff 0 &= 1 + V_{\gamma_{34}}^{-1} + iV_{\gamma_{23}}^{1/2} \\ &= 1 + V_{\gamma_{12}} + iV_{\gamma_{23}}^{1/2} \end{aligned} \tag{4.26}$$

Thus the exact quantization condition is

$$1 + V_{\gamma_{12}} - i\eta V_{\gamma_{23}}^{1/2} = 0 \tag{4.27}$$

where $\eta = -1$ in the antisymmetric(odd) case and $\eta = 1$ in the symmetric(even) case.

4.2.1 Energy splitting

The symmetric double well is a great example where ordinary perturbation theory fails and leads to a degenerate bound state which are not possible in one dimension. The tunneling effects, which are non-perturbative(exponential) in nature, gives the required non-degeneracy and these can be found using the exact quantization condition.

We will calculate the energy and the energy splitting in first order (for more detailed discussion see [28]). To make calculations more simple, we set $b = 1$ in the symmetric double well potential, $V(q) = (q^2 - 1)^2$. The exact quantization condition was

$$1 + V_{\gamma_{12}} - i\eta V_{\gamma_{23}}^{1/2} = 0 \tag{4.28}$$

where the cycles γ_{12} and γ_{23} are show in figure 37. The Voros symbol $V_{\gamma_{ij}}$ in leading order is

$$\begin{aligned}
V_{\gamma_{ij}} &= \exp\left(\oint_{\gamma_{ij}} du\, Q(u,\hbar)\right) \\
&= \exp\left(\oint_{\gamma_{ij}} du \left(\frac{i}{\hbar}p(u) + \hbar Q_1(u) + \hbar^3 Q_3(u) + \cdots\right)\right) \\
&= \exp\left(\frac{i}{\hbar}\oint_{\gamma_{ij}} du \left(p(u) + O(\hbar^2)\right)\right) \\
&= \exp\left(\frac{1}{\hbar}S_{\gamma_{ij}}\right) + O(\hbar^2)
\end{aligned} \tag{4.29}$$

The action integrals $S_{\gamma_{12}}$ and $S_{\gamma_{23}}$ can be written as [28](derivation for the first one in Appendix D)

$$S_{\gamma_{12}}(E) = \frac{i\pi}{2}E\,{}_2F_1\left(\frac{1}{4},\frac{3}{4},2;E\right) \tag{4.30}$$

$$S_{\gamma_{23}}(E) = -\frac{\pi}{\sqrt{2}}(1-E)\,{}_2F_1\left(\frac{1}{4},\frac{3}{4},2;1-E\right) \tag{4.31}$$

$$\tag{4.32}$$

where ${}_2F_1$ is the hypergeometric function [13]. Then in leading order $S_{\gamma_{12}}$ is

$$S_{\gamma_{12}}(E) \sim \frac{i\pi}{2}E \tag{4.33}$$

and $S_{\gamma_{23}}$

$$S_{\gamma_{23}}(E) \sim -\frac{\pi}{\sqrt{2}\Gamma(7/4)\Gamma(5/4)} = -\frac{8}{3} \tag{4.34}$$

so the Voros symbols in leading order are

$$V_{\gamma_{12}} \sim \exp\left(\frac{i\pi}{2\hbar}E\right) \tag{4.35}$$

$$V_{\gamma_{23}} \sim \exp\left(-\frac{8}{3\hbar}\right) \tag{4.36}$$

Perturbatively the quantization condition is $1 + V_{\gamma_{12}} = 0$ and it gives

$$2\pi i\left(n + \frac{1}{2}\right) = \frac{1}{\hbar}S_{\gamma_{12}} \sim \frac{i\pi}{2\hbar}E\,{}_2F_1\left(\frac{1}{4},\frac{3}{4},2;E\right) \sim \frac{i\pi}{2\hbar}E \tag{4.37}$$

from which we find

$$E(n,\hbar) \sim 4\hbar\left(n+\frac{1}{2}\right) \tag{4.38}$$

Because there is no dependence on the parity of the wave functions the perturbative ground state is degenerate.

We have to take into account the non-perturbative part $-i\eta V_{\gamma_{23}}^{1/2}$ in the quantization condition to get the tunneling effects on the ground state and therefore the non-degeneracy. We'll make an ansazt for the energy

$$E(n,\hbar) = 4\hbar\left(n+\frac{1}{2}\right)+\varepsilon \tag{4.39}$$

where ε is a small parameter such that the energy deviates from that of the perturbative harmonic energy. Then the Voros symbol $V_{\gamma_{12}}$ can be written as

$$\begin{aligned} V_{\gamma_{12}} &\sim \exp\left(\frac{i\pi}{2\hbar}E\right) = \exp\left(\frac{i\pi}{2\hbar}\left(4\hbar\left(n+\frac{1}{2}\right)+\varepsilon\right)\right) \\ &= \exp\left(2\pi i\left(n+\frac{1}{2}\right)\right)\exp\left(\frac{i\pi}{2\hbar}\varepsilon\right) \\ &= -\exp\left(\frac{i\pi}{2\hbar}\varepsilon\right) \end{aligned} \tag{4.40}$$

Now the exact quantization condition becomes

$$\begin{aligned} 0 &= 1+V_{\gamma_{12}}-i\eta V_{\gamma_{23}}^{1/2} \\ &\sim 1-\exp\left(\frac{i\pi}{2\hbar}\varepsilon\right)-i\eta\exp\left(-\frac{4}{3\hbar}\right) \end{aligned} \tag{4.41}$$

from which we find that

$$\begin{aligned} \varepsilon &= \frac{2\hbar}{i\pi}\log\left(1-i\eta e^{-4/3\hbar}\right) \\ &\sim \frac{2\hbar}{i\pi}\left(-i\eta e^{-4/3\hbar}\right) \\ &= -\frac{2\hbar}{\pi}\eta e^{-4/3\hbar} \end{aligned} \tag{4.42}$$

and that the energy in leading order is

$$E(n,\hbar) \sim 4\hbar\left(n+\frac{1}{2}\right)-\frac{2\hbar}{\pi}\eta e^{-4/3\hbar} \tag{4.43}$$

Now the energy splitting between even($\eta = 1$) and odd($\eta = -1$) states is

$$\Delta E(n,\hbar) = E_{odd}(n,\hbar) - E_{even}(n,\hbar) \sim \frac{2\hbar}{\pi} e^{-4/3\hbar} \tag{4.44}$$

4.2.2 Stokes phenomena

The Stokes phenomena of the Voros symbols of the double well shown in figure 38 are described by the DDP formula 2.31

$$\mathcal{S}^{\theta^-}[V_{\gamma_{12}}] = \mathcal{S}^{\theta^+}[V_{\gamma_{12}}]\left(1 + \mathcal{S}^{\theta^+}[V_{\gamma_{23}}]\right)^{-(\gamma_{23},\gamma_{12})} \tag{4.45}$$

$$\mathcal{S}^{\theta^-}[V_{\gamma_{34}}] = \mathcal{S}^{\theta^+}[V_{\gamma_{34}}]\left(1 + \mathcal{S}^{\theta^+}[V_{\gamma_{23}}]\right)^{-(\gamma_{23},\gamma_{34})} \tag{4.46}$$

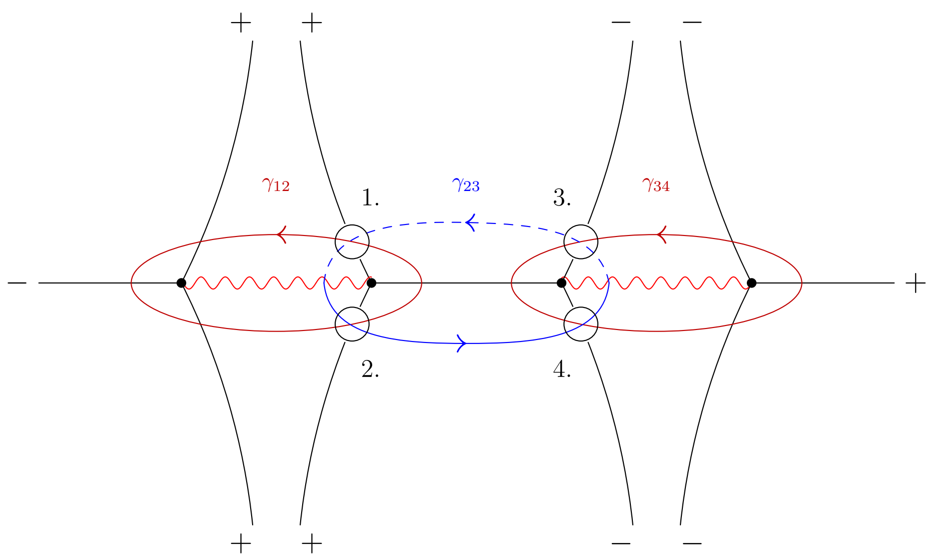

Figure 38. The Stokes graph of the double well with cycles γ_{12},γ_{23} and γ_{34}. The small circles indicate the crossing of cycles.

The intersection numbers are calculated using the rules given in 13 and shown pictorially in figure 39

$$(\gamma_{23},\gamma_{12}) = +1 \tag{4.47}$$

$$(\gamma_{23},\gamma_{34}) = -1 \tag{4.48}$$

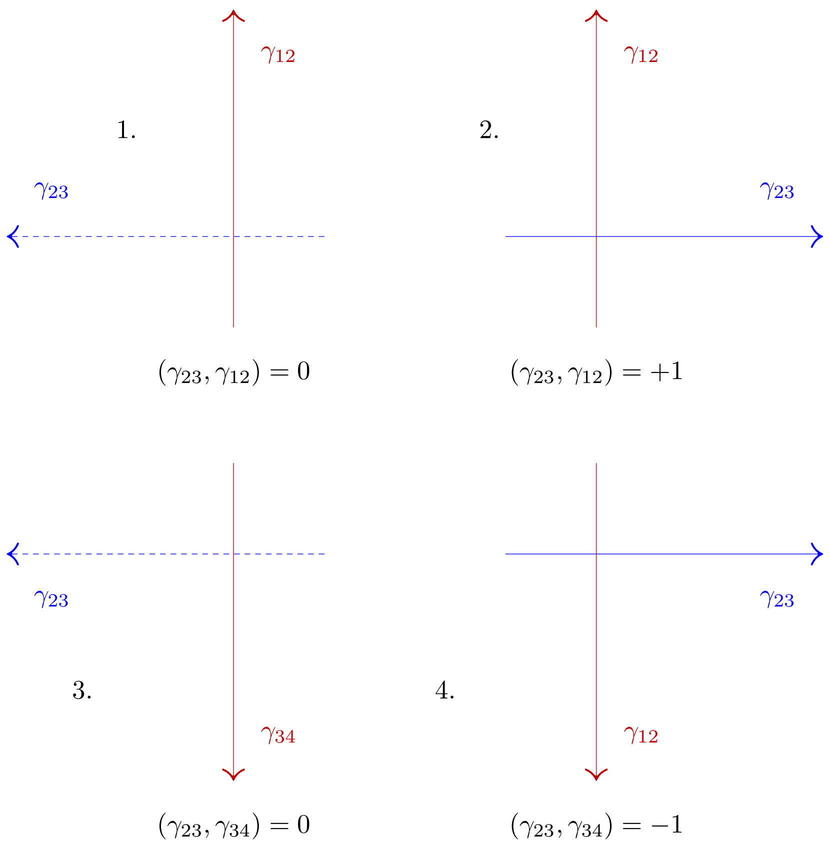

Figure 39. The calculation of intersection numbers of double well cycles pictorially.

Then the DDP formula becomes

$$\mathcal{S}^{\theta^-}\left[V_{\gamma_{12}}\right] = \mathcal{S}^{\theta^+}\left[V_{\gamma_{12}}\right]\left(1 + \mathcal{S}^{\theta^+}\left[V_{\gamma_{23}}\right]\right)^{-1} \tag{4.49}$$

$$\mathcal{S}^{\theta^-}\left[V_{\gamma_{34}}\right] = \mathcal{S}^{\theta^+}\left[V_{\gamma_{34}}\right]\left(1 + \mathcal{S}^{\theta^+}\left[V_{\gamma_{23}}\right]\right)^{+1} \tag{4.50}$$

or using the Stokes automorphism

$$\mathfrak{S}\left[V_{\gamma_{12}}\right] = V_{\gamma_{12}}\left(1 + V_{\gamma_{23}}\right)^{-1} \tag{4.51}$$

$$\mathfrak{S}\left[V_{\gamma_{34}}\right] = V_{\gamma_{34}}\left(1 + V_{\gamma_{23}}\right)^{+1} \tag{4.52}$$

These resurgence relations show that the perturbative expansions of the Voros symbols $V_{\gamma_{12}}$ and $V_{\gamma_{34}}$ are related to the non-perturbative expansion of the Voros symbol $V_{\gamma_{23}}$.

5 Conclusions and outlook

The main goal of this thesis was to learn the basics of resurgent perturbation theory starting with an introductory approach to asymptotics, continuing to basics of the resurgence theory and as an application examine the quantum resurgence of the Airy-type Schröringer equation and the exact quantization conditions arising from the exact WKB analysis. The Airy equation is a standard example in asymptotics and it is also great example in resurgence and a important case in exact WKB.

Why resurgence theory? The resurgence theory gives us a tool to make sense of divergent series and resum them via Borel resummation and makes the theory of asymptotics more rigorous. The Stoke phenomena is fully encoded in the singularities of the Borel plane. Furthermore, via resurgence we can generate the non-perturbative physics from perturbative information. For instance the exact WKB gave us a method to find the non-perturbative level splitting in the symmetric double well, which is not possible via standard perturbation theory. More tools in resurgence which go beyond the scope of this thesis are found using transseries [29] and alien calculus [20, 21, 30] and in exact WKB the case of degenerate turning points [22, 31, 32].

Some of the problems and issues encountered in resurgence theory were that the closure of the algebra of resurgent functions was proved only in cases where the location of Borel singularities were known a priori. Furthermore the resurgence, meaning endless continuability, of the WKB wave functions isn't proved as mentioned in [19, 22, 24]. However Nikolaev mentions in [24] that a proof could be found. More practical problems can be found from the fact the full behaviour of the coefficients of the perturbative expansion is needed to prove asymptocity and the 1-Gevrey class growth condition.

One of the main difficulties writing this thesis was the mathematical complexity of resurgence and the theory being scattered mainly in research papers and many results given without definitive proofs. As such we will end this thesis with a list of recommended references for a further study resurgence theory. For asymptotics the book by Bender and Orszag [4] is a great start. Perhaps the most rigorous works are done by Sauzin and a great, but mathematically rigorous, introduction is found in [21]. More friendly introductions which have more details on alien calculus can be found in [20, 30, 33]. For exact WKB we recommend [19, 25, 32, 34].

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A Asymptotics

A.1 What are asymptotic expansions?

Asymptotic expansions are series expansions that may divergent or convergent. Examples of asymptotic series were discovered in the early 18th century but the theory of asymptotic expansions was started by Stieltjes and Poincaré in late 19th century [16, 35]. The definition given by Poincaré in 1886 was [36]: A divergent series

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \frac{A_n}{x^n} + \cdots \tag{A.1}$$

where the sum of the first $n+1$ terms is S_n, represents asymptotically a function $J(x)$ if

$$x^n(J - S_n) \tag{A.2}$$

tends to 0 if x grows indefinitely. In fact, if x is sufficiently large, we have

$$x^n(J - S_n) < \varepsilon,$$

ε very small. The error

$$J - S_n = \frac{\varepsilon}{x^n}, \tag{A.3}$$

associated with the function J by taking the first $n+1$ terms of the series, is then extremely small.

In this thesis we will use a little modified definition of asymptotic expansions than that of originally given by Poincaré. First, we will define notation describing the relative behaviour of two functions.

Definition A.1. Let $z_0 \in \mathbb{C}$ and $f, g : \mathbb{C} \to \mathbb{C}$ be complex functions, such that $g(z) \neq 0$ in a neighbourhood of z_0, except possibly at z_0. Then we define the following relations:

i) Asymptotically bounded: We say that

$$f(z) = O(g(z)) \quad \text{as} \quad z \to z_0, \tag{A.4}$$

if there exists constants $C \geq 0$ and $\delta > 0$ such that

$$|f(z)| \leq C\,|g(z)| \quad \text{whenever} \quad |z - z_0| < \delta, \tag{A.5}$$

or equivalently if

$$\left|\frac{f(z)}{g(z)}\right| \text{ is bounded.} \tag{A.6}$$

ii) Asymptotically smaller: We say that

$$f(z) = o(g(z)) \quad \text{as} \quad z \to z_0, \tag{A.7}$$

or

$$f(z) \ll g(z) \quad \text{as} \quad z \to z_0, \tag{A.8}$$

if for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(z)| \leq \varepsilon\,|g(z)| \quad \text{whenever} \quad |z - z_0| < \delta, \tag{A.9}$$

Or equivalently if

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0. \tag{A.10}$$

iii) Asymptotic: For $f, g : \mathbb{R} \to \mathbb{R}$ we say that

$$f(x) \sim g(z) \quad \text{as} \quad x \to x_0, \tag{A.11}$$

if the relative error between f and g goes to zero as $x \to x_0$, that is,

$$f(x) - g(x) = o(g(x)) \quad \text{as} \quad x \to x_0, \tag{A.12}$$

or equivalently

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1. \tag{A.13}$$

For example we have

a) $z \sin z = O(z),\ z \to 0$

b) $z = o(z^{-1}),\ z \to 0$

c) $e^x + x \sim e^x,\ x \to +\infty$

d) $x^2 \nsim x, \ x \to 0$

e) $x \nsim 0, \ x \to 0$. Only the zero function can be asymptotic to zero.

If we have any sequence functions where the next term vanishes faster than the previous one, we will call it an asymptotic sequence:

Definition A.2 (Asymptotic sequence)**.** A sequence $\{\phi_0(z), \phi_1(z), \phi_2(z), \dots\}$ of complex functions with a limit point $z_0 \in \mathbb{C} \cup \{\infty\}$ is an asymptotic sequence if there exists neighbourhood U of z_0 such that $\phi_n(z) \neq 0, \ z \in U \setminus \{z_0\}$ and for all n

$$\lim_{z \to z_0} \frac{\phi_{n+1}}{\phi_n} = 0. \tag{A.14}$$

For example the sequence $\{z^n\}$ is an asymptotic sequence for $z_0 = 0$ and the sequence $\{1/z^n\}$ is an asymptotic sequence for $z_0 = \infty$.

Now we can give the definition of an asymptotic expansion. First we give the definition for real valued functions of real variable x. There will be two equivalent definitions. The case for complex valued functions of a complex variable z will be discussed afterwards.

Definition A.3 (Asymptotic expansion)**.** Let $\{\phi_n\}$ be an asymptotic sequence for $x \to x_0$. The formal series $\sum_n a_n \phi_n$ is an asymptotic expansion of $f(x)$ to N as $x \to x_0$ if for all N

$$\lim_{x \to x_0} \frac{f(x) - \sum_{n=0}^{N} a_n \phi_n(x)}{\phi_N(z)} = 0 \tag{A.15}$$

and we write

$$f(x) \sim \sum_{n=0}^{N} a_n \phi_n(x) \quad x \to x_0. \tag{A.16}$$

From the equation (2.18) it can be seen that if an asymptotic expansion for a given asymptotic sequence $\{\phi_n\}$ exists, the coefficients a_n are uniquely determined inductively by

$$a_N = \lim_{x \to x_0} \frac{f(x) - \sum_{n=0}^{N-1} a_n \phi_n(x)}{\phi_N(x)} \tag{A.17}$$

Thus the asymptotic expansion of a function $f(x)$ is unique.

Although the asymptotic expansion of a function $f(x)$ is unique, the asymptotic expansion isn't unique to the function $f(x)$. An asymptotic expansion is asymptotic to class of functions differing by subdominant, or exponentially small, function. This can be seen by the following example:

If a function $f(x)$ has an asymptotic expansion $f(x) \sim \sum a_n x^n,\ x \to 0$, then the function $g(x) = f(x) + e^{-1/x}$ has the same asymptotic expansion $g(z) \sim \sum a_n x^n,\ x \to 0$, since $e^{-1/x}/x^n \to 0,\ x \to 0$, i.e. $e^{-1/x} = o(x^n),\ x \to 0$.

Next we will give and prove an equivalent definition of an asymptotic expansion.

Proposition A.4 (Equivalent asymptotic expansion definition)**.** *Let $\{\phi_n\}$ be an asymptotic sequence for $x \to x_0$. The formal series $\sum_n a_n \phi_n$ is an asymptotic expansion of $f(x)$ to N as $x \to x_0$ if and only if there exists constant $K_N > 0$ such that for all N*

$$\left| f(x) - \sum_{n=0}^{N-1} a_n \phi_n(x) \right| \leq K_N \left| \phi_N(x) \right| \tag{A.18}$$

Proof. Let $f(x)$ have an asymptotic expansion. Then there exist a neighbourhood of x_0 for every $\varepsilon > 0$ such that

$$\begin{aligned} \left| f(x) - \sum_{n=0}^{N-1} a_n \phi_n(x) \right| &= \left| f(x) + \sum_{n=0}^{N} a_n \phi_n(x) + a_N z \phi_N(x) \right| \\ &\leq \underbrace{\left| f(x) + \sum_{n=0}^{N} a_n \phi_n(x) \right|}_{\leq \varepsilon} + \left| a_N \phi_N(x) \right| \\ &\leq \left(|a_N| + \varepsilon \right) |\phi_N| \\ &= K_N \left| \phi_N \right|. \end{aligned} \tag{A.19}$$

Conversely if the equation (**??**) holds then dividing by $|\phi_{N-1}|$ and taking the limit $x \to x_0$ we get

$$\left| \frac{f(x) - \sum_{n=0}^{N-1} a_n \phi_n(x)}{\phi_{N-1}} \right| \leq K_N \left| \frac{\phi_N}{\phi_{N-1}} \right| \longrightarrow 0,\ x \to x_0 \tag{A.20}$$

from which we get the equation (2.18).

$\square$

Remark A.5.

i) The Taylor expansion of a function $f(x)$ with $a_n = f^{(n)}(x_0)/n!$ and $\phi_n(x) = (x - x_0)^n$ fulfills our definition of an asymptotic expansion. Thus the asymptotic expansion can be a convergent expansion.

ii) On the other hand, the Fourier expansion, which is a convergent expansion, is not an asymptotic expansion.

iii) The concept of asymptoticity differs from the concept of convergence such that we are interested in what happens to the partial sum $\sum_{j=0}^{N} a_n\phi_n$ when $x \to x_0$ compared to when $N \to \infty$.

iv) Convergence is a property of the expansion coefficients and convergence can be proved without knowing the function the expansion converges.

v) Asymptoticity is a relative property of the expansion coefficients and the function which the expansion is asymptotic to. Asymptoticity of an expansion doesn't mean anything without knowing the function it is asymptotic to.

Asymptotic expansions on the complex plane

Why do we need a different definition in the complex plane? Let's take a look at the behaviour of the function $\sinh(1/z) = \frac{1}{2}(e^{1/z} - e^{-1/z})$ when $z \to 0$. If $z \in [0, \infty)$, then we can approach 0 only along one path $z \to 0^+$and our definition **??** gives the following asymptotic relation:

$$\sinh(1/z) \sim \frac{1}{2}e^{1/z} \quad \text{as} \quad z \to 0^+. \tag{A.21}$$

Now, on the other hand, if $z \in \mathbb{C}$ then $z \to 0$ means that z can approach 0 along all possible paths in the complex plane - even if our starting point is on the positive real axis. So for the four paths $z = t,\ z = -t,\ z = it$ and $z = -it$ with $t \in [0,\infty)$ our definition gives the following asymptotic relations:

$$\sinh(1/z) \sim \frac{1}{2}e^{1/z} \quad \text{as} \quad z \to 0,\ z = t \tag{A.22}$$

$$\sinh(1/z) \sim -\frac{1}{2}e^{-1/z} \quad \text{as} \quad z \to 0,\ z = -t \tag{A.23}$$

$$\sinh(1/z) \sim -i\sin\left(\frac{1}{t}\right) \quad \text{as} \quad z \to 0,\ z = it \tag{A.24}$$

$$\sinh(1/z) \sim i\sin\left(\frac{1}{t}\right) \quad \text{as} \quad z \to 0,\ z = it. \tag{A.25}$$

This means that the our function $\sinh(1/z)$ doesn't have a unique asymptotic behavior in the complex plane as $z \to 0$.

To make the asymptotic relation unique in the complex plane, we have to restrict the paths along which $z \to z_0$ to a sector of validity $\mathcal{D}(\theta_1, \theta_2)$, as shown in the figure 1, where the angles θ_1 and θ_2 depend on the functions that are asymptotic.

Definition A.6 (Asymptotic relation in the complex plane)**.** Let $z_0 \in \mathbb{C}$ and $f, g : \mathbb{C} \to \mathbb{C}$ be complex functions, such that $g(z) \neq 0$ in a neighbourhood of z_0, except possibly at z_0. Then we say that

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0, \tag{A.26}$$

if

$$f(z) - g(z) = o(g(z)) \quad \text{as} \quad z \to z_0, \tag{A.27}$$

such that $z \to z_0$ along paths that lie in the sector of validity $\mathcal{D}(\theta_1, \theta_2)$ that depend on the functions f and g. An example is shown in figure 1.

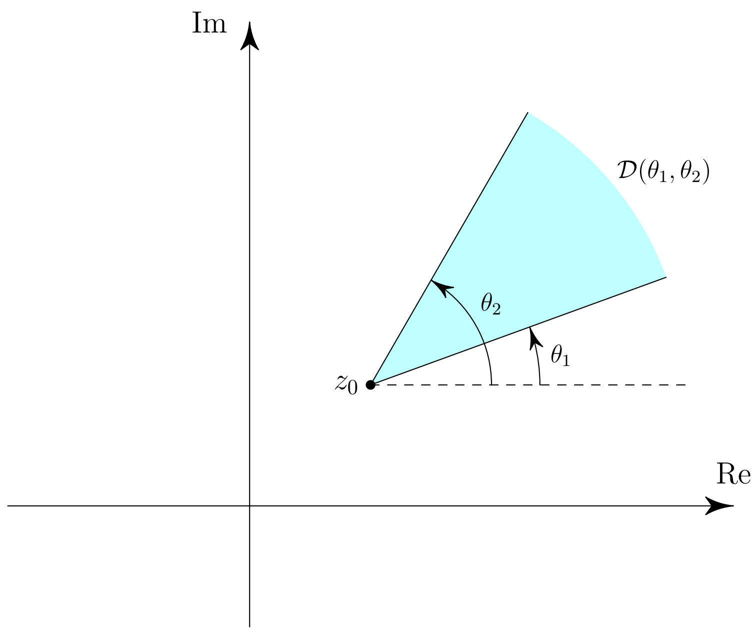

Figure 40. The sector of validity $\mathcal{D}(\theta_1, \theta_2)$, between angles θ_1 and θ_2, where the asymptotic relation is valid. The paths along which $z \to z_0$ must lie in this region.

For our example of $\sinh(1/z)$ the asymptotic relations in the equations (A.22)-

(A.25) are valid in the regions $|\arg z| < \pi/2$, $\pi/2 < \arg z < 3\pi/2$, $\arg z = \pi/2$, $\arg z = -\pi/2$ respectively and the relations are unique in these regions.

Now we are ready to define of the asymptotic expansions in the complex plane:

Definition A.7 (Asymptotic expansion in the complex plane)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be a comlex function and $\{\phi_n\}$ be an asymptotic sequence for $z \to z_0$. The formal series $\sum_n a_n \phi_n$ is an asymptotic expansion of $f(z)$ to N as $z \to z_0$ in the sector of validity $\mathcal{D}(\theta_1,\theta_2)$, as shown in figure 40, if and only if there exists constant $K_N > 0$ such that for all N

$$\left| f(z) - \sum_{n=0}^{N-1} a_n \phi_n(z) \right| \leq K_N \left| \phi_N(z) \right| . \tag{A.28}$$

B Singularities, branches, branch cuts and branch points of functions

B.1 Singularities

Definition B.1 (Singular point of a holomorphic function)**.** Let $f : U \to \mathbb{C}$ be holomorphic in the open set $U \subset \mathbb{C}$. We say that a boundary point $\omega \in \partial U$ is a singular point of f if there doesn't exist a open neighbourhood V of ω, a function g holomorphic in V and a open subset $U' \subset U$ such that $\omega \in \partial U'$ and

$$f_{|U' \cap V} = g_{|U' \cap V}. \tag{B.1}$$

We can further classify singularities into isolated and non-isolated singularities and branch points (section B.4).

Definition B.2 (Isolated singularity)**.** A point $z_0 \in \mathbb{C}$ is an isolated singularity of a function f if f is analytic in the punctured disk $B^*(z_0,r)$ for some $r > 0$.

Definition B.3 (Non-isolated singularity)**.** A point $z_0 \in \mathbb{C}$ is a non-isolated singularity of a function f if z_0 is a singularity of f and for every $r > 0$ there exists another singularity of f in the disk $B^*(z_0,r)$. z_0 is an accumulation point of isolated singularities

Definition B.4 (Types of isolated singularities)**.** Let $z_0 \in \mathbb{C}$ be an isolated singularity of a function f. z_0 is

(i) *a removable singularity* if there exists a function g analytic on $B(z_0,r)$ and $g(z) = f(z)$ for every $z \in B^*(z_0,r)$.
Or if $a_n = 0$ when $n < 0$ in the Laurent series expansion of f

(ii) *a pole* of f if there exists a function g analytic on. $B(z_0,r)$, $g(z_0) \neq 0$ and $n \in \mathbb{N}$ such that $f(z) = \frac{g(z)}{(z-z_0)^n}$ for all $z \in B^*(z_0,r)$. The order of the pole is $\min\left\{n \in \mathbb{N} : f(z) = \frac{g(z)}{(z-z_0)^n}\right\}$.
Or if $a_n \neq 0$ for finitely many $n < 0$ in the Laurent series expansion.

(iii) *an essential singularity* if it is neither a removable singularity nor a pole. Or if $a_n \neq 0$ for infinitely many $n < 0$ in the Laurent series expansion.

Example B.5. i) The point $z = 0$ is a removable singularity of the function $f(z) = \frac{\sin z}{z}$ since

$$\frac{\sin z}{z} = z^{-1}\left(x - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots\right) \tag{B.2}$$

$$= 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots \tag{B.3}$$

ii) The point $\zeta = \omega$ is a third order pole of

$$\frac{1}{(\zeta - \omega)^3}. \tag{B.4}$$

iii) The point $z = 0$ is an essential singularity of the function

$$f(z) = e^{1/z} \tag{B.5}$$

since it is neither a pole or a removable singularity.

B.2 Branches of functions

Let $f : U \to \mathbb{C}$ be holomorphic in the open set $U \subset \mathbb{C}$. If f is not injective, it doesn't have an inverse, that is, there doesn't exist a function g such that $f(g(z)) = z\ \forall z \in U$. An example of such function is the complex exponential function for which the familiar inverse, the logarithm, isn't well-defined on $\mathbb{C}$. But if we were to define the logarithm on a open connected set D such that the exponential function becomes injective, the complex logarithm becomes well defined. A way to do such a restriction is to define D as $D = \{z \in \mathbb{C} : arg(z) \in (\theta, \theta + 2k\pi],\ k \in \mathbb{R}\}$. Then $\exp(\log(z)) = z,\ \forall z \in D$. Noticing that the chosen domain D is not unique (in fact there are uncountable many choices), we'll call the complex logarithm defined on D a branch of the logarithm. This leads to the following definition

Definition B.6 (Branch of a (inverse) function)**.** Let $f : U \to \mathbb{C}$ be holomorphic in the open set $U \subset \mathbb{C}$ and $D \subset f(U)$ be a domain (open and connected). Then a function $g : D \to U$ is a branch of the inverse function f^{-1} if g is continuous on D and $f(g(z)) = z$ for all $z \in D$

B.3 Branch cuts

Let $f(z) = \arg(z)$, that is, the argument of a complex number. Since the complex exponential is $2\pi i$-periodic any complex number z has multiple different representations in polar form, $z = re^{i \arg z}$. In order to have only a single representation of z a common solution is to limit the range of $\arg z$ to an interval $(\theta - 2\pi, \theta]$. But as a consequence, this makes the function $\arg z$ discontinuous along the line $e^{i\theta}[0, \infty)$:

Let $\theta = \pi$, so $\arg z \in (-\pi, \pi]$. Let z be in the negative real axis, $z = -x$ and let's try to approach z along a line parallel to the imaginary axis. Now

$$\lim_{\varepsilon \to 0} \Big(\arg(-x + i\varepsilon) - \arg(-x - i\varepsilon) \Big) = \ \pi - (-\pi) = \ 2\pi \neq 0. \tag{B.6}$$

Thus the argument function is not continuous at any point on the negative real axis. This line of discontinuity is called a branch cut of the function, and we will define it as follows:

Definition B.7 (Branch cut)**.** A branch cut of a holomorphic multi-valued function is a curve in the complex plane across which the function is discontinuous. Most commonly the the branch cut is chosen along the negative real axis and choosing this branch as a principal branch of the function but the branch cut doesn't need to be a straight line.

Equivalently we can say that a branch cut for a multivalued-function f is a curve in the plane on whose complement we can pick a holomorphic branch of f. Thus a branch cut must contain all the branch points

Example B.8. i) The square root function $\sqrt{z}$ and the logarithm $\log z$ can have a branch cut along any direction but the usual choice is the negative real axis and this is called the principal branch.

ii) The function $\sqrt{1 - z^2}$ has a branch cut $[-1,1]$.

B.4 Branch points

Definition B.9 (Branch point)**.** The following are equivalent definitions:

i) A branch point of a multi-valued function f is a point z such that there doesn't exist an open neighbourhood U of z on which the function f has a single-valued branch.

ii) A branch point of a multi-valued function f is a point z such that if the function f is n-valued at that point, all of its neighbourhoods contain a point at which the function has more than $n-$values.

iii) A point z is a branch point of a function f if analytically continuing f along a closed path around the point z leads to a different function.

Let's illustrate these definitions with an examples

Example B.10. i) Let $f(z) = \sqrt{z}$. Then $z = 0$ is a branch point of f: The square root function is single-valued at the origin, $f(0) = 0$. But if we take a neighbourhood of 0, $B(0, \varepsilon)$, then for any $\omega \in B(0, \varepsilon)$, $\sqrt{\omega} = \sqrt{r}e^{i\theta/2} = \sqrt{r}e^{i(\frac{\theta}{2}+n\pi)} = \sqrt{r}e^{i\frac{\theta}{2}}(-1)^n$, $n \in \mathbb{N}$. Thus $\sqrt{z}$ is two-valued at any $\omega \in \mathbb{C} \setminus \{0\}$.

On the other hand if we analytically continue $\sqrt{z}$ along a circle path around $z = 0$ starting at $z = 1$, that is $\arg z$ goes from 0 to 2π, we get $\sqrt{1} = 1 \neq -1 = e^{i\pi} = e^{i\frac{2\pi}{2}} = \sqrt{1}$.

ii) Let $f(z) = \log z$. Taking any branch of the logarithm $\arg z \in (\theta,\, \theta + 2\pi]$, all the neighbourhoods of $z = 0$ contain a point on the branch cut and therefore we can't define single-valued branch of f.

iii) Let $f(z) = e^{-z^{1/k}}$. $z = 0$ is an essential singularity of f and a branch point since analytic continuation around the origin yields a different function.

If we compare the to examples above, we notice that in the case of the square root that going around the origin twice gets us to the same function we started with and with the exponential going around k times gives original function. But with the logarithm we won't ever get to the same function. This leads us to think that there are different types of branch points. Indeed, there are three different classes of branch points, algebraic, transcendental and logarithmic branch points. The branch point of a square root is algebraic, of the logarithm logarithmic and of the exponential transcendental. The transcendental branch point differs from the algebraic because it is also an essential singularity.

C Riemann surfaces

C.1 What are Riemann surfaces?

Let's consider the complex square root function $f(z) = \sqrt{z}$. The point z can be written in polar form as $z = re^{i\theta}$. Because of the periodicity of the complex exponential function the same point can also be written as $z = re^{i\theta+2\pi im}$ for any $m \in \mathbb{Z}$. Now taking the square root of z we get

$$\sqrt{z} = \sqrt{r}e^{i\frac{\theta}{2}} \quad \text{and} \quad \sqrt{z} = \sqrt{r}e^{i\frac{\theta}{2}+\pi im},$$

That is, the square root is multi-valued and therefore not well-defined. This issue lead to the definition of branches of inverse functions.

Another way to make the square root a well-defined function is to replace the complex plane with a new domain which consists of two copies of the complex plane "glued together", such that, $e^{i\theta}$ and $e^{i\theta+2\pi im}$ are now different points on this new domain. This new domain is called the Riemann surface (of the square root) and the two copies are called sheets of the Riemann surface. Now on the Riemann surface, the square root function is single-valued.

C.2 Examples of Riemann surfaces

i) the simplest Riemann surface is the complex plane $\mathbb{C}$

ii) Riemann sphere: compactification of the complex plane $\mathbb{C} \cup \{\infty\}$. The point ∞ can be considered as the north pole of the Riemann sphere

iii) square root: Two sheeted Riemann surface

iv) logarithm: Infinite sheeted Riemann surface

v) torus: The Riemann surface of the double well, $V = (q^2 - 1)^2$ is a torus since $p^2(z) - (E - V(z)) = 0$ is a genus $g = 1$ elliptic curve.

C.3 Abstract Riemann surfaces

Riemann surfaces are two dimensional manifolds with an additional complex structure:

Let X be a two-dimensional manifold (a space that locally looks like the Euclidean space $\mathbb{R}^2$). A *complex chart* on X is the pair (ϕ, U), where the coordinate map $\phi : U \subset X \to V \subset \mathbb{C}$ is a homeomorphism and $\bigcup_i U_i$ is a covering of X. Two complex charts are (ϕ_i, U_i) are *holomorphically compatible* if the transition map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is biholomorphic. A *complex atlas* is the collection of all holomorphically compatible charts, $\{(\phi_i, U_i) : X \subset \bigcup_i U_i\}$. Two complex atlases are called *analytically equivalent* if every chart of the two atlases are holomorphically compatible with each other. Finally, a *complex structure* on X is the equivalence class of analytically equivalent atlases.

Definition C.1 (Riemann surface)**.** A Riemann surface is the pair (X,Σ), where X is a connected two-dimensional manifold and Σ is a complex structure on X.

For more details about abstract Riemann surfaces can be found in [37].

C.4 Riemann surface as a universal covering

Consider the possibly infinite discrete set subset $\Gamma \subset \mathbb{C}$. Then we can define a Riemann surface as a universal covering of $\mathbb{C} \setminus \Gamma$. Consider the set $\mathcal{P}$ of all paths starting from the origin and lying in $\mathbb{C} \setminus \Gamma$ and define an equivalence relation $\sim$ on $\mathcal{P}$ as homotopy with fixed endpoints: $\gamma \sim \gamma_0$ if and only if there exist a continuous function $H : [0,1] \times [0,1] \to \mathbb{C} \setminus \Gamma$ such that

$$H(t,0) = \gamma_0(t), \quad H(t,1) = \gamma(t) \tag{C.1}$$

$$H(0,s) = \gamma_0(0), \quad H(1,s) = \gamma_0(1) \tag{C.2}$$

Then the Riemann surface $\mathcal{R}$ is defined as the set of all equivalence classes

$$\mathcal{R} := \mathcal{P}/\sim \tag{C.3}$$

with the covering map

$$\pi : \mathcal{R} \to \mathbb{C} \setminus \Gamma \tag{C.4}$$

$$\pi(\zeta) = \gamma(1) \tag{C.5}$$

where $\gamma \in \mathcal{P}$ is a representative of the equivalence class $\zeta = [\gamma]$ and the complex structure is the complex structure of $\mathbb{C} \setminus \Gamma$ pulled back by π.

D Double well action integral

In the double well potential $V(q) = (q^2 - 1)^2$ the action in integral is

$$S_{\gamma_{12}} = i \oint_{g_{12}} du\, p(u) = i \oint_{g_{12}} du\, \sqrt{E - (u^2 - 1)^2} \tag{D.1}$$

Expanding the the square root in powers of E it can written as

$$\begin{aligned} \sqrt{E - (u^2 - 1)^2} &= i(u^2 - 1)\left(1 - \frac{E}{(u^2 - 1)}\right) \\ &= i \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(3/2)}{n! \Gamma(3/2 - n)(u^2 - 1)^{2n-1}} E^n \end{aligned} \tag{D.2}$$

Then the action integral is

$$S_{\gamma_{12}} = i^2 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(3/2)}{n! \Gamma(3/2 - n)} E^n \oint_{g_{12}} du\, \frac{1}{(u^2 - 1)^{2n-1}} \tag{D.3}$$

The integrand can be written as

$$\frac{1}{(u^2 - 1)^{2n-1}} = \frac{1}{(u + 1)^{2n-1}(u - 1)^{2n-1}} \tag{D.4}$$

and since $u = -1$ is between the turning points $= -\sqrt{1 \pm \sqrt{E}}$ it is a $(2n - 1)$th order pole for $n \geq 1$. Letting $m = 2n - 1$ the residue at $u = -1$ is

$$\mathrm{Res}(f, -1) = \frac{1}{(m - 1)!} \lim_{u \to -1} \frac{d^{m-1}}{du^{m-1}} \left((u - 1)^{-m}\right) \tag{D.5}$$

and where the derivatives are

$$
\begin{aligned}
&\frac{d^{m-1}}{du^{m-1}}\left((u-1)^{-m}\right) \\
&= \frac{d^{m-2}}{du^{m-2}}\left(-m(u-1)^{-m-1}\right) \\
&= \frac{d^{m-3}}{du^{m-3}}\left((-1)^2 m(m+1)(u-1)^{-m-2}\right) \\
&\vdots \\
&= \frac{d^{m-m}}{du^{m-m}}\left((-1)^{m-1} m(m+1)\cdots(m+(m-2))(u-1)^{-m-(m-1)}\right) \\
&= (-1)^{m-1} m(m+1)\cdots(2m-1)(u-1)^{-2m+1}
\end{aligned}
\tag{D.6}
$$

Using the formula

$$
x^{(n)} = x(x+1)(+2)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(n)} \tag{D.7}
$$

the residue is

$$
\begin{aligned}
\mathrm{Res}(f,-1) &= \frac{1}{(m-1)!}\lim_{u\to -1}\frac{(-1)^{m-1}\Gamma(2m-1)}{\Gamma(m)(u-1)^{2m-1}} \\
&= \frac{\Gamma(4n-3)}{\Gamma(2n-1)^2(-2)^{4n-3}}
\end{aligned}
\tag{D.8}
$$

and the action integral becomes

$$
\begin{aligned}
S_{\gamma_{12}} &= i^2 \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(3/2)}{n!\Gamma(3/2-n)} E^n \frac{-2\pi i \Gamma(4n-3)}{\Gamma(2n-1)^2(-2)^{4n-3}} \\
&= 2\pi i \Gamma(3/2) \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\Gamma(4n+1)E^{n+1}}{(n+1)!\Gamma(1/2-n)\Gamma(2n+1)^2(-2)^{4n+1}} \\
&= \frac{2\pi i \Gamma(3/2) E}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(4n+1) E^n}{(n+1)!\Gamma(1/2-n)\Gamma(2n+1)^2 2^{4n}}
\end{aligned}
\tag{D.9}
$$

Using the Gauss multiplication formula

$$
\prod_{k=0}^{n-1} \Gamma(z+k/n) = (2\pi)^{\frac{1}{2}(n-1)} n^{\frac{1}{2}-nz}\Gamma(nz) \tag{D.10}
$$

we have

$$\Gamma(4n+1) = 4n(2\pi)^{-3/2}4^{4n-1/2}\Gamma(n)\Gamma(n+1/4)\Gamma(n+1/2)\Gamma(n+3/4) \tag{D.11}$$

and

$$\Gamma(2n+1) = 2n(2\pi)^{-1/2}2^{2n-1/2}\Gamma(n)\Gamma(n+1/2) \tag{D.12}$$

and using the Euler reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$

$$\Gamma(1/2-n) = \frac{\pi}{(-1)^n\Gamma(n+1/2)} \tag{D.13}$$

Then the action integral becomes

$$\begin{aligned} S_{\gamma_{12}} &= \pi i\Gamma(3/2)E\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(4n+1)E^n}{(n+1)!\Gamma(1/2-n)\Gamma(2n+1)^2 2^{4n}} \\ &= \pi i\frac{\sqrt{\pi}}{2}E\sum_{n=0}^{\infty}\frac{\Gamma(n+1/4)\Gamma(n+3/4)}{\sqrt{2}\pi^{3/2}n!\Gamma(n+2)}E^n \\ &= \frac{iE}{2\sqrt{2}}\frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(2)}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4},2;E\right) \end{aligned} \tag{D.14}$$

where ${}_2F_1$ is hypergeometric function

$${}_2F_1(a,b,c\,;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(n+a)\Gamma(n+b)}{n!\Gamma(n+c)}z^n \tag{D.15}$$

Finally using $\Gamma(1/4)\Gamma(3/4) = \sqrt{2}\pi$ we get

$$S_{\gamma_{12}} = \frac{iE\pi}{2}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4},2;E\right) \tag{D.16}$$